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Extraction of light quark masses from sum rule analyses of axial vector and vector current Ward identities

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In light of recent lattice results for the light quark masses m_s and m_u+m_d, we reexamine the use of sum rules in the extraction of these quantities, and discuss a number of potential problems with existing analyses. The most important issue is that of the overall normalization of the hadronic spectral functions relevant to the sum rule analyses. We explain why previous treatments, which fix this normalization by assuming complete resonance dominance of the continuum threshold region, can potentially overestimate the resonance contributions to spectral integrals by factors as large as ~ 5. We propose an alternate method of normalization based on an understanding of the role of resonances in chiral perturbation theory which avoids this problem. The second important uncertainty we consider relates to the physical content of the assumed location s_0 of the onset of duality with perturbative QCD. We find that the extracted quark masses depend very sensitively on this parameter. We show that the assumption of duality imposes very severe constraints on the shape of the relevant spectral function in the dual region and present rigorous lower bounds for m_u+m_d as a function of s_0 based on a combination of these constraints and the requirement of positivity of $\rho_5(s)$. In the extractions of m_s, we find that the conventional choice of the value of s_0 is not physical. For a more reasonable choice of s_0, we are not able to find a solution that is stable with respect to variations of the Borel transform parameter. This problem can, unfortunately, be overcome only if the hadronic spectral function is determined up to significantly larger values of s than is currently possible. Finally, we also estimate the error associated with the convergence of perturbative QCD expressions used in the sum rule analyses. Our conclusion is that, taking all of these issues into account, the resulting sum rule estimates for both m_u+m_d and m_s could easily have uncertainties as large as a factor of 2, which would make them compatible with the low estimates obtained from lattice QCD. [S0556-2821(98)00209-4]

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I. INTRODUCTION

The recent lattice results for the light quark masses [1] $m_u+m_d=6.8\pm0.8\pm0.6$ MeV and $m_s=100\pm21\pm10$ MeV in the quenched approximation and the even smaller values $m_u+m_d=5.4\pm0.6\pm0.6$ MeV and $m_s=68\pm12\pm7$ MeV, for the $n_f=2$ flavor theory [all evaluated in the modified minimal subtraction ($\overline{\rm MS}$) scheme at $\mu=2$ GeV], appear to be significantly smaller than results obtained from sum rule analyses. The most recent and complete sum rules analyses are (i) that of Bijnens, Prades, and deRafael (BPR), which yields $m_u+m_d(\mu=2\ \text{GeV})=9.4\pm1.76$ MeV [3], and (ii) that of Chetyrkin, Pirjol, and Schilcher (CPS), which gives $m_s=143\pm14$ MeV [4]. We have translated the original values $m_u+m_d=12\pm2.5$ MeV and $m_s=203\pm20$ MeV, quoted at $\mu=1$ GeV, to $\mu=2$ GeV using the renormalization group running and the preferred value $\Lambda^{(3)}_{\rm QCD}=300$, 380 MeV used, respectively, in the two calculations. (The analysis by CPS is an update of that by Jamin and Münz (JM) [5]; however, since the approach and techniques are the same, we will refer to their work jointly by the abbreviation JM-CPS.) The sum rule results, thus, lie roughly (1–2)σ above the quenched results. The difference between the sum rule and the $n_f=2$ lattice estimates, however, is large and, we feel, significant enough to warrant scrutiny. Both the lattice and sum rules approaches have their share of systematic errors. A recent review of the lattice results is given in [2]. Here we present a reevaluation of the sum rules analyses.

The issues in the sum rule analyses that we shall concentrate on are the convergence of perturbative QCD (PQCD) expressions, the choice of s_0—the scale beyond which quark-hadron duality is assumed to be valid—and the normalization of resonance contributions in the *Ansatz* for the hadronic spectral function for $s\leqslant s_0$.

The first issue is important because both α_s and α_s^2 corrections to the two-point correlation functions used in sum rule analyses are large. This issue has been analyzed in detail by CPS for the extraction of m_s; therefore, we shall only comment on it briefly for the case of m_u+m_d.

The second point is important because, as we will show below, it turns out that the extraction of the quark masses, in particular that of m_u+m_d, is very sensitive to the choice of s_0. This is illustrated by deriving lower bounds on m_u+m_d

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associated with the positivity of $\rho_5(t)$ and by investigating trial spectral functions. Ideally, one would like to pick s_0 large enough so that PQCD, to the order considered, can be shown to be reliable. Unfortunately, for larger s_0, the hadronic spectral function receives contributions from an increasing number of intermediate states and, hence, becomes increasingly hard to model. We discuss the uncertainties introduced by a compromise choice of s_0. In the extraction of m_s by JM-CPS, we argue that an artificially large value of s_0 has been used. For a more reasonable value of s_0, we are not able to find an estimate for m_s that is stable under variations of the Borel transform scale u.

The third issue arises because the continuum part of the hadronic spectral function is typically represented as a sum-of-resonances modulation of a continuum form, the overall normalization of which is fixed by assuming complete resonance dominance of the spectral function near continuum threshold. This turns out to be potentially the most important issue. We in fact show in the case of the vector two-point function, for which experimental information on the spectral function is available in the resonance region, that an analogous extrapolation from threshold to the ρ meson peak would lead to an overestimate of the spectral function in the resonance region by a factor of ~ 5. We then explain the origin of this problem from the point of view of the existing phenomenological understanding of how resonance contributions enter the expressions for low-energy observables as computed in chiral perturbation theory (χPT). Based on this understanding, we propose an alternate method for normalizing the spectral function in the resonance region which requires as input only the expression obtained from χPT to one-loop order, in the near-threshold region. We then employ this method in a reanalysis of the only sum rule treatment for which the relevant χPT expression is known, namely, that of the correlator of the product of two divergences of the strangeness-changing vector current (as used by JM-CPS to obtain the estimate quoted above for m_s) and show that the traditional method of normalization leads to a significant overestimate of m_s.

We find that the size of the corrections suggested by our consideration of the above issues can easily lower the sum rule estimates for both m_u+m_d and m_s by a factor at least as large as 2. In particular, using the corrected normalization for the hadronic spectral function in the JM-CPS analysis alone would lower the extracted value of m_s by almost exactly a factor of 2. Such a change would make the lattice and sum rule estimates consistent. Lowering both estimates by roughly the same factor would, moreover, preserve agreement of the ratio, $r=2m_s/(m_u+m_d)$, with that predicted by χPT.

The paper is organized as follows. In order to make it self-sufficient and to introduce the notation, we reproduce the necessary details from Refs. [3] and [5] in Secs. II and VII. The convergence of PQCD is discussed in Sec. III. In Sec. IV we derive lower bounds on m_u+m_d, as a function of s_0, using the positivity of the relevant spectral function ρ_5. In Sec. V we illustrate the potential sensitivity of the extracted value of m_u+m_d to the choice of s_0 by considering a number of plausible trial spectral functions. The important issue of the overall normalization of the hadronic spectral function is investigated in Sec. VI using the vector current case as an illustrative example. Based on the lessons learned from the vector channel, a reanalysis of the JM-CPS estimate of m_s is presented in Sec. VII. Finally, we end with some conclusions in Sec. VIII.

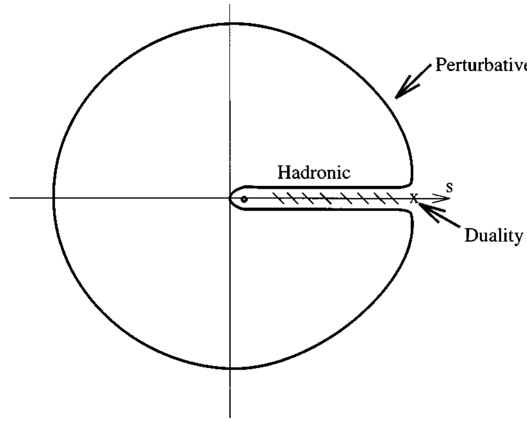

FIG. 1. The contour integral for the FESR's of the text. The ''hadronic'' integral from 0 to s, which includes contributions from the poles and cuts, is obtained using a model for the continuum portion of the spectral function, while the integral over the circle at sufficiently large s ($s>s_0$) is done using the three-loop perturbative result.

