

CLASSICAL SOLUTION TO THE BUCKLING  
OF A THIN CYLINDRICAL SHELL

by

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## SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief the thesis contains no material previously published or written by any other person, except where due reference is made in the text of the thesis.

(N.S. KOSHNITSKY)

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## CHAPTER 1

THE BUCKLING PROBLEM1.1 Problem to be considered

Consider a thin cylindrical shell, of length  $\ell$ , radius  $a$ , and thickness  $t$ , subject to an external force. Elastic equilibrium occurs when basic stresses and displacements occur for a particular load and when the load is removed the shell returns to its original state.

Neutral equilibrium occurs, when for a particular load, additional displacements occur without any additional forces. The smallest load for which neutral equilibrium occurs is the buckling load for the cylindrical shell, and when it happens additional displacements occur spontaneously.

Problems of this type were first introduced to the author by the South Australian Engineering and Water Supply Department. This Department has large cylindrical water tanks and it was their practice to backfill part or all of the outside of the tanks with soil so that they would blend in with the countryside. However the Department was concerned that buckling may take place due to the external soil pressure when the tank would be emptied. A literature search showed that no model could adequately predict the buckling load caused by the triangular load of the soil.

Throughout this thesis the following notation is used.

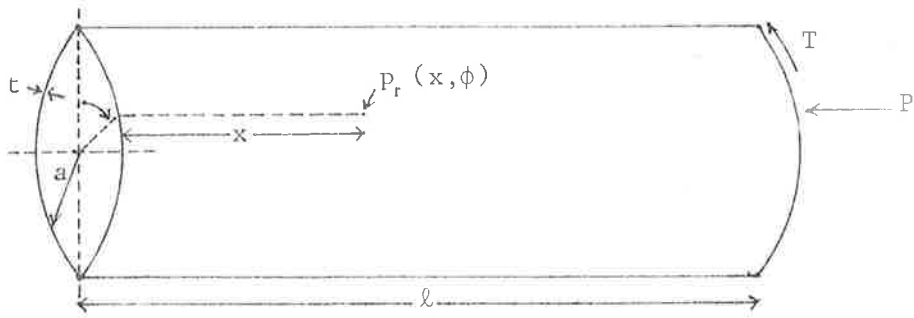


Fig. 1.1.1

Length =  $\ell$

Radius =  $a$

Thickness =  $t$

Young's Modulus =  $E$

Poisson Ratio =  $\nu$

Axial Coordinate =  $x$

Circumferential Coordinate =  $\phi$

Radial Pressure =  $p_r(x, \phi)$

Axial Compression =  $P$

Shearing Force =  $T$

Axial Displacement =  $u$

Circumferential Displacement =  $v$

Radial Displacement =  $w$  (+ve inwards)

The stresses caused by the external forces are as follows:

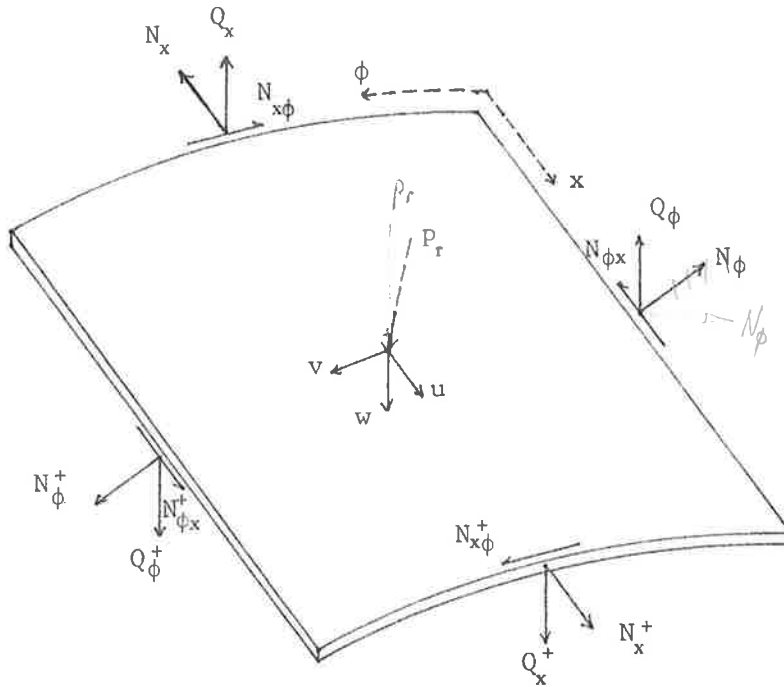


Fig. 1.1.2

The moment diagram for the shell is:

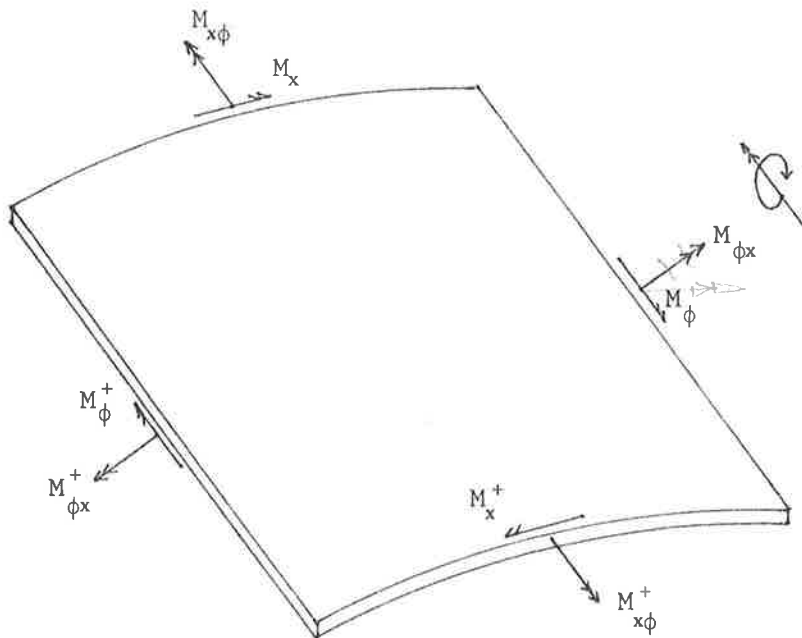


Fig. 1.1.3



## 1.2 Review of previous solutions

The solution to the buckling of a thin cylindrical shell was first approached using the method which is now known as the 'Classical Small Displacement Solution'. From 1908 to 1932 the 'classical' buckling formula was developed and is usually written in the form

$$\sigma_u = \frac{1}{\sqrt{3(1-\nu^2)}} \frac{Et}{a} \quad \dots(1.2.1)$$

where  $\sigma_u$  is the critical uniform axial stress,  $\nu$  Poisson's ratio,  $E$  Young's modulus,  $t$  the thickness and  $a$  the radius of the middle surface of the cylindrical shell. The associated axial wavelength  $\lambda$  of the buckle is given by

$$\lambda = \pi \sqrt{\frac{1}{12(1-\nu^2)}} \sqrt{at} \quad \dots(1.2.2)$$

The buckling pattern predicted by the classical theory is a 'chessboard' pattern as shown in Fig. 1.2.1. However, this buckling pattern has never been observed in experiments. Also the ratio of the experimental buckling stress to the calculated stress from equation (1.2.1) was approximately one half to two thirds. Two causes put forward for these discrepancies were that the boundary conditions are not exactly satisfied in the classical solution and that the experimental cylinder had small initial imperfections which caused the low critical buckling load. However, these explanations did not resolve the poor correlation between the theoretical and experimental loads and consequently the classical theory was assumed to be inadequate for the complex buckling problem and was only used as an approximation to the buckling load.

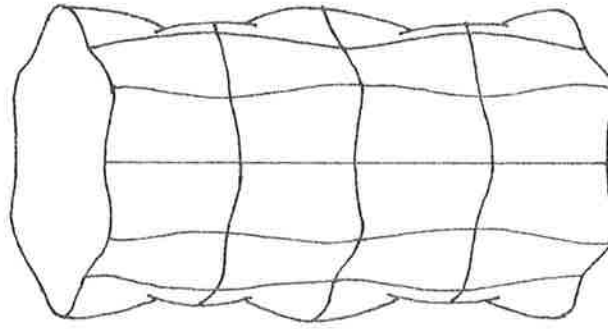


Fig. 1.2.1 'Chessboard' buckling pattern

The major contributors to the classical theory are as follows:

- In 1908, R. Lorenz [3] developed the buckling formula for the axisymmetric deformation of a thin walled cylindrical shell with simply supported edges subjected to uniform axial compression. He represented both the initial elastic displacements and the additional buckling displacements by a Fourier sine series, and hence was modelling a 'chessboard' buckling pattern. The critical buckling load was the load at which the denominator of a particular solution vanished. Essentially, Lorenz derived equation (1.2.1) and (1.2.2) for the particular case of  $\nu = 0$ .
- In 1910, S. Timoshenko [9] derived equation (1.2.1) and (1.2.2) firstly by using an energy method and by solving the eigenvalue problem defined by a fourth order equilibrium equation and the boundary conditions.
- In 1911, R. Lorenz [4] again tackled the problem, but without the restriction to axially symmetric deformations. By assuming the displacements in the axial direction were negligibly small, he derived a sixth order differential equilibrium equation with simply supported boundary conditions.
- In 1914, R. Southwell [8] obtained a buckling formula for a thin cylindrical shell simultaneously subjected to axial compression and

and lateral pressure.

- In 1932, W. Flügge [1] derived a more general solution for the buckling stress for different loading conditions and also obtained formulas defining the axial and circumferential wavelengths in terms of each other. He also explored the discrepancies between theoretical and experimental buckling loads.

### 1.3 Outline of current solution

This analysis is based on the same non-linear classical equilibrium equations, but unlike the previous solutions, the only simplifications involve the deletion of the non-linear terms. Also the solution given in this thesis contains no approximations in the classical thin shell elastic equations. Furthermore it defines a buckling shape to reflect and model actual experimentally observed buckling patterns.

M. Rehn [6], in his Ph.D. thesis, developed an exact general solution for the prebuckling elastic displacements on non symmetric loaded cylindrical shells. Using this, a complete solution to the prebuckling deformation is constructed which can accurately predict deformations resulting from complex loading conditions combined with mixed boundary conditions. Chapter 2 sets out this solution in detail since it is the basis of the solution to the buckling problem.

General non-linear equilibrium equations are used with terms derived from angular displacements of the cylindrical sides and also in-plane stretching of the middle surface. Substitution of general stress-strain relations result in three equilibrium equations involving known prebuckling deformations and unknown incremental buckling displacements. Incremental quadratic terms are then neglected which linearize the equilibrium equation.

The equations are then solved by using a double power series solution for the displacements in the axial and circumferential directions. This results in recurrence relations which are then solved using the boundary conditions of the buckling displacements. A local buckling patch is proposed where buckling occurs and has on its circumference zero incremental displacements. Due to its flexibility, this boundary condition can effectively model the experimentally observed buckling patterns as well as solve the complex equilibrium recurrence relations.

Using the boundary conditions on the buckling 'patch' and the recurrence relations a matrix is constructed and its determinant vanishes at the critical load for the particular 'patch'. A search is made to determine the 'patch' which has its associated determinant vanishing at the lowest load. This load is the buckling load of the cylinder.

The theory is versatile since it can analyse cylinders subjected to complex combinations of lateral pressure, axial load and twist with mixed boundary conditions of free, pinned or fixed at either end of the cylinder.

## CHAPTER 2

ELASTIC PREBUCKLING SOLUTION2.1 Introduction

The aim of Chapter 2 is to rigorously solve for the prebuckling displacements when a thin cylindrical shell is subjected to complex loading conditions. These displacements can then be used in a general classical solution to the buckling phenomena.

The equilibrium equations used are found in most references on this subject [2]. M. Rehn in his Ph.D. thesis [6] solves these equations for the nonaxisymmetric case and also with simple loading conditions.

The analysis which is original in this chapter is contained in Section 2.3 : Axisymmetric Solution :  $\beta = 0$  and Section 2.5 to 2.8 which solve the particular solution under complex loading conditions.

2.2 Prebuckling Equilibrium Equations

The three equations of equilibrium describing a thin cylindrical shell are

$$\dot{N}_x + \dot{N}_{\phi x} - N_x \phi - N_\phi - Q_x - Q_\phi + ap_x = 0 \quad \dots(2.2.1a)$$

$$\dot{N}_\phi + \dot{N}_{x\phi} + N_x + N_{\phi x} - Q_x - Q_\phi + ap_\phi = 0 \quad \dots(2.2.1b)$$

$$\dot{Q}_x + \dot{Q}_\phi + N_{x\phi} + N_x + N_\phi + N_{\phi x} + ap_r = 0 \quad \dots(2.2.1c)$$

where  $N_x$ ,  $N_{\phi x}$ ,  $N_x \phi$ ,  $N_\phi$  are the stress resultants;  $Q_x$ ,  $Q_\phi$  are the transverse shears;  $p_x$ ,  $p_\phi$ ,  $p_r$  are the components of the external surface force;  $a$  is the middle surface radius of the cylinder;  $x$ ,  $\phi$ ,  $z$  are the axial, circumferential and inward radial coordinates;

$$(\ )' = \frac{\partial}{\partial x}; (\ )^\cdot = \frac{\partial}{\partial \phi}.$$

The two equations of moment equilibrium are

$$\dot{M}_{x\phi} - \dot{M}_{\phi} - M_x - M_{\phi x} + aQ_{\phi} = 0 \quad \dots(2.2.2a)$$

$$\dot{M}_{\phi x} + \dot{M}_x + M_{x\phi} - M_{\phi} - aQ_x = 0 \quad \dots(2.2.2b)$$

where  $M_{x\phi}$ ,  $M_{\phi}$ ,  $M_x$ ,  $M_{\phi x}$  are the bending moments.

The generalised elastic law for the cylindrical shell are described by the following set of equations

$$N_{\phi} = \frac{D}{a} (\dot{V}_0 - W_0 + \nu \dot{U}_0) - \frac{K}{a^3} (W_0 + \ddot{W}_0) \quad \dots(2.2.3a)$$

$$N_x = \frac{D}{a} (\dot{U}_0 + \nu \dot{V}_0 - \nu W_0) + \frac{K}{a^3} (\ddot{W}_0) \quad \dots(2.2.3b)$$

$$N_{\phi x} = \frac{D}{a} \left( \frac{1-\nu}{2} \right) (\dot{U}_0 + \dot{V}_0) + \frac{K}{a^3} \left( \frac{1-\nu}{2} \right) (\dot{U}_0 - \dot{W}_0) \quad \dots(2.2.3c)$$

$$N_{x\phi} = \frac{D}{a} \left( \frac{1-\nu}{2} \right) (\dot{U}_0 + \dot{V}_0) + \frac{K}{a^3} \left( \frac{1-\nu}{2} \right) (\dot{V}_0 + \dot{W}_0) \quad \dots(2.2.3d)$$

$$M_{\phi} = -\frac{K}{a^2} (W_0 + \ddot{W}_0 + \nu \ddot{W}_0) \quad \dots(2.2.3e)$$

$$M_x = -\frac{K}{a^2} (\ddot{W}_0 + \nu \ddot{W}_0 + \dot{U}_0 + \nu \dot{V}_0) \quad \dots(2.2.3f)$$

$$M_{\phi x} = -\frac{K}{a^2} \left( \frac{1-\nu}{2} \right) (2\dot{W}_0 - \dot{U}_0 + \dot{V}_0) \quad \dots(2.2.3g)$$

$$M_{x\phi} = -\frac{K}{a^2} \left( \frac{1-\nu}{2} \right) (2\dot{W}_0 + \dot{V}_0) \quad \dots(2.2.3h)$$

where the displacements  $U_0$ ,  $V_0$ ,  $W_0$  are in the  $x$ ,  $\phi$ ,  $z$  directions and the constants are defined as follows.

$$D = \frac{Et}{(1-\nu^2)} \quad \dots(2.2.4a)$$

$$K = \frac{Et^3}{12(1-\nu^2)} \quad \dots(2.2.4b)$$

and  $\nu$  is Poisson's Ratio.

$Q_{\phi}$  and  $Q_x$  can be expressed in terms of the bending moments from equation (2.2.2). The equilibrium equations (2.2.1) can then be expressed

in terms of stress resultants and bending moments. Substituting equations (2.2.3) for these resultants gives the equilibrium equations containing only displacements and external forces. The homogeneous equations are as follows

$$\begin{aligned} \ddot{U}_0 + \frac{1-\nu}{2} \ddot{U}_0 + \frac{(1+\nu)}{2} \dot{V}_0 - \dot{W}_0 \\ + k \left[ \frac{1-\nu}{2} \ddot{U}_0 + \ddot{W}_0 - \frac{1-\nu}{2} \ddot{W}_0 \right] = 0 \quad \dots(2.2.5a) \end{aligned}$$

$$\begin{aligned} \frac{1+\nu}{2} \dot{U}_0 + \dot{V}_0 + \frac{1-\nu}{2} \dot{V}_0 - \dot{W}_0 \\ + k \left[ \frac{3}{2} (1-\nu) \dot{V}_0 + \frac{3-\nu}{2} \dot{W}_0 \right] = 0 \quad \dots(2.2.5b) \end{aligned}$$

$$\begin{aligned} \nu \dot{U}_0 + \dot{V}_0 - W_0 + k \left[ \frac{1-\nu}{2} \ddot{U}_0 - \ddot{U}_0 - \frac{3-\nu}{2} \dot{V}_0 \right. \\ \left. - \ddot{W}_0 - 2\ddot{W}_0 - \ddot{W}_0 - 2\dot{W}_0 - W_0 \right] = 0 \quad \dots(2.2.5c) \end{aligned}$$

where the constant  $k$  is defined as follows

$$k = \frac{t^2}{12a^2} \quad \dots(2.2.6)$$

Since the displacement components vary sinusoidally in the circumferential direction, the following displacement equations are written

$$U_0 = U_0(x) \cos \beta\phi \quad \dots(2.2.7a)$$

$$V_0 = V_0(x) \sin \beta\phi \quad \dots(2.2.7b)$$

$$W_0 = W_0(x) \cos \beta\phi \quad \dots(2.2.7c)$$

where  $\beta = 0, 1, 2, \dots$

Expression (2.2.7) satisfy the geometry of the cylinder and when substituted into equation (2.2.5) a symmetric homogeneous system of three simultaneous equations in  $U_0(x)$ ,  $V_0(x)$ ,  $W_0(x)$  result.

$$\begin{aligned}
& - \left[ D^2 - \frac{1-\nu}{2} \beta^2 (1+k) \right] U_0(x) - \left[ \frac{1+\nu}{2} \beta D \right] V_0(x) \\
& \quad + \left[ \nu D - k \left( D^3 + \frac{1-\nu}{2} \beta^2 D \right) \right] W_0(x) = 0 \quad \dots(2.2.8a)
\end{aligned}$$

$$\begin{aligned}
& \left[ -\frac{1+\nu}{2} \beta D \right] U_0(x) + \left[ -\beta^2 + \frac{1-\nu}{2} (1+3k) D^2 \right] V_0(x) \\
& \quad + \left[ \beta - \frac{3-\nu}{2} k \beta D^2 \right] W_0(x) = 0 \quad \dots(2.2.8b)
\end{aligned}$$

$$\begin{aligned}
& \left[ \nu D - k \left( D^3 + \frac{1-\nu}{2} \beta^2 D \right) \right] U_0(x) + \left[ \beta - \frac{3-\nu}{2} k \beta D^2 \right] V_0(x) \\
& \quad + [-1 - k(D^4 - 2\beta^2 D^2 + (\beta^2 - 1)^2)] W_0(x) = 0 \quad \dots(2.2.8c)
\end{aligned}$$

where  $D = a \frac{\partial}{\partial x}$ . These equations are abbreviated in the following way

$$- [D^2 - K_1] U_0(x) - [K_2 D] V_0(x) - D [K_3 D^2 + K_4] W_0(x) = 0 \quad \dots(2.2.9a)$$

$$- [K_2 D] U_0(x) + [K_5 D^2 + K_6] V_0(x) + [K_9 D^2 + K_8] W_0(x) = 0 \quad \dots(2.2.9b)$$

$$\begin{aligned}
& = D [K_3 D^2 + K_4] U_0(x) + [K_7 D^2 + K_8] V_0(x) \\
& \quad + [K_9 D^4 + K_{10} D^2 + K_{11}] W_0(x) = 0 \quad \dots(2.2.9c)
\end{aligned}$$

where  $K_1 = \frac{1-\nu}{2} \beta^2 (1+k)$  ;  $K_2 = \frac{1+\nu}{2} \beta$  ;  $K_3 = k$  ;  $K_4 = -\nu + \frac{1-\nu}{2} \beta^2 k$  ;

$$K_5 = \frac{1-\nu}{2} (1+3k) ; K_6 = -\beta^2 ; K_7 = -\frac{(3-\nu)}{2} \beta k ; K_8 = \beta ;$$

$$K_9 = -k ; K_{10} = 2\beta^2 k ; K_{11} = -1 - (\beta^2 - 1)^2 k \quad \dots(2.2.10)$$

Writing equation (2.2.9) in matrix form we have

$$\begin{bmatrix}
-(D^2 - K_1) & -K_2 D & -D(K_3 D^2 + K_4) \\
-K_2 D & K_5 D^2 + K_6 & K_7 D^2 + K_8 \\
-D(K_3 D^2 + K_4) & K_7 D^2 + K_8 & K_9 D^4 + K_{10} D^2 + K_{11}
\end{bmatrix}
\begin{bmatrix}
U_0(x) \\
V_0(x) \\
W_0(x)
\end{bmatrix}
= 0 \quad \dots(2.2.11)$$

Assume a solution with the form

$$U_0(x) = A \ell^{\lambda x} ; V_0(x) = B \ell^{\lambda x} ; W_0(x) = C \ell^{\lambda x} \quad \dots(2.2.12)$$



where  $A, B, C, \lambda$  are complex constants to be determined. Substituting (3.8) into the matrix equation (3.8) yields

$$\begin{bmatrix} -(\lambda^2 - K_1) & -K_2\lambda & -\lambda(K_3\lambda^2 + K_4) \\ -K_2\lambda & K_5\lambda^2 + K_6 & K_7\lambda^2 + K_8 \\ -\lambda(K_3\lambda^2 + K_4) & K_7\lambda^2 + K_8 & K_9\lambda^4 + K_{10}\lambda^2 + K_{11} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = 0 \quad \dots(2.2.13)$$

Equation (3.9) has non trivial solutions if the determinant of the  $3 \times 3$  matrix is zero.

