GENERATING INTENSIONAL LOGICS

The application of paraconsistent logics to investigate certain areas of the boundaries of mathematics

By

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ERRATA

p.8. Delete lines 18-20 (after definition 2.1.7).

Substitute 'a' for 'the' in line 14 (twice).

p.14. Delete line 6 ((b) of theorem 2.1.13).

p.16. Line 2, delete the phrase 'and because W is closed under these too'.

p.29. Line 19, replace 'substitution instances' by 'theorems'.

p.38. Line 2, replace 'an admissible' by 'a derived'.

p.147. Line 11, insert the following after 'intensional base,':
'the substitution of equivalents rule (x) (AvB) ≡ A v (x) B (where x is not free in A) to the substitution of equivalents rules permitted in obtaining generalised disjunctive and conjunctive normal forms (definition 2.1.9)'.

p.148. In the statement of the rules (&⊢) and (⊢v) replace 'A&B' by '$\begin{matrix}\{A\&B\}\\\{A\&B\}\end{matrix}$' and 'AvB' by '$\begin{matrix}\{AvB\}\\\{BvA\}\end{matrix}$', respectively.

CONTENTS

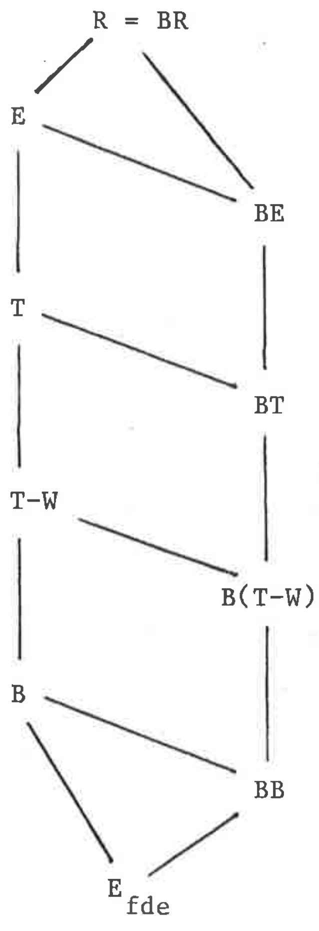
SUMMARY

I use a natural generalisation of Anderson and Belnap's definition of variable sharing, based on similar intuitions, in order to characterise intensional logics which contain theorems of arbitrary degree. This characterisation limits the interplay between the intensional → and extensional connectives &, v to this generalised notion of variable sharing. Thus the extensional/intensional interplay is completely perspicuous, as well as remaining faithful to the intuitions underlying the Tautological Entailments - contrary to the case of the standard relevant logics and their axiomatic formulations.

This new method for characterising logics delivers a very broad class of logics, because the method can be applied to any intensional base (i.e. implication-negation logic) and extensional base (usually just adjunction). This class includes some of the standard relevant logics (notably R) but not others (such as those weaker than E).

I prove that many of the logics so characterised have a corresponding axiomatic formulation, which just involves adding purely extensional and/or purely intensional axioms and rules to those of E_{fde}. Thus the extensional/intensional interplay is grounded, in the axiomatic formulation, in the axioms and rules of E_{fde}. This adds weight to the claim that such interplay is an intuitive generalisation of variable sharing. For those logics which can be formulated as the axioms and rules of B plus purely intensional and/or purely extensional axioms and rules (which includes the standard relevant logics), the process amounts to weakening the axioms $(A \to B) \,\&\, (A \to C) \underset{.}{\to} A \to (B \,\&\, C)$

and $(A \to C) \,\&\, (B \to C) \underset{\cdot}{\to} (A \vee B) \to C$ to the rules $\vdash A \to B$ and $\vdash A \to C \Rightarrow \vdash A \to (B \,\&\, C)$, and $\vdash A \to C$ and $\vdash B \to C \Rightarrow \vdash (A \vee B) \to C$. Thus obtaining a logic with the same purely intensional and purely extensional basis, but which mediates the extensional/intensional interplay via our generalised notion of variable sharing. Letting BL represent the logic obtained when the process is applied to L, the following relationships hold:



I set out the algebraic semantics for BB and its extensions and use these to prove that BB and many of its extensions are decidable, and that BB and a couple of its extensions are both prime and negation-consistent (thus satisfying γ).

I then set out a relational semantics for BB and its extensions, which leans heavily on those for B and on the recipe of ESL. A notion of theoryhood is introduced which is a useful analytical tool for understanding the relational semantics.

I also point out some errors in ALG II and give alternative proofs which provide a partial resurrection of the required results.

Finally, I conclude with some ruminations and open problems.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except when due reference is made in the text of the thesis.

I consent to the thesis being made available for photocopy and loan.

Peter Lavers

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CHAPTER 1

INTRODUCTION

§ 1.0 A new class of logics

The major aim of this work is to describe a new method for generating intensional (and in particular, relevant) logics. This method is a natural extension of the Tautological Entailments (Anderson & Belnap 1962), and so is strongly intuitively motivated. The new class of logics so generated includes some of the standard relevant logics, notably R and some of its extensions. The corresponding alternative characterisation of these logics clarifies the interplay between the extensional connectives (& and v) and the intensional connectives (~ and →) which is more difficult to fathom from the usual axiomatic formulations. In this chapter we set the scene by indicating how the Tautological Entailments can be generalised, and how this generalisation can be exploited in defining intensional logics.

In chapter 2 we more rigorously address these questions and define the logic generating procedure. We also prove, for a large class of such logics, their equivalent axiomatic formulations. This enables a comparison to be made with the standard relevant logics, with the result that we adopt BB, a proper sublogic of B, as a base logic. It also enables determining which of the standard relevant logics (extensions of B) can be formulated by our logic generating procedure. One such logic is R (but not E or sublogics thereof).

In chapter 3 we develop algebraic semantics for BB and its extensions.[1] We then use an appropriate algebra BM_0 which separates E (and its sublogics) from the logic BE obtained when applying our procedure to $E_{\overset{\sim}{\to}}$ (and the corresponding $L_{\overset{\sim}{\to}}$).

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1. By extension I mean throughout axiomatic extension. That is, L' is (continued on page 2)

In chapter 4 we continue to utilise the algebraic semantics to prove that BB and many of its extensions are decidable. And we show that BB and some of its extensions are both prime and negation-consistent, so they have disjunctive syllogism as an admissible rule (γ).

In chapter 5 we develop relational semantics for BB and its extensions. Two types of semantics are needed to cater for all extensions of BB - model structurs and unreduced model structures - as for the standard relevant logics. We follow the strategy of ALG II which, however, is flawed (an essential Priming Lemma for the algebras is in fact false). The unreduced model structures overcome these difficulties. We introduce a new notion of theoryhood which, together with the standard one (closure w.r.t. provable entailment) provides a strong analytical tool for understanding the relationships between the two types of relational semantics and also the algebraic semantics. Finally, we compare our approach with the relational semantics for B and for connexive logics.

In chapter 6 we point out the mistake in ALG II and show how this can be partially remedied by resurrecting the Priming Lemma for B and some of its extensions.

Chapter 7 presents some extensions and open problems which indicate the high degree of generality of our logic generating procedure.

Finally, chapter 8 contains some concluding remarks.

(Footnote continued from page 1)

an extension of L iff L' has an axiomatic formulation which includes the axioms and rules of L. This holds iff the theorems of L are theorems of L' and the rules of L are admissible in L'.

§ 1.1 Setting the scene

Anderson and Belnap's definition of variable sharing (ENT, p.156) involves two components. One is to do with what we signify by extensional & and v, and the other is a basis of independently motivated entailments (c.f. §2.1). The independent basis in this case is just taken to be those $A \rightarrow A$ where A is a propositional variable or the negation of such. The definition can be generalised, making it relative to a set of independent theses W, by taking the basis to be those entailments in W. Hence we arrive at a definition of W-augmented variable sharing. We denote that this is satisfied by $A \rightarrow B$ by (A, B) A.V.S. (W), and say that $A \rightarrow B$ is a W-Augmented Tautological Entailment (W-ATE).

On weak assumptions about W, A.V.S. (W) corresponds to variable sharing plus semisubstitution (directed substitution) (theorem 2.2.3). It simply amounts to, in the definition of variable sharing, replacing the requirement for appropriate identities between atoms to hold, by the requirement that an appropriate entailment be in W.

W-ATEs provide us with a mechanism for generating logics in a hierarchical fashion. This method will be exploited extensively in the following chapters.

Suppose we begin with an implication-negation logic $L_{\tilde{\rightarrow}}$. Then put $W = L_{\tilde{\rightarrow}}$ (the set of substitution-instances thereof) and generate the set of $L_{\tilde{\rightarrow}}$ - ATEs. Close this set under whatever extensional rules are desired (e.g. adjunction), and then under the rules of $L_{\tilde{\rightarrow}}$, giving a set of theorems W_1. Repeat this process: take the W_1-ATEs and close under the rules, giving a set W_2, and so on. In general the series, $W, W_1, W_2, \ldots\ldots$ is strictly increasing (as we will see later), so the process is not trivial. Finally, put $L = \bigcup_{i=1}^{\infty} W_i$.

A logic L generated in this fashion endows the extensional connectives with logical properties which accord with the natural intuitions that lie behind the Tautological Entailments. (Bearing in mind that interaction with $L_{\overset{\sim}{\rightarrow}}$ may distort this somewhat, but then from this perspective stronger logical properties of the extensional connectives can be attributed to strong intensional properties). For such logics the interplay between the extensional and intensional connectives is a clear and natural generalisation of that of the Tautological Entailments. R can be characterised in this alternative manner (building up from $R_{\overset{\sim}{\rightarrow}}$), so it has these desirable properties. Those logics between B and E (inclusive) cannot be characterised in this alternative manner, so the interplay between the intensional and extensional connectives in them is not a natural extension of the Tautological Entailments, hence on this criterion R is to be preferred.

A hierarchical approach to intensional logics is not new. Ross Brady uses such an approach to develop a model structure for dialectical set theory (Priest & Routley 198 ?) and a model structure to investigate depth relevance (Brady 198?). However the levels of the hierarchy correspond to the depth of nesting of $\rightarrow$s. Thus the hierarchy is based on syntactical complexity. The levels of our hierarchical specification of L, starting from $L_{\overset{\sim}{\rightarrow}}$, differ in logical complexity.

CHAPTER 2

B - ing : The basic process

§ 2.0 Introduction

In this chapter I will first give a full account of how the Tautological Entailments and their intuitive underpinning can be generalised (section §2.1 and §2.2), and a more precise definition of the logic generating process described in §1.1 (section §2.3). I will then prove an equivalence theorem which displays the corresponding axiomatic formulations for a broad class of such logics (section §2.4). This will show the relationship between the class of logics generated in this fashion, and the standard class of relevant logics (sections §2.5 and §2.6). The study of the properties of the new class of logics so obtained is the central concern of this thesis.

§ 2.1 Generalising Tautological Entailments

The system FDE of Tautological Entailments is concerned with entailments between extensional wff (so there are no nested arrows) and constitutes the first degree[1] entailments of the usual relevant logics. I will briefly relate Anderson & Belnap's variable sharing characteristion of FDE (ENT Ch. III). It is based on the following intuitions:

A conjunction of atomic sentences entails a disjunction of atomic sentences iff at least one of the former is identical with (in meaning) one of the latter.

A sentence entails a conjunction of sentences iff the former entails each of the conjuncts.

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1. The degree of a wff is the maximum number of nested $\rightarrow$s in it.

A disjunction of sentences entails a sentence iff each of the disjuncts entails the latter sentence.

Definition 2.1.1.

An atom is a propositional variable or the negation of such; A primitive conjunction is a conjunction of atoms; A primitive disjunction is a disjunction of atoms; $A \rightarrow B$ is a primitive entailment if A is a primitive conjunction and B is a primitive disjunction.

Definition 2.1.2.

A primitive entailment is explicitly tautological iff at least one atom of the antecedent is identical with an atom of the consequent. That is $A_1 \& A_2 \& \ldots\ldots \& A_n \rightarrow B_1 \vee B_2 \ldots\ldots \vee B_m$, where each A_i and B_i are atoms, is explicitly tautological iff at least one A_i is identical with some B_k.

The first of the aforementioned intuitions corresponds to the requirement that a primitive entailment is valid iff it is explicitly tautological.

Now consider $A \rightarrow B$ where A is a disjunctive of primitive conjunctions and B is a conjunction of primitive disjunctions, i.e. of the form $A^1 \vee A^2 \vee \ldots\ldots \vee A^r \rightarrow B^1 \& B^2 \& \ldots\ldots \& B^s$ where each A^i is a primitive conjunction and each B^k is a primitive disjunction.

To satisfy the second intuition, we must require that $A \rightarrow B$ is valid iff each $A^1 \vee A^2 \vee \ldots\ldots \vee A^r \rightarrow B^i$ for $i = 1, 2, \ldots\ldots, s$, is valid. And then to satisfy the third intuition, we must require that

$A \rightarrow B$ is valid iff each $A^j \rightarrow B^i$ for $j = 1, \ldots, r$ and $i = 1, \ldots,$ s, is valid. Whence we expand our definition of "explicitly tautological".

Definition 2.1.3.

A first degree entailment is in normal form iff the antecedent is a disjunction of primitive conjunctions and the consequent is a conjunction of primitive disjunctions, i.e. iff the antecedent is in disjunctive normal form and the consequent is in conjunctive normal form.

Definition 2.1.4.

A first degree entailment is explicitly tautological iff it is in normal form and the entailment between each primitive conjunction of the antecedent and every primitive disjunction of the consequent is explicitly tautological.

Finally, we extend this to any first degree entailment.

Definition 2.1.5. Disjunctive normal form and conjunctive normal form.

Where A is an extensional wff (i.e. it contains only ~, &, v and propositional variables) a disjunctive normal form of A, $(A)_{dnf}$, is a wff of the form $(a^1_1 \,\&\, a^1_2 \,\&\, \ldots \,\&\, a^1_{m_1}) \vee (a^2_1 \,\&\, a^2_2 \,\&\, \ldots \,\&\, a^2_{m_2}) \vee \ldots \vee (a^n_1 \,\&\, a^n_2 \,\&\, \ldots \,\&\, a^n_{m_n})$, where each a^i_j is an atom, which is obtained from A by substitution of equivalents using commutation, association, distribution, double negation and De Morgan's laws. The conjunctive normal form of

A, $(A)_{cnf}$, of the form $(a_1^1 \vee a_2^1 \vee \ldots\ldots \vee a_{k_1}^1)$ & $(a_1^2 \vee a_2^2 \vee \ldots\ldots \vee a_{k_2}^2)$ & ..... & $(a_1^r \vee a_2^r \vee \ldots\ldots \vee a_{k_r}^r)$, is defined similarly.

Definition 2.1.6. A first degree entailment $A \rightarrow B$ is a Tautological Entailment iff the antecedent has a disjunctive normal form $(A)_{dnf}$, and the consequent a conjunctive normal form $(B)_{cnf}$, such that $(A)_{dnf} \rightarrow (B)_{cnf}$ is explicitly tautological.

Clearly, accepting that the transformations needed for conversion to normal form preserve co-entailment, the Tautological Entailments exactly capture our three intuitions.

We can encapsulate the path used to arrive at the above definition by the following notion of variable sharing.

Definition 2.1.7. Variable sharing.
Let A and B be extensional wff. Then A and B satisfy variable sharing iff, the disjunctive normal form of A, $(A)_{dnf}$, and the conjunctive normal form of B, $(B)_{cnf}$, are such that every disjunct of $(A)_{dnf}$ and each conjunct of $(B)_{cnf}$ have an atom in common.

(Disjunctive and conjunctive normal forms of a wff are equivalent up to the order and association of conjuncts and disjuncts, which justifies "the" above.)

Clearly, extensional wff A and B satisfy variable sharing iff $A \rightarrow B$ is a Tautological Entailment.

Our aim in this section is to see how the definition of Tautological Entailments can be generalised. So let us consider a formal rendition of our three intuitions. The first intuition corresponds to:

(i) Our basis is only the entailments which hold between atoms, and, moreover, A → B is a theorem, where A and B are atoms, iff A is identical with B.

(ii) If A → C is a theorem, then A & B → C is a theorem.

(iii) If A → B is a theorem, then A → B v C is a theorem.

And the second and third intuitions correspond to:

(iv) A → B & C is a theorem iff both A → B and A → C are theorems.

(v) A v B → C is a theorem iff both A → C and B → C are theorems.

Also, implicit in the definition of Tautological Entailments is the requirement that:

(vi) Substitution of equivalents under commutation, association, distribution, double negation and De Morgan's laws preserves theoremhood.

And implicit in the claim that the Tautological Entailments are the valid first degree entailments is the requirement that:

(vii) A → B is a theorem, where A and B are extensional, only in virtue of (i) - (vi) above.

(ii) - (vi) say nothing more, and nothing less, than that the connectives & and v are to be treated in the purely truth-functional, extensional sense. That is, we can regard (ii) - (vi) as an advertisement of what we take & and v to signify.

The relevantist intuitions at the heart of Anderson and Belnap's ENT are incorporated in (i) and (vii), and these are as much a matter of what is prohibited as of what is allowed. For an entailment to be valid it is required that there be a meaning connection between antecedent and consequent, that is, they overlap in meaning. Anderson and Belnap use variable sharing as the criterion for meaning connection. Thus relevance is ensured by restricting the basis to entailments which hold between atoms (so to all $A \rightarrow A$ where A is an atom) and precluding any other source of valid first degree entailments. For the sake of comparison, to get classical propositional logic we only need to add $A \& \sim A \rightarrow B$ and $B \rightarrow A \vee \sim A$, for all atoms B and propositional variables A, to the basis ((i)). Clearly these do not satisfy relevance, and their exclusion is the key to Tautological Entailment.

Clearly, the set of theorems determined by (i) - (vii) equals the set of Tautological Entailments (where A, B and C in (ii) - (v) are restricted to being extensional wff). In any case, we shall shortly prove this fact (theorem 2.1.16).

The Anderson-Belnap recipe (i) - (vii) is crying out for generalisation to cater for wffs of degree greater than one, thus providing a generalisation of the three intuitions we began with. We now consider how this might be done. (ii) - (vi) are perfectly general, and can be left unchanged, whilst (vii) simply ensures that our basis really is a basis. But (i) is obviously inadequate in a more complex logical environment. Where certain entailments are already independently given, we can use these entailments as a basis to build up from in a manner analogous to the Tautological Entailments.

So, let W be a set of wff, then we consider the class of entailments determined by:

(i') If $A \rightarrow B \in W$ then $A \rightarrow B$ is a theorem. And (ii) - (vii) exactly as before.

This, we will see, corresponds to the following generalisation of explicit tautologyhood, where arbitrary wffs are able to play the role of the atoms in definition 2.1.4.

<u>Definition 2.1.8.</u> <u>W- augmented explicitly tautological</u>

Let W be a set of wff. Then an entailment of the form

$$(A^1_1 \;\&\; A^1_2 \;\&\; \ldots\ldots\; A^1_{n_1}) \vee (A^2_1 \;\&\; A^2_2 \;\&\; \ldots\ldots\; A^2_{n_2}) \vee \ldots\ldots \vee (A^r_1 \;\&\; A^r_2 \;\&\; \ldots\ldots \;\&\; A^r_{n_r})$$

$$\rightarrow (B^1_1 \vee B^1_2 \vee \ldots\ldots \vee B^1_{m_1}) \;\&\; (B^2_1 \vee B^2_2 \vee \ldots\ldots\; B^2_{m_2}) \;\&\; \ldots\ldots \;\&\; (B^k_1 \vee B^k_2 \vee \ldots\ldots B^k_{m_k})$$

is <u>W- augmented explicitly tautological</u> iff for each $i = 1, \ldots\ldots, r$ and $j = 1, \ldots\ldots, k$ there is a pair of wffs $A^i_{n_i}$ and $B^j_{m_j}$ such that $A^i_{n_i} \rightarrow B^j_{m_j} \in W$.

<u>Definition 2.1.9.</u> <u>Generalised disjunctive normal form and generalised conjunctive normal form</u>

Suppose that A is a wff in the full logical vocabulary (propositional variables, &, v, ~, → and conventional symbols such as brackets - see RLR p. 286). Then a <u>generalised disjunctive normal form</u> of A, $(A)_{gdnf}$, is a wff of the form

$$(a^1_1 \;\&\; a^1_2 \;\&\; \ldots\ldots \;\&\; a^1_{m_1}) \vee (a^2_1 \;\&\; a^2_2 \;\&\; \ldots\ldots \;\&\; a^2_{m_2})$$

$$\vee \ldots\ldots \vee (a^n_1 \;\&\; a^n_2 \;\&\; \ldots\ldots\; a^n_{m_n})$$

from which A can be obtained by substitution of equivalents using commutation, association, distribution, double negation and De Morgan's laws - except that no occurrence in a^i_j of a subwff of a^i_j can be substituted for. Generalised conjunctive normal form, $(A)_{gcnf}$, is defined similarly, as an appropriate conjunction of disjunctions.[1]

Definition 2.1.10.

An entailment A → B is a W- Augmented Tautological Entailment (W- ATE) where W is a set of wff, iff A has a generalised disjunctive normal form $(A)_{gdnf}$ and B has a generalised conjunctive normal form $(B)_{gcnf}$ such that $(A)_{gdnf} \rightarrow (B)_{gcnf}$ is W- augmented explicitly tautological.

Note that there is a natural corresponding generalisation of the notion of variable sharing (definition 2.1.7), which amounts to a repetition of definition 2.1.10. So I shall simply remark that whenever A → B is a W- ATE we shall say that A and B satisfy W- augmented variable sharing ((A,B) A.V.S. (W)).

A.V.S. (W) generalises the definition of variable sharing in two ways: Any wff can play the role of the atoms in definition 2.1.7; and the requirement of identity of atoms is weakened to a requirement that an appropriate implication be in W. Hence it will be no surprise that

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1. Note that "normal form" is a misnomer - $(A)_{gdnf}$ is far from unique. Also, substitution of other occurrences of a subwff of a^i_j is permitted, as long as these are not located within some other a^r_s.

the move from variable sharing to A.V.S. (W) corresponds to the move from (i) together with (ii) - (vii), to (i') together with (ii) - (vii), since only our basis has changed. We shall prove that (i') and (ii) - (vii) do generate exactly the W- ATEs, given some minimal restrictions on W, (and whence we have that (i) - (vii) generate exactly the Tautological Entailments). But first, note that (i) - (vii) and (i'), (ii) - (vii) correspond to the following axiomatic formulations.