II. FINITE ENERGY SUM RULES

The standard starting point for the extraction of the light quark mass combination m_u+m_d is the Ward identity relating the divergence of the axial vector current to the pseudoscalar density,

$$\partial^\mu A_\mu^{(\pm)}(x)=(m_d+m_u)\overline{q}(x)i\gamma_5\frac{\lambda_1\pm i\lambda_2}{2}q(x), \qquad (1)$$

where $\overline{q}\equiv(\overline{u},\overline{d},\overline{s})$ and the projections $\pm\equiv(\lambda_1\pm i\lambda_2)/2$ pick out states with the quantum numbers of the $\pi^\pm$. This relation implies, for the two-point function of the product of two such divergences, that

$$\Psi_5(q^2)\equiv i\int d^4x\, e^{iq\cdot x}\langle 0|T\{\partial^\mu A_\mu^{(-)}(x),\partial^\nu A_\nu^{(+)}(0)\}|0\rangle,$$
$$=(m_d+m_u)^2 i\int d^4x\, e^{iq\cdot x}$$
$$\times\langle 0|T\{P^{(-)}(x),P^{(+)}(0)\}|0\rangle. \qquad (2)$$

The idea of the standard analysis [3,6,7,8] is then to consider the finite energy sum rules (FESR's) generated by integrating products of the form $t^n\Psi_5(t)$ over the contour shown in Fig. 1. For n negative the result involves Ψ_5 or its derivatives at $t=0$, while for n greater than or equal to zero, the result is zero. For sufficiently large radii s of the circular portion of the contour, the pseudoscalar two-point function, and hence also its line integral over the circle, can be evaluated using perturbative QCD. Taking the resulting expressions to the right-hand sides (RHS's), one obtains FESR's for the moments of the spectral function $\rho_5(t)\equiv(1/\pi)\mathrm{Im}\,\Psi_5(t)$, on the interval $(0,s)$, for example [3],

$$\int_0^s dt\ \rho_5(t) = \frac{N_c}{8\pi^2}[m_u(s)+m_d(s)]^2\frac{s^2}{2}\times\left\{1+R_1(s)+2\frac{C_4\langle O_4\rangle}{s^2}\right\}; \quad (3)$$

$$\int_0^s dt\ t\rho_5(t) = \frac{N_c}{8\pi^2}[m_u(s)+m_d(s)]^2\frac{s^3}{3}\times\left\{1+R_2(s)-\frac{3}{2}\frac{C_6\langle O_6\rangle}{s^3}\right\}, \quad (4)$$

where $m(s)$ is the running mass evaluated at the scale s, $R_1(s)$ and $R_2(s)$ contain the (higher order in α_s) perturbative corrections, and $C_4\langle O_4\rangle$ and $C_6\langle O_6\rangle$ represent the leading nonperturbative corrections, of dimensions 4 and 6, respectively [9]. They are dominated by the gluon condensate, $C_4\langle O_4\rangle \simeq (\pi/N_c)\langle\alpha_s G^2\rangle$, and the four-quark condensate, which, in the vacuum saturation approximation, is given by $C_6\langle O_6\rangle \simeq (1792/27N_c)\pi^3\alpha_s\langle\overline{q}q\rangle^2$. Since the contribution of the condensates is negligible and we have no new information to add, we simply accept the values quoted by BPR and JM-CPS in the remainder of this paper.

To extract $m_u(s)+m_d(s)$, one then needs to input the scale $s=s_0$, at which one assumes PQCD to have become valid and, second, experimental and/or model information on the hadronic spectral function (and hence its moments) below s_0. Having done so, one may then use either Eq. (3) or (4) to extract $m_u(s)+m_d(s)$ and, from that, the $\overline{\text{MS}}$ combination of the masses at any desired scale μ using the renormalization group running. Most sum rule analyses extract their estimates at $\sqrt{s}\geq 1.7$ GeV and then run down to $\mu = 1$ GeV. We believe that it is unnecessary to introduce an extra uncertainty in the estimates by relying on PQCD over this interval where the running is large. For this reason our final comparisons are at $\mu = 2$ GeV. However, to preserve continuity with existing sum rule analyses, masses quoted without any argument will always refer to the $\overline{\text{MS}}$ values at 1 GeV.

The most up-to-date version of the above analysis was performed by BPR [3], whose treatment we will follow closely below. In this analysis, BPR have used the three-loop PQCD result of Refs. [10,11] for the pseudoscalar two-point function, employing three active quark flavors with $\Lambda^{(3)}_{\overline{\text{MS}}} = 300\pm 150$ MeV [12] and the values

$$C_4\langle O_4\rangle = (0.08\pm 0.04)\ \ \text{GeV}^4, \quad (5)$$

$$C_6\langle O_6\rangle = (0.04\pm 0.03)\ \ \text{GeV}^6 \quad (6)$$

for the nonperturbative, condensate contributions. For the hadronic spectral function on the interval $(0,s)$, they include the pion pole, whose residue is known exactly in terms of f_π and m_π, and a 3π continuum contribution modulated by the π' and π'' resonances. The BPR *Ansatz* is

$$\rho_{\text{hadronic}}(s) = \rho_{\text{pole}} + F(s)\rho^{3\pi}_{\chi\text{PT}}\Theta(s-9m_\pi^2), \quad (7)$$

where the "3π continuum spectral function" $\rho^{3\pi}_{\chi\text{PT}}(t)$ is obtained from the leading-order, tree-level χPT result for $\langle 0|\partial_\mu A^\mu|3\pi\rangle$ and F is a modulating factor which accounts for the presence of the π' and π'' resonances. The form of F is taken to be a superposition of Breit-Wigner terms:

$$F(s) = A\,\frac{|\Sigma_i\xi_i/[s-M_i^2+iM_i\Gamma_i]|^2}{|\Sigma_i\xi_i/[9m_\pi^2-M_i^2+iM_i\Gamma_i]|^2}, \quad (8)$$

with $\xi_1 = 1$.

There remain three unknowns at this point, the overall normalization parameter A, the relative strength and phase ξ_2 of the two resonances, and the value of s_0. BPR find that, if they assume $s_0 \sim 2-3$ GeV^2, duality can be satisfied for a number of values of $[A,\xi_2]$. Their best solution uses the normalization $A=1$ at threshold and then fixes ξ_2 by demanding duality between the hadronic ratio

$$\mathcal{R}_{\text{had}}(s) \equiv \frac{3}{2s}\frac{\int_0^s dt\ t\rho_5(t)}{\int_0^s dt\ \rho_5(t)} \quad (9)$$

and its PQCD counterpart

$$\mathcal{R}_{\text{QCD}}(s) \equiv \frac{1+R_2(s)-\frac{3}{2}\frac{C_6\langle O_6\rangle}{s^3}}{1+R_1(s)+2\frac{C_4\langle O_4\rangle}{s^2}} \quad (10)$$

over the interval between the two resonances, i.e., $2.2 \leq s \leq 3.2$ GeV^2. We have reproduced the results of BPR with their choice of resonance parameters (which differ slightly from those listed in their published version [13]) based on the 1994 Particle Data Group (PDG) book [12]:

$$M_1 = 1300\ \ \text{MeV}, \quad \Gamma_1 = 325\ \ \text{MeV},$$

$$M_2 = 1770\ \ \text{MeV}, \quad \Gamma_2 = 310\ \ \text{MeV}. \quad (11)$$

Their preferred solution (solution 2) is shown in Fig. 2. For $s > s_0 = 3$ GeV^2, we also plot the perturbative spectral function (duality constraint) for $m_u+m_d = 12$ MeV, their extracted value of the quark mass. As is evident from the figure, the rise due to the $\pi''(1800)$ is roughly consistent, both in magnitude and slope, with the perturbative *Ansatz*. This is a consequence of tuning the normalization and relative phase of the second resonance, and leads to approximate duality over the range 2.2 GeV$^2 < s < 3.5$ GeV^2. However, the falloff of the spectral function on the far side of the $\pi''(1800)$ resonance, in contrast to the rising PQCD solution, shows that, in order to preserve duality, further resonances and intermediate states are required to bolster the BPR *Ansatz* beyond the $\pi''(1800)$ peak.

Note that, in the BPR analysis, the threshold behavior of the spectral function is not determined experimentally, but rather obtained from leading-order χPT. To the extent that SU(2)$\times$SU(2) χPT converges well at leading order, the choice $A=1$ then ensures correct normalization of the spectral function near the 3π threshold. However, in the spectral

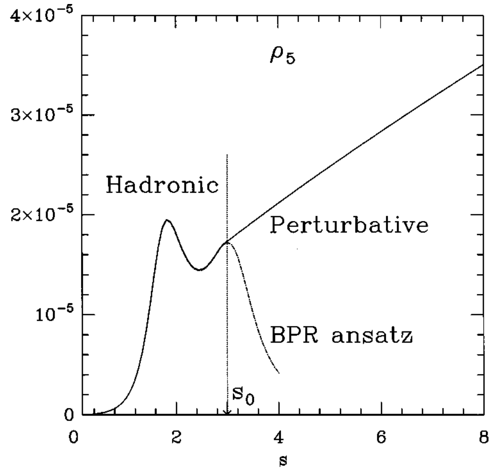

FIG. 2. The hadronic spectral function assuming that the dual region begins at the π'' resonance. The location, widths, and normalization are the same as in the solution found by BPR. The dashed line is the continuation of the BPR solution for $s>s_0$. For $s>s_0$ we also show the spectral function required to satisfy perturbative duality for $m_u+m_d=12$ MeV. The two *Ansätze* are joined smoothly by choosing $s_0=3.0\ \mathrm{GeV}^2$. Here s_0 is in GeV^2 and ρ_5 in GeV^4.

integral appearing in Eq. (3), which determines the light quark mass, the contribution of the near-threshold region is negligible compared to that from the vicinity of the resonance peaks. Correctly normalizing the spectral function in the resonance region is thus much more important than correctly normalizing it near threshold. We will show later, by considering an analogous example (the correlator of two vector currents), that the conventional threshold constraint $A=1$ almost certainly leads to a significant *overestimate* of the spectral function in the resonance region.