### 2.3 Axisymmetric Solution : $\beta = 0$

When  $\beta = 0$ , the circumferential displacement  $V_0 = V_0(x) \sin\beta\phi$  is also zero. Equation (2.2.13) therefore simplifies since

$$\begin{aligned} V_0 = 0 \quad \text{and} \quad K_1 = K_2 = K_6 = K_7 = K_8 = K_{10} = 0 \\ \text{and} \quad K_3 = k ; K_4 = -v ; K_5 = \frac{1-v}{2} (1+3k) ; K_9 = -k ; K_{11} = -1-k \end{aligned} \quad \dots(2.3.1)$$

so that equation (2.2.13) reduces to

$$\begin{bmatrix} -\lambda^2 & \lambda v - k\lambda^3 \\ \lambda v - k\lambda^3 & -(1+k) - k\lambda^4 \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = 0 \quad \dots(2.3.2)$$

The matrix equation (2.3.2) has non trivial solutions when the determinant vanishes. This results in the following characteristic equation in  $\lambda$ .

$$\lambda^2 [k(k-1)\lambda^4 - 2vk\lambda^2 + (v^2-1-k)] = 0 \quad \dots(2.3.3)$$

The solution to this equation is therefore

$$U_0(x) = \sum_{i=1}^4 A_i e^{\lambda_i x} + A_5 + xA_6 \quad \dots(2.3.4a)$$

$$V_0(x) = 0 \quad \dots(2.3.4b)$$

$$W_0(x) = \sum_{i=1}^4 C_i e^{\lambda_i x} + C_5 + xC_6 \quad \dots(2.3.4c)$$

Now  $A_1, \dots, A_6, C_1, \dots, C_6$  are twelve complex constants to be determined by the six boundary conditions. Substituting equations (2.3.4) into (2.2.8) gives the complex matrix equation

$$\sum_{i=1}^4 \begin{bmatrix} -\lambda_i^2 & -v\lambda_i - k\lambda_i^3 \\ -v\lambda_i - k\lambda_i^3 & -(1+k) - k\lambda_i^4 \end{bmatrix} \begin{bmatrix} A_i \\ C_i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -(1+k) \end{bmatrix} \begin{bmatrix} A_5 \\ C_5 \end{bmatrix} + \begin{bmatrix} 0 & -v \\ -v & -(1+k)x \end{bmatrix} \begin{bmatrix} A_6 \\ C_6 \end{bmatrix} = 0 \quad \dots(2.3.5)$$

The determinant of the leftmost matrix is identically zero for each  $i = 1, \dots, 4$  since  $\lambda_i$ 's are the solution to the polynomial produced by the determinant of an identical matrix. Equating coefficients of powers of  $x$  gives the following

$$C_6 = 0 \quad \dots(2.3.6a)$$

and

$$A_6 = -\frac{(1+k)}{v} C_5 = \frac{K_{11}}{K_4} C_5 = \eta_5 C_5 \quad \dots(2.3.6b)$$

The homogeneous equation (2.3.2) for  $i = 1, \dots, 4$  has non trivial solutions and hence the ratios  $A_i / C_i$  can be calculated. Hence

$$\begin{bmatrix} -\lambda_i^2 & -v\lambda_i - k\lambda_i^3 \end{bmatrix} \begin{bmatrix} A_i / C_i \\ 1 \end{bmatrix} = 0 \quad i = 1, \dots, 4 \quad \dots(2.3.7)$$

giving

$$\eta_i = A_i / C_i = -\frac{(v\lambda_i - k\lambda_i^3)}{\lambda_i^2} \quad \dots(2.3.8)$$

The displacement component expressions (2.3.4) can be rewritten with only 6 independent unknowns.

$$U_0(x) = \sum_{i=1}^4 \eta_i C_i e^{\lambda_i x} + \eta_5 C_5 x + C_6 \quad \dots(2.3.9a)$$

$$V_0(x) = 0 \quad \dots(2.3.9b)$$

$$W_0(x) = \sum_{i=1}^4 C_i e^{\lambda_i x} + C_5 \quad \dots(2.3.9c)$$

## 2.4 Non Axisymmetric Solution : $\beta = 1, 2, \dots$

In exactly the same way as the symmetric case  $\beta = 0$ , the displacement component expressions are calculated with eight independent unknowns.

For  $\beta = 1$ , the following is obtained.

$$U_0(x) = \sum_{i=1}^6 \eta_i e^{\lambda_i x} C_i \quad \dots(2.4.1a)$$

$$V_0(x) = \sum_{i=1}^6 \chi_i e^{\lambda_i x} C_i + C_7 + xC_8 \quad \dots(2.4.1b)$$

$$W_0(x) = \sum_{i=1}^5 e^{\lambda_i x} C_i + C_7 + xC_8 \quad \dots(2.4.1c)$$

and for  $\beta = 2, 3, \dots$

$$U_0(x) = \sum_{i=1}^8 \eta_i e^{\lambda_i x} C_i \quad \dots(2.4.2a)$$

$$V_0(x) = \sum_{i=1}^8 \chi_i e^{\lambda_i x} C_i \quad \dots(2.4.2b)$$

$$W_0(x) = \sum_{i=1}^8 e^{\lambda_i x} C_i \quad \dots(2.4.2c)$$

where  $\eta_i$  and  $\chi_i$  are for both equations (2.4.1) and (2.4.2) as follows.

$$\eta_i = \frac{-[K_3 K_5 \lambda_i^5 + (K_4 K_5 + K_6 K_3 - K_7 K_2) \lambda_i^3 + (K_4 K_6 - K_8 K_2) \lambda_i]}{[K_5 \lambda_i^4 + (K_6 - K_1 K_5 + K_2^2) \lambda_i^2 - K_6]} \quad \dots(2.4.3a)$$

$$\chi_i = \frac{-[(K_2 K_3 + K_7) \lambda_i^4 + (K_2 K_4 - K_1 K_7 + K_8) \lambda_i^2 - K_1 K_8]}{[K_5 \lambda_i^4 + (K_6 - K_1 K_5 + K_2^2) \lambda_i^2 - K_1 K_6]} \quad \dots(2.4.3b)$$

## 2.5 Normal Pressure Loading

To allow for general pressure loadings the normal pressure distribution is given by

$$q = q_c \sum_{\beta=0}^{\infty} \sum_{i=0}^p \alpha_i \frac{x}{a} \cos \beta \phi \quad \dots(2.5.1)$$

where  $q_c$  is a constant independent of  $x$  and  $\phi$  and  $\alpha_i$ ,  $i = 0, \dots, P$  are dimensionless constants independent of  $x$  and  $\phi$ , and  $P$  is a series limit.

The equilibrium equations then become

$$\frac{U_0''}{2} + \frac{1-\nu}{2} \ddot{U}_0 + \frac{(1+\nu)}{2} \dot{V}_0' = \nu R \dot{W}_0' + k \left( \frac{1-\nu}{2} \ddot{U}_0 + \dot{W}_0''' - \frac{1-\nu}{2} \dot{W}_0'' \right) = 0 \quad \dots (2.5.2a)$$

$$\frac{1+\nu}{2} \dot{U}_0' + \ddot{V}_0 + \frac{1-\nu}{2} \dot{V}_0'' - \dot{W}_0 + k \left( \frac{3}{2} (1-\nu) \dot{V}_0'' + \frac{3-\nu}{2} \dot{W}_0'' \right) = 0 \quad \dots (2.5.2b)$$

$$\nu \dot{U}_0' + \dot{V}_0 - \dot{W}_0 + k \left( \frac{1-\nu}{2} \ddot{U}_0' - \dot{U}_0''' - \frac{3-\nu}{2} \dot{V}_0'' - \dot{W}_0''' - 2\dot{W}_0'' \right. \\ \left. - \dot{W}_0'' - 2\dot{W}_0' - \dot{W}_0 \right) = -q_c \frac{a^2(1-\nu^2)}{Et} \sum_{\beta=0}^{\infty} \sum_{i=0}^P \alpha_i \frac{x^i}{a^i} \cos \beta \phi \quad \dots (2.5.2c)$$

A particular solution for the displacements  $U_0$ ,  $V_0$ ,  $W_0$  is then found. When  $\beta = 0$ , then  $V_p = 0$  so that these cases have to be considered separately.

## 2.6 Particular Solution : $\beta = 2, 3, \dots$

For the particular solution when  $\beta = 2, 3, \dots$  the following expressions are chosen for the particular solutions.

$$U_{\text{Part}} = \sum_{i=0}^P U_i \frac{x^i}{a^{i-1}} \cos \beta \phi ; \quad \dots (2.6.1a)$$

$$V_{\text{Part}} = \sum_{i=0}^P V_i \frac{x^i}{a^{i-1}} \sin \beta \phi ; \quad \dots (2.6.1b)$$

$$W_{\text{Part}} = \sum_{i=0}^P W_i \frac{x^i}{a^{i-1}} \cos \beta \phi . \quad \dots (2.6.1c)$$

where  $U_i$ ,  $V_i$ ,  $W_i$ ;  $i = 0, \dots, P$  are dimensionless constants.

Substituting expressions (2.6.1) into equations (2.5.2) gives

$$\begin{aligned}
& - \sum_{i=0}^{P-2} (i+1)(i+2)U_{i+2} \frac{x^i}{a^{i-1}} + K_1 \sum_{i=0}^P U_i \frac{x^i}{a^{i-1}} - K_2 \sum_{i=0}^{P-1} (i+1)V_{i+1} \frac{x^i}{a^{i-1}} \\
& - K_3 \sum_{i=0}^{P-3} (i+1)(i+2)(i+3)W_{i+3} \frac{x^i}{a^{i-1}} - K_4 \sum_{i=0}^{P-1} (i+1)W_{i+1} \frac{x^i}{a^{i-1}} = 0
\end{aligned} \quad \dots(2.6.2a)$$

$$\begin{aligned}
& - K_2 \sum_{i=0}^{P-1} (i+1)U_{i+1} \frac{x^i}{a^{i-1}} + K_5 \sum_{i=0}^{P-2} (i+1)(i+2)V_{i+2} \frac{x^i}{a^{i-1}} + K_6 \sum_{i=0}^P V_i \frac{x^i}{a^{i-1}} \\
& + K_7 \sum_{i=0}^{P-2} (i+1)(i+2)W_{i+2} \frac{x^i}{a^{i-1}} + K_8 \sum_{i=0}^P W_i \frac{x^i}{a^{i-1}} = 0
\end{aligned} \quad \dots(2.6.2b)$$

$$\begin{aligned}
& - K_3 \sum_{i=0}^{P-3} (i+1)(i+2)(i+3)U_{i+3} \frac{x^i}{a^{i-1}} - K_4 \sum_{i=0}^{P-1} (i+1)U_{i+1} \frac{x^i}{a^{i-1}} \\
& + K_7 \sum_{i=0}^{P-2} (i+1)(i+2)V_{i+2} \frac{x^i}{a^{i-1}} + K_8 \sum_{i=0}^P V_i \frac{x^i}{a^{i-1}} + K_9 \sum_{i=0}^{P-4} (i+1)(i+2)(i+3)(i+4)W_{i+4} \frac{x^i}{a^{i-1}} \\
& + K_{10} \sum_{i=0}^{P-2} (i+1)(i+2)W_{i+2} \frac{x^i}{a^{i-1}} + K_{11} \sum_{i=0}^P W_i \frac{x^i}{a^{i-1}} = -q_c \frac{a^2(1-v^2)}{Et} \sum_{i=0}^P \alpha_i \frac{x^i}{a^i}
\end{aligned} \quad \dots(2.6.2c)$$

where the constants  $K_1, \dots, K_{11}$  have previously been defined by expressions (2.2.10).

Equating coefficients of powers of  $x$  to zero in equation (2.6.2a) gives

$$K_1 U_P = 0 \quad \dots(2.6.3a)$$

$$K_1 U_{P-1} - K_2(P)V_P = K_4(P)W_P = 0 \quad \dots(2.6.3b)$$

$$K_1 U_{P-2} - K_2(P-1)V_{P-1} - K_4(P-1)W_{P-1} - (P-1)(P)U_P = 0 \quad \dots(2.6.3c)$$

$$\begin{aligned}
& K_1 U_i - K_2(i+1)V_{i+1} - K_4(i+1)W_{i+1} - (i+1)(i+2)U_{i+2} \\
& - K_3(i+1)(i+2)(i+3)W_{i+3} = 0 \quad \text{for } i = 0, \dots, P-3
\end{aligned} \quad \dots(2.6.3d)$$

Equating coefficients of powers of  $x$  to zero in equation (2.6.2b) gives

$$K_6 V_p + V_8 W_p = 0 \quad \dots(2.6.4a)$$

$$K_6 V_{p-1} + K_8 W_{p-1} - K_2(P)U_p = 0 \quad \dots(2.6.4b)$$

$$K_6 V_i + K_8 W_i - K_2(i+1)U_{i+1} + K_5(i+1)(i+2)V_{i+2} + K_7(i+1)(i+2)W_{i+2} = 0 \quad \text{for } i = 0, \dots, p-2 \quad \dots(2.6.4c)$$

Equating coefficients of powers of  $x$  to zero in equation (2.6.2c) gives

$$K_8 V_p + K_{11} W_p = -q_c \frac{a(1-v^2)}{Et} \alpha_p \quad \dots(2.6.5a)$$

$$K_8 V_{p-1} + K_{11} W_{p-1} - K_4(P)U_p = -q_c \frac{a(1-v^2)}{Et} \alpha_{p-1} \quad \dots(2.6.5b)$$

$$K_8 V_{p-2} + K_{11} W_{p-2} - K_4(P-1)U_{p-1} + K_7(P-1)(P)V_p + K_{10}(P-1)(P)W_p = q_c \frac{a(1-v^2)}{E} \alpha_{p-2} \quad \dots(2.6.5c)$$

$$K_8 V_{p-3} + K_{11} W_{p-3} - K_4(P-2)U_{p-2} + K_7(P-2)(P-1)V_{p-1} + K_{10}(P-2)(P-1)W_{p-1} - K_3(P-2)(P-1)(P)U_p = -q_c \frac{a(1-v^2)}{Et} \alpha_{p-3} \quad \dots(2.6.5d)$$

$$K_8 V_i + K_{11} W_i - K_4(i+1)U_{i+1} + K_7(i+1)(i+2)V_{i+2} + K_{10}(i+1)(i+2)W_{i+2} - K_3(i+1)(i+2)(i+3)U_{i+3} + K_9(i+1)(i+2)(i+3)(i+4)W_{i+4} = -q_c \frac{a(1-v^2)}{Et} \alpha_i \quad \text{for } i = 0, \dots, p-4 \quad \dots(2.6.5e)$$

Equations (2.6.3) to (2.6.5) give the following recurrence relations.

$$U_p = 0 \quad \dots(2.6.6a)$$

$$V_p = \frac{K_8}{(K_6 K_{11} - K_8^2)} \left( q_c \frac{a(1-v^2)}{Et} \alpha_p \right) \quad \dots(2.6.6b)$$

$$W_p = \frac{-K_6}{(K_6 K_{11} - K_8^2)} \left( q_c \frac{a(1-v^2)}{Et} \alpha_p \right) \quad \dots(2.6.6c)$$

$$U_{P-1} = P \frac{(K_2 V_P + K_4 W_P)}{K_1} \quad \dots (2.6.6d)$$

$$V_{P-1} = \frac{K_8}{(K_6 K_{11} - K_8^2)} \left( q_c \frac{a(1-\nu^2)}{Et} \alpha_{P-1} \right) \quad \dots (2.6.6e)$$

$$W_{P-1} = \frac{-K_6}{(K_6 K_{11} - K_8^2)} \left( q_c \frac{a(1-\nu^2)}{Et} \alpha_{P-1} \right) \quad \dots (2.6.6f)$$

$$U_{P=2} = (P-1) \frac{(K_2 V_{P-1} + K_4 W_{P-1})}{K_1} \quad \dots (2.6.6g)$$

$$\begin{aligned} V_{P-2} = & \frac{1}{(K_6 K_{11} - K_8^2)} \left[ K_{11} (P-1) (K_2 U_{P-1} - P(K_5 V_P + K_7 W_P)) \right. \\ & - K_8 [(P-1) (K_4 U_{P-1} - P(K_7 V_P + K_{10} W_P))] \\ & \left. - q_c \frac{a(1-\nu^2)}{Et} \alpha_{P-2} \right] \quad \dots (2.6.6h) \end{aligned}$$

$$\begin{aligned} W_{P-2} = & \frac{1}{(K_6 K_{11} - K_8^2)} \left[ -K_8 (P-1) (K_2 U_{P-1} - P(K_5 V_P + K_7 W_P)) \right. \\ & + K_6 [(P-1) (K_4 U_{P-1} - P(K_7 V_P + K_{10} W_P))] \\ & \left. - q_c \frac{a(1-\nu^2)}{Et} \alpha_{P-2} \right] \quad \dots (2.6.6i) \end{aligned}$$

$$U_{P-3} = \frac{(P-2)}{K_1} \left( K_2 V_{P-2} + K_4 W_{P-2} + (P-1) (U_{P-1} + K_3 (P) W_P) \right) \quad \dots (2.6.6j)$$

$$\begin{aligned} V_{P-3} = & \frac{1}{(K_6 K_{11} - K_8^2)} \left[ K_{11} (P-2) (K_2 U_{P-2} - (P-1) (K_5 V_{P-1} + K_7 W_{P-1})) \right. \\ & - K_8 [(P-2) (K_4 U_{P-2} - (P-1) (K_7 V_{P-1} + K_{10} W_{P-1}))] \\ & \left. - q_c \frac{a(1-\nu^2)}{Et} \alpha_{P-3} \right] \quad \dots (2.6.6k) \end{aligned}$$

$$\begin{aligned} W_{P-3} = & \frac{1}{(K_6 K_{11} - K_8^2)} \left[ -K_8 (P-2) (K_2 U_{P-2} - (P-1) (K_5 V_{P-1} + K_7 W_{P-1})) \right. \\ & + K_6 [(P-2) (K_4 U_{P-2} - (P-1) (K_7 V_{P-1} + K_{10} W_{P-1}))] \\ & \left. - q_c \frac{a(1-\nu^2)}{Et} \alpha_{P-3} \right] \quad \dots (2.6.6l) \end{aligned}$$

$$U_{P-4} = \frac{(P-3)}{K_1} \left( K_2 V_{P-3} + K_4 W_{P-3} + (P-2) (U_{P-2} + K_3 (P-1) W_{P-1}) \right) \quad \dots (2.6.6m)$$

and for  $i = 0, \dots, P-4$

$$\begin{aligned}
V_i = & \frac{1}{(K_6 K_{11} - K_8^2)} \left[ K_{11}(i+1) \left( K_2 U_{i+1} - (i+2)(K_5 V_{i+2} + K_7 W_{i+2}) \right) \right. \\
& - K_8 \left[ (i+1) \left( K_4 U_{i+1} - (i+2)(K_7 V_{i+2} + K_{10} W_{i+2}) \right) \right. \\
& + (i+1)(i+2)(i+3) \left( K_3 U_{i+3} - K_9(i+4) W_{i+4} \right) \\
& \left. \left. - q_c \frac{a(1-\nu^2)}{Et} \alpha_i \right] \right] \quad \dots(2.6.6n)
\end{aligned}$$

$$\begin{aligned}
W_i = & \frac{1}{(K_6 K_{11} - K_8^2)} \left[ -K_8(i+1) \left( K_2 U_{i+1} - (i+2)(K_5 V_{i+2} + K_7 W_{i+2}) \right) \right. \\
& + K_6 \left[ (i+1) \left( K_4 U_{i+1} - (i+2)(K_7 V_{i+2} + K_{10} W_{i+2}) \right) \right. \\
& + (i+1)(i+2)(i+3) \left( K_3 U_{i+3} - K_9(i+4) W_{i+4} \right) \\
& \left. \left. - q_c \frac{a(1-\nu^2)}{Et} \alpha_i \right] \right] \quad \dots(2.6.6o)
\end{aligned}$$

and also for  $i = 0, \dots, P-5$

$$U_i = \frac{i}{K_1} \left( K_2 V_i + K_4 W_i + (i+1) \left( U_{i+1} + K_3(i+2) W_{i+2} \right) \right) \quad \dots(2.6.6p)$$

## 2.7 Particular Solution : $\beta = 1$

For the particular solution when  $\beta = 1$  the following expressions are chosen for the particular solution

$$U_{\text{Part}} = \sum_{i=0}^P U_i \frac{x^i}{a^{i-1}} \cos \phi ; \quad \dots(2.7.1a)$$

$$V_{\text{Part}} = 0 ; \quad \dots(2.7.1b)$$

$$W_{\text{Part}} = \sum_{i=0}^P W_i \frac{x^i}{a^{i-1}} \cos \phi . \quad \dots(2.7.1c)$$

$V_{\text{Part}} = 0$  is chosen, since  $K_6 K_{11} - K_8^2 = 0$  when  $\beta = 1$ .