Definition 2.1.11. FDE[1]

Axiom - scheme :	(A1)	$A \rightarrow A$		
Rule - schemata:	(2)	$A \rightarrow C$	$\Rightarrow$	$A \,\&\, B \rightarrow C$
	(3)	$A \rightarrow B$	$\Rightarrow$	$A \rightarrow B \vee C$
	(4)	$A \rightarrow B \,\&\, C$	$\Longleftrightarrow$	$A \rightarrow B$ and $A \rightarrow C$
	(5)	$A \vee B \rightarrow C$	$\Longleftrightarrow$	$A \rightarrow C$ and $B \rightarrow C$
	(6)	Substitution of - commutation, association, distribution, double negation and De Morgan - equivalents.		

Definition 2.1.12. WE

Axioms: (1') All $A \rightarrow B$ such that $A \rightarrow B \in W$.

Rule-schemata: (2) - (6) above.

We now prove that, for a set of wff W which satisfies certain minimal restrictions, WE is exactly equal to the set of W- Augmented Tautological Entailments:

1. The system of First Degree Entailments implicit in ENT, Ch. III.

Theorem 2.1.13.

Let W be a set of wff such that

(a) if $A_1 \to B_1$ and $A_2 \to B_2$ are in W, then neither A_1 occurs extensionally as a sufwff of A_2, nor B_1 occurs extensionally as a subwff of B_2[1]; and

(b) W is closed under the rules (6).

Then a wff $A \to B$ is a W- ATE iff $\vdash_{WE} A \to B$.

Proof. For 'only if', suppose that $A \to B$ is a W- ATE, then there is some $(A)_{gdnf} \to (B)_{gcnf}$ which is W- augmented explicitly tautological, and such that $A \leftrightarrow (A)_{gdnf}$ and $B \leftrightarrow (B)_{gcnf}$.

Each disjunct a^i of $(A)_{gdnf}$ and every conjunct b^k of $(B)_{gcnf}$ have a conjunct a^i_j and disjunct b^k_t, respectively, such that $a^i_j \to b^k_t \in W$.

By (1'), $\vdash_{WE} a^i_j \to b^k_t$, and hence, by (2) and (3), $\vdash_{WE} a^i \to b^k$. And so by (4) and (5) we have $\vdash_{WE} (A)_{gdnf} \to (B)_{gcnf}$, whence by (6) $\vdash_{WE} A \to B$. For 'if', suppose that $\vdash_{WE} A \to B$. We prove by induction on a deduction of $A \to B$ in WE that $A \to B$ is a W- ATE.

Base: If $C \to D$ is an axiom, then $\vdash_{WE} C \to D$, and obviously $C \to D$ is a W- ATE. Thus all premises in a deduction are W- ATEs.

Induction step: For (2), assume that $C \to D$ is a W- ATE, then clearly $E \mathbin{\&} C \to D$ is a W- ATE, so the rule (2) preserves this property. (3) and (4) => and (5) => equally obviously do so too. So consider (4) <=.

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1. A occurs extensionally as a subwff of B, A o.e. B, may be defined as follows: (i) A o.e. ~ A, A o.e. A & B and A o.e. A v B; (ii) if A o.e. B and B o.e. C then A o.e. C; (iii) A o.e. B only in virtue of (i) and (ii).

Suppose that $C \to D$ and $C \to E$ are W- ATEs, then we have $(C)_{gdnf_1} \to (D)_{gcnf}$ and $(C)_{gdnf_2} \to (E)_{gcnf}$ which are W- augmented explicitly tautological. That is,

$$(\underline{C}_1^1 \;\&\; \underline{C}_2^1 \;\ldots\ldots\; \&\; \underline{C}_{n_1}^1) \vee (\ldots\ldots \;\&\; \underline{C}_j^2 \;\&) \vee \ldots\ldots \to$$

$$(D_1^1 \vee D_2^1 \vee \ldots\ldots) \;\&\; (\ldots\ldots \vee D_j^2 \vee \ldots\ldots) \;\&\; \ldots\ldots$$

$$(C_1^1 \;\&\; \ldots\ldots) \vee (C_1^2 \;\&\; \ldots\ldots) \vee \ldots\ldots \to (E_1^1 \vee \ldots\ldots) \;\&$$

$$(E_1^2 \vee \ldots\ldots) \;\&\; \ldots\ldots$$

have the appropriate entailments in W. Now C is equivalent to both of the above antecedents (using the usual transformations), so for every C_j^i, either (a) C_j^i is an extensional subwff of some $\underline{C}_t^k$ (that is, they have the same degree of nesting - namely zero), or (b) C_j^i is an extensional superwff of one or more $\underline{C}_t^k$, or (c) C_j^i is identical with some $\underline{C}_t^k$. The restriction on W allows us to construct an appropriate $(C)_{gdnf}$. Make the following changes to $(C)_{gdnf_2}$. Consider each C_j^i. If (a) is the case then, if $\underline{C}_t^k$ is essential (i.e. $\underline{C}_t^k \to E_s^r \in W$), replace C_j^i by $\underline{C}_t^k$. If (b) is the case and one of the $\underline{C}_t^k$ is essential, replace C_j^i by a $(C_j^i)_{gdnf}$ using the $\underline{C}_t^k$ as "atoms". If (c) is the case, make no change. "Smooth out" the $(C_j^i)_{gdnf}$ introduced under (b), to get $(C)_{gdnf}$, treating the "atoms" of $(C_j^i)_{gdnf}$ as "atoms" of $(C)_{gdnf}$.

Now the restriction on W means that all of the "atoms" in $(C)_{gdnf_1}$ and $(C)_{gdnf_2}$ which were required for W- augmented explicit tautologyhood, remain intact in $(C)_{gdnf}$. The requisite duplicates will also occur in the new disjuncts created by "smoothing out" of wffs introduced under (b). Hence $(C)_{gdnf} \to (D)_{gcnf}$ and $(C)_{gdnf} \to (E)_{gcnf}$ are W- augmented explicitly tautological, and so clearly $(C)_{gdnf} \to (D)_{gcnf} \;\&\; (E)_{gcnf}$ is, and obviously $(D)_{gcnf} \;\&\; (E)_{gcnf}$ is a gcnf of $D \;\&\; E$, so that $C \to D \;\&\; E$ is a W- ATE, as required. (5) $\Leftarrow$ can be proved similarly. Finally, the conversions under (6) preserve W- ATE hood because they are allowed in the gdnf and gcnf process, and because W is closed under these too. Thus the induction step is proved, and so we have the result that any wff deduced in WE is a W- ATE, in particular A → B is a W- ATE, as required. □

We now show that variable sharing is a particular case of A.V.S.(W), thus confirming that A.V.S.(W) is indeed a generalisation of variable sharing.

<u>Theorem 2.1.14.</u>
If W = {all substitution instances of A → A) then (A,B) A.V.S.(W) iff A and B are substitution instances of wffs <u>A</u> and <u>B</u> such that <u>A</u> → <u>B</u> is a Tautological Entailment.

<u>Proof.</u> Suppose that (A,B) A.V.S.(W). Then there are wffs $(A)_{gdnf}$ and $(B)_{gcnf}$ such that each disjunct of the former and every conjunct of the latter contain a conjunct C and disjunct D respectively, where C → D ∈ W. But then it must be that C = D by the definition of W. So clearly, if each "atom" of $(A)_{gdnf}$ and $(B)_{gcnf}$ is replaced by a distinct propositional variable, $(A)_{gdnf} \to (B)_{gcnf}$ becomes an explicit tautology $A^1 \to B^1$ when the replacements are made therein. And A → B becomes <u>A</u> → <u>B</u> of which $A^1 \to B^1$ is a normal form, so <u>A</u> → <u>B</u> is a Tautological Entailment and we have that A → B is a substitution instance of a Tautological Entailment as required. Clearly a substitution instance of a Tautological Entailment satisfies A.V.S.(W) (the substituents are the appropriate "atoms"), so the converse holds. □

Corollary 2.1.15. If {all instances of $A \rightarrow A\} \subseteq W$ then, if $A \rightarrow B$ is a substitution instance of a Tautological Entailment then (A,B) A.V.S.(W).

Proof. Follows from the lemma and the fact that if $X \subseteq Y$ then if (A,B) A.V.S. (X) then (A,B) A.V.S. (Y). □

We now use the previous two theorems to show that, as already claimed, FDE does capture the Tautological Entailments.

Theorem 2.1.16. $\vdash_{FDE} A \rightarrow B$ iff $A \rightarrow B$ is a substitution instance of a Tautological Entailment.

Proof. Let $W_1 = \{p \rightarrow p$: for all propositional variables $p\}$ and let $W_2 = \{$substitution instances of $A \rightarrow A\}$.
Clearly FDE is equal to the set of substitution instances of the system W_1E obtained when (A1) of definition 2.1.11 is replaced by W_1. It is also clear that the set of W_2 - ATEs is identical with the set of substitution instances of the W_1 - ATEs. Now W_1 satisfies the conditions of theorem 2.1.13, so $\vdash_{W_1E} A \rightarrow B$ iff $A \rightarrow B$ is a W_1 - ATE. Hence the set of substitution instances of W_1E is identical with the set of substitution instances of W_1 - ATEs. That is, by the above observations, FDE is equal to the set of W_2 - ATEs. But by theorem 2.1.14 the set of W_2 - ATEs is just the set of substitution instances of the Tautological Entailments, thus the theorem is proved. □

In this section we have seen that the definition of Tautological Entailment can be generalised in a natural and intuitive way. Furthermore, as long as we ensure that the entailments in our basis W satisfy variable sharing, then this will also be true of the W- ATEs. Thus if W is relevant, then so is the class of W- ATEs (WE). The manner by which our generalisation preserves relevance is exactly that by which the Anderson-Belnap definition of variable sharing preserves relevance, from the basis of A → A for all atoms A. It involves separating the machinery of the logic which is simply to do with signifying that we are working with a purely extensional notion of & and v, from the basis of independently given valid entailments. So that we can focus our attention on the basis and ensure that it satisfies relevance. Hence our generalisation is faithful to the intuitions and strategy underlying Anderson and Belnap's definition.

§ 2.2 Semisubstitution and augmented variable sharing

The notion of semisubstitution (or, as we will sometimes say, directed substitution) provides a different perspective from which to view our definition of augmented variable sharing. Zeman has described and investigated semisubstitution and certain variants of the notion in relation to modal logics (Zeman 1973). In this section, we define it and show that, on some weak assumptions about the set of wff W, A.V.S. (W) can be defined in terms of variable sharing and semisubstitution.

Definition 2.2.1. We define antecedent part of a wff A and consequent part of A as follows (ENT pp. 34, 110, 240):

(a) A is a consequent part of A;

(b) if $B \rightarrow C$ is a consequent part of A, then B is an antecedent part of A and C is a consequent part of A;

(c) if $B \rightarrow C$ is an antecedent part of A, then B is a consequent part of A and C is an antecedent part of A;

(d) if $\sim B$ is a consequent part of A then B is an antecedent part of A;

(e) if $\sim B$ is an antecedent part of A then B is a consequent part of A;

(f) if B & C (or B v C) is a consequent part of A then both B and C are consequent parts of A;

(g) if B & C (or B v C) is an antecedent part of A then both B and C are antecedent parts of A.

Definition 2.2.2. Weak semisubstitution for a logic L

A logic L satisfies weak semisubstitution iff, where A and B are wffs everywhere the same except that one subwff A_1 of A is replaced at one location by a wff B_1 in B, then if $\vdash_L A_1 \rightarrow B_1$ we have:

(a) If A_1 is a consequent part of A, then if $\vdash_L$ A then $\vdash_L$ B;

(b) If A_1 is an antecedent part of A, then if $\vdash_L$ B then $\vdash_L$ A.

Weak semisubstitution is evidently a generalisation of rule-prefixing and rule-suffixing (Zeman 1973, pp. 11-13) ($\vdash_L A \to B \Rightarrow \vdash_L C \to A \Rightarrow \vdash_L C \to B$ and $\vdash_L A \to B \Rightarrow \vdash_L B \to C \Rightarrow \vdash_L A \to C$), which holds in even quite weak logical contexts.

<u>Theorem 2.2.3.</u>

If {substitution instances of $A \to A$} $\subseteq$ W, which holds in the cases of interest, then each $A \to B$ such that (A,B)A.V.S.(W) can be obtained by a series of weak semisubstitutions from a substitution instance of some Tautological Entailment.

<u>Proof</u>. Consider the relevant "normal form" $(A)_{gdnf} \to (B)_{gcnf}$:

$$(a^1_1 \,\&\, \ldots\ldots \,\&\, a^1_{n_1}) \vee (a^2_1 \,\&\, \ldots\ldots \,\&\, a^2_{n_2}) \vee \ldots\ldots \vee$$
$$(a^m_1 \,\&\, \ldots\ldots \,\&\, a^m_{n_m})$$
$$\to (b^1_1 \vee \ldots\ldots \vee b^1_{r_1}) \,\&\, (b^2_1 \vee \ldots\ldots \vee b^2_{r_2}) \,\&\, \ldots\ldots \,\&$$
$$(b^k_1 \vee \ldots\ldots \vee b^k_{r_k})$$

Replace each b^i_j by the disjunction of "atoms" of $(A)_{gdnf}$ which imply it in W. For example, suppose that just a^1_1 and a^2_1 imply b^1_1 in W ($a^1_1 \to b^1_1$ and $a^2_1 \to b^1_1$ are members of W), then we replace b^1_1 by $a^1_1 \vee a^2_1$ in the above "normal form". This gives a substitution instance C of a Tautological Entailment, for whereever some $a^i_j \to b^t_u$ held between the ith disjunct and t th conjunct, a^i_j is one of the disjuncts of the new t th conjunct, so the ith disjunct of the antecedent and t th conjunct of the consequent have an "atom", a^i_j, in common. The reverse of this procedure obtains $(A)_{gdnf} \to (B)_{gcnf}$ from C by repeated weak semisubstitution.

For example, where b_1^1 was replaced by $a_1^1 \vee a_1^2$, since a_1^1 and a_1^2 occur in consequent part locations, and $a_1^1 \rightarrow b_1^1$ and $a_1^2 \rightarrow b_1^1$ are in W, two applications of weak semisubstitution gives $b_1^1 \vee b_1^1$. Deleting repetitions, we arrive at $(A)_{gdnf} \rightarrow (B)_{gcnf}$, as required. $\square$

This result shows that, in the usual cases of interest, the definition of A.V.S.(W) has two ingredients - variable sharing (Tautological Entailmenthood) and a suitable class of weak semisubstitutions. Thus, in these cases, its intuitive motivation is just that for variable sharing, and that for requiring theoremhood to be preserved under weak semisubstitution. Weak semisubstitution is an independently desirable feature of a logic, as many have noted (Zeman, Anderson and Belnap and others).

Note, however, that in cases where $\{$substitution instances of $A \rightarrow A\} \not\subseteq W$, not even the Tautological Entailments satisfy A.V.S.(W). (After all, the definition is intended to capture variable sharing relative to W.) Augmented variable sharing may have interesting applications in noncircular logic, where $\not\vdash A \rightarrow A$ (Martin 1984). Noncircular logic (in pure $\rightarrow$ form) satisfies semisubstitution. Extending it to other connectives is an unfinished task.

§ 2.3. The logics

We have defined and motivated augmented variable sharing (as a generalisation of Anderson and Belnap's notion of variable sharing and the relevantist intuitions it is based upon). In this section we show how augmented variable sharing provides us with a general procedure for constructing logics, along the lines suggested in §1.1. In these logics, it is clear that augmented variable sharing mediates the interplay between the extensional and intensional connectives. Subsequently, we shall see how to provide an axiomatic formulation for a large subclass (which includes all of the interesting cases) of these logics.

We think of ourselves as beginning with an implication - negation logic $L_{\tilde{\rightarrow}}$, intended eventually to be the implication - negation fragment of a logic in the full language. $L_{\tilde{\rightarrow}}$ has axiom-schemata (AL_1), (AL_2), ... , (AL_n) and rule-schemata (RL_1), (RL_2), ... , (RL_k). In addition, we are equipped with a zero degree logic in the &, v, ~ vocabulary, specified by axiom-schemata (a_1), (a_2), ... (a_m) and rule-schemata (r_1), (r_2), ... , (r_1). We generate the set of theorems for a logic L as follows.

<u>Definition 2.3.1.</u> <u>The L - Hierarchy</u>

L_o = {all instances of schemata (AL_1), (AL_2), ... , (AL_n)}

L_o* = the closure of L_o under the rules (RL_1), (RL_2), ... (RL_k).

L_1 = the closure of: [L_o* U {all instances of schemata (a_1), (a_2), ... , (a_m)} U {A → B: (A,B) A.V.S. (L_o*)}] under (r_1), (r_2), ... (r_1).

L_1* = the closure of L_1 under (RL_1), ... , (RL_k).

L_2 = the close of: [L_1* U {A → B: (A,B) A.V.S. (L_1*)}] under (r_1), ... , (r_1).

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L_n = the closure of: $[L_{n-1}^* \cup \{A \rightarrow B: (A,B)\ \text{A.V.S.}\ (L_{n-1}^*)\}]$
under $(r_1), \ldots\ldots, (r_l)$.

L_n^* = the closure of L_n under $(RL_1), \ldots\ldots, (RL_k)$.

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Note that $L_o \subseteq L_o^* \subseteq L_1 \subseteq \ldots\ldots L_n \subseteq L_n^* \subseteq L_{n+1} \subseteq \ldots\ldots$

So $\bigcup_{i=o}^{\infty} L_i = \bigcup_{i=o}^{\infty} L_i^*$.

To complete the definition, we put: $L = \bigcup_{i=o}^{\infty} L_i$.

We will see presently that in general the above series is strictly increasing, so the construction is not trivial. But now, let us describe the above process more informally.

We can regard the L-Hierarchy as a recipe for producing theorems, a recipe with two basic processes. The first process is the intensional one, which regulates the behaviour of $\rightarrow$ and $\sim$, but to which & and v are opaque (so wffs of the form A & B and A v B are treated as odd new names for propositional variables). This corresponds to the move from L_i to L_i^*, and involves closure w.r.t. the intensional rules. The second is the extensional process, which regulates the behaviour of &, v and $\sim$; and, most importantly, the interplay between & and v, and $\rightarrow$, through augmented variable sharing. The second process corresponds to the move from L_i^* to L_{i+1}, and involves augmented variable sharing w.r.t. the previously obtained class of theorems, and then closure w.r.t. the extensional rules. (In the cases of interest here there is just one extensional rule - adjunction). Thus the recipe involves alternate applications of the first, and then the second process, indefinitely (as well as throwing in a few ingredients - axioms - early on, in L_0 and L_1).

For a fixed set of extensional axioms and rules the intensional process can be varied, corresponding to the choice of implication - negation logic $L_{\overset{\sim}{\rightarrow}}$, to produce an enormous class of logics. This class of logics expresses a fixed intuition about the behaviour of the extensional connectives and their interplay with $\rightarrow$; differences in these features, expressed by different extensional and mixed theorems, can be regarded as differences in the intensional process, $L_{\overset{\sim}{\rightarrow}}$. Since the behaviour of the extensional connectives is determined by augmented variable sharing, its strong intuitive appeal provides strong support for these logics, in this particular respect. The extent to which the intensional process distorts this feature might be an appropriate criterion for evaluation within the class.

This thesis will focus on a small subclass of the above class of logics. However the L-Hierarchy is clearly a very general process. In fact it can be generalised even further by adding modal operators to the intensional part (although some of the tighter extensional/ intensional interplay of modal logics, e.g. $\Box A \,\&\, \Box B \rightarrow \Box (A \,\&\, B)$, may not follow). Because of the intuitive underpinning of the &, v / $\rightarrow$ interplay of the L-Hierarchy, whether or not a logic has a corresponding L-Hierarchy provides an important evaluative criterion.

§ 2.4. Equivalent axiomatic formulations

In this section we will show, for a very broad subclass of the logics defined in the last section, how to formulate them axiomatically. This will allow us to compare these logics with the standard relevant logics, and to develop their algebraic and relational semantics.

We have seen that if W contains all instances of $A \rightarrow A$, then the set of entailments obtained by A.V.S.(W) contains all instances of the Tautological Entailments (corollary 2.1.15). The Tautological Entailments have an axiomatic formulation which is called E_{fde} (the first degree entailments of E) (ENT Ch. III). Thus where $L \underset{\rightarrow}{\sim}$ has $A \rightarrow A$ as a theorem-scheme, we would expect that the corresponding axiomatic system to the L-Hierarchy includes E_{fde}[1]. In this section we will show that in fact, on the assumption that $L \underset{\rightarrow}{\sim}$ includes $A \rightarrow A$ and rule-contraposition and rule-transitivity, the logic L defined in the last section is equal to

$$E_{fde} + (AL_1) + \ldots\ldots + (AL_n) + (a_1) \ldots\ldots + (a_m) + (RL_1) + \ldots\ldots + (RL_k) + (r_1) + \ldots\ldots + (r_l).$$

Since the only addition to the axioms and rules we originally supposed belonged to L are those of E_{fde}, it is clear that, in the broader context of including entailments of arbitrary degree, E_{fde} corresponds to the augmented variable sharing component of our L-Hierarchy. The fact that A.V.S.(W) is a generalisation of variable sharing which is faithful to the intuitions at the heart of the latter, is here echoed by the fact that E_{fde} is all we need in order to achieve the function of the A.V.S. $(L_i{*})$ steps.

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1. which is expecting too much. We can expect a formulation: FDE or EH, say, which specifies De Morgan laws directly and dispenses with contraposition.

