Topics in Equivariant Cohomology

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Contents

Abstract

The equivariant cohomology of a manifold M acted upon by a compact Lie group G is defined to be the singular cohomology groups of the topological space

$$(M \times EG)/G.$$

It is well known that the equivariant cohomology of M is parametrised by the Cartan model of equivariant differential forms. However, this model has no obvious geometric interpretation – partly because the expression above is not a manifold in general. Work in the 70s by Segal, Bott and Dupont indicated that this space can be constructed as the geometric realisation of a simplicial manifold that is naturally built out of M and G. This simplicial manifold carries a complex of so-called simplicial differential forms which gives a much more natural geometric interpretation of differential forms on the topological space $(M \times EG)/G$.

This thesis provides a model for the equivariant cohomology of a manifold in terms of this complex of simplicial differential forms. Explicit chain maps are constructed, inducing isomorphisms on cohomology, between this complex of simplicial differential forms and the more standard models of equivariant cohomology, namely the Cartan and Weil models.

Signed Statement

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Chapter 1

Introduction

For this chapter, let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. The work of Chern and Weil in the 1940s to generalise the Gauss-Bonnet Theorem led to what is now know as the Chern-Weil homomorphism. Given a principal G-bundle $P \to B$, this is a canonical homomorphism

$$\kappa_G : S(\mathfrak{g}^*)^G \to H^*(B)$$

that is constructed from a connection on $P \to B$ but does not depend on that choice of connection. This classical result can also be interpreted in the language of classifying spaces. That is, there is a principal G-bundle, called the universal bundle and denoted

$$EG \to BG, \tag{1.1}$$

for which the base space BG classifies topological principal G-bundles. More specifically, the isomorphism class of principal G-bundles over a manifold M, is uniquely determined by a map into the classifying space BG

$$f : M \to BG$$

up to homotopy. Indeed, from this perspective the Chern-Weil homomorphism tells us that the cohomology of the classifying space is

$$H^*(BG) \cong S(\mathfrak{g}^*)^G. \tag{1.2}$$

Closely related to this is the subject of equivariant cohomology. The equivariant cohomology groups of a manifold M with right G-action were first explicitly defined by Borel in [Bor60] to be the singular cohomology groups

$$H^*\left((M \times EG)/G\right). \tag{1.3}$$

It is well understood, through the work of Cartan in [Car50a] and [Car50b], that the cohomology of this space is parametrised by the complex of 'equivariant differential forms'

$$\left(S(\mathfrak{g}^*) \otimes \Omega^*(M)\right)^G$$

thought of as symmetric polynomials on the vector space $\mathfrak{g}$ taking values in the differential forms of M

$$p : \mathfrak{g} \to \Omega^*(M)$$

that are invariant under the action of G. Bott provides an excellent and more contemporary proof of this fact in his paper [Bot73]. These equivariant differential forms have no obvious geometric interpretation on the space

$$(M \times EG)/G$$

however, partly because it is not a manifold.

We see in the work of Segal in [Seg68], Bott in [Bot73], Bott-Shulman-Stasheff in [BSS76] and Dupont in [Dup75] that the classifying space and the associated bundle $(M \times EG)/G$ can be built out of a *simplicial manifold* and studied via simplicial methods. This simplicial manifold, which is naturally built out of the manifolds M and G, carries a complex of simplicial differential forms, denoted $A^*(NG)$ and $A^*(N\overline{G}_\bullet \times_G M)$ for the classifying space and associated bundle respectively.

More recently, Guillemin and Sternberg have concisely condensed a lot of the work on equivariant de Rham theory in their text [GS99]. In particular, they isolate conditions for a complex W such that

$$(\Omega^*(M) \otimes W)_{bas} \tag{1.4}$$

is a model for the equivariant cohomology of M. In this construction, W plays the role of the differential forms on EG and the expression in (1.4) is effectively the correct algebraic approximation for 'differential forms' on the space $(M \times EG)/G$.

In the monograph [Dup78], Dupont uses the complex of simplicial differential forms $A^*(NG)$ to calculate the cohomology of the classifying space BG. The purpose of this thesis is to extend the work of Dupont to the case of studying the equivariant cohomology of a manifold using the simplicial principal bundle

$$N\overline{G}_\bullet \times M \to N\overline{G}_\bullet \times_G M. \tag{1.5}$$

Namely, we will construct explicit chain maps from the Cartan model for equivariant cohomology and the Weil model for equivariant cohomology to the complex of simplicial differential forms that induce isomorphisms on cohomology. These results are best summarised by the main two theorems that are proven in Chapter 4.

Theorem 4.78: *Let G be a compact Lie group and M be a manifold with right G-action and let $\alpha \otimes \omega \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G$. There is a chain map*

$$\Phi : (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \to A^*(N\overline{G}_\bullet \times M)_{bas}$$

given explicitly by

$$\Phi(\alpha \otimes \omega) = w(\theta_\bullet)(\alpha) \wedge \mathrm{Hor}(\omega_\bullet)$$

that induces an isomorphism on cohomology.

Theorem 4.82: *Let G be a compact Lie group and M be a manifold with right G-action and let $\alpha \otimes \omega \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}$. There is a chain map*

$$\Psi : (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \to A^*(N\overline{G}_\bullet \times M)_{bas}$$

given explicitly by

$$\Psi(\alpha \otimes \omega) = w(\theta_\bullet)(\alpha) \wedge \omega_\bullet$$

that induces an isomorphism on cohomology.

In these two theorems, $w(\theta_\bullet)$ is the Weil homomorphism associated to a connection $\theta_\bullet$ on the simplicial principal bundle in Equation 1.5 and $\omega_\bullet$ is the natural 'simplicial version' of the differential form $\omega \in \Omega^*(M)$. When M is a point, these maps are precisely the chain maps Dupont describes that induce the isomorphism on cohomology groups

$$\Phi = \Psi : S(\mathfrak{g}^*)^G \xrightarrow{\sim} H^*(BG) \tag{1.6}$$

which is the classical isomorphism in Equation 1.2.

Chapter 2 is a piece of exposition on the classical theory of equivariant cohomology. Firstly, the reader is reminded about the topological definition of equivariant cohomology as first formulated by Borel in [Bor60]. Namely, if X is a topological space on which a group G acts, the equivariant cohomology of X is defined to be

$$H^*\left((X \times E)/G\right) \tag{1.7}$$

where E is a contractible space on which G acts freely. The legitimacy of this definition is then further explored so that it is clear that the definition of equivariant cohomology doesn't depend on the choice of E. In fact, this leads to the construction of the universal bundle

$$EG \to BG$$

which is a principal G-bundle. The differential geometry of smooth principal bundles will also play a role in later chapters. One of our goals will be to build the bundle $EG \to BG$ out of a simplicial manifold and another will be to describe the Chern-Weil homomorphism using this construction. As such, a discussion on the geometry of principal bundles naturally follows the discussion on the universal bundle. The last part of this chapter is used to describe the Weil algebra and develop some of the classical results from equivariant cohomology including a description of the Weil model, Cartan model and the Mathai-Quillen isomorphism from [MQ86]. These two models are both fundamental in this thesis and must be reasonably well understood so that we can construct the explicit chain maps described in the main theorems above, Theorem 4.78 and Theorem 4.82.

The main purpose of Chapter 3 is to construct the universal bundle $EG \to BG$ using manifolds so that we may approximate a de Rham complex for the topological space

$$(M \times EG)/G.$$

The obstruction is that EG is not a manifold in general so we instead define the complex of simplicial differential forms on the simplicial manifold

$$N\overline{G}_\bullet \times_G M$$

and show that by taking geometric realisation of this simplicial manifold one has

$$|N\overline{G}_\bullet \times_G M| = (EG \times M)/G. \tag{1.8}$$

The first half of this chapter is dedicated to building the correct language for this construction and introducing some other important simplicial and cosimplicial objects along the way. Most importantly, the simplicial principal bundle

$$N\overline{G}_\bullet \to NG_\bullet$$

is defined and the last half of the chapter is reserved for checking that the respective map under geometric realisation

$$|N\overline{G}_\bullet| \to |NG_\bullet|$$

corresponds to the projection map of the universal bundle. As Segal remarks upon in [Seg68], the construction of the simplicial principal G-bundle in this way is closely related to the classical Milnor construction of the universal bundle (see [Mil56a] and [Mil56b]). The results naturally extend to showing that Equation 1.8 holds using results of May (see [May72]).

The final chapter of this thesis deals with constructing an approximation of differential forms on the topological space

$$(EG \times M)/G$$

by using the complex of basic simplicial differential forms $A^*(N\overline{G} \times M)_{bas}$. The complex of simplicial differential forms $A^*(M_\bullet)$ was first introduced by Whitney (see [Whi57, Chapter IX]) and used extensively by Dupont to study the Chern-Weil homomorphism in [Dup75] and [Dup78]. Using the fact that $N\overline{G}_\bullet \to NG_\bullet$ is a simplicial principal bundle, Dupont defines simplicial connection and curvature forms such that there is a natural Chern-Weil homomorphism. Moreover, we construct a chain map

$$\Phi : (S(\mathfrak{g}^*) \otimes \Omega^*(M))_{bas} \to A^*(N\overline{G}_\bullet \times_G M)$$

which arises naturally from the associated principal bundle

$$N\overline{G}_\bullet \times M \to N\overline{G}_\bullet \times_G M.$$

The complex of simplicial differential forms on a simplicial manifold $A^*(M_\bullet)$ is a computational powerhouse for the calculations performed in this chapter and we state some well known results to establish the sequence of isomorphisms on cohomology

$$H^*(|M_\bullet|) \cong H^*(\|M_\bullet\|) \cong H^*_{\mathrm{Tot}}(C^{*,*}(M_\bullet)) \cong H^*_{\mathrm{Tot}}(\Omega^{*,*}(M_\bullet)) \cong H^*(A^*(M_\bullet))$$

using the technology of Segal, Dupont and Palais. It is then shown that the complex $A^*(N\overline{G}_\bullet \times M)_{bas}$ is a natural model for the equivariant cohomology of a manifold M with right G-action. Similar work has also been done by Bott, Shulman and Stasheff in [BSS76] but focused on the de Rham double complex associated to a simplicial manifold $\Omega^{*,*}(M_\bullet)$. The work of Guillemin and Sternberg in [GS99] establishes the classical results of Cartan (from [Car50a] and [Car50b]) for a more general G^* algebra. It is shown that the complex $A^*(N\overline{G}_\bullet \times M)$ is a G^* algebra and the complex

$$(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$$

computes the equivariant cohomology of a manifold M acted upon by a compact Lie group G. This all culminates in the construction of a chain map

$$\Phi : (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \to A^*(N\overline{G}_\bullet \times_G M)$$

from Theorem 4.78 which is proven to be an isomorphism on cohomology. Moreover, we see that there is an even more natural chain map

$$\Psi : (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \to A^*(N\overline{G}_\bullet \times_G M)$$

that induces an isomorphism on cohomology given by the much more obvious identification

$$\alpha \otimes \omega \mapsto w(\theta_\bullet)(\alpha) \wedge \omega_\bullet$$

as described above in Theorem 4.82. These chain maps establish alternate proofs of Cartan's theorems as well as describing a simple way to calculate these simplicial differential forms from the classical equivariant differential forms.

Chapter 2

Classical Equivariant Cohomology

2.1 Topological Equivariant Cohomology

2.1.1 Group Actions

Let G be a topological group and X be a topological space. We say that G *acts on the right of* X, or X *has a right G-action*, if there is a map

$$\phi : X \times G \to X$$

such that $\phi(x, 1) = x$ and $\phi(\phi(x, g), h) = \phi(x, gh)$ for every $x \in X$, $g, h \in G$. These conditions can be described by requiring that the following two diagrams commute

$$\begin{tikzcd} X \times G \times G \arrow[r, "\mathrm{id}_X \times m"] \arrow[d, "\phi \times \mathrm{id}_G"'] & X \times G \arrow[d, "\phi"] \\ X \times G \arrow[r, "\phi"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "i_1"] \arrow[dr, "\mathrm{id}_X"'] & X \times G \arrow[d, "\phi"] \\ & X \end{tikzcd}$$

where m is the group multiplication and $i_1 : X \to X \times G$ is the map $x \mapsto (x, 1)$. The notation $\phi_g := \phi(-, g)$ and $x \cdot g = \phi(x, g)$ is more natural in some circumstances and will be used interchangeably. For example, the conditions for G to act on X may be restated as $\phi_1 = \mathrm{id}_x$ and $\phi_h \circ \phi_g = \phi_{gh}$ or $x \cdot 1 = x$ and $(x \cdot g) \cdot h = x \cdot (gh)$.

Similarly, one may define a *left G-action* by instead imposing that $\phi_h \circ \phi_g = \phi_{hg}$ for every $g, h \in G$. For a left G-action we will adopt the notation $g \cdot x = \phi(x, g)$ for which we may restate this condition as $h \cdot (g \cdot x) = (hg) \cdot x$.

Once stated diagrammatically, the definition of a group action can be easily understood for other mathematical objects, X. For example, if X is a smooth manifold, then the diagrams above make sense if G is a Lie group and the action $\phi : X \times G \to X$ is a smooth map.

By letting $Aut(S)$ denote the group of automorphisms of a set S – in this case, the bijections from S to itself – we note that a group action is equivalent to a homomorphism from G to $Aut(S)$.

Proposition 2.1: *Let G be a group and S be a set. Then a G-action on S is equivalent to a group homomorphism*

$$\psi : G \to Aut(S).$$

Proof. Let S have a left G-action with corresponding automorphisms ϕ_g for each $g \in G$. Then define $\psi : G \to Aut(S)$ by

$$\psi(g) = \phi_g.$$

Note that $\psi(g^{-1}) = \phi_{g^{-1}} = \phi_g^{-1}$ since $\phi_g \circ \phi_{g^{-1}}$ is the identity map and $\psi(gh) = \phi_{gh} = \phi_g \circ \phi_h = \psi(g)\psi(h)$ as required.

If G acts on the right, the induced homomorphism is slightly different. Let S have a right G-action with corresponding automorphisms φ_g for each $g \in G$. Then define $\psi : G \to Aut(S)$ by

$$\psi(g) = \varphi_{g^{-1}}.$$

In this case, $\psi(g^{-1}) = \varphi_g = (\varphi_{g^{-1}})^{-1}$ and $\psi(gh) = \varphi_{(gh)^{-1}} = \varphi_{h^{-1}g^{-1}} = \varphi_{g^{-1}} \circ \varphi_{h^{-1}} = \psi(g)\psi(h)$. So if S has a left or right G-action, it determines a homomorphism $\psi : G \to Aut(S)$.

Let $\psi : G \to Aut(S)$ be a group homomorphism. It is clear that $\psi(1) = \mathrm{id}_S$ and that for any group elements g, h, $\psi(gh) = \psi(g)\psi(h)$. So the homomorphism ψ naturally defines a left G action on S. □

For a right G-action, the *orbit* of a point $x \in X$ is the subspace of points

$$x \cdot G := \{\phi_g(x) : g \in G\}$$

and similarly, the orbit of a point under a left G-action is denoted $G \cdot x$. Similarly, the stabiliser group G_x for a point $x \in X$ is the subgroup of G that fixes x,

$$G_x = \{g \in G : \phi_g(x) = x\}.$$

We call the action of G on a space X *free* if the stabiliser group is trivial at every point $x \in X$.

The product of two spaces that admit a G-action also carries a natural G-action. If X, Y are topological spaces with a right G-action then we define the *diagonal* action of $g \in G$ on a point $(x, y) \in X \times Y$ to be

$$(x, y) \cdot g = (x \cdot g, y \cdot g).$$

Unless otherwise stated, it will be assumed that the product of two spaces with G-action is endowed with the diagonal action of G.

Proposition 2.2: *Let G be a topological group and X, Y be topological spaces with a G-action. If X has a free G-action, then the diagonal action of G on $X \times Y$ is also free.*

Proof. Let $(x, y) \in X \times Y$ and $g \in G$. Then the diagonal action $(x, y) \cdot g = (x \cdot g, y \cdot g) = (x, y)$ if and only if $x \cdot g = x$ and $y \cdot g = y$. Since X has a free G-action, this is only true when g is the identity and hence the diagonal action is also free. $\square$

One can consider the set of equivalence classes of orbits $X/G := \{x \cdot G : x \in X\}$. The set of orbits is endowed with the coarsest topology such that the natural projection $X \xrightarrow{q} X/G$ is continuous and the resultant topological space is called the *quotient space*.

Definition 2.3: *Let X, Y be topological spaces with a right G-action. A map $f : X \to Y$ is said to be* equivariant *if*

$$f(x \cdot g) = f(x) \cdot g$$

for every $x \in X$ and $g \in G$.

Remark 2.4: Note that an equivariant map defines a unique map $g : X/G \to Y/G$ such that the diagram below commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X/G \arrow[r, dashed, "g"] & Y/G \end{tikzcd}$$

Remark 2.5: Let X be a topological space with right G-action. If Y is any topological space, it can be endowed with a trivial G-action. An equivariant map $f : X \to Y$ is thus a map where

$$f(x \cdot g) = f(x) \cdot g = f(x)$$

for every $x \in X$. To avoid explicitly stating that Y has been endowed with a trivial G-action, we will instead refer to maps $f : X \to Y$ such that $f(x \cdot g) = f(x)$ as a *G-invariant maps.*

2.1.2 The Borel Construction

When considering a topological group G and a space X with G-action, one might be tempted to study the cohomology of the quotient space X/G. Each point in the space X/G corresponds to an orbit $x \cdot G$ which, so long as the action is free, is homeomorphic to G.

If the action is not free, then the orbit of a point $x \in X$ is homeomorphic to a different space, namely the quotient space G/G_x. In this case, two points in X/G do not always give us the same topological information about the action of G. For this reason, we look for a different topological space to study – one that captures the group action on X but also admits a free G-action. Proposition 2.2 shows that by first finding a topological space Y on which G acts freely, the space $X \times Y$ will admit a free action. In this case, a point in $(X \times Y)/G$ will correspond to an orbit that is homeomorphic to G. So that $X \times Y$ is the same homotopy type as X, it is desirable that the space Y is also contractible.

Showing that there exists a contractible space on which a group G acts freely is a classical result but nonetheless, non-trivial. We will take it on faith that such a space exists in order to exploit its topological properties and return to proving this fact later in Section 3.2.

To motivate Section 2.1.3 we will state a precise definition, known as the *Borel construction*, and sketch a result which uses some terminology that has not been defined; namely the terms *principal bundle*, *numerable* principal bundle and *fibre bundle* (see Definitions 2.8, 2.22, 2.10 respectively).

Definition 2.6: *Let G be a topological group and X be a topological space with right G-action. Then we define the* equivariant cohomology *of X as*

$$H_G^*(X) := H^*((X \times E)/G)$$

where E is contractible and the total space of a numerable principal bundle.

From Borel [Bor60], we also have the following theorem showing that the equivariant cohomology of a topological space on which G acts freely is the singular cohomology of the orbit space X/G.

Theorem 2.7: *Let G be a group and X be a space with free right G-action. Then*

$$H_G^*(X) = H^*(X/G).$$

Sketch proof. The equivariant projection $X \times E \to X$ induces a projection

$$(X \times E)/G \to X/G$$

which is a fibre bundle with fibre E. In this situation, standard results from algebraic topology show the existence of a long exact sequence of homotopy groups

$$\cdots \to \pi_i(E) \to \pi_i\left((X \times E)/G\right) \to \pi_i(X/G) \to \pi_{i-1}(E) \to \cdots$$

for $i > 0$. Since E is a contractible space, $\pi_i(E) = \{0\}$ for every $i > 0$ and thus $\pi_i\left((X \times E)/G\right) \cong \pi_i(X/G)$ for every $i \geq 0$. (The case for $i = 0$ follows from the fact that $X \times E$ and X have the same number of connected components since E is contractible.) Since there is a weak homotopy equivalence between $(X \times E)/G$ and X/G it follows that

$$H_G^*(X) = H^*\left((X \times E)/G\right) = H^*(X/G).$$

□

2.1.3 Principal Bundles and the Classifying Space

The following section will develop the theory of principal bundles as well as justify some of the statements from Section 2.1.2.

Definition 2.8: *Let G be a Lie group, M be a manifold and P be a manifold with right G action. Then a* (smooth) principal G-bundle *is a triple (P, π, M), or $P \xrightarrow{\pi} M$, where the map $\pi : P \to M$ is smooth, surjective and meets the following conditions.*

1. *For every $x \in M$, $\pi^{-1}(x)$ is an orbit.*

2. *Every point $x \in M$ has an open neighbourhood U and a smooth equivariant diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ such that the diagram*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

commutes, where p_U is projection onto U. The pair (U, φ) is called a local trivialisation.

Remark 2.9: The most simple example of a principal G-bundle is the *trivial bundle* $M \times G \xrightarrow{\pi} M$ where π is projection onto M.

A *topological* principal G-bundle is the topological analogue, by letting G be a topological group, P, M topological spaces and

$$\pi : P \to M$$

a surjective map such that the local trivialisations are homeomorphisms. Smooth principal G-bundles will primarily be the examples of study and so the term 'principal G-bundle' will mostly denote a smooth principal G-bundle. In either case, we say that M is the *base space*, P is the *total space*, π is the *projection* and G is the *structure group* of the principal G-bundle. More generally, there is the notion of a *fibre bundle.*

Definition 2.10: *Let F, P, X be topological spaces. Then a* fibre bundle *is a quadruple (F, P, π, X) where the map $\pi : P \to X$ is surjective and meets the following conditions.*

1. *For every $x \in X$, $\pi^{-1}(x)$ is homeomorphic to F.*

2. *Every point $x \in X$ has an open neighbourhood U and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the diagram*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_U"] \\
& U &
\end{tikzcd}$$

commutes, where p_U is projection onto U. The pair (U, φ) is called a local trivialisation.

Similarly, when the spaces F, P, X are smooth manifolds, the projection map π is smooth, and all local trivialisations are smooth diffeomorphisms, we call this construction a *smooth fibre bundle*. Fibre bundles will not be extensively studied in this thesis and will mainly be talked about in the context of what is normally called *the associated fibre bundle*.

Proposition 2.11: *Let $P \xrightarrow{\pi} M$ be a principal G-bundles and N a topological space with right G-action. Then the map $\tau : (P \times N)/G \to M$ defined by*

$$[p, n] \mapsto \pi(p)$$

is a fibre bundle with fibre N. We call this the associated fibre bundle.

Proof. First, consider the following commutative diagram

$$\begin{tikzcd}
P \times N \arrow[r, "p_1"] \arrow[d, "\rho"] & P \arrow[d, "\pi"] \\
(P \times N)/G \arrow[r, "\tau"] & M
\end{tikzcd}$$

where p_1 is the projection onto P and ρ is the quotient by the action of G. A local trivialisation of the principal bundle $P \to M$, say (U, φ), yields a homeomorphism

$$(\pi \circ p_1)^{-1}(U) \xrightarrow{\sim} U \times G \times N.$$

Thus,

$$\tau^{-1}(U) = \rho \circ (\pi \circ p_1)^{-1}(U) \cong U \times N$$

defines local trivialisations (U, ψ) of $\tau : (P \times N)/G \to M$. $\square$

Let $P \xrightarrow{\pi} M$, $Q \xrightarrow{\tau} N$ be topological principal G-bundles. The natural definition of a morphism between principal G-bundles would be a pair of maps $(f, \tilde{f})$ such that the diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "\pi"'] & Q \arrow[d, "\tau"] \\
M \arrow[r, "\tilde{f}"] & N
\end{tikzcd}$$

commutes. Since the quotient P/G is isomorphic to the base space M, an equivariant map $f : P \to Q$ induces a unique map $\tilde{f} : M \to N$ by Remark 2.4. Since $\tilde{f}$ is uniquely determined by f, a *map of principal G-bundles* (or *bundle map*) can thus be defined as an equivariant map between total spaces $f : P \to Q$. We say that f is an isomorphism if there exists a bundle map $g : Q \to P$ such that $g \circ f$ and $f \circ g$ are the respective identity maps on P and Q. A principal bundle is *trivial* if it is isomorphic to the trivial bundle.

Proposition 2.12: *Let $P \xrightarrow{\pi} M$, $Q \xrightarrow{\tau} M$ be smooth principal G-bundles. Let $f : P \to Q$ be a bundle map such that the induced map on base spaces*

$$\tilde{f} : M \to M$$

is the identity. Then f is an isomorphism.

Proof. Let $U \subset M$ be a local trivialisation of $x \in M$. From the definition of a principal bundle, there are diagrams

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi_P"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "p_U"] \\
& U &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\tau^{-1}(U) \arrow[rr, "\varphi_Q"] \arrow[dr, "\tau"'] & & U \times G \arrow[dl, "p_U"] \\
& U &
\end{tikzcd}$$

that commute. The problem reduces to showing that when f is restricted to $\pi^{-1}(U)$, $\varphi_Q \circ f \circ \varphi_P^{-1}$ is an isomorphism for all local trivialisations U. For ease of notation, this map will be denoted

$$f_\varphi = \varphi_Q \circ f \circ \varphi_P^{-1}$$

and the rest of the proof will amount to showing that f_φ is an isomorphism. Since f induces the identity on base spaces, f_φ is an equivariant map such that

$$f_\varphi(u, g) = (u, g\,\sigma(u))$$

for every $u \in U$ and some smooth map $\sigma : U \to G$. It is easy enough to construct an inverse of f_φ by letting

$$f_\varphi^{-1}(u, g) = (u, g\,\sigma(u)^{-1})$$

completing the proof. $\square$

Corollary 2.13: *Let $P \xrightarrow{\pi} X$ be a principal G-bundle. Then $P \cong X \times G$ if and only if $P \xrightarrow{\pi} X$ admits a global section.*

Proof. Suppose $P \xrightarrow{\pi} X$ admits a global section, $s : X \to P$. Consider the map $F : X \times G \to P$

$$F(x, g) = s(p) \cdot g.$$

F is a bundle map that induces the identity on base spaces, hence an isomorphism. Conversely, a global section $s : X \to X \times G$ can easily be constructed via $s(x) = (x, 1)$. $\square$

Corollary 2.14: *Let $P \xrightarrow{\pi} X$ be a principal G-bundle with local trivialisations $\{(U_i, \varphi_i)\}_{i \in I}$. Then there are local sections of the principal bundle*

$$s_i : U_i \to \pi^{-1}(U_i).$$

Proof. Each local trivialisation defines a trivial principal bundle

$$\pi^{-1}(U_i) \xrightarrow{\pi} U_i$$

and the result clearly follows. $\square$

Let $P \xrightarrow{\pi} B$ be a principal G-bundle and X be a topological space. For every map

$$f : X \to B$$

there is a space $f^*(P) := \{(x, p) : f(x) = \pi(p)\} \subset X \times P$ with natural projections onto each factor that cause the diagram

$$\begin{tikzcd} f^*(P) \arrow[r] \arrow[d, "p_X"] & P \arrow[d, "\pi"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

to commute. This carries a right G-action

$$\phi((x, p), g) = (x, p \cdot g)$$

whence we note that for every $x \in X$, $p_X^{-1}(x)$ is clearly an orbit of $f^*(P)$. Furthermore, the map $f : X \to B$ induces a principal G-bundle structure on $f^*(P) \to X$ since a local section of $P \to B$, say $s : U \to \pi^{-1}(U)$, defines a local section of $f^*(P) \to X$, $\tilde{s} : f^{-1}(U) \to (f \circ p_X)^{-1}(U)$ by

$$\tilde{s}(x) = (x, f \circ s(x))$$

which in turn defines local trivialisations by Corollary 2.14. Thus we have proved the following proposition.

Proposition 2.15: *Let $P \xrightarrow{\pi} B$ be a principal G-bundle, X be a topological space and $f : B \to X$ a map. Then the canonical map*

$$f^*(P) \to X$$

is the projection map in a principal G-bundle for the action of G given above. The bundle $f^*(P) \to X$ is called the pullback bundle *of $P \xrightarrow{\pi} B$ along f.*

Remark 2.16: If f is a smooth map then the pullback bundle is a smooth principal G-bundle.

Proposition 2.17: *Let $P \xrightarrow{\pi} M$ be a principal G-bundle and F a topological space with right G-action. To every equivariant map $f : P \to F$ we may associate a section s of the natural projection*

$$(P \times F)/G \to M.$$

Moreover, this correspondence between equivariant maps $f : P \to F$ and sections $s : M \to (P \times F)/G$ is a bijection.

Proof. Let $f : P \to F$ be an equivariant map. Clearly this can be extended to an equivariant map $f' : P \to P \times F$ by

$$f'(p) = (p, f(p))$$

where G acts diagonally on $P \times F$. Then this defines a map on quotient spaces $\tilde{f} : P/G \to (P \times F)/G$ given by

$$\tilde{f}[p] = [p, f(p)].$$

But $P/G \cong M$ so the map $\tilde{f} : M \to (P \times F)/G$ given by

$$\tilde{f}(m) = [p, f(p)]$$

for some $p \in \pi^{-1}(m)$ is well defined. Suppose $f' : P \to F$ is equivariant and $\tilde{f}' = \tilde{f}$. That is, for every $p \in P$

$$(p, f(p)) = (p \cdot g, f'(p) \cdot g)$$

for some $g \in G$. Since G acts freely on P, g is necessarily the identity and thus $f' = f$. Alternatively, suppose there is a section $s : M \to (P \times F)/G$. The section induces a pullback bundle

$$\begin{tikzcd}
s^*(P \times F) \arrow[r, "\tilde{s}"] \arrow[d] & P \times F \arrow[d] \\
M \arrow[r, "s"] & (P \times F)/G
\end{tikzcd}$$

where the map $\tilde{s} : s^*(P \times F) \to P \times F$ is the induced projection of the pullback bundle which is naturally equivariant. Further, $s^*(P \times F) \subset M \times P \times F$ and there is a bundle map $f : s^*(P \times F) \to P$ given by

$$f(m, p, x) = p$$

which induces the identity over M and hence is an isomorphism by Proposition 2.12. Letting $p_2 : P \times F \to F$ denote the equivariant projection onto F, a section $s : M \to (P \times F)/G$ thus induces an equivariant map $f_s = p_2 \circ \tilde{s} \circ f^{-1} : P \to F$. $\square$

The correspondence in Proposition 2.17 is also a correspondence on particular homotopy classes. Accordingly, the notion of homotopy classes for sections of bundles and equivariant homotopy are defined below.