To summarize, we will investigate the following aspects of the BPR solution: the uncertainty in the mass extraction produced by uncertainties in the three-loop PQCD expression, the reliability of the overall normalization of the continuum contribution, and the sensitivity of the results to the value chosen for s_0, the scale characterizing the onset of duality with PQCD. The same issues are also relevant to the extraction of m_s using the Ward identity for the vector current. Our contention is that plausible systematic errors in each are such as to lower the estimates for light quark masses.

III. CONVERGENCE OF TWO-POINT FUNCTIONS IN PQCD

The pseudoscalar two-point function is known to three loops in PQCD [10,11]. The main issue, in applying this expression to the problem at hand, is the question of convergence. If, for example, we write $1+R_i$, with R_i as defined in Eqs. (3) and (4), in the form $1+x\alpha_s/\pi+y(\alpha_s/\pi)^2$, then the coefficients x,y show a geometrical growth; i.e., the growth for R_1 and R_2 is roughly the same and the average values are $x\approx6.5$ and $y\approx46$. As a result, the $\mathcal{O}(\alpha_s,\alpha_s^2)$ correction terms are 0.61 and 0.41, respectively, at $s=3\ \mathrm{GeV}^2$ where $\alpha_s/\pi\approx0.1$. Since these are large, it is important to estimate the sum of the perturbation series. One plausible possibility is to represent the series by the Padé $1/(1-0.63)$, in which case the neglected terms would further increase the PQCD estimate by $\sim35\%$ and consequently lower the BPR result for m_u+m_d by $\sqrt{1.35}$, i.e., from 12 to 10.4 MeV. In fact, historically sum rule estimates have decreased over time precisely because of the increase in the PQCD result. Part of the change has been due to the increase in the value of $\Lambda_{\mathrm{QCD}}^{(3)}$ and part due to the large positive three-loop contribution [3]. If this trend were to continue and the unknown higher-order terms were to continue to grow geometrically and contribute with the same sign (as is the case for the scalar channel discussed below), then the extracted quark mass would be significantly lowered.

The situation in the case of the scalar two-point function analyzed by JM-CPS is somewhat better. A very careful analysis of the stability of the PQCD expressions and of the choice of the expansion parameter has been carried out by CPS [4] who include terms up to α_s^3 in the two-point function and in the running of the coupling and mass. The PQCD result, after Borel transformation, has the expansion $1+4.8\alpha_s/\pi+22(\alpha_s/\pi)^2+53(\alpha_s/\pi)^3$ [5]. Taking $\alpha_s/\pi\approx0.1$, as appropriate for $u=4\ \mathrm{GeV}^2$ with $\Lambda_{\mathrm{QCD}}^{(3)}=380$ MeV, we find that the difference between the PQCD series and a possible Padé representation $1/(1-0.48)$ is only about 9%. This correction would lower the estimate of m_s by $\sim5\%$, consistent with the estimate by JM [5].

IV. CONSTRAINTS ON m_u+m_d FROM THE POSITIVITY OF $\rho_5(s)$

The fact that the spectral function $\rho_5(s)$ is positive definite above threshold allows us to place rigorous lower bounds on m_u+m_d as a function of s_0 [14]. A weak version of this bound (labeled ''pole'') is obtained by ignoring all parts of the spectral function except for the pion pole, whose contribution to the integral in Eq. (3) is $2f_\pi^2m_\pi^4$ [Eq. (4) produces a much less stringent bound and hence is not considered further]. One then finds, assuming the validity of the input three-loop PQCD result,

$$[m_u(s)+m_d(s)]^2\geqslant\frac{2f_\pi^2m_\pi^4}{\dfrac{N_c}{8\pi^2}\dfrac{s^2}{2}\left\{1+R_1(s)+2\dfrac{C_4\langle O_4\rangle}{s^2}\right\}},\tag{12}$$

where s is the upper limit of integration in Eq. (3). A stronger constraint (labeled ''ratio'') is obtained by noting that, for $\rho_5(t)\geqslant0$,

$$\frac{\int_{s_{\mathrm{th}}}^{s}dt\,t\rho_5(t)}{\int_{s_{\mathrm{th}}}^{s}dt\rho_5(t)}\leqslant s,\tag{13}$$

where s_{th} denotes the 3π threshold value. The bound is saturated when the entire spectral strength is concentrated as a delta function at s. If s in Eq. (13) is assumed to be in the dual region, this turns out to place considerably stronger constraints on $m_u(s)+m_d(s)$. To see this, note that the LHS of the inequality in Eq. (13) is, using Eqs. (3) and (4),

$$\frac{\frac{N_c}{8\pi^2}[m_u(s)+m_d(s)]^2\,\frac{s^3}{3}\left\{1+R_2(s)-\frac{3}{2}\,\frac{C_6\langle O_6\rangle}{s^3}\right\}-2\,f_\pi^2 m_\pi^6}{\frac{N_c}{8\pi^2}[m_u(s)+m_d(s)]^2\,\frac{s^2}{2}\left\{1+R_1(s)+2\,\frac{C_4\langle O_4\rangle}{s^2}\right\}-2\,f_\pi^2 m_\pi^4}. \tag{14}$$

From expression (14) we see that, if we start with a large value of $m_u(s)+m_d(s)$ and begin to lower it, keeping s fixed, the inequality (13) will be violated before we reach the value of $m_u(s)+m_d(s)$ corresponding to the pion pole saturation of the spectral function at which point the denominator in Eq. (14) vanishes. Thus the inequality (13) provides a more stringent (larger) lower bound on the extracted quark mass. This is illustrated in Fig. 3 where the dependence of $(m_u+m_d)_{\rm min}$ on s for both of the above constraints is shown.

The ''ratio'' curve shows that if one assumes $s_0 \simeq 2.5\,\mathrm{GeV}^2$, as in the BPR analysis, then $m_u+m_d \geqslant 10\,\mathrm{MeV}$. The fact that the BPR result for the mass extraction, $m_u+m_d \simeq 12\,\mathrm{MeV}$, is close to this lower bound is a reflection of the fact that the spectral strength is concentrated in the region close to the assumed onset of duality. Such a feature is, in fact, rather natural since s_0 is chosen to coincide with the $\pi''(1800)$ peak. However, if s_0 is considerably larger than $3\,\mathrm{GeV}^2$ [to alleviate the problem of large $\mathcal{O}(\alpha_s,\alpha_s^2)$ corrections to R_i at $s\sim 3\,\mathrm{GeV}^2$ discussed above], then considerably smaller masses are allowed by the ''ratio'' constraint, as is evident from the figure. Furthermore, one would, in fact, expect masses not much greater than the ''ratio'' bound to be favored in all cases where spectral functions are characterized by resonance modulation of a rising continuum phase space background and have their spectral strength concentrated in the region near s_0.

These bounds make it clear that the value of the quark mass extracted from FESR's will tend to be very strongly correlated with assumptions about the appropriate value of s_0. In addition, it will, of course, depend on the details of the hadronic spectral function from the 3π threshold up to s_0, which are, at present, not experimentally determined. Since the perturbative $\rho_5(t)$ is known up to the overall normalization, which is given by the quark mass, one test of the validity of the phenomenological *Ansatz* for the hadronic spectral function would be to show that the results for the quark mass remained stable under variations of the upper limit of integration s in Eqs. (3) and (4). This test, however, is meaningful only if one already knows that the values of s being employed are greater than s_0. Unfortunately, the lack of experimental information on the hadronic $\rho_5(s)$ precludes the possibility of making such a test. In the next section we construct plausible spectral functions, all satisfying duality, corresponding to a range of possible values for s_0 lying between 3 and $10\,\mathrm{GeV}^2$, by including higher resonances in the 3π channel. These models illustrate how, in the absence of experimental information, the uncertainty in m_u+m_d might be as large as a factor of 2 if considerably higher values of s_0 are chosen.

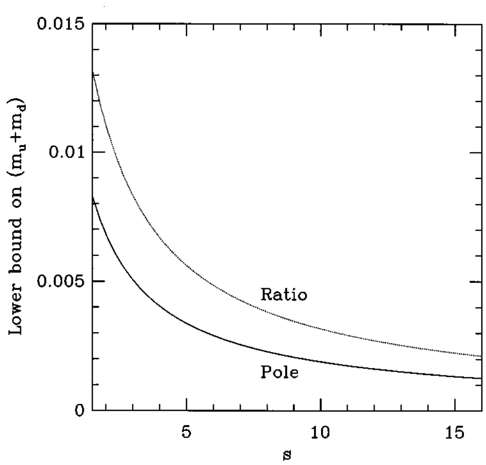

FIG. 3. The lower bounds on m_u+m_d as a function of s. These are obtained by saturating the spectral function with the pion pole contribution and from the ''ratio'' method described in the text. Here s is in GeV^2 and m_u+m_d in GeV.