Equations (2.6.2) are the same except the terms containing  $V_i$ 's are zero. This results in the following recurrence relations



$$U_P = 0 \quad \dots(2.7.2a)$$

$$W_P = -\frac{1}{K_{11}} q_c \frac{a(1-\nu^2)}{Et} \alpha_P \quad \dots(2.7.2b)$$

$$U_{P-1} = \frac{K_4}{K_1} (P)W_P \quad \dots(2.7.2c)$$

$$W_{P-1} = -\frac{1}{K_{11}} q_c \frac{a(1-\nu^2)}{Et} \alpha_{P-1} \quad \dots(2.7.2d)$$

$$U_{P-2} = \frac{K_4}{K_1} (P-1)W_{P-1} \quad \dots(2.7.2e)$$

$$W_{P-2} = \frac{1}{K_{11}} \left[ (P-1) \left( K_4 U_{P-1} - K_{10} (P) W_P \right) - q_c \frac{a(1-\nu^2)}{Et} \alpha_{P-2} \right] \quad \dots(2.7.2f)$$

and for  $i = 0, \dots, P-3$

$$U_i = \frac{1}{K_1} \left[ (i+1) \left( K_4 W_{i+1} + (i+2) U_{i+2} + (i+2)(i+3) K_3 W_{i+3} \right) \right] \quad \dots(2.7.2g)$$

$$W_{P-3} = \frac{1}{K_{11}} \left[ (P-2) K_4 U_{P-2} - K_{10} (P-1) W_{P-1} \right] - q_c \frac{a(1-\nu^2)}{Et} \alpha_{P-3} \quad \dots(2.7.2h)$$

and for  $i = 0, \dots, P-4$

$$\begin{aligned} W_i = \frac{1}{K_{11}} & \left[ (i+1) \left( K_4 U_{i+1} - K_{10} (i+2) W_{i+2} \right) \right. \\ & \left. + (i+1)(i+2)(i+3) \left( K_3 U_{i+3} - K_9 (i+4) W_{i+4} \right) \right. \\ & \left. - q_c \frac{a(1-\nu^2)}{Et} \alpha_i \right] \quad \dots(2.7.2i) \end{aligned}$$

## 2.8 Particular Solution : $\beta = 0$

When  $\beta = 0$ , the constants  $K_1 = K_2 = K_6 = K_7 = K_8 = K_{10} = 0$ .

The following expressions are chosen for the particular solution

$$U_{\text{Part}} = \sum_{i=0}^{P+1} \frac{U_i x^i}{a^{i-1}} ; \quad \dots(2.8.1a)$$

$$V_{\text{Part}} = 0 ; \quad \dots(2.8.1b)$$

$$W_{\text{Part}} = \sum_{i=0}^P \frac{W_i x^i}{a^{i-1}} \quad \dots(2.8.1c)$$

The particular solution results in the following recurrence relations

$$U_{P+1} = \frac{K_4 q_c a(1-\nu^2) \alpha_P}{(K_{11} + K_4^2)(P+1)Et} \quad \dots(2.8.2a)$$

$$W_P = \frac{-q_c a(1-\nu^2) \alpha_P}{(K_{11} + K_4^2)Et} \quad \dots(2.8.2b)$$

$$U_P = \frac{K_4 q_c a(1-\nu^2) \alpha_{P-1}}{(K_{11} + K_4^2)(P)Et} \quad \dots(2.8.2c)$$

$$W_{P-1} = \frac{-q_c a(1-\nu^2) \alpha_{P-1}}{(K_{11} + K_4^2)Et} \quad \dots(2.8.2d)$$

$$U_{P-1} = \frac{1}{(K_{11} + K_4^2)(P-1)} \left[ \frac{K_4 q_c a(1-\nu^2) \alpha_{P-2}}{Et} - K_3 K_{11} (P-1)(P) W_P \right] \quad \dots(2.8.2e)$$

$$W_{P-2} = \frac{1}{(K_{11} + K_4^2)} \left[ \frac{q_c a(1-\nu^2) \alpha_{P-2}}{Et} + K_3 K_4 (P-1)(P) W_P \right] \quad \dots(2.8.2f)$$

$$U_{P-2} = \frac{1}{(K_{11} + K_4^2)(P-2)} \left[ K_4 \left( \frac{q_c a(1-\nu^2) \alpha_{P-3}}{Et} - K_3 (P-2)(P-1)(P) U_P \right) - K_3 K_{11} (P-2)(P-1) W_{P-1} \right] \quad \dots(2.8.2g)$$

$$W_{P-3} = \frac{-1}{(K_{11} + K_4^2)} \left[ \frac{q_c a(1-\nu^2) \alpha_{P-3}}{Et} - K_3 (P-2)(P-1)(P) U_P + K_3 K_4 (P-2)(P-1) W_{P-1} \right] \quad \dots(2.8.2h)$$

and for  $i = 0, \dots, P-4$

$$U_{i+1} = \frac{1}{(K_{11} + K_4^2)(i)} \left[ K_4 \left( \frac{q_c a(1-\nu^2) \alpha_i}{Et} + (i+1)(i+2)(i+3) \left( K_9 (i+4) W_{i+4} - K_3 U_{i+3} \right) \right) - K_3 K_{11} (i)(i+1) W_{i+2} \right] \quad \dots(2.8.2i)$$

$$W_i = \frac{-1}{(K_{11} + K_4^2)} \left[ \frac{q_c a(1-\nu^2) \alpha_i}{Et} + (i+1)(i+2)(i+3) \left( K_9 (i+4) W_{i+4} - K_3 U_{i+3} \right) + K_3 K_{11} (i)(i+1) W_{i+2} \right] \quad \dots(2.8.2j)$$

## 2.9 Boundary Conditions

Boundary conditions at the top and bottom of the cylinder are needed to fully determine the prebuckling displacements. For  $\beta = 0$  and  $\beta = 1$ , only 3 conditions at each end are needed, corresponding to the  $U_0$ ,  $V_0$ ,  $W_0$  displacements and for  $\beta = 2, 3, \dots, 4$  conditions at each end are needed, corresponding to the  $U_0$ ,  $V_0$ ,  $W_0$ ,  $W_0$  displacements.

The stress resultants corresponding to the displacements  $U_0$ ,  $V_0$ ,  $W_0$ ,  $W_0$  are  $N_x$ ,  $T_x$ ,  $S_x$ ,  $M_x$  respectively, where

$$N_x = \frac{Et}{a(1-\nu^2)} \left( \dot{U}_0 + \nu(\dot{V}_0 - \dot{W}_0) + \frac{Et^3 \ddot{W}_0}{12(1-\nu^2)a^3} \right) \quad \dots(2.9.1a)$$

$$T_x = N_x \phi - \frac{M_x \phi}{a} = \frac{Et}{2a(1+\nu)} (\dot{U}_0 + \dot{V}_0) + \frac{Et^3}{8(1+\nu)a^3} (\dot{W}_0 + \dot{V}_0) \quad \dots(2.9.1b)$$

$$S_x = Q_x - \frac{\dot{M}_x \phi}{a} = \frac{Et^3}{12a(1-\nu^2)} \left[ -\ddot{W}_0 - \nu \ddot{W}_0 - \ddot{U}_0 - \nu \dot{V}_0 + (1-\nu) \left( -\ddot{W}_0 + \frac{1}{2} (\ddot{U}_0 - \dot{V}_0) \right) \right] - \frac{Et^3}{12(1+\nu)a^3} (\ddot{W}_0 + \dot{V}_0) \quad \dots(2.9.1c)$$

$$M_x = \frac{-Et^3}{12(1-\nu^2)a^2} \left( \ddot{W}_0 + \dot{U}_0 + \nu(\ddot{W}_0 + \dot{W}_0 + \dot{V}_0 - \dot{W}_0) \right) \quad \dots(2.9.1d)$$

Since all derivatives of  $U, V, W$  can be expressed in terms of the unknown constants  $C_i$ ,  $i = 1, \dots, 8$  using equations (2.3.9) and (2.4.2) the above stress resultants are

$$N_x = \sum_{i=1}^8 N_i C_i ; \quad \dots(2.9.2a)$$

$$T_x = \sum_{i=1}^8 T_i C_i ; \quad \dots(2.9.2b)$$

$$S_x = \sum_{i=1}^8 S_i C_i ; \quad \dots(2.9.2c)$$

$$M_x = \sum_{i=1}^8 M_i C_i \quad \dots(2.9.2d)$$

Boundary conditions for some conditions are

$$\text{Fixed or Clamped} \quad : \quad U_0 = V_0 = W_0 = \dot{W}_0 = 0 \quad \dots(2.9.3)$$

$$\text{Simple or Pinned} \quad : \quad U_0 = V_0 = W_0 = M_x = 0 \quad \dots(2.9.4)$$

$$\text{Free} \quad : \quad N_x = T_x = S_x = M_x = 0 \quad \dots(2.9.5)$$

Axial compression of

$$N \text{ with clamped ends} \quad : \quad \text{End 1} : N_x = N ; V_0 = W_0 = \dot{W}_0 = 0 \quad \dots(2.9.6a)$$

$$\text{End 2} : \quad U_0 = V_0 = W_0 = \dot{W}_0 = 0 \quad \dots(2.9.6b)$$

For  $\beta = 0$  and  $\beta = 1$  only 3 conditions are needed at each end and since  $V_0 = 0$ , the condition for this direction is dropped.

The boundary condition matrix is then assembled and from this the unknown constants  $C_i$  can be calculated. The following example is for a cylinder, free at the top and pinned at the base.

$$\begin{bmatrix} N_{TOP} \\ T_{TOP} \\ S_{TOP} \\ M_{TOP} \\ U_{base} \\ V_{base} \\ W_{base} \\ M_{base} \end{bmatrix} = \begin{bmatrix} N_{1T} & N_{2T} & N_{3T} & N_{4T} & N_{5T} & N_{6T} & N_{7T} & N_{8T} \\ T_{1T} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ S_{1T} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ M_{1T} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ U_{1B} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ V_{1B} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ W_{1B} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ M_{1B} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & M_{8B} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{bmatrix} + \begin{bmatrix} N_{Part T} \\ T_{Part T} \\ S_{Part T} \\ M_{Part T} \\ U_{Part B} \\ V_{Part B} \\ W_{Part B} \\ M_{Part B} \end{bmatrix} = 0 \quad \dots(2.9.7)$$

Each of the elements in the matrix equation can be found from one of the expressions (2.9.1). For example

$U_{1B}$  = Coefficient of  $C_1$  in expression for  $U(x)$  at  $x = \text{base}$

$T_{4T}$  = Coefficient of  $C_4$  in expression for  $T_x$  at  $x = \text{top}$

$N_{\text{Part T}}$  = Particular solution for  $N_x$  at  $x = \text{top}$

$V_{\text{Part B}}$  = Particular solution for  $V$  at  $x = \text{base}$ .

In symbolic form equation (2.9.7) is

$$\delta = G \cdot C + \delta_p \quad \dots(2.9.8)$$

where  $\delta = [N_{\text{TOP}}, T_{\text{TOP}}, S_{\text{TOP}}, M_{\text{TOP}}, U_{\text{base}}, V_{\text{base}}, W_{\text{base}}, M_{\text{base}}]^T$

$G = [8 \times 8]$  matrix in equation (3.47)

$C = [C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8]^T$

$\delta_p = [N_{\text{Part T}}, \dots, M_{\text{Part B}}]$

In equation (2.9.8), all the elements of  $\delta$ ,  $G$ ,  $\delta_p$  can be determined and hence, inversion of matrix  $G$  gives the unknown constants  $C_i$  as follows.

$$C = G^{-1}(\delta - \delta_p) \quad \dots(2.9.9)$$

For  $\beta = 0, 1$ ,  $\delta$ ,  $C$ ,  $\delta_p$  are vectors of length 6 and  $G$  is a  $6 \times 6$  square matrix.

## CHAPTER 3

BUCKLING EQUILIBRIUM EQUATION3.1 Introduction

The equilibrium equations suitable for buckling analysis are constructed in this chapter. These equations are based on the well known equilibrium equations set out in Flügge [2] but they also incorporate angular displacements of the cylindrical sides and also in-plane stretching of the middle surface.

Since these equations are large and complex, most authors neglect terms which have small linear values at the outset of the construction of the equations. For classical solutions, non-linear terms are also neglected.

Chapter 2 of this thesis calculates the exact prebuckling displacements under various single or complex loading conditions. Due to this, all terms containing prebuckling terms are retained in the construction of the equilibrium equations. The only terms which are neglected are non-linear buckling displacements which are omitted all classical solutions.

Based on Flügge original equations, the establishment of the equilibrium equations, contained in this chapter, is original to this thesis.

3.2 General Non-linear Equilibrium Equations

The non-linear equilibrium equations for a thin cylindrical shell which are general enough for the following analysis are given by Flügge [2]. They incorporate angular displacements of the cylindrical sides

and also in-plane stretching of the middle surface, and are as follows.

$$\frac{\dot{\bar{v}}}{\bar{n}_x} + \frac{\dot{\bar{w}}}{\bar{n}_{\phi x}} - \bar{q}_x \frac{\ddot{\bar{v}}}{\bar{a}} - \bar{n}_x \frac{\ddot{\bar{w}}}{\bar{a}} - \bar{q}_\phi \left( \frac{\dot{\bar{v}}}{\bar{a}} + \frac{\dot{\bar{w}}}{\bar{a}} \right) - \bar{n}_\phi \left( \frac{\dot{\bar{v}}}{\bar{a}} - \frac{\dot{\bar{w}}}{\bar{a}} \right) + \bar{a} \bar{p}_x = 0 \quad \dots(3.2.1a)$$

$$\frac{\dot{\bar{v}}}{\bar{n}_\phi} + \frac{\dot{\bar{w}}}{\bar{n}_x \phi} + \bar{n}_x \frac{\ddot{\bar{v}}}{\bar{a}} - \bar{q}_x \left( \frac{\dot{\bar{v}}}{\bar{a}} + \frac{\dot{\bar{w}}}{\bar{a}} \right) + \bar{n}_{\phi x} \left( \frac{\dot{\bar{v}}}{\bar{a}} - \frac{\dot{\bar{w}}}{\bar{a}} \right) - \bar{q}_\phi \left( 1 + \frac{\dot{\bar{v}}}{\bar{a}} + \frac{\dot{\bar{w}}}{\bar{a}} \right) + \bar{a} \bar{p}_\phi = 0 \quad \dots(3.2.1b)$$

$$\frac{\dot{\bar{v}}}{\bar{q}_x} + \frac{\dot{\bar{w}}}{\bar{q}_\phi} + \bar{n}_{x\phi} \left( \frac{\dot{\bar{v}}}{\bar{a}} + \frac{\dot{\bar{w}}}{\bar{a}} \right) + \bar{n}_x \frac{\ddot{\bar{w}}}{\bar{a}} + \bar{n}_\phi \left( 1 + \frac{\dot{\bar{v}}}{\bar{a}} + \frac{\dot{\bar{w}}}{\bar{a}} \right) + \bar{n}_{\phi x} \left( \frac{\dot{\bar{v}}}{\bar{a}} + \frac{\dot{\bar{w}}}{\bar{a}} \right) + \bar{a} \bar{p}_r = 0 \quad \dots(3.2.1c)$$

$$\frac{\dot{\bar{m}}_{x\phi}}{\bar{a}} - \frac{\dot{\bar{m}}_\phi}{\bar{a}} - \frac{\bar{m}_x}{\bar{a}} \frac{\ddot{\bar{v}}}{\bar{a}} - \frac{\bar{m}_{\phi x}}{\bar{a}} \left( \frac{\dot{\bar{v}}}{\bar{a}} - \frac{\dot{\bar{w}}}{\bar{a}} \right) + \bar{q}_\phi + L = 0 \quad \dots(3.2.2a)$$

$$\frac{\dot{\bar{m}}_{\phi x}}{\bar{a}} + \frac{\dot{\bar{m}}_x}{\bar{a}} + \frac{\bar{m}_x \phi}{\bar{a}} \frac{\ddot{\bar{v}}}{\bar{a}} - \frac{\bar{m}_\phi}{\bar{a}} \left( \frac{\dot{\bar{v}}}{\bar{a}} - \frac{\dot{\bar{w}}}{\bar{a}} \right) - \bar{q}_x + M = 0 \quad \dots(3.2.2b)$$

$$\frac{\bar{m}_x}{\bar{a}} \left( \frac{\dot{\bar{v}}}{\bar{a}} + \frac{\dot{\bar{w}}}{\bar{a}} \right) + \frac{\bar{m}_{x\phi}}{\bar{a}} \frac{\ddot{\bar{w}}}{\bar{a}} + \frac{\bar{m}_{\phi x}}{\bar{a}} \left( 1 + \frac{\dot{\bar{v}}}{\bar{a}} + \frac{\dot{\bar{w}}}{\bar{a}} \right) - \frac{\bar{m}_\phi}{\bar{a}} \left( \frac{\dot{\bar{v}}}{\bar{a}} + \frac{\dot{\bar{w}}}{\bar{a}} \right) + \bar{n}_x \phi - \bar{n}_{\phi x} = 0 \quad \dots(3.2.2c)$$

The variables in these equations are defined in Section 1.1. The bar over a variable refers to the sum of the prebuckling variable with the addition of the incremental buckling variable, for example

$$\bar{\frac{u}{a}} = U_0 + u$$

where  $U_0$  is the dimensionless prebuckling displacement in the  $x$  direction,  $u$  is the dimensionless incremental buckling displacement and  $\bar{u}$  is the total displacement in the  $x$  direction.