Definition 2.18: *Let G be a topological group, $P \to M$ be a topological principal G-bundle and F a topological space with right G-action. Let $f_0, f_1 : P \to F$ be equivariant maps. We say f_0 is* equivariantly homotopic *to f_1 if there is a homotopy $h : P \times I \to F$ such that $h(p, 0) = f_0(p)$, $h(p, 1) = f_1(p)$ and*

$$h(p \cdot g, t) = h(p, t) \cdot g$$

for every $p \in P, g \in G$ and $t \in I$.

Definition 2.19: *Let $P \xrightarrow{\pi} M$ and $Q \xrightarrow{\tau} M$ be continuous maps. Let $s_0, s_1 : Q \to P$ be fibrewise maps (or maps over M). That is, $\pi s_0 = \pi s_1$. We say s_0 is* fibrewise homotopic *(or homotopic over M) to s_1 if there is a homotopy $h : M \times I \to P$ such that*

$$\pi h(m, t) = m$$

for every $m \in M$ and $t \in I$. That is, the diagram

$$\begin{tikzcd}
Q \arrow[d, "i_0"] \arrow[dr, "s_0"] & \\
Q \times I \arrow[r, "h"] & P \\
Q \arrow[u, "i_1"] \arrow[ur, "s_1"'] &
\end{tikzcd}$$

commutes where all maps are over M.

Proposition 2.20: *The correspondence above is a bijection on homotopy classes. That is, $f_0, f_1 : P \to F$ are equivariantly homotopic if and only if the corresponding sections $\tilde{f}_0, \tilde{f}_1$ are fibrewise homotopic.*

Proof. Let $\tilde{f}_0, \tilde{f}_1$ be sections and f_0, f_1 the corresponding equivariant maps by Proposition 2.17. Suppose $\tilde{f}_0, \tilde{f}_1$ are fibrewise homotopic with homotopy $\tilde{h} : M \times I \to (P \times F)/G$. This can be extended to a section of the projection

$$((P \times I) \times F)/G \to M \times I \tag{2.21}$$

since G acts trivially on I in this case. This can best be seen by considering the pullback diagram

$$\begin{tikzcd}
(M \times I) \times_M (P \times F)/G \arrow[r] \arrow[d] & (P \times F)/G \arrow[d, "\tau"] \\
M \times I \arrow[r, "p_M"] & M
\end{tikzcd}$$

where p_M is the obvious projection and τ is the projection

$$\tau[p, x] = \pi(p)$$

for $p \in P, x \in F$. The space $(M \times I) \times_M (P \times F)/G$ is naturally homeomorphic to $((P \times I) \times F)/G$ by considering

$$(m, t, [p, x]) \mapsto [(p, t), x].$$

Since $i(m, t) = \tau[(p, t)]$ precisely when $p \in \pi^{-1}(m)$, this map has a continuous inverse

$$[(p, t), x] \mapsto (\pi(p), t, [p, x]).$$

Thus, the maps $\tilde{h}, \mathrm{id}$ induce a map $\tilde{s}$

$$\begin{tikzcd}
M \times I \arrow[dr, dashed, "\tilde{s}"] \arrow[drr, bend left, "\tilde{h}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& ((P \times I) \times F)/G \arrow[r] \arrow[d] & (P \times F)/G \arrow[d, "\tau"] \\
& M \times I \arrow[r, "p_M"] & M
\end{tikzcd}$$

which is clearly the section alluded to in (2.21). This section corresponds to an equivariant map

$$s : P \times I \to F.$$

Since $\tilde{h}(-, 0) = \tilde{f}_0$ and $\tilde{h}(-, 1) = \tilde{f}_1$, it follows that $s(-, 0) = f_0$ and $s(-, 1) = f_1$. Thus s is precisely an equivariant homotopy between the maps f_0, f_1.

Alternatively, an equivariant homotopy $s : P \times I \to F$ between the maps f_0, f_1 defines a section

$$\tilde{s} : M \times I \to ((P \times I) \times F)/G.$$

Composing with the projection $((P \times I) \times F)/G \to (P \times F)/G$, we see that this is precisely a fibrewise homotopy between the sections $\tilde{f}_0, \tilde{f}_1$. $\square$

In [Dol63], Dold defines a class of bundles called *numerable bundles* and shows that a certain class of numerable fibre bundles possess global sections. In particular we have the following definition and theorem.

Definition 2.22: *An open covering $\{U_i\}_{i \in J}$ of a topological space B is numerable provided there exists a locally finite partition of unity $\{u_i\}_{i \in J}$ such that the closure*

$$\overline{u_i^{-1}((0,1])} \subset U_i$$

for each $i \in J$. A fibre bundle $P \xrightarrow{\pi} B$ is a numerable bundle *if there is a numerable covering $\{U_i\}_{i \in J}$ of B such that each U_i admits a local trivialisation.*

Theorem 2.23: *Let $P \to M$ be a principal G-bundle with numerable base M, and E a contractible space with free G-action. Then there is a section of the associated fibre bundle*

$$(P \times E)/G \xrightarrow{\tau} M.$$

Moreover, any two sections of $(P \times E)/G \xrightarrow{\tau} M$ are fibrewise homotopic.

This is a corollary of Dold's result that a numerable bundle with contractible fibre admits a global section (see [Dol63, Cor 2.8 α]). As Husemoller remarks in [Hus66, p. 48-49], every principal G-bundle over a paracompact space is numerable. For example, we have the following inclusions.

$$\left\{\begin{array}{c}\text{Topological principal}\\ G\text{-bundles}\end{array}\right\} \supset \left\{\begin{array}{c}\text{Numerable principal}\\ G\text{-bundles}\end{array}\right\} \supset \left\{\begin{array}{c}\text{Topological principal}\\ G\text{-bundles with}\\ \text{paracompact base}\end{array}\right\} \supset \left\{\begin{array}{c}\text{Topological principal}\\ G\text{-bundles with base a}\\ \text{CW-complex}\end{array}\right\} \supset \left\{\begin{array}{c}\text{Smooth principal}\\ G\text{-bundles}\end{array}\right\}$$

Proofs of the following two lemmas can be found in [Hus66, p. 48-51] and will help us prove the classification theorem that follows.

Lemma 2.24: *Let $P \to M$ be a numerable bundle and $f : N \to M$ a map of topological spaces. Then the pullback bundle*

$$f^*(P) \to N$$

is a numerable bundle.

Lemma 2.25: *Let $P \xrightarrow{\pi} B$ be a numerable principal G-bundle and $f, g : X \to B$ be two homotopic maps. Then there is an isomorphism of pullback bundles*

$$f^*(P) \cong g^*(P).$$

Theorem 2.26: *Let $P \xrightarrow{\pi} M$ be a numerable principal G-bundle. Let $E \to B$ be a numerable principal G-bundle with contractible total space. Then there exists a map $f : M \to B$ such that there is an isomorphism of principal bundles*

$$P \to f^*(E).$$

Moreover, the isomorphism class of bundles containing P determines a unique f up to homotopy.

Proof. It is clear that a map $f_0 : M \to B$ defines a numerable principal G-bundle $f_0^*(E) \to M$ by Proposition 2.15 and Lemma 2.24. Moreover, if $f_1 : M \to B$ is homotopic to f_0 then there is an isomorphism

$$f_0^*(E) \cong f_1^*(E)$$

by Lemma 2.25. On the other hand, let $P \xrightarrow{\pi} M$ be a numerable principal G-bundle. By Theorem 2.23 there is a section s of the associated fibre bundle

$$(P \times E)/G \to M.$$

By Proposition 2.17 there is a corresponding equivariant map $g_s : P \to E$ which induces a map $g : M \to B$ on base spaces. There is a corresponding bundle map $P \to f^*(E)$ given by

$$p \mapsto (\pi(p), g_s(p))$$

which induces the identity over M and is hence an isomorphism. By Proposition 2.20 this homotopy class of g is uniquely determined by s up to fibrewise homotopy. By Theorem 2.23 any two sections of the bundle are fibrewise homotopic and thus determine a unique g up to homotopy. $\square$

Remark 2.27: Let $E \to B$ be a numerable principal bundle with E contractible. Theorem 2.26 classifies numerable principal G-bundles and henceforth we will refer to a map $f : X \to B$ as a *classifying map.*

Corollary 2.28: *Let $E_1 \to B_1$ and $E_2 \to B_2$ be numerable principal bundles with contractible total space. Then there is a G-homotopy equivalence $E_1 \simeq E_2$ and a homotopy equivalence $B_1 \simeq B_2$. Moreover, the definition of equivariant cohomology in Definition 2.6 does not depend on a choice of $E \to B$.*

Proof. Let $E_1 \to B_1, E_2 \to B_2$ be two numerable principal G-bundles with contractible total space. From Theorem 2.26 there is an isomorphism $F : E_1 \to E_2$ and classifying maps $f : B_1 \to B_2$ such that

$$E_1 \cong f^*(E_2).$$

Moreover, the converse is true. That is, there is an isomorphism $H : E_2 \to E_1$ and classifying maps $h : B_2 \to B_1$ such that

$$E_2 \cong h^*(E_1).$$

So we can write $E_2 = (f \circ h)^*(E_2)$ with isomorphism $F \circ H : (f \circ g)^*(E_2) \to E_2$ and classifying map $f \circ h : B_2 \to B_2$. By Theorem 2.26, it must be the case that there is a homotopy $f \circ h \simeq \mathrm{id}_{B_2}$ and a G-equivariant homotopy $F \circ H \simeq \mathrm{id}_{E_2}$. Similarly one shows that there is a homotopy $h \circ f \simeq \mathrm{id}_{B_1}$ and a G-equivariant homotopy $H \circ F \simeq \mathrm{id}_{E_1}$. Hence, given any topological space X with right G action, there is a G-homotopy equivalence

$$X \times E_1 \simeq X \times E_2$$

which descends to a homotopy equivalence

$$(X \times E_1)/G \simeq (X \times E_2)/G.$$

Clearly it follows that the definition of equivariant cohomology does not depend on the choice of E. □

Definition 2.29: *The* universal bundle *is any fixed numerable principal bundle with contractible total space. We will use EG to denote the total space of the universal bundle and BG to denote the base space of the universal bundle.*

2.2 The Geometry of Principal Bundles

2.2.1 The Action of a Lie Algebra

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and M, a manifold with right G-action $\phi : M \times G \to M$. The de Rham complex $\Omega^*(M)$ inherits a natural right G-action from the action on M via ϕ. Explicitly, the induced action $\phi_\Omega : \Omega^*(M) \times G \to \Omega^*(M)$ for $\omega \in \Omega^n(M)$ can be written

$$\phi_\Omega(\omega, g) := \phi_g^* \omega,$$

however we will usually take the shorthand $\omega \cdot g$. A differential form is *G-invariant* if $\omega \cdot g = \omega$ for every $g \in G$.

The Lie algebra of G carries information about the infinitesimal action of G on M. In particular, there is a natural vector field associated to each Lie algebra element $\xi \in \mathfrak{g}$. The curve $\gamma_x : [-1, 1] \to M$ through a point $x \in M$ given by

$$\gamma_x(t) = \phi(x, \exp(t\xi))$$

defines a tangent vector in T_xM by calculating its derivative at zero. Using the notation $\mathfrak{X}(M)$ to denote smooth vector fields on M we see that since the functions ϕ and exp are smooth, the vector field $X_\xi : M \to TM$ defined by

$$X_\xi(x) = \frac{d}{dt}\gamma_x(t)\bigg|_{t=0}$$

is well defined and smooth for each $\xi \in \mathfrak{g}$. If the vector field X_ξ does not vanish, we say the action of G is *locally free*. Defining a basis $\{\xi_1, \ldots, \xi_n\}$ of $\mathfrak{g}$ in turn defines a corresponding basis $\{X_{\xi_1}, \ldots, X_{\xi_n}\}$ of vector fields generated by the infinitesimal action of $\mathfrak{g}$. The Lie bracket on $\mathfrak{g}$ defines associated *structure constants* $c_{ij}^k \in \mathbb{R}$ relative to the chosen basis where

$$[\xi_i, \xi_j] = c_{ij}^k \xi_k$$

and Einstein summation convention is observed. The vector fields generated by the infinitesimal action share a similar structure to the Lie algebra in that

$$[X_{\xi_i}, X_{\xi_j}] = c_{ij}^k X_{\xi_k} = X_{[\xi_i, \xi_j]}.$$

These vector fields are fundamental in the study of equivariant cohomology and possess many properties that allow an analogue of de Rham's theorem to emerge. Perhaps the most natural action that a vector field $X_\xi \in \mathfrak{X}(M)$ has on a differential form is via the interior product, $\iota_\xi : \Omega^n(M) \to \Omega^{n-1}(M)$, given by

$$\iota_\xi \omega(X_1, \ldots, X_{n-1}) = \omega(X_\xi, X_1, \ldots, X_{n-1}), \tag{2.30}$$

for $X_1, \ldots, X_n$ smooth vector fields on M. Note that this notation is a contraction of the more standard ι_{X_ξ} which will be avoided for simplicity. One checks that

$$\iota_\xi(\omega \wedge \mu) = (\iota_\xi \omega) \wedge \mu + (-1)^n \omega \wedge (\iota_\xi \mu)$$

to verify that this is a graded derivation of degree -1. An element of the Lie algebra also determines a one-parameter subgroup of G which naturally acts on differential forms via

$$t \mapsto \omega \cdot \exp(t\xi).$$

Taking the derivative at $t = 0$ of this map is precisely the Lie derivative of ω with respect to the vector field X_ξ, which shall be denoted $\mathcal{L}_\xi$. The Lie derivative observes the identity

$$\mathcal{L}_\xi = d\iota_\xi + \iota_\xi d \tag{2.31}$$

which is known as 'Cartan's magic formula' and is attributed to Élie Cartan (and occasionally Henri Cartan), a proof of which can be found in [CCL99]. This identity immediately implies that $\mathcal{L}_\xi$ is a derivation of degree 0 and commutes with d. Since differential forms are linear in their first term, the sets spanned by these derivations are vector spaces with respective bases $\{\iota_{\xi_1}, \ldots, \iota_{\xi_n}\}$ and $\{\mathcal{L}_{\xi_1}, \ldots, \mathcal{L}_{\xi_n}\}$ corresponding to the basis of $\mathfrak{g}$.

2.2.2 Connections and Curvature

Let $P \xrightarrow{\pi} B$ be a principal G-bundle. The action of G on the total space is free and thus vector fields generated by the infinitesimal action of $\mathfrak{g}$ do not vanish. To each vector field X_{ξ_i} there exists a 1-form $\theta^i \in \Omega^*(P)$ that is in some sense 'dual' in that

$$\theta^i(p; X_{\xi_j}) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for each $p \in P$. If the collection of these forms $\{\theta^i\}$ span a G-invariant subspace of $\Omega^1(P)$, this is precisely the definition of 'connection forms' on the total space P in [GS99]. A differential 1-form that takes values in $\mathfrak{g}$ defined by

$$\theta = \theta^i \otimes \xi_i \in \Omega^1(P) \otimes \mathfrak{g}$$

(and where Einstein summation convention is observed) is an example of the more standard definition of a connection form. One can check that for any $\xi \in \mathfrak{g}$ and every $p \in P$,

$$\theta(p; X_\xi) = \xi.$$

This differential form identifies tangent vectors generated by the Lie algebra and a 'complement' defined by

$$\{u \in T_pP : \theta(p; u) = 0\}$$

in a smooth (but not necessarily unique) way. This differential form can also be realised in a more classical way via parallel transport, as seen in Dupont [Dup78, Ch. 3], which motivates the following discussion. At a point $p \in P$ there is a map $v_p : \mathfrak{g} \to T_pP$ given by

$$v_p(\xi) = X_\xi(p)$$

corresponding to the derivative of the map $g \mapsto p \cdot g$ as seen in the definition of X_ξ. Note that this map is injective since the action of G is free. The projection $\pi : P \to B$ is G-invariant and so results in the following short exact sequence

$$0 \to \mathfrak{g} \xrightarrow{v_p} T_pP \xrightarrow{\pi_*} T_{\pi(p)}B \to 0$$

where $\pi_* : TP \to TB$ is the derivative of π. At each point $p \in P$ the differential form θ defines a linear map

$$\theta_p := \theta(p; -) : T_pP \to \mathfrak{g}$$

such that $\theta_p \circ v_p = \mathrm{id}_{\mathfrak{g}}$. Thus, the connection θ splits the exact sequence at every point and allows us to define the following.

Definition 2.32: *Let $P \xrightarrow{\pi} B$ be a principal G-bundle, $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ a connection on P and $u \in T_pP$ for some $p \in P$. We say u is* vertical *if $u \in \ker(\pi_*)$ and* horizontal *if $u \in \ker(\theta_p)$. Likewise, a vector field X is called* vertical *if all vectors $X(p)$ are vertical and* horizontal *if all vectors $X(p)$ are horizontal. Denote the collection of all vertical vectors at a point $p \in P$ as $\mathcal{V}_p$ and the collection of all vertical vectors at a point $p \in P$ as $\mathcal{H}_p$.*

Remark 2.33: The short exact sequence

$$0 \to \mathfrak{g} \xrightarrow{v_p} T_pP \xrightarrow{\pi_*} T_{\pi(p)}B \to 0$$

means that a vertical vector can equivalently be defined as a tangent vector $u \in T_pP$ such that $u \in \mathrm{im}(\theta_p)$.

Since the connection 1-form determines these subspaces of the tangent space, we would also like the definition of a connection to be 'equivariant' in the sense that it preserves these subspaces under the group action. Put more explicitly, we would like

$$\mathcal{V}_{p\cdot g} = \mathcal{V}_p \cdot g, \quad \mathcal{H}_{p\cdot g} = \mathcal{H}_p \cdot g$$

where the action of $g \in G$ is taken to be $(\phi_g)_*$, the derivative of action ϕ_g. The derivative of the action of $g \in G$ on a vertical vector $X_\xi(p)$ is given by

$$\begin{aligned}
(\phi_g)_* X_\xi(p) &= (\phi_g)_* \left.\frac{d}{dt}\phi(x, \exp(t\xi))\right|_{t=0} \\
&= \left.\frac{d}{dt}\phi(x, \exp(t\xi)g)\right|_{t=0} \\
&= \left.\frac{d}{dt}\phi(x \cdot g, g^{-1}\exp(t\xi)g)\right|_{t=0} \\
&= X_{\mathrm{Ad}(g^{-1})\xi}(p \cdot g) \qquad (2.34)
\end{aligned}$$

where $\mathrm{Ad}(g^{-1}) : \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation of $g^{-1} \in G$, the derivative of the map $h \mapsto g^{-1}hg$ at the identity. This immediately shows that $\mathcal{V}_{p\cdot g} = \mathcal{V}_p \cdot g$, trivially. Let $u = u_v + u_h \in T_pP$ where u_v is vertical and u_h is horizontal. If $\mathcal{H}_{p\cdot g} = \mathcal{H}_p \cdot g$ then it must be true that

$$(\phi_g)_* u_h \in \mathcal{H}_{p\cdot g},$$

or in terms of the connection this can rewritten as

$$\theta_{p\cdot g}\left((\phi_g)_*(u_h)\right) = 0. \qquad (2.35)$$

If we let $u_v = X_\xi(p)$ for some $\xi \in \mathfrak{g}$, the effect that (2.35) has on a connection can thus be calculated by seeing that

$$\begin{aligned}
\theta_{p\cdot g}\left((\phi_g)_* u\right) &= \theta_{p\cdot g}\left((\phi_g)_* u_v\right) + \theta_{p\cdot g}\left((\phi_g)_* u_h\right) \\
&= \theta_{p\cdot g}\left((\phi_g)_* X_\xi(p)\right) \\
&= \theta\left(p \cdot g; X_{\mathrm{Ad}(g^{-1})\xi}(p \cdot g)\right) \\
&= \mathrm{Ad}(g^{-1})\xi \\
&= \mathrm{Ad}(g^{-1})\theta_p(u).
\end{aligned}$$

More concisely, this calculation shows that $\mathcal{H}_{p\cdot g} = \mathcal{H}_p \cdot g$ implies that $\phi_g^*\theta = \mathrm{Ad}(g^{-1}) \circ \theta$. In fact, if $\phi_g^*\theta = \mathrm{Ad}(g^{-1}) \circ \theta$ it implies that $\mathcal{H}_{p\cdot g} = \mathcal{H}_p \cdot g$, so this can be taken to be an equivalent condition. With this in mind, we precisely define what is meant by a connection on a principal bundle.

Definition 2.36: *Let $P \xrightarrow{\pi} B$ be a principal G-bundle. A* connection *on P is a $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ such that*

1. $\theta_p \circ v_p = \mathrm{id}_{\mathfrak{g}}$, *and*

2. $\phi_g^*\theta = \mathrm{Ad}(g^{-1}) \circ \theta$

for all $p \in P$ and every $g \in G$.

Consider the principal G-bundle $G \to \{1\}$ where G acts on itself by a right G-action. Since every vector in TG is vertical and G acts transitively, every vector $u \in T_gG$ can be written as

$$u = g \cdot \xi$$

for some $\xi \in \mathfrak{g}$ where the action of $g \in G$ on $\mathfrak{g}$ corresponds to the derivative of the map $h \mapsto gh$ at the identity. These correspond precisely to the vector fields generated by the infinitesimal action of $\mathfrak{g}$ since

$$X_\xi(g) = \frac{d}{dt} g \cdot \exp(t\xi)\bigg|_{t=0} = (\phi_g)_*\xi = g \cdot \xi.$$

The *left Maurer-Cartan* form θ_L on G can explicitly defined as $\theta_L(g; g \cdot \xi) = \xi$ or, less opaquely,

$$\theta_L(g; u) = (\phi_{g^{-1}})u. \tag{2.37}$$

One checks that $\phi_g^*\theta_L = \mathrm{Ad}(g^{-1}) \circ \theta_L$ to verify this is a connection form. Since we have such an explicit description of this connection we can calculate its exterior derivative

$$\begin{aligned} d\theta_L(X, Y) &= X\theta_L(Y) - Y\theta(X) - \theta_L([X, Y]) \\ &= \mathcal{L}_X\theta_L(Y) - \mathcal{L}_Y\theta(X) - \theta_L([X, Y]). \end{aligned}$$

If we let X, Y be vector fields generated by the infinitesimal action of $\mathfrak{g}$ we notice that it greatly simplifies this expression. For example, for $\xi, \eta \in \mathfrak{g}$,

$$\begin{aligned} d\theta_L(X_\xi, X_\eta) &= \mathcal{L}_\xi\theta_L(X_\eta) - \mathcal{L}_\eta\theta(X_\xi) - \theta_L([X_\xi, X_\eta]) \\ &= -\theta_L([X_\xi, X_\eta]) \\ &= -\theta_L([X_{[\xi,\eta]}) \\ &= -[\xi, \eta] \\ &= -[\theta_L(X_\xi), \theta_L(X_\eta)] \end{aligned}$$

since $\theta_L(X_\xi), \theta_L(X_\eta)$ are constant functions. Now it is enough to note that at a point $g \in G$, $X(g) = g \cdot \xi$ and $Y(g) = g \cdot \eta$ for some $\xi, \eta \in \mathfrak{g}$ and therefore the identity

$$d\theta_L = -\frac{1}{2}[\theta_L, \theta_L],$$

holds for every vector field $X, Y \in \mathfrak{X}(G)$; where for $\omega_1, \omega_2 \in \Omega^1(M) \otimes \mathfrak{g}$,

$$[\omega_1, \omega_2](X, Y) = [\omega_1(X), \omega_2(Y)] - [\omega_1(Y), \omega_2(X)].$$

This identity clearly carries over to the trivial principal bundle $B \times G \to B$ and thus a connection θ on a principal bundle $P \xrightarrow{\pi} B$ is said to be *flat* if

$$d\theta = -\frac{1}{2}[\theta, \theta].$$

The 'curvature' of a principal bundle is then a measure of a connections failure to be flat. That is, for a connection form θ, define the $\mathfrak{g}$-valued *curvature* 2-form Ω by

$$d\theta = \Omega - \frac{1}{2}[\theta, \theta].$$

Recall that by definition, θ vanishes on horizontal vectors. In contrast, the curvature 2-form is *horizontal.*

Definition 2.38: *Let $P \xrightarrow{\pi} B$ be a smooth principal G-bundle. A differential form $\omega \in \Omega^*(P)$ is said to be* horizontal *if it vanishes on vertical vector fields. Equivalently, a differential form is horizontal if*

$$\iota_\xi \omega = 0$$

for every $\xi \in \mathfrak{g}$.

The curvature form is an important ingredient for equivariant cohomology, so the following proposition will be useful for performing calculations.

Proposition 2.39: *Let $P \xrightarrow{\pi} B$ be a smooth principal G-bundle with connection θ and curvature Ω.*

1. *Ω is horizontal,*
2. *Ω is equivariant in the sense that $\phi_g^*\Omega = \mathrm{Ad}(g^{-1}) \circ \Omega$ and*
3. *$d\Omega = [\Omega, \theta]$.*

In the same vein, Dupont's proof that every principal bundle admits a connection will be used in the discussion that follows. For a proof of these two propositions, the reader is directed to [Dup78, p. 47- 49].

Proposition 2.40: *Every principal bundle has a connection.*

Associated to a basis of $\mathfrak{g}$ is a dual basis $\{\xi_1^*, \ldots, \xi_n^*\}$ of $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$. Thus, to each connection $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ are 1-forms

$$\theta^i := \xi_i^* \theta \in \Omega^1(P).$$

Perhaps the most obvious equation these differential forms satisfy is

$$\theta^i(p; X_{\xi_j}) = \delta_{ij}$$

for every $p \in P$, reminiscent of the connection forms as defined in [GS99]. By the same token, for $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$, the curvature of θ, define

$$\mu^i := \xi_i^* \Omega \in \Omega^2(P).$$

In terms of these differential forms, we have the following identities.

Proposition 2.41:

1. $\iota_{\xi_j} \theta^i = \delta_{ij}$.
2. $d\theta^i = \mu^i - \frac{1}{2} c^i_{jk} \theta^j \wedge \theta^k$.
3. $\iota_{\xi_j} \mu^i = 0$.
4. $d\mu^i = c^i_{jk} \mu^j \wedge \theta^k$.

Note that all these identities are corollaries of previous calculations in this section. We will revisit these identities in Section 2.3.1.

2.2.3 Basic Differential Forms

Suppose $P \xrightarrow{\pi} B$ is a principal G-bundle. By de Rham's theorem, the cohomology of B can be calculated using differential forms via calculating the cohomology of the cochain complex $(\Omega^*(B), d)$.

$$\Omega^0(B) \xrightarrow{d} \Omega^1(B) \xrightarrow{d} \Omega^2(B) \xrightarrow{d} \cdots$$

The pullback $\pi^* : \Omega^*(B) \to \Omega^*(P)$ allows us to recover the cohomology of B by only looking at the *basic forms* of P. That is, the subspace of basic differential forms

$$\pi^*(\Omega^*(B)) \subset \Omega^*(P)$$

is a well defined subcomplex of the de Rham complex of P since it is closed under the differential d. This definition is clumsy for the purposes of performing any actual calculations and so a definition of this subcomplex using only information about P is sought after.

Proposition 2.42: *Let $P \xrightarrow{\pi} B$ be a principal G-bundle. A differential form $\omega \in \Omega^*(P)$ is basic if and only if $\iota_\xi \omega = 0$ and $\omega \cdot g = \omega$ for every $\xi \in \mathfrak{g}$ and $g \in G$.*

Proof. Let ω be a basic differential form in $\Omega^q(P)$. That is, there is a form $\nu \in \Omega^q(B)$ such that $\pi_* \nu = \omega$. For any $\xi \in \mathfrak{g}$ the vector field X_ξ is vertical, hence

$$\pi_* X(p) = 0$$

for every $p \in P$ and thus, $\iota_\xi(\pi^* \nu) = 0$. Similarly we could ask how the action of G affects the pullback $\pi^* \nu$. Let $g \in G$ and consider that since $\pi \circ \phi_g = \pi$, we have

$$\begin{aligned}(\pi^* \nu) \cdot g &= \phi_g^* \pi^* \nu \\ &= (\pi \circ \phi_g)^* \nu \\ &= \pi^* \nu.\end{aligned}$$

These two calculations show that if ω is basic, then $\iota_\xi \omega = 0$ and $\omega \cdot g = \omega$.

Now suppose $\iota_\xi \omega = 0$ and $\omega \cdot g = \omega$ for every $\xi \in \mathfrak{g}$ and $g \in G$. Let $s : B \to P$ be a section. We assert that $s^* \omega \in \Omega^*(B)$ satisfies $\pi^* s^* \omega = \omega$. Note too, if $u \in T_p P$ is a tangent vector at $p \in P$, the vector $u - (\phi_g \circ s \circ \pi)_* u$ is vertical since

$$\pi_* (\phi_g \circ s \circ \pi)_* u = \pi_* u$$

for every $g \in G$. To check that $\omega = \pi^* s^* \omega$, it is enough to calculate

$$(\omega - \pi^* s^* \omega)(p; u_1, \ldots, u_q)$$

at an arbitrary point p with arbitrary tangent vectors $u_1, \ldots, u_q \in T_p P$. If

$$(\pi \circ s)(p) = p \cdot g$$

for some $g \in G$ then at the point p

$$\begin{aligned}&(\omega - \pi^* s^* \omega)(p; u_1, \ldots, u_q) \\ &= \omega(p; u_1, \ldots, u_q) - \omega((s \circ \pi)(p); (s \circ \pi)_* u_1, \ldots, (s \circ \pi)_* u_q) \\ &= \omega(p; u_1, \ldots, u_q) - (\omega \cdot g^{-1})((s \circ \pi)(p); (s \circ \pi)_* u_1, \ldots, (s \circ \pi)_* u_q) \\ &= \omega(p; u_1, \ldots, u_q) - \omega(p; (\phi_{g^{-1}} \circ s \circ \pi)_* u_1, \ldots, (\phi_{g^{-1}} \circ s \circ \pi)_* u_q) \\ &= \omega(p; u_1 - (\phi_{g^{-1}} \circ s \circ \pi)_* u_1, \ldots, u_q - (\phi_{g^{-1}} \circ s \circ \pi)_* u_q) \\ &= 0\end{aligned}$$

since ω vanishes on vertical vectors. So ω is basic when $P \to B$ admits a section. If $P \to B$ does not admit a section, it admits local sections by Corollary 2.14. Let $\{U_i\}_{i \in J}$ be local trivialisations with local sections $s_i : B \to P$ and define

$$\nu_i := s_i^* \left(\omega|_{U_i} \right) \in \Omega^q(U_i).$$

By letting $b \in U_i \cap U_j$ and $v_1, \dots, v_q \in T_bB$ and supposing that $s_i(b) = s_j(b) \cdot g$ for some $g \in G$ we see that

$$\begin{aligned}
&(\nu_i - \nu_j)(b; v_1, \dots, v_q) \\
=& \omega(s_i(b); (s_i)_* v_1, \dots, (s_i)_* v_q) - \omega(s_j(b); (s_j)_* v_1, \dots, (s_j)_* v_q) \\
=& \omega(s_i(b); (s_i)_* v_1, \dots, (s_i)_* v_q) - \omega(s_i(b); (\phi_{g^{-1}} \circ s_j)_* v_1, \dots, (\phi_{g^{-1}} \circ s_j)_* v_q) \\
=& \omega(s_i(b); (s_i)_* v_1 - (\phi_{g^{-1}} \circ s_j)_* v_1, \dots, (s_i)_* v_q - (\phi_{g^{-1}} \circ s_j)_* v_q) \\
=& 0
\end{aligned}$$

since $\pi \circ s_i = \pi \circ \phi_{g^{-1}} \circ s_j = \mathrm{id}$. So defining $\nu \in \Omega^*(B)$ by $\nu|_{U_i} = \nu_i$ is well defined since $\nu_i = \nu_j$ on $U_i \cap U_j$ for every $i, j \in J$ which shows that $\pi^* \nu = \omega$ as required. $\square$

Since the set of basic differential forms of a principal bundle $P \xrightarrow{\pi} B$ is closed under the differential d, the set of basic differential forms is also a complex. The following definition of basic will be more helpful for the purposes of calculation.