V. PLAUSIBLE SPECTRAL FUNCTIONS $\rho_5(s)$ IN THE DUAL REGION

The assumption of duality places constraints on the form of the spectral function $\rho_5(s)$. Below $s=s_0$, these constraints amount only to the determination of certain moments of the spectral function on the interval $(s_{\rm th},s_0)$ and, hence, are not particularly strong. In fact, as we illustrate below, the constraints of Eqs. (3) and (4) allow considerable freedom in the choice of $\rho_5(s)$ for $s<s_0$. For $s>s_0$, in contrast, duality determines the ''average'' $\rho_5(s)$, i.e., averaged over some suitable region of s. This average value is given by the perturbative $\rho_5(s)$, which can be obtained straightforwardly by differentiating the RHS of either Eq. (3) or (4). We evaluate these derivatives numerically using either of the two forms, which of course give consistent results. Even if one eliminates the running masses by matching the ratios of Eqs. (9) and (10) for $s>s_0$, it is easy to show that the resulting equation completely determines the perturbative $\rho_5(s)$, up to an overall multiplicative factor, for all $s>s_0$. The result of the duality constraints, in either form, is that $\rho_5(s)$ must be a monotonically increasing function of s, for s in the duality region. Numerically we find that this function is approximately linear as illustrated in Fig. 2.

The hadronic spectral function in this channel is not known experimentally. It receives contributions not only from the pion and its resonances, but also from the resonant and nonresonant portions of the $3\pi, 5\pi, 7\pi$, $K\bar{K}\pi, \ldots, N\bar{N}, \ldots$ intermediate states. Experimentally, only the $\pi'(1300)$ and $\pi''(1800)$ have been observed as distinct resonances [15]. Even so, their decay constants are not known experimentally, and hence the normalization of their contributions to the spectral function have to be treated as free parameters. The number of multiparticle intermediate states one has to consider, moreover, grows with s_0, as does the problem of separating their resonant and nonresonant portions. From dimensional arguments, the contribution of these various intermediate states will grow linearly at sufficiently large s. In the region of resonances, the resonances will modulate the cut contribution and the hadronic spectral function is expected to match the PQCD behavior only after an average over some interval of s. This averaging is crucial if the resonances are narrow and isolated. Alternately, if the widths of subsequent resonances, for example, π''' and π'''', become much greater than the resonance separation, then the overlap of resonances can provide the monotonically rising behavior required by duality and averaging is not crucial.

We illustrate these points by constructing *Ansätze* for the hadronic spectral function which are of the form used by BPR, i.e., involving resonance modulation of the continuum 3π background. To explore values of s_0 as large as 10 GeV^2 with the *Ansatz* above, we include pseudoscalar resonances with masses as large as $\sim\sqrt{10}$ GeV. For the first two such resonances, the $\pi'(1300)$ and $\pi''(1800)$, we use the 1996 Particle Data Group values for the masses and widths. For the remaining two resonances, the π''' and π'''', expected in this range, we are guided by model predictions. The π''' resonance is typically expected to lie around 2400 MeV in models constrained by the lower part of the meson spectrum [16]. In addition, the $^3P_0$ model [17], which has proved to be reasonably successful in estimating decay widths [18], predicts a width for the $\pi'''(2400)$ between 700 and 1900 MeV [16,19] depending on how the relativistic effects are treated. The approach leading to 700 MeV gives 300 MeV for the width of the $\pi''(1800)$, which is larger than the experimental value of 212(37) MeV. We therefore assume the lower limit 700 MeV for the width in this study, even though this may be an overestimate. Similarly, we assume that π'''' lies at 3150 MeV with a width of 900 MeV. In short, we choose

$$M_1=1300\ \text{MeV},\quad \Gamma_1=325\ \text{MeV},$$

$$M_2=1800\ \text{MeV},\quad \Gamma_2=212\ \text{MeV},$$

$$M_3=2400\ \text{MeV},\quad \Gamma_3=700\ \text{MeV},$$

$$M_4=3150\ \text{MeV},\quad \Gamma_4=900\ \text{MeV}. \tag{15}$$

The decay constants of all of these resonances are unknown and will therefore be treated as free parameters. The limitations of such a truncated spectral function are obvious; however, it should be noted that, because we have allowed ourselves some phenomenological freedom in treating the strengths and widths of the last two resonances, our *Ansätze* for the spectral function can also be thought of as providing an approximate means of representing a combination of resonant and nonresonant effects. Our aim is, in any case, to simply demonstrate how the piling up of resonances can give the PQCD behavior, and the nature of plausible spectral functions for which the ''extracted'' quark mass is, as for the BPR case, rather close to the value given by the ''ratio'' bound.

For the resonance-modulated spectral function we adopt, following BPR, the *Ansatz*

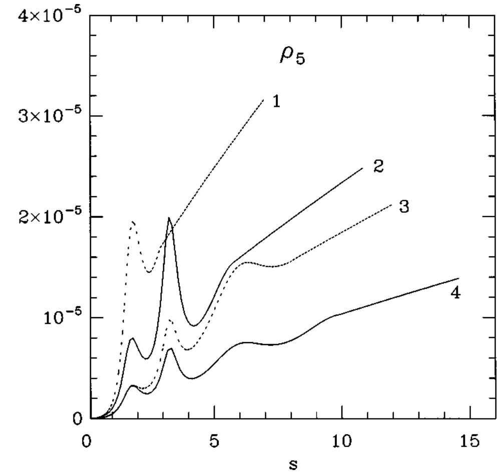

FIG. 4. Four examples of the hadronic spectral function, assuming different resonance structure and point of matching to the perturbative solution. Units are as in Fig. 2. The locations and widths of the resonances used are given in the text. The normalization A and the relative strengths c_i for the four cases are given in Table I, along with the values of s_0 and m_u+m_d used to derive the perturbative solution.

$$\rho_5(s)=F(s)\rho^{3\pi}_{\chi\text{PT}}(s), \tag{16}$$

where $\rho^{3\pi}_{\chi\text{PT}}(s)$ is the spectral function corresponding to the leading-order, tree-level χPT result for $\langle 0|\partial_\mu A^\mu|3\pi\rangle$, and

$$F(s)=A\,\frac{\Sigma_i c_i M_i\Gamma_i/[(s-M_i^2)^2+M_i^2\Gamma_i^2]}{\Sigma_i c_i M_i\Gamma_i/[(s_{\text{th}}-M_i^2)^2+M_i^2\Gamma_i^2]}. \tag{17}$$

The sum in Eq. (17) runs over the appropriate number of resonances, depending on s_0 as described below, with relative strengths c_i. The parameter A is the overall normalization of the resonance contribution to the continuum part of the spectral function at the 3π threshold. We have taken the c_i to be real, in order to simplify the task of searching for suitable spectral functions, whereas BPR, who use a slightly different form for F, as given in Eq. (8), with just the first two resonances, allow the relative strength of the two resonances to be complex.

We display a series of spectral functions in Fig. 4, all satisfying duality and constructed by employing up to four resonances in the *Ansatz* above. The values for A, $\{c_i\}$, s_0, and m_u+m_d used in the construction are given in Table I. As one can see from the figure, there exist perfectly plausible spectral functions corresponding to $m_u+m_d=12$, 9, 8, and 6

TABLE I. The parameters used to generate the spectral functions shown in Fig. 4. The normalization at threshold A and the relative weights c_i assigned to the resonances are defined in Eq. (17).

	s_d (GeV^2)	m_u+m_d (MeV)	c_1	c_2	c_3	c_4	A
Case 1	3.0	12.0	1	$-0.23+0.65i$			1.0
Case 2	5.7	9.0	1	1.0	2.3		1.0
Case 3	8.0	8.0	1	1.2	5.0	6.5	0.7
Case 4	10.0	6.0	1	0.8	2.0	3.68	0.5

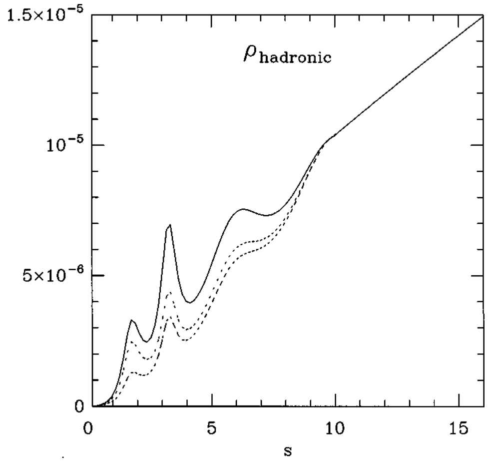

FIG. 5. The hadronic spectral function assuming four resonances with the quantum numbers of the pion. Units are as in Fig. 2. The locations and widths of the resonances are given in the text. The normalization A and relative strengths c_i for the three cases are given in Table II. These have been adjusted to make the hadronic form join smoothly to the duality *Ansatz* at $s_0 = 10\ \text{GeV}^2$. The solid line corresponds to case 1, the dotted line to case 2, and the dashed line to case 3.

