ON A SOLUTION OF THE U(N) ⊃ O(N) STATE LABELLING PROBLEM, FOR TWO-ROWED REPRESENTATIONS

by

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A thesis
submitted for the degree of Master of Science

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March, 1974

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ABSTRACT

Considerations of the number of labelling operators in a general state labelling problem imply that for two-rowed irreducible representations of U(N), labelled $[p,q,o,..,o]$, just one additional commuting labelling operator, Λ, is required for the solution of the $U(N) \supset O(N)$ state labelling problem. A detailed investigation of the $U(N) \supset O(N)$ reduction via tensor representations leads to a proposal of an integral label λ, implying a non-orthogonal labelling scheme. A simple $U(N) \supset O(N)$ branching theorem for two-rowed representations is formulated in terms of λ.

The additional labelling operator Λ, with eigenvalue λ, is to be defined implicitly by an equation of the form $f(\Lambda, \Phi, \cdots) = 0$, where f is a polynomial in Λ, the additional O(N) invariants Φ in the representation subduced by U(N) $[p,q,o,..,o]$, and other known labels and invariants. The O(N) invariants, which are cubic and quartic in the generators of U(N), are discussed. Techniques developed by Green and Bracken[1] for the $U(3) \supset SO(3)$ problem are used to evaluate the single independent cubic invariant, and it is shown how this is used in obtaining the desired implicit operational definition of Λ.

A cubic characteristic polynomial identity, satisfied by the generators of U(N) in two-rowed representations, is also found using these techniques.

Some possible physical applications of the $U(N) \supset O(N)$ state labelling problem are briefly discussed.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university. To the best of the author's knowledge and belief, the thesis contains no material previously published or written by another person, except when due reference is made in the text.

London
April 1974

The author gratefully acknowledges the assistance of Professor H.S. Green, Department of Mathematical Physics, University of Adelaide, under whose guidance the work here presented was carried out.

1. INTRODUCTION

Many problems in different fields of physics can be viewed in a more abstract way as particular cases of certain general problems in group representation theory. Of this kind is the state labelling problem, in which it is required to find commuting labelling operators whose common eigenstates specify a basis for an irreducible representation of a group G, in such a way as to exhibit its irreducible contents as a representation of one of its subgroups, G^o. Such state labelling schemes must take account of the circumstance that a particular irreducible representation of G^o may occur multiply within a given irreducible representation of G; the labels to be defined must distinguish between such equivalent representations, and remove the degeneracy. This is the case for the state labelling problem to be studied, that of U(N), the group of NxN unitary matrices, and its subgroup O(N), the group of NxN orthogonal matrices.

In Sec. 1.1 is given a general discussion of the irreducible representations of U(N) and O(N), and of the counting of state labelling operators. In Sec. 1.2 some physical applications of the $U(N) \supset O(N)$ state labelling problem, are discussed, together with solutions which have been proposed in previous work.

1.1 Irreducible Representations and Abstract Bases

Irreducible representations of semisimple Lie groups are characterised by their highest weights, the components of which coincide with the maximum eigenvalues taken by certain of the group generators in the representation[2]. In the case of U(N), the irreducible representations are labelled $[p_1, \ldots, p_N]$, where the p_i are nonnegative integers satisfying[3]

$$p_1 \geq p_2 \geq \cdots p_{N-1} \geq p_N \geq 0. \tag{1}$$

In the case of O(N), the irreducible representations are labelled $(\ell_1, \ldots, \ell_{[N/2]})$, the square brackets denoting integer part, where the ℓ_i are nonnegative, and either all integers or all half-integers, satisfying[4,5,6]

$$\ell_1 \geq \ell_2 \geq \cdots \geq \ell_{[N/2]} \geq 0. \tag{2}$$

The corresponding labelling operators having these eigenvalues in the irreducible representation, are denoted $P_1, \ldots, P_N$, and $L_1, \ldots, L_{[N/2]}$, for U(N) and O(N), respectively. These labelling operators are given in terms of a complete set of invariants, which are known polynomials in the group generators. For example, in the case of O(3), the total angular momentum operator L is defined in the familiar way through

$$L_x^2 + L_y^2 + L_z^2 = \underline{L}^2 = L(L+1). \tag{3}$$

Sec. 1.2 shows, for the particular case of U(N) and O(N), the physical significance of the general $G \supset G^o$ state labelling problem of finding a complete set of commuting operators, whose common eigenstates provide a basis for the carrier space of a representation of G, such that the subduced representation of G^o is in completely reduced form. The number of labelling operators required for a general abstract basis of G may be determined as follows[7]. Let the group manifold of G have dimension n, so that elements of the group can be specified locally by coordinates $(\xi^1, \xi^2, \cdots, \xi^n)$. The matrix elements $\langle\phi'|U\phi\rangle = \langle\phi'|U(\xi^1, \cdots, \xi^n)\phi\rangle$, of group transformations with respect to abstract basis vectors ϕ, ϕ' in a representation U of G, provide a system of independent functions of n variables (which are orthogonal in a unitary representation), which therefore require n labels for their specification. Now the labelling operators specifying the irreducible representation will provide m such labels, where m is the rank of the group (the dimension of the Cartan sub-algebra). The remaining (n-m) labels will distinguish functions within the same irreducible representation, with $\frac{1}{2}(n-m)$ labels associated with the rows, and $\frac{1}{2}(n-m)$ with the columns, of the matrix elements. The abstract basis vectors will therefore require $m+\frac{1}{2}(n-m) = \frac{1}{2}(n+m)$ labels.

For U(N) and O(N), such abstract bases, with the requisite number of labelling operators, are the Gel'fand bases. For the case of U(N)[3], the labels are those specifying the irreducible representation of U(N), and of its subgroups (U(n), $1 \leq n \leq N$, into which the given representation decomposes upon restriction, according to the chain

$$U(N) \supset U(N-1) \supset \cdots \supset U(n) \supset U(n-1) \supset \cdots \supset U(2) \supset U(1). \qquad (4)$$

The labelling operators commute, and are Hermitian. The corresponding system of common eigenstates (the abstract basis) is therefore orthogonal. The labelling scheme is nondegenerate, for a particular irreducible representation of U(n-1) occurs at most once within a given irreducible representation of U(n). It is complete, for irreducible representations of U(1) are one-dimensional. The abstract basis vectors are denoted

$$\left| \begin{array}{cccccc} p_1^N & p_1^{N-1} & \cdots & & & \\ p_2^N & p_2^{N-1} & \cdots & p_1^3 & & \\ \vdots & \vdots & \cdots & p_2^3 & p_1^2 & p_1^1 \\ \vdots & \vdots & \cdots & p_3^3 & p_2^2 & \\ \vdots & \vdots & \cdots & & & \\ p_N^N & p_{N-1}^{N-1} & \cdots & & & \end{array} \right\rangle , \qquad (5)$$

where column n specifies the irreducible representation $[p_1^n, \ldots, p_n^n]$ of U(n), and where all values of the labels, satisfying the inequalities

$$p_1^n \geq p_1^{n-1} \geq p_2^n \geq \cdots \geq p_{n-1}^n \geq p_{n-1}^{n-1} \geq p_n^n , \qquad (6)$$

occur in the basis.

For the case of O(N)[4,5], the Gel'fand basis is constructed according to the chain

$$O(N) \supset O(N-1) \supset \cdots \supset O(n) \supset \cdots \supset O(3) \supset SO(2). \qquad (7)$$

and has similar properties to the above U(N) basis. The abstract basis vectors are denoted

$$\left| \begin{matrix} \ell_1^N & \ell_1^{N-1} & \cdot & \cdot & & \\ \ell_2^N & \ell_2^{N-1} & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \ell_1^5 \ \ell_1^4 & \\ \cdot & \cdot & \cdot & \cdot & & \ell_1^3 \ \ell_1^2 \\ \cdot & \cdot & \cdot & \cdot & \ell_2^5 \ \ell_2^4 & \\ \ell_{[N/2]}^N & \ell_{[N-1/2]}^{N-1} & \cdot & \cdot & & \end{matrix} \right\rangle , \qquad (8)$$

where column n specifies the irreducible representation $(\ell_1^n, \ldots, \ell_{[n/2]}^n)$ of O(N), where the labels are either all integers or all half-integers, and where all values of the labels, satisfying the inequalities

$$\begin{aligned} \ell_1^{2n+1} \geq \ell_1^{2n} \geq \ell_2^{2n+1} \geq \cdots \geq \ell_n^{2n+1} \geq \ell_n^{2n} \geq 0, \quad 2 \leq n \leq [N/2], \\ \ell_1^{2n} \geq \ell_1^{2n-1} \geq \ell_2^{2n} \geq \cdots \geq \ell_{n-1}^{2n} \geq \ell_{n-1}^{2n-1} \geq \ell_n^{2n}, \quad 2 \leq n \leq [N/2], \\ \ell_1^3 \geq |\ell_1^2| \geq 0, \end{aligned} \qquad (9)$$

occur in the basis.

The numbers of labelling operators for the Gel'fand bases of U(N) and O(N) agree with those prescribed by the counting procedure outlined above. For U(N) and O(N) the dimension, rank and number of labelling operators are

	n	m	$\frac{1}{2}(n+m)$
U(N)	N^2	N	$\frac{1}{2}N(N+1)$
O(N)	$\frac{1}{2}N(N-1)$	[N/2]	[N/2][N+1/2]

(10)

The abstract basis in the U(N) ⊃ O(N) state labelling problem is associated with the non-canonical chain

$$U(N) \supset O(N) \supset O(N-1) \supset \cdots O(3) \supset SO(2) \qquad (11)$$

However, the first link of the chain, the U(N) ⊃ O(N) reduction, is degenerate (examples are given in Secs. 2.4 and 2.5). Hence additional labelling operators, denoted $\Lambda_1, \ldots, \Lambda_f$, which commute with the other labelling operators of this chain, and amongst themselves, must be introduced, to distinguish between equivalent irreducible representations of O(N). The abstract basis vectors will be denoted

$$\left| \begin{array}{cccccc} p_1 & & \ell_1^N & & & \\ \cdot & \lambda_1 & \cdot & \cdot & \cdot & \\ \cdot & \lambda_2 & \cdot & \cdot & \cdot & \ell_1^5 \ell_1^4 \\ \cdot & \vdots & \cdot & \cdot & \cdot & \ell_1^3 \ell_1^2 \\ \cdot & \vdots & \cdot & \cdot & \cdot & \ell_2^5 \ell_2^4 \\ \cdot & \lambda_f & \cdot & \cdot & \cdot & \\ p_N & & \ell_{[N/2]}^N & & & \end{array} \right\rangle , \qquad (12)$$

where $\lambda_1, \lambda_2, \cdots, \lambda_f$ are the eigenvalues of $\Lambda_1, \Lambda_2, \cdots, \Lambda_f$. The number f of additional lables required in the general $G \supset G^0$ state labelling problem can be determined from the requirement

$$\tfrac{1}{2}(n+m) = m + f + \tfrac{1}{2}(n^0 + m^0), \qquad (13)$$

$$f = \tfrac{1}{2}(n - m - n^0 - m^0), \qquad (14)$$

where n^0, m^0 refer to G^0. For the U(N) ⊃ O(N) state labelling problem, from Eq. (10),

$$f = \tfrac{1}{4}(N^2 - N - 2[N/2]), \tag{15}$$

the first few values of which are

$$\begin{array}{llllllllllll} N = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & N & N+1 \\ f = & 0 & 0 & 0 & 1 & 2 & 4 & 6 & 9 & \cdots & f_N & f_N + [N/2] \end{array} . \tag{16}$$

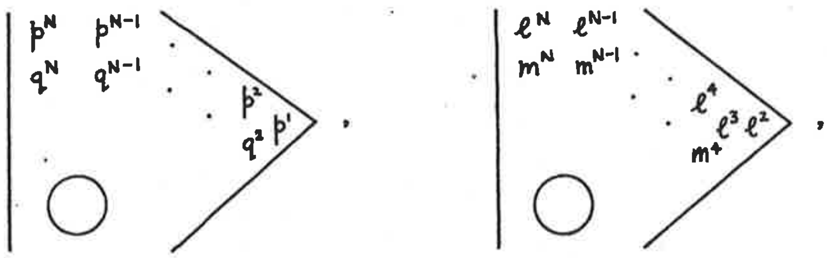
Of primary interest in the following will be a class of irreducible representations of U(N) called here "two-rowed", that is, of the form

$$[p_1, p_2, 0, \cdots, 0] \equiv [p_1, p_2, 0^{N-2}] \equiv [p, q] . \tag{17}$$

The subduced irreducible representations of O(N) will then also be two-rowed, of the form

$$(\ell_1, \ell_2, 0, \cdots, 0) \equiv (\ell_1, \ell_2, 0^{[N/2]-2}) \equiv (\ell, m). \tag{18}$$

The corresponding Gel'fand patterns,



(19)

contain, respectively, 2N-1 and 2N-4 nonvanishing labels. The number $f^{(2)}$ of additional nonvanishing labelling operators, required in the case of two-rowed representations, is therefore given by

$$2N - 1 = 2 + f^{(2)} + (2N - 4), \tag{20}$$

that is, only a single nonvanishing labelling operator Λ is required.

In this respect, the $U(N) \supset O(N)$ state labelling problem for two-rowed representations is similar to the ancient one of $U(3) \supset O(3)$ where, again, only a single additional labelling operator is required; however a considerable generalisation is obtained by allowing arbitrary N.

Green[2] has shown that the matrix of generators of U(N) satisfies a certain characteristic polynomial identity of degree N in an irreducible representation. The restriction to two-rowed representations effectively reduces the rank of the algebra. Correspondingly, it is shown in Sec. 4.2 that the matrix of generators of U(N), in two-rowed representations, satisfies a third degree characteristic polynomial identity, which proves to be simply a factor of the general polynomial identity.

By the same reasoning as led to Eq. (20), if only r-rowed irreducible U(N) representations are considered, then at most

$$f^{(r)} = \tfrac{1}{2}r(r-1), \qquad f^{(r+1)} = f^{(r)} + r, \tag{21}$$

additional nonvanishing commuting labelling operators are required. Also, there should exist a characteristic polynomial identity of reduced degree (r+1) for the matrix of U(N) generators.

The abstract basis vectors for two-rowed irreducible representations U(N) p,q will be denoted

$$\left| \begin{matrix} p & & \ell & \ell^{N-1} & & \\ q & & m & m^{N-1} & \vdots & \\ 0 & & 0 & 0 & \vdots & \ell^4 \\ 0 & \lambda & 0 & 0 & & m^4 \, \ell^3 \, \ell^2 \\ \vdots & & \vdots & \vdots & \vdots & \\ 0 & & 0 & 0 & & \end{matrix} \right\rangle \equiv \left| \begin{matrix} p \\ q \end{matrix} \; \lambda \begin{pmatrix} \ell \\ m \end{pmatrix} \right\rangle , \qquad (22)$$

where λ is the eigenvalue of Λ. The irreducible representations of O(N), occurring in the reduction of U(N) [p,q] $\downarrow$ O(N), will be denoted (p,q; λ ; ℓ,m) or (λ ; ℓ,m).

A suitable choice of the label λ is proposed in Sec. 2.3 from an analysis of the tensor representations. A simple U(N) $\supset$ O(N) branching theorem in this scheme is also derived. λ proves to be integral in the range $0 \leq 2\lambda \leq p$, while the bounds of (ℓ+m) and m are simply related to p,q, and λ.

The additional label λ here proposed has a natural meaning in irreducible tensor representations of U(N). However, in order to ensure a completely representation-independent formulation, and for physical applications, it is necessary to give an abstract definition of the corresponding labelling operator Λ. Although this cannot be done explicitly in terms of the U(N) generators, it is possible to provide an implicit operational definition, of the form

$$f(\Lambda, \Phi, \cdots) = 0 , \qquad (23)$$

where f (Sec. 4.5) is a polynomial in Λ, and other known labelling operators and invariants (compare Eq. (3)).

Chs. 3 and 4 are devoted to the derivation of such an equation. The question of the uniqueness of such a definition is not pursued in the following.

A theorem proved by Racah(7) in discussing the $U(3) \supset O(3)$ state labelling problem, gives important information about the operator Λ, if it is assumed to be valid in general. The result is that, if the state labelling scheme is orthogonal, then the diagonal matrix elements

$$\langle \lambda | x \lambda' \rangle , \quad \langle \lambda | y \lambda' \rangle , \qquad \lambda = \lambda' \tag{24}$$

in this basis of certain independent O(3) invariants x , y , cannot be polynomials in λ . However, this conclusion is contradicted by the evaluation in Sec. 4.4 of what is essentially the cubic invariant x (apart from a term independent of λ). Hence, the state labelling scheme here proposed is non-orthogonal, and the operator Λ is non-Hermitian. Sec. 4.4 shows in addition that the selection rule

$$\Delta\lambda = 0, \pm 1 \tag{25}$$

investigated by Racah(7) as the simplest non-orthogonal case, is satisfied in the scheme proposed.

The $U(N) \supset O(N)$ state labelling problem sometimes appears in applications instead as $U(N) \supset SO(N)$, or $SU(N) \supset SO(N)$, where SU(N) and SO(N) are the groups of NxN unimodular unitary and orthogonal matrices, respectively. With appropriate modifications, a solution of the $U(N) \supset O(N)$

state labelling problem will also provide solutions of these other problems.

Irreducible representations of SU(N) are labelled $[p_1,\ldots,p_{N-1}]$, where the p_i are nonnegative integers satisfying

$$p_1 \geqslant p_2 \geqslant \cdots \geqslant p_{N-1} \geqslant 0. \tag{26}$$

The irreducible representation U(N) $[p_1, p_N, \ldots, p_{N-1}, p_N]$ reduces as [8]:

$$U(N)[p_1,\cdots,p_{N-1},p_N] \downarrow SU(N) = SU(N)[p_1-p_N,\cdots,p_{N-1}-p_N]. \tag{27}$$

Irreducible representations of SO(N) are labelled $(\ell_1, \ell_2, \cdots, \ell_{[N/2]})$, where the ℓ_i are either all integers or all half-integers, satisfying[9]

$$\begin{aligned} &\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{[N/2]} \geqslant 0 \quad (N \text{ odd}), \\ &\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant |\ell_{[N/2]}| \geqslant 0 \quad (N \text{ even}). \end{aligned} \tag{28}$$

The irreducible representation $O(N)(\ell_1, \ell_2, \cdots, \ell_{[N/2]})$ reduces as[3,5]

$$\begin{aligned} O(N)(\ell_1,\cdots,\ell_{[N/2]}) \downarrow SO(N) &= SO(N)(\ell_1,\cdots,\ell_{[N/2]}).\ (N \text{ odd}) \\ O(N)(\ell_1,\cdots,0) \downarrow SO(N) &= SO(N)(\ell_1,\cdots,0).\ (N \text{ even}, \ell_{[N/2]}=0) \\ O(N)(\ell_1,\cdots,\ell_{[N/2]}) \downarrow SO(N) &= SO(N)(\ell_1,\cdots,\ell_{[N/2]}) \quad (N \text{ even}, \\ &\oplus SO(N)(\ell_1,\cdots,-\ell_{[N/2]}). \quad \ell_{[N/2]} \neq 0) \end{aligned} \tag{29}$$

The abstract bases for the U(N) ⊃ SO(N) and SU(N) ⊃ SO(N) state labelling problems are associated with the chains

$$U(N) \supset SO(N) \supset SO(N-1) \supset \cdots \supset SO(3) \supset SO(2) \,, \tag{30}$$

$$SU(N) \supset SO(N) \supset SO(N-1) \supset \cdots \supset SO(3) \supset SO(2) \,, \tag{31}$$

respectively. The abstract basis vectors are given by Eq. (12), where now the labels of column n specify an irreducible representation of SO(n), as in Eq. (9), and satisfy the inequalities

$$\begin{aligned} &\ell_1^{2n+1} \geq \ell_1^{2n} \geq \ell_2^{2n+1} \geq \cdots \geq \ell_n^{2n+1} \geq |\ell_n^{2n}| \geq 0 \,, \\ &\ell_1^{2n} \geq \ell_1^{2n-1} \geq \ell_2^{2n} \geq \cdots \geq \ell_{n-1}^{2n} \geq \ell_{n-1}^{2n-1} \geq \ell_n^{2n} \,. \end{aligned} \tag{32}$$

In the case of $U(N) \supset SO(N)$, the first link in the subgroup chain, Eq. (30), can be expanded as

$$U(N) \supset O(N) \supset SO(N) \,. \tag{33}$$

Thus, in view of Eqs. (29), the additional labelling operators Λ for the $U(N) \supset O(N)$ state labelling problem, will also remove the SO(N) degeneracy in the case of $U(N) \supset SO(N)$.

In the case of $SU(N) \supset SO(N)$, the irreducible representation $SU(N) \, [p_1, \ldots, p_{N-1}, 0]$, with the same SO(N) contents. Hence the problem reduces to one of $U(N) \supset SO(N)$.

Finally, from these observations, and the fact that the diagram

$$\begin{array}{ccc} U(N) & \supset & O(N) \\ \cup & \searrow & \cup \\ SU(N) & \supset & SO(N) \end{array} \tag{34}$$

commutes, in the sense of subduced representations, it should be remarked that the $U(N) \supset O(N)$ state labelling scheme, here proposed for two-rowed representations, can be considered as an $U(N) \supset SO(N)$ state labelling scheme for the special class of irreducible representations of $U(N)$ having N rows, of the form $[p, q, s, s, \ldots, s]$, since

$$U(N)[p,q,s,\cdots,s]\downarrow SO(N) = U(N)[p-s,q-s,0,\cdots,0]\downarrow SO(N). \quad (35)$$

In particular, the complete $U(3) \supset SO(3)$ state labelling problem can be treated.

1.2 Physical Applications

Perhaps the most studied orthogonal groups are SO(3), the group of rotations of three-dimensional Euclidean space, and O(3), which includes the reflections. The first applications of the $U(N) \supset O(N)$ state labelling problem concerned this case.

Elliott[10] pursued the question of the appearance of rotational band structures in certain nuclear shell model calculations (involving oscillator Hamiltonians), which agreed with rotational model predictions and certain observed nuclear spectra. This indicated a classification of states should be sought using as a label the total angular momentum, ℓ, of states arising as mixtures from two or more degenerate levels, with angular momenta ℓ_α, ℓ_β, ... In particular, for simple harmonic oscillator levels, the degeneracy being associated with the U(3) symmetry of the harmonic oscillator Hamiltonian, the U(3) ⊃ SO(3) state labelling problem arose. A whole number label, essentially the λ of Sec. 2.3, was proposed, but this did not appear as the eigenvalue of any operator.

Bargmann and Moshinsky[11] later considered the N-particle simple harmonic oscillator with a quadrupole-quadrupole interaction. In order to specify a complete set of constants of the motion an invariant had to be found commuting with the U(3) and SO(3) invariants, thus solving the state labelling problem. A Hermitian operator, with nondegenerate eigenvalues, was written down explicitly in terms of the generators; however the eigenvalues were in general irrational, and could only be found by solving a system of linear equations (agreeing with the theorem of Racah concerning orthogonal labelling schemes, Sec. 1.1).

Moshinsky and Syamala Devi[12] considered the U(3) ⊃ SO(3) state labelling problem in the context of fractional parentage coefficients, in building states of definite orbital angular momentum and symmetry of N identical, noninteracting particles, from those of N-1 particles. Again an integral label was introduced, which did not appear as an eigenvalue of any operator. It was however related to certain polynomials in operators creating the basic states in the representation from a vacuum state. By this means the transformation coefficients between the canonical U(3) basis and the non-orthogonal basis could be calculated. More recently Louck and Galbraith[3] have used an U(N) ⊃ SO(N) embedding as a means of constructing such N-particle states (realised as homogeneous polynomial solutions of Laplace's equation in N dimensions).

Hughes[13] has given an algorithmic solution of the U(3) ⊃ SO(3) state labelling problem, in an orthogonal basis.

The general state labelling problem for the case N=4 is of importance in relativistic applications, in the form U(3,1) ⊃ O(3,1), where O(3,1) is the group of Lorentz transformations of Minkowski space. (In the analysis of finite dimensional representations, which will be studied in the following, it is necessary only that the metric tensor be nonsingular; no particular signature need be specified.) Thus suppose a physical description in terms of quantities of the form (compare Green and Bracken[1]):

$$t_{\lambda\mu\nu\cdots} = t^a_\lambda t^b_\mu t^c_\nu \cdots S(\cdots) \tag{36}$$

The four-vectors t_μ could be particle space-time coordinates, momentum operators, γ matrices, and so on; S is a given scalar function of the t_μ. The tensors $t_{\lambda\mu\nu\cdots}$ may be regarded as belonging to a tensor representation of U(3,1), whether or not this is present as a symmetry group. The analysis of the $t_{\lambda\mu\nu\cdots}$ into irreducible representations of O(3,1) may therefore be made by firstly projecting out the irreducible U(3,1) representations, and then using the solution of the U(3,1) ⊃ O(3,1) state labelling problem to project further on to the irreducible O(3,1) constituents. For example, if Λ is an O(3,1) labelling operator, the projection on to the subspace labelled by its eigenvalue λ° is

$$\prod_{\lambda \neq \lambda^{\circ}} \frac{(\Lambda - \lambda)}{(\lambda^{\circ} - \lambda)} \tag{37}$$

where λ runs over all possible values occurring within the particular irreducible U(3,1) representation.

Unitary symmetry schemes in models of elementary particles have been much in vogue in recent years. In the fundamental U(3) classification[14], the isospin is associated with the label of the irreducible SU(2) constituents contained within an irreducible representation of U(3), while the hypercharge distinguishes equivalent SU(2) constituents. However, there is an alternative scheme[1,15] in which isospin and hypercharge are related to the labels of the irreducible SO(3) constituents contained within an irreducible representation of U(3), with two values of the hypercharge for each isospin submultiplet[1]. Just such a scheme is required in an order 3 generalised parafermi statistics quark model[15]; an extension exists to order N parastatistics.

There is also the possibility that the U(3,1) $\supset$ O(3,1) state labelling problem may arise in a relativistic elementary particle model, in which the Poincare or Lorentz group is embedded nontrivially in a larger group (whether or not a strict symmetry).

Finally, it should be remarked that certain state labelling problems in the classical groups are interrelated, so that a solution of one provides indirectly solutions of related ones. The connections between various embeddings and irreducible representations have been discussed for example by Quesne[16].

2. THE U(N) ⊃ O(N) REDUCTION VIA TENSOR REPRESENTATIONS.

Irreducible representations of the compact group U(N) are finite dimensional (and may be chosen to be unitary), and may be realised on a space of finite-rank tensors over an N-dimensional vector space. The aim of this chapter is to use the U(N) ⊃ O(N) reduction in this concrete framework, to provide some guidance towards the solution of the abstract U(N) ⊃ O(N) state labelling problem.

Secs. 2.1 and 2.2 give a review of some standard results concerning the irreducible tensor representations of U(N) and O(N) (Hamermesh[8]). In Sec. 2.3, considerations of the decomposition of irreducible tensors into their traceless parts, give rise to a simple U(N) ⊃ O(N) branching theorem for the case of two-rowed representations, involving an additional label, λ , with a natural interpretation. In Sec. 2.4, some results of Littlewood[17] are stated, concerning the characters of U(N) and O(N). These are used to provide an alternative method of carrying out the U(N) ⊃ O(N) reduction. The methods are compared in Sec. 2.5 for a particular example. Finally, Sec. 2.6

gives the explicit reduction to irreducible O(N) constituents, for some simple irreducible U(N) tensors, according to the prescription of Sec. 2.3.

2.1 Irreducible Tensor Representations

A tensor t of rank f is an f-linear mapping of f-tuples of vectors belonging to some space (taken to have finite dimension N), to scalars (complex numbers). Consequently t can be specified, with respect to some chosen basis, by N^f components

$$t^{x_1 \cdots x_f}, \qquad 1 \leq x_1, \cdots, x_f \leq N \tag{1}$$

If a different basis is chosen, related to the original by some nonsingular matrix Φ, then in the new basis t must have components (summation over repeated indices)

$$t'^{x_1 \cdots x_f} = \Phi^{x_1}{}_{x_1'} \cdots \Phi^{x_f}{}_{x_f'} \, t^{x_1' \cdots x_f'} . \tag{2}$$

If only basis transformations corresponding to some matrix subgroup G of GL(N) are considered, t is said to "transform under G".

The rank f tensors themselves form a vector space, denoted T^f. If $\gamma \in G$, the linear transformation

$$U_\gamma : T^f \to T^f, \quad (U_\gamma t)^{x_1 \cdots x_f} = \gamma^{x_1}{}_{x_1'} \cdots \gamma^{x_f}{}_{x_f'} \, t^{x_1' \cdots x_f'} \tag{3}$$

is well-defined, independently of the basis chosen; the mapping $\gamma \mapsto U_\gamma$ is a representation of G on T^f. Also if $\pi \in S_f$, the symmetric group on f symbols, the linear U_π,

$$U_\pi : T^f \to T^f, \quad (U_\pi t)^{x_1 \cdots x_f} = t^{x_{\pi 1} \cdots x_{\pi f}}, \tag{4}$$

is well-defined, independently of the basis chosen, for if Φ is any nonsingular basis transformation,

$$\Phi U_\pi = U_\pi \Phi \quad \forall \pi \in S_f . \tag{5}$$

The mapping $\pi \mapsto U_\pi$ is a representation of S_f on T^f.

The fact that U_π commutes with nonsingular basis transformations has the consequence that

$$U_\pi U_\gamma = U_\gamma U_\pi \quad \forall \pi \in S_f , \gamma \in G \tag{6}$$

and this means that if T^f_* is a subspace of T^f invariant under S_f, it must also be invariant under G. Now the irreducible invariant subspaces of T^f under S_f are those of the form YT^f, where Y, a linear combination of the U_π, is a Young operator, with symmetry corresponding to some partition of f. For G = GL(N), GL(N,R), SL(N), SL(N,R), U(N) and SU(N), the YT^f are also irreducible invariant subspaces under G[8].

