Holomorphic Flexibility Properties of Spaces of Elliptic Functions

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Dedication

To Grandad, for teaching me to count cars.

Abstract

Let X be an elliptic curve and $\mathbb{P}$ the Riemann sphere. Since X is compact, it is a deep theorem of Douady that the set $\mathcal{O}(X, \mathbb{P})$ consisting of holomorphic maps $X \to \mathbb{P}$ admits a complex structure. If R_n denotes the set of maps of degree n, then Namba has shown for $n \geq 2$ that R_n is a $2n$-dimensional complex manifold. We study holomorphic flexibility properties of the spaces R_2 and R_3. Firstly, we show that R_2 is homogeneous and hence an Oka manifold. Secondly, we present our main theorem, that there is a 6-sheeted branched covering space of R_3 that is an Oka manifold. It follows that R_3 is $\mathbb{C}$-connected and dominable. We show that R_3 is Oka if and only if $\mathbb{P}_2 \backslash C$ is Oka, where C is a cubic curve that is the image of a certain embedding of X into $\mathbb{P}_2$.

We investigate the strong dominability of R_3 and show that if X is not biholomorphic to $\mathbb{C}/\Gamma_0$, where Γ_0 is the hexagonal lattice, then R_3 is strongly dominable.

As a Lie group, X acts freely on R_3 by precomposition by translations. We show that R_3 is holomorphically convex and that the quotient space R_3/X is a Stein manifold.

We construct an alternative 6-sheeted Oka branched covering space of R_3 and prove that it is isomorphic to our first construction in a natural way. This alternative construction gives us an easier way of interpreting the fibres of the branched covering map.