MeV. The first case ($s_0 = 3\ \text{GeV}^2$, $m_u + m_d = 12$ MeV) is the BPR solution discussed before. The second case ($s_0 = 5.7\ \text{GeV}^2$, $m_u + m_d = 9$ MeV) corresponds to including three resonances and matching to the duality solution at the top of the third resonance. The assumption here is that the third and higher resonances merge to produce the dual solution above this point. The matching in the third case ($s_0 = 8\ \text{GeV}^2$, $m_u + m_d = 8$ MeV) is at the beginning of the rise of the fourth resonance, while in the fourth case ($s_0 = 10\ \text{GeV}^2$, $m_u + m_d = 6$ MeV) we match at the top of the fourth resonance. In cases where we match at the peak of a resonance, the dual region actually appears to begin somewhat below the input value of s_0. This is because the slope of the rising side of the last resonance tends to match reasonably well the slope of the PQCD version of $\rho_5(s)$.

These spectral functions are, by construction, perfectly dual for $s > s_0$. Duality also requires the low-energy ($s < s_0$) part of $\rho_5(s)$ to have the correct moments to satisfy Eqs. (3) and (4). However, the constraint of duality does not lead to a unique solution. Experimental data (decay constants) are needed to fix the overall normalization A and the relative weights c_i. We illustrate this point in Fig. 5 by constructing three spectral functions that differ for $s < s_0$. In all three cases, s_0 and the input $m_u + m_d$ in the PQCD expression are fixed to be the same, while the values of parameters A and c_i are as defined in Table II. The corresponding output values for $m_u + m_d$ are shown in Fig. 6. As expected, they converge to the input value in the dual region.

There are three features of these *Ansätze* that should be noted. First, the value of $m_u + m_d$ decreases with s_0 in a manner very similar to the ''ratio'' bound. This is because in each case the spectral function is stacked up towards s_0. Second, we find that, to produce spectral functions corresponding to values of $m_u + m_d$ only a few MeV above the ''ratio'' bound, the threshold normalization parameter A has to be decreased with increasing s_0. In the next section we

TABLE II. The parameters used generate the plots shown in Fig. 5. The values of s_0 and $m_u + m_d$ have been fixed to $s_0 = 10\ \text{GeV}^2$ and $m_u + m_d = 6$ MeV, respectively, in each of the three cases.

	c_1	c_2	c_3	c_4	A
Case 1	1	0.8	2.0	3.680	0.5
Case 2	1	0.6	2.0	5.160	0.4
Case 3	1	1.0	4.0	11.75	0.3

will show that values of A significantly smaller than 1 are, in fact, to be expected, based on a consideration of the analogous vector current correlator, for which the normalization in the resonance region is known experimentally. Third, the c_i are large. It is not clear, *a priori*, if this should be considered unreasonable or not. For example, in the narrow width approximation the c_i would scale as $\sim (f_i^2 M_i^4)/(f_\pi^2 M_\pi^4)$ and thus have an explicit dependence on M_i^4. [The BPR model spectral functions, being even larger than ours, of course, correspond to even larger $\pi'(1300)$ and $\pi''(1800)$ decay constants.] Moreover, by leaving the normalizations as free parameters, we are potentially incorporating other nonresonant background effects. Ultimately, this issue can only be resolved by appeal to experimental data which, unfortunately, is not available at present.

The bottom line of the above discussion is that since both the correct value for the location of the onset of duality with PQCD and the correct form of the hadronic spectral function are at present unknown, the value of $m_u + m_d$ extracted using FESR's can easily vary by a factor of 2. As we have pointed out, using $s_0 \sim 3\ \text{GeV}^2$ leads to a perturbation series in which the α_s and α_s^2 terms are large. As soon as one allows significantly larger values of s_0, in order to alleviate this problem, however, considerably smaller values of the extracted quark mass are possible. We will now, furthermore, argue that the conventional method of normalizing the continuum part of the spectral function tends to produce significant overestimates of the resonance contributions and, hence, also significant overestimates of the extracted quark masses.

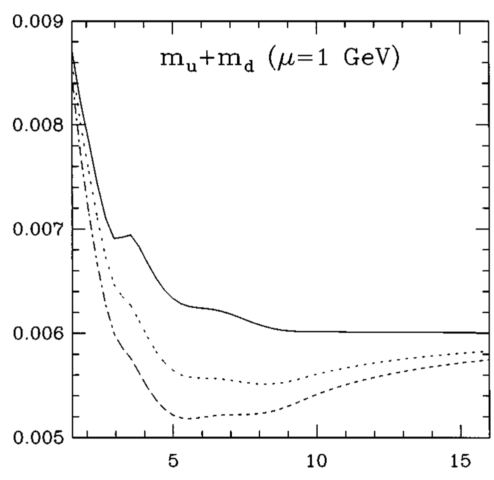

FIG. 6. The output values of $m_u + m_d$ (run to $\mu = 1$ GeV) for the three cases of the spectral function shown in Fig. 5. Here s is in GeV^2 and $m_u + m_d$ in GeV.

VI. NORMALIZATION OF THE RESONANCE-MODULATED SPECTRAL FUNCTION

The assumption that a given spectral function may be written as a Breit-Wigner resonance modulation of a continuum phase space factor, as exemplified in Eqs. (7) and (8), is valid in the vicinity of a narrow resonance. We will assume that this *Ansatz*, used by both BPR and JM-CPS (see Sec. VII) in the pseudoscalar and scalar channels, respectively, is a good approximation. This fixes the general form of the spectral function, but, of course, does not fix the magnitude in the resonance region, since the relevant pseudoscalar and scalar decay constants (with the exception of f_π) are not known experimentally. Both BPR and JM-CPS deal with this problem by assuming that resonance dominance of the relevant spectral function continues to hold all the way down to continuum threshold. Thus, for example, the overall scale of the BPR and JM-CPS *Ansätze* for the continuum part of the spectral function is obtained by choosing $A=1$, i.e., by assuming that the tails of the resonances reproduce the *full* threshold spectral function. BPR, in the absence of experimental data, use the tree-level χPT expression for the spectral function in the threshold region. The JM-CPS treatment differs only in that they normalize the sum-of-resonances *Ansatz* at the $K\pi$ threshold using experimental data (the scalar form factor at threshold is computed using the Omnes representation with experimental $K\pi$ phase shifts as input).

The second key point in the JM-CPS *Ansatz* for the spectral function is the assumption that one can take the ''standard'' s-wave s-dependent widths for the resonance contributions. This assumes that the effective coupling of the strange scalar resonances to $K\pi$ is momentum independent over the whole kinematic range relevant to the spectral integral. We will now show that the combination of this assumption and of resonance saturation threshold can fail badly by studying its exact analogue in the isovector vector channel. In fact, in the vector channel, the analogous set of assumptions produces a significant overestimate of the spectral strength in the region of the resonance peak.

Consider, therefore, the vector correlator

$$\Pi_{33}^{\mu\nu}(q^2)\equiv(q_\mu q_\nu-q^2g_{\mu\nu})\Pi_{33}(q^2)$$
$$=i\int d^4x\, e^{iq\cdot x}\langle 0|T\{V_3^\mu(x)V_3^\nu(0)\}|0\rangle, \quad (18)$$

where V_3^μ is the $I=1$ vector current. In the narrow width approximation, the ρ contributions to the spectral function of Π_{33} and ρ_{33}, at the ρ peak, is known in terms of the ρ decay constant $F_\rho=154$ MeV,

$$[\rho_{33}(m_\rho^2)]_\rho=\frac{F_\rho^2}{\pi\Gamma_\rho m_\rho}=0.0654. \quad (19)$$

Let us now apply the analogue of the BPR and JM-CPS *Ansätze* to the vector channel, by assuming the (trial) spectral function to be given by

$$\rho_{33}^{\rm trial}(s)=\left[\frac{1}{48\pi^2}\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}\theta(s-4m_\pi^2)\right]\left[\frac{c_\rho^{\rm BW}(s)}{c_\rho^{\rm BW}(4m_\pi^2)}\right], \quad (20)$$

where the quantity in the first set of square brackets is the leading-order χPT expression for the spectral function [20] and

$$c_\rho^{\rm BW}(s)=\frac{sm_\rho\Gamma_\rho/[m_\rho^2-4m_\pi^2]^{3/2}}{[(m_\rho^2-s)^2+\Gamma_\rho(s)^2m_\rho^2]}, \quad (21)$$

which follows from employing the p-wave s-dependent width

$$\Gamma_\rho(s)=\frac{m_\rho\Gamma_\rho}{(m_\rho^2-4m_\pi^2)^{3/2}}\left[1-\frac{4m_\pi^2}{s}\right]^{3/2}s. \quad (22)$$