A further reduction is possible for certain groups G corresponding to basis transformations leaving invariant a quadratic form g. If g is symmetric, the subgroup is O(N) (the orthogonal group, or a pseudo-orthogonal group). If g is antisymmetric, the subgroup is Sp(N) (N must be even if g is to be nonsingular). The fundamental property of the group transformations is that

$$g_{x_1' x_2'} \gamma^{x_1'}{}_{x_1} \gamma^{x_2'}{}_{x_2} = g_{x_1 x_2} , \tag{7}$$

and from this it follows that a tensor of rank $f-2$ may be obtained by contraction of, say, indices 1 and 2 of a rank f tensor t, with the metric tensor g :

$$t_{(12)}^{x_3 \cdots x_f} = g_{x_1 x_2} t^{x_1 x_2 x_3 \cdots x_f} , \tag{8}$$

independently of the basis chosen. In fact, if τ denotes any collection of such pairs of index labels, then the operation of contraction with the metric tensor, or taking the trace, over these pairs, satisfies

$$T^f \to T^{f_0}, \quad t \mapsto t_\tau , \quad (U_\gamma t)_\tau = U_\gamma^\circ t_\tau \quad \forall \gamma \in G , \tag{9}$$

where $f-f_0$ is even, t_τ has rank f_0 , and $\gamma \mapsto U_\gamma^\circ$ is the representation of G on T^{f_0} . Moreover if g_τ denotes any product of g's, with rank $f-f_0$, and $t^\circ \in T^{f_0}$ is arbitrary, then

$$U_\gamma(g_\tau t^\circ) = g_\tau(U_\gamma^\circ t^\circ) \quad \forall \gamma \in G . \tag{10}$$

The following general result may be used to obtain further invariant subspaces for the groups O(N) and Sp(N):

<u>Theorem 1</u>:

Let T^f, T^{f_0}, $f_0 \le f$, carry representations $\gamma \mapsto U_\gamma$, $\gamma \mapsto U_\gamma^\circ$, of a group G. Let $T_*^{f_0}$ be an invariant subspace of T^{f_0} under G. Let $\Gamma : T^f \to T^{f_0}$ be a linear transformation such that

$$U_\gamma^\circ \Gamma t = \Gamma U_\gamma t \qquad \forall t \in T^f, \gamma \in G . \tag{11}$$

Then the pre-image $T_*^f = \Gamma^{-1}(T_*^{f_0})$ is also an invariant subspace under G. In particular, the kernel of Γ is an invariant subspace.

Thus from Eq. (9), the space $\bar{T}^f$ of completely traceless tensors, each of whose pair traces vanishes, forms an invariant subspace. For O(N), SO(N) and Sp(N), the irreducible invariant subspaces are simply those of the form $Y\bar{T}^f$, where Y is a Young operator[8].

Not every partition of f yields an independent nonvanishing irreducible invariant subspace YT^f or $Y\bar{T}^f$. For GL(N), and hence for U(N) and SU(N), at most N-rowed partitions suffice. For O(N) and SO(N), at most $^N/_2$ (N even) or $^{N-1}/_2$ (N odd) suffice. Murnaghan[18] provides modification rules for relating other irreducible tensors to the independent ones. (See also Sec. 2.4.) Murnaghan also gives dimension formulae for the irreducible tensor representations.

2.2 Symmetries of Irreducible Tensors

A Young operator Y is obtained[8] for each choice of a standard tableau corresponding to a given partition of f. Y has the form

$$Y = (\Pi Q)(\Pi P), \qquad (12)$$

where the P(Q) are symmetrisation (antisymmetrisation) operators on the index labels occurring in the rows (columns) of the tableau, respectively. The Y operators so obtained are orthogonal, and essentially idempotent. The scalars, ξ , required to make the Y idempotent,

$$(\xi Y)^2 = \xi^2 Y^2 = \xi Y , \qquad (13)$$

are given by Littlewood[17], p. 73.

A subtableau of a given standard tableau will be taken to mean a tableau in standard form obtained by choosing some subset of the rows, or some subset of the columns, of the given tableau. For example, $\left[\begin{smallmatrix}1&2\\3\end{smallmatrix}\right]$ has the following row subtableaux: $\left[\begin{smallmatrix}1&2\\3\end{smallmatrix}\right]$, $[1\ 2]$, $[3]$, and the following column subtableaux: $\left[\begin{smallmatrix}1&2\\3\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$, $[2]$.

In the following, Young operators will be written as $Y\left[\begin{smallmatrix}1&2\\3\end{smallmatrix}\right]$, etc., to indicate their symmetry type. Components of irreducible tensors will be written as $\left[\begin{smallmatrix}x_1&x_2\\x_3\end{smallmatrix}\right]$, etc., to indicate their symmetry type, where labels in the corresponding standard tableau are always assumed to be assigned in the natural order, $\left[\begin{smallmatrix}1&2\\3\end{smallmatrix}\right]$, etc.. All indices are assumed to be contravariant.

<u>Theorem 2</u>:

Any Young operator Y may be written

$$Y = Y_c Z_c = Z_R Y_R , \qquad (14)$$

where Z_c , Z_R are residues, and Y_c , Y_R are Young operators corresponding to any column or row subtableaux of the standard tableau of Y .

<u>Proof</u>:

Let P_c , Q_c be antisymmetrisation and symmetrisation operators for some column subtableau, and $(\Pi' Q)$ the product of the remaining column antisymmetrisation operators. Clearly

$$(\Pi P_c)(\Pi P) \propto (\Pi P) ,$$

and since (ΠP_c), $(\Pi' Q)$ refer to different labels, they commute, so that

$$Y = (\Pi Q)(\Pi P) = (\Pi Q_c)(\Pi' Q)(\Pi P) ,$$

$$Y \propto (\Pi' Q_c)(\Pi' Q)(\Pi P_c)(\Pi P) = (\Pi Q_c)(\Pi P_c)(\Pi' Q)(\Pi P) ,$$

$$Y = Y_c Z_c$$

The proof for $Y = Z_R Y_R$ is similar.

<u>Corollary</u>:

Let $t = Yf$ be an irreducible tensor, with the symmetry type of Y. Let Y_c be the Young operator for any column subtableau of the standard tableau of Y. Let Y_c^o , π^o , π' be such that

$$Y_c^o Y_c = o , \quad U_{\pi^o} Y_c = o , \quad U_{\pi'} Y_c \propto Y_c \qquad (15)$$

respectively. Then

$$Y_c t \propto t , \quad U_{\pi'} t \propto t ;$$

$$Y_c^o t = o , \quad U_{\pi^o} t = o . \qquad (16)$$

In particular, t is symmetric with respect to interchanges of columns of equal length, and antisymmetric with respect to permutations within each column.

The corollary may be used to discover additional symmetries of irreducible tensors. For example since

$$Y\begin{bmatrix} 1 & 2 \\ 3 & \\ 4 & \\ 5 & \end{bmatrix} Y\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{bmatrix} = 0 , \qquad (17)$$

it follows that for rank 5 tensors of symmetry corresponding to the partition (2,2,1),

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & \end{bmatrix}^0 + \begin{bmatrix} x_1 & x_2 \\ x_4 & x_5 \\ x_3 & \end{bmatrix}^0 + \begin{bmatrix} x_1 & x_2 \\ x_5 & x_3 \\ x_4 & \end{bmatrix}^0 = 0 , \quad \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & \end{bmatrix}^0 \equiv \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & \end{bmatrix} + \begin{bmatrix} x_2 & x_1 \\ x_3 & x_4 \\ x_5 & \end{bmatrix} . \qquad (18)$$

Symmetry properties of irreducible tensors belonging to two-rowed representations, that is, having two-rowed tableaux, are specified by the following additional conditions:

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & \end{bmatrix} = \begin{bmatrix} x_1 & x_3 \\ x_2 & \end{bmatrix} - \begin{bmatrix} x_3 & x_1 \\ x_2 & \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} - \begin{bmatrix} x_3 & x_1 \\ x_2 & x_4 \end{bmatrix} . \qquad (19)$$

The following identities, easily derived from Eqs. (19), will find extensive use. The repeated index stands for contraction with the (covariant) metric tensor:

$$\begin{bmatrix} i & i & j \\ x_1 & j & \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i & i & x_1 \\ j & j & \end{bmatrix}$$

$$\begin{bmatrix} i & i & j & k \\ j & k & & \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i & i & k & k \\ j & j & & \end{bmatrix}$$

$$\begin{bmatrix} x_1 & i & i \\ x_2 & x_3 & \end{bmatrix} = \begin{bmatrix} i & i & x_1 \\ x_2 & x_3 & \end{bmatrix} - \begin{bmatrix} i & i & x_2 \\ x_1 & x_3 & \end{bmatrix}$$

$$\begin{bmatrix} x_1 & i & i \\ x_2 & & \end{bmatrix} = \begin{bmatrix} i & i & x_1 \\ x_2 & & \end{bmatrix} - \begin{bmatrix} i & i & x_2 \\ x_1 & & \end{bmatrix}$$

$$\begin{bmatrix} x_1 x_2 & i\,i \\ x_3 x_4 & \end{bmatrix} = \begin{bmatrix} i\,i & x_3 x_4 \\ x_1 x_2 & \end{bmatrix} + \begin{bmatrix} i\,i & x_1 x_2 \\ x_1 x_4 & \end{bmatrix} - \begin{bmatrix} i\,i & x_2 x_3 \\ x_1 x_4 & \end{bmatrix} - \begin{bmatrix} i\,i & x_1 x_4 \\ x_2 x_3 & \end{bmatrix}$$

$$\begin{bmatrix} i\,j & i\,j \\ x_1 x_2 & \end{bmatrix} = \begin{bmatrix} i\,i & j\,j \\ x_1 x_2 & \end{bmatrix} - \frac{1}{2}\begin{bmatrix} i\,i & x_1 x_2 \\ j\,j & \end{bmatrix}$$

$$\begin{bmatrix} i\,x_1 & i\,x_2 \\ j\,j & k\,k \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i\,i & x_1 x_2 \\ j\,j & k\,k \end{bmatrix}$$

$$\begin{bmatrix} i & x_1 & x_2 \\ j & i & j \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i & i & x_1 \\ j & j & x_2 \end{bmatrix}$$

$$\begin{bmatrix} i\,i & x_1 x_2 \\ j\,j & x_3 x_4 \end{bmatrix} = 2\begin{bmatrix} i\,i\,j\,j \\ x_1 x_2 x_3 x_4 \end{bmatrix} - 2\begin{bmatrix} i\,i\,j\,j \\ x_1 x_4 x_2 x_3 \end{bmatrix} . \qquad (20)$$

2.3 U(N) ⊃ O(N) Branching Theorem for Two-Rowed Tensor Representations

As shown in Sec. 2.1, the spaces YT^f of tensors irreducible with respect to U(N) in general admit a further reduction under O(N), because of the properties of the trace operation. The reduction to irreducible O(N) constituents may be pictured in two stages: firstly, the decomposition into a direct sum of subspaces of traceless tensors; and secondly, the application of Young operators to each of these subspaces.

The following result is proved by Weyl[(19)], p. 150:

Theorem 3:

Every tensor $t \in T^f$ can be uniquely decomposed into two summands,

$$t = \bar{t} + u , \qquad (21)$$

where $\bar{t}$ is completely traceless, and u is of the form

$$u^{x_1 \cdots x_f} = g^{x_1 x_2} u_{(12)}^{x_3 \cdots x_f} + \cdots \quad , \qquad (22)$$

with $\frac{1}{2}f(f-1)$ summands; moreover $\bar{t}$ and u are orthogonal, in the sense that

$$\bar{t}^{x_1 \cdots x_f} u_{x_1 \cdots x_f} = 0 \; . \qquad (23)$$

Finally, the corresponding subspaces of T^f are both invariant.

This process can be applied to each of the summands of u , and so on; eventually a decomposition of t into a sum of products of completely traceless tensors with the metric tensor, is obtained.

Let t_τ denote an arbitrary trace of the tensor t . By contraction of both sides of Eq. (21) over the pairs of indices labelled by τ ,

$$t_\tau = u_\tau \quad , \qquad (24)$$

since $\bar{t}$ is completely traceless. By substitution from Eq. (22), a system of linear equations is obtained for the summands of u . The solution leads to a decomposition of t of the form

$$t = \bar{t} + \sum_{\tau,\pi} \xi'_{\tau\pi} \, U_\pi (g_\tau t_\tau) \; , \qquad (25)$$

for some scalars $\xi'_{\tau\pi}$, with $\pi \in S_f$. For example,

$$(g_{(12)(34)} \, t_{(12)(34)})^{x_1 \cdots x_f} = g^{x_1 x_2} g^{x_3 x_4} t_{(12)(34)}^{x_5 \cdots x_f} \; . \qquad (26)$$

Applying this process to each of the t_τ in turn, leads to

$$t = \sum_{\tau\pi} \xi_{\tau\pi} U_\pi (g_\tau \bar{t}_\tau) \quad , \qquad (27)$$

where each of the $\bar{t}_\tau$ is completely traceless:

$$t_\tau = \bar{t}_\tau + \Sigma(\cdots) \quad , \qquad (28)$$

as in Eq. (25) above. By Th. 3, the $\bar{t}_\tau$ are uniquely specified by the corresponding t_τ .

The decomposition of Eq. (27) may not be unique. In particular, symmetry considerations may allow arbitrary traces of t to be expressed in terms of traces in standard form, in which case the summation on τ in Eqs. (25) and (27) runs only over these traces.

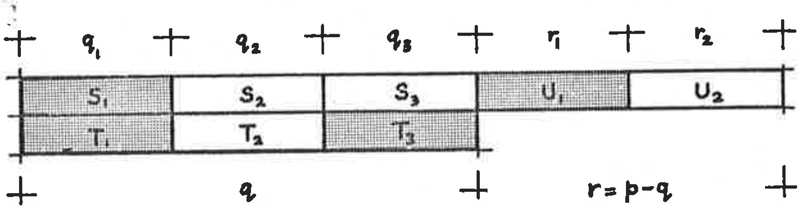
The application of Young operators to each of the $\bar{t}_\tau$ produces a sum of irreducible O(N) tensors. Different $\bar{t}_\tau$ may produce identical irreducible O(N) tensors; after grouping such terms, the various distinct irreducible O(N) constituents (whether equivalent or inequivalent) are found. These constituents are necessarily unique (up to equivalence), whatever choice is made for the decomposition into traceless parts, Eq. (27). A more explicit form for these constituents is given in Sec. 2.6.

Attention is restricted in the following to two-rowed tensor representations, that is, tensors whose symmetry type corresponds to a two-rowed partition. The spaces of irreducible tensor representations of U(N) and O(N) of this type are denoted, respectively, $T[p,q]$ and $T(\ell,m)$.

The following result provides a choice of standard form for the traces of a tensor of two-rowed symmetry type.

Theorem 4:

Each component of an arbitrary trace of a tensor of two-rowed symmetry type, with partition (p,q), can be expressed as a linear combination of components of traces in standard form, specified by three non-negative integers (κ, λ, μ) . Graphically, components of these standard-form traces are represented:



(29)

where:

$$0 \leqslant 2\lambda \leqslant p \ , \quad 0 \leqslant 2\mu \leqslant q \ , \quad \lambda \geqslant \mu \ , \qquad (30)$$

$$0 \leqslant \kappa \leqslant \min(2\lambda - 2\mu \ , p - q) \ , \quad 2\lambda \leqslant q \ ; \qquad (31)$$

$$0 \leqslant \kappa \leqslant \min(q - 2\mu \ , p - 2\lambda) \ , \quad 2\lambda > q \ ;$$

$$(q_1, q_2, q_3, r_1, r_2) = (2\mu, q - 2\lambda + \kappa, 2\lambda - 2\mu - \kappa, \kappa, p - q - \kappa) \ , \quad 2\lambda \leqslant q \ ;$$

$$(q_1, q_2, q_3, r_1, r_2) = (2\mu, \kappa, q - 2\mu - \kappa, 2\lambda + \kappa - q, p - 2\lambda - \kappa) \ , \quad 2\lambda > q \ . \qquad (32)$$

Unshaded portions of the diagram represent free indices not contracted. Shaded portions represent contractions (of adjacent pairs). Here $(\lambda + \mu)$ is the total number of pairs contracted, μ is the number of 2x2 blocks contracted, and $(\lambda - \mu)$ the number of additional pairs contracted. The location of these additional pairs is specified by κ .

For example, the $(1,4,2)$ trace of a tensor of symmetry type (15,10) is:

$$\begin{bmatrix} i_1 i_1\, i_2 i_2\, s_1 s_2 s_3 s_4 s_5 s_6\, j_4\, u_1 u_2 u_3 u_4 \\ j_1 j_1\, j_2 j_2\, t_1 t_2 t_3\, j_3 j_3\, j_4 \end{bmatrix} \qquad (33)$$

and the (2,4,1) trace of a tensor of symmetry type (11,6) is:

$$\begin{bmatrix} i_1 i_1\, s_1 s_2 s_3 s_4\, k_1 k_1 k_2 k_2\, u_1 \\ j_1 j_1\, t_1 t_2\, j_2 j_2 \end{bmatrix} \qquad (34)$$

where the repeated index stands for contraction:

$$[ii] \equiv [ij]\, g_{ij} \qquad (35)$$

Proof:

By induction. Consider the effect of an additional contraction on a trace of a tensor of symmetry type (p,q) which is already in standard form (κ, λ, μ) . In the table below are set out the location of the indices over which the additional (pair) contraction is to be performed, the standard form(s) in terms of which the new trace can be expressed, and the symmetry property required to make this connection, the letters referring to parts of Eq. (20):

S_2S_2 , S_2T_2 or T_2T_2 : $(\,, q_2-2, q_3+2\,,,)$

S_3S_3 : $(q_1+2,, q_3-2\,,,)$ (g)

U_2U_2 : $(\,,,, r_1+2, r_2-2)$

S_2S_3 or T_2S_3 : $(q_1+2\,,, q_3-2\,,,)$ (h)

S_2U_2 or T_2U_2 (q_3 even) : $(, q_2-1, q_3+1, r_1+1, r_2-1)$

S_2U_2 or T_2U_2 (q_3 odd) : $(, q_2-1, q_3+1, r_1+1, r_2-1) + (q_1+2, q_2-1, q_3-1, r_1-1, r_2+1)$ (f)

S_3U_2 : $(q_1+2,, q_3-2, r_1-1, r_2+1)$. (a)

The range of values taken by κ is determined by the range of $q_2 \geqslant 0$, for fixed λ and μ .

The traces in the standard forms (κ,λ,μ) are not all independent. For example, Eqs. (20c,d,e,i) could be used to express all traces in terms of traces in reduced standard form with $\kappa = 0$, and $\mu = 0$ unless $2\lambda \geqslant q-1$. However, as will be seen, the (κ,λ,μ) standard form is closely related to the irreducible O(N) content.

Each of the traces in standard form (κ,λ,μ) can, by the application of Young operators, be further broken up into a sum of traces in symmetrised standard form; which of these occur is given by the following result.

<u>Theorem 5</u>:

Each component of an arbitrary trace of a tensor of two-rowed symmetry type, with partition (p,q), can be expressed as a linear combination of components of traces in symmetrised standard form, specified by three non-negative integers (κ, λ, μ) , with values given by Eqs. (30) and (31), and symmetry type $(q_2+q_3+r_2, q_2)$, in the notation of Eqs. (29) and (32).

For example, the symmetrised (1,4,2) trace of a tensor of symmetry type (15,10) is denoted

$$\begin{bmatrix} - - - - & s_1 s_2 s_3 s_4 s_5 s_6 - u_1 u_2 u_3 u_4 \\ - - - - & t_1 t_2 t_3 - - - \end{bmatrix} \qquad (36)$$

and has symmetry type (10,3) while the symmetrised (2,4,1) trace of a tensor of symmetry type (11,6) is

$$\begin{bmatrix} - - s_1 \; s_2 \; s_3 \; s_4 - - - - u_1 \\ - - t_1 \; t_2 - - \end{bmatrix} , \qquad (37)$$

and has symmetry type (5,2).

Proof:

Not all Young operators corresponding to partitions of length $(p+q) - 2(\lambda+\mu)$, the rank of the unsymmetrised trace (κ,λ,μ), will yield a nonvanishing tensor, when applied to (κ,λ,μ). By Eq. (16), antisymmetrisation over three or more indices of a tensor of two-rowed symmetry type, gives zero. Also, by Eq. (16), a Young operator $Y\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, with $p_2 < q_2$, can be written as ZS, where S denotes symmetrisation over the indices of groups S_2, S_3 and U_2, and some subset of T_2. Therefore S annihilates (κ,λ,μ), since the latter is antisymmetric in corresponding indices of groups S_1 and T_1. Hence the only nonvanishing contributions arise from the application of $Y\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, with $p_2 \geqslant q_2$, to (κ,λ,μ). In particular, $Y\begin{bmatrix} q_2 + q_3 + r_2 \\ q_2 \end{bmatrix}$ produces the symmetrised standard trace (κ,λ,μ). But by Eqs. (20d) and (20i), additional antisymmetrisations between indices of S_3 and U_2, or amongst S_3, corresponding to the effect of $Y\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, with $p_2 > q_2$, applied to (κ,λ,μ), will produce symmetrised standard traces (κ',λ',μ'), with different values of the labels. Hence the distinct symmetrised standard traces occurring are just the symmetrised (κ,λ,μ).

It follows from Th. 3 that to each of the symmetrised standard traces (κ,λ,μ) there will correspond a completely traceless tensor, of the same rank and symmetry type, namely $(q_2+q_3+r_2,\ q_2)$, which therefore belongs to the irreducible representation O(N)(ℓ ,m), where

$$(\ell,m) = (p-2\mu-\kappa,\ q-2\lambda+\kappa),\quad 2\lambda \leq q,$$
$$(\ell,m) = ((p+q)-2(\lambda+\mu)-\kappa,\ \kappa),\quad 2\lambda > q. \qquad (38)$$

Now by the arguments accompanying Eqs. (25) to (27), the irreducible O(N) constituents, occurring in the reduction of the irreducible representation U(N) [p,q], will be products of such symmetrised, completely traceless tensors (κ,λ,μ), with the metric tensor (compare Eq. (51)). Since the latter is invariant under O(N), these irreducible O(N) constituents will belong to essentially the same irreducible representation as (κ,λ,μ) (Eq. (10)). Relating κ and μ to ℓ and m by means of Eq. (32), the irreducible O(N) constituents may be labelled O(N)(λ ; ℓ ,m), and the corresponding irreducible subspaces of tensors, $T(\lambda;\ell,m)$.

Eqs. (30), (31) and (38) lead to the following branching theorem of U(N) ⊃ O(N) for two-rowed tensor representations.

<u>Theorem 6</u>:

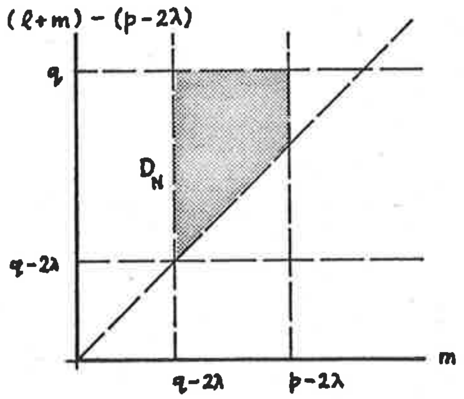
$$N \geq 4:\quad [p,q]\downarrow O(N) = \Sigma^{\oplus}_{D_N}(\lambda;\ell,m) \qquad (39)$$
$$N = 3:\quad [p,q]\downarrow O(3) = (\Sigma^{\oplus}_{D_3}(\lambda;\ell))\oplus(\Sigma^{\oplus}_{D_3^*}(\lambda;\ell)^*)$$
$$N = 2:\quad [p,q]\downarrow O(2) = (\Sigma^{\oplus}_{D_2}(\lambda;\ell))\oplus(\Sigma^{\oplus}_{D_2^*}(\lambda;\ell)^*)\oplus(\Sigma^{\oplus}_{D_2^{o*}}(\lambda;\ell=0)^*).$$

$$D_N: \quad 0 \leq 2\lambda \leq p,$$
$$\max(0, q-2\lambda) \leq (\ell+m)-(p-2\lambda) \leq q,$$
$$\max(0, q-2\lambda) \leq m \leq \min((\ell+m)-(p-2\lambda), p-2\lambda),$$
$$(p+q)-(\ell+m) \text{ even}, \quad m \leq \ell;$$

$$D_3 = D_{N=3}(m=0), \quad D_3^* = D_{N=3}(m=1); \qquad (40)$$
$$D_2 = D_{N=2}(m=0), \quad D_2^* = D_{N=2}(m=2), \quad D_2^{o*} = D_{N=2}(\ell=m=1).$$

Here $(\ell)^*$ is the representation associate to (ℓ), differing by the alternating character (the sign of the determinant). The modification rules (Sec. 2.4) have been used to obtain the branching theorems for N=2 and N=3.

The domain D_N for fixed λ is shown below for the case $2\lambda < q$, $p-2\lambda < q$.



The branching theorem of U(N) ⊃ SO(N) for two-rowed tensor representations is given by Theorem 6, except that associated representations become identical, and for the cases N=2 and N=4,

$$O(4)(\ell,m)\downarrow SO(4) = (\ell,m)\oplus(\ell,-m) \quad (m \neq 0), \qquad (41)$$
$$O(2)(\ell)\downarrow SO(2) = (\ell)\oplus(-\ell) \quad (\ell \neq 0),$$

according to Eq. (1.29).

Theorem 6 bears out the statement of Sec. 1.2 that another independent label, in addition to the invariants (ℓ , m), must be introduced in order to remove the O(N) degeneracy in the U(N) ⊃ O(N) reduction for two-rowed representations. That the label λ introduced here actually distinguishes equivalent representations is clear from the fact that for fixed λ and (ℓ , m), (ℓ+m, m) lies in D_N^λ at most once. In fact, the multiplicity of the irreducible representation O(N)(ℓ , m) within U(N)[p,q] is just the number of λ such that (ℓ+m, m) lies in D_N^λ . By rearrangement of Eqs. (40):

$$\max((p-\ell),(q-m)) \leq 2\lambda \leq \min((p+q)-(\ell+m), p-m). \qquad (42)$$

The U(N) ⊃ O(N) reduction is intrinsically more complicated than, for example, U(N) ⊃ U(N-1) or O(N) ⊃ O(N-1). The branching theorem for two-rowed representations, however, is not unmanageable as formulated here (an example is given in Sec. 2.5). Other branching theorems are given (for three-rowed representations) by Brunet and Resnikoff[20], and (for the case U(3) ⊃ SO(3)) by Elliott[10], and Green and Bracken[1]. Whippman[6] discusses a different approach.

The remaining chapters are devoted to the aim of obtaining an operational definition of the label λ , which so far has a natural significance only in the context of tensor representations. Firstly, however, the relationship is shown between Th. 6 and the methods of character analysis.

2.4 U(N) ⊃ O(N) Reduction by Character Analysis

Purely algebraic methods of character analysis have been applied to the problem of determining branching multiplicities in the classical groups[21]. However, in the context of tensor representations, it is more natural to consider the group approach, in particular as given by Littlewood[17].

Let μ be a partition of f of N rows,

$$\mu = (\mu_1, \cdots, \mu_N),$$
$$\mu_1 + \cdots + \mu_N = f \;, \quad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \geq 0 . \tag{43}$$

To each such μ there corresponds an irreducible tensor representation $[\mu]$ of U(N). The corresponding simple character of U(N), which is a certain symmetrised function ("Schur"-function[17]) of the eigenvalues of elements of U(N) (NxN unitary matrices), is also denoted $[\mu]$. To each irreducible rank f tensor belonging to U(N)$[\mu]$ there corresponds according to Th. 3 a unique completely traceless tensor, of the same rank and symmetry type, which belongs to an irreducible representation of O(N), with simple character (μ).

However, not all of these simple characters are independent. They can be expressed in terms of simple characters (μ) with at most $[N/2]$-rowed partitions,

$$(\mu) = (\mu_1, \cdots, \mu_{[N/2]}) , \tag{44}$$

and the associated simple characters $(\mu)^*$, by means of modification rules, given by Murraghan[18], p. 282. $(\mu)^*$ is defined by

$$(\mu)^* = \epsilon(\mu) , \tag{45}$$

where ϵ is the alternating character of O(N). For uniformity the trivial character of O(N) is denoted (o).

For two-rowed representations the modification rules are necessary for N=2 and N=3. They are:

$$\begin{aligned} N = 2 \ : \ (\mu_1, \mu_2) &= 0 \quad \text{if} \quad \mu_2 > 2 \\ (\mu_1, 2) &= -(\mu_1)^* \\ (\mu_1, 1) &= 0 \quad \text{if} \quad \mu_1 > 1 \\ (1, 1) &= (0)^* \ ; \end{aligned} \tag{46}$$

$$\begin{aligned} N = 3 \ : \ (\mu_1, \mu_2) &= 0 \quad \text{if} \quad \mu_2 \geqslant 2 \\ (\mu_1, 1) &= (\mu_1)^* \end{aligned} \tag{47}$$

<u>Theorem 7</u>:

The simple characters, occurring in the product $[\mu'][\mu]$ of two simple characters of U(N), have Young tableaux which are built from that of $[\mu']$ by adding μ_1 identical symbols 1, μ_2 identical symbols, 2, and so on, subject to:

(i) after the addition of each set of identical symbols, a standard tableau results with no two identical symbols in the same column;

(ii) if the set of additional symbols is read from right to left in the consecutive rows of the final tableau, a lattice permutation of $1^{\mu_1} 2^{\mu_2} 3^{\mu_3} \ldots$ results (the number of symbols 1 $\geq$ the number of symbols 2, and so on, at each point).

Proof:

Littlewood(17), p. 94.

Theorem 8:

Let $[\mu]$ and (μ) be the simple characters of U(N) and O(N) corresponding to a partition μ . Then

$$[\mu] = (\mu) + \sum_{\delta\nu} \xi_{\delta\mu\nu}(\nu) , \qquad (48)$$

where the summation extends over all partitions δ into even parts (2), (4), (2,2), (6), ..., and so on, and over all partitions ν such that $[\mu]$ occurs in the product $[\delta][\nu]$ with multiplicity $\xi_{\delta\mu\nu}$.

Proof:

Littlewood(17), p. 240.

This theorem enables the irreducible O(N) constituents of any irreducible representation of U(N) to be found. The modification rules can be used to express the (μ) and (ν) in the right-hand side of Eq. (48) in terms of simple characters of O(N) of the form given by Eq. (44).

The procedure is illustrated for the case $U(5)[4,3,2,1,0] \downarrow O(5)$.