Definition 2.43: *Let $P \xrightarrow{\pi} B$ be a smooth principal G-bundle. A differential form is said to be* basic *if it is horizontal and G-invariant. That is, $\iota_\xi \omega = 0$ and $\omega \cdot g = \omega$ for every $\xi \in \mathfrak{g}$ and $g \in G$. The* complex of basic differential forms *is denoted $\Omega^*(P)_{bas}$.*

2.3 Equivariant de Rham Theory

2.3.1 The Weil Algebra

Let G be a Lie group and M be a manifold with right G-action. From the previous discussion of basic differential forms, one might wonder whether there is an equivalent way to find a de Rham complex for the total space of the principal bundle

$$M \times EG \to (M \times EG)/G.$$

That is, if we can find a de Rham complex for $M \times EG$ then the basic subcomplex will calculate the equivariant cohomology of M. The space EG does not admit a smooth structure in general, so de Rham theory can not be used in the ordinary sense. Instead, we will define a complex called *the Weil algebra* $W(\mathfrak{g})$ which acts as a surrogate for differential forms on EG. This allows us to form a complex

$$W(\mathfrak{g}) \otimes \Omega^*(M)$$

and study the basic forms of this complex. In this section we will define and study the complex $W(\mathfrak{g})$.

Definition 2.44: *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Define the* Weil Algebra $W(\mathfrak{g})$ *to be the complex*

$$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$$

where Λ, S are the exterior and symmetric algebras over $\mathbb{R}$ and $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$. The grading and differential of $W(\mathfrak{g})$ will be explained further in the exposition below.

Let $\mathfrak{g}$ have basis $\{\xi_1, \dots, \xi_n\}$ and $\mathfrak{g}^*$ have dual basis $\{\xi_1^*, \dots \xi_n^*\}$. Both the exterior and symmetric algebras are graded and generated by their degree 1 elements. The sets $\Lambda^1(\mathfrak{g}^*)$ and $S^1(\mathfrak{g}^*)$ can be naturally identified with $\mathfrak{g}^*$ as vector spaces, hence a choice of basis for $\mathfrak{g}$ induces generating sets for $\Lambda(\mathfrak{g}^*)$ and $S(\mathfrak{g}^*)$ which will be denoted $\{\theta^1, \dots, \theta^n\}$ and $\{\mu^1, \dots, \mu^n\}$ respectively. The Weil algebra is hence generated by the θ^i, μ^i elements and inherits a grading by letting the θ^i have degree 1 and the μ^i have degree 2. Explicitly, one might express the set of degree n elements of $W(\mathfrak{g})$ by

$$W^n(\mathfrak{g}) = \bigoplus_{p+2q=n} \Lambda^p(\mathfrak{g}^*) \otimes S^q(\mathfrak{g}^*).$$

The notation is deliberately suggestive of the remarks following Proposition 2.40. To this end, we define an operator (of degree 1) on the generators of $W(\mathfrak{g})$ by defining

$$d\theta^i = \mu^i - \frac{1}{2} c^i_{jk} \theta^j \theta^k \text{ and } d\mu^i = c^i_{jk} \mu^j \theta^k.$$

Further, we define the linear operator (of degree -1) ι_{ξ_j} similarly by

$$\iota_{\xi_j} \theta^i = \delta_{ij} \text{ and } \iota_{\xi_j} \mu^i = 0$$

so as to mirror the identities in Proposition 2.41. (We take ι_ξ to act trivially on elements of order 0.) By linearity it is clear that for any $\xi = k^i \xi_i \in \mathfrak{g}$ (for constants $k^i \in \mathbb{R}$)

$$\iota_\xi = \iota_{k^i \xi_i} = k^i \iota_{\xi_i}.$$

Clearly these operators can be extended to be derivations on $W(\mathfrak{g})$. The operator ι_ξ clearly satisfies $\iota_\xi^2 = 0$ and leads to an analogous definition of a *horizontal element*.

Definition 2.45: *An element $\alpha \in W(\mathfrak{g})$ is said to be* horizontal *if*

$$\iota_\xi \alpha = 0$$

for every $\xi \in \mathfrak{g}$.

Proposition 2.46: *The operator d above satisfies $d^2 = 0$.*

Proof. The proof that ι_{ξ_j} is a derivation is clear from its definition. As for d, since μ^i can be written as

$$\mu^i = d\theta^i - \frac{1}{2} c^i_{jk} \theta^j \theta^k$$

we need only check that $d^2\theta^i = 0$. The calculation expands out as

$$\begin{aligned}
d^2\theta^i &= d(\mu^i - \frac{1}{2}c^i_{jk}\theta^j\theta^k) \\
&= d\mu^i - \frac{1}{2}c^i_{jk}d\theta^j\theta^k + \frac{1}{2}c^i_{jk}\theta^j d\theta^k \\
&= d\mu^i - \frac{1}{2}c^i_{jk}(\mu^j - \frac{1}{2}c^j_{ab}\theta^a\theta^b)\theta^k + \frac{1}{2}c^i_{jk}\theta^j(\mu^k - \frac{1}{2}c^k_{ab}\theta^a\theta^b).
\end{aligned}$$

Some terms immediately cancel, for example

$$-\frac{1}{2}c^i_{jk}\mu^j\theta^k = -\frac{1}{2}d\mu^i$$

and thus

$$d\mu^i - \frac{1}{2}c^i_{jk}\mu^j\theta^k + \frac{1}{2}c^i_{jk}\mu^k\theta^j = 0$$

since $c^i_{jk} = -c^i_{kj}$. The calculation of $d^2\theta^i = 0$ thus reduces to showing that

$$\begin{aligned}
0 &= \frac{1}{4}c^i_{jk}c^j_{ab}\theta^a\theta^b\theta^k - \frac{1}{4}c^i_{jk}c^k_{ab}\theta^j\theta^a\theta^b \\
&= \frac{1}{2}c^i_{jk}c^j_{ab}\theta^a\theta^b\theta^k. \qquad (2.47)
\end{aligned}$$

This rather tedious calculation can be summarised by observing that the Jacobi identity takes its form in the structure constants by requiring that they satisfy

$$c^i_{jk}c^j_{ab} + c^i_{ja}c^j_{bk} + c^i_{jb}c^j_{ka} = 0 \qquad (2.48)$$

for every fixed i. Further observe that fixing α, β, γ we pull the term

$$\begin{aligned}
&\frac{1}{2}c^i_{j\gamma}c^j_{\alpha\beta}\theta^\alpha\theta^\beta\theta^\gamma + \frac{1}{2}c^i_{j\beta}c^j_{\alpha\gamma}\theta^\alpha\theta^\gamma\theta^\beta + \frac{1}{2}c^i_{j\gamma}c^j_{\beta\alpha}\theta^\beta\theta^\alpha\theta^\gamma + \frac{1}{2}c^i_{j\alpha}c^j_{\beta\gamma}\theta^\beta\theta^\gamma\theta^\beta \\
&+ \frac{1}{2}c^i_{j\beta}c^j_{\gamma\alpha}\theta^\gamma\theta^\alpha\theta^\beta + \frac{1}{2}c^i_{j\alpha}c^j_{\gamma\alpha}\theta^\gamma\theta^\beta\theta^\alpha
\end{aligned}$$

out of the sum in 2.47 (where Einstein summation convention is not being observed). We can simplify this term to get

$$\begin{aligned}
&\frac{1}{2}\theta^\alpha\theta^\beta\theta^\gamma(c^i_{j\gamma}c^j_{\alpha\beta} - c^i_{j\beta}c^j_{\alpha\gamma} - c^i_{j\gamma}c^j_{\beta\alpha} + c^i_{j\alpha}c^j_{\beta\gamma} + c^i_{j\beta}c^j_{\gamma\alpha} - c^i_{j\alpha}c^j_{\gamma\beta}) \\
&= \theta^\alpha\theta^\beta\theta^\gamma(c^i_{j\gamma}c^j_{\alpha\beta} + c^i_{j\alpha}c^j_{\beta\gamma} + c^i_{j\beta}c^j_{\gamma\alpha})
\end{aligned}$$

whence it becomes clear that $d^2 = 0$. □

The algebra $W(\mathfrak{g})$ can thus be thought of a cochain complex with differential d. The de Rham complex of a contractible space is acyclic, so $W(\mathfrak{g})$ should be acyclic in order to mirror this property. The following proposition follows a similar approach to [GS99] and [Mei06].

Theorem 2.49: $W(\mathfrak{g})$ *is an acyclic algebra.*

Proof. Note that the algebra $W(\mathfrak{g})$ can be generated by the elements $\{\theta^i, d\theta^i\}$, since the equation

$$\mu^i = d\theta^i + \frac{1}{2}c^i_{jk}\theta^j\theta^k$$

shows that any element $w \in W^p(\mathfrak{g})$ may be written in the form

$$w = \sum_{n+2m=p} k_{n,m}\, \theta^{\alpha_1}\cdots\theta^{\alpha_n} d\theta^{\beta_1}\cdots d\theta^{\beta_m} = \sum_{n+2m=p} k_{n,m} w_{n,m}$$

where $k_{n,m}$ is some real constant and $w_{n,m} \in W^{n+2m}(\mathfrak{g})$. Note that the set

$$\{w_{n,m} : n + 2m = p\}$$

is a basis for $W^p(\mathfrak{g})$ when thought of as a vector space. On these generators of $W(\mathfrak{g})$ we can define a derivation Q by $Q(\theta^i) = 0$ and $Q(d\theta^i) = \theta^i$. Calculating dQw on its summands we find

$$\begin{aligned} dQ(w_{n,m}) &= dQ(\theta^{\alpha_1}\cdots\theta^{\alpha_n} d\theta^{\beta_1}\cdots d\theta^{\beta_m}) \\ &= d\left(\sum_{i=1}^{m}\theta^{\beta_i}\theta^{\alpha_1}\cdots\theta^{\alpha_n} d\theta^{\beta_1}\cdots d\hat{\theta}^{\beta_i}\cdots d\theta^{\beta_m}\right) \\ &= \sum_{i=1}^{n}\sum_{i=1}^{m}(-1)^j\theta^{\beta_i}\theta^{\alpha_1}\cdots\theta^{\hat{\alpha}_j}\cdots\theta^{\alpha_n} d\theta^{\alpha_j} d\theta^{\beta_1}\cdots d\hat{\theta}^{\beta_i}\cdots d\theta^{\beta_m} \\ &\quad + \sum_{i=1}^{m}\theta^{\alpha_1}\cdots\theta^{\alpha_n} d\theta^{\beta_1}\cdots d\theta^{\beta_m} \end{aligned}$$

where the caret denotes omission from the sequence (e.g. $\theta^{\hat{\alpha}_j}$). Similarly, calculating Qdw on its summands we find

$$\begin{aligned} Qd(w_{n,m}) &= Qd(\theta^{\alpha_1}\cdots\theta^{\alpha_n} d\theta^{\beta_1}\cdots d\theta^{\beta_m}) \\ &= Q\left(\sum_{i=1}^{n}(-1)^j\theta^{\alpha_1}\cdots\theta^{\hat{\alpha}_j}\cdots\theta^{\alpha_n} d\theta^{\alpha_j} d\theta^{\beta_1} d\theta^{\beta_m}\right) \\ &= \sum_{i=1}^{n}\theta^{\alpha_1}\cdots\theta^{\alpha_n} d\theta^{\beta_1}\cdots d\theta^{\beta_m} \\ &\quad + \sum_{i=1}^{m}\sum_{i=1}^{n}(-1)^{j+1}\theta^{\beta_i}\theta^{\alpha_1}\cdots\theta^{\hat{\alpha}_j}\cdots\theta^{\alpha_n} d\theta^{\alpha_j} d\theta^{\beta_1}\cdots d\hat{\theta}^{\beta_i}\cdots d\theta^{\beta_m} \end{aligned}$$

whence it becomes clear that

$$(dQ + Qd)w_{n,m} = (n + m)w_{n,m}$$

for each summand $w_{n,m}$. Since the $w_{n,m}$ elements form a basis for $W^p(\mathfrak{g})$ it is possible to define a linear map $Q_{n,m} : W^p(\mathfrak{g}) \to W^{p-1}(\mathfrak{g})$ by

$$Q_{n,m}(w_{s,t}) = \begin{cases} \frac{1}{n+m} Q w_{s,t} & \text{if } s = n, t = m \\ 0 & \text{otherwise} \end{cases}$$

and hence a chain homotopy K where $K_p = \sum\limits_{n+2m=p} Q_{n,m}$. That is, it has been shown that

$$dK + Kd = \mathrm{id}$$

and hence, $W(\mathfrak{g})$ is acyclic. □

Remark 2.50: In fact, this proof shows that any cochain complex generated by $\{\theta^i, d\theta^i\}$ (where $d\theta^i \neq 0$ for any i) is acyclic.

The complex $W(\mathfrak{g})$ also carries a natural action of G. The action $\phi_{\mathfrak{g}^*} : \mathfrak{g}^* \times G \to \mathfrak{g}^*$ is induced by the *contragredient* (or coadjoint) action of G on $\mathfrak{g}^*$. When considering the natural bilinear product $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, the contragredient action is the unique action such that

$$\langle \xi^* \cdot g, \eta \cdot g \rangle = \langle \xi^*, \eta \rangle = \xi^*(\eta)$$

for every $\xi^* \in \mathfrak{g}^*,\ \eta \in \mathfrak{g},\ g \in G$. Since both $\Lambda(\mathfrak{g}^*)$ and $S(\mathfrak{g}^*)$ are generated by their elements of order 1 (and $\mathfrak{g}^* \cong \Lambda^1(\mathfrak{g}^*) \cong S^1(\mathfrak{g}^*)$ as vector spaces) the contragredient action defines an action on $W(\mathfrak{g})$.

Definition 2.51: *An element of $W(\mathfrak{g})$ is said to be* basic *if it is horizontal and G-invariant. That is, $\iota_\xi \omega = 0$ and $\omega \cdot g = \omega$ for every $\xi \in \mathfrak{g}$ and $g \in G$. The* complex of basic differential forms *is denoted $W(\mathfrak{g})_{bas}$.*

The action of G on $W(\mathfrak{g})$ further induces an action of $\mathfrak{g}$ via the Lie derivative which again agrees with Cartan's magic formula

$$\mathcal{L}_\xi = d\iota_\xi + \iota_\xi d.$$

This formula makes it much easier to calculate the action of $\mathfrak{g}$ on $W(\mathfrak{g})$ and indeed,

$$\begin{aligned} \mathcal{L}_{\xi_j}(\theta^i) &= (d\iota_{\xi_j} + \iota_{\xi_j} d)\theta^i \\ &= \iota_{\xi_j}(\mu^i - \frac{1}{2} c^i_{ab} \theta^a \theta^b) \\ &= -\frac{1}{2} c^i_{jb} \theta^b + \frac{1}{2} c^i_{aj} \theta^a \\ &= -c^i_{jk} \theta^k \end{aligned}$$

since a, b sum over the same numbers and

$$\begin{aligned}
\mathcal{L}_{\xi_k}(\mu^i) &= (d\iota_{\xi_k} + \iota_{\xi_k} d)\mu^i \\
&= \iota_{\xi_k}(c^i_{ab}\mu^a\theta^b) \\
&= c^i_{ak}\mu^a.
\end{aligned}$$

It follows that the differential d may be redefined as

$$d\theta^i = \mu^i - \frac{1}{2}\theta^j \mathcal{L}_{\xi_j}(\theta^i)$$

on the $\Lambda(\mathfrak{g}^*)$ elements and

$$d\mu^i = \theta^k \mathcal{L}_{\xi_k}(\mu^i) \tag{2.52}$$

on the $S(\mathfrak{g}^*)$ elements. With this in mind, the Weil algebra can be thought of as the total complex of a *double complex* $(C^{*,*}, \delta_1, \delta_2)$ where

$$C^{p,q} = \Lambda^{p-q}(\mathfrak{g}^*) \otimes S^q(\mathfrak{g}^*),$$

and $C^{p,q} = 0$ when $p - q < 0$. The map $\delta_2 : C^{p,q} \to C^{p,q+1}$ is defined on generators by

$$\delta_2(\theta^i) = \mu^i, \quad \delta_2(\mu^i) = 0$$

and the map $\delta_1 : C^{p,q} \to C^{p+1,q}$ is defined on generators by

$$\delta_1(\theta^i) = -\frac{1}{2}\theta^j \mathcal{L}_{\xi_j}(\theta^i), \quad \delta_1(\mu^i) = \theta^k \mathcal{L}_{\xi_k}(\mu^i). \tag{2.53}$$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \Lambda^0(\mathfrak{g}^*) \otimes S^2(\mathfrak{g}^*) \arrow[u] \arrow[r, "\delta_1"] & \cdots \\
 & 0 \arrow[u] \arrow[r] & \Lambda^0(\mathfrak{g}^*) \otimes S^1(\mathfrak{g}^*) \arrow[u] \arrow[r, "\delta_1"] & \Lambda^1(\mathfrak{g}^*) \otimes S^1(\mathfrak{g}^*) \arrow[u, "\delta_2"] \arrow[r, "\delta_1"] & \cdots \\
 & \Lambda^0(\mathfrak{g}^*) \otimes S^0(\mathfrak{g}^*) \arrow[u] \arrow[r, "\delta_1"] & \Lambda^1(\mathfrak{g}^*) \otimes S^0(\mathfrak{g}^*) \arrow[u, "\delta_2"] \arrow[r, "\delta_1"] & \Lambda^2(\mathfrak{g}^*) \otimes S^0(\mathfrak{g}^*) \arrow[u, "\delta_2"] \arrow[r, "\delta_1"] & \cdots \\
 \bullet \arrow[uuu, "q"] \arrow[rr, "p"'] & & & &
\end{tikzcd}$$

Lemma 2.54: *Let $\alpha \in W(\mathfrak{g})$. Then*

$$\iota_\xi \alpha = 0$$

for every $\xi \in \mathfrak{g}$ if and only if $\alpha \in \Lambda^0(\mathfrak{g}^*) \otimes S^m(\mathfrak{g}^*)$ for some integer m.

Proof. The statement is trivial for $m = 0$. Showing that $\iota_\xi \alpha = 0$ implies $\alpha \in \Lambda^0(\mathfrak{g}^*) \otimes S^m(\mathfrak{g}^*)$ for $m > 0$ can be done most concisely by observing that the equation

$$\theta^i \iota_{\xi_i} = n \cdot \mathrm{id}$$

holds on $\Lambda^n(\mathfrak{g}^*) \otimes S^m(\mathfrak{g}^*)$ for any $n > 1$. The converse follows by the definition of ι_ξ. □

By Lemma 2.54 there are no horizontal elements in $\Lambda^{p-q}(\mathfrak{g}^*) \otimes S^q(\mathfrak{g}^*)$ where $p - q > 0$. But $C^{p,q} = 0$ when $p - q < 0$ hence the only horizontal elements lie in $C^{q,q} = \Lambda^0(\mathfrak{g}^*) \otimes S^q(\mathfrak{g}^*)$. Since $\iota_\xi \mu^i = 0$ for every $\xi \in \mathfrak{g}$ the horizontal elements are precisely those in $\Lambda^0(\mathfrak{g}^*) \otimes S^q(\mathfrak{g}^*)$. The differential δ_2 is trivial on the horizontal elements. Since the horizontal elements are generated by $S(\mathfrak{g}^*)$, for a basic element $w \in W(\mathfrak{g})_{bas}$

$$\delta_1 w = \theta^k \mathcal{L}_{\xi_k} w = 0$$

since $\mathcal{L}_\xi w = 0$ for all $\xi \in \mathfrak{g}$ if w is G-invariant. Taking the total complex of the basic subalgebra of this double complex yields the complex

$$\Lambda^0(\mathfrak{g}^*) \otimes S^0(\mathfrak{g}^*) \to 0 \to \Lambda^0(\mathfrak{g}^*) \otimes S^1(\mathfrak{g}^*) \to 0 \to \Lambda^0(\mathfrak{g}^*) \otimes S^2(\mathfrak{g}^*) \to \cdots$$

where every arrow is the 0 map. Consequently, we have proved the following theorem.

Theorem 2.55: *$W(\mathfrak{g})_{hor} \cong S(\mathfrak{g}^*)$ and hence $W(\mathfrak{g})_{bas} \cong S^*(\mathfrak{g}^*)^G$. Moreover,*

$$H^*(W(\mathfrak{g})_{bas}) = S(\mathfrak{g}^*)^G$$

since d is trivial on $W(\mathfrak{g})_{bas}$.

2.3.2 The Weil Model

Let G be a compact Lie group and M be a manifold with right G-action. For any principal G-bundle $P \xrightarrow{\pi} B$, the space $P \times M$ is a manifold with free G-action and so

$$(P \times M)/G$$

is also a manifold and hence admits a de Rham complex. By Proposition 2.42 we can identify the differential forms on $(P \times M)/G$ thusly

$$\Omega^*((P \times M)/G) = \Omega^*_{bas}(P \times M) \cong (\Omega^*(P) \otimes \Omega^*(M))_{bas}.$$

Treating $W(\mathfrak{g})$ as a de Rham complex for EG motivates the following definition.

Definition 2.56: *Let G be a compact Lie group and M a manifold with right G-action. The* Weil model *for the equivariant cohomology of M is defined to be the complex*

$$(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}\,.$$

The fact that this is an adequate model for the equivariant cohomology of M is a well understood result and accordingly we have the following classical theorem which will be proved in the final chapter of this thesis.

Theorem 2.57: *Let G be a compact Lie group and M a manifold with right G-action. There is an isomorphism*

$$H_G(M) \cong H^*((W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}).$$

2.3.3 The Chern-Weil Homomorphism

Let $P \xrightarrow{\pi} B$ be a principal G-bundle with connection $\theta_P \in \Omega^1(P) \otimes \mathfrak{g}$ and curvature $\Omega_P \in \Omega^2(P) \otimes \mathfrak{g}$. Recall from Section 2.2.2 that a choice of connection is equivalent to choosing 1-forms

$$\theta^i_P := \xi^*_i \theta \in \Omega^1(P).$$

such that $\theta^i(p; X_{\xi_j}) = \delta_{ij}$ such that the subspace they span is invariant under the action of G. There are also 2-forms

$$\mu^i_P := \xi^*_i \Omega \in \Omega^2(P)$$

related to the curvature form Ω. Associated to the choice of connection is a unique homomorphism $w(\theta_P) : W(\mathfrak{g}) \to \Omega^*(P)$ such that

$$w(\theta_P)(\theta^i) = \theta^i_P, \quad w(\theta_P)(\mu^i) = \mu^i_P.$$

This is the *Weil homomorphism* which is an equivariant chain map for which one can easily check that the identities

1. $dw(\theta_P) - w(\theta_P)d = 0$ and
2. $\iota_\xi w(\theta_P) - w(\theta_P)\iota_\xi = 0$

hold for every $\xi \in \mathfrak{g}$. In particular, this means that $w(\theta_P)$ descends to a chain map on basic differential forms and the following diagram commutes

$$\begin{tikzcd}
W(\mathfrak{g}) \arrow[r, "w(\theta_P)"] & \Omega^*(P) \\
S(\mathfrak{g}^*)^G \arrow[u, hook] \arrow[r, "\bar{w}(\theta_P)"] & \Omega^*(B) \arrow[u, hook]
\end{tikzcd}$$

since $W(\mathfrak{g})_{bas} \cong S(\mathfrak{g}^*)^G$. Moreover, from Theorem 2.55 the basic cohomology ring of $W(\mathfrak{g})$ is $S(\mathfrak{g}^*)^G$ so the map further induces the classical map in cohomology called the *Chern-Weil homomorphism*

$$\kappa_G : S(\mathfrak{g}^*)^G \to H^*(B)$$

which does not depend on the choice of connection θ_P (see [Bot73] or [AB82] for details).

2.3.4 The Mathai-Quillen Isomorphism

Let G be a Lie group with corresponding Lie algebra $\mathfrak{g}$ of dimension n and basis $\{\xi_1, \cdots, \xi_n\}$. Let M be a manifold with right G-action. Consider the operator

$$\zeta := \theta^i \otimes \iota_{\xi_i} : W(\mathfrak{g}) \otimes \Omega^*(M) \to W(\mathfrak{g}) \otimes \Omega^*(M)$$

where for $\alpha \in W(\mathfrak{g})$ and $\omega \in \Omega^*(M)$

$$\zeta(\alpha \otimes \omega) = \sum_{i=1}^{n} \theta^i \alpha \otimes \iota_{\xi_i} \omega. \tag{2.58}$$

Proposition 2.59: *The series*

$$\exp(\zeta) = 1 + \zeta + \frac{1}{2!}\zeta^2 + \cdots$$

converges.

Proof. Since $\theta^i\theta^i = 0$, by the pigeonhole principle one notes that

$$\theta^{i_1} \cdots \theta^{i_{n+1}} = 0$$

and hence $\zeta^{n+1} = 0$. $\square$

Since $\exp(\zeta)$ converges, it has an inverse $\exp(-\zeta)$ and is thus an automorphism of $W(\mathfrak{g}) \otimes \Omega^*(M)$ which is known as *the Mathai-Quillen isomorphism*. One notes that ζ is equivariant and hence $\exp(\zeta)$ is equivariant. The derivations d and ι_ξ are not preserved by $\exp(\zeta)$ and so we turn to calculations of [MQ86], [Kal93] via [GS99, p. 41 - 44].

Theorem 2.60: *The automorphism* $\exp(\zeta)$ *satisfies*

$$\exp(\zeta) d \exp(-\zeta) = d - \mu^k \otimes \iota_{\xi_k} + \theta^k \otimes \mathcal{L}_{\xi_k} \tag{2.61}$$

and

$$\exp(\zeta)(1 \otimes \iota_\xi + \iota_\xi \otimes 1) \exp(-\zeta) = \iota_\xi \otimes 1 \tag{2.62}$$

for every $\xi \in \mathfrak{g}$.

The identity 2.62 shows that the automorphism $\exp(\zeta)$ can be restricted to the horizontal subspace $(W(\mathfrak{g}) \otimes \Omega^*(M))_{hor}$ to yield the following corollary.

Corollary 2.63: $\exp(\zeta) : (W(\mathfrak{g}) \otimes \Omega^*(M))_{hor} \to S(\mathfrak{g}^*) \otimes \Omega^*(M)$.

Proof. Let $\alpha \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{hor}$. Then α satisfies

$$(\iota_\xi \otimes 1) \exp(\zeta)\alpha = 0,$$

thus $\exp(\zeta)\alpha \in W(\mathfrak{g})_{hor} \otimes \Omega^*(M) \cong S(\mathfrak{g}^*) \otimes \Omega^*(M)$ by Theorem 2.55. □

The horizontal subspace is not closed under d, but the *basic* subspace is. Thus the automorphism $\exp \zeta$ restricts further to an algebra automorphism on the basic subalgebra.

2.3.5 The Cartan Model

Following the last section (specifically Corollary 2.63), the Mathai-Quillen isomorphism defines an algebra isomorphism

$$\exp(\zeta) : (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \to (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G .$$

The differential as described in 2.61 can be further simplified on G-invariant elements, too. By the calculation in 2.52,

$$d = d \otimes 1 + 1 \otimes d = \theta^k \mathcal{L}_{\xi_k} \otimes 1 + 1 \otimes d$$

and hence

$$\begin{aligned} \exp(\zeta) d \exp(-\zeta) &= \theta^k \mathcal{L}_{\xi_k} \otimes 1 + 1 \otimes d - \mu^k \otimes \iota_{\xi_k} + \theta^k \otimes \mathcal{L}_{\xi_k} \\ &= 1 \otimes d - \mu^k \otimes \iota_{\xi_k} + \theta^k \otimes 1(\mathcal{L}_{\xi_k} \otimes 1 + 1 \otimes \mathcal{L}_{\xi_k}). \end{aligned}$$

Since the basic subalgebra is G-invariant, $\mathcal{L}_{\xi_k} \otimes 1 + 1 \otimes \mathcal{L}_{\xi_k} = 0$ on $(S(\mathfrak{g}^*) \otimes \Omega^*(M))^G$ and the differential simplifies to

$$d_G := 1 \otimes d - \mu^k \otimes \iota_{\xi_k}.$$

Definition 2.64: *Let M be a manifold and G be a Lie group with Lie algebra $\mathfrak{g}$. Define the* Cartan Model *to be the complex*

$$(S(\mathfrak{g}^*) \otimes \Omega^*(M))^G$$

with differential

$$d_G := 1 \otimes d - \mu^k \otimes \iota_{\xi_k}.$$

Taking Theorem 2.57 for granted, we have proved the following theorem about the Cartan model for equivariant cohomology.