We have chosen this form of the width in analogy to the ''standard'' s-wave s-dependent width of JM-CPS. This *Ansatz* assumes that the effective coupling of the ρ to $\pi\pi$ has the minimal form $g_{\rho\pi\pi}\rho_\mu(\pi^+\partial^\mu\pi^- - \pi^-\partial\pi^+)$ with $g_{\rho\pi\pi}$ independent of momentum over the relevant kinematic range. The threshold factor $[1-4m_\pi^2/s]^{3/2}$ in the numerator of Eq. (22) has been separated out explicitly in writing Eq. (20). The *Ansatz*, Eq. (20), then implies

$$\rho_{33}^{\rm trial}(m_\rho^2)=\frac{1}{192\pi^2}\frac{(m_\rho^2-4m_\pi^2)^{7/2}}{m_\rho^3\Gamma_\rho^2m_\pi^2}=0.27, \quad (23)$$

a factor of 4.1 too large. Had we instead used the normalization given by the full next-to-leading-order χPT expression [20] (which matches well to experimental data near threshold)

$$\rho_{33}^{\chi{\rm PT}}(s)=\frac{1}{48\pi^2}\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}\theta(s-4m_\pi^2)$$
$$\times\left[1+\frac{4L_9^r(\mu)s}{f_\pi^2}+\cdots\right], \quad (24)$$

the peak height would be further increased by a factor of 1.28, the correction being dominated, for $\mu\sim m_\rho$, by the term in the square brackets in Eq. (24) involving the $\mathcal{O}(q^4)$ renormalized low-energy constant (LEC) L_9^r,

$$\frac{4L_9^r(m_\rho)(4m_\pi^2)}{f_\pi^2}=0.24, \quad (25)$$

where we have used $L_9^r(m_\rho)=0.0069(2)$ [21] and $+\cdots$ refers to loop contributions whose form is not important in what follows. Note that, since it is the next-to-leading-order expression, Eq. (24), which matches experimental data, it is this latter normalization which corresponds to the JM-CPS treatment of the scalar channel. The analogue of the JM-CPS *Ansatz*, in the case of the vector correlator, thus overestimates the spectral function at the ρ peak by a factor of 5.1.

The source of this problem is not difficult to identify and, in fact, turns out to be that the crucial assumption that the spectral function can be taken to be completely resonance dominated, even near threshold, is incorrect. This is most easily seen from the perspective of χPT. Indeed, it is known that, when one eliminates resonance degrees of freedom from a general, extended effective Lagrangian, producing in the process the usual effective chiral Lagrangian $\mathcal{L}_{\rm eff}^{\chi{\rm PT}}$ relevant to the low-lying Goldstone boson degrees of freedom alone, the effect of the resonances present in the original theory is to produce contributions to the LEC's appearing in $\mathcal{L}_{\rm eff}^{\chi{\rm PT}}$ [22,23]. There are two important observations about the nature of these contributions which are of relevance to the present discussion. The first is that the resonances do not contribute to the lowest-order $[\mathcal{O}(q^2)]$ LEC's of $\mathcal{L}_{\rm eff}^{\chi{\rm PT}}$; instead, the leading (in the chiral expansion) contributions are to the $\mathcal{O}(q^4)$ LEC's $L_k^r(\mu)$ (where μ is the χPT renormalization scale and we adhere throughout to the notation of Gasser and Leutwyler [24]). The second is the phenomenological observation that, if one takes $\mu\sim m_\rho$, the resonance contributions essentially saturate the $L_k^r(\mu)$ [22,23] (see, for example, Table 2.1 of Ref. [21], for a comparison with recent experimental determinations of the LEC's). An immediate consequence of the first observation is that the correct normalization for the resonance contributions to quantities like ρ_{33} or $\langle 0|\partial_\mu A^\mu|3\pi\rangle$, near threshold, cannot be that coming from the tree-level $[\mathcal{O}(q^2)]$ χPT contributions, since such contributions are associated with the Goldstone boson degrees of freedom alone and contain no resonance contributions whatsoever. Similarly, normalizing to the full threshold value, as obtained, for example, from experiment, would also be incorrect, since this full value necessarily contains both tree-level and leading nonanalytic contributions, neither of which can be associated with the resonance degrees of freedom, in addition to the $\mathcal{O}(q^4)$ LEC contributions which *do* contain resonance contributions. Fortunately, the second observation provides us with an obvious alternative for normalizing resonance contributions near threshold. We propose, therefore, to accept the phenomenological observation above as a general one and identify resonance effects in near-threshold observables with those contributions to the one-loop expressions for these observables involving the appropriate $\mathcal{O}(q^4)$ LEC's $\{L_k^r\}$, evaluated at a scale $\mu\sim m_\rho$. Such an identification, however, requires that the LEC be dominated by the appropriate resonance, as is the case for the vector (L_9) and scalar (L_5) channels, but not for the pseudoscalar channel. This prescription, like that of BPR and JM-CPS, represents a means of using information solely from the near-threshold region (in this case, obtainable from a knowledge of the chiral expansion of the spectral function) to normalize the spectral function in the resonance region. However, we will show below that, in contrast to the analogue of the BPR and JM-CPS *Ansätze*, which was in error by a factor of ~ 5 at the ρ peak, the new prescription normalizes the peak accurate to within a few percent. Based on the success of the prescription in this channel, we will then apply it to a reanalysis of the JM-CPS extraction of m_s involving the correlator of the divergences of the vector current.

Let us return, then, to the spectral function ρ_{33}. According to the discussion above, the ρ meson contributions to ρ_{33}, near threshold, can be obtained by taking just that term in Eq. (24) proportional to L_9^r, evaluated at a scale $\mu\sim m_\rho$. The only change in the above analysis is then a rescaling of $\rho_{33}^{\rm trial}$ in Eq. (23) by a factor of 0.24, the value of the $\mathcal{O}(q^4)$ LEC contribution in Eq. (24) at $\mu=m_\rho$. This leads to a prediction for the spectral function at the ρ peak of

$$\rho_{33}^{\rm LEC}(m_\rho^2)=0.067, \tag{26}$$

in good agreement with the experimental value given in Eq. (19).

Let us stress that the precise numerical aspects of the prescription above, namely, the supposition that the normalization at resonance peak of resonance contributions to the hadronic spectral function can be obtained by evaluating the relevant $\mathcal{O}(q^4)$ LEC contributions appearing in near-threshold χPT expressions, *at a scale* $\mu\sim m_\rho$, is one that is purely phenomenologically motivated [22,23]. While highly successful in the case of the vector channel, it has not been tested outside this channel. The fact that resonance contributions begin only at $\mathcal{O}(q^4)$ in the chiral expansion and, hence, that resonances do not contribute to either lowest-order tree-level or leading nonanalytic terms in the χPT expansions of the relevant spectral functions, however, clearly indicates, independent of the numerical reliability of this prescription, the unsuitability of normalizing the resonance peaks by associating the full χPT or experimental values near threshold with resonance effects. Moreover, as long as the spectral functions of interest have even reasonably normal chiral expansions, with the dominant contributions near threshold coming from the lowest-order tree-level contributions, we can conclude that the standard method of normalization will produce values for these spectral functions in the resonance region that are overestimated by a significant numerical factor.

At this stage we should also mention that Stern and collaborators have suggested that the normalization at threshold could actually be much larger than that given by leading-order χPT, as is expected in ''generalized χPT'' [25]. They then argue that, in that case, the quark masses would be even larger. Our observations are also relevant in this case: We again stress that, since the sum rules we consider are dominated by the resonance region, threshold normalization will only provide useful input if one can disentangle the contributions to threshold amplitudes associated with resonances from those associated with the Goldstone boson degrees of freedom.

VII. REANALYSIS OF THE JM-CPS EXTRACTION OF m_s

In this section we will employ the prescription proposed above to a reanalysis of the JM-CPS extractions of m_s [4,5]. Such a reanalysis is possible in this case because the one-loop χPT expression for the relevant scalar form factor is known [26]. To introduce the notation, we briefly review the analysis of Ref. [5] (that of Refs. [4,27] is similar and need not be discussed separately). These analyses involve a standard QCD sum rule treatment of the correlation function

$$\begin{aligned}\Psi(q^2)&=i\int d^4x e^{iq\cdot x}\langle 0|T\{\partial^\mu V_\mu(x)\partial^\nu V_\nu^\dagger(0)\}|0\rangle\\&=(m_s-m_u)^2 i\int d^4x e^{iq\cdot x}\langle 0|T\{S(x)S^\dagger(0)\}|0\rangle,\end{aligned} \tag{27}$$

where $V_\mu(x)$ is the strangeness-changing vector current and $S(x)$ the corresponding strangeness-changing scalar current. The correlator of scalar currents is evaluated using the operator product expansion (OPE). All terms on this side of the sum rule are proportional to $(m_s-m_u)^2$, and the full α_s^3 PQCD result is known for the predominant contribution Ψ_0'' [4]. The hadronic spectral function in the phenomenological side is again taken to be a sum-of-resonances modulation of the spectral function relevant to the $K\pi$ intermediate state near threshold.