[δ]	[ν]					
[0]	0000 000 00 0					
[2]	0001 001 00 0	0001 000 01 0	0001 000 00 1	0000 001 01 0	0000 001 00 1	0000 000 01 1
[4]	0001 001 01 1					
[22]	0011 002 02 0	0001 012 02 0	0001 002 01 2	0011 002 00 2		
	0001 001 02 2	0001 000 12 2	0000 011 02 2	0000 001 12 2		
[42]	0011 012 02 1	0011 002 11 2	0001 011 12 2			
[222]	0011 022 03 3	0011 002 23 3	0001 012 23 3			
[422]	0111 022 13 3	0011 112 23 3				

$$[4,3,2,1,0] = (4,3,2,1) + (3,2,2,1) + (3,3,1,1) + (3,3,2) + \\ + (4,2,1,1) + (4,2,2) + (4,3,1) + (3,2,1) + \\ + (2,2,1,1) + (3,1,1,1) + (3,2,1) + (2,2,2) + (3,2,1) + \\ + (3,3) + (4,1,1) + (4,2) + (2,1,1) + (2,2) + \\ + (3,1) + (2,1,1) + (2,2) + (3,1) + (1,1) + (2,0).$$

After applying the modificaton rules,

$$U(5)[4,3,2,1,0]\downarrow O(5) = 2(3,2)^* \oplus (3,0)^* \oplus (3,3) \oplus (4,1)^* \oplus (4,2) \oplus \\ 2(2,1)^* \oplus 2(2,2) \oplus 2(3,1) \oplus (2,0) \oplus (1,1) \; ,$$

with dimensional check

$$1024 = 2\times 105^* + 30^* + 84 + 154^* + 220 + 2\times 35^* + 2\times 35 + 2\times 81 + 14 + 10 .$$

2.5 Comparison of Branching Theorem and Character Analysis for the Case U(4)[9,4].

The method of character analysis of Sec. 2.4 gives for the reduction of U(4)[9,4] with respect to O(4):

[δ]	[ν]
[0]	[94]
[20]	[92] + [83] + [74]
[40]	[90] + [81] + [72] + [63] + [54]
[22]	[72]
[60]	[70] + [61] + [52] + [43]
[42]	[70] + [61] + [52]
[80]	[50] + [41]
[62]	[50] + [41] + [32]
[44]	[50]
[82]	[30] + [21]
[64]	[30]
[84]	[10]

The branching theorem of Sec. 2.3 leads to:

$U(4)[9,4] \downarrow O(4) =$

$(0;9,4) \oplus$

$(1;9,2) \oplus (1;8,3) \oplus (1;7,4) \oplus (1;7,2) \oplus$

$(2;9,0) \oplus (2;8,1) \oplus (2;7,2) \oplus (2;6,3) \oplus (2;5,4) \oplus (2;7,0) \oplus (2;6,1) \oplus (2;5,2)$

$(2;5,0) \oplus (3;7,0) \oplus (3;6,1) \oplus (3;5,2) \oplus (3;4,3) \oplus (3;5,0) \oplus (3;4,1) \oplus (3;3,2)$

$(3;3,0) \oplus (4;5,0) \oplus (4;4,1) \oplus (4;3,0) \oplus (4;2,1) \oplus (4;1,0)$.

Here each occurrence of the degenerate representations of O(4) in the reduction is distinguished by a different value of the additional label λ , of Th. 6. Evidently 2λ is just the length of the leading row of the partition δ of Th. 6.

The dimensional check in this example is:

$$
\begin{aligned}
1980 = {} & 168 + \\
& + 192 + 144 + 96 + 120 + \\
& + 100 + 160 + 120 + 80 + 40 + 64 + 96 + 64 + 36 + \\
& + 64 + 96 + 64 + 32 + 36 + 48 + 24 + 16 + \\
& + 36 + 48 + 16 + 16 + 4
\end{aligned}
$$

2.6 Explicit Tensor Reduction for Some Special Cases

The branching theorem of Sec. 2.3 leads to a reduction of the carrier space of U(N)[p,q] into irreducible subspaces (orthogonal when $(\ell,m) \neq (\ell',m')$),

$$T[p,q] = \sum_{D_N}^{\oplus} T(p,q;\lambda;\ell,m) \tag{49}$$

The irreducible O(N) constituents were found to be certain products with the metric tensor of symmetrised, completely traceless tensors associated with certain traces in standard form of tensors of two-rowed symmetry type (p,q). In fact define

$$\Gamma_{(\ell,m)} : T^{\ell+m} \longrightarrow T(\ell,m) \qquad (50)$$

to be the projection operator from the space of tensors of rank (ℓ + m) on to the irreducible subspace of tensors of symmetry type (ℓ,m), the carrier space of O(N)(ℓ,m). Then if $t_{(\ell m)}$ is some standard trace of $t \in T[p,q]$, and $g_{(\ell m)}$ the corresponding product of g's, the associated irreducible O(N) constituent is of the form

$$\sum_{\pi} \xi_{\pi} Y U_{\pi} (g_{(\ell m)} \Gamma t_{(\ell m)}) \qquad (51)$$

The Young operator Y is present as a result of the uniqueness of the decomposition

$$t = \bar{t} + u \quad , \qquad (52)$$

$$Yt = t = Y\bar{t} + Yu \; , \qquad (53)$$

of Th. 3. Therefore each summand must have the symmetry type (p,q). The summation over $\pi \in S_{p+q}$ only extends to permutations resulting in distinct tensors after the application of Y .

This argument leads to a recursive computational procedure for the evaluation of $\Gamma_{(p,q)}$. For if Eq. (52) is written

$$\Gamma_{(p,q)} t = \bar{t} = t - u \; , \qquad (54)$$

and it is supposed that all $\Gamma_{(\ell,m)}$ of lower rank, occurring as summands of u , are known, then $\Gamma_{(p,q)}$ can be found by solving for the ξ_π the system of linear equations obtained by making arbitrary contractions of $\bar{t}$, which is completely traceless.

Now Th. 1 will lead to a direct sum decomposition similar to that of Th. 3 for any invariant subspace. One such is $\bar{\bar{T}}^{p+q}$, the subspace of all double-traceless tensors, that is, tensors which vanish upon contraction of two pairs of indices with the metric tensor. $\bar{T}^{p+q}$ is of course a subspace of $\bar{\bar{T}}^{p+q}$. A decomposition in two stages,

$$\bar{\bar{t}} = t - u_1 ,$$
$$\bar{t} = \bar{\bar{t}} - u_2 = t - u_1 - u_2 , \qquad (55)$$

is then possible, with each stage involving a system of linear equations (in fewer unknown coefficients than for the single-stage decomposition).

The first method can be used for tensors of low rank, while the second is more economical for higher-rank tensors. The decompositions of [2,0], [3,0], [2,1], [4,0], [3,1], [2,2], [3,2] and [3,3] are given in the appendix, Sec. A2.6. The degenerate case [4,2] is treated there by the second method.

3. U(N) AND O(N) INVARIANTS

As shown in Sec. 1.1, the solution of the U(N) ⊃ O(N) state labelling problem for two-rowed representations requires the introduction of a single additional labelling operator Λ which commutes with the O(N) labelling operators L. Now the latter can be given in terms of functions of the O(N) generators[9]. Hence Λ necessarily commutes with the O(N) generators; that is, it must be an O(N) invariant, and therefore a function of a complete set of O(N) invariants. However, if Λ involved invariant functions of only the O(N) generators, it would not be independent of the operators L. Hence Λ must involve certain additional O(N) invariants.

This chapter concerns the determination of additional O(N) invariants, as a preliminary step towards their use in defining an additional labelling operator Λ , with the eigenvalue λ defined in Sec. 2.3.

A class of such invariants, monomials in the U(N) generators, is described in Sec. 3.1. The independent invariants cubic and quartic in the U(N) generators are found in Sec. 3.2. The two-rowed representations are treated as a special case in Sec. 3.3.

3.1 General Properties

The generators $a^i{}_j$ of U(N) have the commutation relations[2]

$$[a^i{}_j, a^k{}_\ell] = [\delta^i{}_\ell a^k{}_j - \delta^k{}_j a^i{}_\ell] \ , \quad 1 \leq i,j,k,\ell \leq N \qquad (1)$$

The generators $\alpha^i{}_j$ of the subgroup $O(N) \subset U(N)$ of all NxN matrices leaving invariant a nonsingular bilinear quadratic form g are given by[2]

$$\alpha^i{}_j = a^i{}_j - \bar{a}^i{}_j , \qquad 1 \leq i,j \leq N , \tag{2}$$

$$\bar{a}^i{}_j = a_j{}^i = g_{jk} g^{i\ell} a^k{}_\ell , \tag{3}$$

with commutation relations

$$[\alpha_{ij}, \alpha_{k\ell}] = g_{kj}\alpha_{i\ell} - g_{i\ell}\alpha_{kj} - g_{ki}\alpha_{j\ell} + g_{j\ell}\alpha_{kj} . \tag{4}$$

The generators may be regarded as matrices a , $\bar{a}$, α . Following Bracken and Green[9], matrix products may be defined, for example

$$(a^2)^i{}_j = a^i{}_k a^k{}_j$$
$$(a^n)^i{}_j = (a^{n-1})^i{}_k a^k{}_j \tag{5}$$

Traces of such matrices are operators, and will be denoted $\langle \cdots \rangle$. Thus

$$\langle \bar{\alpha} \rangle = \bar{\alpha}^i{}_i = \alpha_i{}^i = \langle \alpha \rangle , \tag{6}$$

where summation over repeated indices is understood.

The following result gives a general form of U(N) and O(N) invariants in terms of such traces. Not all of these are independent; nor are all invariants necessarily of this form.

<u>Theorem 1</u>:

(i) If u , v are U(N) tensor operators, having commutation relations

$$[u^i{}_j , a^k{}_\ell] = \delta^i{}_\ell u^k{}_j - \delta^k{}_j u^i{}_\ell \tag{7}$$

with the generators of U(N), then the matrix product uv is also a U(N) tensor operator.

(ii) If u is a U(N) tensor operator, then the trace $\langle u \rangle$ is a U(N) invariant. In particular, traces of the form $\langle a^n \rangle$ are U(N) invariants.

(iii) If u , v are O(N) tensor operators, with commutation relations

$$[u_{ij}, \alpha_{k\ell}] = g_{kj} u_{i\ell} - g_{i\ell} u_{kj} - g_{ki} u_{j\ell} + g_{j\ell} u_{ki} \qquad (8)$$

with the generators of O(N), then the matrix product uv is also an O(N) tensor operator.

(iv) If u is an O(N) tensor operator, then the trace $\langle u \rangle$ is an O(N) invariant. In particular, traces of the form $\langle \alpha^n \rangle$ are O(N) invariants.

(v) Traces of the general form

$$\langle a^m \bar{a}^n a \cdots \rangle , \qquad (9)$$

monomial in the generators of U(N), are O(N) invariants.

Proof:

(iii)

$$\begin{aligned}
[\alpha_{ij}, uv_{k\ell}] &= u_k{}^m [g_{jm} v_{i\ell} + g_{j\ell} v_{mi} - g_{i\ell} v_{mj} - g_{im} v_{j\ell}] + \\
&\quad + [g_{jk} u_{im} + g_{jm} u_{ki} - g_{im} u_{kj} - g_{ik} u_{jm}] v^m{}_\ell \\
&= u_{kj} v_{i\ell} + g_{j\ell}\, uv_{ki} - g_{i\ell}\, uv_{kj} - u_{ki} v_{j\ell} + \\
&\qquad + g_{jk}\, uv_{i\ell} + u_{ki} v_{j\ell} - u_{kj} v_{i\ell} - g_{ik}\, uv_{j\ell}
\end{aligned}$$

$$[\alpha_{ij}, uv_{k\ell}] = g_{jk}\, uv_{i\ell} - g_{i\ell}\, uv_{kj} - g_{ik}\, uv_{j\ell} + g_{j\ell}\, uv_{ki} .$$

(iv) $[\alpha_{ij}, u_{kl}]\, g^{kl} = u_{ij} - u_{ij} - u_{ji} + u_{ji} = 0$

(v) $[\alpha_{ij}, a_{kl}] = [a_{ij} - a_{ji}, a_{kl}] = g_{kj} a_{il} - g_{il} a_{kj} - g_{ki} a_{jl} + g_{jl} a_{ki}$

$[\alpha_{ij}, \bar{a}_{kl}] = [a_{ij} - a_{ji}, a_{lk}] = g_{kj} \bar{a}_{il} - g_{il} \bar{a}_{kj} - g_{ki} \bar{a}_{jl} + g_{jl} \bar{a}_{ki}$.

For example, for the case N=3 the quadratic O(N) invariant $\langle \alpha^2 \rangle$ is just the familiar square of the total angular momentum.

The following result aids the determination of the independent O(N) invariants of the form given by Eq. (9).

<u>Theorem 2:</u>

(i) $\overline{a^{n+1}} = \bar{a}\, \overline{a^n} + N \overline{a^n} - \langle a^n \rangle \qquad \forall n \geq 1$ (10)

(ii) $\langle a^{n+1} - \bar{a}^{n+1} \rangle = \langle \bar{a} (\overline{a^n} - \bar{a}^n) \rangle \qquad \forall n \geq 1$ (11)

(iii) $\langle a^m \bar{a}^n \rangle = \langle \bar{a}^n a^m \rangle \qquad \forall n \geq 1, m \geq 1$ (12)

<u>Proof:</u> <u>Sec. A3.1</u>

Hence $\overline{a^n}$, $\bar{a}^n$ can be expressed as polynomials in $\bar{a}$, a respectively, with coefficients which are O(N) invariants.

3.2 <u>Independent Invariants</u>

An O(N) invariant of the form given by Eq. (9) above can be related to other O(N) invariants of the same form, either by transposition, or by cyclic rearrangement of the factors, using the commutation relations.

For fifth order or lower invariants, the following combinations can be chosen as the independent ones, with respect to these manipulations:

$$\langle 1\rangle = \tfrac{1}{2}\langle a+\bar{a}\rangle = \langle a\rangle = \langle \bar{a}\rangle \qquad (13)$$

$$\langle 2\rangle = \tfrac{1}{2}\langle a^2+\bar{a}^2\rangle = \langle a^2\rangle = \langle \bar{a}^2\rangle$$

$$\langle 12\rangle = \tfrac{1}{2}\langle a\bar{a}+\bar{a}a\rangle = \langle a\bar{a}\rangle = \langle \bar{a}a\rangle \qquad (14)$$

$$\langle 3\rangle = \tfrac{1}{2}\langle a^3+\bar{a}^3\rangle$$

$$\langle 13\rangle = \tfrac{1}{2}\langle a\bar{a}a+\bar{a}a\bar{a}\rangle \qquad (15)$$

$$\langle 4\rangle = \tfrac{1}{2}\langle a^4+\bar{a}^4\rangle$$

$$\langle 14\rangle = \tfrac{1}{2}\langle a\bar{a}a\bar{a}+\bar{a}a\bar{a}a\rangle$$

$$\langle 24\rangle = \tfrac{1}{2}\langle a^2\bar{a}^2+\bar{a}^2a^2\rangle = \langle a^2\bar{a}^2\rangle = \langle \bar{a}^2a^2\rangle$$

$$\langle 34\rangle = \tfrac{1}{2}\langle a\bar{a}^3+\bar{a}a^3\rangle \qquad (16)$$

$$\langle 5\rangle = \tfrac{1}{2}\langle a^5+\bar{a}^5\rangle$$

$$\langle 15\rangle = \tfrac{1}{2}\langle a\bar{a}a\bar{a}a+\bar{a}a\bar{a}a\bar{a}\rangle$$

$$\langle 35\rangle = \tfrac{1}{2}\langle a^2\bar{a}^3+\bar{a}^2a^3\rangle$$

$$\langle 45\rangle = \tfrac{1}{2}\langle a\bar{a}^4+\bar{a}a^4\rangle \qquad (17)$$

It will suffice to treat in detail only the cubic and quartic cases. The results are as follows:

<u>Theorem 3</u>:

(i) <u>Cubic Invariants</u>:

$$\langle a\bar{a}^2\rangle = \langle \bar{a}^2a\rangle = \langle a\bar{a}a\rangle - \langle a^2 - a\bar{a}\rangle$$

$$\langle \bar{a}a^2\rangle = \langle a^2\bar{a}\rangle = \langle \bar{a}a\bar{a}\rangle + \langle a^2 - a\bar{a}\rangle$$

$$\langle a^3 - \bar{a}^3 \rangle = N\langle a^2 \rangle - \langle a \rangle^2$$

$$\langle a\bar{a}a - \bar{a}a\bar{a} \rangle = 2\langle a^2 \rangle - (N+2)\langle a\bar{a} \rangle + \langle a \rangle^2 \qquad (18)$$

$$\langle a^3 \rangle = \tfrac{1}{2}\langle a^3 + \bar{a}^3 \rangle + \tfrac{N}{2}\langle a^2 \rangle - \tfrac{1}{2}\langle a \rangle^2$$

$$\langle \bar{a}^3 \rangle = \tfrac{1}{2}\langle a^3 + \bar{a}^3 \rangle - \tfrac{N}{2}\langle a^2 \rangle + \tfrac{1}{2}\langle a \rangle^2$$

$$\langle a\bar{a}a \rangle = \tfrac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a} \rangle - \tfrac{1}{2}(N+2)\langle a\bar{a} \rangle + \langle a^2 \rangle + \tfrac{1}{2}\langle a \rangle^2$$

$$\langle \bar{a}a\bar{a} \rangle = \tfrac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a} \rangle + \tfrac{1}{2}(N+2)\langle a\bar{a} \rangle - \langle a^2 \rangle - \tfrac{1}{2}\langle a \rangle^2$$

$$\langle a^2\bar{a} \rangle = \langle \bar{a}a^2 \rangle = \tfrac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a} \rangle + \tfrac{N}{2}\langle a\bar{a} \rangle - \tfrac{1}{2}\langle a \rangle^2$$

$$\langle \bar{a}^2 a \rangle = \langle a\bar{a}^2 \rangle = \tfrac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a} \rangle - \tfrac{N}{2}\langle a\bar{a} \rangle + \tfrac{1}{2}\langle a \rangle^2 \qquad (19)$$

$$\overline{a^3} = \bar{a}^3 + 2N\bar{a}^2 + N^2\bar{a} - \langle a \rangle\bar{a} - (\langle a^2 \rangle + N\langle a \rangle) \qquad (20)$$

(ii) <u>Quartic Invariants</u>:

$$\langle a^4 \rangle = \langle \bar{a}^4 \rangle + 2N\langle \bar{a}^3 \rangle + N^2\langle \bar{a}^2 \rangle - \langle a \rangle(2\langle a^2 \rangle + N\langle a \rangle)$$

$$\langle a\bar{a}a^2 \rangle = \langle \bar{a}a^3 \rangle + \langle a^3 \rangle - (N+1)\langle \bar{a}a^2 \rangle + \langle a \rangle\langle a^2 \rangle = \langle a^2\bar{a}a \rangle$$

$$\langle \bar{a}a\bar{a}^2 \rangle = \langle a\bar{a}^3 \rangle - \langle \bar{a}^3 \rangle + (N+1)\langle a\bar{a}^2 \rangle - \langle a \rangle\langle a^2 \rangle = \langle \bar{a}^2 a\bar{a} \rangle$$

$$\langle a\bar{a}^2 a \rangle = \langle \bar{a}^2 a^2 \rangle + \langle \bar{a}^3 \rangle - (N+1)\langle \bar{a}^2 a \rangle + \langle a^2 \rangle\langle a \rangle$$

$$\langle \bar{a}a^2\bar{a} \rangle = \langle a^2\bar{a}^2 \rangle - \langle a^3 \rangle + (N+1)\langle a^2\bar{a} \rangle - \langle a^2 \rangle\langle a \rangle$$

$$\langle a^4 - \bar{a}^4 \rangle = N\langle a^3 + \bar{a}^3 \rangle - 2\langle a \rangle\langle a^2 \rangle$$

$$\langle a\bar{a}^3 - \bar{a}a^3 \rangle = -N\langle a\bar{a}a + \bar{a}a\bar{a} \rangle + \langle a \rangle\langle a\bar{a} \rangle + \langle a \rangle\langle a^2 \rangle \qquad (21)$$

$$\langle a^4\rangle = \tfrac{1}{2}\langle a^4+\bar{a}^4\rangle + \tfrac{N}{2}\langle a^3+\bar{a}^3\rangle - \langle a\rangle\langle a^2\rangle .$$

$$\langle \bar{a}^4\rangle = \tfrac{1}{2}\langle a^4+\bar{a}^4\rangle - \tfrac{N}{2}\langle a^3+\bar{a}^3\rangle + \langle a\rangle\langle a^2\rangle .$$

$$\langle a\bar{a}a\bar{a}\rangle = \langle \bar{a}a\bar{a}a\rangle = \tfrac{1}{2}\langle a\bar{a}a\bar{a} + \bar{a}a\bar{a}a\rangle$$

$$\langle a^2\bar{a}^2\rangle = \langle \bar{a}^2a^2\rangle = \tfrac{1}{2}\langle a^2\bar{a}^2 + \bar{a}^2a^2\rangle$$

$$\langle a\bar{a}^2a\rangle = \langle a^2\bar{a}^2\rangle + \tfrac{1}{2}\langle a^3+\bar{a}^3\rangle - \tfrac{1}{2}(N+1)\langle a\bar{a}a+\bar{a}a\bar{a}\rangle + $$
$$+\tfrac{1}{2}N(N+1)\langle a\bar{a}\rangle - \tfrac{N}{2}\langle a^2\rangle + \langle a\rangle(\langle a^2\rangle - \tfrac{N}{2}\langle a\rangle)$$

$$\langle \bar{a}a^2\bar{a}\rangle = \langle a^2\bar{a}^2\rangle - \tfrac{1}{2}\langle a^3+\bar{a}^3\rangle + \tfrac{1}{2}(N+1)\langle a\bar{a}a+\bar{a}a\bar{a}\rangle + $$
$$+\tfrac{1}{2}N(N+1)\langle a\bar{a}\rangle - \tfrac{N}{2}\langle a^2\rangle - \langle a\rangle(\langle a^2\rangle + \tfrac{N}{2}\langle a\rangle)$$

$$\langle a\bar{a}^3\rangle = \langle \bar{a}^3a\rangle = \tfrac{1}{2}\langle a\bar{a}^3+\bar{a}a^3\rangle - \tfrac{N}{2}\langle a\bar{a}a+\bar{a}a\bar{a}\rangle + \tfrac{1}{2}\langle a\rangle\langle a\bar{a}+a^2\rangle$$

$$\langle \bar{a}a^3\rangle = \langle a^3\bar{a}\rangle = \tfrac{1}{2}\langle a\bar{a}^3+\bar{a}a^3\rangle + \tfrac{N}{2}\langle a\bar{a}a+\bar{a}a\bar{a}\rangle - \tfrac{1}{2}\langle a\rangle\langle a\bar{a}+a^2\rangle$$

$$\langle a\bar{a}a^2\rangle = \langle a^2\bar{a}a\rangle = \tfrac{1}{2}\langle a\bar{a}^3+\bar{a}a^3\rangle - \tfrac{1}{2}\langle a^3+\bar{a}^3\rangle - \tfrac{1}{2}\langle a\bar{a}a+\bar{a}a\bar{a}\rangle + $$
$$+ \tfrac{N}{2}\langle a^2\rangle - \tfrac{1}{2}N(N+1)\langle a\bar{a}\rangle - \langle a\rangle(\tfrac{1}{2}\langle a\bar{a}\rangle - \tfrac{1}{2}\langle a^2\rangle - \tfrac{N}{2}\langle a\rangle)$$

$$\langle \bar{a}a\bar{a}^2\rangle = \langle \bar{a}^2a\bar{a}\rangle = \tfrac{1}{2}\langle a\bar{a}^3+\bar{a}a^3\rangle + \tfrac{1}{2}\langle a^3+\bar{a}^3\rangle + \tfrac{1}{2}\langle a\bar{a}a+\bar{a}a\bar{a}\rangle + $$
$$+ \tfrac{N}{2}\langle a^2\rangle - \tfrac{1}{2}N(N+1)\langle a\bar{a}\rangle + \langle a\rangle(\tfrac{1}{2}\langle a\bar{a}\rangle - \tfrac{1}{2}\langle a^2\rangle + \tfrac{N}{2}\langle a\rangle) \qquad (22)$$

(iii) Invariants in $\alpha = a - \bar{a}$:

$$\langle \alpha\rangle = \langle a-\bar{a}\rangle = 0$$

$$\langle \alpha^2\rangle = \langle (a-\bar{a})^2\rangle = 2\langle a^2 - a\bar{a}\rangle$$

$$\langle \alpha^3\rangle = \langle (a-\bar{a})^3\rangle = \tfrac{1}{2}(N-2)\langle \alpha^2\rangle$$

$$\langle \alpha^4\rangle = \langle (a-\bar{a})^4\rangle = \langle a^4+\bar{a}^4\rangle + \langle a\bar{a}a\bar{a} + \bar{a}a\bar{a}a\rangle + 2\langle a^2\bar{a}^2+\bar{a}^2a^2\rangle - $$
$$-4\langle a\bar{a}^3+\bar{a}a^3\rangle + 3N(\,(N+1)\langle a\bar{a}\rangle - \langle a^2\rangle - \langle a\rangle^2)$$

(23)

Proof:

By direct evaluation of the commutation relations. Working for the quartic case is carried out in Sec. A3.2.

3.3 Invariants for Two-Rowed Representations

In general further relationships exist between the cubic and quartic O(N) invariants given in Eqs. (15) and (16). In particular, the invariants $\langle a^n \rangle$ can be expressed in terms of the U(N) labels (in principle, by direct evaluation). Thus the cubic and quartic invariants $\langle a^3 \rangle$ and $\langle a^4 \rangle$ are in principle known.

Similarly, the invariants $\langle \alpha^n \rangle$ can be expressed in terms of the O(N) labels. Thus $\langle \alpha^4 \rangle$ is in principle known. In fact if $N \leqslant 4$, it may be evaluated explicitly by means of the polynomial identity of degree N satisfied by the O(N) generators α, given by Bracken and Green[9], and Green[2]. For example, in the O(4) representation labelled (ℓ, m),

$$\langle \alpha^2 \rangle = 2\ell(\ell+2) + 2m^2 \tag{24}$$

$$\langle \alpha^3 \rangle = \langle \alpha^2 \rangle$$

and the identity is

$$(\alpha-\ell-2)(\alpha-m-1)(\alpha+m-1)(\alpha+\ell) = 0 \tag{25}$$

from which

$$\langle \alpha^4 \rangle = 2(\ell(\ell+2)+m^2)^2 + 2(\ell(\ell+2)+m^2) - 4m^2(\ell+1)^2 \tag{26}$$

Hence because of Eq. (23d) an additional relationship exists between the quartic O(N) invariants.

The conclusion from the above considerations is that in general there is one independent cubic invariant of O(N), namely $\frac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a}\rangle$, and there are two independent quartic invariants of O(N), say $\frac{1}{2}\langle a\bar{a}a\bar{a} + \bar{a}a\bar{a}a\rangle$, and $\frac{1}{2}\langle a\bar{a}^3 + \bar{a}a^3\rangle$.

For the case of two-rowed representations, the number of independent invariants is further reduced. As will be shown in Sec. 4.2, in this case there is a cubic polynomial identity satisfied by the U(N) generators a. This enables the quartic invariant $\frac{1}{2}\langle a\bar{a}^3 + \bar{a}a^3\rangle$, and incidentally the invariants $\frac{1}{2}\langle a^3 + \bar{a}^3\rangle$, $\frac{1}{2}\langle a^4 + \bar{a}^4\rangle$, to be evaluated explicitly.

From Eqs. (4.34) and (4.35), the cubic identities are

$$a^3 = (\langle a\rangle + 2N - 3)\, a^2 + \tfrac{1}{2}(\langle a^2\rangle - \langle a\rangle^2 - (3N-5)\langle a\rangle - 2(N-1)(N-2))\, a$$

$$\bar{a}^3 = (\langle a\rangle - 3)\,\bar{a}^2 + \tfrac{1}{2}(\langle a^2\rangle - \langle a\rangle^2 - (N-7)\langle a\rangle - 4)\,\bar{a} + (\langle a^2\rangle - \langle a\rangle^2 - (N-3)\langle a\rangle) \qquad (27)$$

Hence, taking care that quantities rearranged actually commute,

$$\langle a^3\rangle = (2N-3)\langle a^2\rangle + \tfrac{1}{2}(3\langle a^2\rangle - \langle a\rangle^2 - (3N-5)\langle a\rangle - 2(N-1)(N-2))\langle a\rangle$$

$$\langle \bar{a}^3\rangle = (N-3)\langle a^2\rangle + \tfrac{1}{2}(3\langle a^2\rangle - \langle a\rangle^2 - (3N-7)\langle a\rangle - 2(N-1)(N-2))\langle a\rangle \qquad (28)$$

$$\langle a\bar{a}^3\rangle = \tfrac{1}{2}(\langle a\rangle - 3)\langle a\bar{a}a + \bar{a}a\bar{a}\rangle + \tfrac{1}{2}(\langle a^2\rangle - \langle a\rangle^2 - (2N-7)\langle a\rangle + (3N-4))\langle a\bar{a}\rangle + \tfrac{1}{2}(2\langle a^2\rangle - \langle a\rangle^2 - (2N-3)\langle a\rangle)\langle a\rangle$$

$$\langle \bar{a}a^3\rangle = \tfrac{1}{2}(\langle a\rangle + 2N - 3)\langle a\bar{a}a + \bar{a}a\bar{a}\rangle + \tfrac{1}{2}(\langle a^2\rangle - \langle a\rangle^2 - (2N-5)\langle a\rangle + (3N-4))\langle a\bar{a}\rangle - \tfrac{1}{2}(\langle a^2\rangle + (2N-3)\langle a\rangle)\langle a\rangle \qquad (29)$$

Therefore:

$$\langle a^3 - \bar{a}^3 \rangle = N\langle a^2 \rangle - \langle a \rangle^2$$

$$\tfrac{1}{2}\langle a^3 + \bar{a}^3 \rangle = \tfrac{3}{2}(N-2)\langle a^2 \rangle + \tfrac{1}{2}(3\langle a^2 \rangle - \langle a \rangle^2 - 3(N-4)\langle a \rangle - 2(N-1)(N-2))\langle a \rangle$$

$$\langle a\bar{a}^3 - \bar{a}a^3 \rangle = -N\langle a\bar{a}a + \bar{a}a\bar{a} \rangle + \langle a \rangle\langle a\bar{a} \rangle + \langle a \rangle\langle a^2 \rangle$$

$$\tfrac{1}{2}\langle a\bar{a}^3 + \bar{a}a^3 \rangle = \tfrac{1}{2}(\langle a \rangle + N-3)\langle a\bar{a}a + \bar{a}a\bar{a} \rangle + \tfrac{1}{2}(\langle a^2 \rangle - \langle a \rangle^2 - 2(N-3)\langle a \rangle + (3N-4))\langle a\bar{a} \rangle$$

$$+ \tfrac{1}{2}(\langle a^2 \rangle - \langle a \rangle^2 - (2N-3)\langle a \rangle)\langle a \rangle . \qquad (30)$$

Similarly, $\langle a^4 \rangle$, $\langle \bar{a}^4 \rangle$ and $\tfrac{1}{2}\langle a^4 \pm \bar{a}^4 \rangle$ could be written down.

For the case of two-rowed representations, therefore, there is one independent cubic invariant of O(N), and one independent quartic invariant of O(N), namely $\tfrac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a} \rangle$, and $\tfrac{1}{2}\langle a\bar{a}a\bar{a} + \bar{a}a\bar{a}a \rangle$, respectively.

4. EVALUATION OF INVARIANTS

In this chapter certain of the U(N) and O(N) invariants, described in Ch. 3, will be evaluated, in the tensor representations T[p,q] and T(p,q; λ ; ℓ ,m).