E_{fde} can be characterised by the following axioms and rules.[1]

Axiom schemata: (A1) $A \rightarrow A$ (A2) $A \& B \rightarrow A$ (A3) $A \& B \rightarrow B$

(A5) $A \rightarrow A \vee B$ (A6) $B \rightarrow A \vee B$

(A8) $A \& (B \vee C) \rightarrow (A \& B) \vee (A \& C)$

(A9) $\sim\sim A \rightarrow A$

Rule schemata: (R5) If $\vdash A \rightarrow \sim B$ then $\vdash B \rightarrow \sim A$

(Rule-contraposition)

(R.A4) If $\vdash A \rightarrow B$ and $\vdash A \rightarrow C$ then $\vdash A \rightarrow B \& C$.

(R.A7) If $\vdash A \rightarrow C$ and $\vdash B \rightarrow C$ then $\vdash A \vee B \rightarrow C$.

(Tr.) If $\vdash A \rightarrow B$ and $\vdash B \rightarrow C$ then $\vdash A \rightarrow C$. (Rule-transitivity).

(There is a point to the odd numbering, which will be explained later.)

Theorem 2.4.1. Let $L_{\rightarrow}^{\sim}$ contain (A1), (R5) and (Tr.), and let L be specified as in the L-Hierarchy process (definition 2.3.1.), then

$L = E_{fde} + (AL_1) + \ldots\ldots + (AL_n) + (a_1) + \ldots\ldots + (a_m) + (RL_1) + \ldots\ldots + (RL_k) + (r_1) + \ldots\ldots + (r_1)$[2].

(By E_{fde} I mean the axiom and rule schemata thereof).

1. ENT p. 158

2. See Footnote 2 on page 27.

<u>Proof</u>. Without loss of generality, we may suppose that (AL_1) is $A \rightarrow A$, (RL_1) is (R5) and (RL_2) is (Tr.). We denote the right-hand side of the above equality by L^1.

I. We prove that $L \subseteq L^1$ by induction on the L_i*s.

For the base case, L_o* is equal to the set of substitution instances of $L_{\tilde{\rightarrow}}$, which is specified by (AL_1), ... , (AL_n) and (RL_1), ... , (RL_k). Since these are also axioms and rules of L^1, clearly $L_o^* \subseteq L^1$.

For the inductive step, we assume that $L_k^* \subseteq L^1$, and we prove that, first $L_{k+1} \subseteq L^1$, and then $L_{k+1}^* \subseteq L^1$.

(i) Consider $\phi \in L_{k+1} - L_k^*$. In the special case k = o it might be that ϕ is a substitution instance of one of (a_1), ... , (a_m); but since these are also axiom-schema of L^1 we then have $\phi \in L^1$.

Otherwise, either (a) $\phi = A \rightarrow B$ and (A,B) A.V.S. (L_k^*), or (b) ϕ follows from $L_k^* \cup \{A \rightarrow B: \ (A,B) \text{ A.V.S. } (L_k^*)\}$ + (a_1)+ ..... + (a_m) by the rules (r_1), ... , (r_1).

Case (a). Consider the corresponding "normal forms" $(a_1^1 \ \& \ a_2^1 \ \& \ .....) \vee (a_1^2 \ \& \ a_2^2 \ \& \ .....) \vee \ ..... \vee (a_1^u \ \& \ a_2^u \ \& \ .....)$ and $(b_1^1 \vee b_2^1 \vee .....) \ \& \ (b_1^2 \vee b_2^2 \vee .....) \ \& \ ..... \ \& \ (b_1^s \vee b_2^s \vee .....)$.

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2. As suggested in a previous footnote, we can use a formulation of the first degree entailments different from E_{fde} to carry the augmented variable sharing burden. In fact we can replace E_{fde} by FDE and the theorem holds with only the assumption that $A \rightarrow A$ is in $L_{\tilde{\rightarrow}}$. Also, the completely general result also holds, where E_{fde} is replaced by WE and no assumptions other than that W is closed under (6) the normal form transformations, are needed.

For every $1 \leqq i \leqq u$ and $1 \leqq j \leqq s$ there is a pair a_1^i and b_m^j such that $a_1^i \rightarrow b_m^j \in L_k{}^*$, by the definition of A.V.S.(L_k^*). Hence $a_1^i \rightarrow b_m^j \in L^1$ by our induction assumption. By (A5), (A6) and (Tr.) of E_{fde} $a_1^i \rightarrow (b_1^j \vee b_2^j \vee \ldots\ldots \vee b_m^j \vee \ldots\ldots)$ $\in L^1$.

By (A2), (A3) and (Tr.) of E_{fde}

$$(a_1^i \;\&\; a_2^i \;\&\; \ldots\ldots\; a_1^i \;\&\; \ldots\ldots) \rightarrow (b_1^j \vee b_2^j \;\ldots\ldots) \in L^1.$$

So for every $(A)_{gdnf}$ - disjunct a^i and $(B)_{gcnf}$- conjunct b^j we have $a^i \rightarrow b^j \in L^1$. So for $1 \leqq i \leqq u$, $a^i \rightarrow b^1 \;\&\; b^2 \;\&\; \ldots\ldots \;\&\; b^s \in L^1$, by repeated applications of (R.A4) of E_{fde}. But then repeated applications of (R.A7) give

$$a^1 \vee a^2 \vee \ldots\ldots \vee a^u \rightarrow b^1 \;\&\; b^2 \;\&\; \ldots\ldots \;\&\; b^s \in L^1.$$

i.e. the "normal form" $(A)_{gdnf} \rightarrow (B)_{gcnf}$ is a member of L^1.

Now this "normal form" can be obtained from ϕ by using substitution of equivalents restricted to commutation, association, distribution, double negation and De Morgan equivalents. These transformations are available in E_{fde}[1] (in the sense that they produce logical equivalents in E_{fde}) and hence in L^1. Thus in L^1 ϕ is logically equivalent to $a^1 \vee a^2 \vee \ldots\ldots\; a^u \rightarrow b^1 \;\&\; b^2 \;\&\; \ldots\ldots \;\&\; b^s$, and since the latter is a theorem of L^1, ϕ is a theorem of L^1.

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1. ENT § 15.1

For case (b), we just need to note that the rules (r_1) , ... , (r_1) are rules of L^1, so since $L_k^* \cup \{A \rightarrow B: (A,B)$ A.V.S. $(L_k^*)\} + (a_1) + ... + (a_m)$ is a subset of L^1 (by the above), then its closure under these rules, i.e. L_{k+1}, is a subset of L^1. So we have $L_{k+1} \subseteq L^1$.

(ii) Consider $\phi \in L_{k+1}^* - L_{k+1}$. L_{k+1}^* is just the closure of L_{k+1} under the rules (RL_1), ... , (RL_k), which are rules of L^1.

So since $L_{k+1} \subseteq L^1$ we have $L_{k+1}^* \subseteq L^1$, which completes the proof of our induction step.

Therefore, by induction $L_k^* \subseteq L^1$ for k = 0, 1, 2, ...

and so $L = \bigcup_{i=o}^{\infty} L_k^* \subseteq L^1$.

II. We prove that $L^1 \subseteq L$ by showing that the axioms of L^1 are all theorem-schema of L, and that L is closed under all of the rules of L^1.

Clearly the axioms (AL_1), ... , (AL_n) and (a_1), ... , (a_m) of L^1 are theorem-schema of L, since all their substitution-instances are added at the L_o and L_1 steps of the L-Hierarchy.

By corollary 2.1.15, all substitution instances of E_{fde} are delivered by A.V.S. $(A \rightarrow A)$. But more formally:

Consider the axioms of E_{fde}. By supposition (A1) $A \rightarrow A$ is equal to (AL_1), so (A1) is a theorem-scheme of L. The remainder, (A2), (A3), (A5), (A6), (A8) and (A9) satisfy A.V.S. (L_o^*) where the only theorem of L_o^* used is $A \rightarrow A$, and where the "atom" sets of each wff just correspond to the sentential variables. Hence every substitution-instance thereof occurs in L_1.

Thus all of the axioms of L^1 are theorem-schemata of L.

The rules (RL_1), ... , (RL_k) of L^1 can be shown to hold for L as follows. Suppose $\vdash_L A_1$, $\vdash_L A_2$, ... , $\vdash_L A_j$ and let (RL_i) be the rule: If $\vdash A_1$, $\vdash A_2$, ... , $\vdash A_j$ then $\vdash B_1$ and $\vdash B_2$ and ..... and $\vdash B_s$. By the definition of the L-Hierarchy, there is a minimum integer u such that A_1, A_2, ... , A_j are all in L_u*. But L_u* is closed under the rules (RL_1), ... (RL_k), and so in particular B_1, B_2, ... , B_s are all in L_u*, and hence in L as required.

The argument is similar for the rules (r_1), ... , (r_1), in this case exploiting the closure of each L_u under these rules.

There remains the rules of E_{fde}. By supposition, (R5) and (Tr.) are equal to (RL_1) and (RL_2), and so L is closed under them by the above argument. This leaves (R.A4) and (R.A7). For (R.A4) suppose that $\vdash_L A \rightarrow B$ and $\vdash_L A \rightarrow C$, then there is a least integer u such that $A \rightarrow B \in L_u*$ and $A \rightarrow C \in L_u*$. But then (A, B & C) A.V.S. (L_u*) where the "atoms" are A, B and C. So $A \rightarrow B \& C \in L_{u+1} \subseteq L$.

Thus L is closed under (R.A4). The argument for (R.A7) is exactly the same.

This completes the proof that all the axioms of L^1 are in L, and that L is closed under the rules of L^1, so $L^1 \subseteq L$. □

So on the assumption of some weak conditions satisfied by the L-Hierarchy, namely that the intensional component includes the axiom $A \rightarrow A$ and the rules rule-contraposition and rule-transitivity, L can be formulated axiomatically by simply adding the axioms and rules of E_{fde} to the purely intensional and purely extensional axioms and rules of L.

§ 2.5. Comparison with the standard relevant logics

The standard relevant logics are extensions of the "basic" logic B, which is not much bigger than E_{fde}. However for our purposes B, we shall see, is too big.

In this section, I first briefly describe B. Then I will put forward a sublogic of B, BB.

From our perspective of restricting the &, v / → interplay to that sanctioned by augmented variable sharing, BB is a much better contender for the role of a basic logic.

B has the following axioms and rules.[1]

Axiom schemata: (A1) $A \to A$; (A2) $A \,\&\, B \to A$; (A3) $A \,\&\, B \to B$;

(A4) $A \to B \,\&\, A \to C \mathrel{\dot{\to}} A \to B \,\&\, C$; (A5) $A \to A \vee B$;

(A6) $B \to A \vee B$; (A7) $A \to C \,\&\, B \to C \mathrel{\dot{\to}} A \vee B \to C$;

(A8) $A \,\&\, (B \vee C) \to (A \,\&\, B) \vee (A \,\&\, C)$;

(A9) $\sim\sim A \to A$.

1. RLR, p. 287.

Rule schemata: (R1) If $\vdash A$ and $\vdash A \rightarrow B$ then $\vdash B$. (modus ponens).

(R2) If $\vdash A$ and $\vdash B$ then $\vdash A \& B$. (Adjunction).

(R3) If $\vdash A \rightarrow B$ then $\vdash B \rightarrow C \mathbin{\dot{\rightarrow}} A \rightarrow C$. (Rule-suffixing).

(R4) If $\vdash A \rightarrow B$ then $\vdash C \rightarrow A \mathbin{\dot{\rightarrow}} C \rightarrow B$. (Rule-prefixing).

(R5) If $\vdash A \rightarrow \sim B$ then $\vdash B \rightarrow \sim A$. (Rule-contraposition).

Extensions of B include T, Anderson and Belnap's system of ticket entailment; their preferred system of entailment E; R, which expunges the modal structure built into E (using $\Box A = df (A \rightarrow A) \rightarrow A$); and the classical sentential calculus.[1]

B is stronger than E_{fde} in that the rules (R1) and (R2) are added (which simply are not applicable when considering first degree entailments); rule-transitivity is strengthened to (R3) and (R4); and the rules (R.A4) and (R.A7) of E_{fde} are strengthened to the axioms (A4) and (A7). Our labelling of the axioms and rules of E_{fde} serves to highlight this relationship between E_{fde} and B.

Note that (R1), (R3) and (R4) are purely intensional, whereas (R2) is purely extensional. Thus the only part of the leap from E_{fde} to B which mixes intensional and extensional connectives is the strengthening of the rules (R.A4) and (R.A7) to their corresponding

1. See RLR, Ch. 4 and Ch. 5, for an investigation of B and its extensions and their relational semantics.

axiom forms. From our standpoint, we must be suspicious of mixed extensions of E_{fde}, for if the mixed axioms and/or rules are not redundant, then from §2.4 we know that such extensions don't have a corresponding L-Hierarchy and hence the &, v / → interplay is not determined by augmented variable sharing. So we consider the system which adds to E_{fde} just the purely intensional and purely extensional components of the move to B. Let us call it BB.

<u>Definition 2.5.1.</u> <u>BB</u>

Axiom schemata: Just those for E_{fde}, i.e. (A1), (A2), (A3) , (A5), (A6), (A8) and (A9).

Rule schemata: Those for B plus (R. A4) and (R. A7) of E_{fde}.

Thus BB = E_{fde} + (R1) + (R2) + (R3) + (R4)

(obviously (Tr.) of E_{fde} is a derived rule in BB, by (R3) and (R4)).

Hence by theorem 2.4.1 BB has an L-Hierarchy whose intensional component has the axiom (A1) A → A and the rules (R1), (R3), (R4) and (R5), and whose extensional component is just (R2) (adjunction). In fact, as we shall see, BB is a proper sublogic of B, and hence B does not have a corresponding L-Hierarchy.

A very large class of logics, which includes the standard relevant logics, can be generated from B by adding just intensional axioms and rules.[1] For such logics L, L = B + $L_{\underset{\to}{\sim}}$ (if L is not a conservative extension of $L_{\underset{\to}{\sim}}$, the nomenclature is bad, but the process still works). Thus

1. RLR p. 288

we can carry out the same procedure for these logics as for B. In effect, this means adding those intensional axioms and rules to BB.

<u>Definition 2.5.2.</u> <u>BL</u> (B - ing)

Let L have a formulation comprising the axioms and rules of B plus purely intensional and purely extensional schemata, then we define BL to be the result of adding those schemata to the axioms and rules of BB.

i.e. where $L = B + L_{\underset{\to}{\sim}}$ + (extensional schemata), $BL = BB + L_{\underset{\to}{\sim}}$ + (extensional schemata).

By theorem 2.4.1 all such logics BL have a corresponding L-Hierarchy. For example, as is well known, E = B + (B3) + (B4) + (B5) + (D3) + (D4) + (BR1)[1] where the axioms and rule are defined:

(B3) $A \to B \underset{\cdot}{\to} B \to C \underset{\cdot}{\to} A \to C$; (B4) $A \to B \underset{\cdot}{\to} C \to A \underset{\cdot}{\to} C \to B$;

(B5) $A \to (A \to B) \underset{\cdot}{\to} A \to B$; (D3) $A \to \sim A \underset{\cdot}{\to} \sim A$;

(D4) $A \to \sim B \underset{\cdot}{\to} B \to \sim A$

(BR1) If A is a theorem then $(A \to B) \to B$ is a theorem.

Hence we have the logic BE = BB + (B3) + (B4) + (B5) + (D3) + (D4) + (BR1) and by theorem 2.4.1 it has the following L-Hierarchy.

E_o = {(A1), (A9), (B3), (B4), (B5), (D3) and (D4)}.

E_o^* = the closure of E_o under (R1) and (BR1) (which equals $E_{\underset{\to}{\sim}}$).

E_1 = the closure of [E_o^* U {A → B: (A,B) A.V.S. (E_o^*)}] under (R2).

E_1^* = the close of E_1 under (R1) and (BR1)[2].

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1. RLR, p. 290.
2. (R3), (R4) and (R5) are made redundant by (B3), (B4) and (D4).

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$$BE = \bigcup_{i=o}^{\infty} E_i^*$$

Clearly, from our perspective of limiting the &, v / → interplay to that sanctioned by augmented variable sharing, the class of BL logics is of primary interest. Since these are extensions of BB, it is more appropriate to regard BB as our basic logic. B imports too much logical machinery into the &, v / → interplay.

It might be argued that B is preferable to BB as a basic relevant logic, on the ground that B and its extensions have a simpler relational semantics[1], and can be conservatively extended to include a primitive fusion operator, which greatly simplifies the algebraic semantics.[2] However a programme seeking a relevant proof theory ought to put less weight on such semantical niceties than on exploiting relevantist intuitions - such as those giving rise to variable sharing and augmented variable sharing as criteria for entailment between truth-functional compounds of wffs. Anderson and Belnap's intuitions place considerable weight on proof theory. Of course both types of considerations might come into play when evaluating logics, since some logics, notably R and some of its extensions, are extensions of the required sort of BB, as well as being extensions of B (i.e. BR = R).

BB can be thought of as being obtained from B by simply "dropping" the axioms (A4) (A → B) & (A → C) $\dot{\to}$ A → (B & C) and (A7) (A → C) & (B → C) $\dot{\to}$ (A v B) → C, down to their weaker rule forms (R.A4) ⊢ A → B and ⊢ A → C => ⊢ A → (B & C) and (R.A7) ⊢ A → C and ⊢ B → C => (A v B) → C. Clearly obtaining BB + L $\underset{\to}{\sim}$ from B + L $\underset{\to}{\sim}$ involves exactly the same

1. RLR, Ch.4.
2. AlG II.

step. Thus a BL logic can be thought of as being obtained from a logic L by simply weakening the axioms (A4) and (A7) to the rules (R.A4) and (R.A7). However, care needs to be taken because the formulation of BL is sensitive to the particular axiomatisation of L. In general a logic L such that $L = B + L_{\tilde{\rightarrow}}$ will have a much "neater" axiomatisation, and simply applying the above procedure to <u>that</u> axiomatisation will very likely lead to error. For example, an axiomatisation of E, neater than the $E = B + E_{\tilde{\rightarrow}}$ one, is given on page 340 of ENT. One of the axioms that would remain after replacing (A4) and (A7) by (R.A4) and (R.A7) is $(\Box A \mathbin{\&} \Box B) \rightarrow \Box (A \mathbin{\&} B)$ (where $\Box A = \text{df.}\ A \rightarrow A \rightarrow A$), which expresses a strong logical connection between & and → which no doubt fails to hold in BE[1]. So in characterising the B-ing process in the above way, we need to keep in mind that this is not true in general for alternative formulations of L.

In ENT it is suggested that we can think of E as E_{fde} (i.e. the Tautological Entailments) plus $E_{\tilde{\rightarrow}}$ (ENT p. 231). The L-Hierarchy clarifies this problem, I suggest, by showing how E_{fde} and $E_{\tilde{\rightarrow}}$ are working within E. Since $BE = BB + E_{\tilde{\rightarrow}}$ (as E is a conservative extension of $E_{\tilde{\rightarrow}}$) and E_{fde} is a sublogic of BB, it is more accurate to say that BE is E_{fde} plus $E_{\tilde{\rightarrow}}$ (all else we need add is adjunction). In fact BE comprises $E_{\tilde{\rightarrow}}$ and entailments sanctioned by augmented variable-sharing, the latter being intuitive generalisations of the Tautological Entailments.

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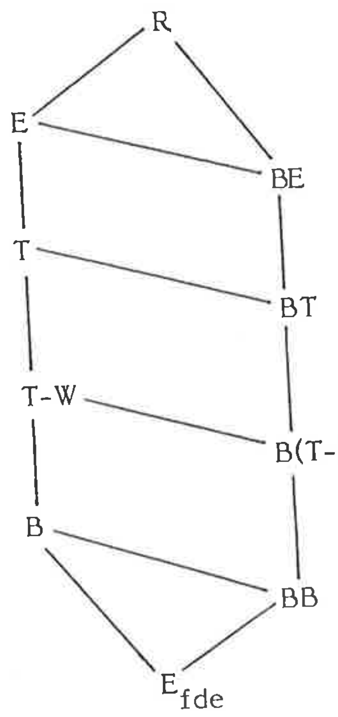
1. I thought I had a proof that this is so, TESTER was its downfall.

§ 2.6. The L-Hierarchy is not trivial and BR = R

In this section I will redeem two earlier promises. First I will show that, in general, the construction of the L-Hierarchy is not trivial, in the sense that the hierarchy collapses into L_1* (say) (promised in §2.3). Then I will show that BR = R (i.e. applying the L-Hierarchy procedure to $R_{\underset{\to}{\sim}}$ gives R) (promised in §1.1).

In light of the fact that BR = R, the question arises whether BE = E, or BB = B, or in general whether B*L* = *L*. As I have already indicated BB ≠ B and BE ≠ E. We defer demonstration of the fact that BE $\subsetneq$ E (and that for *L* $\subseteq$ E where *L* is the required sort of extension of B, B*L* $\subsetneq$ *L*) until §3.2, since it utilises and illustrates the algebraic semantics developed in chapter 3. Thus the relationships between some of the standard relevant logics and their B-ed counterparts are as depicted in the following diagram:



And now the promises.

<u>Theorem 2.6.1.</u> If rule-prefixing is an admissible rule and $A \to A$ is an axiom of $L_{\overset{\sim}{\to}}$ and $L_{\overset{\sim}{\to}}$ is non-trivial, then for all natural numbers i, $L_i \subsetneq L_i{}^* \subsetneq L_{i+1}$ in the L-Hierarchy (definition 2.3.1).

<u>Proof</u>. Since $A \to A$ is in $L_{\overset{\sim}{\to}}$, $p \to p \in L_o{}^*$. Hence $p \to p \vee u$ and $p \to p \vee q$ are in L_1 by A.V.S. ($L_o{}^*$), but they are obviously not in $L_o{}^*$ (= all instances of $L_{\overset{\sim}{\to}}$ theorems) for if they were we would have $p \to r$ in $L_{\overset{\sim}{\to}}$, contrary to the non-triviality of $L_{\overset{\sim}{\to}}$. i.e. $p \to p \vee u$ and $p \to p \vee q$ $\in L_1 - L_o{}^*$. Whence $s \to p \overset{\cdot}{\underset{\cdot}{\to}} s \to p\vee u$ and $s \to p \overset{\cdot}{\underset{\cdot}{\to}} s \to p \vee q \in L_1{}^* - L_1$ by rule-prefixing. (If these were in L_1, since only the antecedents and consequents can be "atoms" because their major connective is $\to$, we would in fact have the entailments in $L_o{}^*$ in order to satisfy A.V.S. ($L_o{}^*$). But then $s \to p \overset{\cdot}{\underset{\cdot}{\to}} s \to r \in L_{\overset{\sim}{\to}}$, contrary to its non-triviality).