The stress-strain relations based on the elastic law for the cylindrical shell are as follows.

$$\frac{\bar{n}_\phi}{\bar{a}} = n_\phi + N_\phi = D(\dot{\bar{v}} - \dot{\bar{w}} + \nu \dot{\bar{u}}) - K(\bar{w} + \ddot{\bar{w}}) + N_\phi \quad \dots(3.2.3a)$$

$$\frac{\bar{n}_x}{\bar{a}} = n_x + N_x = D(\dot{\bar{u}} + \nu \dot{\bar{v}} - \nu \dot{\bar{w}}) + K\ddot{\bar{w}} + N_x \quad \dots(3.2.3b)$$

$$\frac{\bar{n}_{\phi x}}{\bar{a}} = n_{\phi x} + N_{\phi x} = \sigma D(\dot{\bar{u}} + \dot{\bar{v}}) + \sigma K(\dot{\bar{u}} - \dot{\bar{w}}) + N_{\phi x} \quad \dots(3.2.3c)$$

$$\frac{\bar{n}_{x\phi}}{\bar{a}} = n_{x\phi} + N_{x\phi} = \sigma D(\dot{\bar{u}} + \dot{\bar{v}}) + \sigma K(\dot{\bar{v}} + \dot{\bar{w}}) + N_{x\phi} \quad \dots(3.2.3d)$$

$$\frac{\bar{m}_\phi}{a} = m_\phi + M_\phi = -K(w + \dot{w} + \nu \ddot{w}) + M_\phi \quad \dots(3.2.3e)$$

$$\frac{\bar{m}_x}{a} = m_x + M_x = -K(\ddot{w} + \nu \ddot{w} + \dot{u} + \nu \dot{v}) + M_x \quad \dots(3.2.3f)$$

$$\frac{\bar{m}_{\phi x}}{a} = m_{\phi x} + M_{\phi x} = -\sigma K(2\dot{w}' - \dot{u}' + \dot{v}') + M_{\phi x} \quad \dots(3.2.3g)$$

$$\frac{\bar{m}_x \phi}{a} = m_x \phi + M_x \phi = -2\sigma K(\dot{w}' + \dot{v}') + M_x \phi \quad \dots(3.2.3h)$$

$$\text{where } D = \frac{Et}{1-\nu^2} \quad \dots(3.2.4a)$$

$$K = \frac{Et^3}{12(1-\nu^2)} \quad \dots(3.2.4b)$$

$$\sigma = \frac{1-\nu}{2} \quad \dots(3.2.4c)$$

Expressions for  $\bar{q}_x$  and  $\bar{q}_\phi$  are obtained by substituting equations (3.2.3) into equations (3.2.2).

$$\begin{aligned} \bar{q}_x = & -Ku'' + \sigma K\ddot{u} + \frac{M_x \phi}{a} \ddot{v}' - \left( K \frac{(1+\nu)}{2} + \frac{M_\phi}{a} \right) \dot{v}' - 2\sigma K\beta_3 \dot{v}' \\ & - K\ddot{w}''' + K\nu\beta_5 \ddot{w}'' - K\dot{w}'' - 2\sigma K\beta_3 \dot{w}' + \frac{M_\phi}{a} \dot{w}' \\ & + K\beta_5 \ddot{w} + K\beta_5 \dot{w} + Q_x + q_1 \end{aligned} \quad \dots(3.2.5a)$$

$$\begin{aligned} \bar{q}_\phi = & -K\beta_3 \dot{u}' + \sigma K\beta_5 \dot{u}' + \left( 2\sigma K + \frac{M_x}{a} \right) \ddot{v}' + \frac{M_{\phi x}}{a} \dot{v}' \\ & - \sigma K\beta_5 \dot{v}' - K\nu\beta_3 \dot{v}' + K(1-2\nu)\ddot{w}'' - K\beta_3 \ddot{w}' - \frac{M_{\phi x}}{a} \dot{w}' \\ & - 2\sigma K\beta_5 \dot{w}' - K\ddot{w} - K\nu\beta_3 \ddot{w} - K\dot{w} + Q_\phi + q_2 \end{aligned} \quad \dots(3.2.5b)$$

where  $Q_x$  and  $Q_\phi$  are prebuckling stress defined by

$$Q_x = \frac{\dot{M}_{\phi x}}{a} + \frac{\dot{M}_x}{a} + \beta_3 \frac{M_x \phi}{a} - \beta_5 \frac{M_\phi}{a} + M \quad \dots(3.2.6a)$$

$$Q_\phi = -\frac{\dot{M}_x \phi}{a} + \frac{\dot{M}_\phi}{a} + \beta_3 \frac{M_x}{a} + \beta_5 \frac{M_{\phi x}}{a} + L \quad \dots(3.2.6b)$$

and  $q_1$  and  $q_2$  involve only non-linear incremental displacements

$$q_1 = 2\sigma K\ddot{v}'(\dot{w}' + \dot{v}') + K(w + \dot{w} + \nu \ddot{w})(\dot{v}' - \dot{w}') \quad \dots(3.2.7a)$$

$$q_2 = -K\ddot{v}(\ddot{w} + \nu \ddot{w} + \dot{u} + \nu \dot{v}) - \sigma K(2\dot{w}' - \dot{u}' + \dot{v}')(\dot{v}' - \dot{w}') \quad \dots(3.2.7b)$$



### 3.3 Substitution for Displacements

Substitute expressions for  $\bar{q}_x$  and  $\bar{q}_\phi$  and also the stress-strain relations into equations (3.2.1a-c) to give the following 3 equations.

x-direction:

$$\begin{aligned}
 & [D(\ddot{u} + v\dot{v} + v\dot{w}) + K\ddot{w} + \dot{N}_x][1 + E_x \dot{u}] \\
 + & [D(\dot{u} + v\dot{v} - v\dot{w}) + K\dot{w} + N_x][\dot{E}_x \dot{u}] \\
 + & [\sigma D(\ddot{u} + \dot{v}) + \sigma K(\dot{u} - \dot{w}) + \dot{N}_{\phi x}][1 + E_x \dot{u}] \\
 + & [\sigma D(\dot{u} + \dot{v}) + \sigma K(\dot{u} - \dot{w}) + N_{\phi x}][\dot{E}_x \dot{u}] \\
 - & Q_x(\beta_2 + \ddot{w}) - q_x(\beta_2 + \ddot{w}) \\
 - & [\sigma D(\dot{u} + \dot{v}) + \sigma K(\dot{v} + \dot{w}) + N_{x\phi}][(1 + E_\phi)\beta_3 + \beta_3(\dot{v} - w) + (1 + E_\phi)\ddot{v} + \dot{v}(\dot{v} - w)] \\
 - & q_\phi(\beta_1 + \dot{v} + \dot{w}) - Q_\phi(\beta_1 + \dot{v} + \dot{w}) \\
 - & [D(\dot{v} - w + v\dot{u}) - K(w + \ddot{w}) + N_\phi][\beta_5(1 + E_\phi) + (1 + E_\phi)(\dot{v} - \dot{w}) + \beta_5(\dot{v} - w) + (\dot{v} - w)(\dot{v} - \dot{w})] \\
 + & ap_x = 0 \qquad \dots(3.3.1)
 \end{aligned}$$

$\phi$ -direction:

$$\begin{aligned}
& [D(\ddot{v}-\dot{w}+\nu\dot{u}) - K(\dot{w}+\ddot{w}) + \dot{N}_\phi][1+E_\phi+\dot{v}-w] \\
& + [D(\dot{v}-w+\nu\dot{u}) - K(w+\ddot{w}) + N_\phi][\dot{E}_\phi+\ddot{v}-\dot{w}] \\
& + [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w}) + \dot{N}'_{x\phi}][1+E_\phi+\dot{v}-w] \\
& + [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w}) + N'_{x\phi}][\dot{E}_\phi+\dot{v}-\dot{w}] \\
& + [D(\dot{u}+\nu\dot{v}-\nu w) + K\ddot{w} + N_x][[(1+E_x)\beta_3 + \beta_3\dot{u} + (1+E_x)\ddot{v} + \dot{u}\dot{v}]] \\
& - q_x(\beta_1+\dot{v}+\dot{w}) - Q_x(\beta_1+\dot{v}+\dot{w}) \\
& + [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{u}-\dot{w}) + N_{\phi x}][[\beta_5(1+E_x) + (1+E_x)(\dot{v}-\dot{w}) + \beta_5\dot{u} + \dot{u}(\dot{v}-\dot{w})]] \\
& - q_\phi(1+\beta_4+\dot{v}+\dot{w}) - Q_\phi(1+\beta_4+\dot{v}+\dot{w}) + ap_\phi = 0 \quad \dots(3.3.2)
\end{aligned}$$

$z$ -direction:

$$\begin{aligned}
& - K\ddot{u} + \sigma K\dot{u}' - K\beta_3\dot{u} + \sigma K\beta_5\ddot{u} + M_{x\phi}\ddot{v} + \left[ \left( \frac{1-3\nu}{2} \right) K + M_x - M_\phi \right] \ddot{v}' \\
& - 2\sigma K\beta_3\ddot{v} + M_{\phi x}\dot{v}' - \sigma K\beta_5\dot{v}' - K\nu\beta_3\ddot{v} - K\ddot{w} + K\nu\beta_5\ddot{w} - 2\nu K\dot{w}' \\
& - K\beta_3(2\sigma+1)\ddot{w} + M_\phi\ddot{w} + K\beta_5(1-2\sigma)\dot{w}' - M_{\phi x}\dot{w}' + K\beta_5\dot{w}' - K\ddot{w} \\
& - K\nu\beta_3\ddot{w} - K\ddot{w} + \dot{Q}_x + \dot{Q}_\phi + \dot{q}_1 + \dot{q}_2 \\
& + [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w}) + N_{x\phi}][[\beta_1(1+E_\phi) + (1+E_\phi)(\dot{v}+\dot{w}) + \beta_1(\dot{v}-w) + (\dot{v}-w)(\dot{v}+\dot{w})]] \\
& - [D(\dot{u}+\nu\dot{v}-\nu w) + K\ddot{w} + N_x][[(1+E_x)\beta_2 + (1+E_x)\ddot{w} + \beta_2\dot{u} + \dot{u}\dot{w}]] \\
& + [D(\dot{v}-w+\nu\dot{u}) - K(w+\ddot{w}) + N_\phi][[(1+E_\phi)(1+\beta_4) + (1+E_\phi)(\dot{v}+\dot{w}) + (\dot{v}-w)(\dot{v}+\dot{w}) + (1+\beta_4)(\dot{v}-w)]] \\
& + [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{u}-\dot{w}) + N_{\phi x}][[\beta_1(1+E_x) + (1+E_x)(\dot{v}+\dot{w}) + \beta_1\dot{u} + \dot{u}(\dot{v}+\dot{w})]] \\
& + ap_r[(1+E_x)(1+E_\phi) + (1+E_x)(\dot{v}-w) + (1+E_\phi)\dot{u} + \dot{u}(\dot{v}-w)] = 0 \quad \dots(3.3.3)
\end{aligned}$$

### 3.4 Expansion of Equations

Equations (3.3.1-3) are then manipulated to give 3 equations which contain terms of multiples of prebuckling and incremental displacements, prebuckling displacements and products of incremental displacements. Due to the length of the 3 equations, only the manipulations for the x-direction equation (3.3.1) is given.

### 3.5 Expansion of x-direction Equation

Expanding equation (3.3.1) gives

$$\begin{aligned}
 & (1+E_x) [D(\dot{u}+\nu\dot{v}-\nu\dot{w}) + K\ddot{w}] + (1+E_x) \dot{N}_x + \dot{N}_x \dot{u} \\
 & + [D(\dot{u}+\nu\dot{v}-\nu\dot{w}) + K\ddot{w}] \dot{u} + N_x \dot{E}_x + \dot{u} [D(\dot{u}+\nu\dot{v}-\nu\dot{w}) + K\ddot{w}] \\
 & + \dot{E}_x [D(\dot{u}+\nu\dot{v}-\nu\dot{w}) + K\ddot{w}] + N_x \dot{u} + (1+E_x) \dot{N}_{\phi x} + \dot{N}_{\phi x} \dot{u} \\
 & + (1+E_x) [\sigma D(\ddot{u}+\dot{v}) + \sigma K(\ddot{u}-\dot{w}')] + \dot{u} [\sigma D(\ddot{u}+\dot{v}) + \sigma K(\ddot{u}-\dot{w}')] \\
 & + \dot{E}_x [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{u}+\dot{w}')] + \dot{u} [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{u}-\dot{w}')] \\
 & + N_{\phi x} \dot{E}_x + N_{\phi x} \dot{u} - \beta_2 Q_x - Q_x \ddot{w} - q_x \ddot{w} \\
 & - \beta_2 [-K\ddot{u} + \sigma K\ddot{u} + M_{x\phi} \dot{v} - (K \left(\frac{1+\nu}{2}\right) + M_{\phi}) \dot{v} - 2\sigma K\beta_3 \dot{v} - K\ddot{w} + K\nu\beta_5 \ddot{w} \\
 & - K\dot{w}' - 2\sigma K\beta_3 \dot{w}' + M_{\phi} \dot{w}' + K\beta_5 \ddot{w} + K\beta_5 \dot{w}] - \beta_2 q_1 \\
 & - (1+E_x) \beta_3 N_{x\phi} - (1+E_x) \beta_3 [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w}')] \\
 & - N_{x\phi} [\beta_3 (\dot{v}-w) + (1+E_x) \dot{v}] - N_{x\phi} [\dot{v}(\dot{v}-w)] \\
 & - [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w}')] [\dot{v}] [\dot{v}-w] \\
 & - [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w}')] [-\beta_3 (\dot{v}-w) + (1+E_x) \dot{v}] \\
 & - Q_{\phi} \beta_1 - Q_{\phi} (\dot{v}+\dot{w}') - q_{\phi} (\dot{v}+\dot{w}')
 \end{aligned}$$

$$\begin{aligned}
& - \beta_1 [-K\beta_3 \dot{u} + \sigma K\beta_5 \dot{u} + (2\sigma K + M_x) \ddot{v} + M_{\phi x} \dot{v}' - \sigma K\beta_5 \dot{v}' - K\nu\beta_3 \dot{v}' \\
& \quad + K(1-2\nu) \ddot{w} - K\beta_3 \ddot{w} - M_{\phi x} \dot{w}' - 2\sigma K\beta_5 \dot{w}' - K\ddot{w} - K\nu\beta_3 \ddot{w} - K\dot{w}'] \\
& - \beta_1 q_2 - N_{\phi} \beta_5 (1+E_{\phi}) - N_{\phi} (1+E_{\phi}) (\dot{v}' - \dot{w}') = N_{\phi} \beta_5 (\dot{v} - w) \\
& - N_{\phi} (\dot{v} - w) (\dot{v}' - \dot{w}') - \beta_5 (1+E_{\phi}) [D(\dot{v} - w + \nu \dot{u}) - K(w + \ddot{w})] \\
& - [D(\dot{v} - w + \nu \dot{u}) - K(w + \ddot{w})] [(1+E_{\phi}) (\dot{v}' - w') + \beta_5 (\dot{v} - w) + (\dot{v} - w) (\dot{v}' - \dot{w}')] \\
& + a p_x = 0 \quad \dots (3.5.1)
\end{aligned}$$

Grouping the terms of the incremental displacements gives

$$\begin{aligned}
& [D(1+E_x) + N_x + K\beta_2] \ddot{u} + [N_{\phi x}] \dot{u} + [\dot{N}_x + D\dot{E}_x + \dot{N}_{\phi x} + K\beta_1\beta_3 - D(1+E_{\phi})\beta_5\nu] \dot{u} \\
& + [-\sigma D(1+E_{\phi})\beta_3 - \sigma K\beta_1\beta_5] \dot{u} + [\sigma D(1+E_x) + \sigma K(1+E_x) - \sigma K\beta_2] \ddot{u} + [\sigma \dot{E}_x (D+K)] \dot{u} \\
& + [-\beta_2 M_{x\phi} - (1+E_{\phi}) N_{x\phi} - \beta_1 (2\sigma K + M_x)] \ddot{v}' \\
& + [D\nu(1+E_x) + \sigma D(1+E_x) + \beta_2 (K \left(\frac{1+\nu}{2}\right) + M_{\phi}) - \beta_1 M_{\phi x} - (1+E_{\phi}) N_{\phi}] \dot{v}' \\
& + [2\sigma K\beta_2\beta_3 - \sigma D(1+E_{\phi})\beta_3 - \sigma K(1+E_{\phi})\beta_3 - Q_{\phi} + \sigma K\beta_1\beta_5 + \sigma D\dot{E}_x] \dot{v}' \\
& + [D\nu\dot{E}_x - \beta_3 N_{x\phi} + \nu K\beta_1\beta_3 - \beta_5 N_{\phi} - \beta_5 (1+E_{\phi}) D] \dot{v}' \\
& + [K(1+E_x) + K\beta_2] \ddot{w} + [-K\beta_1(1-2\nu)] \ddot{w} + [K\dot{E}_x - Q_x - \nu K\beta_2\beta_5 + K\beta_1\beta_3] \dot{w}' \\
& + [K\beta_2 - \sigma K(1+E_x)] \dot{w}' + [2\sigma K\beta_2\beta_3 - \sigma K(1+E_{\phi})\beta_3 - Q_{\phi} + 2\sigma K\beta_1\beta_5 - \sigma K\dot{E}_x] \dot{w}' \\
& + [\beta_1 M_{\phi x} - \beta_2 M_{\phi} - D\nu(1+E_x)] \dot{w}' + [K\beta_1] \ddot{w} + [(1+E_{\phi}) N_{\phi}] \dot{w}' \\
& + [K\nu\beta_1\beta_3 - K\beta_2\beta_5 + K(1+E_{\phi})\beta_5] \ddot{w} + [K\beta_1] \dot{w}' \\
& + [\beta_3 N_{x\phi} - \nu \dot{E}_x D - K\beta_2\beta_5 + \beta_5 N_{\phi} + D(1+E_{\phi})\beta_5 + K(1+E_{\phi})\beta_5] w \\
& + P^1 + Q^1 = 0 \quad \dots (3.5.2)
\end{aligned}$$

where

$$P^1 = [(1+E_x)N_x]' + [(1+E_x)N_{\phi x}]' - (1+E_\phi)[\beta_3 N_{x\phi} + \beta_5 N_\phi] - \beta_2 Q_x - \beta_1 Q_\phi + ap_x \quad \dots(3.5.3a)$$

and

$$\begin{aligned} Q^1 = & \dot{u}[D(\ddot{u}+\nu\dot{v}-\nu\dot{w}+\sigma(\ddot{u}+\dot{v})) + K(\ddot{w}+\sigma(\ddot{u}-\dot{w}))] \\ & + \ddot{u}[D(\dot{u}+\nu\dot{v}-\nu w) + K\dot{w}] - q_x \ddot{w} - \beta_2 q_1 - \beta_1 q_2 \\ & + [\dot{v}-w][ - N_{x\phi} \ddot{v} - N_\phi (\dot{v}-\dot{w}) - \dot{v}(\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w}))] \\ & - [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w})][ - \beta_3 (\dot{v}-w) + (1+E_\phi) \dot{v}] \\ & - [D(\dot{v}-w+\nu\dot{u}) - K(w+\dot{w})][(1+E_\phi)(\dot{v}-w) + \beta_5 (\dot{v}-w) + (\dot{v}-w)(\dot{v}-\dot{w})] \\ & + \dot{u}[\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{u}-\dot{w})] \quad \dots(3.5.3b) \end{aligned}$$