JM-CPS write the $K\pi$ contribution to the physical spectral function as

$$\rho_{K\pi}(s)=\frac{3}{32\pi^2 s}\,\theta(s-s_+)\sqrt{(s-s_+)(s-s_-)}|d(s)|^2, \tag{28}$$

where $s_\pm=(m_K\pm m_\pi)^2$ and $d(s)$ is the strangeness-changing scalar form factor, measured in K_{l3} for $m_l^2\leqslant s\leqslant s_-$,

$$d(s)\equiv(m_K^2-m_\pi^2)f_0(s)=(m_K^2-m_\pi^2)f_+(s)+sf_-(s), \tag{29}$$

with $f_\pm(s)$ the usual form factors defined by

$$\langle\pi^0(p')|\bar{s}\gamma_\mu u|K^+(p)\rangle=\frac{1}{\sqrt{2}}\,[(p'+p)_u f_+(s)+(p-p')_\mu f_-(s)]. \tag{30}$$

In their analysis, JM-CPS employ the following resonance-modulation *Ansatz* for the spectral function:

$$\rho_{\rm hadronic}(s)=\frac{3}{32\pi^2 s}\,\sqrt{(s-s_+)(s-s_-)}|d(s_+)|^2F(s), \tag{31}$$

where

$$F(s)=\frac{\Sigma_n c_n^{\rm BW}(s)}{\Sigma_n c_n^{\rm BW}(s_+)}, \tag{32}$$

with

$$c_n^{\rm BW}(s)=\frac{f_n^2 m_n^5\Gamma_n}{(m_n-s)^2+m_n^2\Gamma_n^2(s)}. \tag{33}$$

In Eqs. (31)–(33), s_+ is the continuum $K\pi$ threshold, and f_n, m_n, and Γ_n are the decay constant, mass, and width of the nth scalar resonance, $\Gamma_n(s)$ being the usual s-dependent width given in [5]. The s dependence of the width factor occurring in the numerator of the Breit-Wigner resonance forms has already been factored out explicitly in writing Eq. (31). The sum in Eq. (32) is taken to run over two resonances [the $K_0^*(1430)$ and $K_0^*(1950)$], and the duality point s_0 of QCD sum rules (describing the point beyond which the physical spectral function is to be modeled by its perturbative expression) is fixed by a stability analysis. Note that the normalization procedure above assumes that the physical spectral function is completely saturated by resonance contributions near threshold. The threshold value of the scalar form factor, $d(s_+)=0.33\pm0.02\ {\rm GeV}^2$, is obtained using the Omnes representation with experimental $K\pi$ phase shifts as input. This result is, moreover, shown to be consistent with that of χPT to one loop, which can be obtained from the expression for $f_0(s)$ given by Gasser and Leutwyler [26] [$d_0^{\chi{\rm PT}}(s_+)=0.35\ {\rm GeV}^2$]. Last, the master equation used for extracting m_s is [5]

$$u^3\hat{\Psi}''_{\rm OPE}=\int_0^{s_0}e^{-s/u}\rho_{\rm hadronic}ds+\int_{s_0}^{\infty}e^{-s/u}\rho_{\rm PQCD}ds, \tag{34}$$

where both $\hat{\Psi}''_{\rm OPE}$ and $\rho_{\rm PQCD}$ are proportional to $(m_s-m_u)^2$.

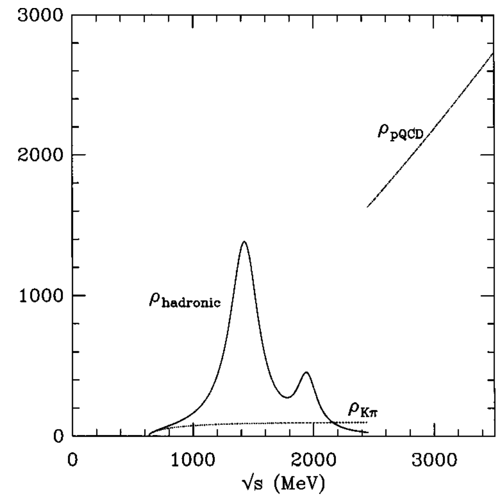

FIG. 7. Plots of the spectral functions $\rho_{\rm PQCD}$ and $\rho_{\rm hadronic}$ used by JM-CPS [5,4]. The scale of matching between the PQCD and hadronic solution is $s_0=6.0\ {\rm GeV}^2$. To highlight the fact that the $\rho_{\rm hadronic}$ is dominated by the resonance contribution, we also show $\rho_{K\pi}$, i.e., $\rho_{\rm hadronic}$ without the Breit-Wigner modulation factor. For convenience, we plot $\rho\times10^5/(m_s-m_u)^2$, and so the units along the y axis are ${\rm GeV}^2$. The values $m_s=189$ MeV and $m_u=5$ MeV have been taken from Ref. [5].

The first of the three issues raised by us, namely, the reliability of PQCD, has already been discussed in Sec. III. We agree with JM-CPS that in this channel the effect of the neglected α_s^4 and higher contributions could, at best, lower estimates of m_s by $\sim5\%$. The remaining two issues, the value of s_0 and the normalization of the hadronic spectral function, are far more serious, as we now explain.

To elucidate the role of s_0 in the JM-CPS analysis we plot, in Fig. 7, both the model JM hadronic spectral function (for $s<s_0$) and the PQCD version of the spectral function (for $s>s_0$). We have used the JM values corresponding to the preferred solution, i.e., $s_0=6.0\ {\rm GeV}^2$, $\Lambda_{\rm QCD}^{(3)}=380$ MeV, and $m_s=189$ MeV. The plot shows very clearly that the *Ansatz* for $\rho_{\rm hadronic}$ is, at best, valid only for $s\leqslant4.0\ {\rm GeV}^2$. Furthermore, as evident from Eqs. (31)–(33), $\rho_{\rm hadronic}$ goes to a constant at large s, whereas $\rho_{\rm PQCD}$ grows linearly (with logarithmic corrections). For this reason there is a large discontinuity between $\rho_{\rm hadronic}$ and $\rho_{\rm PQCD}$ even for s as low as $4\ {\rm GeV}^2$. The only way that $\rho_{\rm hadronic}$ constructed from the $K\pi$ channel can satisfy duality is if there is a piling up of higher resonances, and these have to have large amplitudes (as we illustrated in Sec. V for the pseudoscalar channel). We contend that s_0 should only be chosen in the range where ρ_{hadronic} is known reliably. However, for $s_0 \leqslant 4.0\ \text{GeV}^2$ and using the JM-CPS *Ansatz* for ρ_{hadronic}, we have not been able to find a result for m_s that is stable under variations of the Borel parameter u. It was precisely this lack of stability that forced JM-CPS to choose a larger s_0. Such a choice, we contend, is not reasonable as $\rho_{\text{hadronic}} \ll \rho_{\text{PQCD}}$ over the range $3 < s < 6\ \text{GeV}^2$; i.e., duality is badly violated over this whole range.

Last, we turn to the quantity $d(s_+)$, which sets the overall normalization of the resonance contributions in Eq. (31). This quantity is crucial in the JM-CPS analysis since, as noted by JM, the extracted value of m_s scales directly with $d(s_+)$. The problem is that, just as for the light quark case, the spectral integral appearing on the phenomenological side of the sum rule is dominated, not by the near-threshold region, but by resonance contributions. The *Ansatz* (31)–(33) for the spectral function, however, is designed only to produce the correct overall normalization at the $K\pi$ threshold. From our discussion above of the analogous treatment of the vector current correlator, it is clear that such an *Ansatz* will overestimate the resonance contributions near threshold and, hence, almost certainly significantly overestimate the spectral function in the resonance region. To correct this problem we need to properly rescale the JM-CPS *Ansatz* at threshold. We do so on the basis of the proposal above; i.e., we assume that in the scalar channel, just as in the vector channel, the $\mathcal{O}(q^4)$ LEC's, evaluated at a scale $\mu \sim m_\rho$, give the correct normalization of the scalar resonance contributions at threshold. It is easy to implement this revised normalization of ρ_{hadronic} because, not only is the one-loop χPT expression for $d(s)$ known [26], but, in addition, Jamin and Münz have demonstrated explicitly the accuracy of this expression for $d(s_+)$ [5].