Theorem 2.65: *Let G be a compact Lie group and M a manifold with right G-action. Then there is an isomorphism*

$$H^*_G(M) \cong H^*((S(\mathfrak{g}^*) \otimes \Omega^*(M))^G).$$

Chapter 3

Simplicial Methods

3.1 Simplicial and Cosimplicial Objects

3.1.1 The Simplicial Category

Define *the simplicial category*, Δ, to be the category whose objects are ordered sets of integers $[n] = \{0 < 1 < \cdots < n\}$ (for $n \geq 0$) and whose morphisms $\mu : [n] \to [m]$ are non-decreasing maps (i.e. $\mu(i) \leq \mu(j)$ if $i < j$). In particular, for an injective non-decreasing map $\mu : [n-1] \to [n]$ there are only a total of $n+1$ morphisms to choose from, which may canonically be indexed by $d^i : [n-1] \to [n]$ for $0 \leq i \leq n$ as

$$d^i(k) = j \text{ if } k < i \text{ and } d^i(k) = k+1 \text{ if } k \geq i.$$

These canonical maps are often referred to as the *coface maps* (or *cofaces*). Similarly we may also define non-decreasing maps $s^i : [n+1] \to [n]$ for $0 \leq i \leq n$ as

$$s^i(k) = k \text{ if } k \leq i \text{ and } s^i(k) = k-1 \text{ if } k > i,$$

that can be seen to be, in some sense, complementary to the d^i morphisms and are often referred to as the *codegeneracy maps* (or *codegeneracies*). With these definitions it is not difficult to check that the following *cosimplicial identities* hold whenever their composition is defined.

$$d^j\, d^i = d^i\, d^{j-1} \qquad \text{if } i < j. \tag{3.1}$$

$$s^j\, d^i = \begin{cases} d^i\, s^{j-1} & \text{if } i < j. \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1. \\ d^{i-1}\, s^j & \text{if } i > j+1. \end{cases} \tag{3.2}$$

$$s^j\, s^i = s^i\, s^{j+1} \qquad \text{if } i \leq j. \tag{3.3}$$

The codegeneracies are complementary to the cofaces in the sense that $d^i s^i = \mathrm{id}$ and that any morphism $\mu : [n] \to [m]$ can be written as the composition of these morphisms as seen in the following result of May [May67].

Proposition 3.4: *Let $\mu : [n] \to [m]$ be a map in Δ. Then μ decomposes into a unique sequence of maps*

$$\mu = d^{i_r} \cdots d^{i_1} s^{j_1} \cdots s^{j_t}$$

where $0 \leq i_1 \leq \cdots \leq i_r \leq m$ and $0 \leq j_1 \leq \cdots \leq j_s \leq m$.

Proof. The image of μ contains at most n elements. Let $I = \{i_1, i_2, \ldots i_r\}$ be the set of elements that are not in the image of μ in increasing order and let $J = \{j_i, \ldots, j_t\}$ be the set of elements such that $\mu(j_i) = \mu(j_i + 1)$ in increasing order. We check that the map decomposes into

$$\mu = d^{i_r} \cdots d^{i_1} s^{j_1} \cdots s^{j_t}$$

by applying induction.

First suppose that $\mu(0) = p_0$. Note that $s^i(0) = 0$ for every $i \geq 0$ and hence we need only check how d^i affects the calculation. Since μ is non-decreasing, $0, 1, \ldots, p_0 - 1$ are necessarily in I and one notes that $d^{p_0-1} d^{p_0-2} \cdots d^0(0) = p_0$. But $p_0 \notin I$ and $d^i(p_0) = p_0$ for every $i > p_0$ and so

$$\mu(0) = d^{i_r} \cdots d^{i_1} s^{j_1} \cdots s^{j_t}(0).$$

Now assume that $\mu(k) = d^{i_r} \cdots d^{i_1} s^{j_1} \cdots s^{j_t}(k)$. If $\mu(k) = \mu(k+1)$ then $k \in J$. Note that if $i \geq k+1$ then $s^i(k) = k$ and $s^i(k+1) = k+1$ and in particular, $s^k s^i(k) = k$ and $s^k s^i(k+1) = k$. From this observation we can deduce that induction holds if $\mu(k) = \mu(k+1)$. If we suppose that $\mu(k+1) = \mu(k) + c$ for some $c \neq 0$ then $k \notin J$ and so

$$s^{j_1} \cdots s^{j_t}(k+1) = s^{j_1} \cdots s^{j_t}(k) + 1.$$

Also, it must be true that $k+1, \ldots, k+c-1 \in I$ and so

$$d^{i_r} \cdots d^{i_1} s^{j_1} \cdots s^{j_t}(k+1) = d^{i_r} \cdots d^{i_1} s^{j_1} \cdots s^{j_t}(k) + 1 + c - 1 = \mu(k+1)$$

as required. In fact, since $n - t$ is the number of elements in the image $\mu([n])$ and r is the number of missed elements, it is true that $n - t + r = m$ which confirms that $d^{i_r} \cdots d^{i_1} s^{j_1} \cdots s^{j_t}$ is indeed a morphism from $[n]$ to $[m]$. $\square$

In category theory there is the notion of the *opposite category*. For any category $\mathscr{C}$ there is its opposite category $\mathscr{C}^{op}$ with objects A^{op} for each object $A \in \mathscr{C}$, morphisms $a^{op} : B^{op} \to A^{op} \in \mathrm{Hom}(B^{op}, A^{op})$ for each morphism $a : A \to B \in \mathrm{Hom}(A, B)$ and insisting that $a^{op} b^{op} = (ba)^{op}$ whenever the morphism ba is defined.

Definition 3.5: *A* simplicial object *in a category $\mathscr{C}$ is a functor*

$$F : \Delta^{op} \to \mathscr{C}.$$

If F and G are simplicial objects in $\mathscr{C}$ then a simplicial map *from F to G is a natural transformation*

$$\alpha : F \to G.$$

Thus the simplicial objects in $\mathscr{C}$, together with the simplicial maps between them form a category $s\mathscr{C}$. Moreover, let $d^i : [n-1] \to [n]$ be a coface map of Δ for $0 \le i \le n$. The map $d_i := F(d^i)$ is called a face map *of F. Similarly, let $s^i : [n+1] \to [n]$ be a codegeneracy map of Δ for $0 \le i \le n$. The map $s_i := F(s^i)$ is called a* degeneracy map *of F.*

This definition is helpful in understanding the relationships between simplicial objects later in the discussion, but may not be satisfying to the average reader. To help develop the idea of simplicial objects, some simplicial objects will be defined more explicitly and then shown to be equivalent to their definition as functors. It will thus be useful to have the following corollary of Proposition 3.4.

Corollary 3.6: *Let $\mu^{op} : [m]^{op} \to [n]^{op}$ be a morphism in the category Δ^{op}. Then μ^{op} decomposes into a unique sequence of maps*

$$\mu^{op} = (s^{j_t})^{op} \cdots (s^{j_1})^{op} (d^{i_1})^{op} \cdots (d^{i_r})^{op}.$$

where $0 \le i_1 \le \cdots \le i_r \le m$ and $0 \le j_1 \le \cdots \le j_s \le m$.

This corollary means that a simplicial object $F : \Delta^{op} \to \mathscr{C}$ can be uniquely determined by the images of objects $F_i = F([i])$, face maps d_i and degeneracy maps s_i so long as they meet the *simplicial identities* corresponding to (3.1) — (3.3),

$$d_i\, d_j = d_{j-1}\, d_i \qquad \text{if } i < j. \tag{3.7}$$

$$d_i\, s_j = \begin{cases} s_{j-1}\, d_i & \text{if } i < j. \\ \text{id} & \text{if } i = j \text{ or } i = j+1. \\ s_j\, d_{i-1} & \text{if } i > j+1. \end{cases} \tag{3.8}$$

$$s_i\, s_j = s_{j+1}\, s_i \qquad \text{if } i \le j. \tag{3.9}$$

Since a simplicial map $f : F \to G$ is a natural transformation between simplicial objects, it can simply be checked that

$$d_i f = f d_i \quad \text{and} \quad s_i f = f s_i$$

for every corresponding pair of face and degeneracy map of F and G. Although simplicial objects will be the main focus of later sections, it will be helpful to first have an example of a particular *cosimplicial* object first.

3.1.2 Cosimplicial Objects

Definition 3.10: *A* cosimplicial object *in a category $\mathscr{C}$ is a functor*

$$F : \Delta \to \mathscr{C}.$$

If F and G are cosimplicial objects in $\mathscr{C}$ then a cosimplicial map *from F to G is a natural transformation*

$$\alpha : F \to G.$$

Thus the cosimplicial objects in $\mathscr{C}$, together with the cosimplicial maps between them form a category. Moreover, let $d^i : [n-1] \to [n]$ be a coface map of Δ for $0 \leq i \leq n$. The map $d^i := F(d^i)$ is called a coface map *of F. Similarly, let $s^i : [n+1] \to [n]$ be a codegeneracy map of Δ for $0 \leq i \leq n$. The map $s^i := F(s^i)$ is called a* codegeneracy map *of F.*

Similarly, Proposition 3.4 implies that a cosimplicial object $F : \Delta \to \mathscr{C}$ can be uniquely determined by the images of objects $F_i = F([i])$, face maps d^i and degeneracy maps s^i meeting the cosimplicial identities (3.1) — (3.3). One example of a cosimplicial object that will appear several times in this thesis is the collection of topological n-simplices, $\Delta^\bullet$.

Example 3.11: The *standard topological n-simplex* is the space

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} : t_0, t_1, \cdots, t_n \geq 0, \sum_{i=0}^{n} t_i = 1\}.$$

If we let $\{e_0, \dots, e_n\}$ be the standard basis of $\mathbb{R}^{n+1}$, equivalently Δ^n is the convex hull of this basis. A map $\alpha : [m] \to [n]$ induces a unique linear map $\tilde{\alpha} : \Delta^m \to \Delta^n$ determined by

$$\tilde{\alpha}(e_i) = e_{\alpha(i)}.$$

We can also explicitly define the coface and codegeneracy maps which, as mentioned in 3.1.1, will uniquely determine a functor $F : \Delta^{op} \to \mathscr{C}$. The n-simplex can be canonically embedded in the $(n+1)$-simplex by defining coface maps $d^i : \Delta^n \to \Delta^{n+1}$ for every $0 \leq i \leq n$, which maps the n-simplex into the 'i^{th} boundary component' by

$$d^i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n).$$

Note that the n-simplex has $n+1$ vertices where the i^{th} vertex is at $t_i = 1$, $t_j = 0$ for $j \neq i$. There are then codegeneracy maps $s^i : \Delta^n \to \Delta^{n-1}$ which 'collapse the i^{th} vertex' defined by

$$s^i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_2 \dots, t_n).$$

With the coface and codegeneracy maps thus defined, it becomes apparent that $\Delta^\bullet = \{\Delta^n\}_{n \geq 0}$ is a cosimplicial object by defining a functor $F : \Delta \to \text{Top}$ by the obvious assignment

$$F([n]) = \Delta^n, \quad F(d^i) = d^i, \quad F(s^i) = s^i,$$

and noting that the maps satisfy the cosimplicial identities (3.1) — (3.3).

3.1.3 Simplicial Objects

The following construction of the cosimplicial object $\Delta^\bullet$ leads naturally to the singular simplicial set and singular homology. Let X be a topological space and define a singular n-simplex σ to be a continuous map from the topological n-simplex into X,

$$\sigma : \Delta^n \to X.$$

Denote the set of all singular n-simplices of X by $S_n(X)$. Note that there are natural face and degeneracy maps on the family of sets $\{S_n(X)\}_{n\geq 0}$ induced by the coface and codegeneracy maps on $\Delta^\bullet$. Namely, for an n-simplex σ, we define the face maps $d_i : S_n(X) \to S_{n-1}(X)$ by the composition

$$d_i(\sigma) = \sigma \circ d^i : \Delta^{n-1} \to X \in S_{n-1}(X) \tag{3.12}$$

and respectively define the degeneracy maps $s_i : S_n(X) \to S_{n+1}(X)$ by the composition

$$s_i(\sigma) = \sigma \circ s^i : \Delta^{n+1} \to X \in S_{n+1}(X) \tag{3.13}$$

for every $0 \leq i \leq n$. One checks that these satisfy the simplicial identities (3.7) — (3.9). Denote the family of sets $\{S_n(X)\}_{n\geq 0}$ together with face and degeneracy maps the *singular simplicial set* of a topological space X, which can be thought of as a functor

$$S_\bullet(X) : \Delta^{op} \to \text{Set}.$$

Letting $\sigma : \Delta^n \to X$ be an n-simplex of X, a continuous map $f : X \to Y$ induces a map on the simplicial set $S(f) : S_n(X) \to S_n(Y)$ for each $n \geq 0$ by

$$S(f)\sigma = f \circ \sigma : \Delta^n \to Y.$$

For an n-simplex of X, it is a simple calculation to check that

$$\begin{aligned} d_i\left(S(f)\sigma\right) &= d_i(f\sigma) \\ &= f\sigma d^i \\ &= f(d_i\sigma) \\ &= S(f)(d_i\sigma) \end{aligned}$$

and similarly for the degeneracy maps; thus $S(f)$ is a simplicial map. Leaving the categorical definition behind, we offer an alternative definition of a *simplicial set*.

Definition 3.14: *A* simplicial set $X_\bullet$ *is a family of sets* $X_\bullet = \{X_q\}_{q\geq 0}$ *together with face operators* $d_i : X_q \to X_{q-1}$ *and degeneracy operators* $s_i : X_q \to X_{q+1}$, $i = 0, 1, \ldots, q$, *such that* *the identities (3.7) — (3.9) hold.*

$$d_i\, d_j = d_{j-1}\, d_i \qquad \textit{if } i < j.$$

$$d_i\, s_j = \begin{cases} s_{j-1}\, d_i & \textit{if } i < j. \\ \ \ \mathrm{id} & \textit{if } i = j \ \textit{ or } i = j+1. \\ s_j\, d_{i-1} & \textit{if } i > j+1. \end{cases}$$

$$s_i\, s_j = s_{j+1}\, s_i \qquad \textit{if } i \leq j.$$

A map of simplicial sets $f : X_\bullet \to Y_\bullet$ is then defined to be a sequence of functions $\{f_q : X_q \to Y_q\}_{q\geq 0}$ such that they commute with the face and degeneracy operators. That is, $f_q d_i = d_i f_{q+1}$ and $f_{q+1} s_i = s_i f_q$ for every $q \geq 0$. The collection of simplicial sets forms *the category of simplicial sets* which we will denote sSet.

Remark 3.15: We may also take $X_\bullet$ to be a family of topological spaces, manifolds, groups, etc and replace the face and degeneracy maps with the appropriate morphisms – continuous maps, smooth maps, homomorphisms, etc respectively.

Proposition 3.16: *The category sSet of simplicial sets is equal to the category of functors $F : \Delta^{op} \to \mathrm{Set}$.*

Proof. Let $X_\bullet = \{X_q\}_{q\geq 0}$ be a simplicial set. In Corollary 3.6 we saw that any map $\mu^{op} : [n]^{op} \to [m]^{op}$ in the category Δ^{op} can be decomposed uniquely into the map $(s^{j_t})^{op} \cdots (s^{j_1})^{op} (d^{i_1})^{op} \cdots (d^{i_r})^{op}$. Then define a functor $F : \Delta^{op} \to \mathrm{Set}$ such that

$$F([q]) = X_q, \quad F([n] \xrightarrow{\mu^{op}} [m]) = s_{j_t} \cdots s_{j_1} d_{i_1} \cdots d_{i_r} : X_n \to X_m.$$

Conversely, by using identities (3.7) — (3.9) that any combination of face and degeneracy maps on the simplicial set $X_\bullet$, say $\nu : X_n \to X_m$, may also be rearranged such that

$$\nu = s_{j_t} \cdots s_{j_1} d_{i_1} \cdots d_{i_r}.$$

Now let $Y_\bullet = \{Y_q\}_{q\geq 0}$ be a simplicial set associated to the functor $G : \Delta^{op} \to \mathit{Set}$ and suppose we have a map $f : X_\bullet \to Y_\bullet$. Necessarily the diagrams

$$\begin{tikzcd} X_q \arrow[r, "f_q"] \arrow[d, "d_i"'] & Y_q \arrow[d, "d_i"] \\ X_{q-1} \arrow[r, "f_{q-1}"] & Y_{q-1} \end{tikzcd} \qquad \begin{tikzcd} X_{q+1} \arrow[r, "f_{q+1}"] & Y_{q+1} \\ X_q \arrow[r, "f_q"] \arrow[u, "s_i"] & Y_q \arrow[u, "s_i"'] \end{tikzcd}$$

commute for all $q \geq 0$. This can be rewritten in the language of functors, in which case the diagrams become

$$\begin{array}{ccc} F([q]) & \xrightarrow{f_q} & G([q]) \\ {\scriptstyle F(d_i)}\downarrow & & \downarrow{\scriptstyle G(d_i)} \\ F([q-1]) & \xrightarrow{f_{q-1}} & G([q-1]) \end{array} \qquad \begin{array}{ccc} F([q+1]) & \xrightarrow{f_{q+1}} & G([q+1]) \\ {\scriptstyle F(s_i)}\uparrow & & \uparrow{\scriptstyle G(s_i)} \\ F([q]) & \xrightarrow{f_q} & G([q]) \end{array}$$

after which it becomes clear that $f : X_\bullet \to Y_\bullet$ can be realised as a natural transformation between functors $f : F \to G$ since we can write all morphisms in the simplicial category Δ as a combination of the face and degeneracy maps. $\square$

Denote the free vector space of singular n-simplices over $\mathbb{R}$ by $C_n(X)$. That is, a general element of $C_n(X)$, or a *singular n-chain*, will be a finite formal sum of elements

$$\sum_i c_i \sigma_i \in C_n(X)$$

for n-simplices σ_i and coefficients $c_i \in \mathbb{R}$. We define the face and degeneracy maps to be the linear extensions of d_i, s_i (i.e. (3.12), (3.13)) on the formal sums of n-simplices. Let $\mathrm{Vect}_\mathbb{R}$ denote the category of vector spaces over $\mathbb{R}$, whose morphisms are linear maps. One notes that the family of vector spaces $C_\bullet(X) = \{C_n(X)\}_{n\geq 0}$ can thus be thought of as a functor

$$F : \Delta^{op} \to \mathrm{Vect}_\mathbb{R}.$$

We call $C_\bullet(X)$ the *singular simplicial chain complex* of the topological space, X. Since a continuous map $f : X \to Y$ induces a map $S(f) : C_\bullet(X) \to C_\bullet(Y)$, the linear extension $f_\# : C_\bullet(X) \to C_\bullet(X)$ is a simplicial map. That is, given a singular n-chain $\sum_i c_i\sigma_i$, define

$$f_\# \sum_i c_i \sigma_i = \sum_i c_i S(f) \sigma_i.$$

We recall (e.g. [Hat01]) that there is a differential operator

$$\partial_n : C_n(X) \to C_{n-1}(X),$$

sometimes called the boundary operator, such that an n-simplex is sent to the sum of its boundary components and $\partial_n \partial_{n+1} = 0$. This boundary operator can be constructed using the face maps by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

One checks that using the relation in (3.7)

$$
\begin{aligned}
\partial_n \partial_{n+1} &= \sum_{i,j} (-1)^{i+j} d_i d_j \\
&= \sum_{i<j} (-1)^{i+j} d_{j-1} d_i + \sum_{j \leq i} (-1)^{i+j} d_i d_j \\
&= \sum_{q<p+1} (-1)^{p+q+1} d_p d_q + \sum_{j \leq i} (-1)^{i+j} d_i d_j \\
&= 0
\end{aligned}
$$

since $q < p+1$ is equivalently restated as $q \leq p$ and sums over the same integers.

If M is a manifold, one may also consider a *smooth n-simplex on M* to be a smooth map

$$\sigma : \Delta^n \to M.$$

One may also consider the subspace $S_\bullet^\infty(M) \subset S_\bullet(M)$ of smooth singular simplices on M and hence $C_\bullet^\infty(M) \subset C_\bullet(M)$ the smooth singular chains on M. By the Whitney Approximation Theorem any singular simplex is homotopic to a smooth singular simplex and the inclusion

$$C_n^\infty(M) \to C_n(M)$$

is an isomorphism in homology of chain complexes (see [Lee02, Theorem 16.6] for further details). By dualising the singular complex $C_n(X)$ one obtains the set of *singular n-cochains* $C^n(X)$ whose elements are maps

$$\epsilon : C_n(X) \to \mathbb{R}.$$

These vector spaces naturally carry face maps $d^i : C^n(X) \to C^{n+1}$ by the precomposition

$$d^i \epsilon = \epsilon \circ d_i$$

and degeneracy maps

$$s^i \epsilon = \epsilon \circ s_i.$$

Similarly, $C^n(X)$ is endowed with a natural differential operator $d : C^n(X) \to C^{n+1}(X)$ given by

$$d = \sum_{i=0}^{n} (-1)^i d^i$$

for each $n \geq 0$. For any topological space, one defines the cohomology of X with real coefficients by the sequence of cohomology groups $H^n(X)$ where

$$H^n(X) = \frac{\ker d \cap C^n(X)}{\mathrm{im} d \cap C^n(X)}$$

in the usual way (e.g. [Hat01]). Let M be a manifold and $C^n_\infty(M)$ denote the subspace of smooth singular cochains from dualising $C_n(X)$. The Whitney Approximation Theorem tells us that there is an isomorphism

$$H^n(M) \cong \frac{\ker d \cap C^n_\infty(M)}{\mathrm{im} d \cap C^n_\infty(M)}$$

which we will use later in a proof of de Rham's theorem.

3.1.4 The Nerve of a Category

Definition 3.17: *Let $\mathscr{C}$ be a category with objects* $\mathrm{Ob}(\mathscr{C})$*, morphisms* $\mathrm{Hom}(\mathscr{C})$ *and denote* $\mathrm{Hom}(\mathscr{C}) \circ \mathrm{Hom}(\mathscr{C})$ *by the class of pairs of composable morphisms. A* topological category $\mathscr{C}$ *is a category where* $\mathrm{Ob}(\mathscr{C})$ *and* $\mathrm{Hom}(\mathscr{C})$ *are both topological spaces and*

- *The source map* $S : \mathrm{Hom}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$*,* $S(A \to B) = A$*, is continuous,*
- *the target map* $T : \mathrm{Hom}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$*,* $T(A \to B) = B$*, is continuous and*
- *composition* $C : \mathrm{Hom}(\mathscr{C}) \circ \mathrm{Hom}(\mathscr{C}) \to \mathrm{Hom}(\mathscr{C})$*,* $C(f, g) = g \circ f$*, is continuous*

where $A, B \in \mathrm{Ob}(\mathscr{C})$ *and* $f, g \in \mathrm{Hom}(\mathscr{C})$ *with* $T(f) = S(g)$*.*

Let $\mathscr{C}$ be a topological category. The *nerve* of the category $\mathscr{C}$, $N\mathscr{C}_\bullet = \{N\mathscr{C}_i\}_{i\geq 0}$, is the simplicial space where $N\mathscr{C}_0 = \mathrm{Ob}(\mathscr{C})$, $N\mathscr{C}_1 = \mathrm{Hom}(\mathscr{C})$ and for $n > 1$

$$N\mathscr{C}_n = \mathrm{Hom}(\mathscr{C}) \circ \cdots \circ \mathrm{Hom}(\mathscr{C}) \ \ \text{(n times)}$$

is the subset of n composable morphisms in $\mathscr{C}$. For example, consider the string of morphisms

$$A_{n+1} \xrightarrow{f_n} A_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1.$$

Then $(f_1, \ldots, f_n)$ is an element of $N\mathscr{C}_n$. We define face maps $d_i : N\mathscr{C}_n \to N\mathscr{C}_{n-1}$ by

$$d_i(f_1, \ldots, f_n) = \begin{cases} (f_2, \ldots, f_n) & \text{if } i = 0 \\ (f_1, \ldots, f_i \circ f_{i+1}, \ldots, f_n) & \text{if } 0 < i < n \\ (f_1, \ldots, f_{n-1}) & \text{if } i = n \end{cases} \tag{3.18}$$

and degeneracy maps $s_i : N\mathscr{C}_n \to N\mathscr{C}_{n+1}$ for $i = 0, \ldots, n$ by

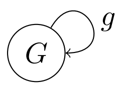
$$s_i(f_1, \ldots, f_n) = (f_1, \ldots, f_{i-1}, \mathrm{id}_{A_i}, f_i, \ldots, f_n) \tag{3.19}$$

where $X = S(f_i) = T(f_{i-1})$.

Remark 3.20: If $\mathscr{C}, \mathscr{D}$ are topological categories, then there is a natural notion of a *continuous functor* from $\mathscr{C}$ to $\mathscr{D}$. The topological categories and continuous functors between them form a category Cat_T. Note that one may consider the nerve $N : \mathrm{Cat}_T \to \mathrm{Top}$ as a functor. For a continuous functor $F : \mathscr{C} \to \mathscr{D}$, $NF : N\mathscr{C}_\bullet \to N\mathscr{D}_\bullet$ is a simplicial map of simplicial topological spaces.

We finish this subsection with an example of a simplicial object that will become ubiquitous in the rest of this thesis.

Example 3.21: A topological group G can be considered a topological category where the only object is the group itself endowed with the trivial topology, $\mathrm{Ob}(G) = \{G\}$, and the morphisms are multiplication by group elements, $\mathrm{Hom}(G) = G$ endowed with the topology of the group. That is, the only object is the group itself and the morphisms of the group are the group elements, $g \in G$.



Since every pair of morphisms in G is composable, $\mathrm{Hom}(G) \circ \mathrm{Hom}(G) = G \times G$ and moreover, the nerve of this category NG is the simplicial space where NG_0 is the single object $\{G\}$ (treated as a point) and in general

$$NG_n = G^n = G \times G \times \cdots \times G \quad (n \text{ times}).$$

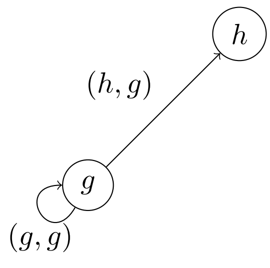
Explicitly, the face maps $d_i : NG_n \to NG_{n-1}$ are

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases} \tag{3.22}$$

and the degeneracy maps $s_i : NG_n \to NG_{n+1}$ are

$$s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_n). \tag{3.23}$$

Perhaps more naturally, one may also consider a topological group as a category $\overline{G}$ whose objects are the group elements, $\mathrm{Ob}(\overline{G}) = G$, endowed with the topology of the group and whose morphisms are the maps $\mathrm{Hom}(\overline{G}) = \{g \to h : g, h \in G\} = G \times G$ endowed with the product topology. We could think of a map $g \to h$ between elements of G as a pair (h, g) to make this clearer.



Two morphisms $g \to h$ and $g' \to h'$ with $h = g'$ may be composed

$$(g \to h) \circ (g' \to h') = g \to h'.$$

Phrasing this in the pair notation, two pairs (h, g) and (h', g') with $h = g'$ may be composed $(h', g') \circ (h, g) = (h', g)$. In this way, an element $g_0 \to g_1 \to g_2$ of $\mathrm{Hom}(\overline{G}) \circ \mathrm{Hom}(\overline{G})$ can be thought of as a triple (g_2, g_1, g_0). Explicitly, the face maps $d_i : N\overline{G}_n \to N\overline{G}_{n-1}$ are

$$d_i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}g_i, \ldots, g_n) \tag{3.24}$$

and the degeneracy maps $s_i : N\overline{G}_n \to N\overline{G}_{n+1}$ are

$$s_i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}, g_i, g_i, \ldots, g_n). \tag{3.25}$$

Consider the functor $\Gamma : \overline{G} \to G$ where $\gamma(g) = G$ for every $g \in G$ and

$$\mathrm{Hom}(\overline{G}) \ni \ (h, g) \xmapsto{\ \Gamma\ } hg^{-1} \ \in \mathrm{Hom}(G).$$

Clearly the functor is continuous and hence $N\Gamma$ is a simplicial map under the nerve. Specifically, the simplicial map $\gamma = N\Gamma : N\overline{G}_\bullet \to NG_\bullet$ is the collection of maps $\{\gamma_n : G^{n+1} \to G^n\}_{n \geq 0}$ where

$$\gamma_n(g_0, \ldots, g_n) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \ldots, g_{n-1} g_n^{-1}).$$

In fact, this is a *simplicial principal G-bundle* in the sense of the following definition.

Definition 3.26: *Let G be a Lie group. A* simplicial principal G bundle $P_\bullet \to B_\bullet$ *is a family of principal G-bundles $P_\bullet = \{P_q\}_{q \geq 0}$ together with* face bundle maps $d_i : P_q \to P_{q-1}$ *and* degeneracy bundle maps $s_i : P_q \to P_{q+1}$, *$i = 0, 1, \ldots, q$, such that the identities (3.7) — (3.9) hold.*

$$\begin{aligned}
d_i\, d_j &= d_{j-1}\, d_i && \textit{if } i < j. \\
d_i\, s_j &= \begin{cases} s_{j-1}\, d_i & \\ \mathrm{id} & \\ s_j\, d_{i-1} & \end{cases} && \begin{aligned} &\textit{if } i < j. \\ &\textit{if } i = j \textit{ or } i = j+1. \\ &\textit{if } i > j+1. \end{aligned} \\
s_i\, s_j &= s_{j+1}\, s_i && \textit{if } i \leq j.
\end{aligned}$$

3.1.5 Geometric Realisation

In this thesis, simplicial methods will be used to prove cohomological results. Since cohomology is a topological property there will be machinery needed to relate a simplicial object to a topological space. In the paper [Mil57], Milnor introduced the notion of *geometric realisation* which assigned a topological space to a simplicial set.

Definition 3.27: *Let* $X_\bullet = \{X_n\}_{n\geq 0}$ *be a simplicial space. The* geometric realisation *of* X *is the topological space*

$$|X_\bullet| = \coprod_{n\geq 0} X_n \times \Delta^n / \sim$$

where, for $x \in X_n, t \in \Delta^{n-1}$*, we have the identification*

$$(d_i x, t) \sim (x, d^i t)$$

and for $x \in X_n, t \in \Delta^{n+1}$*, we have the identification*

$$(s_i x, t) \sim (x, s^i t).$$

The definition of geometric realisation above may also be extended to all simplicial sets by endowing them with the discrete topology.