### 3.6 Expansion of $\phi$ -direction Equation

Proceeding exactly the same as the x-direction equation, the  $\phi$ -direction and z-direction equations (3.5.2) results in the following

$$\begin{aligned} & [K\beta_1] \ddot{u} + [\nu D(1+E_\phi) + \sigma D(1+E_\phi)] \dot{u} \\ & + [\nu D\dot{E}_\phi + D(1+E_x)\beta_3 + \beta_3 N_x + \beta_5 N_{\phi x} + K\beta_3(1+\beta_4)] \dot{u} + [-\sigma K\beta_1] \ddot{u} \\ & + [\sigma D\dot{E}_\phi + \sigma D(1+E_\phi)\beta_5 + \sigma K(1+E_x)\beta_5 - \sigma K(1+\beta_4)\beta_5] \dot{u} \\ & + [\sigma D(1+E_\phi) + \sigma K(1+E_\phi) + (1+E_x)N_x - \beta_1 M_{x\phi} - (1+\beta_4)(2\sigma K + M_x)] \dot{v} \\ & + [N_{x\phi} + \beta_1(K(\frac{1+\nu}{2}) + M_\phi) + (1+E_x)N_{\phi x} - (1+\beta_4)M_{\phi x}] \dot{v} \\ & + [\sigma D\dot{E}_\phi + \sigma K\dot{E}_\phi + 2\sigma K\beta_1\beta_3 - Q_x + \sigma D(1+E_x)\beta_5 + \sigma K(1+\beta_4)\beta_5] \dot{v} \\ & + [D(1+E_\phi) + N_\phi] \dot{v} + [\dot{N}_\phi + D\dot{E}_\phi + \dot{N}_{x\phi} + \nu D(1+E_x)\beta_3 + \nu K\beta_3(1+\beta_4) - Q_\phi] \dot{v} \\ & + [K\beta_1] \ddot{w} + [\sigma K(1+E_\phi) - K(1-2\nu)(1+\beta_4)] \dot{w} \\ & + [K(1+E_x)\beta_3 - \nu K\beta_1\beta_5 + K\beta_3(1+\beta_4)] \ddot{w} + [K\beta_1] \dot{w} \end{aligned}$$

$$\begin{aligned}
& + [\sigma K \dot{E}_\phi + 2\sigma K \beta_1 \beta_3 - Q_x - \sigma K(1+E_x)\beta_5 + 2\sigma K(1+\beta_4)\beta_5] \dot{w}' \\
& + [(1+\beta_4)M_{\phi x} - N_{x\phi} - \beta_1 M_\phi - (1+E_x)N_{\phi x}] \dot{w}' + [K(1+\beta_4) - K(1+E_\phi)] \ddot{w} \\
& + [\nu K \beta_3(1+\beta_4) - K \dot{E}_\phi - K \beta_1 \beta_5 - Q_\phi] \ddot{w} + [K(1+\beta_4) - D(1+E_\phi) - K(1+E_\phi) - N_\phi] \dot{w} \\
& + [-(D+K)\dot{E}_\phi - \dot{N}_{x\phi} - \nu D(1+E_x)\beta_3 - K \beta_1 \beta_5 - \dot{N}_\phi] w \\
& + P^2 + Q^2 = 0 \qquad \dots(3.6.1)
\end{aligned}$$

where

$$\begin{aligned}
P^2 = & [(1+E_\phi)N_\phi]' + [(1+E_\phi)N_{x\phi}]' + (1+E_x)\beta_3 N_x + (1+E_x)\beta_5 N_{\phi x} \\
& - \beta_1 Q_x - (1+\beta_4)Q_\phi + a p_\phi \qquad \dots(3.6.2a)
\end{aligned}$$

and

$$\begin{aligned}
Q^2 = & [D(\dot{v}-\dot{w}+\nu\dot{u}) - K(\dot{w}+\ddot{w})][\dot{v}-w] \\
& + [D(\dot{v}-w+\nu\dot{u}) - K(w+\ddot{w})][\ddot{v}-\dot{w}] \\
& + [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w})][\dot{v}-w] \\
& + [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{v}+\dot{w})][\dot{v}-\dot{w}] + N_x \dot{u} \dot{v} \\
& + [D(\dot{u}+\nu\dot{v}-\nu\dot{w}) + K\ddot{w}][\beta_3 \dot{u} + (1+E_x)\dot{v} + \dot{u} \dot{v}] \\
& - \beta_1 q_1 - q_x [\dot{v}+\dot{w}] + N_{\phi x} \dot{u}(\dot{v}-\dot{w}) \\
& + [\sigma D(\dot{u}+\dot{v}) + \sigma K(\dot{u}+\dot{w})][(1+E_x)(\dot{v}-\dot{w}) + \beta_3 \dot{u} + \dot{u}(\dot{v}-\dot{w})] \\
& - (1+\beta_4)q_2 - q_\phi(\dot{v}+\dot{w}) \qquad \dots(3.6.2b)
\end{aligned}$$

### 3.7 Expansion of z-direction Equation

Expanding, as before, equation (3.5.3) results in the following

$$\begin{aligned}
 & [-K]\dddot{u} + [\sigma K]\ddot{u}' + [-K\beta_3]\dot{u}' \\
 & + [D(1+E_x)\beta_2 - K\beta_3 + \beta_2 N_x + \nu D(1+E_\phi)(1+\beta_4) + \beta_1 N_{\phi x} + a p_r (1+E_\phi)]\dot{u}' \\
 & + [\sigma K\beta_5]\ddot{u} + [\sigma K\beta_5 + \sigma D(1+E_\phi)\beta_1 + \sigma D(1+E_x)\beta_1 + \sigma K(1+E_x)\beta_1]\dot{u} \\
 & + [M_{x\phi}]\dddot{v} + [2\sigma K + M_x - M_\phi - K\left(\frac{1+\nu}{2}\right)]\ddot{v}' + [\dot{M}_x + \dot{M}_{x\phi} - 2\sigma K\beta_3]\dot{v}' \\
 & + [M_{x\phi}]\ddot{v}' + [\dot{M}_{\phi x} - \dot{M}_\phi - \sigma K\beta_5]\dot{v}' \\
 & + [-2\sigma K\beta_3 - \sigma K\beta_5 + \sigma D(1+E_\phi)\beta_1 + \sigma K(1+E_\phi)\beta_1 + (1+E_\phi)N_{x\phi} + \sigma D(1+E_x)\beta_1 + (1+E_x)N_{\phi x}]\dot{v}' \\
 & + [-\nu K\beta_3]\ddot{v} + [-\nu K\beta_3 + \beta_1 N_{x\phi} + \nu D(1+E_x)\beta_2 + D(1+E_\phi)(1+\beta_4) + (1+\beta_4)N_\phi] \\
 & \quad + (1+E_\phi)N_\phi + a p_r (1+E_x)]\dot{v}' \\
 & + [-K]\dddot{w} + [\nu K\beta_5]\ddot{w}' + [-2\nu K]\dot{w}' + [-K\beta_3(1+2\sigma)]\dot{w}' \\
 & + [\nu K\beta_5 + M_\phi - K\beta_3 + K(1+E_x)\beta_2 + N_x(1+E_x)]\dot{w}' + [K\beta_5(1-2\sigma)]\dot{w}' \\
 & + [-M_{\phi x} - 2\sigma K\beta_3 - 2\sigma K\beta_5 + \sigma K(1+E_\phi)\beta_1 + (1+E_\phi)N_{x\phi} - \sigma K(1+E_x)\beta_1 + (1+E_x)N_{\phi x}]\dot{w}' \\
 & + [\dot{M}_\phi - \dot{M}_{\phi x} + K\beta_5]\dot{w}' + [-K]\ddot{w} + [-\nu K\beta_3]\ddot{w} \\
 & + [K\beta_5 - \nu K\beta_3 - K - K(1+E_\phi)(1+\beta_4) + (1+E_\phi)N_\phi]\ddot{w} \\
 & + [K\beta_5 - \beta_1 N_{x\phi} - \nu D(1+E_x)\beta_2 - D(1+E_\phi)(1+\beta_4) - K(1+E_\phi)(1+\beta_4) - (1+\beta_4)N_\phi \\
 & \quad - a p_r (1+E_x)]\ddot{w} + P^3 + Q^3 = 0 \quad \dots(3.7.1)
 \end{aligned}$$

where

$$\begin{aligned}
 P^3 &= \dot{Q}_x + \dot{Q}_\phi + (1+E_\phi)\beta_1 N_{x\phi} + (1+E_x)\beta_2 N_x + (1+E_\phi)(1+\beta_4)N_\phi \\
 & \quad + (1+E_x)\beta_1 N_{\phi x} + a p_r (1+E_x)(1+E_\phi) \quad \dots(3.7.2a)
 \end{aligned}$$

and

$$\begin{aligned}
 Q^3 = & \dot{q}_1 + \dot{q}_2 + N_{x\phi}(\dot{v}-w)(\dot{v}+\dot{w}) + N_x \dot{u}\dot{w} + N_\phi(\dot{v}-w)(\dot{v}+\dot{w}) \\
 & + N_{\phi x} \dot{u}\dot{w} + a p_r \dot{u}(\dot{v}-w) \\
 & + n_{x\phi} [\beta_1(\dot{v}-w) + (1+E_\phi)(\dot{v}+\dot{w}) + (\dot{v}-w)(\dot{v}+\dot{w})] \\
 & + n_x [\beta_2 \dot{u} + (1+E_x)\dot{w} + \dot{u}\dot{w}] \\
 & + n_\phi [(1+\beta_4)(\dot{v}-w) + (1+E_\phi)(\dot{v}+\dot{w}) + (\dot{v}-w)(\dot{v}+\dot{w})] \\
 & + n_{\phi x} [\beta_1 \dot{u} + (1+E_x)(\dot{v}+\dot{w}) + \dot{u}\dot{w}] \quad \dots (3.7.2b)
 \end{aligned}$$



## CHAPTER 4

POWER SERIES SOLUTION4.1 Introduction

In this chapter, the general non-linear equilibrium equations, established in Chapter 3, are linearized by making only one assumption. This is to set the terms involving multiples of the incremental displacements to zero. This step is employed by all authors who solve the buckling problem using the classical method. The other alternative is to neglect the majority of linear and non-linear terms and solve the resulting non-linear equilibrium equation. (See Flügge [2].)

Having linearized the equilibrium equations the coefficients of the incremental displacements are set out in a form which can be solved by a computer.

The equations are then solved by using a double power series solution in the  $x$  and  $\phi$  directions. This results in 3 recurrence relations for the unknown coefficients  $A, B, C$ . Due to the complexity of the recurrence relations, the method of using these recurrence relations is set out so that the solution can be solved by a computer.

The basis of this solution is a standard power series solution to a differential equation, however the equations set out in this chapter are original to this thesis.

4.2 Linearized Equilibrium Equations

Equations (3.5.2), (3.6.1) and (3.7.1) are the generalized non-linear equilibrium equations in terms of the buckling displacement  $u, v, w$  and their derivatives.

Now  $P^1$ ,  $P^2$ ,  $P^3$  (equations (3.5.3a), (3.6.2b) and (3.7.2b)) are the exact prebuckling linear equilibrium equations so that

$$P^1 = P^2 = P^3 = 0 \quad \dots(4.2.1)$$

$Q^1$ ,  $Q^2$ ,  $Q^3$  (equations (3.5.3b), (3.6.2b) and (3.7.2b)) are the terms containing multiples of buckling displacements. To solve the set of equations we set these terms to zero.

$$Q^1 = Q^2 = Q^3 = 0 \quad \dots(4.2.2)$$

The 3 equations are now linear partial differential equations in the unknown buckling displacements of  $u, v, w$ . To solve we assume a power series solution and obtain recurrence relations.

Let

$$u = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A(i, j) x^i \phi^j \quad \dots(4.2.3a)$$

$$v = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} B(i, j) x^i \phi^j \quad \dots(4.2.3b)$$

$$w = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C(i, j) x^i \phi^j \quad \dots(4.2.3c)$$

Also define coefficients for the derivatives of  $u, v, w$  and use the following notation. ( $L \equiv u$ ;  $M \equiv v$ ;  $N \equiv w$ )

$L^*(1, r, p)$  = coefficient of  $\frac{\partial^{r+p} u}{\partial x^r \partial \phi^p}$  in equation x-direction.

$N^*(3, 2, 1)$  = coefficient of  $\ddot{w}$  in equation in z-direction.

$M^*(2, 1, 1)$  = coefficient of  $\dot{u}$  in equation in  $\phi$ -direction.

4.3 Coefficients for x-direction Equation (3.5.2)

$$\dot{L}^*(1,2,0) = D(1+E_x) + N_x + K\beta_2 \quad \dots(4.3.1a)$$

$$\dot{L}^*(1,1,1) = N_{\phi x} \quad \dots(4.3.1b)$$

$$\dot{L}^*(1,1,0) = \dot{N}_x + \dot{N}_{\phi x} + K\beta_1\beta_3 + D[\dot{E}_x - \nu\beta_5(1+E_\phi)] \quad \dots(4.3.1c)$$

$$\dot{L}^*(1,0,2) = \sigma[K(1+E_x - \beta_2) + D(1+E_x)] \quad \dots(4.3.1d)$$

$$\dot{L}^*(1,0,1) = \sigma[\dot{E}_x(D+K) - D\beta_3(1+E_\phi) - K\beta_1\beta_5] \quad \dots(4.3.1e)$$

$$\dot{M}^*(1,2,0) = -\beta_2 M_{x\phi} - \beta_1 M_x - (1+E_\phi)N_{x\phi} - 2\sigma K\beta_1 \quad \dots(4.3.2a)$$

$$\dot{M}^*(1,1,1) = \beta_2 M_\phi - \beta_1 M_{\phi x} - (1+E_\phi)N_\phi + \left(\frac{1+\nu}{2}\right)[D(1+E_x) + K\beta_2] \quad \dots(4.3.2b)$$

$$\dot{M}^*(1,1,0) = -Q_\phi + \sigma[D\dot{E}_x + \beta_1\beta_5K + \beta_3(2K\beta_2 - (1+E_\phi)(D+K))] \quad \dots(4.3.2c)$$

$$\dot{M}^*(1,0,1) = \nu D\dot{E}_x - \beta_3 N_{x\phi} - \beta_5 N_\phi + \nu K\beta_1\beta_3 - D(1+E_\phi)\beta_5 \quad \dots(4.3.2d)$$

$$\dot{N}^*(1,3,0) = K(1+E_x + \beta_2) \quad \dots(4.3.3a)$$

$$\dot{N}^*(1,2,1) = -K\beta_1(1-2\nu) \quad \dots(4.3.3b)$$

$$\dot{N}^*(1,2,0) = K\dot{E}_x - Q_x + K(\beta_1\beta_3 - \nu\beta_2\beta_5) \quad \dots(4.3.3c)$$

$$\dot{N}^*(1,1,2) = K[\beta_2 - \sigma(1+E_x)] \quad \dots(4.3.3d)$$

$$\dot{N}^*(1,1,1) = -Q_\phi + \sigma K[2\beta_2\beta_3 - \dot{E}_x - (1+E_\phi)\beta_3 + 2\beta_1\beta_5] \quad \dots(4.3.3e)$$

$$\dot{N}^*(1,1,0) = \beta_1 M_{\phi x} - \beta_2 M_\phi + (1+E_\phi)N_{\phi\phi} - \nu D(1+E_x) \quad \dots(4.3.3f)$$

$$\dot{N}^*(1,0,3) = K\beta_1 \quad \dots(4.3.3g)$$

$$\dot{N}^*(1,0,2) = K[\beta_5(1+E_\phi - \beta_2) + \nu\beta_1\beta_3] \quad \dots(4.3.3h)$$

$$\dot{N}^*(1,0,1) = K\beta_1 \quad \dots(4.3.3i)$$

$$\dot{N}^*(1,0,0) = \beta_3 N_{x\phi} + \beta_5 N_\phi - \nu D\dot{E}_x - K\beta_2\beta_5 + \beta_5(1+E_\phi)(D+K) \quad \dots(4.3.3j)$$

4.4 Coefficients for  $\phi$ -direction Equation (3.6.1)

$$\dot{L}^*(2,2,0) = K\beta_1 \quad \dots(4.4.1a)$$

$$\dot{L}^*(2,1,1) = D(1+E_\phi)(\nu+\sigma) \quad \dots(4.4.1b)$$

$$\dot{L}^*(2,1,0) = \nu D \dot{E}_\phi + D(1+E_x)\beta_3 + \beta_3 N_x + \beta_5 N_{\phi x} + K\beta_3(1+\beta_4) \quad \dots(4.4.1c)$$

$$\dot{L}^*(2,0,2) = -\sigma K\beta_1 \quad \dots(4.4.1d)$$

$$\dot{L}^*(2,0,1) = \sigma[D \dot{E}_\phi + (1+E_x)(D+K)\beta_5 - K(1+\beta_4)\beta_5] \quad \dots(4.4.1e)$$

$$\dot{M}^*(2,2,0) = \sigma(1+E_\phi)(D+K) + (1+E_x)N_x - \beta_1 M_{\phi x} - (1+\beta_4)(2\sigma K + M_x) \quad \dots(4.4.2a)$$

$$\dot{M}^*(2,1,1) = N_{\phi x} + \beta_1 \left( M_\phi + K \left( \frac{1+\nu}{2} \right) \right) + (1+E_x)N_{\phi x} - (1+\beta_4)M_{\phi x} \quad \dots(4.4.2b)$$

$$\dot{M}^*(2,1,0) = -Q_x + \sigma[(D+K)\dot{E}_\phi + 2K\beta_1\beta_3 + D(1+E_x)\beta_5 + K(1+\beta_4)\beta_5] \quad \dots(4.4.2c)$$

$$\dot{M}^*(2,0,2) = N_\phi + D(1+E_\phi) \quad \dots(4.4.2d)$$

$$\dot{M}^*(2,0,1) = -Q_\phi + \dot{N}_\phi + D \dot{E}_\phi + \dot{N}_{\phi x} + \nu D(1+E_x)\beta_3 + \nu K\beta_3(1+\beta_4) \quad \dots(4.4.2e)$$

$$\dot{N}^*(2,3,0) = K\beta_1 \quad \dots(4.4.3a)$$

$$\dot{N}^*(2,2,1) = \sigma K(1+E_\phi) - K(1-2\nu)(1+\beta_4) \quad \dots(4.4.3b)$$

$$\dot{N}^*(2,2,0) = K[(1+E_x)\beta_3 - \nu\beta_1\beta_5 + \beta_3(1+\beta_4)] \quad \dots(4.4.3c)$$

$$\dot{N}^*(2,1,2) = K\beta_1 \quad \dots(4.4.3d)$$

$$\dot{N}^*(2,1,1) = -Q_x + \sigma[K \dot{E}_\phi + 2K\beta_1\beta_3 - K(1+E_x)\beta_5 + 2K(1+\beta_4)\beta_5] \quad \dots(4.4.3e)$$

$$\dot{N}^*(2,1,0) = -N_{\phi x} - \beta_1 M_\phi - (1+E_x)N_{\phi x} + (1+\beta_4)M_{\phi x} \quad \dots(4.4.3f)$$

$$\dot{N}^*(2,0,3) = K(\beta_4 - E_\phi) \quad \dots(4.4.3g)$$

$$\dot{N}^*(2,0,2) = -Q_\phi + K[\nu\beta_3(1+\beta_4) - \dot{E}_\phi - \beta_1\beta_5] \quad \dots(4.4.3h)$$

$$\dot{N}^*(2,0,1) = -\dot{N}_{\phi} - (1+E_{\phi})(D+K) + K(1+\beta_4) \quad \dots(4.4.3i)$$

$$\dot{N}^*(2,0,0) = -\dot{N}'_{x\phi} - \dot{N}_{\phi} - (D+K)\dot{E}_{\phi} - \nu D(1+E_x)\beta_3 - K\beta_1\beta_5 \quad \dots(4.4.3j)$$