And so $s \to p \overset{\cdot}{\underset{\cdot}{\to}} [\,(s \to p \vee u) \;\&\; (s \to p \vee q)] \vee {}^{u}_{q} \in L_2 - L_1{}^*$,

and $s \to (s \to p) \overset{\cdot}{\underset{\cdot}{\to}} s \to [\;\;] \vee u,\; s \to (s \to p) \overset{\cdot}{\underset{\cdot}{\to}} s \to [\;\;] \vee q \in L_2{}^* - L_2$.

And so $s \to (s \to p) \overset{\cdot}{\underset{\cdot}{\to}} |[\,(s \to [\;\;] \vee u) \;\&\; (s \to [\;\;] \vee q)\,]| \vee {}^{u}_{q} \in L_3 - L_2{}^*$

and $s \to (s \to (s \to p)) \overset{\cdot}{\underset{\cdot}{\to}} s \to (|[\;\;]| \vee {}^{u}_{q}) \in L_3{}^* - L_3$,

and the process doesn't stop. □

We prove that BR = R via a more general result:

<u>Lemma 2.6.2.</u> Suppose that unrestricted rule-permutation, $\vdash A \dot{\rightarrow} B \rightarrow C \Leftrightarrow\ \vdash B \dot{\rightarrow} A \rightarrow C$, and (Tr.) rule-transitivity are included in the intensional rules (those of $L_{\tilde{\rightarrow}}$), and that (A1) $A \rightarrow A$ and (D4) contraposition are in $L_{\tilde{\rightarrow}}$, then the L-Hierarchy (definition 2.3.1) has the following theorems:

(A4) $A \rightarrow B \ \&\ A \rightarrow C \ \dot{\rightarrow}\ A \rightarrow B \ \&\ C$

(A7) $A \rightarrow C \ \&\ B \rightarrow C \ \dot{\rightarrow}\ A \vee B \rightarrow C$

<u>Proof.</u> By theorem 2.4.1 the logic L obtained by the L-Hierarchy has an axiomatic formulation which includes the axioms and rules of E_{fde}.
Hence $\vdash_L A \rightarrow B \ \&\ A \rightarrow C \ \dot{\rightarrow}\ A \rightarrow B$ (by (A2))
and so $\vdash_L A \dot{\rightarrow} (A \rightarrow B \ \&\ A \rightarrow C) \rightarrow B$ using rule permutation.

Similarly $\vdash_L A \dot{\rightarrow} (A \rightarrow B \ \&\ A \rightarrow C) \rightarrow C$, and so by (D4) and (Tr.), $\vdash_L A \dot{\rightarrow} \sim B \rightarrow \sim(\ )$ and $\vdash_L A \dot{\rightarrow} \sim C \rightarrow \sim(\ )$. Whence we get $\vdash_L \sim B \dot{\rightarrow} A \rightarrow \sim(\ )$ and $\vdash_L \sim C \dot{\rightarrow} A \rightarrow \sim(\ )$ using rule-permutation and so by (R.A7) $\vdash_L \sim B \vee \sim C \dot{\rightarrow} A \rightarrow \sim(\ )$.[1] Reversing the procedure and using De Morgan equivalents, this gives $\vdash_L A \rightarrow B \ \&\ A \rightarrow C \ \dot{\rightarrow}$ $A \rightarrow B \ \&\ C$ as required.

(A7) is easier: We have $\vdash_L A \rightarrow C \ \&\ B \rightarrow C \ \dot{\rightarrow}\ A \rightarrow C$ (by (A2))
and so $\vdash_L A \dot{\rightarrow} (A \rightarrow C \ \&\ B \rightarrow C) \rightarrow C$ and $\vdash_L B \dot{\rightarrow} (A \rightarrow C \ \&\ B \rightarrow C) \rightarrow C$,as before. By (R.A7) $\vdash_L A \vee B \dot{\rightarrow} (\ ) \rightarrow C$ and so

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1. We could have done an extra contraposition step and used (R.A4).

$\vdash_L$ $A \rightarrow C$ & $B \rightarrow C \,\dot{\rightarrow}\, A \vee B \rightarrow C$ as required.[1] □

<u>Theorem 2.6.3.</u> Suppose that *L* is an extension of B which can be formulated as the axioms and rules of B plus purely intensional and/or purely extensional axioms and rules. Then, if $L \tilde{\rightarrow}$ has both unrestricted rule-permutation and full contraposition (or, alternatively, $\vdash A \,\dot{\rightarrow}\, B \rightarrow C \Leftrightarrow\ \vdash \sim (A \rightarrow \sim B) \rightarrow C$ and full contraposition[1]) B*L* = *L*.

<u>Proof</u>. By the lemma, (A4) and (A7) are theorems of B*L*. But BB ⊆ B*L* and BB + (A4) + (A7) = B, and so B ⊆ B*L*. Hence, since the <u>extra</u> axioms and rules of the relevant formulation of *L* are in B*L*, *L* ⊆ B*L*. Also, it is obvious from definition 2.5.2 that B*L* ⊆ *L*, in general. Thus B*L* = *L* as required. □

<u>Corollary 2.6.4.</u> BR = R[2]

<u>Proof</u>. R satisfies the antecedent conditions of theorem 2.6.3. □

This is a big payoff for R. By theorem 2.4.1 R has an alternative characterisation, given by the L-Hierarchy for BR. This alternative formulation of R shows clearly the interplay in R between the extensional

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1. Note that only rule-permutation and (R.A7) are needed for (A7), whereas full contraposition is also needed for (A4) (we are really using intensional conjunction). Now if, rather than rule-permutation we had used its dual: $\vdash A \,\dot{\rightarrow}\, B \rightarrow C \Leftrightarrow\ \vdash \sim (A \rightarrow \sim B) \rightarrow C$, (A4) would have come easy but (A7) would have needed full contraposition (D4). When we have (D4), both of these rules are interderivable.

2. Pointed out to me by Bob Meyer.

and intensional connectives, and the motivation for that interplay (augmented variable sharing). The standard formulations of R make this interplay much less perspicuous. Furthermore, it *is* accurate to describe R as E_{fde} plus $R_{\tilde{\rightarrow}}$, in line with the Anderson-Belnap intuitions about E which, I have argued (§2.5), actually hold for BE.

CHAPTER 3

The algebraic semantics for BB and its extensions

§ 3.0 Introduction

In this chapter we will investigate the algebraic semantics for the BL systems defined in Chapter 2 (definition 2.5.2). In doing so we will follow Routley and Meyer's approach in ALG II. The algebras provide a useful theoretical tool for the study of these logics, which we will exploit in this and later chapters. In the first section we define BB-algebras and prove their adequacy for BB, then in §3.2 we extend the definition to cater for extensions of BB. Finally, in §3.3 we will reap some dividends by proving that BE ≠ E (and hence that B$L \neq L$ for $L \subseteq$ E), as promised in §2.6, thus confirming that there is a real distinction between B-ed and un B-ed logics in even quite strong logical contexts.

§ 3.1. BB-algebras

The approach of ALG II leans heavily on the fusion operator o, which can be conservatively added to B and its extensions. (In the case of R fusion corresponds to intensional conjunction: $\overline{A \to \overline{B}}$.) Fusion is intended as a tighter (relevant) premiss-bunching connective than &, in the sense that if $\vdash A \circ B \to C$, then both of A and B are <u>used</u> in the corresponding deduction A, B $\vdash$ C.[1] Hence it is required to satisfy residuation ($\vdash A \circ B \to C$ iff $\vdash A \to . B \to C$). This in turn produces a relevant deduction theorem:

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1. R.K. Meyer, <u>Arithmetic formulated relevantly</u>, typescript forthcoming, pp. 32-34.

Interpreting $A_1, A_2, \ldots\ldots, A_n \vdash B$ as $(\ldots\ldots((A_1 \circ A_2) \circ A_3)\ldots\ldots) \circ A_n \vdash B$, we have $A_1, A_2, \ldots\ldots, A_n \vdash B$ iff $A_1, \ldots\ldots, A_{n-1} \vdash A_n \to B$.

(In general, the order of the A_is <u>does</u> matter.)

But no such connective can be conservatively added to those *BL* such that $BL \subseteq BE$. And here we take the fact that $BE \subsetneq E$, proved in §3.3, on credit. For suppose we add o to *BL*, call the augmented logic BL^o, and require it to satisfy the above property, then (A4) is a theorem of BL^o, since:

$\vdash_{BL^o}$ $(A \to B \;\&\; A \to C) \mathrel{\dot{\to}} A \to B$ (A2)

$\vdash_{BL^o}$ $((A \to B \;\&\; A \to C) \circ A) \to B$ Residuation

$\vdash_{BL^o}$ $((A \to B \;\&\; A \to C) \circ A) \to C$ Similarly,

$\vdash_{BL^o}$ $((A \to B \;\&\; A \to C) \circ A) \to B \;\&\; C$ by (R.A4)

$\vdash_{BL^o}$ $(A \to B \;\&\; A \to C) \mathrel{\dot{\to}} A \to B \;\&\; C$ Residuation

That is, (A4) holds in BL^o. (We will see that (A4) does not hold in any $BL \subseteq BE$.) Now by (A4) we also have

$\vdash_{BL^o}$ $(\sim C \to \sim A \;\&\; \sim C \to \sim B) \mathrel{\dot{\to}} \sim C \to \sim A \;\&\; \sim B$

so if full contraposition holds in *BL*, then using that plus rule-prefixing and rule-suffixing, and De Morgan properties, gives

$\vdash_{BL^o}$ $(A \to C \;\&\; B \to C) \mathrel{\dot{\to}} A \vee B \to C$ which is (A7).[1]

1. This provides an alternative proof of theorem 2.6.2, using fusion, as originally shown by Bob Meyer.

Thus fusion reinstates (A4), and on the addition of contraposition, (A7) as well, which negates the whole B-ing process (which can be thought of as dropping these axioms - §2.5). So we have to do without fusion.

Definition 3.1.1.

A BB-algebra is a structure $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \rightarrow \rangle$ where G is a set; $\cap$, $\cup$ and $\rightarrow$ are binary operations on G; $-$ is a unary operation on G and F is a subset of G such that, for all a, b and c in G:-

where $a \leqq b \equiv_{df} a \cap b = a$,

(1) $\langle G, \cap, \cup, - \rangle$ is a De Morgan lattice
(i.e. $\langle G, \cap, \cup, - \rangle$ is a distributive lattice s.t. for $a, b \in G$, $--a = a$ and $a \leqq -b$ entails that $b \leqq -a$.)

(2) F is a filter on this lattice (i.e. $a \cap b \in F$ iff $a \in F$ and $b \in F$).
(Note: this ensures that F is closed upward w.r.t. $\leqq$)

(3) $a \rightarrow b \in F$ iff $a \leqq b$.

(4) $a \leqq b$ only if $b \rightarrow c \leqq a \rightarrow c$.

(5) $a \leqq b$ only if $c \rightarrow a \leqq c \rightarrow b$.

Definition 3.1.2.

An interpretation I is a function from the set of wffs into G such that:

(i) $I(A \rightarrow B) = I(A) \rightarrow I(B)$ (ii) $I(\sim A) = -I(A)$

(iii) $I(A \& B) = I(A) \cap I(B)$ (iv) $I(A \vee B) = I(A) \cup I(B)$

Definition 3.1.3.

A wff A is true on I in $\underset{\sim}{G}$ iff $I(A) \in F$, and otherwise A is false on I in $\underset{\sim}{G}$.

A is valid in $\underset{\sim}{G}$ ($\underset{\sim}{G}$-valid) iff A is true on all I in $\underset{\sim}{G}$.

A is BB-valid iff A is valid in all BB-algebras.

We now prove that BB is sound and complete w.r.t. the set of all BB-algebras.

Theorem 3.1.4. The basic algebraic adequacy theorem.

$\vdash_{BB}$ A iff A is BB-valid.

Proof (a) Soundness: $\vdash_{BB}$ A only if A is BB-valid.

That the axioms of BB (definition 2.5.1) are BB-valid is immediate from lattice properties and (3). And the rules preserve validity:-

(R1) If $I(A) \in F$ and $I(A \to B) \in F$, so $I(A) \to I(B) \in F$, then $I(A) \leq I(B)$ by (3), and so $I(B) \in F$ by (2).

(R2) If $I(A) \in F$ and $I(B) \in F$, then $I(A) \cap I(B) \in F$ by (2), but $I(A) \cap I(B) = I(A \,\&\, B)$.

(R3) Rule-suffixing follows similarly using (4) and (3).

(R4) Rule-prefixing follows similarly using (5) and (3).

(R5) $I(A \to \sim B) \in F$ iff $I(A) \leq -I(B)$ iff $I(B) \leq -I(A)$ iff $I(B \to \sim A) \in F$.

(R.A4) If $I(A \to B) \in F$ and $I(A \to C) \in F$ then $I(A) \leqq I(B)$ and $I(A) \leqq I(C)$ by (3). So $I(A) \leqq I(B) \cap I(C)$ by (1). Whence $I(A \to B \,\&\, C) \in F$.

(R.A7) Similarly.

This completes the proof of (a).

(b) Completeness: $\vdash_{BB} A$ if A is BB-valid.

We prove this in the usual way, showing that A is not BB-valid if $\nvdash_{BB} A$, using the Lindenbaum algebra.

For a wff A, define $|A| = \{C : \vdash_{BB} A \leftrightarrow C\}$. Set $G = \{|A|;\ A \text{ is a wff}\}$. Set $F = \{|A|;\ \vdash_{BB} A\}$ and define the operations on G as follows: $|A| \cap |B| =_{df} |A \,\&\, B|$; $|A| \cup |B| =_{df} |A \vee B|$; $-|A| =_{df} |{\sim}A|$; $|A| \to |B| =_{df} |A \to B|$.

We remark that obviously provable co-entailment is an equivalence relation on the set of wffs, and that substitutivity w.r.t. provable co-entailment holds for BB, so the above are well defined.

We verify that $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \to \rangle$ is a BB-algebra. That $\cap$ and $\cup$ are idempotent ($|A| \cap |A| = |A|$), commutative and associative is trivial. We check absorption:-

$|A| \cap (|A| \cup |B|) = |A| \cap |A \vee B| = |A \,\&\, (A \vee B)| =$

$\{C : \vdash_{BB} A \,\&\, (A \vee B) \leftrightarrow C\}$

But $\vdash_{BB} A \,\&\, (A \vee B) \leftrightarrow A$, so R.H.S. $= \{C : \vdash_{BB} A \leftrightarrow C\} = |A|$

Similarly $|A| \cup (|A| \cap |B|) = |A|$.

The above properties ensure we have a lattice (Skornjakov 1977, p. 42).

Note that $|A| \leq |B|$ iff $\vdash_{BB} A \rightarrow B$ (where, as usual, $|A| \leq |B|$ $\equiv_{df}$ $|A| \cap |B| = |A|$). The proof is exactly that of ALG II which I reproduce here (ALG II, p. 7). First, suppose that $|A| \leq |B|$, i.e. $|A| \cap |B| = |A|$, then $|A \& B| = |A|$ by our definition, so that $\vdash_{BB} A \& B \leftrightarrow C$ iff $\vdash_{BB} A \leftrightarrow C$. Whence $\vdash_{BB} A \& B \leftrightarrow A$ and since $\vdash_{BB} A \& B \rightarrow B$ ((A3)) by transitivity we have $\vdash_{BB} A \rightarrow B$. Conversely, if $\vdash_{BB} A \rightarrow B$, then by (A1) and (R.A4) $\vdash_{BB} A \rightarrow A \& B$, so since $\vdash_{BB} A \& B \rightarrow A$, we have $\vdash_{BB} A \& B \leftrightarrow A$. Whence $\vdash_{BB} A \& B \leftrightarrow C$ iff $\vdash_{BB} A \leftrightarrow C$ for every wff C, and so $|A \& B| = |A|$, that is to say $|A| \cap |B| = |A|$ as required.

We continue our verification that the Lindenbaum algebra $\underset{\sim}{G}$ is a BB-algebra.

Distributivity follows from $\vdash_{BB} A \& (B \vee C) \leftrightarrow (A \& B) \vee (A \& C)$. ($\vdash_{BB} (A \& B) \vee (A \& C) \rightarrow A \& (B \vee C)$ is provable (as in E_{fde}) using (R.A4) and (R.A.7).) $--|A| = -|{\sim}A| = |{\sim}{\sim}A| = |A|$ since $\vdash_{BB} {\sim}{\sim}A \leftrightarrow A$ (by (A1) and (R5)).

$$|A| \leq -|B| \text{ iff } |A| \leq |{\sim}B| \text{ iff } \vdash_{BB} A \rightarrow{\sim} B \text{ iff} \vdash_{BB} B \rightarrow{\sim} A$$
$$\text{iff } |B| \leq |{\sim}A| \text{ iff } |B| \leq -|A| .$$

So $\langle G, \cap, \cup, - \rangle$ is a De Morgan lattice.

F is a filter, by the axioms (A2) $\vdash A \& B \rightarrow A$ and (A3) $\vdash A \& B \rightarrow B$, and the rule adjunction. Since $|A| \rightarrow |B| = |A \rightarrow B|$, $|A| \rightarrow |B| \in F$ iff $\vdash_{BB} A \rightarrow B$, which holds iff $|A| \leq |B|$, so (3) is satisfied.

Similarly (4) and (5) follow from the rules (R3) and (R4), and the fact that $|A| \leq |B|$ iff $\vdash_{BB} A \rightarrow B$.

This completes the proof that $\underset{\sim}{G}$ is a BB-algebra. We define the interpretation I by: I(A) = df |A| .

By the definition of the operations on G, I is an interpretation. But if $\nvdash_{BB}$ A then I(A) $\not\in$ *F* by the definition of F, thus A is false on I in G, and so is not BB-valid. So (b), and the theorem, is proved. □

§ 3.2. Extending the BB-algebras to extensions of BB

For the extensions B*L* of BB, we simply add the algebraic analogues of the extra axioms and rules to the definition of a BB-algebra, and we call the resulting structure a B*L*-algebra. The proof of the corresponding soundness and completeness theorem requires just verifying the extra cases, which will obviously carry through. I offer the following as a reason why we know it will work:

Digression on the relationship between L-algebras and the Lindenbaum algebra of L.

Given any logic L, define the Lindenbaum algebra as usual. Define L-algebras in the manner indicated for BB. Now each interpretation I is a homomorphism from the set of wffs to G, in the sense that I(A * B) = I(A) *' I(B) where *' is the corresponding algebraic operator to the connective * . Denote the Lindenbaum algebra by $|\underset{\sim}{G}| = \langle |G|, |F|, \cap, \cup, -, \to\rangle$. Now *H*: |G| => G where *H* (|A|) = I(A) (where I is an interpretation on $\underset{\sim}{G}$) is a homomorphism:

$$
\begin{aligned}
H(|A| *' |B|) &= H(|A * B|) \\
&= I(A * B) \\
&= I(A) *' I(B) \\
&= H(|A|) *' H(|B|), \text{ etc.}
\end{aligned}
$$

H is well-defined, since $|A| = |B|$ iff $\vdash_L A \leftrightarrow B$, which holds only if $I(A) \leq_G I(B)$ and $I(B) \leq_G I(A)$, i.e. $I(A) = I(B)$. (The L-algebras being manufactured to ensure that $\vdash_L A \rightarrow B$ entails that $I(A) \leq I(B)$.)

Definition 3.2.1. An interpretation I on $\underset{\sim}{G}$ is redundant if it is not onto $\underset{\sim}{G}$, otherwise irredundant.

Suppose that I on $\underset{\sim}{G}$ is irredundant, then $H : |G| \Rightarrow G$ is onto, so by the Homomorphism Theorem there exists an isomorphism $\mathcal{I}$ from $\underset{\sim}{G}$ onto the quotient lattice $\frac{|\underset{\sim}{G}|}{\delta}$, where δ is the kernel of H (Skornjakov 1977, p. 46). (So δ is the relation: $|A| \; \delta \; |B|$ iff $H(|A|) = H(|B|)$, i.e. iff $I(A) = I(B)$.)

$\mathcal{I} : \underset{\sim}{G} \Rightarrow \frac{|\underset{\sim}{G}|}{\delta}$ is defined: Where $I(A) = a \in G$, then $\mathcal{I}(a)$ equals the congruence class of $|A|$ under δ .

Since the L-algebras are defined to ensure that $\vdash_L A$ only if $I(A) \in F$, for all I on all $\underset{\sim}{G}$, every element of $|F|$ has an image in F under H. So under the isomorphism $\mathcal{I} : \underset{\sim}{G} \Rightarrow \frac{|\underset{\sim}{G}|}{\delta}$, the image of F includes $\frac{|F|}{\delta}$, i.e. $\frac{|F|}{\delta} \subseteq \mathcal{I}\,(F)$.

This shows that for every pair <I, $\underset{\sim}{G}$>, where I is an irredundant interpretation on the L-algebra $\underset{\sim}{G}$, there is an isomorphism $\mathcal{I}$ to a quotient lattice of the Lindenbaum algebra $\frac{|\underset{\sim}{G}|}{\delta_I}$, where δ_I is the congruence relation : $|A|\delta_I|B|$ iff $I(A) = I(B)$. Furthermore $\frac{|F|}{\delta_I} \subseteq \mathcal{I}(F)$.

The converse also holds. For suppose we have some quotient lattice $\frac{|\underset{\sim}{G}|}{\delta}$ for some congruence relation δ (i.e. a relation such that if $|A|\delta|B|$ and $|C|\delta|D|$, then $(|A| \cup |C|) \; \delta \; (|B| \cup |D|)$ and $(|A| \cap |C|) \; \delta \; (|B| \cap |D|)$.), then we can define interpretation I_δ as follows:

$I_\delta(A)$ is equal to the congruence class of $|A|$ under δ. This gives $\langle I_\delta, \frac{|\underset{\sim}{G}|}{\delta}\rangle$ where I_δ is an irredundant interpretation on the L-algebra $\frac{|G|}{\delta}$. (We define the filter of $\frac{|\underset{\sim}{G}|}{\delta}$ to be just $\frac{|F|}{\delta}$, i.e. the set of congruence classes of $|A|$ where $\vdash_L A$.)

