

# Topics in Equivariant Cohomology

Luke Keating Hughes

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School of Mathematical Sciences



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# Contents

<b>Abstract</b>	<b>v</b>
<b>Signed Statement</b>	<b>vii</b>
<b>Acknowledgements</b>	<b>ix</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Classical Equivariant Cohomology</b>	<b>7</b>
2.1 Topological Equivariant Cohomology . . . . .	7
2.1.1 Group Actions . . . . .	7
2.1.2 The Borel Construction . . . . .	9
2.1.3 Principal Bundles and the Classifying Space . . . . .	11
2.2 The Geometry of Principal Bundles . . . . .	20
2.2.1 The Action of a Lie Algebra . . . . .	20
2.2.2 Connections and Curvature . . . . .	21
2.2.3 Basic Differential Forms . . . . .	26
2.3 Equivariant de Rham Theory . . . . .	28
2.3.1 The Weil Algebra . . . . .	28
2.3.2 The Weil Model . . . . .	34
2.3.3 The Chern-Weil Homomorphism . . . . .	35
2.3.4 The Mathai-Quillen Isomorphism . . . . .	36
2.3.5 The Cartan Model . . . . .	37
<b>3 Simplicial Methods</b>	<b>39</b>
3.1 Simplicial and Cosimplicial Objects . . . . .	39
3.1.1 The Simplicial Category . . . . .	39
3.1.2 Cosimplicial Objects . . . . .	41
3.1.3 Simplicial Objects . . . . .	43
3.1.4 The Nerve of a Category . . . . .	47
3.1.5 Geometric Realisation . . . . .	49

3.2	A Simplicial Construction of the Universal Bundle . . . . .	53
3.2.1	Basic Properties of $ N\overline{G}_\bullet $ . . . . .	53
3.2.2	Principal Bundles and Local Trivialisations . . . . .	56
3.2.3	The Homotopy Extension Property and NDR Pairs . . . . .	57
3.2.4	Constructing Local Sections . . . . .	61
<b>4</b>	<b>Simplicial Equivariant de Rham Theory</b>	<b>65</b>
4.1	Dupont's Simplicial de Rham Theorem . . . . .	65
4.1.1	The Double Complex of a Simplicial Space . . . . .	65
4.1.2	Geometric Realisation of a Simplicial Manifold . . . . .	68
4.1.3	Simplicial Differential Forms . . . . .	70
4.2	Simplicial Chern-Weil Theory . . . . .	72
4.2.1	Basic Simplicial Differential Forms . . . . .	72
4.2.2	Simplicial Connections and Curvature . . . . .	74
4.2.3	The Simplicial Chern-Weil Homomorphism . . . . .	76
4.3	An Analogue of the Weil Model . . . . .	80
4.3.1	$G^*$ Algebras . . . . .	80
4.3.2	The Cohomology of a $G^*$ Algebra . . . . .	84
4.3.3	The Cartan Model for $G^*$ Algebras . . . . .	85
4.4	Simplicial Equivariant Cohomology and Cartan's Theorem . . . . .	86
4.4.1	Constructing a $G^*$ Homomorphism . . . . .	86
4.4.2	Cartan's Theorem via Simplicial Differential Forms . . . . .	89
4.4.3	The Weil Model via Simplicial Differential Forms . . . . .	96
	<b>Bibliography</b>	<b>99</b>

# Abstract

The equivariant cohomology of a manifold  $M$  acted upon by a compact Lie group  $G$  is defined to be the singular cohomology groups of the topological space

$$(M \times EG)/G.$$

It is well known that the equivariant cohomology of  $M$  is parametrised by the Cartan model of equivariant differential forms. However, this model has no obvious geometric interpretation – partly because the expression above is not a manifold in general. Work in the 70s by Segal, Bott and Dupont indicated that this space can be constructed as the geometric realisation of a simplicial manifold that is naturally built out of  $M$  and  $G$ . This simplicial manifold carries a complex of so-called simplicial differential forms which gives a much more natural geometric interpretation of differential forms on the topological space  $(M \times EG)/G$ .

This thesis provides a model for the equivariant cohomology of a manifold in terms of this complex of simplicial differential forms. Explicit chain maps are constructed, inducing isomorphisms on cohomology, between this complex of simplicial differential forms and the more standard models of equivariant cohomology, namely the Cartan and Weil models.



# Signed Statement

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