#### 4.5 Coefficients for z-direction Equation (3.7.1)

$$\dot{L}^*(3,3,0) = -K \quad \dots(4.5.1a)$$

$$\dot{L}^*(3,1,2) = \sigma K \quad \dots(4.5.1b)$$

$$\dot{L}^*(3,1,1) = -K\beta_3 \quad \dots(4.5.1c)$$

$$\begin{aligned} \dot{L}^*(3,1,0) &= \beta_1 \dot{N}_{\phi x} + \beta_2 \dot{N}_x - K\dot{\beta}_3 + D(1+E_x) \\ &\quad + \nu D(1+E_{\phi})(1+\beta_4) + a p_r (1+E_{\phi}) \end{aligned} \quad \dots(4.5.1d)$$

$$\dot{L}^*(3,0,2) = \sigma K\beta_5 \quad \dots(4.5.1e)$$

$$\dot{L}^*(3,0,1) = \sigma [K\dot{\beta}_5 + D(1+E_{\phi})\beta_1 + D(1+E_x)\beta_1 + K(1+E_x)\beta_1] \quad \dots(4.5.1f)$$

$$\dot{M}^*(3,3,0) = \dot{M}_{x\phi} \quad \dots(4.5.2a)$$

$$\dot{M}^*(3,2,1) = \dot{M}_x - \dot{M}_{\phi} + \left(\frac{1-3\nu}{2}\right) K \quad \dots(4.5.2b)$$

$$\dot{M}^*(3,2,0) = \dot{M}_x + \dot{M}'_x - 2\sigma K\beta_3 \quad \dots(4.5.2c)$$

$$\dot{M}^*(3,1,2) = \dot{M}_{\phi x} \quad \dots(4.5.2d)$$

$$\dot{M}^*(3,1,1) = \dot{M}_{\phi x} - \dot{M}'_{\phi} = \sigma K\beta_5 \quad \dots(4.5.2e)$$

$$\begin{aligned} \dot{M}^*(3,1,0) &= (1+E_{\phi})\dot{N}_{x\phi} + (1+E_x)\dot{N}_{\phi x} + \sigma [D(1+E_x)\beta_1 \\ &\quad + (D+K)(1+E_{\phi})\beta_1 - 2K\dot{\beta}_3 - K\dot{\beta}_5] \end{aligned} \quad \dots(4.5.2f)$$

$$\dot{M}^*(3,0,2) = -\nu K\beta_3 \quad \dots(4.5.2g)$$

$$\begin{aligned} \dot{M}^*(3,0,1) &= \beta_1 \dot{N}_{x\phi} + (1+\beta_4)\dot{N}_{\phi} + (1+E_{\phi})\dot{N}_{\phi} + a p_r (1+E_x) \\ &\quad = \nu K\dot{\beta}_3 + \nu D(1+E_x)\beta_2 + D(1+E_{\phi})(1+\beta_4) \end{aligned} \quad \dots(4.5.2h)$$

$$\dot{N}(3,4,0) = -K \quad \dots(4.5.3a)$$

$$\dot{N}(3,3,0) = \nu K \beta_5 \quad \dots(4.5.3b)$$

$$\dot{N}(3,2,2) = -2\nu K \quad \dots(4.5.3c)$$

$$\dot{N}(3,2,1) = -K \beta_3 (1+2\sigma) \quad \dots(4.5.3d)$$

$$\dot{N}(3,2,0) = \dot{M}_\phi + (1+E_x) N_x + \nu K \beta'_5 - K \beta_3 + K(1+E_x) \beta_2 \quad \dots(4.5.3e)$$

$$\dot{N}(3,1,2) = \nu K \beta_5 \quad \dots(4.5.3f)$$

$$\dot{N}(3,1,1) = -\dot{M}_{\phi x} + (1+E_\phi) N_{x\phi} + (1+E_x) N_{\phi x} \quad \dots(4.5.3g)$$

$$\dot{N}(3,1,0) = \dot{M}_\phi - \dot{M}_{\phi x} + K \beta_5 \quad \dots(4.5.3h)$$

$$\dot{N}(3,0,4) = -K \quad \dots(4.5.3i)$$

$$\dot{N}(3,0,3) = -\nu K \beta_3 \quad \dots(4.5.3j)$$

$$\dot{N}(3,0,2) = (1+E_\phi) N_\phi + K[\beta'_5 - \nu \beta_3 - 1 - (1+E_\phi)(1+\beta_4)] \quad \dots(4.5.3k)$$

$$\begin{aligned} \dot{N}(3,0,0) &= -\beta_1 N_{x\phi} - (1+\beta_4) N_\phi - a p_r (1+E_x) + K \beta'_5 \\ &\quad - \nu D(1+E_x) \beta_2 - (D+K)(1+E_\phi)(1+\beta_4) \quad \dots(4.5.3l) \end{aligned}$$

#### 4.6 Summary of Linear Buckling Equilibrium Equations

The 3 equilibrium equations (3.5.2), (3.6.1) and (3.7.1) are rewritten with the coefficients defined in equations (4.3.1-3), (4.4.1-3) and (4.5.1-3).

x-direction

$$\begin{aligned}
& \check{L}^*(1,2,0)\check{u} + \check{L}^*(1,1,1)\check{u} + \check{L}^*(1,1,0)\check{u} + \check{L}^*(1,0,2)\check{u} + \check{L}^*(1,0,1)\check{u} \\
& + \check{M}^*(1,2,0)\check{v} + \check{M}^*(1,1,1)\check{v} + \check{M}^*(1,1,0)\check{v} + \check{M}^*(1,0,1)\check{v} \\
& + \check{N}^*(1,3,0)\check{w} + \check{N}^*(1,2,1)\check{w} + \check{N}^*(1,2,0)\check{w} + \check{N}^*(1,1,2)\check{w}' + \check{N}^*(1,1,1)\check{w}' \\
& + \check{N}^*(1,1,0)\check{w} + \check{N}^*(1,0,3)\check{w} + \check{N}^*(1,0,2)\check{w} + \check{N}^*(1,0,1)\check{w} + \check{N}^*(1,0,0)w = 0 \\
& \dots(4.6.1)
\end{aligned}$$

 $\phi$ -direction

$$\begin{aligned}
& \check{L}^*(2,2,0)\check{u} + \check{L}^*(2,1,1)\check{u} + \check{L}^*(2,1,0)\check{u} + \check{L}^*(2,0,2)\check{u} + \check{L}^*(2,0,1)\check{u} \\
& + \check{M}^*(2,2,0)\check{v} + \check{M}^*(2,1,1)\check{v} + \check{M}^*(2,1,0)\check{v} + \check{M}^*(2,0,2)\check{v} + \check{M}^*(2,0,1)\check{v} \\
& + \check{N}^*(2,3,0)\check{w} + \check{N}^*(2,2,1)\check{w} + \check{N}^*(2,2,0)\check{w} + \check{N}^*(2,1,2)\check{w}' + \check{N}^*(2,1,1)\check{w}' \\
& + \check{N}^*(2,1,0)\check{w} + \check{N}^*(2,0,3)\check{w} + \check{N}^*(2,0,2)\check{w} + \check{N}^*(2,0,1)\check{w} + \check{N}^*(2,0,0)w = 0 \\
& \dots(4.6.2)
\end{aligned}$$

z-direction

$$\begin{aligned}
& \check{L}^*(3,3,0)\check{u} + \check{L}^*(3,1,2)\check{u} + \check{L}^*(3,1,1)\check{u} + \check{L}^*(3,1,0)\check{u} + \check{L}^*(3,0,2)\check{u} + \check{L}^*(3,0,1)\check{u} \\
& + \check{M}^*(3,3,0)\check{v} + \check{M}^*(3,2,1)\check{v} + \check{M}^*(3,2,0)\check{v} + \check{M}^*(3,1,2)\check{v} + \check{M}^*(3,1,1)\check{v} + \check{M}^*(3,1,0)\check{v} \\
& + \check{M}^*(3,0,2)\check{v} + \check{M}^*(3,0,1)\check{v} + \check{N}^*(3,4,0)\check{w} + \check{N}^*(3,3,0)\check{w} + \check{N}^*(3,2,2)\check{w}'' \\
& + \check{N}^*(3,2,1)\check{w} + \check{N}^*(3,2,0)\check{w} + \check{N}^*(3,1,2)\check{w}' + \check{N}^*(3,1,1)\check{w}' + \check{N}^*(3,1,0)\check{w}' \\
& + \check{N}^*(3,0,4)\check{w} + \check{N}^*(3,0,3)\check{w} + \check{N}^*(3,0,2)\check{w} + \check{N}^*(3,0,0)w = 0 \quad \dots(4.6.3)
\end{aligned}$$

where  $L, M, N$  are defined by equations (4.3.1-3), (4.4.1-3) and (4.5.1-3) and terms containing multiples of buckling displacements are neglected.

#### 4.7 Prebuckling Power Series Expansion

Since the prebuckling terms  $L^*(m,i,j)$ ,  $M^*(m,i,j)$  and  $N^*(m,i,j)$  are known functions of  $x$  and  $\phi$  (section 4.3-5) they can be expressed by power series such as

$$\sum_{p=0}^P \sum_{q=0}^Q L(m,i,j,p,q) x^p \phi^q \doteq L^*(m,i,j) \quad \dots(4.7.1)$$

$$\sum_{p=0}^P \sum_{q=0}^Q M(m,i,j,p,q) x^p \phi^q \doteq M^*(m,i,j) \quad \dots(4.7.2)$$

$$\sum_{p=0}^P \sum_{q=0}^Q N(m,i,j,p,q) x^p \phi^q \doteq N^*(m,i,j) \quad \dots(4.7.3)$$

where  $P$  and  $Q$  are limits to power series.

#### 4.8 Recurrence Relations : x-direction

Substitute the power series into (4.6.1) to give

x-direction

$$\begin{aligned} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^P \sum_{q=0}^Q x^{i+p} \phi^{j+q} \left\{ \right. & L(1,2,0,p,q) * A(i+2,j) * (i+2) * (i+1) \\ & + L(1,1,1,p,q) * A(i+1,j+1) * (i+1)(j+1) \\ & + L(1,1,0,p,q) * A(i+1,j) * (i+1) \\ & + L(1,0,2,p,q) * A(i,j+2) * (j+2)(j+1) \\ & + L(1,0,1,p,q) * A(i,j+1) * (j+1) \\ & + M(1,2,0,p,q) * B(i+2,j) * (i+2)(i+1) \\ & + M(1,1,1,p,q) * B(i+1,j+1) * (i+1)(j+1) \\ & + M(1,1,0,p,q) * B(i+1,j) * (i+1) \\ & \left. + M(1,0,1,p,q) * B(i,j+1) * (j+1) \right\} \end{aligned}$$



$$\begin{aligned}
& + N(1,3,0,p,q) * C(i+3,j) * (i+3)(i+2)(i+1) \\
& + N(1,2,1,p,q) * C(i+2,j+1) * (i+2)(i+1)(j+1) \\
& + N(1,2,0,p,q) * C(i+2,j) * (i+2)(i+1) \\
& + N(1,1,2,p,q) * C(i+1,j+2) * (i+1)(j+2)(j+1) \\
& + N(1,1,1,p,q) * C(i+1,j+1) * (i+1)(j+1) \\
& + N(1,1,0,p,q) * C(i+1,j) * (i+1) \\
& + N(1,0,3,p,q) * C(i,j+3) * (j+3)(j+2)(j+1) \\
& + N(1,0,2,p,q) * C(i,j+2) * (j+2)(j+1) \\
& + N(1,0,1,p,q) * C(i,j+1) * (j+1) \\
& + N(1,0,0,p,q) * C(i,j) \} = 0 \quad \dots(4.8.1)
\end{aligned}$$

Equating coefficients of power of  $x$  and  $\phi$  to zero in equation (4.8.1) gives the following

x-direction , for  $i = 0, \dots, \infty$  ,  $j = 0, \dots, \infty$  .

$$\begin{aligned}
\sum_{p=\max(0, i-P)}^i \sum_{q=\max(0, j-Q)}^j \left\{ \right. & L(1,2,0,i-p,j-q) * A(p+2,q) * (p+2)(p+1) \\
& + L(1,1,1,i-p,j-q) * A(p+1,q+1) * (p+1)(q+1) \\
& + L(1,1,0,i-p,j-q) * A(p+1,q) * (p+1) \\
& + L(1,0,2,i-p,j-q) * A(p,q+2) * (q+2)(q+1) \\
& + L(1,0,1,i-p,j-q) * A(p,q+1) * (q+1) \\
& + M(1,2,0,i-p,j-q) * B(p+2,q) * (p+2)(p+1) \\
& + M(1,1,1,i-p,j-q) * B(p+1,q+1) * (p+1)(q+1) \\
& \left. + M(1,1,0,i-p,j-q) * B(p+1,q) * (p+1) \right\}
\end{aligned}$$

$$\begin{aligned}
& + M(1,0,1,i-p,j-q) * B(p,q+1) * (q+1) \\
& + N(1,3,0,i-p,j-q) * C(p+3,q) * (p+3)(p+2)(p+1) \\
& + N(1,2,1,i-p,j-q) * C(p+2,q+1) * (p+2)(p+1)(q+1) \\
& + N(1,2,0,i-p,j-q) * C(p+2,q) * (p+2)(p+1) \\
& + N(1,1,2,i-p,j-q) * C(p+1,q+2) * (p+1)(q+2)(q+1) \\
& + N(1,1,1,i-p,j-q) * C(p+1,q+1) * (p+1)(q+1) \\
& + N(1,1,0,i-p,j-q) * C(p+1,q) * (p+1) \\
& + N(1,0,3,i-p,j-q) * C(p,q+3) * (q+3)(q+2)(q+1) \\
& + N(1,0,2,i-p,j-q) * C(p,q+2) * (q+2)(q+1) \\
& + N(1,0,1,i-p,j-q) * C(p,q+1) * (q+1) \\
& + N(1,0,0,i-p,j-q) * C(p,q) \} = 0 \quad \dots(4.8.2)
\end{aligned}$$

for  $i, j = 0, \dots, \infty$ .

Express  $A(i, j+2)$  in terms of other  $A, B, C$  as follows.

$$\begin{aligned}
A(i, j+2) = & - \left\{ \sum_{p=\max(0, i-p)}^i \sum_{q=\max(0, j-q)}^j \left[ L(1,2,0,i-p,j-q)A(p+2,q)(p+2)(p+1) \right. \right. \\
& + L(1,1,1,i-p,j-q)A(p+1,q+1)(p+1)(q+1) + L(1,1,0,i-p,j-q)A(p+1,q)(p+1) \\
& + L(1,0,1,i-p,j-q)A(p,q+1)(q+1) + M(1,2,0,i-p,j-q)B(p+2,q)(p+2)(p+1) \\
& + M(1,1,1,i-p,j-q)B(p+1,q+1)(p+1)(q+1) + M(1,1,0,i-p,j-q)B(p+1,q)(p+1) \\
& + M(1,0,1,i-p,j-q)B(p,q+1)(q+1) + N(1,3,0,i-p,j-q)C(p+3,q)(p+3)(p+2)(p+1) \\
& + N(1,2,1,i-p,j-q)C(p+2,q+1)(p+2)(p+1)(q+1) + N(1,2,0,i-p,j-q)C(p+2,q)(p+2)(p+1) \\
& \left. \left. + N(1,1,2,i-p,j-q)C(p+1,q+2)(p+1)(q+2)(q+1) + N(1,1,1,i-p,j-q)C(p+1,q+1)(p+1)(q+1) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + N(1,1,0,i-p,j-q)C(p+1,q)(p+1) + N(1,0,3,i-p,j-q)C(p,q+3)(q+3)(q+2)(q+1) \\
& + N(1,0,2,i-p,j-q)C(p,q+2)(q+2)(q+1) + N(1,0,1,i-p,j-q)C(p,q+1)(q+1) \\
& + N(1,0,0,i-p,j-q)C(p,q) \quad ] \\
& \sum_{p=\max(0,i-p)}^{i-1} \sum_{q=\max(0,j-q)}^j \left[ L(1,0,2,i-p,j-q)A(p,q+2)(q+2)(q+1) \right] \\
& + L(1,0,2,0,1)A(i,j+1)(j+1)(j) \} / \left\{ L(1,0,2,0,0)(j+2)(j+1) \right\} \dots(4.8.3)
\end{aligned}$$

#### 4.9 Recurrence Relations : $\phi$ -direction

In exactly the same way, by substituting power series expansions into (4.6.2) and then equating coefficients of power of  $x$  and  $\phi$  to zero, and then expressing  $B(i,j+2)$  in terms of other  $A,B,C$  gives the following.

$$\begin{aligned}
B(i,j+2) = & - \left\{ \sum_{p=\max(0,i-p)}^i \sum_{q=\max(0,j-q)}^j \left[ L(2,2,0,i-p,j-q)A(p+2,q)(p+2)(p+1) \right. \right. \\
& \left. \left. + L(2,0,2,i-p,j-q)A(p,q+2)(q+2)(q+1) \right. \right. \\
& + L(2,1,1,i-p,j-q) * A(p+1,q+1) * (p+1)(q+1) \\
& + L(2,1,0,i-p,j-q) * A(p+1,q) * (p+1) \\
& + L(2,0,1,i-p,j-q) * A(p,q+1) * (q+1) \\
& + M(2,2,0,i-p,j-q) * B(p+2,q) * (p+2)(p+1) \\
& + M(2,1,1,i-p,j-q) * B(p+1,q+1) * (p+1)(q+1) \\
& + M(2,1,0,i-p,j-q) * B(p+1,q) * (p+1) \\
& + M(2,0,1,i-p,j-q) * B(p,q+1) * (q+1) \\
& + N(2,3,0,i-p,j-q) * C(p+3,q) * (p+3)(p+2)(p+1) \\
& \left. \left. + N(2,2,1,i-p,j-q) * C(p+2,q+1) * (p+2)(p+1)(q+1) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + N(2,2,0,i-p,j-q) * C(p+2,q) * (p+2)(p+1) \\
& + N(2,1,2,i-p,j-q) * C(p+1,q+2) * (p+1)(q+2)(q+1) \\
& + N(2,1,1,i-p,j-q) * C(p+1,q+1) * (p+1)(q+1) \\
& + N(2,1,0,i-p,j-q) * C(p+1,q) * (p+1) \\
& + N(2,0,3,i-p,j-q) * C(p,q+3) * (q+3)(q+2)(q+1) \\
& + N(2,0,2,i-p,j-q) * C(p,q+2) * (q+2)(q+1) \\
& + N(2,0,1,i-p,j-q) * C(p,q+1) * (q+1) \\
& + N(2,0,0,i-p,j-q) * C(p,q) \quad ] \\
& + \sum_{p=\max(0,i-P)}^{i-1} \sum_{q=\max(0,j-Q)}^j \left[ M(2,0,2,i-p,j-q)B(p,q+2)(q+2)(q+1) \right] \\
& + M(2,0,2,0,1)B(i,j+1)(j+1)(j) \Big/ \left\{ M(2,0,2,0,0)(j+2)(j+1) \right\} \\
& \dots(4.9.1)
\end{aligned}$$

#### 4.10 Recurrence Relation : z-direction

Substitute power series expansions into equation (4.6.3) and equate coefficients of  $x$  and  $\phi$  to zero and then express  $C(i,j+4)$  in terms of other  $A,B,C$  gives for  $i,j = 0,1,\dots,\infty$ .

$$\begin{aligned}
C(i,j+4) = & - \left\{ \sum_{p=\max(0,i-P)}^i \sum_{q=\max(0,j-Q)}^j \right. \\
& \left[ L(3,3,0,i-p,j-q)A(p+3,q)(p+3)(p+2)(p+1) \right. \\
& \quad + L(3,1,2,i-p,j-q)A(p+1,q+2)(p+1)(q+2)(q+1) \\
& \quad + L(3,1,1,i-p,j-q) * A(p+1,q+1) * (p+1)(q+1) \\
& \quad + L(3,1,0,i-p,j-q) * A(p+1,q) * (p+1) \\
& \quad \left. + L(3,0,2,i-p,j-q) * A(p,q+2) * (q+2)(q+1) \right.
\end{aligned}$$

$$\begin{aligned}
& + L(3,0,1,i-p,j-q) * A(p,q+1) * (q+1) \\
& + M(3,3,0,i-p,j-q) * B(p+3,q) * (p+3)(p+2)(p+1) \\
& + M(3,2,1,i-p,j-q) * B(p+2,q+1) * (p+2)(p+1)(q+1) \\
& + M(3,2,0,i-p,j-q) * B(p+2,q) * (p+2)(p+1) \\
& + M(3,1,2,i-p,j-q) * B(p+1,q+2) * (p+1)(q+2)(q+1) \\
& + M(3,1,1,i-p,j-q) * B(p+1,q+1) * (p+1)(q+1) \\
& + M(3,1,0,i-p,j-q) * B(p+1,q) * (p+1) \\
& + M(3,0,2,i-p,j-q) * B(p,q+2) * (q+2)(q+1) \\
& + M(3,0,1,i-p,j-q) * B(p,q+1) * (q+1) \\
& + N(3,4,0,i-p,j-q) * C(p+4,q) * (p+4)(p+3)(p+2)(p+1) \\
& + N(3,3,0,i-p,j-q) * C(p+3,q) * (p+3)(p+2)(p+1) \\
& + N(3,2,2,i-p,j-q) * C(p+2,q+2) * (p+2)(p+1)(q+2)(q+1) \\
& + N(3,2,1,i-p,j-q) * C(p+2,q+1) * (p+2)(p+1)(q+1) \\
& + N(3,2,0,i-p,j-q) * C(p+2,q) * (p+2)(p+1) \\
& + N(3,1,2,i-p,j-q) * C(p+1,q+2) * (p+1)(q+2)(q+1) \\
& + N(3,1,1,i-p,j-q) * C(p+1,q+1) * (p+1)(q+1) \\
& + N(3,1,0,i-p,j-q) * C(p+1,q) * (p+1) \\
& + N(3,0,3,i-p,j-q) * C(p,q+3) * (q+3)(q+2)(q+1) \\
& + N(3,0,2,i-p,j-q) * C(p,q+2) * (q+2)(q+1) \\
& + N(3,0,0,i-p,j-q) * C(p,q) \quad ]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p=\max(0, i-P)}^{i-1} \sum_{q=\max(0, j-Q)}^j \left[ N(3,0,4, i-p, j-q) C(p, q+4) (q+4)(q+3)(q+2)(q+1) \right] \\
& + N(3,0,4,0,1) C(i, j+3) (j+3)(j+2)(j+1)(j) \Bigg/ \left\{ N(3,0,4,0,0) (j+4)(j+3)(j+2)(j+1) \right\} \\
& \dots (4.10.1)
\end{aligned}$$

#### 4.11 Summary of Recurrence Relations

Equations (4.8.3), (4.9.1) and (4.10.1) give expressions for  $A(i, j+2)$ ,  $B(i, j+2)$ ,  $C(i, j+4)$  for  $i, j = 0, \dots, \infty$  in terms of other  $A, B, C$ .