Proposition 3.28: *Geometric realisation,* $|-| : s\mathrm{Top} \to \mathrm{Top}$*, is a functor.*

Proof. Clearly for $X_\bullet = \{X_n\}_{n\geq 0}$ a simplicial space $|X_\bullet|$ is a topological space. Consider $f : X_\bullet \to Y_\bullet$ a family of maps $\{f_n : X_n \to Y_n\}_{n\geq 0}$. The map $|f| : |X_\bullet| \to |Y_\bullet|$ is the map

$$[x, t] \mapsto [f(x), t]$$

where $[x, t]$ denotes an equivalence class in $|X_\bullet|$. This is well defined because $f : X_\bullet \to Y_\bullet$, being a map of simplicial spaces, commutes with the face and degeneracy operators. From this construction it is immediately clear that $|\mathrm{id}_{X_\bullet}| = \mathrm{id}_{|X_\bullet|} : |X_\bullet| \to |X_\bullet|$.
Now let $Y = \{Y_n\}_{n\geq 0}$, $Z = \{Z_n\}_{n\geq 0}$ be simplicial sets and f, g be simplicial maps

$$X_\bullet \xrightarrow{f} Y_\bullet \xrightarrow{g} Z_\bullet.$$

First we need to show that given $f : X_\bullet \to Y_\bullet$, $|f| : |X_\bullet| \to |Y_\bullet|$ is continuous. First consider the diagram

$$\begin{tikzcd}
\coprod_{n\geq 0} X_n \times \Delta^n \arrow[r, "\tilde{f}"] \arrow[d] & \coprod_{n\geq 0} Y_n \times \Delta^n \arrow[d] \\
|X_\bullet| \arrow[r, "|f|"] & |Y_\bullet|
\end{tikzcd}$$

where $\tilde{f}(x, t) = (f(x), t)$. A set U is open in $|Y_\bullet|$ if and only if there is an open set

$$\tilde{U} \subset \coprod_{n\geq 0} Y_n \times \Delta^n$$

that is sent to U under the quotient topology. Then, for an open set $U \subset |Y_\bullet|$,

$$|f|^{-1}(U) = \{[x,t] \in |X_\bullet| : [f(x),t] \in U\}$$

is open if and only if

$$\tilde{f}^{-1}(U) = \{(x,t) \in |X_\bullet| : (f(x),t) \in U\}$$

is open in $\coprod_{n\geq 0} X_n \times \Delta^n$. Since f is a map of simplicial spaces, $\tilde{f}^{-1}(U)$ will be open in the product topology and hence $|f| : |X_\bullet| \to |Y_\bullet|$ is a continuous map. Finally,

$$\begin{aligned} |g \circ f|[x,t] &= [(g \circ f)(x), t] \\ &= [g(f(x)), t] \\ &= |g|[f(x), t] \\ &= (|g| \circ |f|)[x,t]. \end{aligned}$$

Hence $|g \circ f| = |g| \circ |f|$. □

In Dupont's monograph [Dup78] he uses the *fat realisation.* There will be more exposition comparing these two topological constructions in Section 4.1.2.

Definition 3.29: *Let $X_\bullet = \{X_n\}_{n\geq 0}$ be a simplicial space. The* fat realisation *of $X_\bullet$ is the topological space*

$$\|X_\bullet\| = \coprod_{n\geq 0} X_n \times \Delta^n / \sim$$

where, for $x \in X_n, t \in \Delta^{n-1}$, we have the identification

$$(d_i x, t) \sim (x, d^i t).$$

Corollary 3.30: *Fat realisation, $\| - \| : s\mathrm{Top} \to \mathrm{Top}$, is a functor.*

Proof. This proof immediately follows from Proposition 3.28. □

For the following theorems regarding the geometric realisation of products require that we must restrict ourselves to the category $\mathcal{U}$ of compactly generated Hausdorff spaces. Specifically we turn our attention to the category $\mathcal{SU}$, a subcategory of $\mathcal{S}\mathrm{Top}$, whose objects are functors

$$F : \Delta^{op} \to \mathcal{U}.$$

For more details, the reader is directed to [May72, Ch. 11].

Theorem 3.31: *Let $X_\bullet, Y_\bullet$ be simplicial spaces in the category $\mathcal{SU}$. Then there is a homeomorphism*

$$|X_\bullet \times Y_\bullet| \to |X_\bullet| \times |Y_\bullet|.$$

Proof. The reader is directed to [May72, Theorem 11.5] for a proof of this statement. $\square$

Corollary 3.32: *Let $f : X_\bullet \to B_\bullet$ and $p : Y_\bullet \to B_\bullet$ be maps in the category $\mathcal{SU}$. Then $|X_\bullet \times_{B_\bullet} Y_\bullet|$ is naturally homeomorphic to $|X_\bullet| \times_{|B_\bullet|} |Y_\bullet|$ via the map*

$$|p_X| \times |p_Y| : |X_\bullet \times_{B_\bullet} Y_\bullet| \to |X_\bullet| \times_{|B_\bullet|} |Y_\bullet|$$

where p_X, p_Y are the projections onto $X_\bullet, Y_\bullet$ respectively.

Proof. The reader is directed to [May72, Corollary 11.6] for a proof of this statement. $\square$

Example 3.33: As Segal illustrates in [Seg68], if $X_\bullet$ is a simplicial space its geometric realisation $|X_\bullet|$ has a natural filtration

$$|X_\bullet|_0 \subset |X_\bullet|_1 \subset |X_\bullet|_2 \subset \cdots \subset |X_\bullet|$$

where $|X_\bullet|_i$ is the image of $\Delta^i \times X_i$ in $|X_\bullet|$. Applying this to the simplicial principal G-bundle

$$N\overline{G}_\bullet \xrightarrow{\gamma} NG_\bullet$$

one obtains the natural filtration below.

$$\begin{tikzcd}
{|N\overline{G}_\bullet|_0} \arrow[r, phantom, "\subset"] \arrow[d, "|\gamma|_0"] & {|N\overline{G}_\bullet|_1} \arrow[r, phantom, "\subset"] \arrow[d, "|\gamma|_1"] & {|N\overline{G}_\bullet|_2} \arrow[r, phantom, "\subset \cdots \subset"] \arrow[d, "|\gamma|_2"] & {|N\overline{G}_\bullet|} \arrow[d, "|\gamma|"] \\
{|NG_\bullet|_0} \arrow[r, phantom, "\subset"] & {|NG_\bullet|_1} \arrow[r, phantom, "\subset"] & {|NG_\bullet|_2} \arrow[r, phantom, "\subset \cdots \subset"] & {|NG_\bullet|}
\end{tikzcd}$$

One can also 'reverse engineer' the geometric realisation of a simplicial space $X_\bullet$ given enough data. Since the geometric realisation is a gluing along degeneracies and faces, the space $|X_\bullet|_n$ can be built from $|X_\bullet|_{n-1}$, the space $\Delta^n \times X_n$ and the gluing instructions. The gluing instructions for the faces are contained in the image of

$$\partial\Delta^n \times X_n$$

and the gluing instructions for the degeneracies are contained in the image of

$$\Delta^{n-1} \times \bigcup_{i=0}^{n} s_i(X_n).$$

We will be referring to the union of the sets $s_i(X_n)$ with some frequency and so the notation

$$s(X_n) = \bigcup_{i=0}^{n} s_i(X_n) \tag{3.34}$$

will be adopted for clarity and brevity. The construction alluded to above is best described in the following proposition.

Proposition 3.35: *If $X_\bullet$ is a simplicial space, the following diagram is a pushout diagram of topological spaces.*

$$\begin{tikzcd}
\partial\Delta^n \times X_n \cup \Delta^{n-1} \times s(X_n) \arrow[r] \arrow[d] & {|X_\bullet|_{n-1}} \arrow[d] \\
\Delta^n \times X_n \arrow[r] & {|X_\bullet|_n}
\end{tikzcd}$$

Proof. The reader is directed to [May74, Theorem A.4] or [GJ99, Proposition 1.7, Ch. VII] for a proof of this statement. □

3.2 A Simplicial Construction of the Universal Bundle

In section 2.1.2 equivariant cohomology of a topological space X with right G-action was defined as the singular cohomology of the quotient

$$H^*\left((X \times EG)/G\right)$$

where EG is any model for the universal G-bundle. The following section will motivate a model for the universal bundle constructed via a *simplicial principal G-bundle*. The advantage of this model will be that the model for the universal bundle, which is typically not a smooth principal G-bundle, will be built from a sequence of smooth principal G-bundles which preserve cohomological properties under geometric realisation. This section will primarily focus on the construction of $EG \to BG$ which will then be used extensively in the last chapter.

3.2.1 Basic Properties of $|N\overline{G}_\bullet|$

Let G be a topological group and X be a topological space with right G-action. The space

$$X \times_{X/G} X \subset X \times X$$

is the set of pairs $(x, y) \in X \times X$ such that $x = y \cdot g$ for some $g \in G$. We say that the action of G on X is *strongly free* if there is a continuous function $\tau : X \times_{X/G} X \to G$ satisfying

$$y = x \cdot \tau(x, y). \tag{3.36}$$

Example 3.37: Let G act on itself by right multiplication. Then for any $g, h \in G$, clearly

$$\tau(g, h) = g^{-1}h.$$

This map is clearly continuous since the inverse map and right multiplication map are both continuous on a topological group.

Lemma 3.38: *Let G be a topological group. Then the action of G is strongly free on the space $|N\overline{G}_\bullet|$.*

Proof. Firstly note that since $N\overline{G}_\bullet$ is the total space of the simplicial principal G-bundle over $NG_\bullet$, we may associate the orbits of $N\overline{G}_n$ with the space NG_n. The action of G on the space $N\overline{G}_n$ is clearly strongly free for every $n \geq 0$. Specifically, there is a map

$$\tau_n : N\overline{G}_n \times_{NG_n} N\overline{G}_n \to G$$

that satisfies Equation 3.36. In fact, this defines a simplicial map

$$\tau : N\overline{G}_\bullet \times_{NG_\bullet} N\overline{G}_\bullet \to G$$

since the face and degeneracy maps are equivariant. Taking geometric realisation, there is a continuous map

$$|\tau| : |N\overline{G}_\bullet \times_{NG_\bullet} N\overline{G}_\bullet| \to G.$$

and by Corollary 3.32, there is a homeomorphism induced by the respective projections

$$|N\overline{G}_\bullet \times_{NG_\bullet} N\overline{G}_\bullet| \xrightarrow{\sim} |N\overline{G}_\bullet| \times_{|NG_\bullet|} |N\overline{G}_\bullet|$$

which completes the proof. $\square$

In order to show $|N\overline{G}_\bullet|$ is contractible, we will use the following theorem without proof. For a complete proof of this fact, the reader is directed to [Eng89, p. 151].

Theorem 3.39: *For every locally compact space X and any quotient mapping $g : Y \to Z$, the Cartesian product $f = \mathrm{id}_X \times g : X \times Y \to X \times Z$ is a quotient mapping.*

Lemma 3.40: *The space $|N\overline{G}_\bullet|$ is contractible.*

Proof. Let $[g_0, \ldots, g_n; u_0, \ldots, u_n]$ denote an equivalence class in $N\overline{G}_\bullet$ with $g_i \in G$ and $(u_0, \ldots, u_n) \in \Delta^n$. Let $h : |N\overline{G}_\bullet| \times I \to |N\overline{G}_\bullet|$ be the homotopy

$$h([g_0, \ldots, g_n; u_0, \ldots, u_n], t) = [1, g_0, \ldots, g_n; t, (1-t)u_0, \ldots, (1-t)u_n].$$

Note that by the construction of h,

$$\begin{aligned} h([g_0, \ldots, g_n; u_0, \ldots, u_n], 0) &= [1, g_0, \ldots, g_n; d^0(u_0, \ldots, u_n)] \\ &= [g_0, \ldots, g_n; u_0, \ldots, u_n] \end{aligned}$$

and also

$$\begin{aligned} h([g_0, \ldots, g_n; u_0, \ldots, u_n], 1) &= [1, g_0, \ldots, g_n; d^n \cdots d^1(1)] \\ &= [1; 1]. \end{aligned}$$

It is perhaps easiest to show that the homotopy h is continuous by noting that it lifts to a continuous map

$$\tilde{h} : \coprod_{n\geq 0} G^{n+1} \times \Delta^n \times I \to \coprod_{n\geq 0} G^{n+1} \times \Delta^n$$

given by the obvious choice of lift, for $(g_0, \ldots, g_n; u_0, \ldots, u_n) \in \coprod_{n\geq 0} G^{n+1} \times \Delta^n$ and $t \in I$,

$$\tilde{h}((g_0, \ldots, g_n; u_0, \ldots, u_n), t) = (1, g_0, \ldots, g_n; t, (1-t)u_0, \ldots, (1-t)u_n).$$

Consider the commutative diagram below.

$$\begin{tikzcd} \coprod_{n\geq 0} \Delta^n \times G^{n+1} \times I \arrow[r, "\tilde{h}"] \arrow[d, "q \times \mathrm{id}_I"] & \coprod_{n\geq 0} \Delta^n \times G^{n+1} \arrow[d, "q"] \\ |N\overline{G}_\bullet| \times I \arrow[r, "h"] & |N\overline{G}_\bullet| \end{tikzcd}$$

The space I is locally compact so by Theorem 3.39 the map $q \times \mathrm{id}_I$ is a quotient map and thus an open mapping. Hence, for an open set $U \subset |N\overline{G}_\bullet|$,

$$h^{-1}(U) = (q \times \mathrm{id}_I \circ \tilde{h}^{-1} \circ q^{-1})(U)$$

which is clearly open and thus h is continuous. $\square$

Lemma 3.41: *Under the quotient of a topological group G there is a homeomorphism*

$$|N\overline{G}_\bullet|/G \cong |NG_\bullet|.$$

Proof. Define $\gamma_n : G^{n+1} \to G^n$ by the map

$$\gamma_n(g_0, \ldots, g_n) = (g_0 g_1^{-1}, \ldots, g_{n-1} g_n^{-1}) \tag{3.42}$$

and note that the collection of maps $\{\gamma_n\}_{n\geq 0} =: \gamma : N\overline{G}_\bullet \to NG_\bullet$ is a map of simplicial manifolds. Moreover, γ is equivariant since

$$\begin{aligned} \gamma_n((g_0, \ldots, g_n) \cdot h) &= \gamma_n((g_0 h, \ldots, g_n h)) \\ &= (g_0 h (g_1 h)^{-1}, \ldots, g_{n-1} h (g_n h)^{-1}) \\ &= \gamma_n(g_0, \ldots, g_n) \end{aligned}$$

for every $n \geq 0$ and every $h \in G$. The map γ descends to a map

$$\gamma_G : N\overline{G}_\bullet/G \to NG_\bullet$$

under the identifications of the quotient by G since it is a G-invariant map. Moreover, we define a map $\psi_G : NG_\bullet \to N\overline{G}_\bullet/G$ defined by the collection of maps

$$\psi_n(g_1, \cdots, g_n) = [g_1 g_2 \cdots g_n,\, g_2 g_3 \cdots g_n, \ldots,\, g_n,\, 1].$$

Notice that ψ_G is a simplicial map and a continuous inverse to γ_G. Since γ_G and ψ_G are simplicial maps, they descend again to continuous maps under geometric realisation and we thus have shown that

$$|N\overline{G}_\bullet/G| \cong |NG_\bullet|.$$

To show that $|N\overline{G}_\bullet/G| = |N\overline{G}_\bullet|/G$ it suffices to note that for $x \in N\overline{G}_\bullet$, $|x| \in |N\overline{G}_\bullet|$ its corresponding equivalence class and any $g \in G$

$$|x \cdot g| = |x| \cdot g.$$

Hence, we have shown that $|N\overline{G}_\bullet|/G = |N\overline{G}_\bullet/G| \cong NG_\bullet$. □

3.2.2 Principal Bundles and Local Trivialisations

Lemma 3.43: *Let G be a topological group and X be a topological space with strongly free G-action. Then $X \xrightarrow{\pi} X/G$ is a trivial bundle if and only if $\pi : X \to X/G$ has a continuous section.*

Proof. Suppose $\pi : X \to X/G$ is a trivial bundle. That is, there is a homeomorphism $\varphi : X \to (X/G) \times G$ such that the induced map on base spaces $X/G \to X/G$ is the identity. Let $[x]$ denote the equivalence class in X/G such that $\pi(x) = [x]$. There is an obvious choice of section for the bundle $(X/G) \times G \to (X/G)$, $s : (X/G) \to (X/G) \times G$ given by

$$s([x]) = ([x], e)$$

after which we note that $\varphi \circ s : (X/G) \to X$ is also a continuous global section.

Now suppose instead that $\pi : X \to X/G$ has a continuous section, $s : X/G \to X$. Since G is a strongly free action, there is a continuous map $\tau : X^* \to G$ and another map $Q : X \to (X/G) \times G$ defined by

$$Q(x) = ([x], \tau(s([x]), x)).$$

Note that Q is an equivariant homeomorphism with continuous inverse defined by

$$Q^{-1}([x], g) = s([x]) \cdot g.$$

Thus $X \cong (X/G) \times G$ and $X \xrightarrow{\pi} X/G$ is a trivial bundle. □

Corollary 3.44: *Let G be a topological group and X be a topological space with strongly free G-action. Then $X \xrightarrow{\pi} X/G$ is a Principal G-bundle if and only if $\pi : X \to X/G$ has continuous local sections.*

Proof. Consider $U_i \subset X/G$. It follows from the above lemma that local sections $s_i : U_i \to \pi^{-1}(U_i)$ correspond to trivialisations of $\pi^{-1}(U_i)$ since G is strongly free on $\pi^{-1}(U_i)$. $\square$

Lemma 3.45: *Let G be a group and X be a topological set with right G-action. If U is a G-stable subset of X then an equivariant map*

$$f : U \to G$$

is equivalent to a local section $s : U/G \to U$.

Proof. Suppose there exists such a map $f : U \to G$ and let $[u]$ be an equivalence class in U/G. Define the map $s : U/G \to U$ by

$$s[u] = u \cdot f(u)^{-1}.$$

The map is well defined since for any representative $u \cdot g \in [u]$,

$$\begin{aligned} s[u \cdot g] &= u \cdot g \cdot f(u \cdot g)^{-1} \\ &= u \cdot g \cdot (f(u) \cdot g)^{-1} \\ &= u \cdot f(u)^{-1}. \end{aligned}$$

$\square$

Remark 3.46: Note that the converse of this lemma is true when the action of G is strongly free.

3.2.3 The Homotopy Extension Property and NDR Pairs

Definition 3.47: *Let (X, A) be a pair of spaces with $A \overset{i}{\hookrightarrow} X$. We say that (X, A) has the* Homotopy Extension Property (HEP)*, or that A has the HEP with respect to X, if for every map $F : X \to Y$ and for every homotopy $h : A \times I \to Y$ such that $h(a, 0) = F \circ i(a)$ there exists a map*

$$H : X \times I \to Y$$

such that $H \circ (i \times 1) = h$ and $H(x, 0) = F(x)$.

If we let $i_0(x) = (x, 0)$ we can rephrase this definition by requiring that there exists a map H that makes the diagram below commute.

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[dd, hook, "i"'] & A \times I \arrow[dl, "h"] \arrow[dd, "i \times \mathrm{id}"] \\ Y & \\ X \arrow[u, "F"] \arrow[r, "i_0"'] & X \times I \arrow[ul, dashed, "H"'] \end{tikzcd}$$

The following theorem of Palais leads to a great number of examples of pairs (X, A) that have the HEP.

Theorem 3.48: *Let X be a metrisable manifold and let A be a closed subspace of X which is also a manifold. Then (X, A) has the HEP.*

Proof. The reader is directed to [Pal65, Theorem 6] for a proof of this statement. $\square$

Remark 3.49: If G is a Lie group then this theorem implies that $(G, \{1\})$ has the HEP.

Lemma 3.50 (Pasting Lemma)**:** *Let X, Y be closed subspaces of a topological space $A = X \cup Y$. If a function $f : A \to B$ is continuous when restricted to both X and Y, then f is continuous.*

Proof. Let C be a closed subset of B. Then since f is continuous when restricted to X, $f^{-1}(C) \cap X$ is closed and similarly $f^{-1}(C) \cap Y$ is closed. Then the set

$$f^{-1}(C) = \big(f^{-1}(C) \cap X\big) \cup \big(f^{-1}(C) \cap Y\big)$$

is also closed, hence $f : A \to B$ is continuous. $\square$

Proposition 3.51: *Let X be a topological space and $A \subset X$ be a closed subspace. The pair of spaces (X, A) has the HEP if and only $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.*

Proof. Suppose $A \times I \cup X \times \{0\}$ is a retract of $X \times I$ with retraction

$$r : X \times I \to A \times I \cup X \times \{0\}.$$

If there is a map $F : X \to Y$ and a homotopy $h : A \times I \to Y$ such that $F \circ i = h \circ i_0$ then we define a new function $g : A \times I \cup X \times \{0\} \to Y$ where

$$g(x,t) = \begin{cases} F(x) & \text{if } t = 0 \\ h(x,t) & \text{if } t \neq 0 \end{cases}.$$

The function g is well defined since $x \in A$ if $t \neq 0$ and continuous by the previous pasting lemma. Then we may define a homotopy $H = g \circ r : X \times I \to Y$.

Now suppose (X, A) has the HEP. Then by letting $F : X \to A \times I \cup X \times \{0\}$ be the function

$$F(x) = (x, 0)$$

and the homotopy $h : A \times I \to A \times I \cup X \times \{0\}$ be the inclusion

$$h(a,t) = (a,t)$$

there is a function $H : X \times I \to A \times I \cup X \times \{0\}$ such that $H|_{A \times I} = h$ and $H(x, 0) = F(x)$. Note that H is precisely a retract of $X \times I$ since

$$H(a,t) = h(a,t) = (a,t)$$

for every $(a,t) \in A \times I$. $\square$

Example 3.52: For any $n \geq 1$, consider the pair (D^n, S^{n-1}) where

$$D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

and $S^{n-1} = \partial D^n$. We wish to find a retraction $r : D^n \times I \to D^n \times \{0\} \cup S^{n-1} \times I$. First, embed $D^n \times I \hookrightarrow D^n \times \mathbb{R}$ and fix a point $N = (x, t) \in D^n \times \mathbb{R}$ where x is in the interior of D^n and $t > 1$. Then for any point $M = (y, s) \in D^n \times I$ consider the straight line L passing through N and (y, s). Then define $r(M)$ to be the point

$$r(M) = L \cap (D^n \times \{0\} \cup S^{n-1} \times I).$$

One may check that this map is well defined, continuous and the identity on

$$D^n \times \{0\} \cup S^{n-1} \times I,$$

hence a retraction. Thus, the pair (D^n, S^{n-1}) has the HEP. Since the topological n-simplex is homeomorphic to D^n, and similarly its boundary is homeomorphic to S^{n-1}, we may conclude that the pair $(\Delta^n, \partial\Delta^n)$ has the HEP.

Definition 3.53: *A pair (X, A) is a* neighbourhood deformation retract pair*, or an* NDR pair*, if there is a map $u : X \to I$ such that $u^{-1}(0) = A$ and a homotopy $h : X \times I \to X$ such that:*

- $h(-, 0) = \mathrm{id}_X$*,*
- $h(a, t) = a$ *for all* $a \in A, t \in I$ *and*
- $h(x, 1) \in A$ *if* $u(x) < 1$.

We say an NDR pair (X, A) is a deformation retract pair*, or a* DR pair*, if $u(x) < 1$ for all $x \in X$.*

The following three results are proven in [May99] but first proved by Puppe in [Pup67].

Proposition 3.54: *Let (X, A) and (Y, B) be NDR pairs. Then*

$$(X \times Y, A \times Y \cup X \times B)$$

is an NDR pair. Moreover, if (X, A) or (Y, B) is a DR pair, then

$$(X \times Y, A \times Y \cup X \times B)$$

is a DR pair.

The proof is technical and will be omitted for brevity, but the reader is referred to [May99, p. 45] for a complete proof.

Corollary 3.55: *Let X be a topological space and $A \subset X$ a closed subspace. Then (X, A) is an NDR pair if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.*

Proof. This follows from the fact that $(I, \{0\})$ is a DR pair. This is shown by letting $u = \mathrm{id}_I$ and $h : I \times I \to I$ be defined by

$$h(s,t) = t(1-s).$$

Thus, from the proposition above, $(X \times I, A \times I \cup X \times \{0\})$ is a DR pair with homotopy $k : (X \times I) \times I \to X \times I$. Then there is a retraction $r : X \times I \to A \times I \cup X \times \{0\}$ given by

$$r(x,t) = k\left((x,t), 1\right).$$

Now suppose there is a retraction $r : X \times I \to A \times I \cup X \times \{0\}$. Let $p_X : X \times I \to X$ and $p_I : X \times I \to I$ be the projections and define $u : X \to I$ by

$$u(x) = \sup\{t - (p_I \circ r)(x,t) : t \in I\}$$

and $h : X \times I \to X$ by

$$h(x,t) = p_X \circ r(x,t).$$

Then $h(-,0) = \mathrm{id}_X$, $h(a,t) = a$ for all $a \in A, t \in I$ and $h(x,1) \in A$ if $u(x) < 1$. We observe that $u^{-1}(0) \supset A$ since if $a \in A$ then

$$p_I(a,t) = p_I r(a,t) \implies t = p_I r(a,t) \text{ for every } t \in I.$$

Suppose $x \in X \backslash A$. Since r is continuous there is a neighbourhood V of x and an $\epsilon > 0$ such that $r(V \times [0,\epsilon)) \subset X \times \{0\}$. But then if $(x,t) \in V \times (0,\epsilon)$, $p_I(x,t) > p_I r(x,t)$ in which case $u(x) > 0$. □

Corollary 3.56: *Let X be a topological space and $A \subset X$ be a closed subspace. Then (X, A) has the HEP if and only if it is an NDR pair.*

This corollary means that for a Lie group G, $(G, \{1\})$ is an NDR pair. We will use this fact in conjunction with the following lemma.

Lemma 3.57: *If (X, A) is an NDR pair then there is an open set $U \subset X$ such that $A \subset U$ and there is a retraction*

$$r : U \to A.$$

Proof. Recall that since (X, A) is an NDR pair, there is a continuous map $u : X \to I$ and a homotopy $h : X \times I \to X$ satisfying certain properties. By taking $\epsilon \in (0,1]$ and letting

$$U = u^{-1}\left((0,\epsilon)\right)$$

there is a retraction $r : U \to A$ given by $r(x) = h(x,1)$ for every $x \in U$. □

3.2.4 Constructing Local Sections

We are particularly interested in $s(NG_n)$ which, by Equation 3.34, is the space

$$s(NG_n) = \bigcup_{i=0}^{n+1} s_i(NG_n) = \bigcup_{i=0}^{n+1} G^i \times \{1\} \times G^{n-i}.$$

The pair $(NG_{n+1}, s(NG_n))$ has the HEP by Theorem 3.48 and hence is an NDR pair by Corollary 3.56. Also note the space

$$s(N\overline{G}_n) = \bigcup_{i=0}^{n+1} s_i(N\overline{G}_n)$$

contains elements $(g_0, \dots, g_n)$ where $g_i = g_{i+1}$ for some $i = 0, 1, \dots, n-1$. Since $(NG_{n+1}, s(NG_n))$ and $(\Delta^{n+1}, \partial\Delta^{n+1})$ are both NDR pairs, Proposition 3.54 and Corollary 3.56 implies that

$$\big(\Delta^{n+1} \times NG_{n+1}, \partial\Delta^{n+1} \times NG_{n+1} \cup \Delta^{n+1} \times s(NG_{n+1})\big) \tag{3.58}$$

has the HEP. Moreover, we have a pullback diagram

$$\begin{tikzcd}
\partial\Delta^{n+1} \times N\overline{G}_{n+1} \cup \Delta^{n+1} \times s(N\overline{G}_{n+1}) \arrow[r, hook] \arrow[d] & \Delta^{n+1} \times N\overline{G}_{n+1} \arrow[d, "\gamma_{n+1}"] \\
\partial\Delta^{n+1} \times NG_{n+1} \cup \Delta^{n+1} \times s(NG_{n+1}) \arrow[r, hook] & \Delta^{n+1} \times NG_{n+1}
\end{tikzcd}$$

and hence both vertical maps are the projections of trivial principal G-bundles. Because of this, we have equivariant homeomorphisms

$$\varphi : \Delta^{n+1} \times N\overline{G}_{n+1} \xrightarrow{\sim} (\Delta^{n+1} \times NG_{n+1}) \times G$$

and

$$\varphi' : \partial\Delta^{n+1} \times N\overline{G}_{n+1} \cup \Delta^{n+1} \times s(NG_{n+1}) \xrightarrow{\sim} (\partial\Delta^{n+1} \times NG_{n+1} \cup \Delta^{n+1} \times s(NG_{n+1})) \times G.$$

Since the pair in Equation 3.58 is an NDR pair, Lemma 3.57 implies that there is an open set $U \subset \Delta^{n+1} \times NG_{n+1}$ and a retraction

$$r : U \to \partial\Delta^{n+1} \times NG_{n+1} \cup \Delta^{n+1} \times s(NG_{n+1}). \tag{3.59}$$

Let $V = \gamma_{n+1}^{-1}(U)$ and notice that this set is G-stable. We may construct an equivariant retraction

$$\rho : V \to \partial\Delta^{n+1} \times N\overline{G}_{n+1} \cup \Delta^{n+1} \times s(N\overline{G}_{n+1}) \tag{3.60}$$

defined by

$$\rho = \varphi'^{-1} \circ (r \times 1) \circ \varphi.$$

Since the maps $\varphi'^{-1}, (r \times 1), \varphi$ are equivariant, so too is the map ρ. We first start by constructing the G-stable open set $U_0 \subset |N\overline{G}_\bullet|_0 = \Delta^0 \times G$ and equivariant map $h_0 : U_0 \to G$. This is clearly achieved by letting $U_0 = |N\overline{G}_\bullet|_0$ and projecting onto G,

$$h_0(t, g) = g.$$

Now, assume inductively that we have a G-stable set $U_n \subset |N\overline{G}_\bullet|_n$ and an equivariant map $h_n : U_n \to G$. Note that there is a map

$$\partial\Delta^{n+1} \times N\overline{G}_{n+1} \cup \Delta^{n+1} \times s(N\overline{G}_{n+1}) \xrightarrow{m_n} |N\overline{G}_\bullet|_n$$

where

$$\begin{aligned} &m_n(d^i t; g_0, \ldots, g_n) = [t; g_0, \ldots, \hat{g}_i, \ldots, g_n] \quad \& \\ &m_n(t; g_0, \ldots, g_i, g_i, \ldots, g_n) = [t; g_0, \ldots, g_i, g_{i+1}, \ldots, g_n]. \end{aligned}$$

which is clearly a continuous equivariant map. Hence, the set

$$W_n = m_n^{-1}(U_n) \subset \partial\Delta^{n+1} \times N\overline{G}_{n+1} \cup \Delta^{n+1} \times s(N\overline{G}_{n+1})$$

is a G-stable set. That is, since U_n is G-stable, and for $w \in W_n$, $m(w \cdot g) = m(w) \cdot g \in U_n$ and therefore, $w \cdot g \in W_n$. By Equation 3.60, there is a G-equivariant retraction from an open set $A \subset \Delta^{n+1} \times N\overline{G}_{n+1}$ to $\partial\Delta^{n+1} \times N\overline{G}_{n+1} \cup \Delta^{n+1} \times s(N\overline{G}_{n+1})$

$$r : A \to \partial\Delta^{n+1} \times N\overline{G}_{n+1} \cup \Delta^{n+1} \times s(N\overline{G}_{n+1}).$$

Hence, the retraction may be restricted to the set W_n by letting $V_n = r^{-1}(W_n) \subset \Delta^n \times N\overline{G}_{n+1}$

$$r_n = r|_{V_n} : V_n \to W_n.$$

Hence, we have a G-equivariant function $k_n = h_n \circ m_n \circ r_n : V_n \to G$. Then let $U_{n+1} = V_n \cup_{W_n} U_n$ be the pushout and $h_{n+1} : U_{n+1} \to G$ be the unique map defined by the diagram below.

$$\begin{tikzcd}
W_n \arrow[r, "m_n"] \arrow[d, hook] & U_n \arrow[d] \arrow[ddr, bend left, "h_n"] & \\
V_n \arrow[r] \arrow[drr, bend right, "k_n"'] & V_n \cup_{W_n} U_n \arrow[dr, dashed, "h_{n+1}"'] & \\
 & & G
\end{tikzcd}$$

It follows that U_{n+1} is G-stable since U_n and V_n are G-stable and $h_{n+1} : U_{n+1} \to G$ is equivariant since k_n and h_n are equivariant. Finally, the set U_{n+1} can naturally be thought of as a subset of $|N\overline{G}_\bullet|_{n+1}$ and is open since both U_n and V_n are open. Thus, we have proven the proposition.