Let us write the one-loop χPT expression for $d(s_+)$ in the form

$$d_{\chi\text{PT}}(s_+) = d_{\text{tree}}(s_+) + d_{\text{res}}(s_+,\mu) + d_{\text{loop}}(s_+,\mu), \quad (35)$$

where $d_{\text{tree}}(s_+)$ is the leading, $\mathcal{O}(q^2)$ tree-level contribution, $d_{\text{res}}(s_+,\mu)$ contains the $\mathcal{O}(q^4)$ LEC contributions, and $d_{\text{loop}}(s_+,\mu)$ contains the contributions associated with one-loop graphs generated from the $\mathcal{O}(q^2)$ part of the effective chiral Lagrangian. The latter two terms are separately scale dependent. According to the prescription introduced above, resonance contributions to $d(s_+)$ are to be identified with $d_{\text{res}}(s_+,m_\rho)$. Resonance contributions to $|d(s_+)|^2$, consistent to one-loop order, are thus given by

$$|d(s_+)|^2_{\text{res}} \simeq 2\,d_{\text{tree}}(s_+)\,d_{\text{res}}(s_+,m_\rho). \quad (36)$$

Using [26]

$$d_{\text{tree}}(s_+) = (m_K^2 - m_\pi^2) = 0.22\ \ \text{GeV}^2,$$

$$d_{\text{res}}(s_+,m_\rho) = 4 s_+ (m_K^2 - m_\pi^2) L_5^r(m_\rho)/f_\pi^2, \quad (37)$$

with $L_5^r(m_\rho) = 0.0014 \pm 0.0005$, we find that $|d(s_+)|^2_{\text{res}} \sim 0.23 |d(s_+)|^2$. With no changes to the JM-CPS analysis other than the corresponding rescaling of the continuum spectral function, the value of m_s would thus be lowered by almost exactly a factor of 2. However, as discussed above, there are problems of consistency with using the JM-CPS *Ansatz* for the spectral function with values of s_0 as large as $6\ \text{GeV}^2$.

In light of the above corrections, the question before us is whether it is possible to get a stable estimate of m_s by repeating the JM-CPS analysis with $s_0 \approx 4\ \text{GeV}^2$ and an overall normalization of ρ_{hadronic} of $A \approx 0.25$. To answer this question we have varied $\Lambda^{(3)}_{\text{QCD}}$ in the range 200–450 MeV, the relative strength f_2/f_1 of the two Breit-Wigner resonances in the modulating factor over 0.2–1, and A over the range 0.2–1. Despite this, we have failed to find a solution that is stable under variations in the Borel scale u. The cause of this failure is the *Ansatz* for ρ_{hadronic} and the small range of s over which it can be evaluated. It is our contention that reliable results for m_s using sum rules can only be obtained if ρ_{hadronic} is determined to high precision over a sufficiently large range of scales, say, from s_{th} to $8\ \text{GeV}^2$. If s_0 is ''small,'' then limitations of the operator product expansion, convergence of perturbation theory at small s, and details of resonances contributions make it difficult to test the reliability of the results.

For completeness, we should also mention the alternate JM determination of m_s via an analysis of the analogous strangeness-changing axial correlator. Their result in this case is $m_s(\overline{\text{MS}},\ 2\ \text{GeV}) = 91$ MeV, significantly smaller than that obtained from the scalar channel via the treatment of the vector current correlator. They, however, consider this analysis incomplete because it employs, for the normalization of the continuum spectral function at threshold, the leading-order, tree-level χPT result. They contend, based on the expectation that the full normalization will, as in the scalar channel, significantly exceed that given by tree-level χPT [$d(s_+) = 1.5 d_{\text{tree}}(s_+)$ for the scalar channel], that the true normalization will likely be significantly larger. If true, this would mean that m_s would be correspondingly increased. They thus expect their two analyses to become consistent once they employ a normalization at threshold corresponding to the one-loop expression for the continuum spectral function in the pseudoscalar channel. Our contention is that, in fact, the ''correct'' normalization is given, not by the full threshold spectral function, but rather by the appropriate $\mathcal{O}(q^4)$ LEC contributions to the one-loop result and that it should hence be significantly *smaller* than that corresponding to the tree-level result. Further progress on this issue, and that of the consistency of the two different extractions for m_s, will be possible only once the one-loop expression for $\langle 0|\partial_\mu A^\mu|K\pi\pi\rangle$ is known [28]. A reanalysis of the BPR FESR treatment of $m_u + m_d$ is similarly stymied by the absence of one-loop expressions for the matrix elements $\langle 0|\partial^\mu A^{(\pm)}_\mu|3\pi\rangle$ and by the lack of association of the L_i involved with just the pseudoscalar resonances.

In the past, of course, the agreement of the ratio $r \sim 2(180)/12 = 30$ obtained from the different sum rule analyses with that (24.4 ± 1.5) obtained from χPT [29] has been taken to provide *a posteriori* support for the validity of the sum rule treatments. Our contention is that a self-consistent sum rule analysis would yield estimates of both m_s and $m_u + m_d$ that are lower by a factor of ~ 2, thus maintaining the consistency with the χPT value of r.

VIII. CONCLUSIONS

We have shown that the ability to make reliable extractions of m_s and m_u+m_d using sum rule analyses rests on three key features of these analyses: the degree of reliability of PQCD, a knowledge of the scale s_0 at which quark-hadron duality becomes valid, and an ability to construct hadronic spectral functions which are correctly normalized in the resonance region, even in channels where experimental data on the relevant decay constants is not available.

We find that, in the relevant PQCD expressions, the $\alpha_s, \alpha_s^2, \ldots$ corrections are large both in the scalar and pseudoscalar channels. Including reasonable estimates for the unknown higher-order terms lowers the sum rule estimates of quark masses. The largest effect is in the extraction of m_u+m_d, which we estimate would be lowered by $\approx 20\%$ compared to the value quoted by BPR [3]. The correction in the case of m_s extracted from the scalar channel is roughly 5%, and this has been accounted for by JM-CPS.

Second, the estimates obtained for the quark masses are potentially very sensitive to the choice of s_0. We have illustrated this through an analysis of rigorous lower bounds and the use of a variety of plausible spectral functions in the case of m_u+m_d. Current sum rule analyses are forced to choose low values of s_0 due to lack of experimental information. The FESR extraction of m_u+m_d, for example, is based on rather low values of $s_0 \leq 3\ \mathrm{GeV}^2$, and so no tests of the stability of the estimates under variations of s_0 can be made. In the case of the JM-CPS analysis of m_s, the value chosen, $s_0 = 6.0\ \mathrm{GeV}^2$, is artificially large. This choice arises from an attempt to achieve stability of the Borel-transformed sum rule with respect to the Borel parameter u. Since, however, the phenomenological *Ansatz* for the spectral function breaks down for $s \gtrsim 4.0\ \mathrm{GeV}^2$, it is clear that such a choice of s_0 is not physical. For reasonable choices of s_0 we are also not able to find a solution that is stable with respect to variations in u. We therefore conclude that no reliable estimates of m_s can be made unless ρ_{hadronic} is known accurately over a significantly larger range of s.

Third, we have shown that the method employed in previous analyses for fixing the overall normalization of the resonance-modulated model spectral functions leads to significant overestimates of the continuum contributions to the relevant spectral integrals and hence to significant overestimates of the quark masses. The source of this problem is the fact that normalizing the resonance-modulated *Ansatz* [see Eqs. (31)–(33)] to either the experimental value or to the χPT value for the spectral function in the near-threshold region results in the inclusion of near-threshold contributions of the Goldstone-boson degrees of freedom in addition to the desired resonance contributions. Overestimating the size of the resonance tail in this manner, of course, leads to a corresponding overestimate of the resonance contributions at resonance peak. Since it is the resonance peak region, and not the threshold region, which dominates the phenomenological side of the sum rules, the conventional procedure produces significant overestimates of the quark masses. In the case of the vector current correlator, where the normalization of the spectral function at the ρ peak is known experimentally, we have shown that the magnitude of this overestimate is large: The conventional method of normalization produces a spectral function which, at the ρ peak, is a factor 4.1–5.1 larger than that given by experiment. We have explained, based on an understanding of the manner in which resonance effects manifest themselves in χPT, why the conventional method of normalization cannot be correct and have proposed an alternate phenomenological prescription for normalizing the spectral function, designed to provide estimates which are reliable, not so much in the threshold region, but in the resonance region relevant to the sum rule quark mass extractions. We verify that this prescription reproduces the experimental result for the vector (ρ) channel. This method is straightforward to apply to the scalar channel as the one-loop [$\mathcal{O}(q^4)$] χPT corrections are known, and the revised estimate for the normalization could reduce the estimate of m_s by as much as a factor of ~ 2 over the values found in previous analyses. We argue that a similar overestimate of the normalization will exist in the pseudoscalar channels, though we are unable to estimate its magnitude at present.

The bottom line is that unless the hadronic spectral function is known accurately over a large range of scales, say, up to $s = 8\ \mathrm{GeV}^2$, reliable extraction of quark masses from sum rules considered is not possible. Even though the lattice QCD estimates have their share of statistical and systematic errors [2], we claim that at present they represent the most reliable means of estimating the quark masses. Our estimates of the systematic errors in sum rule analysis suggest that revised sum rule estimates could easily be smaller by a factor of 2, in which case these would be consistent with the small values obtained from lattice QCD.

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