In Sec. 4.1, some results are proved which facilitate these evaluations, using Secs. 2.3 and 2.6. In Sec. 4.2, a technique used by Green and Bracken[1] for the evaluation of invariants in tensor representations, for the U(3) $\supset$ SO(3) problem, is adapted for the case of general two-rowed representations, and a short-hand notation introduced. The cubic characteristic polynomial identity, satisfied by the matrix of U(N) generators in two-rowed representations (Sec. 1.1), is found by this means. As a further check of the method, the quadratic O(N) invariant $\langle a\bar{a}\rangle = \langle a^2\rangle - \frac{1}{2}\langle\alpha^2\rangle$ is found by direct evaluation in Sec. 4.3. In Sec. 4.4 the cubic O(N) invariant $\frac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a}\rangle$ is evaluated. Finally in Sec. 4.5 it is shown how this evaluation is used, in obtaining an implicit definition of the labelling operator Λ.

Most of the details of the calculations are given in the appendix, Secs. A4.2 - A4.4.

4.1 Preliminaries

The arguments of Secs. 2.3 and 2.6 showed that an arbitrary tensor $t \in T[p,q]$ can be reduced into a sum of irreducible O(N) constituents belonging to the invariant subspaces $T(\lambda;\ell,m)$, corresponding to traces $t_{(\lambda;\ell m)}$ of t in standard form. Each irreducible constituent is a linear combination of tensors of the form (Eq. (2.51))

$$\Gamma^{\circ}_{(\ell m)} t_{(\lambda;\ell m)} = Y(g_{(\lambda;\ell m)}\Gamma_{(\ell m)} t_{(\lambda;\ell m)}), \qquad (1)$$

where $\Gamma^{\circ}_{(\ell m)}$ stands for the process of projection on to the subspace of completely traceless tensors, effected by $\Gamma_{(\ell m)}$, appropriate combination with the metric tensor, and symmetrisation. The space $T(\lambda;\ell,m)$ is therefore spanned by tensors of this form. The shift operators are defined on $T[p,q]$ through their actions on these tensors:

$$\Lambda^{\pm}(\Gamma^{\circ}_{(\ell m)} t_{(\lambda;\ell m)}) = \Gamma^{\circ}_{(\ell m)} t_{(\lambda\pm 1,\ell m)} \quad (0 \leq 2(\lambda\pm 1) \leq p),$$

$$\Lambda^{\pm}(\Gamma^{\circ}_{(\ell m)} t_{(\lambda;\ell m)}) = 0 \qquad (\text{otherwise}), \qquad (2)$$

using Eq. (2.40a).

If Φ is an arbitrary invariant of O(N) in the representation carried by $T[p,q]$, then a modified invariant Φ° is defined on $T[p,q] = \Sigma \oplus T(\lambda;\ell,m)$ by

$$\Phi^{\circ} t = \Gamma^{\circ}_{(\ell m)} (\Phi t)_{(\lambda;\ell m)} \quad , t \in T(\lambda;\ell,m). \qquad (3)$$

The proof is given by the following theorem.

Theorem 1:

Φ°, defined by Eq. (3), is an O(N) invariant.

Proof:

Let $t \in T(\lambda;\ell,m)$, and $\gamma \in O(N)$. Then by definition

$$U_{\gamma} \Phi^{\circ} t = U_{\gamma} \Gamma^{\circ}_{(\ell m)} (\Phi t)_{(\lambda;\ell m)} \qquad (4)$$

Now $\Gamma^{\circ}_{(\ell m)}$ is some combination of permutations, contractions, and products with the metric tensor. In view of Eqs. (2.6), (2.9) and (2.10), respectively,

$$\begin{aligned} U_{\gamma} \Phi^{\circ} t &= \Gamma^{\circ}_{(\ell m)} (U_{\gamma} \Phi t)_{(\lambda;\ell m)} \\ &= \Gamma^{\circ}_{(\ell m)} (\Phi U_{\gamma} t)_{(\lambda;\ell m)} = \Phi^{\circ} U_{\gamma} t \ , \end{aligned} \qquad (5)$$

using the fact that Φ is an O(N) invariant.

Since Φ° is an O(N) invariant, the image $\Phi^{\circ} T(\lambda;\ell,m)$ is an invariant subspace. Moreover Φ° can only mix the label λ, leaving the O(N) labels fixed. Therefore

$$\Phi^{\circ} : T(\lambda;\ell,m) \longrightarrow \sum_{\lambda} \oplus T(\lambda;\ell,m) \qquad (6)$$

The Φ°, rather than the Φ, are more readily evaluated in the framework developed here. When Φ is a function only of the O(N) generators, or is an U(N) invariant, then

$$\Phi t \propto t \ , \ \Phi^{\circ} = \Phi . \qquad (7)$$

The following results pertain to situations arising in the subsequent calculations.

<u>Theorem 2</u>:

Let $t \in T(\lambda;\ell,m)$ and $t_{(\lambda;\ell m)}$ the corresponding standard trace. Let Φ be an arbitrary O(N) invariant in the representation carried by $T[p,q]$, and Φ° the related invariant. Let ξ be some scalar factor. Then

(i) if $(\Phi t)_{(\lambda;\ell m)} = \xi t_{(\lambda;\ell m)}$, then $\Phi^{\circ} t = \xi t$.

(ii) if $(\Phi t)_{(\lambda;\ell m)} = \xi U_{\pi} t_{(\lambda;\ell m)}$, where $\Gamma_{(\ell m)} t_{(\lambda;\ell m)}$ is symmetric under the permutation $\pi \in S_{\ell+m}$, then $\Phi^{\circ} t = \xi t$.

(iii) if $(\Phi t)_{(\lambda;\ell m)}$ is a (non-standard) trace, or combination of standard traces, antisymmetric in some pair of indices which are symmetrised by the application of $\Gamma_{(\ell m)}$, then $\Phi^{\circ} t = 0$.

(iv) if $(\Phi t)_{(\lambda;\ell m)} = \xi g_{\sigma} t_{(\lambda;\ell m)\sigma}$, where σ denotes a trace disjoint from the standard trace $(\lambda;\ell,m)$, then $\Phi^{\circ} t = 0$.

(v) if $(\Phi t)_{(\lambda;\ell m)} = \xi U_{\pi} t_{(\lambda\pm1,\ell m)}$, where $\Gamma_{(\ell m)} t_{(\lambda;\ell m)}$ is symmetric under the permutation $\pi \in S_{\ell+m}$, then $\Phi^{\circ} t = \xi \Lambda^{\pm} t$.

<u>Proof</u>:

(i), (ii) and (iii): by substitution, using Eqs. (1) and (3).

(iv): let $u \in T^{\ell+m}$; the contraction of $g_\sigma t_{(\lambda;\ell m)\sigma}$ with $\Gamma_{(\ell m)} u$, a completely traceless tensor, must vanish because of the presence of the factor g_σ: since u is arbitrary, $\Gamma_{(\ell m)} g_\sigma t_{(\lambda;\ell m)\sigma} = 0$.

(v) : by substitution, using Eqs. (1), (2) and (3).

In general, it will be found that

$$(\Phi t)_{(\lambda;\ell m)} = \xi U_\pi t_{(\lambda;\ell m)} + \xi^+ U_{\pi^+} t_{(\lambda+1;\ell m)} + \xi^- U_{\pi^-} t_{(\lambda-1;\ell m)}, \quad (8)$$

in the subsequent calculations, where $\pi, \pi^\pm \in S_{\ell+m}$ all act on indices which are symmetrised by the application of $\Gamma_{(\ell m)}$. Therefore from Th. 2(ii), (v),

$$\Phi^\circ = \xi + \xi^+\Lambda^+ + \xi^-\Lambda^-, \quad (9)$$

where the scalar factors ξ, $\xi^\pm$ are polynomials in the parameters N; p,q; λ; ℓ,m.

4.2 U(N) Invariants and Cubic Polynomial Identity

The action of the generators of U(N) in tensor representations is given by[2]

$$(a^i{}_j t)^{\varkappa_1\cdots\varkappa_f} = \delta^{\varkappa_1}{}_j t^{i\varkappa_2\cdots\varkappa_f} + \cdots + \delta^{\varkappa_f}{}_j t^{\varkappa_1\cdots\varkappa_{f-1}i}. \quad (10)$$

Such substitution operators, and their products, may be represented in the following way[1]:

$$\begin{aligned} 1 &= (\,), \\ a^i{}_j &= \delta^\varkappa{}_j \binom{i}{\varkappa}, \\ a^i{}_j a^k{}_\ell &= \delta^{\varkappa'}{}_j \delta^\varkappa{}_\ell \binom{i\,k}{\varkappa'\varkappa} + \delta^k{}_j \delta^\varkappa{}_\ell \binom{i}{\varkappa}, \end{aligned}$$

$$a^i{}_j a^k{}_\ell a^m{}_n = \delta^{x''}{}_j \delta^{x'}{}_\ell \delta^{x}{}_n \begin{pmatrix} i & k & m \\ x'' & x' & x \end{pmatrix} + \delta^k{}_j \delta^m{}_\ell \delta^x{}_n \begin{pmatrix} i \\ x \end{pmatrix} +$$
$$\delta^k{}_j \delta^{x'}{}_\ell \delta^x{}_n \begin{pmatrix} i & m \\ x' & x \end{pmatrix} + \delta^m{}_j \delta^{x'}{}_\ell \delta^x{}_n \begin{pmatrix} k & i \\ x' & x \end{pmatrix} + \delta^{x'}{}_j \delta^m{}_\ell \delta^x{}_n \begin{pmatrix} i & k \\ x' & x \end{pmatrix}, \qquad (11)$$
$$x \neq x' \neq x'',$$

where the index label x runs over the indices of the index set, namely $x_1, \cdots, x_f$, and where the bracket $\begin{pmatrix} i \\ x \end{pmatrix}$ indicates that for each of these values, the label i is to be substituted in the appropriate location.

It is convenient also to introduce a more contracted notation for substitution operators, involving only the index sets $X, Y, Z, \cdots$, and the labels to be substituted:

$$(X)^i{}_j = \delta^x{}_j \begin{pmatrix} i \\ x \end{pmatrix},$$
$$(X|Y)^i{}_j = \delta^x{}_j \begin{pmatrix} i & y \\ y & x \end{pmatrix},$$
$$(X|X)^i{}_j = \delta^x{}_j \begin{pmatrix} i & x' \\ x' & x \end{pmatrix},$$
$$(X|YZ)^i{}_j = \delta^x{}_j \begin{pmatrix} i & y & z \\ y & z & x \end{pmatrix},$$
$$(X|Y) = (X|Y)^i{}_i = \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$
$$(XY) = g^{xy} \begin{pmatrix} i & j \\ x & y \end{pmatrix} g_{ij},$$
$$(XY|Z) = g^{xy} \begin{pmatrix} i & z & j \\ z & x & y \end{pmatrix} g_{ij} + g^{xy} \begin{pmatrix} i & z & j \\ z & y & x \end{pmatrix} g_{ij}, \qquad (12)$$

and so on, where $x \in X$, $y \in Y$, ..., and indices $x, x' \in X$ running over the same index set, do not take the same value simultaneously. Index sets separated by a vertical stroke contribute only internal summations.

As shown by Bracken and Green[9], and Green[2], the generators of U(N) satisfy a polynomial characteristic identity of degree N in a particular irreducible representation. For the two-rowed representations U(N)[p,q], however, there exists a cubic polynomial identity, of reduced degree (Sec. 1.1), as will be shown by direct evaluation of a^3 and $\bar{a}^3$. The calculation also yields $\langle a \rangle$, $\langle \bar{a} \rangle$, $\langle a^2 \rangle$, $\langle a^3 \rangle$, $\langle \bar{a}^3 \rangle$, and $\frac{1}{2}\langle a^3 + \bar{a}^3 \rangle$. Details are given in Sec. A4.2. Eqs. (2.19) are used extensively in the proofs.

According to Eqs. (11a) and (11b) above,

$$a^i{}_j = \delta^x{}_j \binom{i}{x} , \qquad (13)$$

$$a^{2i}{}_\ell = a^i{}_j a^j{}_\ell = N \delta^x{}_\ell \binom{i}{x} + \delta^x{}_\ell \left(\begin{smallmatrix} i & x' \\ x' & x \end{smallmatrix}\right) , \qquad (14)$$

which may be written, in the scheme of Eq. (47),

$$a^i{}_j = \delta^s{}_j \binom{i}{s} + \delta^t{}_j \binom{i}{t} + \delta^u{}_j \binom{i}{u} , \qquad (15)$$

$$\begin{aligned} a^{2i}{}_\ell = {} & N\delta^s{}_\ell \binom{i}{s} + N\delta^t{}_\ell \binom{i}{t} + N\delta^u{}_\ell \binom{i}{u} + \\ & + \delta^s{}_\ell \left[\left(\begin{smallmatrix} i & s' \\ s' & s \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & t \\ t & s \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & u \\ u & s \end{smallmatrix}\right)\right] + \delta^t{}_\ell \left[\left(\begin{smallmatrix} i & s \\ s & t \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & t' \\ t' & t \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & u \\ u & t \end{smallmatrix}\right)\right] + \\ & + \delta^u{}_\ell \left[\left(\begin{smallmatrix} i & s \\ s & u \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & t \\ t & u \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & u' \\ u' & u \end{smallmatrix}\right)\right] , \end{aligned} \qquad (16)$$

and using appropriate results of Sec. A4.2,

$$a^{2i}{}_\ell = (q+N-2)\left[\delta^s{}_\ell \binom{i}{s} + \delta^t{}_\ell \binom{i}{t}\right] + (p+N-1)\delta^u{}_\ell \binom{i}{u} + \left[\delta^s{}_\ell \left(\begin{smallmatrix} i & u \\ u & s \end{smallmatrix}\right) + \delta^t{}_\ell \left(\begin{smallmatrix} i & u \\ u & t \end{smallmatrix}\right)\right] , \qquad (17)$$

$$a^{2i}{}_\ell = (q+N-2)\, a^i{}_\ell + (p-q+1)\delta^u{}_\ell \binom{i}{u} + \left[\delta^s{}_\ell \left(\begin{smallmatrix} i & u \\ u & s \end{smallmatrix}\right) + \delta^t{}_\ell \left(\begin{smallmatrix} i & u \\ u & t \end{smallmatrix}\right)\right] , \qquad (18)$$

$$a^{3k}{}_\ell = (q+N-2)\, a^{2k}{}_\ell + (p-q+1)\delta^u{}_\ell a^k{}_i \binom{i}{u} + \left[\delta^s{}_\ell a^k{}_i \left(\begin{smallmatrix} i & u \\ u & s \end{smallmatrix}\right) + \delta^t{}_\ell a^k{}_i \left(\begin{smallmatrix} i & u \\ u & t \end{smallmatrix}\right)\right] , \qquad (19)$$

whence, from Sec. A4.2 again,

$$a^{3k}{}_\ell = (q+N-2)a^{2k}{}_\ell + (p-q+1)(p+N-1)\delta^u{}_\ell\binom{k}{u} + (p+N-1)\left[\delta^s{}_\ell\binom{k\ u}{u\ s} + \delta^t{}_\ell\binom{k\ u}{u\ t}\right] \qquad (20)$$

Comparing with Eq. (18) gives

$$a^3 = (p+q+2N-3)a^2 - (p+N-1)(q+N-2)a\,. \qquad (21)$$

The corresponding calculation for the conjugate is given without comment:

$$\bar{a}_i{}^j = a^j{}_i = \delta^s{}_i\binom{j}{s} + \delta^t{}_i\binom{j}{t} + \delta^u{}_i\binom{j}{u}\,, \qquad (22)$$

$$\bar{a}^2{}_j{}^k = a^\ell{}_j a^k{}_\ell = (p+q)\delta^k{}_j(\,) + \delta^x{}_j\binom{k\ x'}{x'\ x}\,, \qquad (23)$$

$$\bar{a}^2{}_j{}^k = (p+q)\delta^k{}_j(\,) + (q-2)\left[\delta^s{}_j\binom{k}{s} + \delta^t{}_j\binom{k}{t}\right] + (p-1)\delta^u{}_j\binom{k}{u} +$$

$$+\left[\delta^s{}_j\binom{k\ u}{u\ s} + \delta^t{}_j\binom{k\ u}{u\ t}\right], \qquad (24)$$

$$\bar{a}^3{}_i{}^k = (p+q)\bar{a}_i{}^k + (q-2)\bar{a}^2{}_i{}^k + (p-q+1)\delta^u{}_j\bar{a}_i{}^j\binom{k}{u} +$$

$$+\left[\delta^s{}_j\bar{a}_i{}^j\binom{k\ u}{u\ s} + \delta^t{}_j\bar{a}_i{}^j\binom{k\ u}{u\ t}\right], \qquad (25)$$

$$\bar{a}^3{}_i{}^k = (p+q)\bar{a}_i{}^k + (q-2)\bar{a}^2{}_i{}^k + (p-q)(p+1)\delta^k{}_i(\,) +$$

$$(p-q+1)(p-1)\delta^u{}_i\binom{k}{u} + (p-1)\left[\delta^s{}_i\binom{k\ u}{u\ s} + \delta^t{}_i\binom{k\ u}{u\ t}\right], \qquad (26)$$

$$\bar{a}^3 = (p+q-3)\bar{a}^2 - (p(q-1) - 2(q-1) - 2p)\bar{a} - 2p(q-1)\,, \qquad (27)$$

and is carried out as for a^3, using the results of Sec. A4.2.

Bracken and Green[(9)] give the general characteristic polynomial identity, of degree N, satisfied by the generators in an arbitrary irreducible representation of U(N). For two-rowed representations, of the form [p,q,o, ..., o], this reduces to

$$(a-o)(a-o-1)(\cdots)(a-o-N+3)(a-q-N+2)(a-p-N+1) = 0,$$

$$(\bar{a}-o+N-1)(\bar{a}-o+N-2)(\cdots)(\bar{a}-o+2)(\bar{a}-q+1)(\bar{a}-p) = 0, \qquad (28)$$

while from Eqs. (21) and (27) above,

$$a(a-q-N+2)(a-p-N+1) = 0,$$

$$(\bar{a}+2)(\bar{a}-q+1)(\bar{a}-p) = 0. \qquad (29)$$

Thus for the two-rowed representations, a cubic characteristic polynomial identity exists, which is a factor of the general identity of degree N.

For the irreducible representations U(N)[p,p,o, .., o], in a similar manner, quadratic identities are found:

$$a(a-p-N+2) = 0, \qquad (30)$$

$$(\bar{a}+2)(\bar{a}-p) = 0. \qquad (31)$$

The values of the quadratic and cubic invariants of U(N) (Sec. 3.1) may also be found from the above (compare Sec. 3.3):

$$\langle a\rangle = \langle\bar{a}\rangle = p+q, \qquad (32)$$

$$\langle a^2\rangle = \langle\bar{a}^2\rangle = p(p+N-1) + q(q+N-3), \qquad (33)$$

$$a^3 = (\langle a\rangle + 2N-3)a^2 + \tfrac{1}{2}(\langle a^2\rangle - \langle a\rangle^2 - (3N-5)\langle a\rangle - 2(N-1)(N-2))a, \qquad (34)$$

$$\bar{a}^3 = (\langle a\rangle - 3)\bar{a}^2 + \tfrac{1}{2}(\langle a^2\rangle - \langle a\rangle^2 - (N-7)\langle a\rangle - 4)\bar{a} + (\langle a^2\rangle - \langle a\rangle^2 - (N-3)\langle a\rangle), \qquad (35)$$

$$\langle a^3\rangle = (2N-3)\langle a^2\rangle + \tfrac{1}{2}(3\langle a^2\rangle - \langle a\rangle^2 - (3N-5)\langle a\rangle - 2(N-1)(N-2))\langle a\rangle, \quad (36)$$

$$\langle \bar{a}^3\rangle = (N-3)\langle a^2\rangle + \tfrac{1}{2}(3\langle a^2\rangle - \langle a\rangle^2 - (3N-7)\langle a\rangle - 2(N-1)(N-2))\langle a\rangle. \quad (37)$$

4.3 O(N) Invariant $\langle a\bar{a}\rangle = \langle a^2\rangle - \frac{1}{2}\langle \alpha^2\rangle$.

From Eq. (11b), the O(N) invariant $\langle a\bar{a}\rangle$ is given in terms of substitution operators by

$$\langle a\bar{a}\rangle = g_{ij}\, a^i{}_k\, a^j{}_\ell\, g^{k\ell} ,$$

$$\langle a\bar{a}\rangle = \delta^{x}{}_{i}\binom{i}{x} + g^{xx'}\binom{ij}{xx'} g_{ij} ,$$

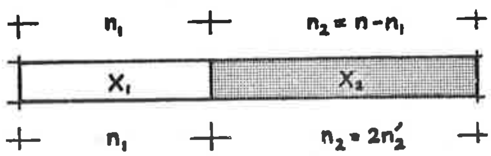
$$\langle a\bar{a}\rangle = \langle a\rangle + (xx). \quad (38)$$

The direct evaluation of $\langle a\bar{a}\rangle$ should be consistent with the identity

$$\langle a\bar{a}\rangle = \langle a^2\rangle - \tfrac{1}{2}\langle \alpha^2\rangle . \quad (39)$$

The evaluation is carried out using Eq. (3) and Th. 2, Sec. 4.1, and the form of the standard trace labelled $(\lambda;\ell,m)$.

For example, consider the special case of the irreducible representations O(N)$(n_1,0)$ contained within the irreducible representation U(N)[n,0]. The corresponding tensors are completely symmetrical. The associated standard trace has the form



(40)

where shaded portions represent contractions. The substitution operator to be found is

$$(XX) = (X_1X_1) + (X_2X_2) + (X_1X_2) , \qquad (41)$$

and is evaluated by considering special cases. Tensor components appear in square brackets; all indices are taken to be contravariant. Repeated indices stand for contractions. The general results follow from

$$(X_1X_1)[x_1x_2] = 2g^{x_1x_2}[\ell\ell] \qquad (42)$$

$$(X_1X_?)[x_1x_2i_1i_2] = 2g^{x_1i_1}[\ell x_2\ell i_2] + 2g^{x_1i_2}[\ell x_2 i_1\ell] + $$
$$+ 2g^{x_2i_1}[x_1\ell\ell i_2] + 2g^{x_2i_2}[x_1\ell i_1\ell] . \qquad (43)$$

whence, applying the standard trace, and using Th. 2,

$(X_1X_1)[x_1x_2] \to 0, \quad (X_1X_1) \to 0 \;(\text{Th.}2(iv));$

$(X_2X_2)[i_1i_2] \to 2N[\ell\ell] \;(n_2' = 1),$

$(X_2X_2)[i_1i_2i_3i_4] \to (4N+8)[\ell\ell mm] \;(n_2' = 2),$

$(X_2X_2) \to n_2(n_2+N-2) ; \;(\text{Th.}2(i));$

$$(X_1X_2)[x_1x_2i_1i_2] \to 8[x_1x_2\ell\ell] ; \; (X_1X_2) \to 2n_1n_2 . \qquad (44)$$

where the notation $\to$ indicates the result after applying the standard trace, and the appropriate part of Th. 2.

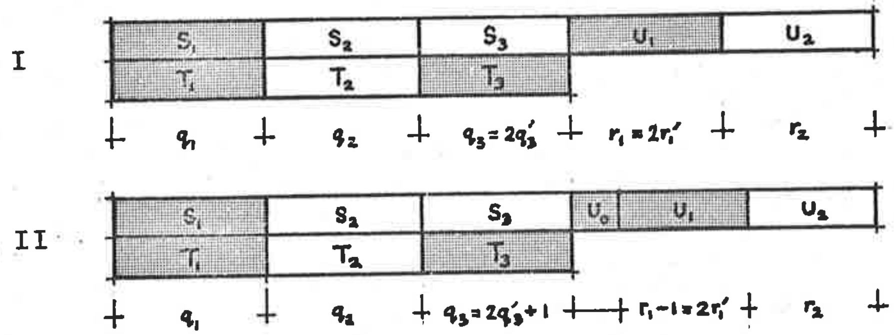
Adding these results, and using $\langle a\rangle = n$, gives

$$(XX) = n_2^2 + 2n_1n_2 + (N-2)n_2 , \qquad (45)$$

$$\langle a\bar{a}\rangle = n(n+N-1) - n_1(n_1+N-2), \tag{46}$$

in agreement with the general result, Eq. (55).

In general the standard trace labelled $(\lambda;\ell,m)$ has the form given by Eq. (2.29),



(47)

depending upon whether q_3, r_1 are both even or both odd, where shaded portions indicate contractions, and from Eqs. (2.32), (2.38),

$$q_1 + q_2 + q_3 = q\ , \tag{48}$$
$$r_1 + r_2 = r = p - q\ ;$$

$$q_1 = 2\mu = (p+q) - (\ell+m) - 2\lambda\ ,$$
$$q_2 = \kappa + \max(0, q-2\lambda) = m\ ,$$
$$q_3 = q - 2\mu - \kappa - \max(0,q-2\lambda) = \ell - (p-2\lambda),$$
$$r_1 = \kappa + \max(0,q-2\lambda) - (q-2\lambda) = m - (q-2\lambda),$$
$$r_2 = p - 2\lambda - \kappa - \max(0,q-2\lambda) = (p-2\lambda) - m\ ; \tag{49}$$

$$\ell = q_2 + q_3 + r_2\ ,$$
$$m = q_2\ . \tag{50}$$

In terms of substitution operators in this scheme, to be evaluated is

$$\Phi = g^{ss'}\binom{i\,j}{s\,s'}g_{ij} + g^{tt'}\binom{i\,j}{t\,t'}g_{ij} + g^{uu'}\binom{i\,j}{u\,u'}g_{ij} + 2g^{st}\binom{i\,j}{s\,t}g_{ij} + 2g^{su}\binom{i\,j}{s\,u}g_{ij} + 2g^{tu}\binom{i\,j}{t\,u}g_{ij}\ , \qquad (51)$$

$$\Phi_I = \binom{S_1S_1}{T_1T_1} + \binom{S_2S_2}{T_2T_2} + \binom{S_3S_3}{T_3T_3} + \binom{S_1S_2}{T_1T_2} + \binom{S_1S_3}{T_1T_3} + \binom{S_2S_3}{T_2T_3} + (UU) + \binom{S_1U_1}{T_1} + \binom{S_1U_2}{T_1} + \binom{S_2U_1}{T_2} + \binom{S_2U_2}{T_2} + \binom{S_3U_1}{T_3} + \binom{S_3U_2}{T_3}. \qquad (52)$$

$$\Phi_{II} = \binom{S_1S_1}{T_1T_1} + \binom{S_2S_2}{T_2T_2} + \binom{S_3U_0S_3U_0}{T_3\ \ T_3} + \binom{S_1S_2}{T_1T_2} + \binom{S_1S_3U_0}{T_1T_3} + \binom{S_2S_3U_0}{T_2T_3} + (UU) + \binom{S_1U_1}{T_1} + \binom{S_1U_2}{T_1} + \binom{S_2U_1}{T_2} + \binom{S_2U_2}{T_2} + \binom{S_3U_0U_1}{T_3} + \binom{S_3U_0U_2}{T_3}. \qquad (53)$$

The details of the calculation are given in Sec. A4.3. The method used is the same as in the above example. The symmetry properties of traces of two-rowed tensors, Eqs. (2.20), are also needed. The result for both cases is

$$\Phi = 2q_1^2 + q_3^2 + 4q_1q_2 + 4q_1q_3 + 2q_2q_3 + 2(N-3)q_1 + (N-4)q_3 + r_1^2 + 2r_1r_2 + (N-2)r_1 + 2q_1r_1 + 2q_2r_1 + 2q_3r_1 + 2q_1r_2\ , \qquad (54)$$

$$\langle a\bar{a}\rangle = (q_1+q_2+q_3+r_1+r_2)(q_1+q_2+q_3+r_1+r_2+N-1) + (q_1+q_2+q_3)(q_1+q_2+q_3+N-3) - (q_2+q_3+r_2)(q_2+q_3+r_2+N-2) - q_2(q_2+N-4),$$

or, using Eqs. (48), (50),

$$\langle a\bar{a}\rangle = p(p+N-1) + q(q+N-3) - \ell(\ell+N-2) - m(m+N-4). \qquad (55)$$

The result is, as required, in agreement with the identity

$$\langle a\bar{a}\rangle = \langle a^2\rangle - \tfrac{1}{2}\langle\alpha^2\rangle\ , \qquad (56)$$

since the right-hand terms are[2,9]

$$\langle a^2 \rangle = p(p+N-1) + q(q+N-3),$$

$$\langle \alpha^2 \rangle = 2\ell(\ell+N-2) + 2m(m+N-4), \qquad (57)$$

in two-rowed representations.