Proposition 3.61: *Let G be a topological group. Then*

$$|N\overline{G}_\bullet| \xrightarrow{|\gamma|} |NG_\bullet|$$

is a principal G-bundle with contractible total space.

A proof that $|N\overline{G}_\bullet| \to |NG_\bullet|$ is numerable can be found in [Dol63] and hence Proposition 3.61 yields a model for the universal bundle $EG \to BG$.

Let G be a Lie group and M be a manifold with right G-action. Note that M can be turned into a simplicial manifold $M_\bullet$ by letting $M_q = M$ for every $q \geq 0$ and setting each face and degeneracy map to the identity

$$d_i = \mathrm{id}_M : M_n \to M_{n-1}, s_i = \mathrm{id}_M : M_n \to M_{n+1}$$

for every $0 \leq i \leq n$. It is simple to check that $|M_\bullet| = M$ in this case. If $P_\bullet \to B_\bullet$ is a simplicial principal G-bundle then, as a slight abuse of notation, we will write $P_\bullet \times M$ instead of $P_\bullet \times M_\bullet$ and the corresponding simplicial principal G-bundle as

$$P_\bullet \times M \to P_\bullet \times_G M$$

where we define $P_\bullet \times_G M := (P_\bullet \times M)/G$ for simpler notation.

Corollary 3.62: *Let G be a Lie group and M be a manifold with right G-action. Then*

$$|N\overline{G}_\bullet \times M| = EG \times M$$

and moreover

$$|N\overline{G}_\bullet \times_G M| = (EG \times M)/G.$$

Proof. This is simply application of Theorem 3.31 to Proposition 3.61. □

Chapter 4

Simplicial Equivariant de Rham Theory

4.1 Dupont's Simplicial de Rham Theorem

4.1.1 The Double Complex of a Simplicial Space

As seen in Chapter 3, associated to every topological space X there is a singular cochain complex $C^*(X)$. For any simplicial topological space $X_\bullet$ there is thus a sequence of associated singular cochain complexes $C^*(X_p)$. The face maps $d_i : X_p \to X_{p-1}$ for $0 \leq i \leq p$ of $X_\bullet$ induce coface maps $d_i^\# : C^*(X_{p-1}) \to C^*(X_p)$ from which we can construct a differential $\delta : C^*(X_{p-1}) \to C^*(X_p)$ by defining

$$\delta = \sum_{i=0}^{p} (-1)^i d_i^\# \tag{4.1}$$

and noting that (3.7) — (3.9) implies that $\delta^2 = 0$ in keeping with the calculation in Section 3.1.3. So, for a fixed $p \geq 0$ we obtain a cochain complex $C^q(X_\bullet)$. Since we have two differential operators on $C^{p,q}(X_\bullet) := C^q(X_p)$ we have a double complex $(C^{p,q}(X_\bullet), d, \delta)$ which can be drawn diagrammatically as

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^{0,2}(X_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & C^{1,2}(X_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & C^{2,2}(X_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
C^{0,1}(X_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & C^{1,1}(X_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & C^{2,1}(X_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
C^{0,0}(X_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & C^{1,0}(X_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & C^{2,0}(X_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

in which every square commutes. If $M_\bullet$ is a simplicial manifold, there is also a de Rham complex $\Omega^*(M_p)$ associated to every M_p and $p \geq 0$. The face and degeneracy maps on $M_\bullet$ induce face and degeneracy maps on $\Omega^*(M_p)$ and once again we define a 'horizontal' differential

$$\delta = \sum_{i=0}^{p} (-1)^i d_i^* : \Omega^*(M_p) \to \Omega^*(M_{p+1}) \tag{4.2}$$

such that we obtain a cochain complex $\Omega^q(M_\bullet)$ for a fixed $q \geq 0$. Clearly the differential operators d and δ commute (as pullbacks commute through d) so in the same way we define $\Omega^{p,q}(M_\bullet) := \Omega^q(M_p)$ and define the associated double complex $(\Omega^{p,q}(M_\bullet), d, \delta)$.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^{0,2}(M_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^{1,2}(M_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^{2,2}(M_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
\Omega^{0,1}(M_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^{1,1}(M_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^{2,1}(M_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
\Omega^{0,0}(M_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^{1,0}(M_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^{2,0}(M_\bullet) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

Let $(B^{*,*}, d_B, \delta_B), (C^{*,*}, d_C, \delta_C)$ be two double complexes. A morphism of double complexes or *chain map* is a homomorphism $f : B^{*,*} \to C^{*,*}$ such that

$$f \circ d_B = d_C \circ f \text{ and } f \circ \delta_B = \delta_C \circ f.$$

Associated to every double complex $(B^{*,*}, d, \delta)$ is the *total complex* whose elements of degree n are

$$\mathrm{Tot}^n(B) = \bigoplus_{p+q=n} B^{p,q}.$$

Since d, δ commute

$$D = \delta + (-1)^p d : \bigoplus_{p+q=n} B^{p,q} \to \bigoplus_{p+q=n} B^{p,q} \tag{4.3}$$

naturally defines a differential operator on $\mathrm{Tot}^*(B)$. Note that a map of double complexes $f : B^{*,*} \to C^{*,*}$ determines a chain map $f : \mathrm{Tot}^*(B) \to \mathrm{Tot}^*(C)$ since

$$f \circ (\delta + (-1)^p d) = (\delta + (-1)^p d) \circ f.$$

Thus, to each double complex we may associate cohomology groups where the n^{th} cohomology group is defined to be

$$H^n_{\mathrm{Tot}}(B) := H^n(\mathrm{Tot}^*(B)) = \frac{\ker D \cap \mathrm{Tot}^n(B)}{\mathrm{im} D \cap \mathrm{Tot}^n(B)}$$

and a map of double complexes induces a map $f^* : H^n_{\mathrm{Tot}}(B) \to H^n_{\mathrm{Tot}}(C)$. The classical result of de Rham's gives us a result about the singular and de Rham cohomology groups of a manifold M.

Theorem 4.4 (de Rham's Theorem)**:** *Let M be a manifold and define the map $I(-) : \Omega^q(M) \to C^q_\infty(M)$ by*

$$I(\omega)\,(\sigma) = \int_{\Delta^q} \sigma^*\omega$$

where $\omega \in \Omega^q(M)$ and $\sigma : \Delta^q \to M$ is a smooth map. It is clear that $I(\omega)$ extends linearly to a singular cochain in $C^q_\infty(M)$. Then the map I induces an isomorphism of cohomology groups

$$H^n(\Omega^*(M)) \cong H^n(C^*(M))$$

for every non-negative integer n.

We refer the reader to [Lee02] or [Dup78] for the proof of this theorem. For a simplicial manifold $M_\bullet$, we thus have the following application.

Corollary 4.5: *Let $M_\bullet$ be a simplicial manifold and consider the map $I(-) : \Omega^q(M_p) \to C^q_\infty(M_p)$ defined by*

$$I(\omega)\,(\sigma) = \int_{\Delta^q} \sigma^*\omega$$

where $\omega \in \Omega^q(M_p)$ and $\sigma : \Delta^q \to M_p$ is a smooth map. Then the map I induces an isomorphism of cohomology groups

$$H^n(\Omega^{p,*}(M_\bullet)) \cong H^n(C^{p,*}(M_\bullet))$$

for every non-negative integer n and p.

Lemma 4.6: *Let $B^{*,*}, C^{*,*}$ be two double chain complexes and $f : B^{*,*} \to C^{*,*}$ be a chain map of double complexes. Then f restricts to a chain map on columns*

$$f_p : B^{p,*} \to C^{p,*}.$$

If f_p induces an isomorphism $H^*(B^{p,*}) \cong H^*(C^{p,*})$ for every $p \geq 0$ then f induces an isomorphism

$$H^*_{\mathrm{Tot}}(B) \cong H^*_{\mathrm{Tot}}(C).$$

Proof. The reader is directed to Proposition 9.13 in [BT82] or Lemma 2.7.3 of [Wei94]. □

Letting $f : M \to N$ be a smooth map, it is a simple calculation to show that

$$\begin{aligned} I(f^*\omega)\left(\sum_i c_i\sigma_i\right) &= \sum_i c_i \int_{\Delta^p} \sigma_i^* f^*\omega \\ &= \sum_i c_i \int_{\Delta^p} (f \circ \sigma_i)^*\omega \\ &= I(\omega)\left(\sum_i c_i(f \circ \sigma_i)\right) \\ &= f^{\#} I(\omega)\left(\sum_i c_i\sigma_i\right), \end{aligned}$$

establishing that $I \circ f^* = f^{\#} \circ I$. In particular, this means that

$$I \circ \delta = \delta \circ I$$

which implies that I is a map of double complexes and yields the following corollary of Theorem 4.4.

Corollary 4.7: *The map I can be thought of as a map*

$$I : \Omega^{*,*}(M_\bullet) \to C^{*,*}(M_\bullet)$$

that induces an isomorphism in cohomology

$$H^*_{\mathrm{Tot}}(\Omega^{*,*}(M_\bullet)) \cong H^*_{\mathrm{Tot}}(C^{*,*}(M_\bullet)).$$

4.1.2 Geometric Realisation of a Simplicial Manifold

The goal of this section is to show that the double complex $\Omega^{*,*}(M_\bullet)$ calculates the cohomology of the geometric realisation $|M_\bullet|$. Firstly we have the following result of Dupont.

Lemma 4.8: *Let $X_\bullet = \{X_p\}_{p\geq 0}$ be a simplicial topological space. Then there is an isomorphism in cohomology*

$$H^*(\|X_\bullet\|) \cong H^*_{\mathrm{Tot}}(C^{*,*}(X_\bullet)).$$

Proof. The reader is directed to Proposition 5.15 in [Dup78]. $\square$

Secondly, we observe that the work of Segal establishes a homotopy equivalence between the fat and geometric realisation under suitable conditions.

Definition 4.9: *A simplicial space $X_\bullet$ is* good *if every inclusion*

$$s_i(X_n) \hookrightarrow X_{n+1}$$

is a closed cofibration for every degeneracy map $s_i : X_n \to X_{n+1}$ and $n \geq 0$.

Lemma 4.10: *Let $X_\bullet$ be a simplicial topological space. If $X_\bullet$ is good, then there is a homotopy equivalence*

$$\|X_\bullet\| \xrightarrow{\simeq} |X_\bullet|.$$

Proof. The reader is directed to proposition A.1 of Segal in [Seg72]. $\square$

Remark 4.11: If the pairs $(X_{n+1}, s_i(X_n))$ have the HEP and $s_i(X_n)$ is closed in X_{n+1} for every $n \geq 0$ and $0 \leq i \leq n$ then $X_\bullet$ is good.

Example 4.12: Let G be a Lie group. Then the simplicial manifold $N\overline{G}_\bullet$ is good. The image of s_i on $N\overline{G}_n$ is precisely

$$s_i(G^{n+1}) = G^i \times \{1\} \times G^{n+1-i}$$

as demonstrated in Chapter 3, which is clearly closed in $N\overline{G}_{n+1}$. It remains to show that the inclusion is a cofibration. This is also demonstrated in Chapter 3 since this is an NDR pair and hence a HEP pair. A pair $(s_i(N\overline{G}_n), N\overline{G}_{n+1})$ has the HEP precisely when the inclusion is a cofibration.

It remains to show that a simplicial manifold is always good in order to prove the original stated goal of this section.

Lemma 4.13: *A simplicial manifold $M_\bullet$ is good.*

Proof. The reader is directed to [Pal65] for proof that if M is a smooth manifold and $A \subset M$ is closed, then the pair (M, A) has the HEP. Since $d_i s_i = \mathrm{id}$ and $s_i d_i : M_{n+1} \to s_i(M_n)$ is a retraction, the map

$$s_i : M_n \to s_i(M_{n+1})$$

is a homeomorphism. Both M_n, M_{n+1} are smooth manifolds and are thus completely metrizable. Combining these facts means that $s_i(M_n)$ is closed in M_{n+1} since M_n is closed. Hence $(M_{n+1}, s_i(M_n))$ has the HEP, and $s_i(M_n)$ is closed for every $n \geq 0$ and $0 \leq i \leq n$ which completes the proof by Remark 4.11. $\square$

Theorem 4.14: *Let $M_\bullet = \{M_p\}_{p\geq 0}$ be a simplicial manifold. Then there is an isomorphism in cohomology*

$$H^*(|M_\bullet|) \cong H^*_{\mathrm{Tot}}(\Omega^{*,*}(M_\bullet)).$$

Proof. From Lemma 4.10 and Lemma 4.13 we have an isomorphism of cohomology

$$H^*(|M_\bullet|) \cong H^*(\|M_\bullet\|).$$

By also considering Lemma 4.8 and Lemma 4.7, the sequence of isomorphisms

$$H^*(|M_\bullet|) \cong H^*(\|M_\bullet\|) \cong H^*_{\mathrm{Tot}}(C^{*,*}(M_\bullet)) \cong H^*_{\mathrm{Tot}}(\Omega^{*,*}(M_\bullet))$$

becomes apparent. □

4.1.3 Simplicial Differential Forms

In the final section of Chapter 3, we saw that we can build $M \times EG$ by taking the geometric realisation of a simplicial space $M \times N\overline{G}_\bullet$. In [Dup78], Dupont defines geometric objects called *simplicial differential forms* as a way of defining sequences of differential forms on simplicial spaces to approximate a de Rham complex for the topological fat realisation of a simplicial manifold $\|M_\bullet\|$.

Definition 4.15: *Let $M_\bullet = \{M_p\}_{p\geq 0}$ be a simplicial manifold. A* simplicial (differential) q-form *$\omega_\bullet$ on $M_\bullet$ is a sequence of differential q-forms $\omega_p \in \Omega^q(\Delta^p \times M_p)$ such that*

$$(d^i \times \mathrm{id})^*\omega_{p+1} = (\mathrm{id} \times d_i)^*\omega_p \tag{4.16}$$

for every $p \geq 0$ and $i = 0, 1, \ldots, p$. The vector space of simplicial q-forms will be denoted $A^*(M_\bullet)$. Moreover, the differential d on each complex $\Omega^*(\Delta^p \times M_p)$ commutes with pullbacks and so $A^*(M_\bullet)$ is naturally a complex with differential d.

The wedge product also commutes with pullbacks and so there is a natural wedge product defined on $A^*(M_\bullet)$,

$$\wedge : A^n(M_\bullet) \otimes A^m(M_\bullet) \to A^{n+m}(M_\bullet).$$

Let M and F be manifolds and $M \times F \to M$ be a trivial fibre bundle. If F is compact, oriented and of dimension k, one may *integrate along the fibre*

$$\int_F : \Omega^q(M \times F) \to \Omega^{q-k}(M).$$

Consequently, there is a sensible notion of integrating a differential form $\omega_p \in \Omega^q(\Delta^p \times M_p)$ over Δ^p to be left with a differential form

$$\int_{\Delta^p} \omega_p \in \Omega^{q-p}(M_p).$$

For any simplicial differential form $\omega_\bullet = \{\omega_p\} \in A^*(M_\bullet)$, each differential form ω_p can be thought of as a sum

$$\omega_p = \sum_{k,l} \omega_p^{k,l} \tag{4.17}$$

where $\omega_p^{k,l}$ is thought of loosely as a k-form on Δ^p and a l-form on M_p. This is in the sense that $\omega_p^{k,l}$ can locally be written in the form

$$\omega_p^{k,l} = \sum f_{k,l} \cdot dt_{i_1} \cdots dt_{i_k} d\varphi_{j_1} \cdots d\varphi_{j_k}.$$

where $f : \Delta^p \times M \to \mathbb{R}$ and $\{t_i\}, \{\phi_i\}$ are the local coordinates of Δ^p, M_p respectively. The decomposition of the simplicial form $\omega_\bullet$ in Equation 4.17 satisfies

$$\sum_{k,l} (d^i \times \mathrm{id})^* \omega_{p+1}^{k,l} = \sum_{k,l} (\mathrm{id} \times d_i)^* \omega_p^{k,l}$$

for every $0 \leq i \leq p$ by Equation 4.16 and hence

$$(d^i \times \mathrm{id})^* \omega_{p+1}^{k,l} = (\mathrm{id} \times d_i)^* \omega_p^{k,l}$$

for every $p \geq 0$ by linear independence.

Moreover, one defines a differential $d_\Delta : A_\bullet^{k,l}(M_\bullet) \to A_\bullet^{k+1,l}(M_\bullet)$ by restricting d to the coordinates of Δ^p and $d_{M_\bullet} : A_\bullet^{k,l}(M_\bullet) \to A_\bullet^{k,l+1}(M_\bullet)$ by restricting d to the coordinates on M_p. This means we have a natural decomposition of $A^*(M_\bullet)$ into a double complex $A^{k,l}(M_\bullet)$ and from construction it is clear that

$$\mathrm{Tot}^*(A^{*,*}(M_\bullet)) \cong A^*(M_\bullet).$$

Letting $\omega_\bullet = \{\omega_p\} \in A^{k,l}(M_\bullet)$ be a simplicial form, we have a map of double complexes $I_\Delta : A^{k,l}(M_\bullet) \to \Omega^{k,l}(M_\bullet)$

$$I_\Delta \omega_\bullet = \int_{\Delta^k} \omega_k \in \Omega^l(M_k).$$

By Stokes' theorem and the identifications in Equation 4.16

$$\begin{aligned}
I_\Delta d_\Delta \omega_\bullet &= \int_{\Delta^{k+1}} d_{\Delta^k} \omega_{k+1} \\
&= \int_{\partial \Delta^{k+1}} \omega_{k+1} \\
&= \sum_{i=0}^{k} (-1)^i \int_{\Delta^k} (d^i \times \mathrm{id})^* \omega_{k+1} \\
&= \sum_{i=0}^{k} (-1)^i \int_{\Delta^k} (\mathrm{id} \times d_i)^* \omega_k \\
&= \sum_{i=0}^{k} (-1)^i d_i^* \int_{\Delta^k} \omega_k
\end{aligned}$$

and hence $I_\Delta d_\Delta = \delta I_\Delta$. Also, it is easy for one to check

$$I_\Delta d_M = dI_\Delta$$

and thus I_Δ is a chain map.

Dupont also constructs a chain map of double complexes $E_\Delta : \Omega^{*,*}(M_\bullet) \to A^{*,*}(M_\bullet)$ where

$$E_\Delta(\omega)_p = p! \sum_{s=0}^{p} (-1)^s t_{i_s}\, dt_{i_0} \cdots \hat{dt_{i_s}} \cdots dt_{i_p} \wedge p_M^* \omega \tag{4.18}$$

and $\omega_\bullet = \{E_\Delta(\omega)_p\}$ defines a simplicial differential form. The reader is directed to [Dup75] for more details on the map E_Δ in which he establishes the identity

$$I_\Delta \circ E_\Delta = \mathrm{id}. \tag{4.19}$$

An important theorem proved in [Dup78] is the following.

Theorem 4.20: *Let $M_\bullet$ be a simplicial manifold. Then the maps I_Δ and E_Δ induce natural isomorphisms on cohomology*

$$H^*(A^*(M_\bullet)) \cong H^*_{\mathrm{Tot}}(\Omega^{*,*}(M_\bullet)) \cong H^*(|M_\bullet|).$$

4.2 Simplicial Chern-Weil Theory

4.2.1 Basic Simplicial Differential Forms

Let G be a Lie group and suppose $M_\bullet$ is a simplicial manifold with right G-action

$$\phi_\bullet : M_\bullet \times G \to M_\bullet.$$

Letting G act trivially on Δ^p, there is a natural right action on each manifold $\Delta^p \times M_p$ and thus the de Rham complex $\Omega^*(\Delta^p \times M_p)$. Moreover, the simplicial identities (4.16) clearly commute with the action of G and so $A^*(M_\bullet)$ inherits an action induced by $\phi_\bullet$,

$$\phi_A : A^*(M_\bullet) \times G \to A^*(M_\bullet).$$

The complex $A^*(M_\bullet)$ inherits an action of the Lie algebra of G, $\mathfrak{g}$, in the usual sense too. That is, for a simplicial differential form $\omega_\bullet \in A^k(M_\bullet)$ and $\xi \in \mathfrak{g}$ we define an operator $\iota_\xi : A^k(M_\bullet) \to A^{k-1}(M_\bullet)$ defined by

$$\iota_\xi \omega_\bullet := \{\iota_\xi \omega_n\}_{n \geq 0}.$$

To show this is a simplicial differential form, we will first prove the following lemma.

Lemma 4.21: *Let $f : M \to N$ be an equivariant map between manifolds M, N with right G-action with derivative $f_* : TM \to TN$. Let $X_\xi \in \mathfrak{X}(M)$ and $Y_\xi \in \mathfrak{X}(N)$ be the vector fields generated by the infinitesimal action of $\xi \in \mathfrak{g}$ on M, N respectively. Then*

$$f_* X_\xi(x) = Y_\xi(f(x))$$

for every $x \in M$ and every $\xi \in \mathfrak{g}$. Put diagrammatically, this square

$$\begin{tikzcd} TM \arrow[r, "f_*"] & TN \\ M \arrow[u, "X_\xi"] \arrow[r, "f"] & N \arrow[u, "Y_\xi"'] \end{tikzcd}$$

commutes for every $\xi \in \mathfrak{g}$.

Proof. By definition of X_ξ, the left hand side of the equation is precisely

$$f_* X_\xi(x) = f_* \left.\frac{d}{dt}\phi(x, \exp(t\xi))\right|_{t=0}$$

where $\phi : M \times G \to M$ is the right action of G on M. By the chain rule, this is precisely the derivative of the composition $t \mapsto f \circ \phi(x, \exp(t\xi))$ at $t = 0$, hence

$$f_* X_\xi(x) = \left.\frac{d}{dt} f \circ \phi(x, \exp(t\xi))\right|_{t=0}.$$

By equivariance, $f(x) \cdot g = f(x \cdot g)$ – or in the notation above $f \circ \phi(x, \exp(t\xi)) = \phi(f(x), \exp(t\xi))$ – and hence

$$f_* X_\xi(x) = \left.\frac{d}{dt}\phi(f(x), \exp(t\xi))\right|_{t=0}$$

at which point it is noted that the right hand side is precisely the definition of $Y_\xi(f(x))$. □

Corollary 4.22: *Let $f : M \to N$ be an equivariant map. Then*

$$f^* \iota_\xi = \iota_\xi f^*$$

for every $\xi \in \mathfrak{g}$.

Proof. Let X_ξ, Y_ξ be the vector fields generated by the infinitesimal action of $\xi \in \mathfrak{g}$ on M, N respectively. Let $\omega_\bullet = \{\omega_q\}_{q \in \mathbb{N}}$ and $x \in M$. Then

$$\begin{aligned} f^*(\iota_\xi \omega_q)(x; Y_0, \ldots, Y_{n-1}) &= \iota_\xi \omega_q(f(x); f_* Y_0, \ldots, f_* Y_{n-1}) \\ &= \omega_q(f(x); X_\xi, f_* Y_0, \ldots, f_* Y_{n-1}) \\ &= f^* \omega_q(x; Y_\xi, Y_0, \ldots, Y_{n-1}) \\ &= \iota_\xi (f^* \omega_q)(x; Y_0, \ldots, Y_{n-1}) \end{aligned}$$

by Corollary 4.22, where f_* is the derivative of f. □

So ι_ξ commutes with the equivariant maps (4.16) on $M_\bullet$ and can thus be considered a well defined derivation (of degree -1) on $A^*(M_\bullet)$. We say a simplicial differential form $\omega_\bullet$ is *horizontal* if

$$\iota_\xi \omega_\bullet = 0$$

for every $\xi \in \mathfrak{g}$. From Cartan's Magic Formula, it follows that

$$\mathcal{L}_\xi = d\iota_\xi + \iota_\xi d$$

is a derivation (of degree 0) on $A^*(M_\bullet)$ and may be explicitly be defined to be

$$\mathcal{L}_\xi \omega_\bullet := \{\mathcal{L}_\xi \omega_n\}_{n \geq 0}.$$

Suppose $P_\bullet \xrightarrow{\pi_\bullet} B_\bullet$ is a simplicial principal G-bundle. From Proposition 2.42 we know that for every $n \geq 0$ there is an isomorphism $\Omega^*(B_n) \cong \Omega^*(P_n)_{bas}$ and hence it follows that there is an isomorphism

$$A^*(P_\bullet)_{bas} \cong A^*(B_\bullet).$$

Definition 4.23: *Let $P_\bullet \xrightarrow{\pi_\bullet} B_\bullet$ be a simplicial principal G-bundle. A simplicial differential form $\omega_\bullet$ is said to be* basic *if it is horizontal and G-invariant. That is, $\iota_\xi \omega_\bullet = 0$ and $\omega_\bullet \cdot g = \omega_\bullet$ for every $\xi \in \mathfrak{g}$ and $g \in G$. The* complex of basic differential forms *is denoted $A^*(P_\bullet)_{bas}$.*

4.2.2 Simplicial Connections and Curvature

The complex $A^*(N\overline{G})$ gives us a model for differential forms on EG and thus we would like to further define connection and curvature elements on this complex in keeping with previous calculations in Chapter 2.

Definition 4.24: *Let $P_\bullet \xrightarrow{\pi} B_\bullet$ be a simplicial principal G-bundle. A* simplicial connection *on $P_\bullet$ is a $\mathfrak{g}$-valued 1-form*

$$\theta_\bullet = \{\theta_n\}_{n\geq 0} \in A^1(P_\bullet) \otimes \mathfrak{g}$$

such that each θ_n is a connection form on the bundle $\Delta^n \times P_n \to \Delta^n \times B_n$.

Recall the connection θ_L on the trivial G-bundle $G \to \{1\}$ as defined in (2.37). When considering the manifold $N\overline{G}_n = G^{n+1}$ we could define projections $p_i : N\overline{G}_n \to G$ where

$$p_i(g_0, \ldots, g_n) = g_i$$

for $i = 0, 1, \ldots, n$. We would like to show that $p_i^* \theta_L$ is a connection on the bundle $N\overline{G}_n \to NG_n$ for each i which amounts to proving the following lemma.

Corollary 4.25: *Let $f : P \to Q$ be an equivariant map of principal bundles and θ be a connection on Q. Then $f^*\theta \in \Omega^1(P) \otimes \mathfrak{g}$ is a connection on P.*

Proof. Let X_ξ be the vector field generated by the infinitesimal action of ξ on P. By the previous lemma, we have that

$$f^*\theta(x; X_\xi) = \theta(f(x), Y_\xi) = \xi$$

and hence $f^*\theta$ is itself a connection. $\square$

The projection maps p_i are clearly equivariant bundle maps, and hence define connections $p_i^*\theta_L$ on $\Omega^1(N\overline{G}_n) \otimes \mathfrak{g}$. In fact, we are trying to build connection forms on the bundle

$$N\overline{G}_n \times \Delta^n \to NG_n \times \Delta^n$$

where G acts trivially on Δ^n. We now sketch a standard result about connections.

Proposition 4.26: *Any convex combination of connections is again a connection.*

Proof. Let $\theta_1, \ldots, \theta_k$ be connections on a principal bundle P and $\lambda_1, \ldots, \lambda_k$ be smooth, real valued functions on P such that $\sum_{i=0}^{k} \lambda_i = 1$. For any vector field $\xi \in \mathfrak{g}$ one calculates that

$$\sum_{i=0}^{k} \lambda_i \theta_i(x; X_\xi) = \sum_{i=0}^{k} \lambda_i(x)\xi = \xi.$$

It is also clear that for any $g \in G$,

$$\sum_{i=0}^{k} \phi_g^* \lambda_i \theta_i(x; X_\xi) = \sum_{i=0}^{k} \lambda_i(x \cdot g)\mathrm{Ad}(g^{-1})\xi = \mathrm{Ad}(g^{-1})\xi.$$

These two calculations verify that $\sum_{i=0}^{k} \lambda_i \theta_i$ is indeed a connection. $\square$

On the standard n-simplex Δ^n we use the coordinates $(t_0, \ldots, t_n)$ where $\sum_{i=0}^{n} t_i = 1$. From this and the previous proposition the differential form defined by

$$\theta_n = \sum_{i=0}^{n} t_i \, p_i^* \theta_L \in \Omega^*(N\overline{G}_n \times \Delta^n) \otimes \mathfrak{g}$$

is a connection form. Moreover, by construction it is true that

$$(d^i \times \mathrm{id})^* \theta_n = (\mathrm{id} \times d_i)^* \theta_{n-1}$$

for each $i = 0, 1, \ldots, n$ and hence

$$\theta_\bullet = \{\theta_n\}_{n \geq 0} \in A^1(N\overline{G}_\bullet) \otimes \mathfrak{g} \tag{4.27}$$

is a well defined simplicial connection on $N\overline{G}_\bullet$. Similarly, we can define the simplicial curvature 2-form by

$$\Omega_\bullet = \{d\theta_n - \frac{1}{2}[\theta_n, \theta_n]\}_{n \geq 0} \in A^2(N\overline{G}_\bullet) \otimes \mathfrak{g} \tag{4.28}$$

which we will use shortly.