Now  $A(i, j+2)$  is a function of  $A(i_1, j_1)$ ,  $B(i_2, j_2)$ ,  $C(i_3, j_3)$  where  $j_1 < j+2$ ,  $j_2 < j+2$ ,  $j_3 < j+4$  for  $i, j = 0, \dots, \infty$  e.g. for  $i=0$  and  $j=0$ .

$$\begin{aligned}
A(0,2) = f_A \{ & A(0,1), A(1,0), A(1,1), A(2,0), B(0,1), B(1,0), \\ & B(1,1), B(2,0), C(0,0), C(0,1), C(0,2), C(0,3), \\ & C(1,0), C(1,1), C(1,2), C(2,0), C(2,1), C(3,0) \}.
\end{aligned}$$

Also  $B(i, j+2)$  is a function of  $A(i_4, j_4)$ ,  $B(i_5, j_5)$ ,  $C(i_6, j_6)$  where  $j_4 \leq j+2$ ,  $j_5 < j+2$ ,  $j_6 < j+4$  for  $i, j = 0, \dots, \infty$  e.g. for  $i=0$  and  $j=0$ .

$$\begin{aligned}
B(0,2) = f_B \{ & A(0,1), A(0,2), A(1,0), A(1,1), A(2,0), B(0,1), \\ & B(1,0), B(1,1), B(2,0), B(2,1), C(0,0), C(0,1), \\ & C(0,2), C(0,3), C(1,0), C(1,1), C(1,2), C(2,0), \\ & C(2,1), C(3,0) \}.
\end{aligned}$$

For  $C(i, j+4)$  is a function of  $A(i_7, j_7)$ ,  $B(i_8, j_8)$ ,  $C(i_9, j_9)$  where  $j_7 \leq j+2$ ,  $j_8 \leq j+2$ ,  $j_9 < j+4$  for  $i, j = 0, \dots, \infty$  e.g. for  $i=0$  and  $j=0$ .

$$c(0,4) = f_c \{A(0,1), A(0,2), A(1,0), A(1,1), A(1,2), A(3,0), \\ B(0,1), B(0,2), B(1,0), B(1,1), B(1,2), B(2,0), \\ B(2,1), B(3,0), C(0,0), C(0,2), C(0,3), C(1,0), \\ C(1,1), C(1,2), C(2,0), C(2,1), C(2,2), C(3,0), \\ C(4,0)\}.$$

#### 4.12 Method to Solve Recurrence Relations

To solve the recurrence relations the procedure is as follows.

(a) Select limits to buckling power series e.g.  $L_x, L_\phi$  so that

$$u \doteq \sum_{i=0}^{L_x} \sum_{j=0}^{L_\phi} A(i,j)x^i \phi^j ; v \doteq \sum_{i=0}^{L_x} \sum_{j=0}^{L_\phi} B(i,j)x^i \phi^j ; w \doteq \sum_{i=0}^{L_x} \sum_{j=0}^{L_\phi} C(i,j)x^i \phi^j$$

(b) For each  $i = 0, \dots, L_x$  there are 8 independent constants.

$A(i,0), A(i,1), B(i,0), B(i,1), C(i,0), C(i,1), C(i,2), C(i,3)$   
and the other A,B,C can be expressed in terms of these 8 constants.

(c) For each  $i$  set one of 8 constants to unity and the rest to zero.  
This gives  $8(L_x + 1)$  cases.

(d) Use the recurrence relations to find A,B,C in the following

$$(1) C(L_x, 4) \quad (\text{producing } B(L_x, 2), A(L_x, 2))$$

$$(2) B(L_x, 2) \quad (\text{producing } A(L_x, 2))$$

$$(3) A(L_x, 2)$$

$$\text{then (4) } C(L_x - 1, 4)$$

$$(5) B(L_x - 1, 2)$$

$$(6) A(L_x - 1, 2)$$

and continuing until

$$(7) C(0, 4)$$

$$(8) B(0, 2)$$

$$(9) A(0, 2)$$



then increment  $j$  and calculate

$$(10) \quad C(L_x, 5)$$

$$(11) \quad B(L_x, 3)$$

$$(12) \quad A(L_x, 3)$$

proceeding in this manner we can calculate  $A(i, j)$ ,  $B(i, j)$ ,  $C(i, j)$  for  $i = 0, \dots, L_x$  and  $j = 0, \dots, L_\phi$ .

Once  $A, B, C$  are known the  $u, v, w$  are known. Since there are  $8(L_x + 1)$  unknowns we need the same number of boundary condition equations to solve the system.



## CHAPTER 5

BOUNDARY CONDITIONS5.1 Introduction

To solve the recurrence relations established in Chapter 4, boundary conditions are needed which reflect experimentally observed buckling patterns. A local buckling patch is proposed where buckling occurs and has on its circumference zero incremental displacement. The number of points chosen on the circumference is dependent on the number of terms in the double power series expansion outlined in Chapter 4. The shape of the local buckling patch is dependent on the boundary conditions of the cylinder, and also the applied external load. A number of examples are given for the more frequently analysed buckling problems.

Once the shape of the local buckling patch is decided upon, a boundary condition matrix is set up. Buckling occurs for the particular local buckling patch chosen, when the determinant of the boundary condition matrix vanishes.

All the work outlined in this chapter is original to this thesis.

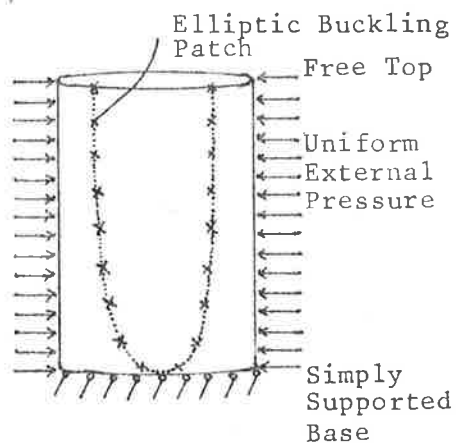
5.2 Local Buckling Patch

In this theory any buckling pattern can be adopted. A local buckle is assumed to occur over an area of the shell and defined by a boundary on which the buckling displacements are zero. This is achieved by considering a number of points on the boundary and at each point setting the buckling displacements  $u = v = w = \frac{dw}{dz} = 0$ .

It was found that an ellipse was the best shape for the local buckling patch since it had no corners and could approximate both the classical theoretical and the Yoshimura Buckling Pattern (10) and the results of buckling patterns from experiments.

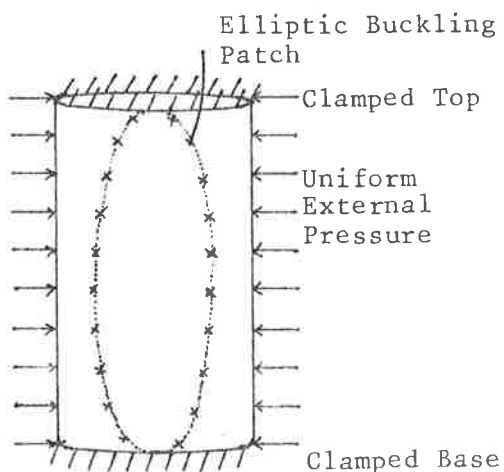
Difference loading cases and boundary conditions decide the choice of the buckling patch.

External Pressure - Simple Free B.C.



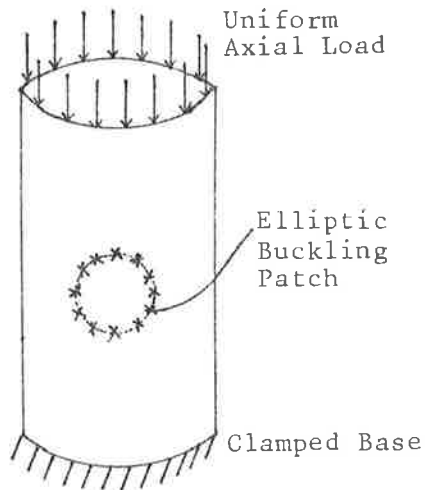
The diagram shows a cylinder simply supported at one end and free at the other with a uniform external pressure. The buckling shape chosen is an ellipse with circumferential axis on the free top edge and with this axis dividing the circumference of the cylinder integrally.

External Pressure Clamped - Clamped B.C.



The diagram shows a cylinder with clamped edges with a uniform external pressure. The buckling shape chosen is an ellipse touching the edges and with the circumferential axis being an integral submultiple of the circumference.

Axial Load - One end clamped the other held circular



The diagram shows a cylinder axially loaded. The buckling patch chosen is an ellipse with circumferential axis an integral submultiple of the circumference and the axial axis an integral submultiple of the length of the cylinder.

5.3 Boundary Condition Equation

The local buckling patch is approximated by a number of points on the boundary of the patch at which the buckling displacement is zero. Within this patch the buckling displacements are non-zero. At each point we have

$$u = v = w = \frac{\partial w}{\partial z} = 0 \quad \dots(5.3.1)$$

This is 4 equations for each boundary point. Since there are  $8(L_x + 1)$  unknowns (see section 4.12) we need  $2(L_x + 1)$  boundary points which give  $8(L_x + 1)$  equations. A matrix equation is then set up.



## CHAPTER 6

COMPUTER RESULTS6.1 Introduction

A computer program was written which was based on the buckling analysis of this thesis. The program was written specifically for problems encountered by the Engineering and Water Supply Department although it could be adapted for more general loading conditions. The specific problem of interest was whether or not it is safe to backfill a cylindrical water tank, and if there is any likelihood of buckling occurring, then by how much does the tank wall need to be thickened.

This chapter describes how the results from the computer program were analysed and then sets out the analysis of two cylindrical tanks. The program was written in FORTRAN IV on a CYBER 6400.

All the work described in this chapter was completed by myself and is original to this thesis.

6.2 Computer Program

The following table outlines the program modules and the amount of code involved.

STAGE	Amount of Code	Description
INPUT	353 lines	Input - dimensions of cylinder - elasticity of cylinder - loading of cylinder - boundary conditions at each end of cylinder - local buckling patch shape.
PREBUCKLING	750 lines	Calculates prebuckling deformation, and stress resultants as outlined in Chapter 2.
BUCKLING	2227 lines	Converts prebuckling results to double power series. Sets up recurrence relations outlined in Chapter 4. Forms bounding condition matrix from the points on the buckling patch. Calculates the determinant of the local buckling patch boundary condition matrix.
OUTPUT	76 lines	Output results.

The dimensions of the local buckling patch was fed in and not automatically incremented since the cost of running the computer program was high. For 10 terms in both the  $x$  and  $\phi$  directions of the double power series expansion, calculating the determinant took 65 CPU sec.

### 6.3 Determining the Buckling Load

The buckling analysis "assumes" that buckling occurs and based on this assumption an  $H$  matrix (section 5.3) is formed from the equations of equilibrium and the boundary condition on points on the circumference of the local buckling patch. Each particular local buckling patch has a buckling load which is found by incrementing the applied load until the determinant of the  $H$  matrix vanishes. The buckling load of the cylinder is found by calculating the buckling load for various buckling patches and choosing the lowest of these buckling loads.

### 6.4 Normalizing the Determinant

For each size buckling patch the order of magnitude of the determinant is quite different. However, when the determinant plotted against the load, it found that as the load increased from zero, the corresponding determinant gradually increased or gradually decreased. The determinant gradually increased when the buckling patch was large and then did not vanish. This may be due to the radius of convergence of the power series.

For smaller buckling patches, the determinant gradually decreased and as the load was increased the determinant would dramatically decrease, then change sign, and then increase. The load at which this plunge to zero occurred was the buckling load for the particular buckling patch.

To compare the results for different sized buckling patches, the determinant is normalized by dividing the determinant at a particular load by the determinant for a zero load. The zero load determinant was chosen since buckling could not occur at zero load and hence the determinant would be stable and only reflect the magnitude of the determinant and not the buckling phenomena.

### 6.5 Size of the Buckling Patch

There exists an optimum sized buckling patch for a cylinder which gives the lowest load. To find the optimum size, the following method was used.

- (i) Choose the shape of the buckling patch based on the external load and the boundary conditions of the cylinder. Section 5.2 has a few examples of choices of buckling patch shapes.
- (ii) Fix the axial diameter of the buckling patch.
- (iii) Let

$$\lambda = \frac{\text{circumference of cylinder}}{\text{circumferential diameter of buckling patch}} \quad \dots(6.5.1)$$

Start at  $\lambda = 2, 3, 4, \dots$  and calculate the determinant for zero load and then a small positive load, say  $\delta$ . Plot the graph of  $\log \left[ \frac{\det(\text{load}=\delta)}{\det(\text{load}=0)} \right]$  against  $\lambda$ . The graph appears unstable for small  $\lambda$ 's until at  $\lambda = \lambda_c$ , say, and then for  $\lambda \geq \lambda_c$  the graph is smooth and stable. By calculating the buckling load for  $\lambda = 1, 2, 3, \dots$  it was found that the lowest buckling load occurred for the buckling patch when  $\lambda = \lambda_c$ , the largest buckling patch which has a stable determinant. Thus  $\lambda_c$  is the critical circumferential diameter for the particular axial diameter of the buckling patch.

- (iv) By varying the size of the axial diameter a set of  $\lambda_c$  are formed. The  $\lambda_c$  which has the largest value of  $\log \left[ \frac{\det(\text{load}=\delta)}{\det(\text{load}=0)} \right]$  gave the correspondingly lower buckling load.

### 6.6 Buckling Load

Having determined the size and shape of the buckling patch as outlined in 6.5 the critical buckling load is determined as follows.



The load is gradually increased and the determinant calculated. The determinant gradually decreases, until for increasing loads, rapidly decreases, and then changes sign, and then increases. The buckling load for the cylinder is when the determinant changes sign or vanishes.

A graphical method helps to see the above behaviour. The load is on the horizontal axis and the logarithm of the normalized determinant on the vertical axis. For buckling to occur the logarithm of the normalized determinant dips to  $-\infty$ .

### 6.7 Example 1 - Failure of a Prestressed Concrete Tank

In 1966 the failure of a circular prestressed concrete reservoir, backfilled over one half of its circumference, was described and analysed [5]. The conclusion reached was that the collapse of the reservoir was caused by the pressure exerted by the backfill on a part of the circular wall.

This example was analysed with buckling theory, outlined in this thesis to see if the backfilled load was close to the buckling load of the cylinder. If so the collapse may have been caused by a buckling phenomena.

The basic data needed for the buckling analysis is as follows:

Radius = 20.0914 m ; Thickness = 0.254 m ; Height = 9.2964 m ;

Young's Modulus =  $2.3715 \times 10^7$  KN/m<sup>2</sup> ; Poisson's Ratio = 0.1

Boundary Conditions : Top - Free :  $N_x = T_x = S_x = M_x = 0$  .

Base - Simple :  $U_0 = V_0 = W_0 = M_x = 0$  ,

Loading Conditions : The buckling analysis outlined in this thesis can accommodate a triangular backfill pressure over only part of the circular wall. However, when this example was analysed, the computer

program had only the facility for a triangular load which was non-zero at the top and uniform around the cylinder. The buckling load which resulted from this loading condition would be greater than the buckling load resulting from the actual backfill pressure over part of the cylinder.

For a soil load all around the tank, up to the top of the tank, 9.3 m high, the active pressure at the base of the wall is  $52 \text{ KN/m}^2$ .

At the time of the collapse, a truck was also on top of the soil which would result in an additional  $6 \text{ KN/m}^2$  active pressure at the base of the wall. The loading condition which is taken is a triangular load, uniform around the circumference, zero at the top and  $58 \text{ KN/m}^2$  at the base.

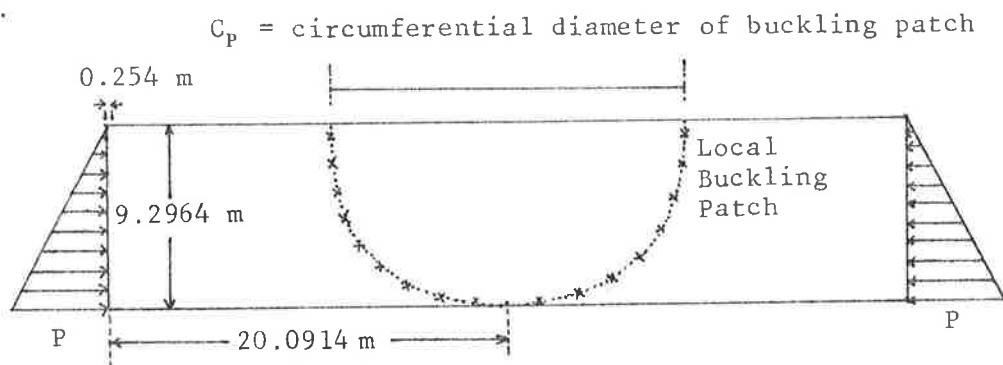


Fig. 6.7.1

The local buckling patch used is half an ellipse, with one end touching the base of the tank and the circumferential diameter at the top of the cylinder, as in the above diagram. The length of the cylinder's circumference enclosed by the circumferential diameter of the buckling patch is denoted by  $C_p$  which is an integral submultiple of the circumference of the cylinder.

Define

$$\lambda = \frac{\text{Circumference of Cylinder}}{C_p} \quad \dots(6.7.1)$$

then,  $\lambda$  is an integer. Ten terms were taken in the double power series expansion, so that 22 points are used on the boundary of the local buckling patch to build the boundary condition matrix (section 5.3).

The load used to determine the buckling load for each sized buckling patch is a triangular load, zero at the top of cylinder, and  $P$  KN/m<sup>2</sup> at the base. For each buckling patch,  $P$  is gradually incremented until it vanishes, which is the buckling load for the patch.

The smallest buckling load was obtained for  $\lambda=4$ , four buckling lobes around the cylinder, with the associated buckling load of 82 KN/m<sup>2</sup> at the base of the cylinder. This buckling load is 41% greater than the approximated load of 58 KN/m<sup>2</sup> which caused the collapse of the cylinder. Since the actual load was only on a portion of the cylinder, the buckling load would be reduced, so that the collapse of the tank is likely to be caused by buckling of the cylinder.

The following table shows the determinant for increasing values of  $\lambda$  at loads of 0 and 0.0001 KN/m<sup>2</sup>.

$\lambda$	Load	Determinant	$\log \left[ \frac{\det(\text{load}=0.001)}{\det(\text{load}=0.0)} \right]$
2	0.0	$- 0.362973 \times 10^{288}$	
	0.0001	$- 0.360258 \times 10^{288}$	- 0.003260
3	0.0	$- 0.614120 \times 10^{237}$	
	0.0001	$- 0.614029 \times 10^{237}$	- 0.000064
4	0.0	$0.238376 \times 10^{218}$	
	0.0001	$0.238359 \times 10^{218}$	- 0.000032
5	0.0	$0.122188 \times 10^{206}$	
	0.0001	$0.122181 \times 10^{206}$	- 0.000026
6	0.0	$0.694866 \times 10^{196}$	
	0.0001	$0.694833 \times 10^{196}$	- 0.000020
7	0.0	$0.248119 \times 10^{189}$	
	0.0001	$0.248110 \times 10^{189}$	- 0.000016
8	0.0	$0.132855 \times 10^{183}$	
	0.0001	$0.132851 \times 10^{183}$	- 0.000013

When the load was incremented for  $\lambda=2$  and  $\lambda=3$  the determinant did not vanish but rose dramatically. The following graph shows the behaviour of the determinants for  $\lambda=4,5,6$  and how the determinant vanishes for  $\lambda=4$  at  $80 \text{ KN/m}^2$ , for  $\lambda=5$  at approximately  $105 \text{ KN/m}^2$ , and for  $\lambda=6$  at approximately  $120 \text{ KN/m}^2$ . A total of 36 loads were used in the calculation which took a total of 2300 CPU sec. (\$540.00) in computing time.