4.4 O(N) Invariant $\frac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a} \rangle$

From Sec. 3.2, there is a single independent additional O(N) invariant which is cubic in the U(N) generators a and $\bar{a}$, taken to be $\frac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a} \rangle$. For computational purposes it is more convenient to evaluate $\langle \bar{a}a^2 \rangle$, for which, from Eq. (3.19),

$$\tfrac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a} \rangle = \langle \bar{a}a^2 \rangle - \tfrac{N}{2}\langle a\bar{a} \rangle + \tfrac{1}{2}\langle a \rangle^2. \qquad (58)$$

In terms of substitution operators, from Eq. (18),

$$a^{2\,i}{}_j = (q+N-2)a^i{}_j + (p-q+1)\delta^u{}_j\left(\begin{smallmatrix} i \\ u \end{smallmatrix}\right) + \left[\delta^s{}_j\left(\begin{smallmatrix} i & u \\ u & s \end{smallmatrix}\right) + \delta^t{}_j\left(\begin{smallmatrix} i & u \\ u & t \end{smallmatrix}\right)\right], \qquad (59)$$

$$\langle \bar{a}a^2 \rangle = a^i{}_j\, a^{2\,k}{}_\ell\, g^{j\ell} g_{ik},$$

$$\langle \bar{a}a^2 \rangle = (q+N-2)\langle a\bar{a} \rangle + (p-q+1)\, g^{uj} a^i{}_j \left(\begin{smallmatrix} k \\ u \end{smallmatrix}\right) g_{ik} + \left[g^{sj} a^i{}_j \left(\begin{smallmatrix} k & u \\ u & s \end{smallmatrix}\right) + g^{tj} a^i{}_j \left(\begin{smallmatrix} k & u \\ u & t \end{smallmatrix}\right)\right] g_{ik}, \qquad (60)$$

$$g^{uj} a^i{}_j \left(\begin{smallmatrix} k \\ u \end{smallmatrix}\right) g_{ik} = (p-q) + \left[g^{us}\left(\begin{smallmatrix} i & k \\ u & s \end{smallmatrix}\right) g_{ik} + g^{ut}\left(\begin{smallmatrix} i & k \\ u & t \end{smallmatrix}\right) g_{ik}\right] + \left[g^{uu'}\left(\begin{smallmatrix} i & k \\ u & u' \end{smallmatrix}\right) g_{ik}\right], \qquad (61)$$

$$g^{sj} a^i{}_j \left(\begin{smallmatrix} k & u \\ u & s \end{smallmatrix}\right) g_{ik} + g^{tj} a^i{}_j \left(\begin{smallmatrix} k & u \\ u & t \end{smallmatrix}\right) g_{ik} =$$

$$\left[g^{ss'}\left(\begin{smallmatrix} i & u & j \\ u & s & s' \end{smallmatrix}\right) + g^{st}\left(\begin{smallmatrix} i & u & j \\ u & s & t \end{smallmatrix}\right) + g^{ts}\left(\begin{smallmatrix} i & u & j \\ u & t & s \end{smallmatrix}\right) g_{ij} + g^{su}\left(\begin{smallmatrix} i & u' & j \\ u' & s & u \end{smallmatrix}\right) + g^{tu}\left(\begin{smallmatrix} i & u' & j \\ u' & t & u \end{smallmatrix}\right) + g^{tt'}\left(\begin{smallmatrix} i & u & j \\ u & t & t' \end{smallmatrix}\right)\right] g_{ij} +$$

$$\left[g^{su}\left(\begin{smallmatrix} i & j \\ s & u \end{smallmatrix}\right) + g^{tu}\left(\begin{smallmatrix} i & j \\ t & u \end{smallmatrix}\right)\right] g_{ij} + \left[\left(\begin{smallmatrix} s & u \\ u & s \end{smallmatrix}\right) + \left(\begin{smallmatrix} t & u \\ u & t \end{smallmatrix}\right)\right], \qquad (62)$$

and obtaining the last term from Sec. A4.2,

$$\langle \bar{a}a^2 \rangle = (p-q)(p+1) + (q+N-2)\langle a\bar{a} \rangle + (p-q+1)\Phi_I + (p+q-2)\Phi_{II} + \Phi_{III} , \quad (63)$$

$$\tfrac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a} \rangle = (p-q)(p+1) + \tfrac{1}{2}(p+q)^2 + (q+\tfrac{N}{2}-2)\langle a\bar{a} \rangle + (p-q+1)\Phi_I + (p+q-2)\Phi_{II} + \Phi_{III} , \quad (64)$$

$$\Phi_I = [g^{uu'}(\begin{smallmatrix} i\,j \\ u\,u' \end{smallmatrix}) g_{ij}] , \quad (65)$$

$$\Phi_{II} = [g^{su}(\begin{smallmatrix} i\,j \\ s\,u \end{smallmatrix}) + g^{tu}(\begin{smallmatrix} i\,j \\ t\,u \end{smallmatrix})] g_{ij} ,$$

$$\Phi_{III} = [g^{ss'}(\begin{smallmatrix} i\,u\,j \\ u\,s\,s' \end{smallmatrix}) + g^{st}(\begin{smallmatrix} i\,u\,j \\ u\,s\,t \end{smallmatrix}) + g^{ts}(\begin{smallmatrix} i\,u\,j \\ u\,t\,s \end{smallmatrix}) + g^{su}(\begin{smallmatrix} i\,u'\,j \\ u'\,s\,u \end{smallmatrix}) + g^{tu}(\begin{smallmatrix} i\,u'\,j \\ u'\,t\,u \end{smallmatrix}) + g^{tt'}(\begin{smallmatrix} i\,u\,j \\ u\,t\,t' \end{smallmatrix})] g_{ij}.$$

Φ_I and Φ_{II} have been found in Sec. 4.3. Details of the calculation of Φ_{III} are given in Sec. A4.4 for the case when q_3, r_1 are both even. In the scheme of Eq. (47),

$$\Phi_I = (UU) = (U_1U_1) + (U_2U_2) + (U_1U_2),$$

$$2\Phi_{II} = (\begin{smallmatrix} S_1U_1 \\ T_1 \end{smallmatrix}) + (\begin{smallmatrix} S_1U_2 \\ T_1 \end{smallmatrix}) + (\begin{smallmatrix} S_2U_1 \\ T_2 \end{smallmatrix}) + (\begin{smallmatrix} S_2U_2 \\ T_2 \end{smallmatrix}) + (\begin{smallmatrix} S_3U_1 \\ T_3 \end{smallmatrix}) + (\begin{smallmatrix} S_3U_2 \\ T_3 \end{smallmatrix}) ,$$

$$\Phi_{III} = (\begin{smallmatrix} S_1S_1 \\ T_1T_1 \end{smallmatrix}|U) + (\begin{smallmatrix} S_2S_2 \\ T_2T_2 \end{smallmatrix}|U) + (\begin{smallmatrix} S_3S_3 \\ T_3T_3 \end{smallmatrix}|U) + (\begin{smallmatrix} S_1S_2 \\ T_1T_2 \end{smallmatrix}|U) + (\begin{smallmatrix} S_1S_3 \\ T_1T_3 \end{smallmatrix}|U) + (\begin{smallmatrix} S_2S_3 \\ T_2T_3 \end{smallmatrix}|U) +$$
$$(\begin{smallmatrix} S_1U_1 \\ T_1 \end{smallmatrix}|U) + (\begin{smallmatrix} S_1U_2 \\ T_1 \end{smallmatrix}|U) + (\begin{smallmatrix} S_2U_1 \\ T_2 \end{smallmatrix}|U) + (\begin{smallmatrix} S_2U_2 \\ T_2 \end{smallmatrix}|U) + (\begin{smallmatrix} S_3U_1 \\ T_3 \end{smallmatrix}|U) + (\begin{smallmatrix} S_3U_2 \\ T_3 \end{smallmatrix}|U), \quad (66)$$

with values

$$\Phi_I = r_1^2 + 2r_1r_2 + (N-2)r_1 , \quad (67)$$

$$\Phi_{II} = q_1r_1 + q_2r_1 + q_3r_1 + q_1r_2 , \quad (68)$$

$$\Phi_{III} = (q_1^2r_1 + q_1^2r_2 + q_3^2r_1 + q_1r_1^2 + q_2r_1^2 + q_3r_1^2 + 2q_1q_2r_1 + 2q_1q_2r_2 + 2q_1q_3r_1 +$$
$$2q_1q_3r_2 + 2q_2q_3r_1 + 2q_1r_1r_2 + q_2r_1r_2 + q_3r_1r_2 + (N-3)q_1r_1 + (N-3)q_1r_2 + (N-4)q_3r_1)$$
$$+ 2\Lambda^+(q_1r_2^2 - q_1r_2) - \tfrac{1}{2}\Lambda^-(q_3^2r_1 + 2q_2q_3r_1 + (N-4)q_3r_1), \quad (69)$$

Φ_{III} being evaluated in accordance with Eqs. (2), (3) and Th. 2. The parameters q_1, q_2, q_3, r_1, r_2 are linear functions of $p, q; \lambda; \ell, m$, given by Eqs. (49).

4.5 Operational Definition of Λ

The evaluation in Sec. 4.4 of the cubic O(N) invariant $\frac{1}{2}\langle a\bar{a}\bar{a} + \bar{a}a\bar{a}\rangle$, by the techniques developed in this chapter, gives an expression of the form

$$\Phi = \Lambda^+ f^+(p,q;\lambda;\ell,m) + \Lambda^- f^-(p,q;\lambda;\ell,m), \qquad (70)$$

where the lengthy constant term in Eq. (69) has been absorbed in the invariant term Φ on the left hand side. The functions $f^{\pm}$ are polynomials in the labels $N; p,q;\lambda;\ell,m;$ in terms of the related parameters $N; q_1,q_2,q_3,r_1,r_2$,

$$f^+ = 2q_1 r_2(r_2-1) = 2(p-2\lambda-m)(p-2\lambda-m-1)(p+q-2\lambda-\ell-m),$$

$$f^- = -\tfrac{1}{2} q_3 r_1(q_3+2q_2+N-4) = \tfrac{1}{2}(p-2\lambda-\ell)(q-2\lambda-m)(p-2\lambda-\ell-2m-N+4). \qquad (71)$$

Another equation in the shift operators $\Lambda^{\pm}$ can be found by commuting Eq. (70) with the labelling operator Λ:

$$[\Lambda, \Lambda^{\pm}] = \pm\Lambda^{\pm}. \qquad (72)$$

$$[\Lambda, \Phi] = \Lambda^+ f^+(p,q;\lambda;\ell,m) - \Lambda^- f^-(p,q;\lambda;\ell,m). \qquad (73)$$

Eqs. (70) and (73) give

$$4\Lambda^+\Lambda^- f^+ f^- + ([\Lambda,\Phi] + \Phi)([\Lambda,\Phi] - \Phi) = 0 \qquad (74)$$

Now in the abstract basis $\left|{}^{p}_{q}\lambda\binom{\ell}{m}\right\rangle$ for the irreducible representation O(N)(p,q;λ ; ℓ ,m),

$$\langle {}^{p}_{q}\lambda'({}^{\ell'}_{m'})|\Lambda^{+}\Lambda^{-}|{}^{p}_{q}\lambda({}^{\ell}_{m})\rangle = \langle {}^{p}_{q}\lambda'({}^{\ell'}_{m'})|{}^{p}_{q}\lambda({}^{\ell}_{m})\rangle \quad (2\lambda > 0,\ 2\lambda' < p-1)$$

$$\langle {}^{p}_{q}\lambda'({}^{\ell'}_{m'})|\Lambda^{+}\Lambda^{-}|{}^{p}_{q}\lambda({}^{\ell}_{m})\rangle = 0 \quad (\text{otherwise}) . \tag{75}$$

from the definition of $\Lambda^{\pm}$. However, since from Eq. (49)

$$\begin{aligned} q_3 &= 0 \ (\lambda = 0) , \\ r_2 &= 0 \ (2\lambda = p) , \\ r_2 &= 1 \ (2\lambda = p-1) , \end{aligned} \tag{76}$$

then the product $f^{+}f^{-}$ vanishes, whenever λ takes on its maximum or minimum value:

$$f^{+}f^{-} = 0 \quad (2\lambda = 0,\ 2\lambda = p \text{ or } 2\lambda = p-1). \tag{77}$$

Hence, in the irreducible representation O(N)(p,q;λ ; ℓ ,m),

$$\langle {}^{p}_{q}\lambda'({}^{\ell'}_{m'})|\Lambda^{+}\Lambda^{-}f^{+}f^{-}|{}^{p}_{q}\lambda({}^{\ell}_{m})\rangle = \langle {}^{p}_{q}\lambda'({}^{\ell'}_{m'})|f^{+}f^{-}|{}^{p}_{q}\lambda({}^{\ell}_{m})\rangle \tag{78}$$

so that Eq. (74) becomes

$$4f^{+}f^{-}(P,Q;\Lambda;L,M) + ([\Lambda,\Phi] + \Phi)([\Lambda,\Phi] - \Phi) = 0 \tag{79}$$

providing an implicit operational definition of Λ .

Another, somewhat more explicit, definition can be given when there exists a second equation

$$\Psi = \Lambda^{+}g^{+}(p,q;\lambda;\ell,m) + \Lambda^{-}g^{-}(p,q;\lambda;\ell,m) \tag{80}$$

of the type of Eq. (70). For example, Sec. 3.3 shows that, for two-rowed representations, there is just one independent additional quartic O(N) invariant, taken to be

$\frac{1}{2}\langle a\bar{a}a\bar{a} + \bar{a}a\bar{a}a\rangle$; an evaluation of this along the lines of Sec. 4.4 would produce such an equation. Then combining Eqs. (70) and (80),

$$\Lambda^{+}\Lambda^{-}(f^{+}g^{-} - g^{+}f^{-})^{2} + (\Phi g^{-} - \Psi f^{-})(\Phi g^{+} - \Psi f^{+}) = 0 , \qquad (81)$$

leading to an implicit operational definition of Λ in the irreducible representation $O(N)(p,q;\lambda;\ell,m)$,

$$(f^{+}g^{-}(\Lambda) - g^{+}f^{-}(\Lambda))^{2} + (\Phi g^{-}(\Lambda) - \Psi f^{-}(\Lambda))(\Phi g^{+}(\Lambda) - \Psi f^{+}(\Lambda)) = 0 , \qquad (82)$$

provided that the products $f^{+}g^{-}$, $g^{+}f^{-}$ satisfy the condition of Eq. (77).

A disclaimer must be added in connection with Eqs. (79) and (82). Since a definite choice of normalisation of the abstract basis $\left|{}^{p}_{q}\lambda\binom{\ell}{m}\right\rangle$ has not been given, the operator Λ appearing in Eqs. (79) and (82) should possibly be accompanied by a normalisation factor which is a further polynomial function of the labels. This could also be accomplished by a subsequent redefinition of the unnormalised labelling operator Λ .

Finally, it should be remarked also that, by the arguments of Sec. 2.3, and the explicit construction of common eigenstates of Sec. 2.6, the operator Λ , introduced by Eq. (79) or (82), does indeed commute with the other labelling operators, as required.

APPENDICES

A2.6

The following results show the reduction of several low-rank irreducible U(N) tensors of two-rowed symmetry type, to their irreducible O(N) constituents, according to the method outlined in Sec. 2.6.

Tensor components are arranged between brackets $[\;]$ to indicate the symmetry type. Traceless and double-traceless tensors are written $\overline{[\;]}$ and $\overline{\overline{[\;]}}$ respectively. All indices are assumed to be contravariant. Repeated indices indicate contraction with the metric tensor. Dashes indicate that the tensor is obtained by application of an appropriate Young operator to a trace of a higher-rank tensor.

The Young operators have variable (but definite) normalisation. Thus $Y\left[\begin{smallmatrix}4\\0\end{smallmatrix}\right][g^{x_1x_2}[x_3x_4]]$ has only $6 = \frac{4!}{2!\,2!}$ distinct terms, and the highest common factor 4 is divided out.

$\underline{[2,0] = (2,0) \oplus (0,0)}$

$$\overline{[x_1x_2]} = [x_1x_2] - \xi^{(0)} g^{x_1x_2}\overline{[--]}$$

$$\overline{[--]} = [ii]$$

$$\xi^{(0)} = +\frac{1}{N}$$

$\underline{[3,0] = (3,0) \oplus (1,0)}$

$$\overline{[x_1x_2x_3]} = [x_1x_2x_3] - \xi^{(1)} Y[g^{x_2x_3}\overline{[x_1--]}]$$

$$\overline{[x_1--]} = [x_1ii]$$

$$\xi^{(1)} = +\frac{1}{(N+2)}$$

$[2,1] = (2,1) \oplus (1,0)$

$$\overline{\left[\begin{smallmatrix} x_1 & x_2 \\ x_3 & \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 & x_2 \\ x_3 & \end{smallmatrix}\right] - \xi^{(1)} Y \left[g^{x_3 x_2} \overline{\left[\begin{smallmatrix} x_1 & - \\ - & \end{smallmatrix}\right]} \right]$$

$$\overline{\left[\begin{smallmatrix} x_1 & - \\ - & \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 & i \\ i & \end{smallmatrix}\right]$$

$$\xi^{(1)} = + \frac{1}{(N-1)}$$

$[4,0] = (4,0) \oplus (2,0) \oplus (0,0)$

$$\overline{[x_1 x_2 x_3 x_4]} = [x_1 x_2 x_3 x_4] - \xi^{(2)} Y \left[g^{x_3 x_4} \overline{[x_1 x_2 - -]} \right] - \xi^{(0)} Y \left[g^{x_1 x_2} g^{x_3 x_4} \overline{[- - - -]} \right]$$

$$\overline{[x_1 x_2 - -]} = [x_1 x_2 i\, i] - \frac{1}{N} g^{x_1 x_2} [i\, i\, j\, j]$$

$$\overline{[- - - -]} = [i\, i\, j\, j]$$

$$\xi^{(2)} = + \frac{1}{(N+4)} \qquad \xi^{(0)} = + \frac{2}{N(N+2)(N+4)}$$

$[3,1] = (3,1) \oplus (2,0) \oplus (1,1)$

$$\overline{\left[\begin{smallmatrix} x_1 & x_2 & x_3 \\ x_4 & & \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 & x_2 & x_3 \\ x_4 & & \end{smallmatrix}\right] - \xi^{(20)} Y \left[g^{x_4 x_3} \overline{\left[\begin{smallmatrix} x_1 & x_2 & - \\ - & & \end{smallmatrix}\right]} \right] - \xi^{(11)} Y \left[g^{x_2 x_3} \overline{\left[\begin{smallmatrix} x_1 & - & - \\ x_4 & & \end{smallmatrix}\right]} \right]$$

$$\overline{\left[\begin{smallmatrix} x_1 & x_2 & - \\ - & & \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 & x_2 & i \\ i & & \end{smallmatrix}\right] + \left[\begin{smallmatrix} x_2 & x_1 & i \\ i & & \end{smallmatrix}\right]$$

$$\overline{\left[\begin{smallmatrix} x_1 & - & - \\ x_4 & & \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 & i & i \\ x_4 & & \end{smallmatrix}\right]$$

$$\xi^{(20)} = + \frac{1}{2N} \qquad \xi^{(11)} = + \frac{1}{2(N+2)}$$

$[2,2] = (2,2) \oplus (2,0) \oplus (0,0)$

$$\overline{\left[\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right] - \xi^{(20)} Y \left[g^{x_3 x_4} \overline{\left[\begin{smallmatrix} x_1 & x_2 \\ - & - \end{smallmatrix}\right]} \right] - \xi^{(0)} Y \left[g^{x_1 x_2} g^{x_3 x_4} \overline{\left[\begin{smallmatrix} - & - \\ - & - \end{smallmatrix}\right]} \right]$$

$$\overline{\left[\begin{smallmatrix} x_1 & x_2 \\ - & - \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 & x_2 \\ i & i \end{smallmatrix}\right] - \frac{1}{N} g^{x_1 x_2} \left[\begin{smallmatrix} i & i \\ j & j \end{smallmatrix}\right]$$

$$\overline{\left[\begin{smallmatrix} - & - \\ - & - \end{smallmatrix}\right]} = \left[\begin{smallmatrix} i & i \\ j & j \end{smallmatrix}\right]$$

$$\xi^{(2)} = + \frac{1}{(N-2)} \qquad \xi^{(0)} = + \frac{1}{N(N-1)}$$

$[3,2] = (3,2) \oplus (3,0) \oplus (2,1) \oplus (1,0)$

$$\overline{\left[\begin{smallmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & \end{smallmatrix}\right] - \xi^{(30)} Y \left[g^{x_4 x_5} \overline{\left[\begin{smallmatrix} x_1 & x_2 & x_3 \\ - & - & \end{smallmatrix}\right]} \right] - \xi^{(21)} Y \left[g^{x_5 x_3} \overline{\left[\begin{smallmatrix} x_1 & x_2 & - \\ x_4 & - & \end{smallmatrix}\right]} \right] -$$
$$- \xi^{(10)} Y \left[g^{x_1 x_2} g^{x_4 x_5} \overline{\left[\begin{smallmatrix} - & - & x_3 \\ - & - & \end{smallmatrix}\right]} \right]$$

$$\overline{\left[\begin{smallmatrix} x_1 & x_2 & x_3 \\ - & - & \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 & x_2 & x_3 \\ i & i & \end{smallmatrix}\right] + \left[\begin{smallmatrix} x_2 & x_3 & x_1 \\ i & i & \end{smallmatrix}\right] + \left[\begin{smallmatrix} x_3 & x_1 & x_2 \\ i & i & \end{smallmatrix}\right] -$$
$$- \frac{2}{(N+2)} \left(g^{x_1 x_2} \left[\begin{smallmatrix} i & i & x_3 \\ j & j & \end{smallmatrix}\right] + g^{x_2 x_3} \left[\begin{smallmatrix} i & i & x_1 \\ j & j & \end{smallmatrix}\right] + g^{x_3 x_1} \left[\begin{smallmatrix} i & i & x_2 \\ j & j & \end{smallmatrix}\right] \right)$$

$$\overline{\left[\begin{smallmatrix} x_1 x_2 - \\ x_4 - \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 x_2 i \\ x_4 i \end{smallmatrix}\right] - \frac{1}{2(N-1)}\left(g^{x_1x_2}\left[\begin{smallmatrix} i\,i\; x_4 \\ j\,j \end{smallmatrix}\right] - g^{x_4x_2}\left[\begin{smallmatrix} i\,i\; x_1 \\ j\,j \end{smallmatrix}\right] \right)$$

$$\overline{\left[\begin{smallmatrix} -\,-\; x_3 \\ -\,- \end{smallmatrix}\right]} = \left[\begin{smallmatrix} i\,i\; x_3 \\ j\,j \end{smallmatrix}\right]$$

$$\xi^{(30)} = +\frac{1}{3(N-2)} \qquad \xi^{(21)} = +\frac{1}{3(N+1)} \qquad \xi^{(10)} = +\frac{1}{2(N-1)(N+2)}$$

$[3,3] = (3,3) \oplus (3,1) \oplus (1,1)$

$$\overline{\left[\begin{smallmatrix} x_1 x_2 x_3 \\ x_4 x_5 x_6 \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 x_2 x_3 \\ x_4 x_5 x_6 \end{smallmatrix}\right] - \xi^{(31)}\, Y\left[g^{x_5x_6} \overline{\left[\begin{smallmatrix} x_1 x_2 x_3 \\ x_4 -\,- \end{smallmatrix}\right]} \right] - \xi^{(11)}\, Y\left[g^{x_2x_3} g^{x_5x_6} \overline{\left[\begin{smallmatrix} x_1 -\,- \\ x_4 -\,- \end{smallmatrix}\right]} \right]$$

$$\overline{\left[\begin{smallmatrix} x_1 x_2 x_3 \\ x_4 -\,- \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 x_2 x_3 \\ x_4\, i\;\, i \end{smallmatrix}\right] - \frac{1}{2(N+2)}\left(g^{x_2x_3}\left[\begin{smallmatrix} x_1\, i\,i \\ x_4\, j\,j \end{smallmatrix}\right] + g^{x_1x_3}\left[\begin{smallmatrix} x_2\, i\,i \\ x_4\, j\,j \end{smallmatrix}\right] + g^{x_3x_4}\left[\begin{smallmatrix} x_1\, i\,i \\ x_2\, j\,j \end{smallmatrix}\right] + \right.$$
$$\left. + g^{x_2x_3}\left[\begin{smallmatrix} x_1\, i\,i \\ x_4\, j\,j \end{smallmatrix}\right] + g^{x_1x_2}\left[\begin{smallmatrix} x_3\, i\,i \\ x_4\, j\,j \end{smallmatrix}\right] + g^{x_2x_4}\left[\begin{smallmatrix} x_1\, i\,i \\ x_3\, j\,j \end{smallmatrix}\right] \right)$$

$$\overline{\left[\begin{smallmatrix} x_1 -\,- \\ x_4 -\,- \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1\, i\,i \\ x_4\, j\,j \end{smallmatrix}\right]$$

$$\xi^{(31)} = +\frac{1}{N} \qquad \xi^{(11)} = +\frac{2}{2(N+1)(N+2)}$$

$[4,2] = (4,2) \oplus (4,0) \oplus (3,1) \oplus (2,2) \oplus (2,0)^2 \oplus (0,0)$

$$\overline{\overline{\left[\begin{smallmatrix} x_1 x_2 x_3 x_4 \\ x_5 x_6 \end{smallmatrix}\right]}} = \left[\begin{smallmatrix} x_1 x_2 x_3 x_4 \\ x_5 x_6 \end{smallmatrix}\right] - \xi_1^{(1;20)}\, Y\left[g^{x_1x_2} g^{x_5x_6} \overline{\left[\begin{smallmatrix} -\,-\; x_3 x_4 \\ -\,- \end{smallmatrix}\right]} \right] - \xi_2^{(1;20)}\, Y\left[g^{x_1x_6} g^{x_2x_5} \overline{\left[\begin{smallmatrix} -\,-\; x_3 x_4 \\ -\,- \end{smallmatrix}\right]} \right]$$
$$- \xi_1^{(2;20)}\, Y\left[g^{x_3x_4} g^{x_5x_6} \overline{\left[\begin{smallmatrix} x_1 x_2 -\,- \\ -\,- \end{smallmatrix}\right]} \right] - \xi_2^{(2;20)}\, Y\left[g^{x_3x_4} g^{x_1x_2} \overline{\left[\begin{smallmatrix} x_5 x_6 -\,- \\ -\,- \end{smallmatrix}\right]} \right]$$
$$- \xi^{(00)}\, Y\left[g^{x_1x_2} g^{x_3x_4} g^{x_5x_6} \overline{\left[\begin{smallmatrix} -\,-\,-\,- \\ -\,- \end{smallmatrix}\right]} \right]$$

$$\overline{\left[\begin{smallmatrix} -\,-\; x_3 x_4 \\ -\,- \end{smallmatrix}\right]} = \left[\begin{smallmatrix} i\,i\; x_3 x_4 \\ j\,j \end{smallmatrix}\right] - \frac{1}{N}\, g^{x_3x_4}\left[\begin{smallmatrix} i\,i\; k\,k \\ j\,j \end{smallmatrix}\right]$$

$$\overline{\left[\begin{smallmatrix} x_1 x_2 -\,- \\ -\,- \end{smallmatrix}\right]} = \left[\begin{smallmatrix} x_1 x_2\, j\,j \\ i\;\, i \end{smallmatrix}\right] - \frac{1}{N}\, g^{x_1x_2}\left[\begin{smallmatrix} k\,k\; j\,j \\ i\,i \end{smallmatrix}\right]$$

$$\overline{\left[\begin{smallmatrix} -\,-\,-\,- \\ -\,- \end{smallmatrix}\right]} = \left[\begin{smallmatrix} i\,i\; k\,k \\ j\,j \end{smallmatrix}\right]$$

$$\xi_1^{(1;20)} = -\frac{1}{N(N+1)(N+4)} \qquad \xi_2^{(1;20)} = -\frac{1}{2N(N+1)}$$

$$\xi_1^{(2;20)} = +\frac{2}{N(N+1)(N-2)(N+4)} \qquad \xi_2^{(2;20)} = +\frac{(N-1)}{N(N+1)(N-2)} \qquad \xi^{(00)} = +\frac{1}{2N(N-1)(N+2)}$$

$$\overline{\left[\begin{smallmatrix} x_1 x_2 x_3 x_4 \\ x_5 x_6 \end{smallmatrix}\right]} = \overline{\overline{\left[\begin{smallmatrix} x_1 x_2 x_3 x_4 \\ x_5 x_6 \end{smallmatrix}\right]}} - \xi^{(40)}\, Y\left[g^{x_5x_6} \overline{\left[\begin{smallmatrix} x_1 x_2 x_3 x_4 \\ -\,- \end{smallmatrix}\right]} \right] - \xi^{(31)}\, Y\left[g^{x_6x_4} \overline{\left[\begin{smallmatrix} x_1 x_2 x_3 - \\ x_5 - \end{smallmatrix}\right]} \right] -$$
$$- \xi^{(22)}\, Y\left[g^{x_3x_4} \overline{\left[\begin{smallmatrix} x_1 x_2 -\,- \\ x_5 x_6 \end{smallmatrix}\right]} \right]$$

$$\overline{\left[\begin{smallmatrix} x_1 x_2 x_3 x_4 \\ -\,- \end{smallmatrix}\right]} = \overline{\overline{\left[\begin{smallmatrix} x_1 x_2 x_3 x_4 \\ i\; i \end{smallmatrix}\right]}} + \overline{\overline{\left[\begin{smallmatrix} x_3 x_4 x_1 x_2 \\ i\; i \end{smallmatrix}\right]}} + \overline{\overline{\left[\begin{smallmatrix} x_1 x_3 x_2 x_4 \\ i\; i \end{smallmatrix}\right]}} + \overline{\overline{\left[\begin{smallmatrix} x_2 x_4 x_1 x_3 \\ i\; i \end{smallmatrix}\right]}} + \overline{\overline{\left[\begin{smallmatrix} x_1 x_4 x_2 x_3 \\ i\; i \end{smallmatrix}\right]}} + \overline{\overline{\left[\begin{smallmatrix} x_2 x_3 x_1 x_4 \\ i\; i \end{smallmatrix}\right]}}$$

$$\overline{\left[\begin{smallmatrix} x_1 x_2 x_3 - \\ x_5 - \end{smallmatrix}\right]} = \overline{\overline{\left[\begin{smallmatrix} x_1 x_2 x_3\, i \\ x_5\, i \end{smallmatrix}\right]}} + \overline{\overline{\left[\begin{smallmatrix} x_1 x_3 x_2\, i \\ x_5\, i \end{smallmatrix}\right]}}$$

$$\overline{\left[\begin{smallmatrix} x_1 x_2 -\,- \\ x_5 x_6 \end{smallmatrix}\right]} = \overline{\overline{\left[\begin{smallmatrix} x_1 x_2\, i\, i \\ x_5 x_6 \end{smallmatrix}\right]}}$$

$$\xi^{(40)} = +\frac{1}{6(N-2)} \qquad \xi^{(31)} = +\frac{1}{8(N+2)} \qquad \xi^{(22)} = +\frac{1}{6(N+4)}$$

A3.1

Theorem 2:

$$\overline{a^{n+1}} = \bar{a}\,\overline{a^n} + N\overline{a^n} - \langle a^n\rangle \qquad \forall n \geq 1$$

$$\langle a^{n+1} - \bar{a}^{n+1}\rangle = \langle \bar{a}(\overline{a^n} - \bar{a}^n)\rangle \qquad \forall n \geq 1$$

$$\langle \bar{a}^m a^n\rangle = \langle a^n \bar{a}^m\rangle \qquad \forall m,n \geq 1$$

Proof:

(i)

$$\overline{a^{n+1}}{}^i{}_k = a^{n+1}{}_k{}^i$$
$$= a^n{}_k{}^j a_j{}^i$$
$$= a_j{}^i a^n{}_k{}^j - [a_j{}^i, a^n{}_k{}^j]$$
$$= \bar{a}\,\overline{a^n}{}^i{}_k - [\delta^i{}_k\langle a^n\rangle - N\overline{a^n}{}^i{}_k]$$

(ii)

$$\langle\overline{a^{n+1}}\rangle = \langle\bar{a}\,\overline{a^n}\rangle + \langle N\overline{a^n} - \langle a^n\rangle\rangle = \langle\bar{a}\,\overline{a^n}\rangle$$

(iii)

$$\langle a^m\bar{a}\rangle = a^m{}_{ij}a^{ij} = a^{ij}a^m{}_{ij} - (g^{ij}a^m{}_{ij} - g^{ij}a^m{}_{ij}) = \langle\bar{a}a^m\rangle.$$

$$\langle a^m\bar{a}^{n'}\rangle = \langle\bar{a}^{n'}a^m\rangle \quad \forall n' \leq n-1.$$

$$\langle a^m\bar{a}^n\rangle = a^m{}_{ij}a^{kj}(\bar{a}^{n-1})_k{}^i$$
$$= a^{kj}a^m{}_{ij}(\bar{a}^{n-1})_k{}^i - [(a^m)^k{}_i(\bar{a}^{n-1})_k{}^i - (a^m)_i{}^k(\bar{a}^{n-1})_k{}^i]$$
$$= a^{kj}a^m{}_{ij}a^\ell{}_k(\bar{a}^{n-2})_\ell{}^i - \langle\overline{a^m}\,\bar{a}^{n-1}\rangle + \langle\overline{a^m}\,\overline{\bar{a}^{n-1}}\rangle$$
$$= (\bar{a}^2)^{j\ell}a^m{}_{ij}(\bar{a}^{n-2})_\ell{}^i - \bar{a}^{jk}[g_{ik}(a^m)^\ell{}_j - \delta^\ell{}_j(a^m)_{ik}](\bar{a}^{n-2})_\ell{}^i - (\cdots)$$
$$= (\bar{a}^2)^{j\ell}a^m{}_{ij}(\bar{a}^{n-2})_\ell{}^i - \langle\bar{\bar{a}}\,\overline{a^m}\,\bar{a}^{n-2}\rangle + \langle\bar{a}\,\overline{a^m}\,\overline{\bar{a}^{n-2}}\rangle - (\cdots)$$
$$= \cdots\cdots\cdots\cdots\cdots\cdots$$
$$= \langle\bar{a}^n a^m\rangle - \langle\overline{\bar{a}^{n-1}\,\overline{a^m}}\rangle + \langle\bar{a}^{n-1}\overline{a^m}\rangle$$
$$- \langle\overline{\bar{a}^{n-2}}\,\overline{a^m}\,\bar{a}\rangle + \langle\bar{a}^{n-2}\,\overline{a^m}\,\bar{\bar{a}}\rangle$$
$$- \langle\overline{\bar{a}^{n-3}}\,\overline{a^m}\,\bar{a}^2\rangle + \langle\bar{a}^{n-3}\,\overline{a^m}\,\overline{\bar{a}^2}\rangle$$
$$- \quad \cdots \quad + \quad \cdots$$
$$- \langle\overline{\bar{a}^2}\,\overline{a^m}\,\bar{a}^{n-3}\rangle + \langle\bar{a}^2\,\overline{a^m}\,\overline{\bar{a}^{n-3}}\rangle$$
$$- \langle\bar{\bar{a}}\,\overline{a^m}\,\bar{a}^{n-2}\rangle + \langle\bar{a}\,\overline{a^m}\,\overline{\bar{a}^{n-2}}\rangle$$
$$- \langle\overline{a^m}\,\bar{a}^{n-1}\rangle + \langle\overline{a^m}\,\overline{\bar{a}^{n-1}}\rangle.$$

The expansion above can be written as a sum of pairs of the form

$$\langle \overline{\bar{a}^{n'} \overline{a^m}}\, \bar{a}^{n-n'-1} \rangle \ - \ \langle \bar{a}^{n-n'-1}\, \overline{\overline{a^m}\, \bar{a}^{n'}} \rangle .$$

Using the result of (i), since $\overline{a^m}$ can be expressed as a polynomial of degree m in $\bar{a}$, with O(N) invariant coefficients,

$$\bar{a}^{n'} \overline{a^m} = \overline{a^m}\, \bar{a}^{n'}$$

whence by the induction hypothesis, since $n-n'-1 \leq n-1$, it follows that each of these pairs cancels.