Hence the class of $\langle I, \underset{\sim}{G}\rangle$ where I is irredundant, can just be regarded as the class of $\langle I_\delta, \frac{|\underset{\sim}{G}|}{\delta}\rangle$ where it is supposed that the filter F of $\frac{|\underset{\sim}{G}|}{\delta}$ includes $\frac{|F|}{\delta}$. So the algebraic models are just quotient lattices of the Lindenbaum algebra.

Obviously, L is sound and complete w.r.t. the $\langle I_\delta, \frac{|\underset{\sim}{G}|}{\delta}\rangle$. Soundness follows from the fact that $\frac{|F|}{\delta} \subseteq F$ and completeness from the fact that the Lindenbaum algebra itself, with the usual interpretation , is a member of this class (just use the identity relation on $|\underset{\sim}{G}|$, for δ).

So L is sound and complete w.r.t. the class of $\langle I, \underset{\sim}{G}\rangle$ where I is irredundant on $\underset{\sim}{G}$. Now suppose we have some I redundant on some $\underset{\sim}{G}$, then we can define a new L-algebra $\underset{\sim}{G}'$ by putting $G' = \{I(A) : A \text{ is a wff}\}$ and using the operations of $\underset{\sim}{G}$ restricted to G', and putting $F' = G' \cap F$. This will be well-defined because if the application of the operations eventually took us out of G', then the corresponding wff A (built up using the corresponding connectives) would have $I(A) \not\in G'$, contradicting the definition of G'. Clearly the elements in G - G' don't figure, under this I, in considerations of soundness and completeness - all we need look at is $\langle I, \underset{\sim}{G}'\rangle$ where I is irredundant on $\underset{\sim}{G}'$.

Therefore, L is sound and complete w.r.t. all interpretations I on all L-algebras $\underset{\sim}{G}$; given that these are correctly defined, so that the class of $\langle I, \underset{\sim}{G}\rangle$ can be regarded as the class of $\langle I_\delta, \frac{|\underset{\sim}{G}|}{\delta}\rangle$.

Now suppose that L satisfies the above proviso, and consider extensions L' of L. Suppose that L'-algebras are defined just by adding algebraic analogues of the new axioms and rules to the definition of L-algebras. But the class of $<I'_\delta, \frac{|\underset{\sim}{G}|'}{\delta}>$ for L' will just be that subset of $<I_\delta, \frac{|\underset{\sim}{G}|}{\delta}>$ for L which satisfies the algebraic analogues of the extra rules and axioms. Hence the correspondence will be maintained – the class of $<I', \underset{\sim}{G}'>$ for L' can be regarded as the class of $<I'_\delta, \frac{|\underset{\sim}{G}|'}{\delta}>$ for L'. And so L' will also be sound and complete w.r.t. all interpretations I' on all L'-algebras $\underset{\sim}{G}'$.

BB satisfies the above proviso. (Examination of the argument p. 48 – 50 shows that BB-algebras have the required properties for it to go through.) Hence extensions of BB are sound and complete w.r.t. the subclass of BB-algebras which satisfy the algebraic analogues of the extra rules and axioms. <u>End of digression</u>.

After the long digression, here is the short proof. For extra axioms, simply add the corresponding algebraic condition on *F* and the corresponding operators. For example, on adding $\vdash A \rightarrow B \dot{\rightarrow} C \rightarrow A \dot{\rightarrow} C \rightarrow B$, simply require that: $a \rightarrow b \dot{\rightarrow} c \rightarrow a \dot{\rightarrow} c \rightarrow b \in F$ for all a, b, c $\in$ G, similarly, for $\vdash A \vee \sim A$, require that $a \cup - a \in F$ for all $a \in G$, etc. For extra rules, do the same. For example, on adding 'if $\vdash$ A then $\vdash (A \rightarrow B) \rightarrow B$', simply require that: 'if $a \in F$ then $(a \rightarrow b) \rightarrow b \in F$ for all a, b $\in$ G'. Clearly this ensures soundness, and since the corresponding axioms and rules are in *BL*, the Lindenbaum algebra will remain a *BL*-algebra, and so completeness holds. Let us state the result as a theorem:

Theorem 3.2.2. Extensions of BB, B*L*, are sound and complete w.r.t. the subclass of BB-algebras which satisfies the algebraic analogues of the extra rules and axioms of B*L*, i.e. w.r.t. the B*L*-algebras.[1]

Our algebraic models are, of course, blatant copies of their corresponding logics. However, they do reassure us of one fact. Namely, that as for the standard relevant logics, the extensional connectives behave very classically in the B*L* systems - the algebras are De Morgan lattices, after all. Thus, one should be cautious about rejecting B-ed logics on the grounds that they have "funny" conjunction and disjunction. This is an illusion which might follow from an over-emphasis on the simplicity of the relational semantics for B compared to that for BB. But from the present algebraic point of view and also the previous proof-theoretic point of view as exemplified in the L-Hierarchy and associated WE formulation discussed in the previous chapter (definition 2.1.12), conjunction and disjunction have their ordinary extensional sense in the B-ed logics. The differences are in the interplay between the intensional and extensional connectives. Moreover, this interplay in the case of the B*L* systems is very natural, being an intuitive extension of that for the Tautological Entailments.

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1. Obviously, the theorem holds for all extensions of BB, whether-or-not they satisfy the B*L* requirements (see definition 2.5.2).

§ 3.3. Some algebraic payoffs

In this section we reap some further algebraic dividends. As promised earlier I will prove that BE $\neq$ E. To begin with, let's consider T-W. T-W is B plus prefixing, suffixing, contraposition and reductio axioms (RLR, p.289). So we add these to BB, and the corresponding algebraic requirements to the definition of a BB-algebra:

Definition 3.3.1.

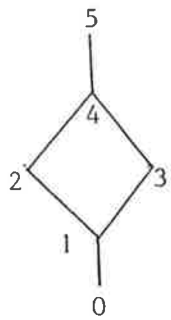
A B(T-W)-algebra is a BB-algebra, plus:

For all a, b, c $\in$ G, (6) $a \to b \leqq c \to a \to. c \to b$,

(7) $a \to b \leqq b \to c \to. a \to c$, (8) $a \to -b \leqq b \to -a$,

(9) $a \to -a \leqq -a$.

The following crystal lattice is a B(T-W)-algebra (and an example of a 6-valued logic).



$\to$	0	1	2	3	*4	*5
0	4	4	4	4	4	4
1	1	4	4	4	4	4
2	1	1	4	1	4	4
3	1	1	1	4	4	4
*4	0	1	1	1	4	4
*5	0	0	1	1	1	4

x	-x
0	5
1	4
2	3
3	2
4	1
5	0

$F = \{4,5\}$

$\cap$ and $\cup$ are represented in the usual way in the Hasse diagram.

TESTER verified that (6) prefixing and (7) suffixing are satisfied.[1]

---

1. Many thanks to Igor Urbas, who introduced me to TESTER, and to its creators - Anderson, Belnap and Dale Isner.

(8) follows from symmetry about the left-bottom to right-top diagonal of the $\rightarrow$ table. (9) corresponds to every element in the left-bottom to right-top diagonal being less than or equal to the column-name. The requirements (1) to (5) (Definition 3.1.1) are trivial to verify.

Theorem 3.3.2.

$\nvdash_{B(T-W)} \quad A \rightarrow B \;\&\; A \rightarrow C \;\dot{\rightarrow}\; A \rightarrow B \;\&\; C$ and

$\nvdash_{B(T-W)} \quad A \rightarrow C \;\&\; B \rightarrow C \;\dot{\rightarrow}\; A \vee B \rightarrow C$.

Proof. The above wff is not B(T-W)-valid, as the assignment A = 5, B = 2, C = 3 in the above algebra gives:

$$((5 \rightarrow 2) \cap (5 \rightarrow 3)) \rightarrow (5 \rightarrow (2 \cap 3))$$

$$= 1 \cap 1 \rightarrow 0$$

$$= 1 \rightarrow 0 \quad \notin F$$

$A \rightarrow C \;\&\; B \rightarrow C \;\dot{\rightarrow}\; A \vee B \rightarrow C$ can be falsified putting A = 3, B = 2 and C = 0 . □

Note that contraction $(A \;\dot{\rightarrow}\; A \rightarrow B \;\dot{\rightarrow}\; A \rightarrow B)$ is falsified because of $1 \rightarrow 0 = 1$, since then $1 \rightarrow (1 \rightarrow o) = 4$ but $4 \nleq 1$. Unfortunately it is necessary to move to a slightly more complex lattice in order to overcome this and accommodate $E_{\tilde{\rightarrow}}$. The strong relationships between the various elements comprising the counter-example have to be loosened. (Without TESTER's help, the diligent reader will be sorry to hear.) However, we are obviously already in a position to conclude that the distinction between B*L* and *L* for all sublogics *L* of T-W (of the requisite sort), holds good, including BB $\subsetneq$ B.

To obtain BE from B(T-W) we just need to add contraction and rule-assertion ($\vdash A \Rightarrow\ \vdash (A \to B) \to B$) (RLR, p. 290).

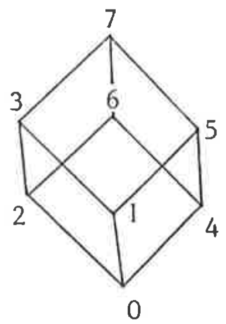
<u>Definition 3.3.3.</u> A BE-algebra satisfies all of the conditions for a B(T-W)-algebra as well as, for all a, b $\in$ G :-

(10) $a \to (a \to b) \leq a \to b$

(11) If $a \in F$ then $(a \to b) \to b \in F$.

<u>Definition 3.3.4.</u> Call the following algebra $\underline{BM_o}$.

x	0	1	2	3	4	5	6	7
-x	7	6	5	4	3	2	1	0



$F = \{3,7\}$

$\to$	0	1	2	*3	4	5	6	*7
0	3	3	3	3	3	3	3	3
1	0	3	0	3	0	3	0	3
2	0	1	3	3	1	1	3	3
*3	0	1	0	3	0	1	0	3
4	0	1	0	1	3	3	3	3
5	0	0	0	0	0	3	0	3
6	0	1	0	1	1	1	3	3
*7	0	0	0	0	0	0	0	3

Theorem 3.3.5.

BM_o is a BE-algebra:-

ad (2): F is clearly a filter .

ad (3): By inspection of the $\to$ table, $a \to b = 3$ iff $a \leqq b$.

ad (8): Symmetry about the left-bottom to right-top diagonal shows that $a \to - b = b \to - a$.

ad (9): $a \to - a \leqq - a$ since every element in that diagonal is less than or equal to its column-name.

ad (1): The lattice is distributive and $- - a = a$.
$a \leqq - b$ iff $b \leqq - a$ follows from (3) and (8) .
Thus the lattice is de Morgan .

ad (10): To show $a \underset{\cdot}{\to} a \to b \leqq a \to b$. By inspection, this fails only if $a \to b = 0$ or $a \to b = 1$.

If $a \to b = 1$, then by inspection $a \underset{\cdot}{\to} a \to b = 1$, and similarly if $a \to b = 0$, then $a \underset{\cdot}{\to} a \to b = 0$.

ad (11): By inspection, if x equals 3 or 7, $x \to b \leqq b$, so by (3) $x \to b \underset{\cdot}{\to} b = 3 \in F$ as required.

There remains (6) and (7), prefixing and suffixing, to prove.[1] We first note that the rule forms (4) and (5) hold: These (loosely) correspond to nonincreasing values as you move down the columns, and nondecreasing values as you move right along the rows. (Loosely because there is no need for this to hold in the case that the row values being compared

1. Thanks again to Igor Urbas, who confirmed, using TESTER, that BM_o satisfies (6) and (7).

are in rows corresponding to incomparable elements, and similarly for columns. e.g. $6 \to 1 = 1$ whilst $5 \to 1 = 0$ is O.K., since 5 and 6 are incomparable.) So now to (6) $a \to b \leqq c \to a \underset{\cdot}{\to} c \to b$ and (7) $a \to b \leqq b \to c \underset{\cdot}{\to} a \to c$:

If $a \to b = 3$ then $a \leqq b$ and so by (4) and (5) we have that $c \to a \leqq c \to b$ and $b \to c \leqq a \to c$ for every $c \in BM_o$. And so $c \to a \underset{\cdot}{\to} c \to b = 3$ and $b \to c \underset{\cdot}{\to} a \to c = 3$, by (3), thus in these cases (6) and (7) are satisfied. Clearly (6) and (7) are also satisfied if $a \to b = 0$, since 0 is the least element of BM_o. There remains the cases where $a \to b = 1$. Of these, the subcases where $c \to a \underset{\cdot}{\to} c \to b$ (or $b \to c \underset{\cdot}{\to} a \to c$) are equal to 1 or 3 satisfy (6) (or (7)), since 1 is less than or equal to these values. So we only need to look for cases where $a \to b = 1$ and $c \to a \underset{\cdot}{\to} c \to b = 0$ (or $b \to c \underset{\cdot}{\to} a \to c = 0$).

Now $d \to e = 0$ only if either (i) $d = 3$ and $e = o$
or (ii) $d = 1$ and $e = 0$.

<u>Cases ($a \to b = 1$)</u> :-

$2 \to 1$ (a) Check the 2-column for any $c \to 2$ equal to 3 or 1. Note that for every such c, $c \to 1 \neq 0$ (just look across at the corresponding locations in the 1-column). Thus prefixing is satisfied in this case.

(b) Check the 1-row for any $1 \to c$ equal to 3 or 1. Note that for every such c, $2 \to c \neq 0$ (just look down at the corresponding location in the 2-row). Thus suffixing is satisfied in this case.

Repeat for $2 \to 4$, $2 \to 5$, $3 \to 1$, $3 \to 5$, $4 \to 1$, $4 \to 3$, $6 \to 1$, $6 \to 3$, $6 \to 4$ and $6 \to 5$.

This completes the proof of (6) and (7), and proof of the fact that BM_o is a BE-algebra. □

Theorem 3.3.6. (A4) and (A7) are not theorems of BE, that is $\nvdash_{BE} A \to B \;\&\; A \to C \mathrel{\dot{\to}} A \to B \;\&\; C$ and $\nvdash_{BE} A \to C \;\&\; B \to C \mathrel{\dot{\to}} A \vee B \to C$.

Proof. Assign A = 4, B = 3 and C = 6 in BM_o :

$$(4 \to 3) \cap (4 \to 6) \mathrel{\dot{\to}} 4 \to (3 \cap 6) = (1 \cap 3) \to (4 \to 2)$$
$$= 1 \to 0$$
$$= 0 \quad \notin F$$

So (A4) is not BE-valid and hence by theorem 3.2.2 is not a theorem of BE. For (A7) we simply need the dual assignments to the above (since we have contraposition, $((a \to b) \cap (a \to c)) \to (a \to (b \cap c)) = ((-b \to -a) \cap (-c \to -a)) \to ((-b \cup -c) \to -a))$.

Thus putting A = 4, B = 1 and C = 3 falsifies (A7) in BM_o. □

Corollary 3.3.7. If $BL \subseteq BE$, then $BL \neq L$, where L is an extension of B (including the trivial case where L = B).

Proof. BM_o validates all of the theorems of any logic BL such that $BL \subseteq$ BE. Hence (A4) and (A7) are not theorems of such BL, by the theorem 3.3.6. But (A4) and (A7) are theorems of B and hence of any extension thereof. □

This confirms that the diagram depicting the relationship between some of the B-ed and un B-ed systems in chapter 2 is correct.

We also have the following slightly stronger corollary, which applies to <u>all</u> sublogics of E.

<u>Corollary 3.3.8</u>. For any $L \subseteq$ E, the distinction between the rule forms and axiom forms of

$$A \to B \;\&\; A \to C \underset{\cdot}{\to} A \to B \;\&\; C \quad \text{and}$$

$$A \to C \;\&\; B \to C \underset{\cdot}{\to} A \vee B \to C$$

can be maintained.

So what is it that distinguishes E from R, or more particularly $E_{\overset{\sim}{\to}}$ from $R_{\overset{\sim}{\to}}$, such that the latter is strong enough to facilitate the derivation of (A4) and (A7) from the rules (R.A4) and (R.A7), whereas the former is not ? Well, obviously it is unrestricted permutation. Unrestricted rule-permutation is what is used in the proof of theorem 2.6.2 - all else used to prove that $\vdash$ (A7) is available in BB. In fact $R_{\overset{\sim}{\to}}$ is just $E_{\overset{\sim}{\to}}$ plus unrestricted permutation (ENT, p. 144).

(A4) and (A7) arise from a desire for deducibility for the system to correspond with entailment within the system. So, good first-degree intuitions resulting in (R.A4) and (R.A7) are shunted to (A4) and (A7) by the above desire. We ought to be suspicious of how the above desire will, or should, cash out in practice, given the somewhat complex nature of deducibility in the relevant systems. Recall that when a relevant deduction theorem is available (using fusion) then in fact (A4) and (A7) can be proved from (R.A4) and (R.A7) (contraposition is also needed for (A7))(§3.1). So in a system where deduction behaves somewhat normally, (R.A4) and (R.A7) are all that is needed. R is the first logic, in its chain from B, where a fusion operater arises "naturally" and again, (R.A4) and (R.A7) are all that is needed.

Note that, if our deduction theorem is even more "normal" then the fusion operator is commutative, i.e. $A \circ B \vdash C$ iff $B \circ A \vdash C$; which, given that $A \circ B \vdash C$ iff $\vdash A \div B \rightarrow C$, results in the intensional component of the logic satisfying unrestricted rule-permutation. Hence again (A4) and (A7) follow from just (R.A4) and (R.A7) (theorem 2.6.2).

In contexts where it appears that our intuitions about deducibility might apply, (A4) and (A7) follow from (R.A4) and (R.A7), anyway.

The Relevant deduction theorem holds for B and its extensions <u>only</u> because (A4) and (A7) are axioms of B. Otherwise, as the argument of §3.1 shows, a fusion operator cannot be conservatively added to B and its extensions. So, in effect, including (A4) and (A7) in B is just a tool for forcing through the deduction theorem.

Rather than distort the extensional/intensional interplay away from good first degree intuitions in order to force through a deduction theorem, might it not be more perspicuous (and honest) to let it come when it comes naturally, out of the intensional part of the logic. For surely, the rightful guarantor of a deduction theorem <u>is</u> the intensional component of the logic.

CHAPTER 4

Decidability, primeness, consistency and γ

§ 4.0 Introduction

In this chapter we will reap some more algebraic dividends. First, I show that BB is decidable, although the proof will not deliver an efficient decision procedure, and I shall comment on this later. I will also show that a large class of extensions of BB are also decidable, and discuss where the general type of proof breaks down. In section §4.2 I will prove that BB and some of its extensions are both prime and negation-consistent, and hence γ holds forthem (if $\vdash \sim A$ and $\vdash A \vee B$ then $\vdash B$). That is to say, modus ponens for material implication, $\supset$E, is an admissible rule of these logics. I will indicate why γ is an important requirement for logics to satisfy.

§ 4.1 BB is decidable

We prove that BB is dedicable by the standard method of showing that BB satisfies the Finite Model Property; namely, that any wff which is not a theorem of BB is falsified by a finite BB-algebra, the number of elements of which having an upper bound determined by the number of subwffs in the wff. That BB is decidable then follows from the fact that there is a calculable finite number of BB-algebras with a fixed upper bound of elements, and just a finite number of possible interpretation functions I from the subwffs of the wff in question into each.[1] Hence the wff in question will, in a determined finite number of steps, either be shown to be valid in all such BB-algebras, and so, by the

1. In fact we do not need to display an upper bound, since the theorems are recursively enumerable.

Finite Model Property, to be a theorem, or be shown to be invalid, and so not a theorem. That BB is decidable is not surprising, since its close relative B is also decidable (RLR, p. 401).

Theorem 4.1.1. BB has the Finite Model Property.

Proof. Suppose that $\nvdash_{BB} A$, then by theorem 3.1.4 there is a BB-algebra $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \rightarrow \rangle$ and an interpretation I on $\underset{\sim}{G}$ such that $I(A) \notin F$. We generate a finite BB-algebra from $\underset{\sim}{G}$, which also falsifies A, as follows.

Let e be any element of F. Let $\overline{G}$ be the set of elements in the smallest sublattice of $\langle G, \cap, \cup, - \rangle$ which contains $\{I(B) : B$ is a subwff of $A\} \cup \{e\}$. We proceed to construct a finite BB-algebra $\underset{\sim}{\overline{G}} = \langle \overline{G}, \overline{F}, \cap, \cup, -, \Rightarrow \rangle$. The operations $\cap, \cup, -$ on $\underset{\sim}{\overline{G}}$ are just the restrictions of those of $\underset{\sim}{G}$ to $\overline{G}$.

Note that $\overline{G}$ is finite, since it is a finitely generated (De Morgan) lattice. Reason: Every element in $\langle G, \cap, \cup, - \rangle$ is identical with its disjunctive normal form, because a De Morgan lattice satisfies the required identities - commutation, association, distribution, double negation and De Morgan's laws. Suppose that A has m subwffs, then the maximum possible number of atoms needed to generate $\overline{G}$ equals 2 (m + 1) (including negations), since there can be at most m propositional variables in A and $\overline{G}$ includes e. Hence the maximum possible n$\overline{o}$ of different disjuncts, $(x_1^i \cap x_2^i \cap \ldots\ldots \cap x_{r_i}^i)$, is

$$K = \sum_{r=o}^{2(m+1)} \binom{2(m+1)}{r} = \sum_{r=o}^{2(m+1)} \frac{2(m+1)\,!}{r!\,(2(m+1)-r)\,!} = 2^{2(m+1)} - 1$$

(= total number of possible subsets of the set of atoms, excluding the empty set) and so the maximum possible no of d.n.f.s of elements in $\overline{G}$

$(x_1^1 \cap x_2^1 \cap \ldots\ldots \cap x_{r_1}^1) \cup (x_1^2 \cap \ldots\ldots \cap x_{r_2}^2) \cup \ldots\ldots \cup$

$(x_1^n \cap \ldots\ldots \cap x_{r_n}^n)$, is

$$N = \sum_{n=o}^{K} \binom{K}{n} = 2^K - 1 = 2^{(2^{2(m+1)}-1)} - 1 .$$

N is the maximum possible size of $\overline{G}$.