4.2.3 The Simplicial Chern-Weil Homomorphism

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\theta_\bullet$ be a connection form on a principal G-bundle $P_\bullet \to B_\bullet$. Since θ_n is a connection on $\Delta^n \times P_n \to \Delta^n \times B_n$ it is clear that for any $\xi \in \mathfrak{g}$ and $p \in P_n$,

$$\theta_n(p; X_\xi) = \xi.$$

Let $\{\xi_1, \dots, \xi_n\}$ be a basis for $\mathfrak{g}$ and $\{\xi_1^*, \dots, \xi_n^*\}$ the corresponding dual basis for $\mathfrak{g}^*$. Similarly to the discussion in Section 2.2.2, we define a series of 1-forms

$$\theta_\bullet^i := \{\xi_i^* \theta_n\}_{n \geq 0} \in A^1(P_\bullet).$$

Likewise, if $\Omega_\bullet$ is the curvature 2-form associated to $\theta_\bullet$ we define a corresponding series of 2-forms

$$\mu_\bullet^i := \{\xi_i^* \Omega_n\}_{n \geq 0} \in A^2(P_\bullet).$$

Associated to the choice of connection $\theta_\bullet$ is a unique homomorphism

$$w(\theta_\bullet) : W(\mathfrak{g}) \to A^*(P_\bullet)$$

given by the natural identifications

$$w(\theta_\bullet)(\theta^i) = \theta_\bullet^i, \quad w(\theta_\bullet)(\mu^i) = \mu_\bullet^i.$$

As previously calculated in Section 2.3.3, for each $n \geq 0$ this homomorphism can be thought of as the ordinary Weil homomorphism

$$w(\theta_n) : W(\mathfrak{g}) \to \Omega^*(P_n)$$

and hence the identities

1. $dw(\theta_\bullet) - w(\theta_\bullet)d = 0$
2. $\iota_\xi w(\theta_\bullet) - w(\theta_\bullet)\iota_\xi = 0$

are satisfied accordingly. First, a useful lemma will be proved.

Lemma 4.29: *Let M, N be manifolds and F be a compact manifold of dimension k. If $g : M \to N$ is a smooth function then the diagram*

$$\begin{tikzcd}
\Omega^q(N \times F) \arrow[r, "(g \times \mathrm{id})^*"] \arrow[d, "\int_F"] & \Omega^q(M \times F) \arrow[d, "\int_F"] \\
\Omega^{q-k}(N) \arrow[r, "g^*"] & \Omega^{q-k}(M)
\end{tikzcd}$$

commutes.

Proof. If ω is a differential n-form on $N \times F$ it can locally be written as

$$\omega = \sum f_{i_1 \cdots i_q} dx_{i_1} \cdots dx_{i_n} d\varphi_{j_1} \cdots d\varphi_{j_{q-n}}$$

where $f_{i_1 \cdots i_q} : N \times F \to \mathbb{R}$ and $\{x_i\}, \{\varphi_i\}$ are local coordinates on F and N respectively. Then one notes that for any $y \in N$, we may consider the smooth map

$$f_{i_1 \cdots i_q}(y, -) : F \to \mathbb{R}.$$

If $n = k$ we can thus consider

$$\int_F f_{i_1 \cdots i_q} dx_{i_1} \cdots dx_{i_k} : N \to \mathbb{R}$$

as a smooth map which, at a point $y \in N$, takes the value

$$\left(\int_F f_{i_1 \cdots i_q} dx_{i_1} \cdots dx_{i_k} \right)(y) = \int_F f_{i_1 \cdots i_q}(y, -) dx_{i_1} \cdots dx_{i_k}.$$

It follows that if $g : M \to N$ is a smooth map,

$$\begin{aligned}
\left(g^* \int_F f_{i_1 \cdots i_q} dx_{i_1} \cdots dx_{i_k} \right)(y) &= \left(\int_F f_{i_1 \cdots i_q}\, dx_{i_1} \cdots dx_{i_k} \right)(g(y)) \\
&= \int_F f_{i_1 \cdots i_q}(g(y), -) dx_{i_1} \cdots dx_{i_k} \\
&= \int_F \left(f_{i_1 \cdots i_q} \circ (g \times \mathrm{id}) \right) dx_{i_1} \cdots dx_{i_n}
\end{aligned}$$

and hence

$$\begin{aligned}
g^* \int_F \omega &= \sum \int_F \left(f_{i_1 \cdots i_q} \circ (g \times \mathrm{id}^*) \right) dx_{i_1} \cdots dx_{i_n} d(\varphi_{j_1} \circ g) \cdots d(\varphi_{j_{q-n}} \circ g) \\
&= \int_F (g \times \mathrm{id}^*) \omega
\end{aligned}$$

proving the lemma. □

Lemma 4.30: *Let $M_\bullet$ be a simplicial manifold and F be a compact manifold of dimension m. Define $M_\bullet \times F$ to be the sequence of simplicial manifolds*

$$M_n \times F$$

where the face and degeneracy maps are simply

$$\bar{d_i} := d_i \times \mathrm{id}_F : M_n \times F \to M_{n-1} \times F \quad \& \quad \bar{s_i} := s_i \times \mathrm{id}_F : M_n \times F \to M_{n+1} \times F$$

where $0 \leq i \leq n$ and $n \geq 0$. Then the integration map over the fibre F,

$$\int_F : A^q(M_\bullet \times F) \to A^{q-k}(M_\bullet),$$

is well defined.

Proof. Let $\omega_\bullet = \{\omega_p\}$ be a simplicial differential form on $A^*(M_\bullet \times F)$. By Equation 4.16, we have that

$$(d^i \times \mathrm{id}_{M_{p+1}} \times \mathrm{id}_F)^*\omega_{p+1} = (\mathrm{id}_{\Delta^p} \times \bar{d}_i)^*\omega_p$$

for every $p \geq 0$. By two applications of Lemma 4.29, we see that

$$(d^i \times \mathrm{id}_{M_{p+1}})^* \int_F \omega_{p+1} = \int_F (d^i \times \mathrm{id}_{M_{p+1}} \times \mathrm{id}_F)^*\omega_{p+1} \tag{4.31}$$

$$= \int_F (\mathrm{id}_{\Delta^p} \times \bar{d}_i)^*\omega_p \tag{4.32}$$

$$= (\mathrm{id}_{\Delta^p} \times d_i)^* \int_F \omega_p \tag{4.33}$$

and hence $\int_F \omega_\bullet$ is well defined. □

Theorem 4.34: *Let $P_\bullet \to B_\bullet$ be a simplicial principal bundle and $\theta_\bullet$, $\theta'_\bullet$ be two connections on $P_\bullet$. Then there is a chain homotopy*

$$w(\theta_\bullet) \simeq w(\theta'_\bullet).$$

Moreover, since the Weil homomorphism descends to a map on basic differential forms, the induced maps on cohomology

$$w(\theta_\bullet), w(\theta'_\bullet) : S(\mathfrak{g}^*) \to H^*(A^*(P_\bullet))$$

are equal.

Proof. Let $P_\bullet \to B_\bullet$ be a simplicial principal G-bundle with simplicial connections $\theta_\bullet, \theta'_\bullet$. Associated to each connection are the respective simplicial curvature 2-forms $\Omega_\bullet, \Omega'_\bullet$. We restrict our attention to individually considering connections θ_n, θ'_n on the principal bundle

$$\Delta^n \times P_n \to \Delta^n \times B_n.$$

Associated to this bundle is the bundle $\Delta^n \times P_n \times I \to \Delta^n \times B_n \times I$ and projection bundle maps

$$p_n : \Delta^n \times P_n \times I \to \Delta^n \times P_n$$

given by $p_n(t, p, s) = (t, p)$ and inclusion bundle maps

$$i_n^s : \Delta^n \times P_n \to \Delta^n \times P_n \times I$$

given by $i_n^s(t, p) = (t, p, s)$ for every $s \in I$. Note that these maps commute with the maps defined in (4.16) and hence for any simplicial differential form $\omega_\bullet \in A^*(P_\bullet)$ the simplicial differential form

$$p_\bullet^*\omega_\bullet = \{p_n^*\omega_n\}_{n\geq 0} \in A^*(P_\bullet \times I)$$

is well defined as well as the simplicial differential form

$$i_\bullet^{s*}\nu_\bullet = \{i_n^{s*}\nu_n\}_{n\geq 0} \in A^*(P_\bullet)$$

for $\nu_\bullet \in A^*(P_\bullet \times I)$ and any $s \in I$. Then define a connection $\tilde{\theta}_\bullet \in A^1(P_\bullet \times I) \otimes \mathfrak{g}$ by

$$\tilde{\theta}_\bullet := (1-s)p_\bullet^*\theta_\bullet + sp_\bullet^*\theta'_\bullet$$

and note that $i_\bullet^{0*}\tilde{\theta}_\bullet = \theta_\bullet$ and $i_\bullet^{1*}\tilde{\theta}_\bullet = \theta'_\bullet$. The connection $\tilde{\theta}_\bullet$ has curvature $\tilde{\Omega}_\bullet$ which also satisfies that $i_\bullet^{0*}\tilde{\Omega}_\bullet = \Omega_\bullet$ and $i_\bullet^{1*}\tilde{\Omega}_\bullet = \Omega'_\bullet$ since curvature commutes with pullbacks.

The connection $\tilde{\theta}_\bullet$ also induces a simplicial Weil homomorphism

$$w(\tilde{\theta}_\bullet) : W(\mathfrak{g}) \to A^*(P_\bullet).$$

Note that from the identities above,

$$i_\bullet^{0*}w(\tilde{\theta}_\bullet) = w(\theta_\bullet) \quad \text{and} \quad i_\bullet^{1*}w(\tilde{\theta}_\bullet) = w(\theta'{}_\bullet). \tag{4.35}$$

By Lemma 4.30, we can define an operator $h_\bullet : A^q(P_\bullet \times I) \to A^{q-1}(P_\bullet)$ for every q by

$$h_\bullet\omega_\bullet = \int_I \omega_\bullet$$

The map $h_\bullet$ is a homotopy operator in the sense that for every $n \geq 0$

$$(dh_n + h_n d)\tilde{\omega}_n = i_n^{1*}\tilde{\omega}_n - i_n^{0*}\tilde{\omega}_n$$

and hence $dh_\bullet + h_\bullet d = i_\bullet^{1*} - i_\bullet^{0*}$. It follows that

$$\begin{aligned}(dh_\bullet + h_\bullet d)w(\tilde{\theta}_\bullet) &= (i_\bullet^{1*} - i_\bullet^{0*})w(\tilde{\theta}_\bullet) \\ &= w(\theta_\bullet)(\alpha) - w(\theta'_\bullet)(\alpha) \end{aligned} \tag{4.36}$$

for every $n \geq 0$. Since $w(\tilde{\theta}_\bullet)$ is a chain map it follows by Equation 4.36 that for any $\alpha \in W(\mathfrak{g})$

$$\left(d\left(h_\bullet w(\tilde{\theta}_\bullet)\right) + \left(h_\bullet w(\tilde{\theta}_\bullet)\right)d\right)\alpha = w(\theta_\bullet)(\alpha) - w(\theta'_\bullet)(\alpha).$$

The homomorphisms $w(\theta_\bullet), w(\theta'_\bullet)$ are thus chain homotopic. The map $w(\tilde{\theta}_\bullet)$ descends to a map on basic differential forms and hence if α is a closed and basic

$$d\left(h_\bullet w(\tilde{\theta}_\bullet)(\alpha)\right) = w(\theta_\bullet)(\alpha) - w(\theta'_\bullet)(\alpha)$$

and hence the simplicial Chern-Weil homomorphism is independent of the choice of connection. $\square$

4.3 An Analogue of the Weil Model

4.3.1 G^* Algebras

The term G^* algebra is due to [GS99] but the idea can be traced back to Cartan's paper [Car50b]. Let G be a Lie group and A be a chain complex that has a right G-action. The general idea of a G^* algebra is to isolate the conditions on a chain complex A such that it can be thought of as an algebraic model for equivariant cohomology and is motivated by the *graded derivations* d, ι_ξ and $\mathcal{L}_\xi$ on $\Omega^*(M)$ that were encountered in Section 2.2.1. We first recall the definition of a graded commutative algebra and some facts about these graded derivations.

Definition 4.37: A $(\mathbb{Z}-)$graded commutative algebra A *is an algebra*

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

equipped with a multiplication satisfying

$$a \cdot b = (-1)^{ij} b \cdot a$$

for $a \in A_i$ and $b \in A_j$ such that

$$A_i \cdot A_j \subset A_{i+j}.$$

Definition 4.38: *Let A be a graded commutative algebra. A graded derivation D of degree k is an endomorphism $D : A \to A$ satisfying the graded Leibniz identity*

$$D(a \cdot b) = (Da) \cdot b + (-1)^{ki} a \cdot (Db) \in A_{i+j+k}$$

for $a \in A_i$ and $b \in A_j$.

The vector space of graded derivations of degree k is denoted $\mathrm{Der}_k(A)$. The collection of all derivations of A is defined as

$$\mathrm{Der}(A) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Der}_k(A).$$

Because of this natural grading on the derivations of A, there is a bracket

$$[-,-] : \mathrm{Der}_k(A) \otimes \mathrm{Der}_l(A) \to \mathrm{Der}_{k+l}(A)$$

called the *graded commutator* which is defined by

$$[C, D] = CD - (-1)^{kl} DC$$

where C is a graded derivation of degree k and D is a graded derivation of degree l. With this bracket, it follows that $\mathrm{Der}_0(A)$ has a natural Lie algebra structure.

Example 4.39: Note that if M is a manifold and $M_\bullet$ is a simplicial manifold, the wedge product on the graded algebras $\Omega^*(M)$ and $A^*(M_\bullet)$ satisfy the condition in Definition 4.37. Moreover, the de Rham differential d is a graded derivation of degree 1 satisfying Definition 4.38. The fact that $d^2 = 0$ means this is a *differential operator* and the notion of a *differential graded commutative algebra* follows.

Definition 4.40: *A* differential graded commutative algebra (A, d) *is a graded commutative algebra with a graded derivation d of degree 1 which is also a differential operator (i.e. $d^2 = 0$).*

From this definition it is clear that $(\Omega^*(M), d)$ and $(A^*(M_\bullet), d)$ are differential graded commutative algebras. We now recall some properties about the graded derivations d, ι_ξ and $\mathcal{L}_\xi$ on $\Omega^*(M)$ which we will carry over to the definition of a G^* algebra.

The graded derivation ι_ξ of degree -1

The derivation ι_ξ can be thought of more abstractly as a map from $\mathfrak{g}$ to the derivations of $\Omega^*(M)$

$$\iota : \mathfrak{g} \to \mathrm{Der}_{-1}(\Omega^*(M))$$

and hence can be acted upon by G. Firstly note that $\iota_\xi^2 = 0$ for every $\xi \in \mathfrak{g}$. The natural action of G on $\mathfrak{g}$ is the adjoint action and so one would expect that the identity

$$\phi_g^* \iota_\xi \phi_{g^{-1}}^* = \iota_{\mathrm{Ad}(g)\xi} \tag{4.41}$$

is held.

Proof. Let $\xi \in \mathfrak{g}$, $g \in G$ and consider an n-form $\omega \in \Omega^n(M)$. First recall

$$(\phi_g)_* X_\xi(p) = X_{\mathrm{Ad}(g^{-1})\xi}(p)$$

from Equation 2.34. Then consider

$$(\phi_g^* \iota_\xi \phi_{g^{-1}}^* \omega)(x; X_1, \ldots, X_{n-1})$$

where $X_1, \ldots, X_{n-1}$ are any $n-1$ vector fields on M. The calculation is straightforward and yields

$$\begin{aligned}
\phi_g^*(\iota_\xi \phi_{g^{-1}}^* \omega)(x; X_1, \ldots, X_{n-1}) &= (\iota_\xi \phi_{g^{-1}}^* \omega)(x \cdot g; (\phi_g)_* X_1, \ldots, (\phi_g)_* X_{n-1}) \\
&= (\phi_{g^{-1}}^* \omega)(x \cdot g; X_\xi, (\phi_g)_* X_1, \ldots, (\phi_g)_* X_{n-1}) \\
&= \omega(x; (\phi_g^{-1})_* X_\xi, X_1, \ldots, X_{n-1}) \\
&= \iota_{\mathrm{Ad}(g)\xi} \omega(x; X_1, \ldots, X_{n-1})
\end{aligned}$$

which verifies that the identity in Equation 4.41 holds. $\square$

The graded derivation $\mathcal{L}_\xi$ of degree 0

Because the derivations of degree 0 are naturally a Lie algebra, the derivation $\mathcal{L}_\xi$ can be thought of more abstractly as a Lie algebra homomorphism

$$\mathcal{L} : \mathfrak{g} \to \mathrm{Der}_0(\Omega^*(M))$$

and hence can be acted upon by G. We defined $\mathcal{L}_\xi$ in 2.31 to be the Lie derivative

$$\mathcal{L}_\xi \omega = \frac{d}{dt} \phi^*_{\exp t\xi} \omega \bigg|_{t=0}. \tag{4.42}$$

Since this is thought of as 'differentiation in the direction ξ', one would expect this derivation to agree with the group action in this way. Similarly to the case of ι_ξ, one would expect that the identity

$$\phi^*_g \mathcal{L}_\xi \phi^*_{g^{-1}} = \mathcal{L}_{\mathrm{Ad}(g)\xi} \tag{4.43}$$

is held.

Proof. Cartan's magic formula may be invoked to show that

$$\begin{aligned}
\phi^*_g \mathcal{L}_\xi \phi^*_{g^{-1}} &= \phi^*_g (d\iota_\xi + \iota_\xi d) \phi^*_{g^{-1}} \\
&= d\phi^*_g \iota_\xi \phi^*_{g^{-1}} + \phi^*_g \iota_\xi \phi^*_{g^{-1}} d \\
&= d\iota_{\mathrm{Ad}(g)\xi} + \iota_{\mathrm{Ad}(g)\xi} d \\
&= \mathcal{L}_{\mathrm{Ad}(g)\xi}
\end{aligned}$$

which verifies that the identity in Equation 4.43 holds. □

The graded derivation d of degree 1

The commutativity identity

$$\phi^*_g d \phi^*_{g^{-1}} = d \tag{4.44}$$

holds since pullbacks commute with d.

We combine these identities associated to the differential graded commutative algebra $\Omega^*(M)$ as a blueprint to form the definition of a G^* algebra.

Definition 4.45: *A* G^*-algebra *is a differential graded commutative algebra* (A, d) *with a right* G*-action*

$$\phi : A \times G \to A$$

together with maps

$$\iota\colon \mathfrak{g} \to \mathrm{Der}_{-1}(A)$$
$$\mathcal{L}\colon \mathfrak{g} \to \mathrm{Der}_0(A)$$

such that $(\iota_\xi)^2 = 0$ *for all* $\xi \in \mathfrak{g}$, $\mathcal{L}$ *is a Lie algebra homomorphism, and the following identities are satisfied:*

$$\mathcal{L}_\xi = d\iota_\xi + \iota_\xi d,\ [\mathcal{L}_\xi, \iota_\eta] = \iota_{[\xi,\eta]}$$

for all $\xi, \eta \in \mathfrak{g}$. *Additionally the following identities involving the map* ϕ *are satisfied:*

1. $\frac{d}{dt}\phi_{\exp t\xi}\big|_{t=0} = \mathcal{L}_\xi$,
2. $\phi_g \mathcal{L}_\xi \phi_{g^{-1}} = \mathcal{L}_{\mathrm{Ad}(g)\xi}$,
3. $\phi_g \iota_\xi \phi_{g^{-1}} = \iota_{\mathrm{Ad}(g)\xi}$ *and*
4. $\phi_g d \phi_{g^{-1}} = d$.

Corollary 4.46: *Let* G *be a Lie group and* M *a manifold with right* G*-action. Then* $\Omega^*(M)$ *is a* G^* *algebra.*

The natural notion of a homomorphism of G^* algebra arises as an algebra homomorphism that commutes with the action of G and the graded derivations $d, \iota_\xi, \mathcal{L}_\xi$.

Definition 4.47: *Let* A, B *be* G^* *algebras. A* G^* homomorphism

$$f : A \to B$$

is an algebra homomorphism such that

1. $\phi_g \circ f = f \circ \phi_g$,
2. $d \circ f = f \circ d$,
3. $\iota_\xi \circ f = f \circ \iota_\xi$ *and* $\mathcal{L}_\xi \circ f = f \circ \mathcal{L}_\xi$

for every $g \in G$ *and* $\xi \in \mathfrak{g}$.

Remark 4.48: Note that a G^* homomorphism $f : A \to B$ induces a natural chain map $f : A_{bas} \to B_{bas}$ since it commutes with d, ι_ξ (for every $\xi \in \mathfrak{g}$) and is equivariant.

Let G be a Lie group and $M_\bullet$ be a simplicial manifold with right G-action. The complex $A^*(M_\bullet)$ is built out of the complexes $\Omega^*(\Delta^p \times M_p)$ which are G^* algebras from Corollary 4.46. Because of this we only need to show is that the graded derivations $\mathcal{L}_\xi, \iota_\xi$ and d are well defined on $A^*(M_\bullet)$ by showing that they commute with the face maps defined in 4.16. Recall that the pullbacks of equivariant maps commute with ι_ξ for all $\xi \in \mathfrak{g}$ by Proposition 4.22. It is clear that they also commute with $\mathcal{L}_\xi$ from the identity

$$\mathcal{L}_\xi = d\iota_\xi + \iota_\xi d.$$

Hence, we have the following corollary.

Corollary 4.49: *Let $f : M \to N$ be an equivariant map. Then*

$$f^* \mathcal{L}_\xi = \mathcal{L}_\xi f^*$$

for every $\xi \in \mathfrak{g}$.

From these last two corollaries, it is evident that since the face maps $d^i \times \mathrm{id}$ and $\mathrm{id} \times d_i$ are equivariant the derivations $\mathcal{L}_\xi, \iota_\xi, d$ commute with them and hence $A^*(M_\bullet)$ is a G^* algebra for any simplicial manifold with right G-action.

Proposition 4.50: *Let $M_\bullet$ be a simplicial manifold with right G-action. Then $A^*(M_\bullet)$ is a G^* algebra.*

4.3.2 The Cohomology of a G^* Algebra

Since a G^* algebra A has a differential operator d there is a well defined notion of cohomology for A. Namely, the n^{th} cohomology group of A is defined to be

$$H^n(A) = \frac{\ker d \cap A}{\operatorname{im} d \cap A}$$

for every $n \geq 0$. We can also define the *basic subalgebra* of a G^* algebra.

Definition 4.51: *Let A be a G^* algebra. The element $a \in A$ is said to be* basic *if it is horizontal and G-invariant. That is, $\iota_\xi a = 0$ and $a \cdot g = a$ for every $\xi \in \mathfrak{g}$ and $g \in G$. The* basic subcomplex *is the complex that contains all the basic elements of A and is denoted A_{bas}.*

Note that an implication of Cartan's magic formula is that for any $a \in A_{bas}$, $da \in A_{bas}$. This is because the action of G commutes with d,

$$\iota_\xi(da) = \mathcal{L}_\xi a - d(\iota_\xi a),$$

$\mathcal{L}_\xi a = 0$ when a is G-invariant and $\iota_\xi a = 0$ by the definition of a basic element of A. Thinking of (A, d) as a complex, this means that (A_{bas}, d) is a well defined subcomplex. One defines the *basic cohomology* of a A to be the cohomology of the basic subcomplex

$$H^*(A_{bas}).$$

4.3.3 The Cartan Model for G^* Algebras

Recall from Section 2.3.4 that there was an operator

$$\zeta = \theta^i \otimes \iota_{\xi_i}$$

such that $\exp(\zeta) : (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \to (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G$ was an isomorphism. If A is a G^* algebra, then Definition 4.45 gives a sensible notion of ζ on the complex $W(\mathfrak{g}) \otimes A$. Namely, we define the operator

$$\zeta = \theta^i \otimes \iota_{\xi_i} : W(\mathfrak{g}) \otimes A \to W(\mathfrak{g}) \otimes A. \tag{4.52}$$

Since the results of Secion 2.3.4 and 2.3.5 only rely on the G^* structure of $\Omega(M)$, the fact that the map

$$\exp(\zeta) : (W(\mathfrak{g}) \otimes A)_{bas} \to (S(\mathfrak{g}^*) \otimes A)^G$$

is an isomorphism can also be proven in a similar fashion. We will refer to the complex

$$(S(\mathfrak{g}^*) \otimes A^*(M_\bullet))^G$$

as the *Cartan Model* of A. We state the results and direct the reader to [GS99, Ch. 4] for further reading on the matter.

Proposition 4.53: *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and A be a G^* algebra. Then the* Mathai-Quillen Isomorphism

$$\exp(\zeta) : (W(\mathfrak{g}) \otimes A)_{bas} \to (S(\mathfrak{g}^*) \otimes A)^G$$

satisfies the following identities.

1. $d_G := \exp(\zeta) d \exp(-\zeta) = d - \mu^k \otimes \iota_{\xi_k}$.
2. $\exp(\zeta)(1 \otimes \iota_\xi + \iota_\xi \otimes 1) \exp(-\zeta) = \iota_\xi \otimes 1$.

Example 4.54: Let $\alpha \otimes b \in (S(\mathfrak{g}^*) \otimes A)^G$. Then

$$d_G(\alpha \otimes b) = \alpha \otimes db - \mu^k \alpha \otimes \iota_{\xi_k} b$$

where Einstein summation convention is observed.

We exploit the technology of Guillemin and Sternberg to further prove a result about homomorphisms between G^* algebras and the induced homomorphisms between their Cartan models.

Theorem 4.55: *Let G be a compact group. Let A, B be G^* algebras and $f : A \to B$ a G^* homomorphism that induces an isomorphism in cohomology. Then the map*

$$1 \otimes f : \{S(\mathfrak{g}^*) \otimes A\}^G \to \{S(\mathfrak{g}^*) \otimes B\}^G$$

is an isomorphism on cohomology with respect to the differential operator d_G.

Proof. The reader is directed to [GS99, Ch. 6] for a proof of this statement. □

4.4 Simplicial Equivariant Cohomology and Cartan's Theorem

4.4.1 Constructing a G^* Homomorphism

At the end of the last section, we stated a result of Guillemin and Sternberg regarding induced maps on the Cartan models of G^* Algebras. In this section we will construct a G^* homomorphism

$$\Omega^*(M) \to A^*(N\overline{G}_\bullet \times M)$$

such that there is a chain map

$$\{S(\mathfrak{g}^*) \otimes \Omega^*(M)\}^G \to \{S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M)\}^G$$

that induces an isomorphism in cohomology via Theorem 4.55.

Recall that any manifold M can be turned into a simplicial manifold $\{M_n\}_{n\geq 0}$ by letting $M_n = M$ and setting each face and degeneracy map to the identity

$$d_i = \mathrm{id}_M : M_n \to M_{n-1}, s_i = \mathrm{id}_M : M_n \to M_{n+1}$$

for every $0 \leq i \leq n$. As a slight abuse of notation, for the rest of the section we will consider $\Omega^{*,*}(M)$ and $A^*(M)$ to refer to the chain complexes associated to the trivial simplicial manifold constructed from the manifold M.

Lemma 4.56: *The canonical inclusion*

$$\Omega^*(M) \hookrightarrow \Omega^{0,*}(M)$$

of $\Omega^*(M)$ into the first column of $\Omega^{*,*}(M)$ induces an isomorphism in cohomology

$$H^*(\Omega^*(M)) \cong H^*_{\mathrm{Tot}}(\Omega^{*,*}(M)).$$

Proof. An element $\nu \in \mathrm{Tot}^q(\Omega^{*,*}(M))$ is of the form

$$\nu = (\nu_0, \nu_1, \ldots, \nu_q)$$

where $\nu_i \in \Omega^{q-i}(M) = \Omega^{i,q-i}(M)$. From the definition of δ in Equation 4.2 we see that

$$\delta\nu_i = \begin{cases} \nu_i & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases} \tag{4.57}$$

for each ν_i. It is perhaps easiest to see this as a diagram.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\Omega^2(M) & \xrightarrow{0} & \Omega^2(M) & \xrightarrow{\mathrm{id}^*} & \Omega^2(M) & \xrightarrow{0} & \Omega^2(M) & \xrightarrow{\mathrm{id}^*} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\Omega^1(M) & \xrightarrow{0} & \Omega^1(M) & \xrightarrow{\mathrm{id}^*} & \Omega^1(M) & \xrightarrow{0} & \Omega^1(M) & \xrightarrow{\mathrm{id}^*} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\Omega^0(M) & \xrightarrow{0} & \Omega^0(M) & \xrightarrow{\mathrm{id}^*} & \Omega^0(M) & \xrightarrow{0} & \Omega^0(M) & \xrightarrow{\mathrm{id}^*} & \cdots
\end{array}$$

Looking again at Equations 4.3 and 4.57,

$$D\nu = \begin{cases} (d\nu_0, -d\nu_1, \nu_1 + d\nu_2, -d\nu_3, \dots, -d\nu_q, \nu_q) \text{ if } q \text{ is odd} \\ (d\nu_0, -d\nu_1, \nu_1 + d\nu_2, -d\nu_3, \dots, \nu_{q-1} + d\nu_q, 0) \text{ if } q \text{ is even} \end{cases} \tag{4.58}$$

for a general element ν. If ν is a cocycle then the condition that $d\nu_i = 0$ and $\nu_i = -d\nu_{i+1}$ for each even odd $i < q$. If q is odd, we also have that $\nu_q = 0$. This means we can write any cocycle ν as

$$\nu = (\nu_0, 0, \dots, 0) + D(0, \nu_2, 0, \nu_4, \dots)$$

and thus any cocycle can be represented by it's first entry ν_0. Ultimately, this means the inclusion

$$\omega \mapsto (\omega, 0, \dots, 0)$$

is a surjection on cohomology. Clearly this map is an injection on cohomology as well and thus the map

$$\Omega^*(M) \hookrightarrow \Omega^{0,*}(M)$$

defines an isomorphism on cohomology. □

Proposition 4.59: *The map* $p_M : N\overline{G}_\bullet \times M \to M$ *induces a map*

$$\Omega^*(M) \to A^*(N\overline{G}_\bullet \times M)$$

which is an isomorphism in cohomology.