A3.2

Commutation Relations:

$$[a_{ij}, a_{k\ell}] = g_{kj} a_{i\ell} - g_{i\ell} a_{kj}$$
$$[a_{ik}, a_j{}^k] = a_{ij} - a_{ji}$$
$$[a_{ki}, a^k{}_j] = a_{ij} - a_{ji}$$
$$[a_{ik}, a^k{}_j] = N a_{ij} - g_{ij} \langle a \rangle$$
$$[a_{ki}, a_j{}^k] = g_{ij} \langle a \rangle - N a_{ji}$$

$$[a_{ij}, \bar{a}_{k\ell}] = g_{j\ell} \bar{a}_{ki} - g_{ki} \bar{a}_{j\ell}$$
$$[a_{ik}, \bar{a}_j{}^k] = N \bar{a}_{ji} - g_{ij} \langle a \rangle$$
$$[a_{ki}, \bar{a}^k{}_j] = g_{ij} \langle a \rangle - N \bar{a}_{ij}$$
$$[a_{ik}, \bar{a}^k{}_j] = \bar{a}_{ji} - \bar{a}_{ij}$$
$$[a_{ki}, \bar{a}_j{}^k] = \bar{a}_{ji} - \bar{a}_{ij}$$

$$[a_{ij}, a\bar{a}_{k\ell}] = g_{j\ell}\, a\bar{a}_{ki} - a_{ki}a_{\ell j} + g_{jk}\, a\bar{a}_{i\ell} - a_{kj}a_{\ell i}$$
$$[a_{ik}, a\bar{a}_j{}^k] = N\, a\bar{a}_{ji} - \langle a\rangle a_{ji} + a\bar{a}_{ij} - a^2{}_{ji}$$
$$[a_{ki}, a\bar{a}^k{}_j] = g_{ij}\langle a\bar{a}\rangle - \langle a\rangle a_{ji} + a\bar{a}_{ij} - a^2{}_{ij}$$
$$[a_{ik}, a\bar{a}^k{}_j] = a\bar{a}_{ji} - a^2{}_{ij} + N\, a\bar{a}_{ij} - \langle a\rangle a_{ji}$$
$$[a_{ki}, a\bar{a}_j{}^k] = a\bar{a}_{ji} - a^2{}_{ji} + g_{ij}\langle a\bar{a}\rangle - \langle a\rangle a_{ji}$$

$$[a_{ij}, \bar{a}a_{k\ell}] = a_{jk}a_{i\ell} - g_{i\ell}\,\bar{a}a_{kj} + a_{ik}a_{j\ell} - g_{ik}\,\bar{a}a_{j\ell}$$
$$[a_{ik}, \bar{a}a_j{}^k] = \bar{a}^2{}_{ji} - \bar{a}a_{ji} + \langle a\rangle a_{ij} - g_{ij}\langle\bar{a}a\rangle$$
$$[a_{ki}, \bar{a}a^k{}_j] = a^2{}_{ij} - \bar{a}a_{ji} + \langle a\rangle a_{ij} - N\,\bar{a}a_{ij}$$
$$[a_{ik}, \bar{a}a^k{}_j] = \langle a\rangle a_{ij} - g_{ij}\langle\bar{a}a\rangle + a^2{}_{ij} - \bar{a}a_{ij}$$
$$[a_{ki}, \bar{a}a_j{}^k] = \langle a\rangle a_{ij} - N\,\bar{a}a_{ji} + \bar{a}^2{}_{ji} - \bar{a}a_{ij}$$

Quartic Invariants:

$$\overline{a^3} = \bar{a}^3 + 2N\bar{a}^2 + N^2\bar{a} - \langle a\rangle\bar{a} - (\langle a^2\rangle + N\langle a\rangle)$$
$$\langle\bar{a}a^3\rangle = \langle a\overline{a^3}\rangle = \langle a\bar{a}^3\rangle + 2N\langle a\bar{a}^2\rangle + N^2\langle a\bar{a}\rangle - \langle a\rangle\langle a\bar{a}\rangle - \langle a\rangle(\langle a^2\rangle + N\langle a\rangle)$$
$$\langle a\bar{a}^3 - \bar{a}a^3\rangle = -N\langle a\bar{a}a + \bar{a}a\bar{a}\rangle + \langle a\rangle\langle a\bar{a}\rangle + \langle a\rangle\langle a^2\rangle\,;$$

$$\overline{a^4} = \bar{a}^4 + 3N\bar{a}^3 + (3N^2 - \langle a\rangle)\bar{a}^2 + (N^3 - 2N\langle a\rangle - \langle a^2\rangle)\bar{a} - (\langle a^3\rangle + N\langle a^2\rangle + N^2\langle a\rangle)$$
$$\langle a^4\rangle = \langle\bar{a}^4\rangle + 2N\langle\bar{a}^3\rangle + N^2\langle\bar{a}^2\rangle - \langle a\rangle(2\langle a^2\rangle + N\langle a\rangle)$$
$$\langle a^4 - \bar{a}^4\rangle = N\langle a^3 + \bar{a}^3\rangle - 2\langle a\rangle\langle a^2\rangle\,;$$

$$
\begin{aligned}
\langle a^2\bar{a}a\rangle &= a^i{}_j\, a\bar{a}^j{}_\ell\, a^\ell{}_i \\
&= a\bar{a}^j{}_\ell\, a^i{}_j\, a^\ell{}_i + (a\bar{a}_\ell{}^i - \bar{a}^{2\,i}{}_\ell + N\, a\bar{a}^i{}_\ell - \langle a\rangle a_\ell{}^i)\, a^\ell{}_i \\
&= a\bar{a}^j{}_\ell\, a^i{}_j\, a^\ell{}_i + \langle a\bar{a}^2\rangle - \langle\bar{a}^2 a\rangle + N\langle a\bar{a}a\rangle - \langle a\rangle\langle a\bar{a}\rangle \\
&= a\bar{a}^j{}_\ell\, a^\ell{}_i\, a^i{}_j + a\bar{a}^j{}_\ell(\delta^\ell{}_j\langle a\rangle - N a^\ell{}_j) + N\langle a\bar{a}a\rangle - \langle a\rangle\langle a\bar{a}\rangle \\
&= \langle a\bar{a}a^2\rangle + \langle a\bar{a}\rangle\langle a\rangle - N\langle a\bar{a}a\rangle + N\langle a\bar{a}a\rangle - \langle a\rangle\langle a\bar{a}\rangle \\
&= \langle a\bar{a}a^2\rangle + [\langle a\bar{a}\rangle, \langle a\rangle]\,;
\end{aligned}
$$

$$\langle a\bar{a}a^2\rangle = a^i{}_j a_k{}^j a^{2k}{}_i$$
$$= a_k{}^j a^i{}_j a^{2k}{}_i + (a^i{}_k - a_k{}^i) a^{2k}{}_i$$
$$= a_k{}^j a^i{}_j a^{2k}{}_i + \langle a^3\rangle - \langle \bar{a}a^2\rangle$$
$$= a_k{}^j a^{2k}{}_i a^i{}_j + a_k{}^j(\delta^k{}_j \langle a^2\rangle - N a^{2k}{}_j) + \langle a^3\rangle - \langle \bar{a}a^2\rangle$$
$$= \langle \bar{a}a^3\rangle + \langle a^3\rangle - (N+1)\langle \bar{a}a^2\rangle + \langle a\rangle\langle a^2\rangle ;$$

$$\langle \bar{a}^2 a\bar{a}\rangle = a_j{}^i a_k{}^j a\bar{a}^k{}_i$$
$$= a_k{}^j a_j{}^i a\bar{a}^k{}_i + (\delta^i{}_k \langle a\rangle - N a_k{}^i) a\bar{a}^k{}_i$$
$$= a_k{}^j a_j{}^i a\bar{a}^k{}_i + \langle a\rangle\langle a\bar{a}\rangle - N\langle \bar{a}a\bar{a}\rangle$$
$$= a_k{}^j a\bar{a}^k{}_i a_j{}^i + a_k{}^j(N a\bar{a}^k{}_j - a^k{}_j\langle a\rangle + a\bar{a}_j{}^k - \bar{a}^{2k}{}_j) + (\cdots)$$
$$= \langle \bar{a}a\bar{a}^2\rangle + N\langle \bar{a}a\bar{a}\rangle - \langle a\bar{a}\rangle\langle a\rangle + \langle a^2\bar{a}\rangle - \langle \bar{a}a^2\rangle + \langle a\rangle\langle \bar{a}a\rangle - N\langle \bar{a}a\bar{a}\rangle$$
$$= \langle \bar{a}a\bar{a}^2\rangle ;$$

$$\langle \bar{a}a\bar{a}^2\rangle = a_j{}^i a^j{}_k \bar{a}^{2k}{}_i$$
$$= a^j{}_k a_j{}^i \bar{a}^{2k}{}_i + (a^i{}_k - a_k{}^i)\bar{a}^{2k}{}_i$$
$$= a^j{}_k a_j{}^i \bar{a}^{2k}{}_i + \langle a\bar{a}^2\rangle - \langle \bar{a}^3\rangle$$
$$= a^j{}_k \bar{a}^{2k}{}_i a_j{}^i + a^j{}_k(N\bar{a}^{2k}{}_j - \delta^k{}_j\langle \bar{a}^2\rangle) + \langle a\bar{a}^2\rangle - \langle \bar{a}^3\rangle$$
$$= \langle a\bar{a}^3\rangle - \langle \bar{a}^3\rangle + (N+1)\langle a\bar{a}^2\rangle - \langle a\rangle\langle a^2\rangle ;$$

$$\langle a\bar{a}^2 a\rangle = a^i{}_j \bar{a}^{2j}{}_k a^k{}_i$$
$$= \bar{a}^{2j}{}_k a^i{}_j a^k{}_i + (\bar{a}^2{}_k{}^i - \bar{a}^{2i}{}_k) a^k{}_i$$
$$= \bar{a}^{2j}{}_k a^k{}_i a^i{}_j + \bar{a}^{2j}{}_k(\delta^k{}_j\langle a\rangle - N a^k{}_j) + \langle \bar{a}^3\rangle - \langle \bar{a}^2 a\rangle$$
$$= \langle \bar{a}^2 a^2\rangle + \langle \bar{a}^3\rangle - (N+1)\langle \bar{a}^2 a\rangle + \langle a^2\rangle\langle a\rangle ;$$

$$\langle \bar{a}a^2\bar{a}\rangle = a_j{}^i a^{2j}{}_k a_i{}^k$$
$$= a^{2j}{}_k a_j{}^k a_i{}^k + (a^{2i}{}_k - a^2{}_k{}^i) a_i{}^k$$
$$= a^{2j}{}_k a_i{}^k a_j{}^i + a^{2j}{}_k(N a_j{}^k - \delta^k{}_j\langle a\rangle) + \langle a^2\bar{a}\rangle - \langle a^3\rangle$$
$$= \langle a^2\bar{a}^2\rangle - \langle a^3\rangle + (N+1)\langle a^2\bar{a}\rangle - \langle a^2\rangle\langle a\rangle ;$$

$$
\begin{aligned}
\langle \bar{a} a \bar{a} a \rangle &= a_j{}^i \, a\bar{a}^j{}_k \, a^k{}_i \\
&= a\bar{a}^j{}_k \, a_j{}^i \, a^k{}_i + (\delta^i{}_k \langle a\bar{a} \rangle - \langle a \rangle a_k{}^i + a\bar{a}^i{}_k - \bar{a}^{2i}{}_k) \, a^k{}_i \\
&= a\bar{a}^j{}_k \, a_j{}^i \, a^k{}_i + \langle a\bar{a} \rangle \langle a \rangle - \langle a \rangle \langle a\bar{a} \rangle + \langle a\bar{a}a \rangle - \langle \bar{a}^2 a \rangle \\
&= a\bar{a}^j{}_k \, a^k{}_i \, a_j{}^i + a\bar{a}^j{}_k (a_j{}^k - a^k{}_j) + \langle a\bar{a}a \rangle - \langle \bar{a}^2 a \rangle \\
&= \langle a\bar{a}a\bar{a} \rangle + \langle a\bar{a}^2 \rangle - \langle a\bar{a}a \rangle + \langle a\bar{a}a \rangle - \langle \bar{a}^2 a \rangle \\
&= \langle a\bar{a}a\bar{a} \rangle .
\end{aligned}
$$

A4.2

The various substitution operators occurring in the calculation of a^2, $\bar{a}^2$, a^3, $\bar{a}^3$ are found by evaluating them in various special cases, using the symmetry properties of tensors with two-rowed symmetry type, Eqs. (2.19). Since the scalar factors occurring are polynomial in the parameters p and q (at most quadratic, for these calculations), they can be found from a limited number of special cases.

The notation for substitution operators introduced in Sec. 4.2 is followed; tensor components appear in square brackets, indicating the symmetry type. In the proofs, external indices, appearing in δ-factors, are held fixed, and the others are summed over their appropriate index set.

Substitutions for a^2, $\bar{a}^2$, a^3, $\bar{a}^3$:

$$(U|U)^i{}_j = \delta^u{}_j \left(\begin{smallmatrix} i & u' \\ u' & u \end{smallmatrix}\right) = (p-q-1)\,\delta^u{}_j \left(\begin{smallmatrix} i \\ u \end{smallmatrix}\right)$$

$$\left(S\middle|{S \atop T}\right)^i{}_j = \delta^s{}_j\left[\left(\begin{smallmatrix} i & s' \\ s' & s \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & t \\ t & s \end{smallmatrix}\right)\right] = (q-2)\,\delta^s{}_j\left(\begin{smallmatrix} i \\ s \end{smallmatrix}\right)$$

$$\left(T\middle|{S \atop T}\right)^i{}_j = \delta^t{}_j\left[\left(\begin{smallmatrix} i & t' \\ t' & t \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & s \\ s & t \end{smallmatrix}\right)\right] = (q-2)\,\delta^t{}_j\left(\begin{smallmatrix} i \\ t \end{smallmatrix}\right)$$

$$\left(U\middle|{S \atop T}\right)^i{}_j = \delta^u{}_j\left[\left(\begin{smallmatrix} i & s \\ s & u \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & t \\ t & u \end{smallmatrix}\right)\right] = q\,\delta^u{}_j\left(\begin{smallmatrix} i \\ u \end{smallmatrix}\right)$$

$$\left(U\middle|{S \atop T}\right) = \left[\left(\begin{smallmatrix} u & s \\ s & u \end{smallmatrix}\right) + \left(\begin{smallmatrix} u & t \\ t & u \end{smallmatrix}\right)\right] = q(p-q)(\,)$$

$$\left(S\middle|{U\,S \atop T}\right)^i{}_j = \delta^s{}_j\left[\left(\begin{smallmatrix} i & u & s' \\ u & s' & s \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & u & t \\ u & t & s \end{smallmatrix}\right)\right] = (q-2)\,\delta^s{}_j\left(\begin{smallmatrix} i & u \\ u & s \end{smallmatrix}\right)$$

$$\left(T\middle|{U\,S \atop T}\right)^i{}_j = \delta^t{}_j\left[\left(\begin{smallmatrix} i & u & s \\ u & s & t \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & u & t' \\ u & t' & t \end{smallmatrix}\right)\right] = (q-2)\,\delta^t{}_j\left(\begin{smallmatrix} i & u \\ u & t \end{smallmatrix}\right)$$

$$\left(U\middle|{U\,S \atop T}\right)^i{}_j = \delta^u{}_j\left[\left(\begin{smallmatrix} i & u' & s \\ u' & s & u \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & u' & t \\ u' & t & u \end{smallmatrix}\right)\right] = q(p-q-1)\,\delta^u{}_j\left(\begin{smallmatrix} i \\ u \end{smallmatrix}\right)$$

$$\left(S\middle|UU\right)^i{}_j = \delta^s{}_j\left(\begin{smallmatrix} i & u & u' \\ u & u' & s \end{smallmatrix}\right) = (p-q-1)\,\delta^s{}_j\left(\begin{smallmatrix} i & u \\ u & s \end{smallmatrix}\right)$$

$$\left(S\middle|{S\,U \atop T}\right)^i{}_j = \delta^s{}_j\left[\left(\begin{smallmatrix} i & s' & u \\ s' & u & s \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & t & u \\ t & u & s \end{smallmatrix}\right)\right] = q\,\delta^s{}_j\left(\begin{smallmatrix} i & u \\ u & s \end{smallmatrix}\right) - (p-q)\,\delta^s{}_j\left(\begin{smallmatrix} i \\ s \end{smallmatrix}\right)$$

$$\left(T\middle|{S\,U \atop T}\right)^i{}_j = \delta^t{}_j\left[\left(\begin{smallmatrix} i & s & u \\ s & u & t \end{smallmatrix}\right) + \left(\begin{smallmatrix} i & t' & u \\ t' & u & t \end{smallmatrix}\right)\right] = q\,\delta^t{}_j\left(\begin{smallmatrix} i & u \\ u & t \end{smallmatrix}\right) - (p-q)\,\delta^t{}_j\left(\begin{smallmatrix} i \\ t \end{smallmatrix}\right),$$

$$a^u{}_i\left(\begin{smallmatrix} j \\ u \end{smallmatrix}\right) = (p-q)\,\delta_i{}^j(\,) + (p-q-1)\,\delta^u{}_i\left(\begin{smallmatrix} j \\ u \end{smallmatrix}\right) + \left[\delta^s{}_i\left(\begin{smallmatrix} j & u \\ u & s \end{smallmatrix}\right) + \delta^t{}_i\left(\begin{smallmatrix} j & u \\ u & t \end{smallmatrix}\right)\right]$$

$$a^s{}_i\left(\begin{smallmatrix} j \\ s \end{smallmatrix}\right) + a^t{}_i\left(\begin{smallmatrix} j \\ t \end{smallmatrix}\right) = 2q\,\delta_i{}^j(\,) + q\,\delta^u{}_i\left(\begin{smallmatrix} j \\ u \end{smallmatrix}\right) + (q-2)\left[\delta^s{}_i\left(\begin{smallmatrix} j \\ s \end{smallmatrix}\right) + \delta^t{}_i\left(\begin{smallmatrix} j \\ t \end{smallmatrix}\right)\right]$$

$$a^s{}_i\left(\begin{smallmatrix} j & u \\ u & s \end{smallmatrix}\right) + a^t{}_i\left(\begin{smallmatrix} j & u \\ u & t \end{smallmatrix}\right) = q(p-q)\,\delta_i{}^j(\,) + q(p-q+1)\,\delta^u{}_i\left(\begin{smallmatrix} j \\ u \end{smallmatrix}\right) + (q-2)\left[\delta^s{}_i\left(\begin{smallmatrix} j & u \\ u & s \end{smallmatrix}\right) + \delta^t{}_i\left(\begin{smallmatrix} j & u \\ u & t \end{smallmatrix}\right)\right]$$

$$a^k{}_i\,\delta^s{}_j\left(\begin{smallmatrix} i \\ s \end{smallmatrix}\right) = (q+N-2)\,\delta^s{}_j\left(\begin{smallmatrix} k \\ s \end{smallmatrix}\right) + \delta^s{}_j\left(\begin{smallmatrix} k & u \\ u & s \end{smallmatrix}\right)$$

$$a^k{}_i\,\delta^t{}_j\left(\begin{smallmatrix} i \\ t \end{smallmatrix}\right) = (q+N-2)\,\delta^t{}_j\left(\begin{smallmatrix} k \\ t \end{smallmatrix}\right) + \delta^t{}_j\left(\begin{smallmatrix} k & u \\ u & t \end{smallmatrix}\right)$$

$$a^k{}_i\,\delta^u{}_j\left(\begin{smallmatrix} i \\ u \end{smallmatrix}\right) = (p+N-1)\,\delta^u{}_j\left(\begin{smallmatrix} k \\ u \end{smallmatrix}\right)$$

$$a^i{}_k\,\delta^s{}_j\left(\begin{smallmatrix} k & u \\ u & s \end{smallmatrix}\right) = (p+N-1)\,\delta^s{}_j\left(\begin{smallmatrix} i & u \\ u & s \end{smallmatrix}\right)$$

$$a^i{}_k\,\delta^t{}_j\left(\begin{smallmatrix} k & u \\ u & t \end{smallmatrix}\right) = (p+N-1)\,\delta^t{}_j\left(\begin{smallmatrix} i & u \\ u & t \end{smallmatrix}\right).$$

Proof:

$$(U|U)^i{}_j\,[u_1u_2u_3] = \delta^{u_1}{}_j\left([u_2\,i\,u_3] + [u_3u_2\,i]\right) = 2\,\delta^{u_1}{}_j\,[i\,u_2u_3] \qquad (r=3)$$

$$\left(S\middle|{S \atop T}\right)^i{}_j \left[\begin{smallmatrix} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{smallmatrix}\right] = \delta^{s_1}{}_j\left(\left[\begin{smallmatrix} s_2 & i & s_3 \\ \cdot & \cdot & \cdot \end{smallmatrix}\right]_1 + \left[\begin{smallmatrix} s_3 & s_2 & i \\ \cdot & \cdot & \cdot \end{smallmatrix}\right]_2 + \left[\begin{smallmatrix} t_1 & \cdot & \cdot \\ i & \cdot & \cdot \end{smallmatrix}\right]_3 + \left[\begin{smallmatrix} \cdot & t_2 & \cdot \\ \cdot & i & \cdot \end{smallmatrix}\right]_1 + \left[\begin{smallmatrix} \cdot & \cdot & t_3 \\ \cdot & \cdot & i \end{smallmatrix}\right]_2\right)$$

$$= \delta^{s_1}{}_j\left(\left[\begin{smallmatrix} s_2 & i & \cdot \\ t_2 & t_1 & \cdot \end{smallmatrix}\right]_4 + \left[\begin{smallmatrix} i & \cdot & s_3 \\ t_1 & \cdot & t_3 \end{smallmatrix}\right] + \left[\begin{smallmatrix} t_1 & \cdot & \cdot \\ i & \cdot & \cdot \end{smallmatrix}\right]_4\right) = \delta^{s_1}{}_j\left[\begin{smallmatrix} i & \cdot & \cdot \\ t_1 & \cdot & \cdot \end{smallmatrix}\right] \quad (q=3)$$

$$\left(U \middle| {S \atop T}\right)^i{}_j \genfrac{[}{]}{0pt}{}{s_1\, u_1}{t_1} = \delta^{u_1}{}_j\left(\genfrac{[}{]}{0pt}{}{i\, s_1}{t_1} + \genfrac{[}{]}{0pt}{}{s_1\, t_1}{i}\right) = \delta^{u_1}{}_j \genfrac{[}{]}{0pt}{}{s_1\, i}{t_1} \quad (q = 1)$$

$$\left(S \middle| {U\,S \atop T}\right)^i{}_j \genfrac{[}{]}{0pt}{}{s_1\, u_1}{t_1} = -\delta^{s_1}{}_j \genfrac{[}{]}{0pt}{}{u_1\, i}{t_1} \quad (q = 1)$$

$$\left(S \middle| {U\,S \atop T}\right)^i{}_j \genfrac{[}{]}{0pt}{}{s_1 s_2\, u}{t_1 t_2} = \delta^{s_1}{}_j\left(\genfrac{[}{]}{0pt}{}{s_2 u_1 i}{t_1 t_2}_2 + \genfrac{[}{]}{0pt}{}{t_1 s_2 i}{u_1 t_2}_1 + \genfrac{[}{]}{0pt}{}{t_2 s_2 i}{t_1 u_1}_1\right) = 0 \quad (q = 2)$$

$$\left(S \middle| {U\,S \atop T}\right)^i{}_j \genfrac{[}{]}{0pt}{}{s_1 s_2 s_3\, u_1}{t_1 t_2 t_3} = \delta^{s_1}{}_j\left(\genfrac{[}{]}{0pt}{}{s_2 u_1 \cdot i}{\cdot\ \cdot\ \cdot}_3 + \genfrac{[}{]}{0pt}{}{s_3 \cdot u_1 i}{\cdot\ \cdot\ \cdot}_2 + \genfrac{[}{]}{0pt}{}{t_1 \cdot \cdot\, i}{u_1 \cdot\ \cdot}_1 + \genfrac{[}{]}{0pt}{}{t_2 \cdot \cdot\, i}{\cdot\ u_1 \cdot}_1 + \genfrac{[}{]}{0pt}{}{t_3 \cdot \cdot\, i}{\cdot\ \cdot\ u_1}_2\right)$$

$$= \delta^{s_1}{}_j\left(\genfrac{[}{]}{0pt}{}{t_1 u_1 \cdot i}{s_2 t_2 \cdot}_4 + \genfrac{[}{]}{0pt}{}{t_3 \cdot t_1 i}{s_3 \cdot u_1}_5 + \genfrac{[}{]}{0pt}{}{s_2 u_1 \cdot i}{t_1 t_2 \cdot}_4\right) = \delta^{s_1}{}_j \genfrac{[}{]}{0pt}{}{u_1 s_2 s_3\, i}{t_1 t_2 t_3}$$

$$\left(U \middle| {U\,S \atop T}\right)^i{}_j \genfrac{[}{]}{0pt}{}{s_1\, u_1 u_2}{t_1} = \delta^{u_1}{}_j\left(\genfrac{[}{]}{0pt}{}{u_2 s_1 i}{t_1} + \genfrac{[}{]}{0pt}{}{s_1 t_1 i}{u_2}\right) = \delta^{u_1}{}_j \genfrac{[}{]}{0pt}{}{s_1 i\, u_2}{t_1} \quad (r = 2)$$

$$\left(S \middle| U\,U\right)^i{}_j \genfrac{[}{]}{0pt}{}{s_1\, u_1 u_2}{t_1} = \delta^{s_1}{}_j\left(\genfrac{[}{]}{0pt}{}{u_1 u_2 i}{t_1} + \genfrac{[}{]}{0pt}{}{u_2 i\, u_1}{t_1}\right) = \delta^{s_1}{}_j\left(\genfrac{[}{]}{0pt}{}{u_1 i\, u_2}{t_1} + \genfrac{[}{]}{0pt}{}{u_2 u_1 i}{t_1}\right)$$

$$\left(S \middle| {S\,U \atop T}\right)^i{}_j \genfrac{[}{]}{0pt}{}{s_1 s_2\, u_1}{t_1 t_2} = \delta^{s_1}{}_j\left(\genfrac{[}{]}{0pt}{}{u_1 i\, s_2}{t_1 t_2}_1 + \genfrac{[}{]}{0pt}{}{u_1 s_2 t_1}{i\ t_2} + \genfrac{[}{]}{0pt}{}{u_1 s_2 t_2}{t_1\ i}\right)$$

$$= \delta^{s_1}{}_j\left(\genfrac{[}{]}{0pt}{}{u_1 s_2 i}{t_1 t_2} + \genfrac{[}{]}{0pt}{}{u_1 s_2 t_1}{i\ t_2}\right) = \delta^{s_1}{}_j\left(2\genfrac{[}{]}{0pt}{}{u_1 \cdot i}{\cdot\ \cdot} - \genfrac{[}{]}{0pt}{}{i \cdot u_1}{\cdot\ \cdot}\right)$$

$$\delta^s{}_j\, a^i{}_h \genfrac{(}{)}{0pt}{}{k\ u}{u\ s} = \delta^s{}_j\left[\delta^{s'}{}_k \genfrac{(}{)}{0pt}{}{k\ u\ i}{u\ s\ s'} + \delta^u{}_k \genfrac{(}{)}{0pt}{}{k\ i}{u\ s} + \delta^t{}_k \genfrac{(}{)}{0pt}{}{k\ u\ i}{u\ s\ t} + \delta^{u'}{}_k \genfrac{(}{)}{0pt}{}{k\ u\ i}{u\ s\ u'} + \delta^k{}_h \genfrac{(}{)}{0pt}{}{i\ u}{u\ s}\right]$$

$$= \delta^s{}_j\left[N \genfrac{(}{)}{0pt}{}{i\ u}{u\ s} + (p-q)\genfrac{(}{)}{0pt}{}{i}{s} + \genfrac{(}{)}{0pt}{}{i\ s'\ u}{s'\ u\ s} + \genfrac{(}{)}{0pt}{}{i\ t\ u}{t\ u\ s} + \genfrac{(}{)}{0pt}{}{i\ u'\ u}{u'\ u\ s}\right]$$

$$= (p + N - 1)\, \delta^s{}_j \genfrac{(}{)}{0pt}{}{i\ u}{u\ s}$$

A4.3

Substitutions for <aā> :

I ($q_3 = 2q_3'$; $r_1 = 2r_1'$)

$$\genfrac{(}{)}{0pt}{}{S_1 S_1}{T_1 T_1} \longrightarrow 2q_1(q_1 + N - 3)$$

$$\genfrac{(}{)}{0pt}{}{S_2 S_2}{T_2 T_2} \longrightarrow 0$$

$$\genfrac{(}{)}{0pt}{}{S_3 S_3}{T_3 T_3} \longrightarrow q_3(q_3 + N - 4)$$

$$(U_1U_1) + (U_2U_2) + (U_1U_2) \longrightarrow r_1(r_1 + 2r_2 + N - 2)$$

$$\genfrac{(}{)}{0pt}{}{S_1 S_2}{T_1 T_2} \longrightarrow 4q_1q_2$$

$$\genfrac{(}{)}{0pt}{}{S_1 S_3}{T_1 T_3} \longrightarrow 4q_1q_3$$

$$\genfrac{(}{)}{0pt}{}{S_2 S_3}{T_2 T_3} \longrightarrow 2q_2q_3$$

$$\genfrac{(}{)}{0pt}{}{S_1 U_1}{T_1} \longrightarrow 2q_1r_1$$

II ($q_3 = 2q_3' + 1$; $r_1 = 2r_1' + 1$)

$$\genfrac{(}{)}{0pt}{}{S_1 S_1}{T_1 T_1} \longrightarrow 2q_1(q_1 + N - 3)$$

$$\genfrac{(}{)}{0pt}{}{S_2 S_2}{T_2 T_2} \longrightarrow 0$$

$$\genfrac{(}{)}{0pt}{}{S_3 U_0 S_3 U_0}{T_3 \quad\ T_3} \longrightarrow (q_3 - 1)(q_3 - 1 + N - 4) + 4q_3 + 2N - 6$$

$$(U_1U_1) + (U_2U_2) + (U_1U_2) \longrightarrow (r_1 - 1)(r_1 - 1 + 2r_2 + N - 2)$$

$$\genfrac{(}{)}{0pt}{}{S_1 S_2}{T_1 T_2} \longrightarrow 4q_1q_2$$

$$\genfrac{(}{)}{0pt}{}{S_1 S_3 U_0}{T_1 T_3} \longrightarrow 4q_1(q_3 - 1) + 6q_1$$

$$\genfrac{(}{)}{0pt}{}{S_2 S_3 U_0}{T_2 T_3} \longrightarrow 2q_2(q_3 - 1) + 4q_2$$

$$\genfrac{(}{)}{0pt}{}{S_1 U_1}{T_1} \longrightarrow 2q_1(r_1 - 1)$$

$\binom{S_1 U_2}{T_1}$	$\to 2q_1 r_2$	$\binom{S_1 U_2}{T_1}$	$\to 2q_1 r_2$
$\binom{S_2 U_1}{T_2}$	$\to 2q_2 r_1$	$\binom{S_2 U_1}{T_2}$	$\to 2q_2(r_1 - 1)$
$\binom{S_2 U_2}{T_2}$	$\to 0$	$\binom{S_2 U_2}{T_2}$	$\to 0$
$\binom{S_3 U_1}{T_3}$	$\to 2q_3 r_1$	$\binom{S_3 U_0 U_1}{T_3}$	$\to 2(q_3 - 1)(r_1 - 1) + 4(r_1 - 1)$
$\binom{S_3 U_2}{T_3}$	$\to 0$	$\binom{S_3 U_0 U_2}{T_3}$	$\to 0 + 2r_2$

<u>Proof I:</u>

The terms are calculated in special cases, and the general formulae deduced therefrom. The symmetry properties of traces of two-rowed tensors, Eqs. (2.20), are used throughout. The appropriate section of Th. 2 being invoked is indicated in each case.