Put $\overline{F} = \overline{G} \cap F$ (set-theoretic intersection) and $1 =_{df}$ g.l.b. of $\overline{F}$ (1 is just the finite intersection of all of the elements of $\overline{F}$).

All that remains is to define the arrow for $\overline{G}$:

For all x, y $\in \overline{G}$,

$$x \Rightarrow y =_{df} \begin{cases} (\text{l.u.b. } \{\alpha \in \overline{G} : \alpha \leqq x \to y\}) \cup 1 & \text{if } x \to y \in F, \\ \text{l.u.b. } \{\alpha \in \overline{G} : \alpha \leqq x \to y\} & \text{otherwise.} \end{cases}$$

where we suppose that l.u.b. { } = g.l.b. of $\overline{G}$.

We needed to define a new arrow on $\overline{G}$, because a lattice generated by a finite number of elements of G which is also required to be closed under $\to$ on G may not be finite. Since as we shall shortly prove $\overline{F}$ is a filter, $1 \in \overline{F}$. So the above definition ensures that if $x \to y \in F$, then $x \Rightarrow y \in \overline{F}$. ($\overline{F}$ and 1 are well-defined, because e ensures that $\overline{F}$ is nonempty, which is all e is needed for - the process is effective as we could put $e = I(A) \to I(A)$.)

We now prove that our defined $\underset{\sim}{\overline{G}} = < \overline{G}, \overline{F}, \cap, \cup, -, \Rightarrow >$ is a BB-algebra.

ad (1): Clearly $< \overline{G}, \cap, \cup, - >$ is a De Morgan lattice, for any counter-example would also contradict the fact that $< G, \cap, \cup, - >$ is a De Morgan lattice.

ad (2): $\overline{F}$ is a filter on $\overline{G}$: $a \in \overline{G} \cap F$ and $b \in \overline{G} \cap F$ iff $a \cap b \in \overline{G} \cap F$, by the definition of $\overline{G}$ and the fact that F is a filter on G.

ad (3) : (i) Suppose that $a \Rightarrow b \in \overline{F}$. Now if $\{\alpha \in \overline{G} : \alpha \leq a \rightarrow b\}$ is empty, then it must be that $a \rightarrow b \in F$ and $a \Rightarrow b = 1$ (since otherwise l.u.b. $\{a \rightarrow b\}^{\nabla}$[1] = g.l.b. $\overline{G}$, which is not an element of $\overline{F}$ as at least one element of $\overline{G}$, namely I(A), is not an element of F). But by (3) for $\underset{\sim}{G}$, we have $a \leq b$. The remaining case is where $\{a \rightarrow b\}^{\nabla}$ is nonempty. By our supposition $1 \leq a \Rightarrow b$. Now if $a \rightarrow b \in F$, then $a \leq b$ as required; otherwise $a \rightarrow b \notin F$ and $a \Rightarrow b \leq a \rightarrow b$ in $\underset{\sim}{G}$, but then $1 \leq a \rightarrow b$ and since $1 \in F$, $a \rightarrow b \in F$ contrary to our supposition.

ad (3) : (ii) Suppose that $a \leq b$ for $a, b \in \overline{G}$. Then by (3) for $\underset{\sim}{G}$ $a \rightarrow b \in F$. (The partial order on $\overline{G}$ is the restriction of that on G). Hence $a \Rightarrow b =$ l.u.b. $\{a \rightarrow b\}^{\nabla} \cup 1$, so $1 \leq a \Rightarrow b$, and since $\overline{F}$ is a filter, $a \Rightarrow b \in \overline{F}$.

---

1. I shall abbreviate $\{\alpha \in \overline{G} : \alpha \leq x\}$ by $\{x\}^{\nabla}$.

<u>ad (4)</u>: Suppose that a, b, c $\in \overline{G}$ and that a $\leqq$ b.

Then by (4) for $\underset{\sim}{G}$, $b \rightarrow c \leqq a \rightarrow c$.

Hence $\{b \rightarrow c\}^{\nabla} \subseteq \{a \rightarrow c\}^{\nabla}$ and so

l.u.b. $\{b \rightarrow c\}^{\nabla} \leqq$ l.u.b. $\{a \rightarrow c\}^{\nabla}$.

If $a \rightarrow c \notin F$, then $b \rightarrow c \notin F$ and $b \Rightarrow c \leqq a \Rightarrow c$.

So suppose that $a \rightarrow c \in F$.

If both $b \rightarrow c \in F$ and $a \rightarrow c \in F$, then

$$b \Rightarrow c = (\text{l.u.b. } \{b \rightarrow c\}^{\nabla}) \cup 1 \leqq (\text{l.u.b. } \{a \rightarrow c\}^{\nabla}) \cup 1 = a \Rightarrow c .$$

If $a \rightarrow c \in F$ but $b \rightarrow c \notin F$, then

$$b \Rightarrow c = \text{l.u.b. } \{b \rightarrow c\}^{\nabla} \leqq \text{l.u.b. } \{a \rightarrow c\}^{\nabla} \cup 1 = a \Rightarrow c.$$

This exhausts the possible cases.

<u>ad (5)</u>: Similarly.

This completes the proof that $\overline{\underset{\sim}{G}}$ is a BB-algebra.

We now define an interpretation I on $\overline{\underset{\sim}{G}}$ as follows:-

For each propositional variable p_i of A, $I(p_i) = \mathrm{I}(p_i)$.

For wfffs A and B, $I\,(A \,\&\, B) = I(A) \cap I(B)$

$I\,(A \vee B) = I(A) \cup I(B)$

$I\,(\sim A) = -\,I(A)$

$I\,(A \rightarrow B) = I(A) \Rightarrow I(B)$.

By the manner of its definition, I is an interpretation.

We now prove that $\mathcal{I}(A) = I(A)$, by induction on the complexity of A. Clearly this holds for the propositional variables of A. Suppose that subwffs B and C of A satisfy $\mathcal{I}(B) = I(B)$ and $\mathcal{I}(C) = I(C)$. Now for $D = B \mathbin{\&} C$, $D = B \vee C$ and $D = \sim B$, $I(D) \in \overline{G}$ by the definition of $\overline{G}$ and the fact that I is an interpretation. Also, for the latter reason, $\mathcal{I}(D) = I(D)$. Now $I(B \to C) = I(B) \to I(C)$, where $B \to C$ is a subwff of A, is in $\overline{G}$ by the definition of $\overline{G}$. Now $\mathcal{I}(B \to C) = \mathcal{I}(B) \Rightarrow \mathcal{I}(C)$

$$= I(B) \Rightarrow I(C)$$

By our induction assumption.

But clearly where $a \to b \in \overline{G}$, $a \Rightarrow b = a \to b$, since then l.u.b. $\{x \in \overline{G} : x \leqq a \to b\} = a \to b$; and if $a \to b \in F$, it is in $\overline{F}$, then $1 \leqq a \to b$, by the definition of 1, so $a \to b \cup 1 = a \to b$.

Thus $\mathcal{I}(B \to C) = I(B) \to I(C)$

$$= I(B \to C) \quad \text{as required.}$$

This completes the proof of the inductive step, hence we have the result : $\mathcal{I}(A) = I(A)$.

Now $I(A) \notin F$ by supposition, so $I(A) \notin \overline{F}$ (as $\overline{F} = \overline{G} \cap F$). So A is falsified by $\mathcal{I}$ on $\underset{\sim}{\overline{G}}$. □

<u>Theorem 4.1.2.</u> BB is decidable.

<u>Proof</u> Follows immediately from theorem 4.1.1. □

The theorem can be extended to cover a limited class of logics $BL \supseteq BB$. The only impediment is in the proof of the extra clauses needed to show that $\overline{\underset{\sim}{G}}$ is a BL-algebra. If triple or more nesting of arrows is involved, then $\Rightarrow$ is not sensitive enough to do the job. For example, prefixing cannot be added, for then we need $a \to b \underset{\cdot}{\to} c \to a \underset{\cdot}{\to} c \to b \in F$, i.e. $a \to b \leqq c \to a \underset{\cdot}{\to} c \to b$; and while it is the case that $a \Rightarrow b \leqq c \to a \Rightarrow c \to b$ (if $c \to a \in \overline{G}$ and $c \to b \in \overline{G}$), in general $a \Rightarrow b \leqq c \Rightarrow a \underset{\cdot}{\Rightarrow} c \Rightarrow b$ fails to hold. But where there are no nested arrows on either side of $\leqq$, the requirement carries over, for example $a \to b \leqq -b \to -a$ entails that $a \Rightarrow b \leqq -b \Rightarrow -a$. So the logics obtained from augmenting BB with one or more of reductio $(A \to \sim A \underset{\cdot}{\to} \sim A)$, contraposition $(A \to \sim B \underset{\cdot}{\to} B \to \sim A)$, and the rule $\vdash A \Rightarrow \vdash (A \to B) \to B$, are decidable. Hence we have the following theorem.

<u>Theorem 4.1.3.</u> Let L be any logic which can be specified by the schemata of BB plus axiom and rule schemata none of which are of degree greater than two. Then L is decidable.

<u>Proof</u> As indicated, the extra requirements for L-algebras follow because $\Rightarrow$ is order preserving on first degree sentences. That is, if $a \to b \leqq c \to d$ then $a \Rightarrow b \leqq c \Rightarrow d$. So theorem 4.1.1 carries over and L has the finite model property, and thus is decidable.

□

So BB can be augmented by many of the extensional and mixed rules and axioms listed in RLR, and theorem 4.1.1 - suitably modified - will carry through. However in most of these cases logics which don't have an L-Hierarchy, or which entertain nasty paradoxes of relevance, will ensue. In fact, that B is decidable follows from theorem 4.1.3. A fact which, as we have noted, is already known.

Another manner in which BB can be augmented and for which the result carries over, is where the sentential constant t is added to the language, and the following axioms to the formulation of BB giving BB^t:

$$\vdash t \quad \text{and} \quad \vdash t \underset{\cdot}{\rightarrow} A \rightarrow A \qquad \text{(RLR, p. 350)}$$

As usual, t can be regarded as the conjunction of all logical truths. For all entailments $A \rightarrow B$, if $\vdash_{BB^t} A \rightarrow B$ then $\vdash_{BB^t} A \rightarrow A \underset{\cdot}{\rightarrow} A \rightarrow B$ by rule-prefixing, and hence $\vdash_{BB^t} t \underset{\cdot}{\rightarrow} A \rightarrow B$. A swift proof follows, by induction on the length of a proof of a theorem in BB^t, of the fact that, if $\vdash_{BB^t} A$ then $\vdash_{BB^t} t \rightarrow A$. (All cases, except the rules modus ponens and adjunction, follow from the above fact, and the latter cases follow from rule-transitivity and (R.A4).) Now BB^t-algebras are just algebras $< G, F, 1, \cap, \cup, -, \rightarrow >$ defined as for BB-algebras with the additional requirement that 1 is the g.l.b. of F and that $1 \in F$. Interpretations on BB^t-algebras are just required to satisfy the extra condition that $I(t) = 1$. Obviously, BB^t is sound and complete w.r.t. the class of BB^t-algebras. And, the finite BB-algebras of theorem 4.1.1 are also BB^t-algebras. Hence we have the following corollaries:

<u>Corollary 4.1.4</u>. BB^t is decidable.

<u>Corollary 4.1.5</u>. BB^t is a conservation extension of BB.

<u>Proof</u> By theorem 4.1.1, any non-theorem of BB is falsified in a BB^t-algebra, hence is not a theorem of BB^t. □

Obviously these results can be extended in the manner indicated for BB:

<u>Corollary 4.1.6.</u> Let L^t be any logic which can be specified by the schemata of BB^t plus axiom and rule schemata none of which are of degree greater than two. Then L^t is decidable.

<u>Corollary 4.1.7</u> Let L be as in theorem 4.1.3 and let L^t be the corresponding extension of BB^t. Then L^t is a conservative extension of L.

As indicated earlier, the decision procedure is very inefficient. My feeling is that a more efficient decision procedure could be based on the L-Hierarchy of BB. Attempts along these lines in the case of BE have so far been unsuccessful.[1]

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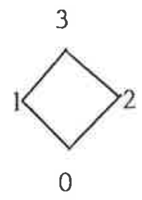
1. In fact I believed there was a proof that BE and BR are decidable, before it was shown (by Bob Meyer) that BR= R (Michael McRobbie pointed out that distribution was its downfall). Alasdair Urquhart has shown that T, E and R are undecidable (Urquhart, 198?), however inspection of his argument shows that it fails to apply when (A4) and (A7) are not theorems (as in the case of BE).

§ 4.2 Constructing prime and consistent algebras

In this section we will prove that BB and some of its extensions are both prime ($\vdash$ A v B iff either $\vdash$ A or $\vdash$ B) and negation - consistent (not both $\vdash$ A and $\vdash \sim$ A). We also indicate where the proof breaks down for others of its extensions. This will provide one route for proving the completeness of the relational semantics for BB (investigated in chapter 5).

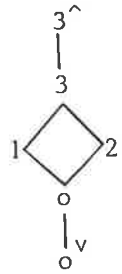
Our strategy is to show that given an interpretation I on a BB-algebra $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \rightarrow \rangle$ such that $I(A) \notin F$, we can construct a BB-algebra $\underset{\sim}{G}' = \langle G', F', \cap', \cup', -', \rightarrow' \rangle$ which is both prime and negation-consistent (i.e. if $a \cup' b \in F'$ then either $a \in F'$ or $b \in F'$ and for no $a \in G'$ is $a \in F'$ and $-' a \in F'$), and define an I' on $\underset{\sim}{G}'$ such that $I'(A) \notin F'$. The idea is to add a new filter F' on "top" of F, where F' is isomorphic to F. F' "leaves behind" any $a \vee b \in F$ such that both $a \notin F$ and $b \notin F$. An appropriate redefinition $\rightarrow'$ of the arrow ensures that the required intensional properties continue to hold. A simple example of the process is as follows:



$F = \{3\}$

Add $3^{\wedge}$ (and $-3^{\wedge} = 0^{\vee}$) and put $F' = \{3^{\wedge}\}$.

$1 \cup 2$ is no longer "true".



We begin by defining the prime extension $\underset{\sim}{G}'$ of $\underset{\sim}{G}$. But first we need the following definition.

<u>Definition 4.2.1</u> The <u>superscript algebra</u>, $\underset{\sim}{S} = \langle S, \cap, \cup, - \rangle$ consists of $S = \{\wedge, o, \vee\}$ and operations $\cap$ and $\cup$ determined in the usual way from the Hasse diagram, and $-$ as according to the table.

x	$\wedge$	o	$\vee$
$-x$	$\vee$	o	$\wedge$

In what follows o represents the absence of a superscript.

<u>Definition 4.2.2</u> <u>Prime extension</u>

Let $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \rightarrow \rangle$ be a BB-algebra. Then the <u>prime extension</u> of $\underset{\sim}{G}$, $\underset{\sim}{G}' = \langle G', F', \cap', \cup', -', \rightarrow' \rangle$, is defined as follows. Put $F' = \{a^{\wedge} : a \in F\}$. Each $a^{\wedge}$, for each $a \in F$, is a distinct new element. Put $-F' = \{a^{\vee} : -a \in F\}$. Ditto. Read $a^{\wedge}$ as "a-up" and $a^{\vee}$ as "a-down". Put $G' = G \cup F' \cup -F'$.

For all $a, b \in G$ and $\alpha, \beta \in S$:

$$a^{\alpha} \cap' b^{\beta} =_{df} (a \cap b)^{\alpha \cap \beta} \quad ;$$

$$a^{\alpha} \cup' b^{\beta} =_{df} (a \cup b)^{\alpha \cup \beta} \quad ;$$

$$-' a^{\alpha} = (-a)^{-\alpha} \quad ;$$

$$a^{\alpha} \rightarrow' b^{\beta} = \begin{cases} (a \rightarrow b)^{\wedge} \text{ if } a \leqq b \text{ and } \alpha \leqq \beta \text{ (i.e. if } a^{\alpha} \leqq' b^{\beta}) , \\ a \rightarrow b \quad \text{otherwise.} \end{cases}$$

<u>Lemma 4.2.3</u> For any BB-algebra $\underset{\sim}{G}$, the prime extension $\underset{\sim}{G}'$ is well-defined.

<u>Proof</u> We only need to check that the operations are well-defined. Consider $\cap'$: (i) $\alpha \cap \beta = \wedge$, then $\alpha = \wedge$ and $\beta = \wedge$, so both $a \in F$ and $b \in F$; thus, since F is a filter, $a \cap b \in F$ and $(a \cap b)^{\wedge} \in F$ as required. (ii) $\alpha \cap \beta = o$, then $(a \cap b)^{\alpha \cap \beta} = a \cap b$. (iii) $\alpha \cap \beta = v$, then either $\alpha = v$ or $\beta = v$. Suppose w.l.o.g. that $\alpha = v$, so $-a \in F$ and hence $-a \cup -b \in F$, since F is a filter; i.e. $-(a \cap b) \in F$, so that $(a \cap b)^{v} \in -F'$ as required.

The argument for $\cup'$ is similar to the above.

Consider $-'$: (i) $-\alpha = \wedge$, then $\alpha = v$, so $-a \in F$ and $(-a)^{\wedge} \in F'$. (ii) $-\alpha = 0$ then $-' a^{\alpha} = a$. (iii) $-\alpha = v$, then $\alpha = \wedge$, so $a \in F$ and $(-a)^{v} \in -F'$.

Consider $\rightarrow'$: $a^{\alpha} \rightarrow' b^{\beta} = (a \rightarrow b)^{\wedge}$ only if $a \leqq b$, i.e. only if $a \rightarrow b \in F$, so that $(a \rightarrow b)^{\wedge} \in F'$. Otherwise $a^{\alpha} \rightarrow' b^{\beta} = a \rightarrow b \in G$. □

<u>Lemma 4.2.4</u> For all $a, b \in G$ and $\alpha, \beta \in S$,
$a^{\alpha} \leqq' b^{\beta}$ iff $a \leqq b$ in $\underset{\sim}{G}$ and $\alpha \leqq \beta$ in $\underset{\sim}{S}$.

<u>Proof</u> By definition, $a^{\alpha} \leqq' b^{\beta}$ iff $a^{\alpha} \cap' b^{\beta} = a^{\alpha}$, which holds iff $(a \cap b)^{\alpha \cap \beta} = a^{\alpha}$, which holds iff $a \cap b = a$ and $\alpha \cap \beta = \alpha$, that is $a \leqq b$ and $\alpha \leqq \beta$. □

Lemma 4.2.5 For any BB-algebra, $\underset{\sim}{G}$, the prime extension $\underset{\sim}{G}'$ is also a BB-algebra.

Proof ad(1): $(G', \cap', \cup', -' \rangle$ is a lattice. For clearly $\leqq'$ is an order relation on G' (reflexive, antisymmetric and transitive), by lemma 4.2.4, so G' is a partially ordered set under $\leqq'$, and for all $x, y \in G'$, $x \cap' y$ and $x \cup' y$ exist. The lattice is distributive, since: For all $a, b, c \in G$ and $\alpha, \beta, \gamma \in S$,

$$\begin{aligned}(a^{\alpha} \cap' b^{\beta}) \cup' c^{\gamma} &= (a \cap b)^{\alpha \cap \beta} \cup' c^{\gamma} \\ &= ((a \cap b) \cup c)^{(\alpha \cap \beta) \cup \gamma} \\ &= ((a \cup c) \cap (b \cup c))^{(\alpha \cup \gamma) \cap (\beta \cup \gamma)}\end{aligned}$$

Since the two corresponding lattices are distributive. But,

$$\begin{aligned}\text{R.H.S.} &= ((a \cup c)^{\alpha \cup \gamma}) \cap' ((b \cup c)^{\beta \cup \gamma}) \\ &= (a^{\alpha} \cup' c^{\gamma}) \cap' (b^{\beta} \cup' c^{\gamma})\end{aligned}$$

and hence distribution holds.

Now for $a, b \in G$, $a^{\alpha} \leqq' -' b^{\beta}$ iff $a^{\alpha} \leqq' (-b)^{-\beta}$, which holds iff $a \leqq -b$ and $\alpha \leqq -\beta$ by lemma 4.2.4 (by the definition of $-'$, 4.2.2);
and hence by contraposition for $\underset{\sim}{G}$ and $\underset{\sim}{S}$, $b \leqq -a$ and $\beta \leqq -\alpha$, which holds iff $b^{\beta} \leqq' (-a)^{-\alpha}$, i.e. iff $b^{\beta} \leqq' -' a^{\alpha}$.

Also $-' -' a^{\alpha} = -' (-a)^{-\alpha} = (--a)^{--\alpha} = a^{\alpha}$, completing the proof that $\langle G', \cap', \cup', -' \rangle$ is a De Morgan lattice.

<u>ad (2)</u>: F' is a filter. We have $x \in F'$ and $y \in F'$ iff $x = a^{\wedge}$ and $y = b^{\wedge}$ where $a \in F$ and $b \in F$. Thus $a \cap b \in F$, since F is a filter, and so $(a \cap b)^{\wedge}$ exists and is an element of F', by the definition of F'. But $(a \cap b)^{\wedge} = a^{\wedge} \cap' b^{\wedge}$, so $x \cap' y \in F'$ as required.

<u>ad (3)</u>: $x \to' y \in F'$ iff $x \leqq' y$. For $a, b \in G$ and $\alpha, \beta \in S$, $a^{\alpha} \to' b^{\beta} \in F'$ iff $a \to b \in F$ and $\alpha \leqq \beta$, by the definition of $\to'$. But $a \to b \in F$ iff $a \leqq b$ in $\underset{\sim}{G}$, hence $a^{\alpha} \to' b^{\beta} \in F'$ iff $a \leqq b$ and $\alpha \leqq \beta$, which holds iff $a^{\alpha} \leqq' b^{\beta}$ by lemma 4.2.4.