Proof. From Lemma 4.56 we have a natural inclusion

$$\Omega^*(M) \xhookrightarrow{i} \Omega^{0,*}(M).$$

The map p_M induces a map

$$\Omega^{*,*}(M) \xrightarrow{p_M^*} \Omega^{*,*}(N\overline{G}_\bullet \times M)$$

which is an isomorphism in cohomology by Theorem 4.14 and Corollary 3.62. From Theorem 4.20, the chain map

$$E_\Delta : \Omega^{*,*}(N\overline{G}_\bullet \times M) \to A^*(N\overline{G}_\bullet \times M)$$

induces an isomorphism in cohomology. Hence the map $E_\Delta \circ p_M^* \circ i$, which may be explicitly written as

$$\omega \mapsto \left\{ p! \sum_{s=0}^{p} (-1)^s t_{i_s}\, dt_{i_0} \cdots \hat{dt_{i_s}} \cdots dt_{i_p} \wedge p_M^* \omega \right\}_{p \geq 0},$$

induces an isomorphism in cohomology. □

Proposition 4.60: *The map $E_\Delta \circ p_M^* \circ i : \Omega^*(M) \to A^*(N\overline{G}_\bullet \times M)$ is a G^* morphism.*

Proof. Firstly we note that each map E_Δ, p_M^*, i is a chain map and hence

$$d(E_\Delta \circ p_M^* \circ i) = (E_\Delta \circ p_M^* \circ i)d.$$

The maps p_M^*, i are equivariant so it remains to verify that E_Δ is equivariant. This is trivially true since

$$\begin{aligned} E_\Delta(\omega)_p \cdot g &= p! \left(\sum_{s=0}^{p} (-1)^s t_{i_s}\, dt_{i_0} \cdots \hat{dt_{i_s}} \cdots dt_{i_p} \wedge \omega \right) \cdot g \\ &= p! \sum_{s=0}^{p} (-1)^s t_{i_s}\, dt_{i_0} \cdots \hat{dt_{i_s}} \cdots dt_{i_p} \wedge (\omega \cdot g) \\ &= E_\Delta(\omega \cdot g)_p \end{aligned}$$

for every $g \in G$ since G acts trivially on Δ^p. Since $E_\Delta \circ p_M^* \circ i$ is equivariant, by Definition 4.45 (1) it is also true that

$$\mathcal{L}_\xi(E_\Delta \circ p_M^* \circ i) = (E_\Delta \circ p_M^* \circ i)\mathcal{L}_\xi.$$

for every $\xi \in \mathfrak{g}$. Finally, the fact that $E_\Delta \circ p_M^* \circ i$ commutes with ι_ξ follows from the calculation

$$\begin{aligned} \left(\iota_\xi(E_\Delta \circ p_M^* \circ i)\omega\right)_p &= \iota_\xi p! \sum_{s=0}^{p} (-1)^s t_{i_s}\, dt_{i_0} \cdots \hat{dt_{i_s}} \cdots dt_{i_p} \wedge p_M^* \omega \\ &= p! \sum_{s=0}^{p} (-1)^s t_{i_s}\, dt_{i_0} \cdots \hat{dt_{i_s}} \cdots dt_{i_p} \wedge \iota_\xi p_M^* \omega \\ &= p! \sum_{s=0}^{p} (-1)^s t_{i_s}\, dt_{i_0} \cdots \hat{dt_{i_s}} \cdots dt_{i_p} \wedge p_M^* \iota_\xi \omega \\ &= \left((E_\Delta \circ p_M^* \circ i)\iota_\xi \omega\right)_p \end{aligned}$$

for every $\xi \in \mathfrak{g}$ and every $p \geq 0$ since ι_ξ vanishes on Δ^p and commutes with pullbacks by Proposition 4.22. □

4.4.2 Cartan's Theorem via Simplicial Differential Forms

Lemma 4.61: *Let $P \to B$ be a principal G-bundle and $\mathfrak{g}$ be the Lie algebra of G with basis $\{\xi_1, \dots, \xi_n\}$. Let P have connection θ and elements $\{\theta^1, \dots, \theta^n\}$ respectively to the basis of $\mathfrak{g}$. Then a differential form ω can be written uniquely as a sum*

$$\omega = \sum \theta^{i_1} \dots \theta^{i_k} \nu_{i_1,\dots,i_k}$$

where $\nu_{i_1,\dots,i_k}$ is a horizontal differential form on P.

Proof. Let ω be a differential form on P. Define the components $\alpha_i = \iota_{\xi_i}\omega$ and $\beta_i = \omega - \theta^i\alpha_i$ such that $\omega = \theta^i\alpha_i + \beta_i$. Clearly α satisfies $\iota_{\xi_i}\alpha_i = 0$. Moreover, β_i satisfies

$$\begin{aligned} \iota_{\xi_i}\beta_i &= \iota_{\xi_i}\omega - \iota_{\xi_i}(\theta^i\iota_{\xi_i}\omega) \\ &= \iota_{\xi_i}\omega - (\iota_{\xi_i}\theta^i)\iota_{\xi_i}\omega + \theta^i\iota_{\xi_i}(\iota_{\xi_i}\omega) \\ &= 0. \end{aligned}$$

Since any differential form ω can be written as

$$\omega = \theta^i\alpha_i + \beta_i,$$

the process can be repeated on α_i and β_i for a different index $j \neq i$ to yield the decomposition

$$\alpha_i = \theta^j\alpha_{ij} + \beta_{ij}, \quad \beta_i = \theta^j\alpha'_{ij} + \beta'_{ij}$$

and thus $\omega = \theta^i\theta^j\alpha_{ij} + \theta^i\beta_{ij} + \theta^j\alpha'_{ij} + \beta'_{ij}$ and each of the differential forms $\alpha_{ij}, \beta_{ij}, \alpha'_{ij}, \beta'_{ij}$ vanish on ι_{ξ_i} and ι_{ξ_j}. Repeating this for each index $i = 1, \dots, n$ we obtain the unique decomposition

$$\omega = \sum \theta^{i_1} \dots \theta^{i_k} \nu_{i_1,\dots,i_k}$$

as required. $\square$

Since the lemma above relies only on the existence of the operator ι_ξ and connection elements, the above calculation can be repeated in the simplicial case to yield the following corollary.

Corollary 4.62: *Let $P_\bullet$ have simplicial connection $\theta_\bullet$ and respectively elements $\{\theta^1_\bullet, \dots, \theta^n_\bullet\}$. Then a simplicial differential form $\omega_\bullet$ can be written uniquely as a sum*

$$\omega_\bullet = \sum \theta^{i_1}_\bullet \dots \theta^{i_k}_\bullet \nu_{i_1,\dots,i_k\bullet}$$

where $\nu_{i_1,\dots,i_k\bullet}$ is a horizontal simplicial differential form on $P_\bullet$.

Let $A^*(N\overline{G}_\bullet \times M)_{vert}$ be the set of elements generated by $\{\theta^1_\bullet, \dots, \theta^n_\bullet\}$. Clearly from the corollary we have the decomposition

$$A^*(N\overline{G}_\bullet \times M) = A^*(N\overline{G}_\bullet \times M)_{vert} \otimes A^*(N\overline{G}_\bullet \times M)_{hor}.$$

There is a natural projection onto the *horizontal component* of $\omega_\bullet$,

$$\mathrm{Hor}(\omega_\bullet) = \mathrm{Hor}(\sum \theta_\bullet^{i_1} \dots \theta_\bullet^{i_k} \nu_{i_1,\dots,i_k \bullet}) = \nu_0$$

where ν_0 is the single summand in $A^*(N\overline{G}_\bullet \times M)_{hor}$. We are interested in the complex $(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$ which can thus be decomposed

$$(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G = \bigoplus C^{p,q} \tag{4.63}$$

where $C^{p,q} = (S^p(\mathfrak{g}^*) \otimes A^q(N\overline{G}_\bullet \times M)_{vert} \otimes A^*(N\overline{G}_\bullet \times M)_{hor})^G$. That is, an element of $c \in C^{p,q}$ can be written as

$$c = \sum \alpha \otimes \theta_\bullet^{i_1} \cdots \theta_\bullet^{i_q} \nu_{i_1,\dots,i_q \bullet}$$

where $\alpha \in S^p(\mathfrak{g}^*)$ and $\nu_{i_1,\dots,i_q \bullet} \in A^*(N\overline{G}_\bullet \times M)_{hor}$ by Corollary 4.62. Note that $C^{0,0}$ can naturally be identified with $A^*(N\overline{G}_\bullet \times M)_{bas}$ which we will occasionally refer to as a subcomplex

$$A^*(N\overline{G}_\bullet \times M)_{bas} \subset (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G.$$

The following two results establish the existence of an explicit chain map that induces an isomorphism in cohomology

$$H^*\left(\left(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M)\right)^G \right) \cong H^*(A^*(N\overline{G}_\bullet \times M)_{bas})$$

and then an explicit description of the chain map. The results follow the proof of Theorem 5.9 of [GS99] closely.

Proposition 4.64: *There is a chain map*

$$\chi : (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G \to (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$$

such that

$$\mathrm{im}(\chi) = A^*(N\overline{G}_\bullet \times M)_{bas} \subset (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$$

and it is an isomorphism on cohomology.

Proof. Firstly let $\mathfrak{g}$ have basis $\{\xi_1, \dots, \xi_n\}$ and correspondingly $S(\mathfrak{g}^*)$ has generators $\{\mu_1, \dots, \mu_n\}$. Define an operator $\partial_i : S^1(\mathfrak{g}^*) \to S^0(\mathfrak{g}^*)$ by

$$\partial_i \mu^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for every $i = 1, \dots, n$. Since ∂_i is defined on the generators of $S(\mathfrak{g}^*)$, this operator can be extended as a map $\partial_i : S^p(\mathfrak{g}^*) \to S^{p-1}(\mathfrak{g}^*)$ for every $p \geq 0$. Moreover, it is clear that for $\alpha \in S^p(\mathfrak{g}^*)$,

$$(\mu^i \partial_i)\alpha = \begin{cases} p \cdot \alpha & \text{if } p > 0 \\ 0 & \text{if } p = 0 \end{cases}$$

where Einstein summation convention is observed. Also, from Corollary 4.62, we know that any simplicial differential form $\omega \in A^*(N\overline{G}_\bullet \times M)$ can be written as a sum

$$\omega_\bullet = \sum \theta_\bullet^{i_1} \dots \theta_\bullet^{i_q} \nu_{i_1,\dots,i_q \bullet}.$$

One observes that for each summand $\beta = \theta_\bullet^{i_1} \dots \theta_\bullet^{i_q} \nu_{i_1,\dots,i_q \bullet}$ we have a similar relation

$$(\theta_\bullet^i \iota_{\xi_i})\beta = \begin{cases} q \cdot \beta & \text{if } q > 0 \\ 0 & \text{if } q = 0 \end{cases}$$

as before. The two operators $(\mu^i \partial_i)$ and $(\theta_\bullet^i \iota_{\xi_i})$ can naturally be thought of as operators on the tensor product $(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$ and thus on $C^{p,q}$ (as described in Equation 4.63) one observes that

$$(\mu^i \partial_i) + (\theta_\bullet^i \iota_{\xi_i}) = (p+q) \cdot \mathrm{id}$$

and is hence invertible for every p, q such that $p + q > 0$. We construct an operator $z : (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G \to (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$ by

$$z\alpha := \begin{cases} \alpha & \text{if } \alpha \in A^*(N\overline{G}_\bullet \times M)_{bas} \\ 0 & \text{otherwise} \end{cases} \tag{4.65}$$

for every $\alpha \in (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$. One can restate this identity in the language of the decomposition $C^{p,q}$ by saying that the restriction of z to $C^{p,q}$ is defined by

$$z = \begin{cases} \mathrm{id} & \text{if } p + q = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.66}$$

for every $p, q \geq 0$. We define z so that we can define an operator $E : C^{p,q} \to C^{p,q}$

$$E := (\mu^i \partial_i) + (\theta_\bullet^i \iota_{\xi_i}) + z = \begin{cases} (p+q) \cdot \mathrm{id} & \text{if } p + q > 0 \\ \mathrm{id} & \text{if } p + q = 0 \end{cases} \tag{4.67}$$

that is clearly invertible on all of $(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$. Alternatively, define an operator $R := (d\theta_\bullet^i)\partial_i$. By direct calculation, we can show $E - R$ is invertible. That is, since

$$E - R = (\mathrm{id} - RE^{-1})E$$

we can write

$$(E - R)^{-1} = E^{-1}(\mathrm{id} + RE^{-1} + (RE^{-1})^2 + \cdots) \tag{4.68}$$

as long as the sum

$$id + RE^{-1} + (RE^{-1})^2 + \cdots$$

terminates. This sum must terminate however, since RE^{-1} lowers the degree of $S(\mathfrak{g}^*)$ by 1 and thus for $\alpha \in C^{p,q}$

$$(RE^{-1})^{p+1}\alpha = 0.$$

Let $\chi = z(E-R)^{-1}$ and $Q = (-\theta^i_\bullet \partial_i)(E-R)^{-1}$. We will show, following the proof in [GS99, Theorem 5.9], that

$$d_G Q + Q d_G = \mathrm{id} - \chi$$

holds. I.e. that Q is a chain homotopy from id to χ and hence χ is an isomorphism on cohomology. Thinking of $A^*(N\overline{G}_\bullet \times M)_{bas}$ as a subspace of $(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$, clearly $\mathrm{im}(\chi) = A^*(N\overline{G}_\bullet \times M)_{bas}$ by the definition of z in Equation 4.66. One checks by direct calculation that

$$\begin{aligned}\left(d_G(-\theta^i_\bullet\partial_i) + (-\theta^i_\bullet\partial_i)d_G\right)\alpha &= (\mu^i\partial_i)\alpha + (\theta^i_\bullet\iota_{\xi_i})\alpha - (d\theta^i_\bullet\partial_i)\alpha \\ &= (E-R-z)\alpha \end{aligned} \tag{4.69}$$

for any $\alpha \in (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$. In particular, this means that

$$d_G(E-R) - (E-R)d_G = d_G z - z d_G. \tag{4.70}$$

From above, the operator $(E-R)$ is invertible so conjugating both sides of Equation 4.70 by its inverse yields

$$d_G(E-R)^{-1} - (E-R)^{-1}d_G = (E-R)^{-1}(d_G z - z d_G)(E-R)^{-1}. \tag{4.71}$$

Note that $d_G|_{C^{0,0}}$ is the regular de Rham differential d since $\mu^i\iota_{\xi_i}$ vanishes on $A^*(N\overline{G}_\bullet \times M)_{bas}$ and hence we can consider the operator $d_G z - z d_G$ as a map onto the basic forms of $A^*(N\overline{G}_\bullet \times M)$

$$d_G z - z d_G : C^{p,q} \to C^{0,0}.$$

Also, since R vanishes on $A^*(N\overline{G}_\bullet \times M)_{bas}$, $(E-R)|_{C^{0,0}} = \mathrm{id}$ and thus

$$(E-R)^{-1}(d_G z - z d_G) = \mathrm{id}(d_G z - z d_G).$$

Thus one simplification we can make to Equation 4.71 is that

$$d_G(E-R)^{-1} - (E-R)^{-1}d_G = (d_G z - z d_G)(E-R)^{-1}. \tag{4.72}$$

With these facts, we now expand the expression

$$\begin{aligned} d_G Q + Q d_G =& d_G(-\theta^i_\bullet\partial_i)(E-R)^{-1} + (-\theta^i_\bullet\partial_i)(E-R)^{-1}d_G \\ =& d_G(-\theta^i_\bullet\partial_i)(E-R)^{-1} \\ & + (-\theta^i_\bullet\partial_i)\left((d_G z - z d_G)(E-R)^{-1} + d_G(E-R)^{-1}\right) \end{aligned}$$

from Equation 4.72. Since $(-\theta^i_\bullet \partial_i)$ lowers the degree of $S(\mathfrak{g}^*)$,

$$(-\theta^i_\bullet \partial_i)(d_G z - z d_G) = 0$$

which yields the equation

$$\begin{aligned} d_G Q + Q d_G &= d_G(-\theta^i_\bullet \partial_i)(E-R)^{-1} + (-\theta^i_\bullet \partial_i) d_G (E-R)^{-1} \\ &= (d_G(-\theta^i_\bullet \partial_i) + (-\theta^i_\bullet \partial_i) d_G)(E-R)^{-1} \\ &= (E-R-z)(E-R)^{-1} \\ &= \mathrm{id} - \chi \end{aligned} \tag{4.73}$$

proving the assertion that χ is an isomorphism on cohomology. □

The chain map χ is studied further and we adapt the proof of [GS99, Theorem 5.2.1] to our case.

Theorem 4.74: *The map* $\chi : (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G \to (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$ *is the map*

$$\chi(\alpha \otimes \omega_\bullet) = 1 \otimes w(\theta_\bullet)(\alpha) \wedge \mathrm{Hor}(\omega_\bullet).$$

Proof. Firstly, consider the series

$$\chi = zE^{-1}(\mathrm{id} + RE^{-1} + (RE^{-1})^2 + \cdots)$$

given by the definition of χ and Equation 4.68. The map E does not change the degree of an element in $(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$ and z projects onto $A^*(N\overline{G}_\bullet \times M)_{bas}$ so it follows that

$$zE^{-1} = z$$

since E is the identity on $A^*(N\overline{G}_\bullet \times M)_{bas}$. Moreover, by the definition of R,

$$\begin{aligned} zR &= z(d\theta^i_\bullet \partial_i) \\ &= z(\mu^i_\bullet - \frac{1}{2} c^i_{jk} \theta^j_\bullet \theta^k_\bullet)\partial_i \\ &= z\mu^i_\bullet \partial_i \end{aligned} \tag{4.75}$$

since $z = 0$ on elements of $C^{p,q}$ with $q > 0$. Define the operator $K = \mu^i_\bullet \partial_i$. From Equation 4.75 we have that $zR = zK$ and hence

$$zRE^{-1} = zKE^{-1}.$$

Since E does not decrease the index of q on $C^{p,q}$, the operator $z(RE^{-1})^n$ vanishes on $C^{p,q}$ with $q > 0$ for any $n \geq 1$ and it follows that we can write

$$z(RE^{-1})^n = z(KE^{-1})^n.$$

From the definition of E in Equation 4.67, one can directly calculate that

$$\begin{aligned}(KE^{-1})^n\beta &= \frac{1}{p+q}(KE^{-1})^{n-1}K(\beta)\\ &= \frac{1}{p+q}\cdot\frac{1}{p+q-1}(KE^{-1})^{n-2}K^2(\beta)\\ &\vdots\\ &= \frac{(p+q-n)!}{(p+q)!}K^n(\beta)\end{aligned}\tag{4.76}$$

where $\beta \in C^{p,q}$. But $K^n(\beta) \in C^{p-n,q}$ and so $zK^n(\beta)$ is only non-zero when $q = 0$ and $p = n$. This fact combined with Equations 4.75 and 4.76 yield

$$\chi = z(\mathrm{id} + K + \frac{1}{2!}K^2 + \cdots) = z\exp(K).$$

If $\beta \in C^{p,0}$ then we may write $\beta = \alpha \otimes \omega_\bullet$ for some $\alpha \in S^p(\mathfrak{g}^*)$. So the problem reduces to evaluating $K^p(\alpha \otimes \omega_\bullet)$ for which we assert that

$$K^p(\alpha \otimes \omega_\bullet) = p! \cdot 1 \otimes w(\theta_\bullet)(\alpha)\omega_\bullet.$$

Since $S^p(\mathfrak{g}^*)$ is generated by $\{\mu^1, \cdots, \mu^n\}$, we can write

$$\alpha \otimes \omega_\bullet = \sum \mu^{i_1}\cdots\mu^{i_p} \otimes \omega_\bullet.$$

If $p = 1$ then it is easy to see that

$$K(\mu^{i_1} \otimes \omega) = 1 \otimes w(\theta_\bullet)(\mu^i) \wedge \omega_\bullet$$

and thus by the linearity of K, $K(\alpha \otimes \omega_\bullet) = 1 \otimes w(\theta_\bullet)(\alpha) \wedge \omega_\bullet$. Assume that

$$K^{p-1}(\alpha \otimes \omega_\bullet) = (p-1)! \cdot 1 \otimes w(\theta_\bullet)(\alpha) \wedge \omega_\bullet$$

for every $\alpha \in S^{p-1}(\mathfrak{g}^*)$. Then clearly in the case where $\alpha = \sum \mu^{i_1}\cdots\mu^{i_p}$

$$\begin{aligned}K^p(\mu^{i_1}\cdots\mu^{i_p} \otimes \omega_\bullet) &= K^{p-1}\left(\sum_{k=1}^{p}\mu^{i_1}\cdots\hat{\mu^{i_k}}\cdots\mu^{i_p} \otimes w(\theta_\bullet)(\mu^{i_k}) \wedge \omega_\bullet\right)\\ &= \sum_{k=1}^{p}(p-1)! \cdot 1 \otimes w(\theta_\bullet)(\mu^{i_1}\cdots\hat{\mu^{i_k}}\cdots\mu^{i_p})w(\theta_\bullet)(\mu^{i_k}) \wedge \omega_\bullet\\ &= p! \cdot 1 \otimes w(\theta_\bullet)(\mu^{i_1}\cdots\mu^{i_p}) \wedge \omega_\bullet\end{aligned}$$

and thus we have shown

$$\frac{1}{p!}K^p(\alpha \otimes \omega_\bullet) = \begin{cases}1 \otimes w(\theta_\bullet)(\alpha) \wedge \omega_\bullet & \text{if } \alpha \in S^p(\mathfrak{g}^*)\\ 0 & \text{otherwise}\end{cases}$$

via induction. Since z vanishes on forms with any vertical component, only the horizontal component of $\omega_\bullet$ is preserved. As a result

$$\chi(\alpha \otimes \omega_\bullet) = 1 \otimes w(\theta_\bullet)(\alpha) \wedge \mathrm{Hor}(\omega_\bullet).$$

$\square$

Corollary 4.77: *The map* $\bar{\chi} : (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G \to A^*(N\overline{G}_\bullet \times M)_{bas}$ *given by*

$$\bar{\chi}(\alpha \otimes \omega_\bullet) = w(\theta_\bullet)(\alpha) \wedge \mathrm{Hor}(\omega_\bullet)$$

induces an isomorphism in cohomology. Moreover, the inclusion map

$$i : A^*(N\overline{G}_\bullet \times M)_{bas} \to (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G$$

also induces an isomorphism in cohomology and is the chain homotopy inverse to $\bar{\chi}$*.*

Proof. The fact that $\bar{\chi} : (S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M))^G \to A^*(N\overline{G}_\bullet \times M)_{bas}$ is an isomorphism in cohomology is an immediate corollary of Theorem 4.74.

One notices that $i \circ \bar{\chi} = \bar{\chi}$ and so, looking back at Equation 4.73,

$$d_G Q + Q d_G = \mathrm{id} - i \circ \chi.$$

One can consider $\bar{\chi} \circ i = \mathrm{id}$ on $A^*(N\overline{G}_\bullet \times M)_{bas}$ and so clearly there is a respective chain homotopy between these two maps. Thus we have established that i and $\bar{\chi}$ are chain homotopy inverses. $\square$

Theorem 4.78: *Let G be a compact Lie group and M be a manifold with right G-action and let $\alpha \otimes \omega \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G$. There is a chain map*

$$\Phi : (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \to A^*(N\overline{G}_\bullet \times M)_{bas}$$

given by

$$\Phi(\alpha \otimes \omega) = w(\theta_\bullet)(\alpha) \wedge \mathrm{Hor}(\omega_\bullet)$$

that induces an isomorphism on cohomology.

Proof. From Theorem 4.55 and Proposition 4.59 and Proposition 4.60 the map

$$1 \otimes (E_\Delta \circ p_M^* \circ i) : (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \to \big(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M)\big)^G$$

induces an isomorphism in cohomology. If we let

$$\omega_\bullet = (E_\Delta \circ p_M^* \circ i)(\omega)$$

then from Theorem 4.74 the map $\Phi := \bar{\chi} \circ 1 \otimes (E_\Delta \circ p_M^* \circ i)$ which can be explicitly written as

$$\Phi(\alpha \otimes \omega) = w(\theta_\bullet)(\alpha) \wedge \mathrm{Hor}(\omega_\bullet)$$

is an isomorphism on cohomology. $\square$

This theorem derives the classical result of Cartan – that the Cartan model calculates the equivariant cohomology of a manifold M acted upon by a compact Lie group G.

Corollary 4.79 (Cartan's Theorem)**:** *Let G be a compact Lie group and M be a manifold with right G-action. The Cartan model*

$$(S(\mathfrak{g}^*) \otimes \Omega^*(M))^G$$

computes the equivariant cohomology of M.

Proof. This follows as a result from Theorem 4.78, Theorem 4.14 and Corollary 3.62. □

4.4.3 The Weil Model via Simplicial Differential Forms

One may also derive results about the relation between the ordinary Weil model and Dupont's simplicial model for equivariant cohomology.

Proposition 4.80: *Let G be a compact Lie group and M be a manifold with right G-action. The chain map*

$$\psi : \big(W(\mathfrak{g}) \otimes A^*(N\overline{G}_\bullet \times M)\big)_{bas} \to A^*(N\overline{G}_\bullet \times M)_{bas}$$

given by

$$\psi(\alpha \otimes \omega_\bullet) = w(\theta_\bullet)(\alpha) \wedge \omega_\bullet$$

induces an isomorphism on cohomology.

Proof. Consider the sequence of chain maps below.

$$\big(W(\mathfrak{g}) \otimes A^*(N\overline{G}_\bullet \times M)\big)_{bas} \xrightarrow{\exp(\zeta)} \big(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M)\big)^G \underset{i}{\overset{\bar{\chi}}{\rightleftarrows}} A^*(N\overline{G}_\bullet \times M)_{bas}$$

The map $\exp(\zeta)$ is an isomorphism and hence induces an isomorphism in cohomology. That is

$$\bar{\chi} \circ \exp(\zeta) : \big(W(\mathfrak{g}) \otimes A^*(N\overline{G}_\bullet \times M)\big)_{bas} \to A^*(N\overline{G}_\bullet \times M)_{bas}$$

is an isomorphism in cohomology. As well, we have the map

$$\psi(\alpha \otimes \omega_\bullet) = w(\theta_\bullet)(\alpha) \wedge \omega_\bullet$$

which satisfies $\bar{\chi} \circ \exp(\zeta) \circ \psi = \psi$. Recall from Corollary 4.77 that the inclusion

$$i : A^*(N\overline{G}_\bullet \times M)_{bas} \to \big(S(\mathfrak{g}^*) \otimes A^*(N\overline{G}_\bullet \times M)\big)^G$$

is an isomorphism in cohomology and hence

$$\exp(-\zeta) \circ i : A^*(N\overline{G}_\bullet \times M)_{bas} \to \big(W(\mathfrak{g}) \otimes A^*(N\overline{G}_\bullet \times M)\big)_{bas}$$

is an isomorphism in cohomology. But for any $\omega_\bullet \in A^*(N\overline{G}_\bullet \times M)_{bas}$

$$(\zeta \circ i)\omega_\bullet = (\theta^i \otimes \iota_{\xi_i})(1 \otimes \omega_\bullet) = 0$$

since $\omega_\bullet$ is basic. So the map

$$\exp(-\zeta) \circ i : A^*(N\overline{G}_\bullet \times M)_{bas} \to \big(W(\mathfrak{g}) \otimes A^*(N\overline{G}_\bullet \times M)\big)_{bas}$$

is a similar inclusion

$$(\exp(-\zeta) \circ i)\, \omega_\bullet = 1 \otimes \omega_\bullet$$

and also an isomorphism on cohomology. One notes that

$$\psi \circ (\exp(-\zeta) \circ i) = \mathrm{id}$$

and so ψ induces an isomorphism on cohomology. □

Lemma 4.81: *Consider the map*

$$1 \otimes (E_\Delta \circ p_M^* \circ i) : (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \to \big(S(\mathfrak{g}^*) \otimes A^*(N\overline{G} \times M)\big)^G$$

from Theorem 4.78. This map commutes with the Mathai-Quillen operator ζ from Equation 4.52.

Proof. This result naturally follows from Proposition 4.59 which establishes that the map $E_\Delta \circ p_M^* \circ i$ is a G^* morphism. □

Theorem 4.82: *Let G be a compact Lie group and M be a manifold with right G-action and let $\alpha \otimes \omega \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}$. There is a chain map*

$$\Psi : (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \to A^*(N\overline{G}_\bullet \times M)_{bas}$$

given by

$$\Psi(\alpha \otimes \omega) = w(\theta_\bullet)(\alpha) \wedge \omega_\bullet$$

that induces an isomorphism on cohomology.

Proof. Firstly we note that since $1 \otimes (E_\Delta \circ p_M^* \circ i)$ commutes with ζ, the diagram

$$\begin{tikzcd}
(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \arrow[r, "\exp(\zeta)"] \arrow[d, "1 \otimes (E_\Delta \circ p_M^* \circ i)"'] & (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \arrow[d, "1 \otimes (E_\Delta \circ p_M^* \circ i)"] \\
\big(W(\mathfrak{g}) \otimes A^*(N\overline{G} \times M)\big)_{bas} \arrow[r, "\exp(\zeta)"] & \big(S(\mathfrak{g}^*) \otimes A^*(N\overline{G} \times M)\big)^G
\end{tikzcd}$$

is commutative. Accordingly,

$$1 \otimes (E_\Delta \circ p_M^* \circ i) : (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \to W(\mathfrak{g}) \otimes A^*(N\overline{G} \times M)_{bas}$$

is an isomorphism on cohomology from Proposition 4.59. Thus the composition

$$\Psi := \psi \circ 1 \otimes (E_\Delta \circ p_M^* \circ i) : (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \to A^*(N\overline{G} \times M)_{bas}$$

is an isomorphism in cohomology from Proposition 4.80. □

Corollary 4.83: *Let G be a compact Lie group and M be a manifold with right G-action. The Weil model*

$$(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}$$

computes the equivariant cohomology of M.

Proof. This follows as a result from Theorem 4.82, Theorem 4.14 and Corollary 3.62. □

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