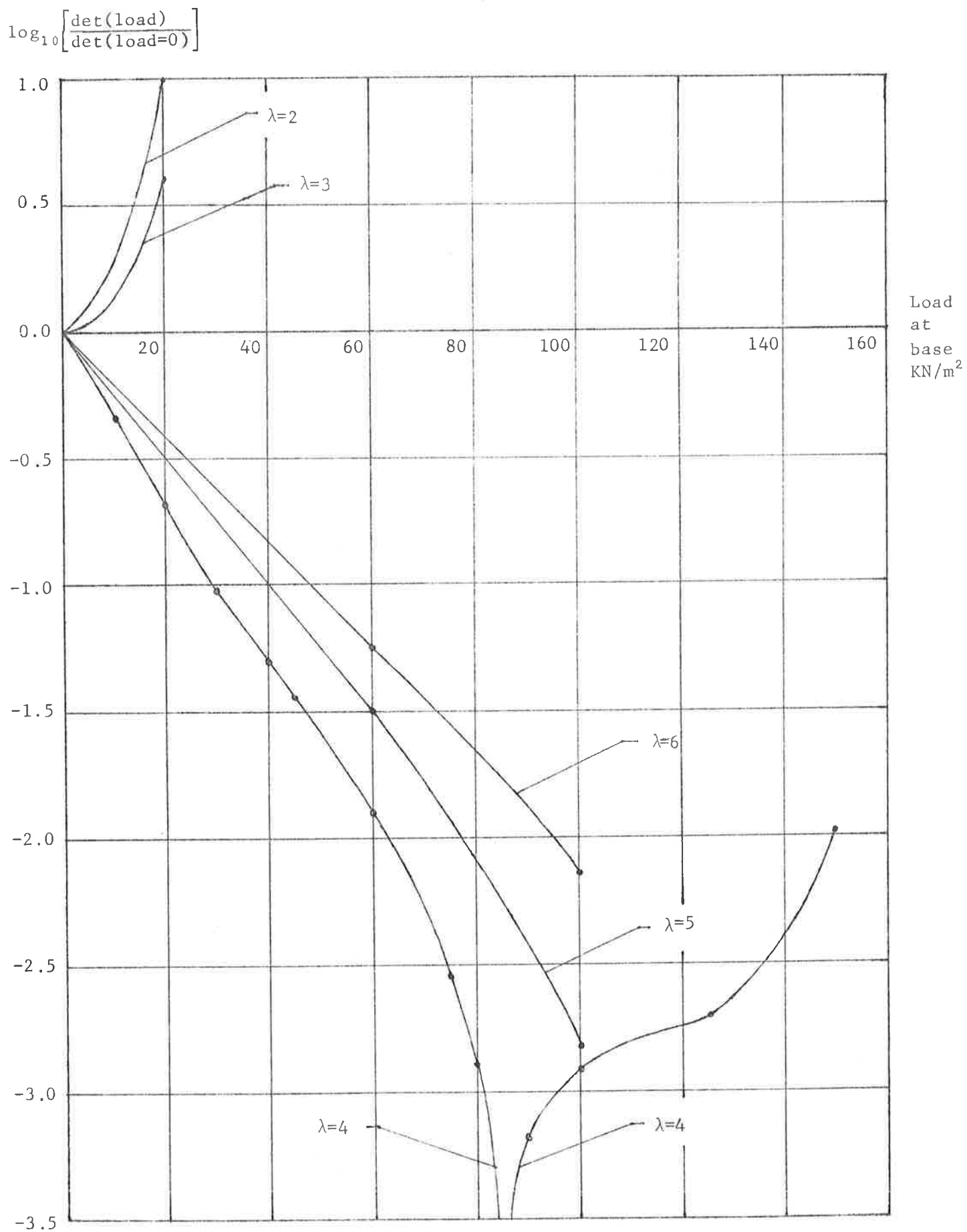


Fig. 6.7.2 Failure of a Prestressed Concrete Tank

### 6.8 Example 2 - Backfilling a Cylindrical Water Tank

The following analysis was done to see if a cylindrical water tank could be safely backfilled. The dimensions of the tank were as follows.

Radius = 23.0 m ; Height = 5.8 m ; Thickness = 0.26 m ;

Young's Modulus =  $2.3715 \times 10^{10}$  KN/m<sup>2</sup> ; Poisson's Ratio = 0.1.

Boundary Condition : Free at Top ( $N_x = T_x = S_x = M_x = 0$ )

Pinned at Base ( $U_0 = V_0 = W_0 = M_x = 0$ )

Loading Conditions - Uniform backfill to height of tank, which gives an active pressure at the base of 35 KN/m<sup>2</sup>.

To find an approximate buckling load, consider a cylinder pinned at both ends, twice the length of the cylinder in question, and with a uniform load. The approximate formula is given by Roark [7] as

$$q' = 0.807 \frac{Et^2}{\lambda r} \sqrt{\left(\frac{1}{1-\nu^2}\right)^2 \frac{t^2}{r^2}} = 518 \text{ KN/m}^2$$

The buckling patch chosen is half an ellipse as in the previous example. The buckling analysis produced the minimum buckling load of 400 KN/m<sup>2</sup> for  $\lambda=17$ .

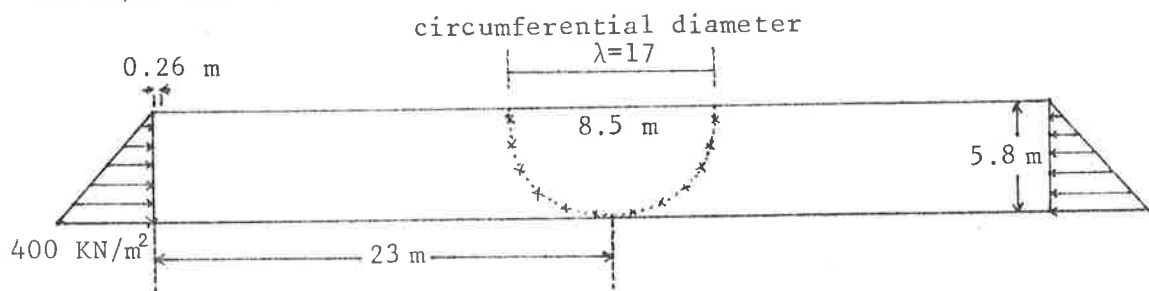


Fig. 6.8.1

Figure 6.8.1 shows the dimensions of the tank and the relative size of the buckling patch.

The following table lists the determinant for a range of  $\lambda$ 's and loads of 0 and 1 N/m<sup>2</sup>.

$\lambda$	Load (N/m <sup>2</sup> )	Determinant	$\log \left[ \frac{\det(\text{load}=1)}{\det(\text{load}=0)} \right]$
3	0.	$- 0.1213783 \times 10^{98}$	
	1.	$- 0.1212834 \times 10^{98}$	- 0.0003399
5	0.	$0.1087677 \times 10^{66}$	
	1.	$0.1227758 \times 10^{66}$	+ 0.0526126
6	0.	$0.1150098 \times 10^{64}$	
	1.	$0.1147137 \times 10^{64}$	- 0.0011197
7	0.	$0.5127585 \times 10^{64}$	
	1.	$0.5126050 \times 10^{64}$	- 0.0001300
8	0.	$0.1463673 \times 10^{65}$	
	1.	$0.1463672 \times 10^{65}$	- 0.0000002
9	0.	$0.2048534 \times 10^{65}$	
	1.	$0.2048609 \times 10^{65}$	+ 0.0000159
10	0.	$0.1487267 \times 10^{65}$	
	1.	$0.1487317 \times 10^{65}$	+ 0.0000143
11	0.	$0.6020696 \times 10^{64}$	
	1.	$0.6020846 \times 10^{64}$	+ 0.0000108
12	0.	$0.1420836 \times 10^{64}$	
	1.	$0.1420862 \times 10^{64}$	+ 0.0000081
13	0.	$0.1990595 \times 10^{63}$	
	1.	$0.1990624 \times 10^{63}$	+ 0.0000063
14	0.	$0.1570853 \times 10^{62}$	
	1.	$0.1570874 \times 10^{62}$	+ 0.0000056
15	0.	$0.4856390 \times 10^{60}$	
	1.	$0.4856477 \times 10^{60}$	+ 0.0000077
16	0.	$- 0.1510289 \times 10^{57}$	
	1.	$- 0.1510324 \times 10^{57}$	+ 0.0000101
17	0.	$0.9397369 \times 10^{58}$	
	1.	$0.9397303 \times 10^{58}$	- 0.0000031
18	0.	$0.3188807 \times 10^{58}$	
	1.	$0.3188795 \times 10^{58}$	- 0.0000017
19	0.	$0.5961152 \times 10^{57}$	
	1.	$0.5961135 \times 10^{57}$	- 0.0000013
20	0.	$0.8532799 \times 10^{56}$	
	1.	$0.8532776 \times 10^{56}$	- 0.0000012

Table 6.8.1

Three types of buckling patches occur which is observed when the  $\log_{10} \left[ \text{abs} \left[ \frac{\log(\det(1))}{\log(\det(0))} \right] \right]$  is plotted against  $\lambda$ , the number of buckling lobes around the cylinder. These are

Type A :  $\det(0) > \det(1)$  and  $\det(0) > 1$

Type B :  $\det(0) < \det(1)$  and  $\det(0) > 1$

Type C :  $\det(0) < \det(1)$  and  $\det(0) < 1$

For  $\lambda$  less than 17 the determinant appears unstable, however for  $\lambda$  17 and greater, the determinant is stable.  $\lambda=17$  is then chosen for the lowest stable buckling patch size.

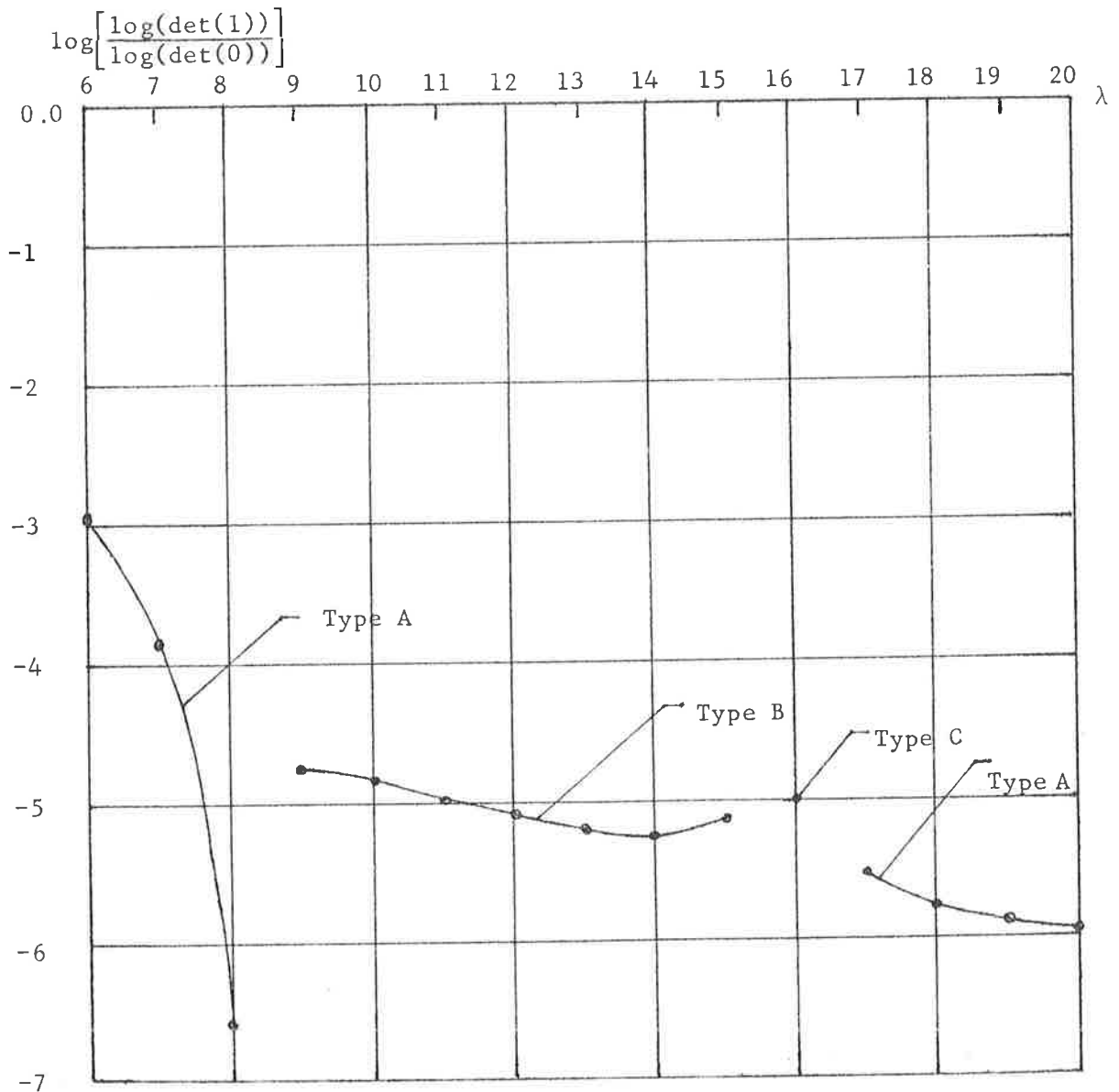


Fig. 6.8.1



For  $\lambda \geq 17$  the determinant decreases for increasing loads, and for  $\lambda=17$  the determinant decreases faster than for  $\lambda > 17$ . The load is then incremented for  $\lambda=17$  until it vanishes. The following table sets out the results of this incrementation of the load.

Load (KN/m <sup>2</sup> )	Determinant ( $\times 10^{58}$ )	$\log \left[ \frac{\det(\text{load})}{\det(\text{load}=0)} \right]$
0.0	0.93974	
0.0001	.93973	- 0.000003
10.0	.87539	- 0.030802
110.0	.40711	- 0.363299
250.0	.10658	- 0.945334
300.0	.05654	- 1.220684
350.0	.02429	- 1.587547
400.0	.00466	- 2.304415
450.0	- 0.00625	- 2.176839
500.0	- 0.01135	- 1.918135
750.0	- 0.00474	- 1.297107
1000.0	0.00314	- 1.476418

The  $\log \left[ \frac{\det(\text{load})}{\det(\text{load}=0)} \right]$  is then graphed to determine the buckling load, which is 400 KN/m<sup>2</sup>.

A total of 48 loads were used in the calculation, which at 65 CPU sec. per load, took 3120 CPU sec. (\$720.00).

Fig. 6.8.3 shows that for  $\lambda=17$  buckling lobes the critical load is approximately  $400 \text{ KN/m}^2$ . This occurs when the determinant first vanishes or the  $\log(\det(\text{load}))$  tends to  $-\infty$ .

In the graph Type A has  $\det(\text{load}) > 0$  and Type B has  $\det(\text{load}) < 0$ .

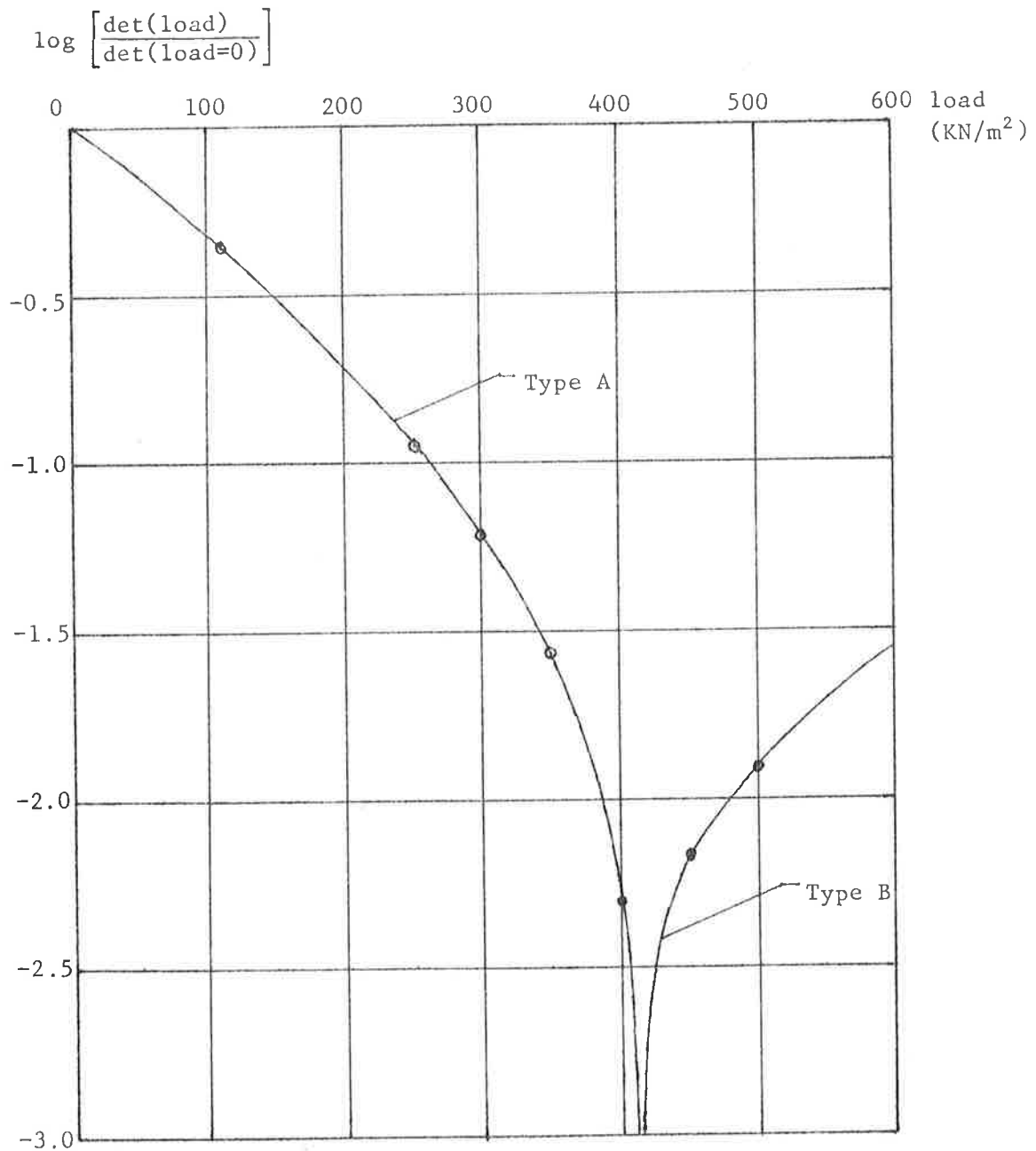


Fig. 6.8.3

## CHAPTER 7

CONCLUSIONS

The method of solution to the buckling of a thin walled cylindrical shell is based on an exact solution to the prebuckling elastic deformation of the cylinder. Once these prebuckling deformations are calculated a hypothetical incremental displacement is superimposed and tested for its existence. The smallest load for which the incremental displacement can exist is the buckling load for the cylindrical shell.

Since the prebuckling deformations are calculated independently, the method of solution can be used for complex loading conditions as well as complex boundary conditions of the cylinder. The inability of the classical buckling solution to accurately model these two fundamental conditions may be the main reason for the classical solution overestimating the actual buckling load.

The method of solution attempts to minimize the approximations to the governing equations. The major approximation is the linearization of the general non-linear equilibrium equations which is a standard approach to the classical solution.

The solution of the buckling equilibrium equations involves the introduction of a buckling 'patch'. The shape and size of the 'patch' is unrestricted and is chosen to correspond with observed buckling patterns. Previous classical solutions do not have this approach and instead constrain the buckling displacements to conform with the boundary conditions of the cylinder, which result in chessboard patterns which have never been observed experimentally.

The method of solution accurately calculates the effects of complex loading conditions and complex boundary conditions as well as modelling experimentally observed buckling patterns. The examples given are not extensive or conclusive but do indicate that the analysis predicts buckling loads well below the classical buckling load which is generally much greater than the actual experimental buckling load.

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