<u>ad(4)</u>: Suffixing. For $a, b, c \in G$ and $\alpha, \beta, \gamma \in S$, suppose that $a^{\alpha} \leqq' b^{\beta}$. Then $a \leqq b$ in $\underset{\sim}{G}$ and $\alpha \leqq \beta$. So $b \to c \leqq a \to c$ in $\underset{\sim}{G}$. Now if $b^{\beta} \to' c^{\gamma} = (b \to c)^{\wedge}$ then both $b \to c \in F$ and $\beta \leqq \gamma$, hence $a \to c \in F$ and $\alpha \leqq \gamma$, so that $a^{\alpha} \to' c^{\gamma} = (a \to c)^{\wedge}$. And then $(b \to c)^{\wedge} \leqq' (a \to c)^{\wedge}$ by lemma 4.2.4, i.e. $b^{\beta} \to' c^{\gamma} \leqq' a^{\alpha} \to' c^{\gamma}$ as required. Alternatively, if $b^{\beta} \to' c^{\gamma} = b \to c$, then since $a^{\alpha} \to' c^{\gamma} \geqq' a \to c$ (it is either $a \to c$ or $(a \to c)^{\wedge}$), it follows that $b^{\beta} \to' c^{\gamma} \leqq' a^{\alpha} \to' c^{\gamma}$.

<u>ad (5)</u>: Prefixing. For $a, b, c \in G$ and $\alpha, \beta, \gamma \in S$, suppose that $a^{\alpha} \leqq' b^{\beta}$. As above $c \to a \leqq c \to b$ in $\underset{\sim}{G}$ and $\alpha \leqq \beta$. Now if $c^{\gamma} \to' a^{\alpha} = (c \to a)^{\wedge}$ then both $c \to a \in F$ and $\gamma \leqq \alpha$, hence $c \to b \in F$ and $\gamma \leqq \beta$, so that $c^{\gamma} \to' b^{\beta} = (c \to b)^{\wedge}$ whence, as above, by lemma 4.2.4 $c^{\gamma} \to' a^{\alpha} \leqq' c^{\gamma} \to' b^{\beta}$. Alternatively, if $c^{\gamma} \to' a^{\alpha} = c \to a$, then, as above, $c^{\gamma} \to' a^{\alpha} \leqq' c^{\gamma} \to' b^{\beta}$.

This completes the proof that $\underset{\sim}{G}$ is a BB-algebra. □

Lemma 4.2.6. For any BB-algebra $\underset{\sim}{G}$, its prime extension $\underset{\sim}{G}'$ is both prime and negation consistent. That is:

(i) If $x \cup' y \in F'$ then either $x \in F'$ or $y \in F'$.

(ii) For no $x \in G'$ do we have $x \in F'$ and $-' x \in F'$.

Proof ad(i): Suppose that $a^{\alpha} \cup' b^{\beta} \in F'$, then by the definition of $\cup'$ (4.2.2), it must be that $a^{\alpha} \cup' b^{\beta} = (a \cup b)^{\wedge}$ and at least one of α, β is equal to $\wedge$. But then either $a^{\wedge}$ exists and is an element of F', or $b^{\wedge}$ exists and is an element of F' (or both).

ad(ii): Suppose that $x \in F'$, then $x = a^{\wedge}$ where $a \in F$. Now $-' a^{\wedge} = (-a)^{\vee}$, and clearly $(-a)^{\vee} \notin F'$ because $F' = \{b^{\wedge} : b \in F\}$; that is, $-' x \notin F'$. □

Theorem 4.2.7. Suppose that we have a BB-algebra $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \to \rangle$. Then there is a BB-algebra $\underset{\sim}{G}' = \langle G', F', \cap', \cup', -', \to' \rangle$ such that:

(i) F' is a prime filter;

(ii) for no $x \in G'$, do we have $x \in F'$ and $-' x \in F'$;

(iii) for every interpretation I on $\underset{\sim}{G}$ there is an interpretation I' on $\underset{\sim}{G}'$, such that for all wffs A, if $I(A) \notin F$ then $I'(A) \notin F'$.

Proof We set $\underset{\sim}{G}'$ as the prime extension of $\underset{\sim}{G}$ (definition 4.2.2). That $\underset{\sim}{G}'$ is a BB-algebra follows from lemmas 4.2.3 and 4.2.5. (i) and (ii) follow immediately from lemma 4.2.6. It only remains to prove (iii).

Let I be any interpretation on $\underset{\sim}{G}$. We define I' on $\underset{\sim}{G}'$ as follows:

For all propositional variables p, $I'(p) = I(p)$, and, for all wffs A and B:

$$I'(A \,\&\, B) = I'(A) \cap' I'(B) \quad ;$$

$$I'(A \vee B) = I'(A) \cup' I'(B) \quad ;$$

$$I'(\sim A) = -' I'(A) \quad ;$$

$$I'(A \rightarrow B) = I'(A) \rightarrow' I'(B) \quad .$$

We now prove that, for all wffs A, I'(A) is equal to either I(A), $I(A)^{\wedge}$ (and $I(A) \in F$), or $I(A)^{\vee}$ (and $-I(A) \in F$). The proof is by induction on the complexity of A. Clearly the result holds for propositional variables. For our induction assumption, we assume that the result holds for wffs A and B.

(i) $I'(A \,\&\, B) = I'(A) \cap' I'(B)$

$= I(A)^{\alpha} \cap' I(B)^{\beta}$ where $\alpha, \beta \in S$,

by the induction assumption. However,

R.H.S. $= (I(A) \cap I(B))^{\alpha \cap \beta}$

$= I(A \,\&\, B)^{\alpha \cap \beta}$ as required.

(ii) $I'(A \vee B) = I'(A) \cup' I'(B)$

$= I(A)^{\alpha} \cup' I(B)^{\beta}$ by assumption.

$= (I(A) \cup I(B))^{\alpha \cup \beta}$

$= I(A \vee B)^{\alpha \cup \beta}$

(iii) $I'(\sim A) = -' I'(A)$

$= -' I(A)^{\alpha}$ by assumption.

$= (- I(A))^{-\alpha}$

$= I(\sim A)^{-\alpha}$

(iv) $I'(A \rightarrow B) = I'(A) \rightarrow' I'(B)$

$= I(A)^{\alpha} \rightarrow' I(B)^{\beta}$ by assumption

$$= \begin{cases} (I(A) \rightarrow I(B))^{\wedge} & \text{if } I(A) \leqq I(B) \text{ and } \alpha \leqq \beta, \\ I(A) \rightarrow (B) & \text{otherwise.} \end{cases}$$

$$\text{So } I'(A \rightarrow B) = \begin{cases} I(A \rightarrow B)^{\wedge} & \text{if } I(A) \leqq I(B) \text{ and } \alpha \leqq \beta, \\ I(A \rightarrow B) & \text{otherwise.} \end{cases}$$

This completes the proof of the induction step, hence the result holds for all wffs A.

So now suppose that $I(A) \notin F$ for some wff A. But in such a case $I(A)^{\wedge}$ simply doesn't exist, by the definition of G' and F' and $-F'$. So it cannot be that $I'(A) = a^{\wedge} \in F'$, for any $a^{\wedge} \in F'$ $(a \in G)$, since F' consists only of "up" elements. Hence either $I'(A) = I(A)$ or $I'(A) = I(A)^{\vee}$, and in either case $I'(A) \notin F'$. □

We are now in a position to prove four further significant properties of BB.

<u>Theorem 4.2.8.</u> BB is both prime and negation-consistent.

<u>Proof</u> We simply apply the above process to the Lindenbaum algebra. For all wffs A, let $|A| = \{B : \vdash_{BB} A \leftrightarrow B\}$, $G = \{|A| : A \text{ is a wff}\}$, $F = \{|A| : \vdash_{BB} A\}$ and define the operations as follows:

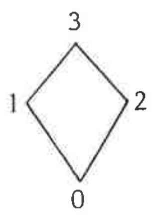
$|A| \cap |B| = |A \,\&\, B|$, $|A| \cup |B| = |A \vee B|$, $-|A| = |{\sim}A|$, $|A| \rightarrow |B| = |A{\rightarrow}B|$. Standardly, $\underset{\sim}{G}$ is a BB-algebra (as proved in theorem 3.1.4). Applying theorem 4.2.7 we have a BB-algebra $\underset{\sim}{G}' = \langle G', F', \cap', \cup', -', \rightarrow' \rangle$ where F' is both prime and negation-consistent. Using the canonical interpretation ($I(A) = |A|$) we have that for all non-theorems A, $I(A) \notin F$. Using the corresponding interpretation I' on $\underset{\sim}{G}'$ (as given by theorem 4.2.7) we also have that if $\nvdash_{BB} A$ then $I'(A) \notin F'$. Now suppose that $\vdash_{BB} A \vee B$ and that $\nvdash_{BB} A$ and $\nvdash_{BB} B$. Then $I'(A \vee B) \in F'$ because BB is sound w.r.t. all BB-algebras, and both $I'(A) \notin F'$ and $I'(B) \notin F'$. But $I'(A \vee B) = I'(A) \cup' I'(B)$ and the above supposition entails that F' is not prime, hence by reductio there is no such theorem of BB.

Similarly, if both $\vdash_{BB} A$ and $\vdash_{BB} {\sim} A$, then we have $I'(A) \in F'$ and $I'({\sim}A) \in F'$, i.e. $I'(A) \in F'$ and $-' I'(A) \in F'$, contrary to the fact that F' is negation-consistent. □

Theorem 4.2.9. BB satisfies γ; i.e. if $\vdash_{BB} {\sim} A$ and $\vdash_{BB} A \vee B$, then $\vdash_{BB} B$.

Proof This follows immediately from theorem 4.2.8 - the class of theorems of BB is both prime and negation-consistent. □

The recipe of definition 4.2.2, for generating prime and negation-consistent BB-algebras, is of limited application because the new arrow →' is not very sensitive to arrow properties of the original algebra. Hence, the γ proof doesn't extend very far to bigger logics (which is not surprising since primeness doesn't either). For example, consider the following BB-algebra which also satisfies reductio ($A \rightarrow {\sim} A \,\dot{\rightarrow}\, {\sim} A$):

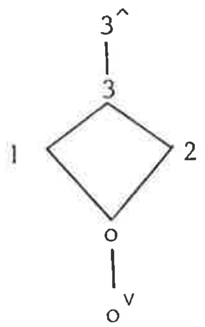


→	0	1	2	3*
0	3	3	3	3
1	2	3	2	3
2	1	1	3	3
* 3	0	1	2	3

$F = \{3\}$

a	0	1	2	3
−a	3	2	1	0

Applying theorem 4.2.7 we get:



And $o \to' - o = o \to' 3 = 3^{\wedge}$,

so $o \to' - o \not\leq' - o$, since $3^{\wedge} \not\leq' 3$.

Thus reductio fails in the corresponding prime extension.

The procedure also fails to preserve, in general, rule-contraction and rule-permutation. However, the recipe can be extended to cater for the axioms prefixing, suffixing and contraposition.

Theorem 4.2.10. Where *BL* is BB augmented by one or more of

$\vdash A \to B \underset{\cdot}{\to} C \to A \underset{\cdot}{\to} C \to B$, $\vdash A \to B \underset{\cdot}{\to} B \to C \underset{\cdot}{\to} A \to C$ and

$\vdash A \to \sim B \underset{\cdot}{\to} B \to \sim A$,

then for every *BL*-algebra $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \to \rangle$ there is a *BL*-algebra $\underset{\sim}{G}' = \langle G', F', \cap', \cup', -', \to' \rangle$ such that (i), (ii) and (iii) (of theorem 4.2.7) hold.

Proof The only amendment we need make to the proof of theorem 4.2.7 is to check that if the following properties hold in $\underset{\sim}{G}$, then they also hold in $\underset{\sim}{G}'$ (i.e. that $\underset{\sim}{G}'$ remains a *BL*-algebra).

(4') $a \to b \leqq b \to c \underset{\cdot}{\to} a \to c$

(5') $a \to b \leqq c \to a \underset{\cdot}{\to} c \to b$

(6) $a \to - b \leqq b \to - a$

ad (4'): Suppose that $a, b, c \in G$ and $\alpha, \beta, \gamma \in S$, and that $a \to b \leqq b \to c \underset{\cdot}{\to} a \to c$ in $\underset{\sim}{G}$.

Consider $a^{\alpha} \to' b^{\beta}$ and $b^{\beta} \to' c^{\gamma} \underset{\cdot}{\to}' a^{\alpha} \to' c^{\alpha}$.

Now $$a^{\alpha} \to' b^{\beta} = \begin{cases} (a \to b)^{\wedge} & \text{if } a^{\alpha} \leqq' b^{\beta}, \\ a \to b & \text{otherwise.} \end{cases}$$

Suppose first that $a^{\alpha} \leqq' b^{\beta}$, i.e. $a \leqq b$ in $\underset{\sim}{G}$ and $\alpha \leqq \beta$ in $\underset{\sim}{S}$. Then $b \to c \leqq a \to c$ in $\underset{\sim}{G}$. Now the possibilities for $b^{\beta} \to' c^{\gamma} \underset{\cdot}{\to}' a^{\alpha} \to' c^{\gamma}$ are:

(1) $(b \to c) \to' (a \to c)$ (2) $(b \to c)^{\wedge} \to' (a \to c)$

(3) $(b \to c) \to' (a \to c)^{\wedge}$ (4) $(b \to c)^{\wedge} \to' (a \to c)^{\wedge}$

But since $\alpha \leqq \beta$ we cannot have $\beta \leqq \gamma$ and $\alpha \nleqq \gamma$. Nor, since $a \leqq b$ can we have $b \leqq c$ and $a \nleqq c$. Hence our supposition precludes case (2). In the remaining cases (1), (3) and (4), $b^{\beta} \to' c^{\gamma} \underset{\cdot}{\to}' a^{\alpha} \to' c^{\gamma} =$ $((b \to c) \to (a \to c))^{\wedge}$. Since $a \to b \leqq b \to c \underset{\cdot}{\to} a \to c$ in $\underset{\sim}{G}$ we have $(a \to b)^{\wedge}$ $\leqq' (b \to c \underset{\cdot}{\to} a \to c)^{\wedge}$, i.e. $a^{\alpha} \to' b^{\beta} \leqq' b^{\beta} \to' c^{\gamma} \underset{\cdot}{\to}' a^{\alpha} \underset{\cdot}{\to}' c^{\gamma}$.

Finally, suppose that $a^{\alpha} \nleqq' b^{\beta}$ and $a^{\alpha} \to' b^{\beta} = a \to b$.

Now $b^{\beta} \to' c^{\gamma} \underset{\cdot}{\to}' a^{\alpha} \to' c^{\gamma}$ is either $b \to c \underset{\cdot}{\to} a \to c$ or $(b \to c \underset{\cdot}{\to} a \to c)^{\wedge}$. But both of these are $\geqq' b \to c \underset{\cdot}{\to} a \to c$ in $\underset{\sim}{G}'$. So since $a \to b \leqq' b \to c$ $\underset{\cdot}{\to} a \to c$, $a^{\alpha} \to' b^{\beta} \leqq' b^{\beta} \to' c^{\gamma} \underset{\cdot}{\to}' a^{\alpha} \to' c^{\gamma}$ in this case too.

ad(5'): Prefixing. Similar to (4'), the proof exploits the same properties of $\underset{\sim}{G}'$ pluse (5') in $\underset{\sim}{G}$.

ad(6): Contraposition. Suppose that $a, b \in G$ and $\alpha, \beta \in S$, and that $a \to -b \leqq b \to -a$ in $\underset{\sim}{G}$. Consider $a^{\alpha} \to' -' b^{\beta}$ and $b^{\beta} \to' -' a^{\alpha}$. Suppose first that $a \leqq -b$ and $\alpha \leqq -\beta$. Then $a^{\alpha} \to' -' b^{\beta} = a^{\alpha} \to' (-b)^{-\beta} = (a \to -b)^{\wedge}$, and $b \leqq -a$ and $\beta \leqq -\alpha$. Hence $b^{\beta} \to' (-a)^{-\alpha} = (b \to -a)^{\wedge}$, and since $a \to -b \leqq b \to -a$, $(a \to -b)^{\wedge} \leqq' (b \to -a)^{\wedge}$ by lemma 4.2.4, i.e. $a^{\alpha} \to' -' b^{\beta} \leqq' b^{\beta} \to' -' a^{\alpha}$. The remaining case is where $a^{\alpha} \to' -' b^{\beta} = a \to -b$, but $b \to -a \leqq' b^{\beta} \to' (-a)^{-\alpha}$, so since $a \to -b \leqq' b \to -a$, we have $a^{\alpha} \to' -' b^{\beta} \leqq' b^{\beta} \to' -' a^{\alpha}$. □

So we also have, where *BL* is BB plus one or more of suffixing, prefixing and contraposition:

Theorem 4.2.11. *BL* is both prime and negation-consistent.

Proof Exactly as in theorem 4.2. 8 – use the Lindenbaum algebra. □

Theorem 4.2.12. *BL* satisfies γ.

Proof This is just a corollary of theorem 4.2.11 above. □

We shall see later that this process can also be applied to B and its corresponding extensions (chapter 6).

Bob Meyer and Mike Dunn proved that γ holds for E and R, as well as many other systems (Meyer and Dunn, 1969). Proof, using the relational semantics, that γ holds for these and other logics is given in RLR section 5.6.

The γ problem is closely related to the thorny issue of the status of disjunctive syllogism apropos relevant logics. I will not delve into the "debate" over the "rejection" of disjunctive syllogism by relevant logics,[1] except as to point out the significance of γ for one of the major positions.[2]

This position (expounded in Mortensen 1983) is that it is perfectly correct to use disjunctive syllogism in normal, everyday contexts, but that for validity we need to take into account negation-inconsistent and non-prime contexts (such as classical Peano arithmetic). If we assume that our meta-theory is such a normal context, then we can prove (using disjunctive syllogism in the meta-theory) that negation-consistency and primeness ensure closure w.r.t. disjunctive syllogism. From this perspective it is highly desirable that the set of theses of logic are closed w.r.t. disjunctive syllogism, for the following reasons.

The set of logical theses ought reflect what obtains in the broadest class of possible reasoning contexts. That is, what is logically true ought remain invariant w.r.t. the different reasoning contexts. But if the logical theses were not closed under disjunctive syllogism, then in normal contexts they would be augmented by extra "logical truths", contrary to the above invariance requirement. Hence γ ought hold for the logic.

What seems a natural outgrowth of this position appears to be implicit in Meyer 198?? (where it is argued that relevant logics do not reject disjunctive syllogism): Define the class of $\supset$E-theories of the

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1. See Anderson and Belnap 1962, Belnap and Dunn 1981, Stephensen 1976, Burgess 1981, 1983 and 1984, Mortensen 1983 and 198? , Read 1983.
2. Which don't include the fourth, correct position.

logic to be those theories closed under provable material implications. That is, if A is in the theory and $A \supset B$ is a theorem of the logic, then B is in the theory. Then the class of $\supset$E-theories corresponds to the class of presumed normal contexts. The distinction between rules of the logic and admissible rules is somewhat muted in the case of relevant logics when notions of theoryhood are taken into account. Standardly, theories are just required to be closed under provable entailment, and not under modus ponens for $\rightarrow$ ($\rightarrow$ E). That is, closed under: If A is in the theory and $\vdash A \rightarrow B$ then B is in the theory. But not: If A is in the theory and $A \rightarrow B$ is in the theory then B is in the theory. Thus the theories aren't required to be closed under the rules[1] of the logic. The distinction between rules and admissible rules in the case of the logical theses, that the former can be used to generate theorems, whereas the latter cannot, no longer holds when we consider theories.

So why not regard the logic as describing two sorts of implication, and include the classes of theories closed under both versions of provable implication as part of our formal apparatus. Whence we are on the path towards a universal logic (why stop at just two types of implication).

So suppose we have various notions of implication, which can be correctly applied in corresponding particular types of reasoning contexts. For example entailment $\rightarrow$ which is supposed applies in all reasoning contexts, and material implication $\supset$ which is supposed applies in normal reasoning contexts. (By the sense in which a notion of implication can

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1. And maybe this isn't such a good thing. For more on theories see §5.4.

be correctly applied, I mean merely that modus ponens for that type of implication is a correct rule of inference in the corresponding type of reasoning contexts - not that we have an otherwise adequate notion of implication.) Thus whether-or-not modus ponens for a type of implication is a rule (not just admissible) of the logic is simply an indication of whether-or-not that type of implication corresponds to logical entailment.

For the same invariance of the logical theses considerations applied to disjunctive syllogism, it is desirable for the set of theses of the logic to be closed under modus ponens for each type of implication. In particular, the set of logical theses ought be closed under modus ponens for material implication, since the logical truths ought obtain in normal contexts, where modus ponens for material implication is permissible. That is, once again we have that γ ought hold for the logic.

Irrespective of the merit of these positions, it is clear that a requirement that γ holds for logics is an important consideration in the philosophical problem of the status of disjunctive syllogism apropos relevant logics.

CHAPTER 5

Relational semantics for BB and its extensions

§ 5.0 Introduction

In this chapter we will develop a relational "worlds" semantics for BB and its extensions. However, these semantics, as will be seen, are almost directly induced by the algebra of the logics. They are of the form of the semantics described in ESL, and owe a heavy debt to the relational semantics for the standard relevant logics.[1] We follow Routley and Meyer's strategy in ALG II much of the way, which displays the lovely connections between the algebraic and relational semantics.

Unfortunately, there is a bug in that strategy. A representation theorem which shows how to convert <u>prime</u> algebras into relational model structures, is crucially used in proving completeness w.r.t. the relational semantics. In order to do so, we need a result that all non-theorems can be falsified in <u>prime</u> algebras. Unfortunately we have proved in the last section that this is so only for a limited number of extensions of BB. In fact it is not true for some extensions of BB, and indeed of B, as we will show in theorem 5.5.8.

To overcome this we will develop two types of relational semantics. The first (stronger) notion is that of an L-model structure, which faithfully characterises those logics $L \supseteq BB$ whose non-theorems can be falsified in prime L-algebras. The second notion is that of an unreduced L-model structure which faithfully characterises <u>all</u> extensions of BB. The dichotomy is similar to that between reduced and "unreduced" relational L-models in RLR.

1. Richard Routley & Bob Meyer, "The Semantics of Entailment I", <u>Truth, Syntax, Modality</u>, ed. H. Leblanc; also RLR, Ch. 4.