All terms can be deduced from three general evaluations:

$$\binom{S_2 S_2}{T_2 T_2}\begin{bmatrix} s_1 s_2 \\ t_1 t_2 \end{bmatrix} = 2g^{s_1 s_2}\begin{bmatrix} \ell \ell \\ t_1 t_2 \end{bmatrix} + 2g^{t_1 t_2}\begin{bmatrix} s_1 s_2 \\ \ell \ell \end{bmatrix} + 2g^{s_1 t_2}\begin{bmatrix} \ell s_2 \\ t_1 \ell \end{bmatrix} + 2g^{s_2 t_1}\begin{bmatrix} s_1 \ell \\ \ell s_2 \end{bmatrix};$$

$$\begin{aligned}\binom{S_1 S_2}{T_1 T_2}\begin{bmatrix} i_1 i_2 s_1 s_2 \\ j_1 j_2 t_1 t_2 \end{bmatrix} = {} & 2g^{i_1 s_1}\begin{bmatrix} \ell i_2 \ell s_2 \\ j_1 j_2 t_1 t_2 \end{bmatrix} + 2g^{i_1 s_2}\begin{bmatrix} \ell i_2 s_1 \ell \\ j_1 j_2 t_1 t_2 \end{bmatrix} + 2g^{i_2 s_1}\begin{bmatrix} i_1 \ell \ell s_2 \\ j_1 j_2 t_1 t_2 \end{bmatrix} + 2g^{i_2 s_2}\begin{bmatrix} i_1 \ell s_1 \ell \\ j_1 j_2 t_1 t_2 \end{bmatrix} + \\ & + 2g^{j_1 t_1}\begin{bmatrix} i_1 i_2 s_1 s_2 \\ \ell j_2 \ell t_2 \end{bmatrix} + 2g^{j_1 t_2}\begin{bmatrix} i_1 i_2 s_1 s_2 \\ \ell j_2 t_1 \ell \end{bmatrix} + 2g^{j_2 t_1}\begin{bmatrix} i_1 i_2 s_1 s_2 \\ j_1 \ell \ell t_2 \end{bmatrix} + 2g^{j_2 t_2}\begin{bmatrix} i_1 i_2 s_1 s_2 \\ j_1 \ell t_1 \ell \end{bmatrix} + \\ & + 2g^{i_1 t_1}\begin{bmatrix} \ell i_2 s_1 s_2 \\ j_1 j_2 \ell t_2 \end{bmatrix} + 2g^{i_1 t_2}\begin{bmatrix} \ell i_2 s_1 s_2 \\ j_1 j_2 t_1 \ell \end{bmatrix} + 2g^{i_2 t_1}\begin{bmatrix} i_1 \ell s_1 s_2 \\ j_1 j_2 \ell t_2 \end{bmatrix} + 2g^{i_2 t_2}\begin{bmatrix} i_1 \ell s_1 s_2 \\ j_1 j_2 t_1 \ell \end{bmatrix} + \\ & + 2g^{j_1 s_1}\begin{bmatrix} i_1 i_2 \ell s_2 \\ \ell j_2 t_1 t_2 \end{bmatrix} + 2g^{j_1 s_2}\begin{bmatrix} i_1 i_2 s_1 \ell \\ \ell j_2 t_1 t_2 \end{bmatrix} + 2g^{j_2 s_1}\begin{bmatrix} i_1 i_2 \ell s_2 \\ j_1 \ell t_1 t_2 \end{bmatrix} + 2g^{j_2 s_2}\begin{bmatrix} i_1 i_2 s_1 \ell \\ j_1 \ell t_1 t_2 \end{bmatrix};\end{aligned}$$

$$\begin{aligned}\binom{S_2 U_2}{T_2}\begin{bmatrix} s_1 s_2 u_1 u_2 \\ t_1 t_2 \end{bmatrix} = {} & 2g^{s_1 u_1}\begin{bmatrix} \ell s_2 \ell u_2 \\ t_1 t_2 \end{bmatrix} + 2g^{s_1 u_2}\begin{bmatrix} \ell s_2 u_1 \ell \\ t_1 t_2 \end{bmatrix} + 2g^{s_2 u_1}\begin{bmatrix} s_1 \ell \ell u_2 \\ t_1 t_2 \end{bmatrix} + 2g^{s_2 u_2}\begin{bmatrix} s_1 \ell u_1 \ell \\ t_1 t_2 \end{bmatrix} + \\ & + 2g^{t_1 u_1}\begin{bmatrix} s_1 s_2 \ell u_2 \\ \ell t_2 \end{bmatrix} + 2g^{t_1 u_2}\begin{bmatrix} s_1 s_2 u_1 \ell \\ \ell t_2 \end{bmatrix} + 2g^{t_2 u_1}\begin{bmatrix} s_1 s_2 \ell u_2 \\ t_1 \ell \end{bmatrix} + 2g^{t_2 u_2}\begin{bmatrix} s_1 s_2 u_1 \ell \\ t_1 \ell \end{bmatrix};\end{aligned}$$

and applying the standard trace appropriately, following the prescription of Eq. (4.3). The notation (...)

indicates terms whose general form has already been deduced.

$$\begin{pmatrix} S_1 & S_1 \\ T_1 & T_1 \end{pmatrix}\begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix} \to 2N\begin{bmatrix} i & i \\ j & j \end{bmatrix} + 2N\begin{bmatrix} i & i \\ j & j \end{bmatrix} + 2\begin{bmatrix} i & j \\ j & i \end{bmatrix} + 2\begin{bmatrix} j & i \\ i & j \end{bmatrix} \to (4N-4)\,q'_1 .$$

$$\begin{pmatrix} S_1 & S_1 \\ T_1 & T_1 \end{pmatrix}\begin{bmatrix} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} \to (\cdots) + 16\begin{bmatrix} i & j & i & j \\ k & k & \ell & \ell \end{bmatrix} + 16\begin{bmatrix} i & j & k & k \\ \ell & \ell & i & j \end{bmatrix} \to (4N-4)\,q'_1 + 8q'_1(q'_1-1).$$

$$\begin{pmatrix} S_3 & S_3 \\ T_3 & T_3 \end{pmatrix}\begin{bmatrix} s_1 & s_2 \\ j_1 & j_2 \end{bmatrix} \to 0 + 2N\begin{bmatrix} s_1 & s_2 \\ j & j \end{bmatrix} + 4\begin{bmatrix} j & s_2 \\ s_1 & j \end{bmatrix} \to (2N-4)\,q'_3 ,$$

$$\begin{aligned} \begin{pmatrix} S_3 & S_3 \\ T_3 & T_3 \end{pmatrix}\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} \to (\cdots) &+ 4\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ j & k & j & k \end{bmatrix} + 4\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ j & k & k & j \end{bmatrix} + \\ &+ 2\left(\begin{bmatrix} k & s_2 & s_3 & s_4 \\ j & j & k & s_1 \end{bmatrix} + \begin{bmatrix} k & s_2 & s_3 & s_4 \\ j & j & s_1 & k \end{bmatrix} + \begin{bmatrix} s_1 & k & s_3 & s_4 \\ j & j & k & s_2 \end{bmatrix} + \begin{bmatrix} s_1 & k & s_3 & s_4 \\ j & j & s_2 & k \end{bmatrix}\right) + \\ &+ 2\left(\begin{bmatrix} s_1 & s_2 & j & s_4 \\ j & s_3 & k & k \end{bmatrix} + \begin{bmatrix} s_1 & s_2 & j & s_4 \\ s_3 & j & k & k \end{bmatrix} + \begin{bmatrix} s_1 & s_2 & s_3 & j \\ j & s_4 & k & k \end{bmatrix} + \begin{bmatrix} s_1 & s_2 & s_3 & j \\ s_4 & j & k & k \end{bmatrix}\right), \end{aligned}$$

$$\begin{pmatrix} S_3 & S_3 \\ T_3 & T_3 \end{pmatrix}\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} \to (\cdots) + 8\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ j & j & k & k \end{bmatrix} + 0 \to 4q'_3(q'_3-1) + 2(N-2)\,q'_3 \to (\text{Th. } 2(ii),(iii)).$$

$$(U_1U_1) + (U_2U_2) + (U_1U_2) \to r_1(r_1 + 2r_2 + N - 2) \quad (\text{Eq. }(4.45)).$$

$$\begin{pmatrix} S_1 & S_2 \\ T_1 & T_2 \end{pmatrix}\begin{bmatrix} i_1 & i_2 & s_1 & s_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} \to 16\begin{bmatrix} i & i & s_1 & s_2 \\ j & j & t_1 & t_2 \end{bmatrix} \to 8q'_1q_2 .$$

$$\begin{pmatrix} S_2 & S_3 \\ T_2 & T_3 \end{pmatrix}\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ t_1 & t_2 & j_1 & j_2 \end{bmatrix} \to 8\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ t_1 & t_2 & j & j \end{bmatrix} \to 4q_2q'_3 .$$

$$\begin{pmatrix} S_1 & S_3 \\ T_1 & T_3 \end{pmatrix}\begin{bmatrix} i_1 & i_2 & s_1 & s_2 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} \to 32\begin{bmatrix} i & i & s_1 & s_2 \\ j & k & j & k \end{bmatrix} \to 16\begin{bmatrix} i & i & s_1 & s_2 \\ j & j & k & k \end{bmatrix} \to 16\,q'_1q'_3 .$$

$$\begin{pmatrix} S_1 & U_1 \\ T_1 & \end{pmatrix}\begin{bmatrix} i_1 & i_2 & k_1 & k_2 \\ j_1 & j_2 & & \end{bmatrix} \to 16\begin{bmatrix} i & k & i & k \\ j & j & & \end{bmatrix} \to 8\begin{bmatrix} i & i & k & k \\ j & j & & \end{bmatrix} \to 8\,q'_1r'_1 .$$

$$\begin{pmatrix} S_2 & U_1 \\ T_2 & \end{pmatrix}\begin{bmatrix} s_1 & s_2 & k_1 & k_2 \\ t_1 & t_2 & & \end{bmatrix} \to 8\begin{bmatrix} s_1 & s_2 & k & k \\ t_1 & t_2 & & \end{bmatrix} \to 4\,q_2r'_1$$

$$\begin{pmatrix} S_1 & U_2 \\ T_1 & \end{pmatrix}\begin{bmatrix} i_1 & i_2 & u_1 & u_2 \\ j_1 & j_2 & & \end{bmatrix} \to 16\begin{bmatrix} i & u_2 & i & u_1 \\ j & j & & \end{bmatrix} \to 8\begin{bmatrix} i & i & u_1 & u_2 \\ j & j & & \end{bmatrix} \to 4q'_1r_2 .$$

$$\begin{pmatrix} S_3 & U_1 \\ T_3 & \end{pmatrix}\begin{bmatrix} s_1 & s_2 & k_1 & k_2 \\ j_1 & j_2 & & \end{bmatrix} \to 8\begin{bmatrix} s_1 & s_2 & j & k \\ j & k & & \end{bmatrix} + 4\begin{bmatrix} k & k & s_1 & s_2 \\ j & j & & \end{bmatrix} \to 8\begin{bmatrix} s_1 & s_2 & k & k \\ j & j & & \end{bmatrix} \to 8\,q'_3r'_1$$

$$\begin{pmatrix} S_3 & U_2 \\ T_3 & \end{pmatrix}\begin{bmatrix} s_1 & s_2 & u_1 & u_2 \\ j_1 & j_2 & & \end{bmatrix} \to 2\begin{bmatrix} s_1 & s_2 & j & u_2 \\ j & u_1 & & \end{bmatrix} + 2\begin{bmatrix} s_1 & s_2 & u_1 & j \\ j & u_2 & & \end{bmatrix} + 2\begin{bmatrix} s_1 & s_2 & j & u_2 \\ u_1 & j & & \end{bmatrix} + 2\begin{bmatrix} s_1 & s_2 & u_1 & j \\ u_2 & j & & \end{bmatrix} \to 0 .$$

<u>Proof II:</u>

The results of I are used (with modified q_3 , r_1) except where the index set U_0 appears. Three additional general evaluations are needed:

$$\begin{pmatrix} S_3 & U_0 & S_3 & U_0 \\ T_3 & & T_3 & \end{pmatrix}\begin{bmatrix} s_0 & u_0 \\ t_0 & \end{bmatrix} = 2g^{s_0u_1}\begin{bmatrix} \ell & \ell \\ t_0 & \end{bmatrix} + 2g^{t_0u_0}\begin{bmatrix} s_0 & \ell \\ \ell & \end{bmatrix} ,$$

$$\begin{aligned} \begin{pmatrix} S_3 & U_0 & U_1 \\ T_3 & & \end{pmatrix}\begin{bmatrix} s_0 & u_0 & k_1 & k_2 \\ t_0 & & & \end{bmatrix} = {} & 2g^{s_0k_1}\begin{bmatrix} \ell & u_0 & \ell & k_2 \\ t_0 & & & \end{bmatrix} + 2g^{u_0k_1}\begin{bmatrix} s_0 & \ell & \ell & k_2 \\ t_0 & & & \end{bmatrix} + 2g^{t_0k_1}\begin{bmatrix} s_0 & u_0 & \ell & k_2 \\ \ell & & & \end{bmatrix} + \\ & + 2g^{s_0k_2}\begin{bmatrix} \ell & u_0 & k_1 & \ell \\ t_0 & & & \end{bmatrix} + 2g^{u_0k_2}\begin{bmatrix} s_0 & \ell & k_1 & \ell \\ t_0 & & & \end{bmatrix} + 2g^{t_0k_2}\begin{bmatrix} s_0 & u_0 & k_1 & \ell \\ \ell & & & \end{bmatrix} . \end{aligned}$$

$$\binom{s_1 s_3\; u_0}{T_1 T_3}\begin{bmatrix} i_1 i_2 s_0 u_0 \\ j_1 j_2 t_0 \end{bmatrix} = 2g^{i_1 s_0}\begin{bmatrix} \ell\, i_2 \ell\, u_0 \\ j_1 j_2 t_0 \end{bmatrix} + 2g^{i_1 u_0}\begin{bmatrix} \ell\, i_2 s_0 \ell \\ j_1 j_2 t_0 \end{bmatrix} + 2g^{i_1 t_0}\begin{bmatrix} \ell\, i_2 s_0 u_0 \\ j_1 j_2 \ell \end{bmatrix} +$$
$$+ 2g^{i_2 s_0}\begin{bmatrix} i_1 \ell\, \ell\, u_0 \\ j_1 j_2 t_0 \end{bmatrix} + 2g^{i_2 u_0}\begin{bmatrix} i_1 \ell\, s_0 \ell \\ j_1 j_2 t_0 \end{bmatrix} + 2g^{i_2 t_0}\begin{bmatrix} i_1 \ell\, s_0 u_0 \\ j_1 j_2 \ell \end{bmatrix} +$$
$$+ 2g^{j_1 s_0}\begin{bmatrix} i_1 i_2 \ell\, u_0 \\ \ell\, j_2 t_0 \end{bmatrix} + 2g^{j_1 u_0}\begin{bmatrix} i_1 i_2 s_0 \ell \\ \ell\, j_2 t_0 \end{bmatrix} + 2g^{j_1 t_0}\begin{bmatrix} i_1 i_2 s_0 u_0 \\ \ell\, j_2 \ell \end{bmatrix} +$$
$$+ 2g^{j_2 s_0}\begin{bmatrix} i_1 i_2 \ell\, u_0 \\ j_1 \ell\, t_0 \end{bmatrix} + 2g^{j_2 u_0}\begin{bmatrix} i_1 i_2 s_0 \ell \\ j_1 \ell\, t_0 \end{bmatrix} + 2g^{j_2 t_0}\begin{bmatrix} i_1 i_2 s_0 u_0 \\ j_1 \ell\, \ell \end{bmatrix}.$$

$$\binom{s_3 u_0 s_3 u_0}{T_3 \quad T_3}\begin{bmatrix} s_0 j_2 \\ j_1 \end{bmatrix} \rightarrow (2N-2)\begin{bmatrix} s_0 j \\ j \end{bmatrix} \rightarrow (2N-2)$$

$$\binom{s_3 u_0 s_3 u_0}{T_3 \quad T_3}\begin{bmatrix} s_1 s_2 s_0 k_2 \\ j_1 j_2 k_1 \end{bmatrix} \rightarrow ((\cdots) + (2N-2))\begin{bmatrix} s_1 s_2 s_0 k \\ j\; j\; k \end{bmatrix} + 4\left(\begin{bmatrix} s_0 s_2 s_1 k \\ j\; j\; k \end{bmatrix} + \begin{bmatrix} s_1 s_0 s_2 k \\ j\; j\; k \end{bmatrix}\right)$$
$$\rightarrow (q_3-1)(q_3-1+N-4) + (2N-2) + 4(q_3-1) \quad (\text{Th. } 2(ii)).$$

$$\binom{s_1 s_3\; u_0}{T_1 T_3}\begin{bmatrix} i_1 i_2 s_0 k_2 \\ j_1 j_2 k_1 \end{bmatrix} \rightarrow (\cdots) + 12\begin{bmatrix} i\, i\, s_0 k \\ j\, j\, k \end{bmatrix} \rightarrow 4q_1(q_3-1) + 6q_1.$$

$$\binom{s_2 s_3\; u_0}{T_2 T_3}\begin{bmatrix} s_1 s_2 s_0 k_2 \\ t_1 t_2 k_1 \end{bmatrix} \rightarrow (\cdots) + 8\begin{bmatrix} s_1 s_2 s_0 k \\ t_1 t_2 k \end{bmatrix} \rightarrow 2q_2(q_3-1) + 4q_2.$$

$$\binom{s_3\; u_0 u_1}{T_3}\begin{bmatrix} s_0 j_2 k_1 k_2 \\ j_1 \end{bmatrix} \rightarrow (\cdots) + 8\begin{bmatrix} s_0 j\, k\, k \\ j \end{bmatrix} \rightarrow 2(q_3-1)(r_1-1) + 4(r_1-1).$$

$$\binom{s_3\; u_0 u_2}{T_3}\begin{bmatrix} s_0 j_2 u_1 u_2 \\ j_1 \end{bmatrix} \rightarrow (\cdots) + 4\begin{bmatrix} s_0 j\, u_1 u_2 \\ j \end{bmatrix} + 4\begin{bmatrix} s_0 u_2 j\, j \\ u_1 \end{bmatrix} \rightarrow 0 + 2r_2 + 0 \quad (\text{Th. } 2(iii)).$$

A4.4

Substitutions for $\frac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a}\rangle$ $(q_3 = 2q_3', r_1 = 2r_1')$:

$$\binom{s_1 s_1}{T_1 T_1}\Big| u\Big) \rightarrow q_1^2(r_1+r_2) + (N-3)\,q_1(r_1+r_2)$$

$$\binom{s_2 s_2}{T_2 T_2}\Big| u\Big) \rightarrow 0$$

$$\binom{s_3 s_3}{T_3 T_3}\Big| u\Big) \rightarrow q_3^2 r_1 + (N-4)q_3 r_1 - \tfrac{1}{2}\Lambda^-(q_3^2 r_1 + (N-4)q_3 r_1)$$

$$\binom{s_1 s_2}{T_1 T_2}\Big| u\Big) \rightarrow 2q_1 q_2 r_1 + 2q_1 q_2 r_2$$

$$\binom{s_1 s_3}{T_1 T_3}\Big| u\Big) \rightarrow 2q_1 q_3 r_1 + 2q_1 q_3 r_2$$

$$\binom{s_2 s_3}{T_2 T_3}\Big| u\Big) \rightarrow 2q_2 q_3 r_1 - \Lambda^- q_2 q_3 r_1$$

$$\binom{s_1 u_1}{T_1}\Big| u\Big) \rightarrow q_1 r_1^2 + q_1 r_1 r_2$$

$$\binom{s_2 u_1}{T_2}\Big| u\Big) \rightarrow q_2 r_1^2 + q_2 r_1 r_2$$

$$\binom{s_3 u_1}{T_3}\Big| u\Big) \rightarrow q_3 r_1^2 + q_3 r_1 r_2$$

$$\binom{s_1 u_2}{T_1}\Big| u\Big) \rightarrow q_1 r_1 r_2 + 2\Lambda^+ q_1 r_2(r_2-1)$$

$$\binom{s_2 u_2}{T_2}\Big| u\Big) \rightarrow 0$$

$$\binom{s_3 u_2}{T_3}\Big| u\Big) \rightarrow 0$$

Proof:

Again the terms are calculated in special cases, using the symmetry properties of traces of two-rowed tensor, Eq. (2.20), and the general formulae deduced therefrom. The appropriate section of Th. 2 being invoked is indicated in each case. Internal summations over the U indices are evaluated in two stages, over the U_1 and U_2 separately.

All terms can be deduced from four general evaluations:

$$
\begin{aligned}
\left(\begin{smallmatrix} S_1 S_2 \\ T_1 T_2 \end{smallmatrix}\middle| U\right)\begin{bmatrix} i_1 & i_2 & s_1 & s_2 & u_1 & u_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} = {} & g^{i_1 s_1}\left(\begin{bmatrix} u_1 & i_2 & \ell & s_2 & \ell & u_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} \ell & i_2 & u_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} u_2 & i_2 & \ell & s_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} \ell & i_2 & u_2 & s_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix}\right) \\
& + g^{i_1 s_2}\left(\begin{bmatrix} u_1 & i_2 & s_1 & \ell & \ell & u_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} \ell & i_2 & s_1 & u_1 & \ell & u_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} u_2 & i_2 & s_1 & \ell & u_1 & \ell \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} \ell & i_2 & s_1 & u_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix}\right) \\
& + g^{i_2 s_1}\left(\begin{bmatrix} i_1 & u_1 & \ell & s_2 & \ell & u_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & \ell & u_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & u_2 & \ell & s_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & \ell & u_2 & s_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix}\right) \\
& + g^{i_2 s_2}\left(\begin{bmatrix} i_1 & u_1 & s_1 & \ell & \ell & u_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & \ell & s_1 & u_1 & \ell & u_2 \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_2 & u_2 & s_1 & \ell & u_1 & \ell \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & \ell & s_1 & u_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & t_2 \end{bmatrix}\right) \\
& + g^{i_1 t_1}\left(\begin{bmatrix} u_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & \ell & t_2 \end{bmatrix} + \begin{bmatrix} \ell & i_2 & s_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & u_1 & t_2 \end{bmatrix} + \begin{bmatrix} u_2 & i_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & j_2 & \ell & t_2 \end{bmatrix} + \begin{bmatrix} \ell & i_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & j_2 & u_2 & t_2 \end{bmatrix}\right) \\
& + g^{i_1 t_2}\left(\begin{bmatrix} u_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & t_1 & \ell \end{bmatrix} + \begin{bmatrix} \ell & i_2 & s_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & t_1 & u_1 \end{bmatrix} + \begin{bmatrix} u_2 & i_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & \ell \end{bmatrix} + \begin{bmatrix} \ell & i_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & u_2 \end{bmatrix}\right) \\
& + g^{i_2 t_1}\left(\begin{bmatrix} i_1 & u_1 & s_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & \ell & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & \ell & s_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & u_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & u_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & j_2 & \ell & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & \ell & s_1 & s_2 & u_1 & \ell \\ j_1 & j_2 & u_2 & t_2 \end{bmatrix}\right) \\
& + g^{i_2 t_2}\left(\begin{bmatrix} i_1 & u_1 & s_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & t_1 & \ell \end{bmatrix} + \begin{bmatrix} i_1 & \ell & s_1 & s_2 & \ell & u_2 \\ j_1 & j_2 & t_1 & u_1 \end{bmatrix} + \begin{bmatrix} i_1 & u_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & \ell \end{bmatrix} + \begin{bmatrix} i_1 & \ell & s_1 & s_2 & u_1 & \ell \\ j_1 & j_2 & t_1 & u_2 \end{bmatrix}\right) \\
& + g^{j_1 s_1}\left(\begin{bmatrix} i_1 & i_2 & \ell & s_2 & \ell & u_2 \\ u_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & u_1 & s_2 & \ell & u_2 \\ \ell & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & \ell & s_2 & u_1 & \ell \\ u_2 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & u_2 & s_2 & u_1 & \ell \\ \ell & j_2 & t_1 & t_2 \end{bmatrix}\right) \\
& + g^{j_1 s_2}\left(\begin{bmatrix} i_1 & i_2 & s_1 & \ell & \ell & u_2 \\ u_1 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & u_1 & \ell & u_2 \\ \ell & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & \ell & u_1 & \ell \\ u_2 & j_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & u_2 & u_1 & \ell \\ \ell & j_2 & t_1 & t_2 \end{bmatrix}\right) \\
& + g^{j_1 t_1}\left(\begin{bmatrix} i_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ u_1 & j_2 & \ell & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ \ell & j_2 & u_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & u_1 & \ell \\ u_2 & j_2 & \ell & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & u_1 & \ell \\ \ell & j_2 & u_2 & t_2 \end{bmatrix}\right) \\
& + g^{j_1 t_2}\left(\begin{bmatrix} i_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ u_1 & j_2 & t_1 & \ell \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ \ell & j_2 & t_1 & u_1 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & u_1 & \ell \\ u_2 & j_2 & t_1 & \ell \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & u_1 & \ell \\ \ell & j_2 & t_1 & u_2 \end{bmatrix}\right) \\
& + g^{j_2 s_1}\left(\begin{bmatrix} i_1 & i_2 & \ell & s_2 & \ell & u_2 \\ j_1 & u_1 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & u_1 & s_2 & \ell & u_2 \\ j_1 & \ell & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & \ell & s_2 & u_1 & \ell \\ j_1 & u_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & u_2 & s_2 & u_1 & \ell \\ j_1 & \ell & t_1 & t_2 \end{bmatrix}\right) \\
& + g^{j_2 s_2}\left(\begin{bmatrix} i_1 & i_2 & s_1 & \ell & \ell & u_2 \\ j_1 & u_1 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & u_1 & \ell & u_2 \\ j_1 & \ell & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & \ell & u_1 & \ell \\ j_1 & u_2 & t_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & u_2 & u_1 & \ell \\ j_1 & \ell & t_1 & t_2 \end{bmatrix}\right) \\
& + g^{j_2 t_1}\left(\begin{bmatrix} i_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ j_1 & u_1 & \ell & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ j_1 & \ell & u_1 & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & u_2 & \ell & t_2 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & \ell & u_2 & t_2 \end{bmatrix}\right) \\
& + g^{j_2 t_2}\left(\begin{bmatrix} i_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ j_1 & u_1 & t_1 & \ell \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & \ell & u_2 \\ j_1 & \ell & t_1 & u_1 \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & u_2 & t_1 & \ell \end{bmatrix} + \begin{bmatrix} i_1 & i_2 & s_1 & s_2 & u_1 & \ell \\ j_1 & \ell & t_1 & u_2 \end{bmatrix}\right),
\end{aligned}
$$

$$\left(\begin{smallmatrix} S_2 S_2 \\ T_2 T_2 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} s_1 s_2 u_1 u_2 \\ t_1 t_2 \end{bmatrix} = g^{s_1 s_2}\left(\begin{bmatrix} u_1 \ell \ell u_2 \\ t_1 t_2 \end{bmatrix} + \begin{bmatrix} \ell u_1 \ell u_2 \\ t_1 t_2 \end{bmatrix} + \begin{bmatrix} u_2 \ell u_1 \ell \\ t_1 t_2 \end{bmatrix} + \begin{bmatrix} \ell u_2 u_1 \ell \\ t_1 t_2 \end{bmatrix}\right)$$
$$+ g^{s_1 t_2}\left(\begin{bmatrix} u_1 s_2 \ell u_2 \\ t_1 \ell \end{bmatrix} + \begin{bmatrix} \ell s_2 \ell u_2 \\ t_1 u_1 \end{bmatrix} + \begin{bmatrix} u_2 s_2 u_1 \ell \\ t_1 \ell \end{bmatrix} + \begin{bmatrix} \ell s_2 u_1 \ell \\ t_1 u_2 \end{bmatrix}\right)$$
$$+ g^{s_2 t_1}\left(\begin{bmatrix} s_1 u_1 \ell u_2 \\ \ell t_2 \end{bmatrix} + \begin{bmatrix} s_1 \ell \ell u_2 \\ u_1 t_2 \end{bmatrix} + \begin{bmatrix} s_1 u_2 u_1 \ell \\ \ell t_2 \end{bmatrix} + \begin{bmatrix} s_1 \ell u_1 \ell \\ u_2 t_2 \end{bmatrix}\right)$$
$$+ g^{t_1 t_2}\left(\begin{bmatrix} s_1 s_2 \ell u_2 \\ u_1 \ell \end{bmatrix} + \begin{bmatrix} s_1 s_2 \ell u_2 \\ \ell u_1 \end{bmatrix} + \begin{bmatrix} s_1 s_2 u_1 \ell \\ u_2 \ell \end{bmatrix} + \begin{bmatrix} s_1 s_2 u_1 \ell \\ \ell u_2 \end{bmatrix}\right),$$

$$\left(\begin{smallmatrix} S_1 U_1 \\ T_1 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} i_1 i_2 k_1 k_2 u_1 u_2 \\ j_1 j_2 \end{bmatrix} = g^{i_1 k_1}\left(\begin{bmatrix} u_1 i_2 \ell k_2 \ell u_2 \\ j_1 j_2 \end{bmatrix} + \begin{bmatrix} u_2 i_2 \ell k_2 u_1 \ell \\ j_1 j_2 \end{bmatrix}\right) + g^{i_1 k_2}\left(\begin{bmatrix} u_1 i_2 k_1 \ell \ell u_2 \\ j_1 j_2 \end{bmatrix} + \begin{bmatrix} u_2 i_2 k_1 \ell u_1 \ell \\ j_1 j_2 \end{bmatrix}\right)$$
$$+ g^{j_1 k_1}\left(\begin{bmatrix} i_1 i_2 \ell k_2 \ell u_2 \\ u_1 j_2 \end{bmatrix} + \begin{bmatrix} i_1 i_2 \ell k_2 u_1 \ell \\ u_2 j_2 \end{bmatrix}\right) + g^{j_1 k_2}\left(\begin{bmatrix} i_1 i_2 k_1 \ell \ell u_2 \\ u_1 j_2 \end{bmatrix} + \begin{bmatrix} i_1 i_2 k_1 \ell u_1 \ell \\ u_2 j_2 \end{bmatrix}\right)$$
$$+ g^{i_2 k_1}\left(\begin{bmatrix} i_1 u_1 \ell k_2 \ell u_2 \\ j_1 j_2 \end{bmatrix} + \begin{bmatrix} i_1 u_2 \ell k_2 u_1 \ell \\ j_1 j_2 \end{bmatrix}\right) + g^{i_2 k_2}\left(\begin{bmatrix} i_1 u_1 k_1 \ell \ell u_2 \\ j_1 j_2 \end{bmatrix} + \begin{bmatrix} i_1 u_2 k_1 \ell u_1 \ell \\ j_1 j_2 \end{bmatrix}\right)$$
$$+ g^{j_2 k_1}\left(\begin{bmatrix} i_1 i_2 \ell k_2 \ell u_2 \\ j_1 u_1 \end{bmatrix} + \begin{bmatrix} i_1 i_2 \ell k_2 u_1 \ell \\ j_1 u_2 \end{bmatrix}\right) + g^{j_2 k_2}\left(\begin{bmatrix} i_1 i_2 k_1 \ell \ell u_2 \\ j_1 u_1 \end{bmatrix} + \begin{bmatrix} i_1 i_2 k_1 \ell u_1 \ell \\ j_1 u_2 \end{bmatrix}\right),$$

$$\left(\begin{smallmatrix} S_1 U_1 \\ T_1 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} i_1 i_2 k_1 k_2 \\ j_1 j_2 \end{bmatrix} = g^{i_1 k_1}\begin{bmatrix} k_2 i_2 \ell \ell \\ j_1 j_2 \end{bmatrix} + g^{i_1 k_2}\begin{bmatrix} k_1 i_2 \ell \ell \\ j_1 j_2 \end{bmatrix} + g^{i_2 k_1}\begin{bmatrix} i_1 k_2 \ell \ell \\ j_1 j_2 \end{bmatrix} + g^{i_2 k_2}\begin{bmatrix} i_1 k_1 \ell \ell \\ j_1 j_2 \end{bmatrix}$$
$$+ g^{j_1 k_1}\begin{bmatrix} i_1 i_2 \ell \ell \\ k_2 j_2 \end{bmatrix} + g^{j_1 k_2}\begin{bmatrix} i_1 i_2 \ell \ell \\ k_1 j_2 \end{bmatrix} + g^{j_2 k_1}\begin{bmatrix} i_1 i_2 \ell \ell \\ j_1 k_2 \end{bmatrix} + g^{j_2 k_2}\begin{bmatrix} i_1 i_2 \ell \ell \\ j_1 k_1 \end{bmatrix},$$

and applying the standard trace appropriately, following the prescription of Eq. (4.3). The notation (...) indicates terms whose general form has already been deduced.

$$\left(\begin{smallmatrix} S_1 S_1 \\ T_1 T_1 \end{smallmatrix}\middle| U_1\right) \rightarrow (8N-8)\begin{bmatrix} i_1 k_1 i_1 k_1 \\ j_1 j_1 \end{bmatrix} = (4N-4)\begin{bmatrix} i_1 i_1 k_1 k_1 \\ j_1 j_1 \end{bmatrix} \rightarrow (4N-4)\, q_1' r_1'.$$
$$\left(\begin{smallmatrix} S_1 S_1 \\ T_1 T_1 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} i_1 i_2 i_3 i_4 k_1 k_2 \\ j_1 j_2 j_3 j_4 \end{bmatrix} \rightarrow (\cdots) + 64\begin{bmatrix} i_1 k_1 i_1 i_2 k_1 i_2 \\ j_1 j_1 j_2 j_2 \end{bmatrix} \rightarrow (\cdots) + 8 q_1'(q_1'-1) r_1'.$$
$$\left(\begin{smallmatrix} S_1 S_1 \\ T_1 T_1 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} i_1 i_2 u_1 \\ j_1 j_2 \end{bmatrix} \rightarrow (4N-4)\begin{bmatrix} i_1 u_1 i_1 \\ j_1 j_1 \end{bmatrix} = (2N-2)\begin{bmatrix} i_1 i_1 u_1 \\ j_1 j_1 \end{bmatrix} \rightarrow (2N-2)\, q_1' r_2.$$
$$\left(\begin{smallmatrix} S_1 S_1 \\ T_1 T_1 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} i_1 i_2 i_3 i_4 u_1 \\ j_1 j_2 j_3 j_4 \end{bmatrix} \rightarrow (\cdots) + 32\begin{bmatrix} u_1 i_1 i_2 i_1 i_2 \\ j_1 j_1 j_2 j_2 \end{bmatrix} \rightarrow (\cdots) + 4 q_1'(q_1'-1) r_2.$$

$$\left(\begin{smallmatrix} S_2 S_2 \\ T_2 T_2 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} s_1 s_2 k_1 k_2 \\ t_1 t_2 \end{bmatrix} \rightarrow 4\left(g^{s_1 s_2}\begin{bmatrix} \ell \ell k_1 k_1 \\ t_1 t_2 \end{bmatrix} + g^{t_1 t_2}\begin{bmatrix} \ell \ell k_1 k_1 \\ s_1 s_2 \end{bmatrix} - g^{s_1 t_2}\begin{bmatrix} \ell \ell k_1 k_1 \\ s_2 t_1 \end{bmatrix} - g^{s_2 t_1}\begin{bmatrix} \ell \ell k_1 k_1 \\ s_1 t_2 \end{bmatrix}\right)$$
$$-2\left(g^{s_1 s_2}\begin{bmatrix} k_1 k_1 t_1 t_2 \\ \ell \ell \end{bmatrix} + g^{t_1 t_2}\begin{bmatrix} k_1 k_1 s_1 s_2 \\ \ell \ell \end{bmatrix} - g^{s_1 t_2}\begin{bmatrix} k_1 k_1 s_2 t_1 \\ \ell \ell \end{bmatrix} - g^{s_2 t_1}\begin{bmatrix} k_1 k_1 s_1 t_2 \\ j_1 j_2 \end{bmatrix}\right).$$
$$\left(\begin{smallmatrix} S_2 S_2 \\ T_2 T_2 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} s_1 s_2 u_1 \\ t_1 t_2 \end{bmatrix} \rightarrow g^{s_1 s_2}\left(\begin{bmatrix} u_1 \ell \ell \\ t_1 t_2 \end{bmatrix} + \begin{bmatrix} \ell u_1 \ell \\ t_1 t_2 \end{bmatrix}\right) + g^{t_1 t_2}\left(\begin{bmatrix} u_1 \ell \ell \\ s_1 s_2 \end{bmatrix} + \begin{bmatrix} \ell u_1 \ell \\ s_1 s_2 \end{bmatrix}\right)$$
$$-g^{s_1 t_2}\left(\begin{bmatrix} u_1 \ell \ell \\ s_2 t_1 \end{bmatrix} + \begin{bmatrix} \ell u_1 \ell \\ s_2 t_1 \end{bmatrix}\right) - g^{s_2 t_1}\left(\begin{bmatrix} u_1 \ell \ell \\ s_1 t_2 \end{bmatrix} + \begin{bmatrix} \ell u_1 \ell \\ s_1 t_2 \end{bmatrix}\right).$$
$$\left(\begin{smallmatrix} S_2 S_1 \\ T_2 T_2 \end{smallmatrix}\middle| U_1\right) + \left(\begin{smallmatrix} S_2 S_2 \\ T_2 T_2 \end{smallmatrix}\middle| U_2\right) \rightarrow 0 + 0 \quad (\text{Th. } 2(\text{iv})).$$

$$\left(\begin{smallmatrix} S_3 S_3 \\ T_3 T_3 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} s_1 s_2 k_1 k_2 \\ j_1 j_2 \end{bmatrix} \rightarrow 2g^{s_1 s_2}\begin{bmatrix} \ell \ell k_1 k_1 \\ j_1 j_1 \end{bmatrix} + (4N-8)\begin{bmatrix} s_1 s_2 k_1 k_1 \\ j_1 j_1 \end{bmatrix} - (2N-4)\begin{bmatrix} k_1 k_1 s_1 s_2 \\ j_1 j_1 \end{bmatrix}.$$

$$\left(\begin{smallmatrix} S_3 S_3 \\ T_3 T_3 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} s_1 s_2 s_3 s_4 k_1 k_2 \\ j_1 j_2 j_3 j_4 \end{bmatrix} \rightarrow (\cdots) + \left(g^{s_1 s_3}\left(\begin{bmatrix} \ell \ell s_2 s_4 k_1 k_1 \\ j_1 j_1 j_2 j_2 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} \ell \ell k_1 k_1 s_2 s_4 \\ j_1 j_1 j_2 j_2 \end{bmatrix}\right) + \cdots\right) +$$

$$+4\left(\begin{bmatrix} s_1 s_2 k_1 s_4 j_1 k_1 \\ s_3 j_1 j_2 j_2 \end{bmatrix} + \begin{bmatrix} s_1 s_2 s_3 k_1 j_1 k_1 \\ s_4 j_1 j_2 j_2 \end{bmatrix} + \begin{bmatrix} s_1 s_2 k_1 s_4 j_1 k_1 \\ j_1 s_3 j_2 j_2 \end{bmatrix} + \cdots\right) +$$

$$+4\left(\begin{bmatrix} s_1 s_3 s_2 s_4 j_1 k_1 \\ j_1 k_2 j_2 j_2 \end{bmatrix} + \begin{bmatrix} s_1 s_4 s_2 s_3 j_1 k_1 \\ j_1 k_1 j_2 j_2 \end{bmatrix} + \begin{bmatrix} s_2 s_3 s_1 s_4 j_1 k_1 \\ j_1 k_1 j_2 j_2 \end{bmatrix} + \begin{bmatrix} s_2 s_4 s_1 s_3 j_1 k_1 \\ j_1 k_1 j_2 j_2 \end{bmatrix}\right).$$

$$\left(\begin{smallmatrix} S_3 S_3 \\ T_3 T_3 \end{smallmatrix}\middle| U_1\right) \rightarrow 0 + (4N-8)q_3' r_1' - (2N-4)\bar{\Lambda} q_3' r_1' + 0 + 0 + 8q_3'(q_3'-1)r_1' - 4\bar{\Lambda} q_3'(q_3'-1)r_1'.$$

$$\left(\begin{smallmatrix} S_3 S_3 \\ T_3 T_3 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} s_1 s_2 u_1 \\ j_1 j_2 \end{bmatrix} \rightarrow 2g^{s_1 s_2}\begin{bmatrix} u_1 \ell \ell \\ j_1 j_1 \end{bmatrix} + (N-2)\left(\begin{bmatrix} s_1 s_2 \ell \\ u_1 \ell \end{bmatrix} + \begin{bmatrix} s_1 s_2 \ell \\ \ell u_1 \end{bmatrix}\right) \rightarrow 0 + 0 \quad (\text{Th. } 2(\text{iii}),(\text{iv})).$$

$$\left(\begin{smallmatrix} S_3 S_3 \\ T_3 T_3 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} s_1 s_2 s_3 s_4 u_1 \\ j_1 j_2 j_3 j_4 \end{bmatrix} \rightarrow \left(g^{s_1 s_3}\begin{bmatrix} u_1 s_2 \ell s_4 \ell \\ j_1 j_1 j_2 j_2 \end{bmatrix} + g^{s_1 s_3}\begin{bmatrix} \ell s_2 u_1 s_4 \ell \\ j_1 j_1 j_2 j_2 \end{bmatrix} + \cdots\right) + \left(\begin{bmatrix} u_1 s_2 s_3 s_4 \ell \\ j_1 j_2 \ell s_1 \end{bmatrix} + \cdots\right) +$$

$$+\left(\begin{bmatrix} i_1 i_2 s_1 s_2 \ell \\ u_1 j_2 \ell j_2 \end{bmatrix} + \begin{bmatrix} i_1 i_2 s_1 s_2 \ell \\ \ell j_2 u_1 j_2 \end{bmatrix} + \begin{bmatrix} i_1 i_2 s_1 s_2 \ell \\ u_1 j_2 j_2 \ell \end{bmatrix} + \cdots\right).$$

$$\left(\begin{smallmatrix} S_3 S_3 \\ T_3 T_3 \end{smallmatrix}\middle| U_2\right) \rightarrow 0 + 0 + 0 \quad (\text{Th. } 2(\text{iii}),(\text{iv})).$$

$$\left(\begin{smallmatrix} S_1 S_2 \\ T_1 T_2 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} i_1 i_2 s_1 k_1 k_2 \\ j_1 j_2 t_1 \end{bmatrix} \rightarrow 16\begin{bmatrix} i_1 k_1 s_1 i_1 k_1 \\ j_1 j_1 t_1 \end{bmatrix} = 8\begin{bmatrix} i_1 i_1 s_1 k_1 k_1 \\ j_1 j_1 \end{bmatrix} \rightarrow 8q_1' q_2 r_1'.$$

$$\left(\begin{smallmatrix} S_1 S_2 \\ T_1 T_2 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} i_1 i_2 s_1 u_1 \\ j_1 j_2 t_1 \end{bmatrix} \rightarrow 8\begin{bmatrix} i_1 u_1 s_1 i_1 \\ j_1 j_1 t_1 \end{bmatrix} = 4\begin{bmatrix} i_1 i_1 s_1 u_1 \\ j_1 j_1 \end{bmatrix} \rightarrow 4q_1' q_2 r_2.$$

$$\left(\begin{smallmatrix} S_1 S_3 \\ T_1 T_3 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} i_1 i_2 s_1 s_2 k_1 k_2 \\ j_1 j_2 j_3 j_4 \end{bmatrix} \rightarrow 32\begin{bmatrix} i_1 k_1 s_1 s_2 i_1 k_1 \\ j_1 j_1 j_2 j_2 \end{bmatrix} = 16\begin{bmatrix} i_1 i_1 s_1 s_2 k_1 k_1 \\ j_1 j_1 j_2 j_2 \end{bmatrix} \rightarrow 16q_1' q_3' r_1'.$$

$$\left(\begin{smallmatrix} S_1 S_3 \\ T_1 T_3 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} i_1 i_2 s_1 s_2 u_1 \\ j_1 j_2 j_3 j_4 \end{bmatrix} \rightarrow 16\begin{bmatrix} i_1 u_1 s_1 s_2 i_1 \\ j_1 j_1 j_2 j_2 \end{bmatrix} = 8\begin{bmatrix} i_1 i_2 s_1 s_2 u_1 \\ j_1 j_1 j_2 j_2 \end{bmatrix} \rightarrow 8q_1' q_3' r_2.$$

$$\left(\begin{smallmatrix} S_2 S_3 \\ T_2 T_3 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} s_1 s_2 s_3 k_1 k_2 \\ t_1 j_1 j_2 \end{bmatrix} \rightarrow 4\left(g^{s_1 s_2}\begin{bmatrix} k_1 \ell s_3 \ell k_1 \\ t_1 j_1 j_1 \end{bmatrix} + g^{s_1 s_3}\begin{bmatrix} k_1 s_2 \ell \ell k_1 \\ t_1 j_1 j_1 \end{bmatrix} + \cdots\right) + 8\begin{bmatrix} s_1 s_2 s_3 j_1 k_1 \\ t_1 j_1 k_1 \end{bmatrix}.$$

$$\left(\begin{smallmatrix} S_2 S_3 \\ T_2 T_3 \end{smallmatrix}\middle| U_1\right) \rightarrow 0 + 8q_2 q_3' r_1' - 4\bar{\Lambda} q_2 q_3' r_1'. \quad (\text{Th. } 2(\text{v});\ \text{Eq. } (2.20f)).$$

$$\left(\begin{smallmatrix} S_2 S_3 \\ T_2 T_3 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} s_1 s_2 s_3 u_1 \\ t_1 j_1 j_2 \end{bmatrix} \rightarrow g^{s_1 s_2}\left(\begin{bmatrix} u_1 \ell s_3 \ell \\ t_1 j_1 j_1 \end{bmatrix} + \begin{bmatrix} \ell u_1 s_3 \ell \\ t_1 j_1 j_1 \end{bmatrix}\right) + g^{s_1 s_3}\left(\begin{bmatrix} u_1 s_2 \ell \ell \\ t_1 j_1 j_1 \end{bmatrix} + \begin{bmatrix} \ell s_2 u_1 \ell \\ t_1 j_1 j_1 \end{bmatrix}\right) +$$

$$+g^{t_1 s_2}\left(\begin{bmatrix} s_1 \ell s_3 \ell \\ u_1 j_1 j_1 \end{bmatrix} + \begin{bmatrix} s_1 u_1 s_3 \ell \\ \ell j_1 j_1 \end{bmatrix}\right) + g^{t_1 s_3}\left(\begin{bmatrix} s_1 s_2 \ell \ell \\ u_1 j_1 j_1 \end{bmatrix} + \begin{bmatrix} s_1 s_2 u_1 \ell \\ \ell j_1 j_1 \end{bmatrix}\right) +$$

$$+2\left(\begin{bmatrix} s_1 s_2 s_3 \ell \\ t_1 u_1 \ell \end{bmatrix} + \begin{bmatrix} s_1 s_2 s_3 \ell \\ t_1 \ell u_1 \end{bmatrix}\right) \rightarrow 0 + 0 \quad (\text{Th. } 2(\text{iii}),(\text{iv})).$$

$$\left(\begin{smallmatrix} S_1 U_1 \\ T_1 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} i_1 i_2 k_1 k_2 \\ j_1 j_2 \end{bmatrix} \rightarrow 8\begin{bmatrix} i_1 i_1 k_1 k_1 \\ j_1 j_1 \end{bmatrix} \rightarrow 8q_1' r_1'.$$

$$\left(\begin{smallmatrix} S_1 U_1 \\ T_1 \end{smallmatrix}\middle| U_1\right)\begin{bmatrix} i_1 i_2 k_1 k_2 k_3 k_4 \\ j_1 j_2 \end{bmatrix} \rightarrow (\cdots) + 16\begin{bmatrix} i_1 k_1 i_1 k_1 k_2 k_2 \\ j_1 j_1 \end{bmatrix} + 16\begin{bmatrix} i_1 k_2 k_1 k_1 i_1 k_2 \\ j_1 j_1 \end{bmatrix}$$

$$\rightarrow (\cdots) + 8q_1' r_1'(r_1'-1).$$

$$\left(\begin{smallmatrix} S_1 U_1 \\ T_1 \end{smallmatrix}\middle| U_2\right)\begin{bmatrix} i_1 i_2 k_1 k_2 u_1 \\ j_1 j_2 \end{bmatrix} \rightarrow 8\begin{bmatrix} u_1 i_1 k_1 k_1 i_1 \\ j_1 j_1 \end{bmatrix} = 4\begin{bmatrix} i_1 i_1 k_1 k_1 u_1 \\ j_1 j_1 \end{bmatrix} \rightarrow 4q_1' r_1' r_2.$$

$\left(\begin{smallmatrix}S_2U_1\\T_2\end{smallmatrix}\middle|U_1\right)\left[\begin{smallmatrix}s_1k_1k_2\\t_1\end{smallmatrix}\right] \to 4\left[\begin{smallmatrix}s_1k_1k_1\\t_1\end{smallmatrix}\right] \to 4q_2r_1'$.

$\left(\begin{smallmatrix}S_2U_1\\T_2\end{smallmatrix}\middle|U_1\right)\left[\begin{smallmatrix}s_1k_1k_2k_3k_4\\t_1\end{smallmatrix}\right] \to (\cdots) + 8\left(\left[\begin{smallmatrix}k_2k_1k_1s_1k_2\\t_1\end{smallmatrix}\right] + \left[\begin{smallmatrix}s_1k_1k_1t_1k_2\\k_2\end{smallmatrix}\right]\right)$

$\to (\cdots) + 4q_2r_1'(r_1'-1)$.

$\left(\begin{smallmatrix}S_2U_1\\T_2\end{smallmatrix}\middle|U_2\right)\left[\begin{smallmatrix}s_1k_1k_2u_1\\t_1\end{smallmatrix}\right] \to 2\left(\left[\begin{smallmatrix}u_1k_1k_1s_1\\t_1\end{smallmatrix}\right] + \left[\begin{smallmatrix}s_1k_1k_1t_1\\u_1\end{smallmatrix}\right]\right) = 2\left[\begin{smallmatrix}s_1k_1k_1u_1\\t_1\end{smallmatrix}\right] \to 2q_2r_1'r_2$.

$\left(\begin{smallmatrix}S_3U_1\\T_3\end{smallmatrix}\middle|U_1\right)\left[\begin{smallmatrix}s_1s_2k_1k_2\\j_1j_2\end{smallmatrix}\right] \to 8\left[\begin{smallmatrix}s_1s_2k_1k_2\\j_1j_1\end{smallmatrix}\right] \to 8q_3'r_1'$

$\left(\begin{smallmatrix}S_3U_1\\T_3\end{smallmatrix}\middle|U_1\right)\left[\begin{smallmatrix}s_1s_2k_1k_2k_3k_4\\j_1j_2\end{smallmatrix}\right] \to (\cdots) + \left(16\left[\begin{smallmatrix}s_1s_2k_1k_1j_1k_2\\j_1k_2\end{smallmatrix}\right] + 8\left[\begin{smallmatrix}k_1k_1k_2k_2s_1s_2\\j_1j_1\end{smallmatrix}\right]\right)$

$\to (\cdots) + 16\left[\begin{smallmatrix}i_1i_2k_1k_1k_2k_2\\j_1j_1\end{smallmatrix}\right]$

$\left(\begin{smallmatrix}S_3U_1\\T_3\end{smallmatrix}\middle|U_1\right) \to (\cdots) + 8q_3'r_1'(r_1'-1)$

$\left(\begin{smallmatrix}S_3U_1\\T_3\end{smallmatrix}\middle|U_2\right)\left[\begin{smallmatrix}s_1s_2k_1k_2u_1\\j_1j_2\end{smallmatrix}\right] \to 2\left(\left[\begin{smallmatrix}s_1u_1k_1k_1s_2\\j_1j_1\end{smallmatrix}\right] + \left[\begin{smallmatrix}u_1s_2k_1k_1s_1\\j_1j_1\end{smallmatrix}\right] + \left[\begin{smallmatrix}s_1s_2k_1k_1j_1\\u_1j_1\end{smallmatrix}\right] + \left[\begin{smallmatrix}s_1s_2k_1k_1j_1\\j_1u_1\end{smallmatrix}\right]\right)$

$\to 4q_3'r_1'r_2$

$\left(\begin{smallmatrix}S_1U_2\\T_1\end{smallmatrix}\middle|U_1\right)\left[\begin{smallmatrix}i_1i_2k_1k_2u_1\\j_1j_2\end{smallmatrix}\right] \to 8\left[\begin{smallmatrix}i_1u_1k_1k_1i_1\\j_1j_1\end{smallmatrix}\right] = 4\left[\begin{smallmatrix}i_1i_1k_1k_1u_1\\j_1j_1\end{smallmatrix}\right] \to 4q_1'r_1'r_2$

$\left(\begin{smallmatrix}S_1U_2\\T_1\end{smallmatrix}\middle|U_2\right)\left[\begin{smallmatrix}i_1i_2u_1u_2\\j_1j_2\end{smallmatrix}\right] \to 8\left[\begin{smallmatrix}u_1u_2i_1i_1\\j_1j_1\end{smallmatrix}\right]$

$\left(\begin{smallmatrix}S_1U_2\\T_1\end{smallmatrix}\middle|U_2\right)\left[\begin{smallmatrix}i_1i_2u_1u_2u_3\\j_1j_2\end{smallmatrix}\right] \to 8\left[\begin{smallmatrix}u_1u_2i_1i_2u_3\\j_1j_1\end{smallmatrix}\right] + 8\left[\begin{smallmatrix}u_2u_3i_1i_1u_1\\j_1j_1\end{smallmatrix}\right] + 8\left[\begin{smallmatrix}u_1u_3i_1i_1u_2\\j_1j_1\end{smallmatrix}\right]$

$\left(\begin{smallmatrix}S_1U_2\\T_1\end{smallmatrix}\middle|U_2\right) \to 4\Lambda^+q_1'r_2(r_2-1)$ (Th. 2(v)).

$\left(\begin{smallmatrix}S_2U_2\\T_2\end{smallmatrix}\middle|U_1\right)\left[\begin{smallmatrix}s_1k_1k_2u_1\\t_1\end{smallmatrix}\right] \to 2g^{s_1u_1}\left[\begin{smallmatrix}\ell k_1k_1\ell\\t_1\end{smallmatrix}\right] + 2g^{t_1u_1}\left[\begin{smallmatrix}s_1k_1k_1\ell\\\ell\end{smallmatrix}\right] \to 0$ (Th. 2(iv)).

$\left(\begin{smallmatrix}S_2U_2\\T_2\end{smallmatrix}\middle|U_2\right)\left[\begin{smallmatrix}s_1u_1u_2\\t_1\end{smallmatrix}\right] \to g^{s_1u_1}\left[\begin{smallmatrix}u_2\ell\ell\\t_1\end{smallmatrix}\right] + g^{s_1u_2}\left[\begin{smallmatrix}u_1\ell\ell\\t_1\end{smallmatrix}\right] + g^{t_1u_1}\left[\begin{smallmatrix}s_1\ell\ell\\u_2\end{smallmatrix}\right] + g^{t_1u_2}\left[\begin{smallmatrix}s_1\ell\ell\\u_1\end{smallmatrix}\right]$

$\left(\begin{smallmatrix}S_2U_2\\T_2\end{smallmatrix}\middle|U_2\right) \to 0$ (Th. 2(iv)).

$\left(\begin{smallmatrix}S_3U_2\\T_3\end{smallmatrix}\middle|U_1\right)\left[\begin{smallmatrix}s_1s_2k_1k_2u_1\\j_1j_2\end{smallmatrix}\right] \to 2\left(g^{s_1u_1}\left[\begin{smallmatrix}\ell s_2k_1k_1\ell\\j_1j_1\end{smallmatrix}\right] + g^{s_2u_1}\left[\begin{smallmatrix}s_1\ell k_1k_1\ell\\j_1j_1\end{smallmatrix}\right]\right) + 2\left(\left[\begin{smallmatrix}s_1s_2k_1k_1j_1\\j_1u_1\end{smallmatrix}\right] + \left[\begin{smallmatrix}s_1s_2k_1k_1j_1\\u_1j_1\end{smallmatrix}\right]\right)$.

$\left(\begin{smallmatrix}S_3U_2\\T_3\end{smallmatrix}\middle|U_1\right) \to 0+0$ (Th. 2(iii), (iv)).

$\left(\begin{smallmatrix}S_3U_2\\T_3\end{smallmatrix}\middle|U_2\right)\left[\begin{smallmatrix}s_1s_2u_1u_2\\j_1j_2\end{smallmatrix}\right] \to \left(g^{s_1u_1}\left[\begin{smallmatrix}u_2s_2\ell\ell\\j_1j_1\end{smallmatrix}\right] + g^{s_1u_2}\left[\begin{smallmatrix}u_1s_2\ell\ell\\j_1j_1\end{smallmatrix}\right] + \cdots\right) + 2\left(\left[\begin{smallmatrix}s_1s_2j_1j_1\\u_1u_2\end{smallmatrix}\right] + \left[\begin{smallmatrix}s_1s_2j_1j_1\\u_2u_1\end{smallmatrix}\right]\right)$.

$\left(\begin{smallmatrix}S_3U_2\\T_3\end{smallmatrix}\middle|U_2\right) \to 0+0$ (Th. 2(iii),(iv)).

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