The bug in the strategy of ALG II is not only a hindrance to us. It lurks in ALG II and in fact Theorem 7[1] of that paper, which expresses the above priming result for extensions of B, is false. In chapter 6 I will show why the theorem fails in general, demonstrate a counter-example to an essential lemma, and then resurrect a proof for B and a couple of its extensions (along the lines of theorem 4.2.7).

In the first section of this chapter we define L-model structures and prove that each extension L of BB is sound w.r.t. the class of L-model structures. In the second section we continue to follow the strategy of ALG II and prove completeness w.r.t. model structures for those L which satisfy the required priming property.

It is instructive to prove the same completeness theorem using the canonical model structure, so in the third section we do that, adapting methods from ESL.

The fourth section is the key to providing comprehensive relational semantics for all extensions of BB. In it we distinguish between two notions of theoryhood, the standard one (closure under provable entailment and adjunction, which we call I. theories) and closure under the rules (which we call D. theories - D for deductive). We will show that the class of regular D. theories corresponds to the class of truth filters F of the algebras. Thus the approach of ALG II will be successful when priming works for the D. theories, which it doesn't in general. However priming does hold in general for the I. theories. So to provide completely general relational semantics we define unreduced model structures, the completeness result for which only requires priming to hold for I. theories.

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1. Where L is any one of the extensions of B, or its parts introduced, A is a theorem of L iff A is valid in every prime L groupoid.

We will see that the notion of D. theoryhood plays an important explanatory role. In section §5.5 we investigate the inter-relationships between the two notions of theoryhood and the two types of relational semantics. These investigations clearly indicate that the notion of D. theoryhood introduced in §5.4 merits further study.

Finally in §5.6 we compare the BB relational semantics with the standard B semantics, and with alternative approaches suggested by the relational semantics for connexive logics.

§ 5.1 Model structures and soundness.

We begin by defining model structures and interpretations on model structures.

Definition 5.1.1 A BB-model structure (BB-m.s.) is a structure $\underset{\sim}{M} = \langle 0, K, \nabla, R, * \rangle$ where K is a set, $0 \in K$, ∇ is a subset of the power set of K, R is a relation on $K \times \nabla \times \nabla$ and $*$ is a unary operation on ∇, such that:

1. $R\,0\,\alpha\,\beta$ iff $\alpha \subseteq \beta$.
2. For all $\alpha, \beta \in \nabla$, $\{a \in K : R\,a\,\alpha\,\beta\} \in \nabla$.
3. If $R\,0\,\alpha\,\beta$ then (i) if $R\,a\,\beta\,\gamma$ then $R\,a\,\alpha\,\gamma$, and (ii) if $R\,a\,\gamma\,\alpha$ then $R\,a\,\gamma\,\beta$.
4. $\alpha ** = \alpha$.
5. $R\,0\,\alpha\,\beta$ iff $R\,0\,\beta^*\,\alpha^*$.
6. ∇ is closed under set-theoretical intersection and union.

We can regard K as a set of worlds (or setups, or theories) and 0 as the actual world. Each element of ∇ can be regarded as the set of worlds at which a particular sentence is true - its "range".

* determines the negation properties. The fact that * is needed to provide a loose enough connection between the class of worlds at which a sentence is true and the class of worlds at which its negation is true, shows the intensionality of negation. Otherwise inconsistent and incomplete worlds cannot be catered for, and the paradoxes of suppression ensue (where premises which are used are not acknowledged, because they are supposed true at all worlds).[1] R determines the implication properties:

1. says that, from the perspective of the actual world, α implies β iff at every world where α is true, β is true;

3. says that, if α implies β at the actual world, then, if from the perspective of world a, β implies γ, then from the perspective of a α implies γ - and similarly for prefixing;

5. says that α implies β at the actual world iff the negation of β implies that of α at the actual world.

Definition 5.1.2. An interpretation I on $\underset{\sim}{M}$ is a function from {wffs} X K into {T, F} generated as follows. Define, for any wff A, $|A|$ to be $\{a \in K : I(A, a) = T\}$.

First assign to each propositional variable at each world a value : $I(p, a) \in \{T, F\}$,

satisfying the restriction that for every propositional variable p, $|p| \in \nabla$.[2]

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1. For a discussion of suppression, see Urquart 1972 and Routley and Routley 1972.

2. This requirement, and the requirement on model structures that $R\ 0\ \alpha\ \beta$ iff $\alpha \subseteq \beta$, is analogous to the Hereditary Condition of the standard relational semantics for relevant logics.

Now extend I to all wffs according to the following:-

(i) $I(A \,\&\, B, a) = T$ iff $I(A, a) = T$ and $I(B, a) = T$;

(ii) $I(A \vee B, a) = T$ iff either $I(A, a) = T$ or $I(B, a) = T$;

(iii) $I(\sim A, a) = T$ iff $a \in |A|^*$;

(iv) $I(A \rightarrow B, a) = T$ iff $Ra|A|\,|B|$.

Note that I is well-defined, since it is inductively defined on the complexity of wffs. Thus the possible trouble-spots (iii) and (iv) are o.k. since, e.g. $|A|^*$ is well-defined because $I(A, a)$ is, on the appropriate inductive hypothesis.

The following lemma justifies the manner of speaking about the elements of ∇.

> <u>lemma 5.1.3</u>. For any wff A, $|A| \in \nabla$. Informally, this says that the ranges of wffs <u>are</u> in ∇.

<u>Proof</u> By induction on the complexity of A.

The result obviously holds, by the restriction on I, for propositional variables.

Assume that $|A|$ and $|B|$ are in ∇, for our induction assumption.

By (i) of definition 5.1.2 $|A \,\&\, B| = |A| \cap |B|$, which is in ∇ by 6 of definition 5.1.1.

By (ii) $|A \vee B| = |A| \cup |B|$, which is in ∇ by 6 of definition 5.1.1.

By (iii) $|\sim A| = |A|^*$, which is in ∇, since by our assumption $|A| \in \nabla$, and * is an operation on ∇.

Now by (iv) $|A \to B| = \{a \in K : R\, a\, |A|\, |B|\}$, which is in ∇ by 2.
This completes the proof of the inductive step, so by induction $|A| \in \nabla$ for all wffs A. □

Definition 5.1.4. A wff A is *true on an interpretation I in a BB - m.s.* $\mathbf{M}$ iff $I(A, 0) = T$.
A wff A is *M-valid* iff A is true on all interpretations I in $\mathbf{M}$.
A wff A is *BB - m.s. valid* iff A is $\mathbf{M}$-valid for all BB-m.s. $\mathbf{M}$.

We shall prove that BB is sound w.r.t. the class of BB-m.s., as already indicated, by a representation theorem which shows how to generate a BB-algebra $\mathbf{G}^M$ from an arbitrary BB-m.s. $\mathbf{M}$, such that if a wff A is $\mathbf{G}^M$- valid then A is also M-valid. The result is then immediate, for since BB is sound w.r.t. the class of BB-algebras, all its theorems are $\mathbf{G}^M$-valid and hence $\mathbf{M}$-valid.

Definition 5.1.5. The *algebra of ranges* $\mathbf{G}^M$ of a model structure $\mathbf{M} = \langle 0, K, \nabla, R, * \rangle$ is the algebra $\mathbf{G}^M = \langle \nabla, F, \cap, \cup, -, \to \rangle$ where $\cap$ and $\cup$ have their set-theoretical sense as operations on ∇, and for all $\alpha, \beta \in \nabla$: $-\alpha = \alpha^*$; $\alpha \to \beta = \{x \in K : R\, x\, \alpha\, \beta\}$; and $F = \{\alpha \in \nabla : 0 \in \alpha\}$.

Clearly $\mathbf{G}^M$ is well-defined, we check that it is indeed a BB-algebra.

Lemma 5.1.6. For every BB-model structure $\mathbf{M}$, the corresponding algebra of ranges $\mathbf{G}^M$ is a BB-algebra.

Proof *ad(1)* : $\langle \nabla, \cap, \cup, - \rangle$ is a distributive lattice. It is just the set-inclusion lattice of ∇, which by 6. (definition 5.1.1) is closed under $\cap$ and $\cup$. Also, by 1. and 5., $\alpha \leqq \beta$ (i.e. $\alpha \subseteq \beta$) iff $-\beta = \beta^* \leqq \alpha^* = -\alpha$; and by 4., $- - \alpha = \alpha^{**} = \alpha$, so the lattice is De Morgan.

ad(2): F is a filter since $0 \in \alpha$ and $0 \in \beta$ iff $0 \in \alpha \cap \beta$.

ad(3): $\alpha \leqq \beta$ iff $R\,0\,\alpha\,\beta$, which holds iff $0 \in \alpha \to \beta$, i.e. iff $\alpha \to \beta \in F$, as required.

ad(4): Suffixing. Suppose that $\alpha \leqq \beta$. Then $R\,0\,\alpha\,\beta$, so

$$\{a \in K : R\,a\,\beta\,\gamma\} \subseteq \{a \in K : R\,a\,\alpha\,\gamma\} \quad \text{for all } \gamma \in \nabla,$$

by 3.(i). So we have $\beta \to \gamma \leqq \alpha \to \gamma$ as required.

ad(5): Prefixing. Similarly, using 3.(ii).

This completes the proof that $\underset{\sim}{G}^M$ is a BB-algebra. □

Theorem 5.1.7. For any BB-model structure $\underset{\sim}{M}$, a wff A is $\underset{\sim}{M}$-valid iff A is $\underset{\sim}{G}^M$-valid.

Proof We show that for each interpretation I on $\underset{\sim}{M}$ there is an interpretation I' on $\underset{\sim}{G}^M$, and vice-versa, such that for every wff A, A is true on I in $\underset{\sim}{M}$ iff A is true on I' in $\underset{\sim}{G}^M$, as follows.

Where $|A| = \{x \in K : I(A, x) = T\}$ for all wffs A, we put $I'(A) = |A|$. By lemma 5.1.3 I' is well-defined, and the proof of that lemma shows that the function $|A|$, and hence I', satisfies the assignment rules for interpretations on BB-algebras.

Now $I(A, 0) = T$ iff $0 \in I'(A)$, so A is true on I in $\underset{\sim}{M}$ iff $I'(A) \in F$, i.e. iff A is true on I' in $\underset{\sim}{G}^M$.

Suppose that we have an interpretation I' on $\underset{\sim}{G}^{M}$.

Define I on $\underset{\sim}{M}$ as follows:

I(p, a) = T iff a ∈ I'(p) for all propositional variables p;

extend I as according to the definition of interpretations (definition 5.1.2).

We prove by induction on the complexity of a wff A that $|A| = I'(A)$ for all wffs A:-

The result holds by the definition of I in the case of propositional variables.

Assume that $|A| = I'(A)$ and $|B| = I'(B)$.

$$I(A \,\&\, B, a) = T \text{ iff } I(A,a) = T \text{ and } I(B, a) = T$$
$$\text{iff } a \in |A| \cap |B| = I'(A) \cap I'(B) = I'(A \,\&\, B)$$

$$I(A \vee B, a) = T \text{ iff either } I(A, a) = T \text{ or } I(B, a) = T$$
$$\text{iff either } a \in |A| \text{ or } a \in |B|$$
$$\text{iff } a \in |A| \cup |B| = I'(A) \cup I'(B) = I'(A \vee B)$$

(Primeness is not a worry in going in the reverse direction because the partial order on ∇ is the subset algebra, so if a $\in |A| \cup |B|$ then either $a \in |A|$ or $a \in |B|$.)

$$I(\sim A\ a) = T \text{ iff } a \in |A|^{*} = I'(A)^{*} = -I'(A) = I'(\sim A)$$

$I(A \rightarrow B, a) = T$ iff $R\ a\ |A|\ \ |B|$ which holds iff

$$a \in |A| \rightarrow |B| = I'(A) \rightarrow I'(B) = I'(A \rightarrow B) .$$

This completes the proof of the inductive step, so by induction $|A| = I'(A)$ for all wffs of A. Hence, as before,

$I(A, 0) = T$ iff $0 \in I'(A)$, giving the required result. □

<u>Theorem 5.1.8.</u> BB is sound w.r.t. BB-model structures.

<u>Proof</u> This is a corollary of theorem 5.1.7. and theorem 3.1.4.

If $\vdash_{BB}$ A then A is BB-m.s. valid. □

The theorem can easily be modified to cater for arbitrary extensions of BB. In the manner of ESL we can use the algebraic analogues of the extra axioms and rules to generate the appropriate restrictions on the model structures. (Of course this will not in general deliver the most efficient specification or the most lucid class of model structures which can do the job.) For example:

If we augment BB by the axiom prefixing, then in the corresponding BL-models we require $\alpha \to \beta \subseteq \gamma \to \alpha \dot{\to} \gamma \to \beta$,

so we need: If $R\, x\, \alpha\, \beta$ then $R\, x\, (\gamma \to \alpha)\ \ (\gamma \to \beta)$.

i.e. if $R\, x\, \alpha\, \beta$ then $R\, x\, \{a \in K : R\, a\, \gamma\, \alpha\}\ \ \{a \in K : R\, a\, \gamma\, \beta\}$.

Which supercedes 3(ii) of definition 5.1.1.

This process can be formally specified by induction on the complexity of the expressions involved. Both axioms and rules correspond to conditions on the membership of 0 in the ranges of particular wffs. For an axiom A we require $0 \in |A|$, and for a rule $\vdash A \Rightarrow\ \vdash B$ we require $0 \in |A| \Rightarrow 0 \in |B|$.

Now, first replace all of the sentential variables of the schema by variables representing arbitrary members of ∇. Then replace the conditions $\vdash \alpha$ by $0 \in \alpha$, and begin to remove the connectives as follows:

$0 \in \alpha \to \beta$ becomes $R\ 0\ \alpha\ \beta$;

$0 \in \alpha \ \&\ \beta$ becomes $0 \in \alpha \cap \beta$;

$0 \in \alpha \vee \beta$ becomes $0 \in \alpha \cup \beta$;

$0 \in \sim \alpha$ becomes $0 \in \alpha^*$.

And then continue to delete the logical connectives as follows:

$\alpha \to \beta$ becomes $\{x \in K : R\ x\ \alpha\ \beta\}$;

$\alpha \ \&\ \beta$ becomes $\alpha \cap \beta$;

$\alpha \vee \beta$ becomes $\alpha \cup \beta$;

$\sim \alpha$ becomes α^* .

Until all of the logical connectives are removed. Finally, replace $R\ 0\ \alpha\ \beta$ by the requirement that $\alpha \subseteq \beta$.

e.g. (i) For the rule (BR1) $\vdash A \Rightarrow\ \vdash (A \to B) \to B$, we get:

$\vdash \alpha \quad \Rightarrow \quad \vdash (\alpha \to \beta) \to \beta$

$0 \in \alpha \quad \Rightarrow \quad 0 \in (\alpha \to \beta) \to \beta$

$0 \in \alpha \quad \Rightarrow \quad R\ 0\ (\alpha \to \beta)\ \beta$

$0 \in \alpha \quad \Rightarrow \quad R\ 0\ \{x \in K : R\ x\ \alpha\ \beta\}\ \beta$

$0 \in \alpha \quad \Rightarrow$ If $R\ x\ \alpha\ \beta$ then $x \in \beta$; for all $\alpha, \beta \in \nabla$ and $x \in K$.

e.g. (ii) For contraposition $\vdash A \to \sim B \mathrel{\dot{\to}} B \to \sim A$ we get:

$\vdash \alpha \to \sim \beta \mathrel{\dot{\to}} \beta \to \sim \alpha$

$0 \in \alpha \to \sim \beta \mathrel{\dot{\to}} \beta \to \sim \alpha$

$R\ 0\ (\alpha \to \sim \beta)\ (\beta \to \sim \alpha)$

$R\ 0\ \{x \in K : R\ x\ \alpha \sim \beta\}\ \{x \in K : R\ x\ \beta \sim \alpha\}$

$R\ 0\ \{x \in K : R\ x\ \alpha\ \beta^*\}\ \{x \in K : R\ x\ \beta\ \alpha^*\}$

$\{x \in K : R\ x\ \alpha\ \beta^*\} \subseteq \{x \in K : R\ x\ \beta\ \alpha^*\}$

i.e. for all $\alpha, \beta \in \nabla$ and $x \in K$, if $R\ x\ \alpha\ \beta^*$ then $R\ x\ \beta\ \alpha^*$.

Definition 5.1.9. An L-model structure where BB $\subseteq$ L is a BB-model structure which satisfies the conditions, according to the above recipe, corresponding to the extra axioms and rules of L.

Interpretations on L-m.s. and the associated notions are defined in exactly the same way as for BB-m.s.

Our representation theorem carries over:

Theorem 5.1.10. If we replace BB by L throughout the statement of theorem 5.1.3, then the ensuing statement also holds, where L is an extension of BB.

Proof All we need to do is check that the extra requirements on the L-algebra $\underset{\sim}{G}^M$ hold. But this is obvious, since for extra axioms A our procedure ensures that $0 \in |A|$ under all interpretations I on $\underset{\sim}{M}$, so that $A \in F$ under all interpretations I' on $\underset{\sim}{G}^M$, that is, A is valid in $\underset{\sim}{G}^M$, as required. Similarly the rules correspond to appropriate conditionals of sentences of the above sort, leading to the required conditional statement being true for $\underset{\sim}{G}^M$. □

Theorem 5.1.11. L is sound w.r.t. the class of L-m.s., where BB $\subseteq$ L.

Proof The result follows, just as in the case of BB, from theorem 5.1.10 and theorem 3.2.2. □

§ 5.2. The other way : the reverse representation theorem.

In this section we prove a limited completeness theorem for model structures. We continue to follow the strategy of ALG II, and show how to generate model structures from prime algebras. The desired completeness theorem ensues, but only for those extensions of BB which satisfy the requirement that every non-theorem can be falsified in a prime algebra. We will see that some extensions of BB (and of B) do not satisfy this requirement (theorem 5.5.8).

Definition 5.2.1. Let $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \rightarrow \rangle$ be a prime BB-algebra. The associated model structure, $\underset{\sim}{M}^G$, is defined as follows: $O = F$; $K = \{$prime filters of $\underset{\sim}{G}\}$; denoting $\{P \in K : a \in P\}$ by $|a|$ for all $a \in G$, $\nabla = \{|a| : a \in G\}$; $|a|^* = |-a|$; and $R\,P\,|a|\,|b|$ iff $a \rightarrow b \in P$.

Clearly $\underset{\sim}{M}^G$ is well-defined, we check that it is indeed a BB-model structure.

Lemma 5.2.2. For every prime BB-algebra $\underset{\sim}{G}$, the associated model structure $\underset{\sim}{M}^G$ is a BB-model structure.

Proof ad 1. : $R\,O\,|a|\,|b|$ iff $a \rightarrow b \in O$, i.e. iff $a \rightarrow b \in F$, which holds

iff $a \leq b$ in $\underset{\sim}{G}$,

* iff for all $P \in K$, $a \in P$ entails that $b \in P$,

iff $|a| \subseteq |b|$.

(The reverse direction of * follows from the Stone theorem[1].)

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1. In any distributive lattice, if $a \nleq b$ then there is a prime filter which contains a and doesn't contain b. Grätzer 1978, pp.63-64.

ad 2. : $\{P \in K : R\,P\,|a|\;\;|b|\} = \{P \in K : a \to b \in P\} = |a \to b|$

$\in \nabla$ by the definition of ∇.

ad 3. : Suppose that $R\,0\,|a|\;\;|b|$:

(i) If $R\,P\,|b|\;|c|$, then $b \to c \in P$; and since our supposition entails that $a \leqq b$ in $\underset{\sim}{G}$ we have $b \to c \leqq a \to c$, hence $a \to c \in P$, i.e. $R\,0\,|a|\;|c|$.

(ii) Similarly, using rule-prefixing of $\underset{\sim}{G}$.

ad 4. : $|a|^{**} = |-a|^{*} = |--a| = |a|$

ad 5. : $R\,0\,|a|\;|b|$ iff $a \to b \in F$, which holds iff $-b \to -a \in F$, which holds iff $R\,0\,|-b|\;|-a|$, i.e. $R\,0\,|b|^{*}\;|a|^{*}$.

ad 6. : $|a| \cup |b| = \{P \in K : a \in P \text{ or } b \in P\}$

$= \{P \in K : a \cup b \in P\}$

Since each P is a filter (so $\left\{\begin{matrix} b \\ a \end{matrix}\right\} \in P \Rightarrow a \cup b \in P$),
and each is prime ($a \cup b \in P \Rightarrow$ either $a \in P$ or $b \in P$).

Thus $|a| \cup |b| = |a \cup b|$, which is an element of ∇.

$|a| \cap |b| = \{P \in K : a \in P \text{ and } b \in P\}$

$= \{P \in K : a \cap b \in P\}$

since each P is a filter.

Thus $|a| \cap |b| = |a \cap b|$, which is an element of ∇ .

So $\underset{\sim}{M}^{G}$ is a BB-m.s. □

We show that a wff A is $\underset{\sim}{M}^G$-valid iff A is $\underset{\sim}{G}$-valid by proving that $\underset{\sim}{G}$ is equal to (up to structural isomorphism) the algebra of ranges $\underset{\sim}{G}^{MG}$ of $\underset{\sim}{M}^G$.

Theorem 5.2.3. For all prime BB-algebras $\underset{\sim}{G}$ there is a structure-preserving isomorphism from $\underset{\sim}{G}$ to $\underset{\sim}{G}^{MG}$, the algebra of ranges of M^G.

Proof Recall that $\underset{\sim}{G}^{MG} = \langle \nabla, F_1, \cap, \cup, -, \rightarrow \rangle$ where ∇ is that of $\underset{\sim}{M}^G$, $\cap$ and $\cup$ have their set-theoretical sense,

$F_1 = \{|a| \in \nabla : 0 \in |a|\}$, $-|a| = |a|^*$ and

$|a| \rightarrow |b| = \{P \in K : R\ P\ |a|\ |b|\}$ (definition 5.1.5).

Consider $Q : G \mapsto \nabla$ defined by $Q(a) = |a|$ for all $a \in G$. Clearly Q is onto ∇. Suppose that $Q(a) = Q(b)$, that is $|a| = |b|$, so every prime filter which contains a also contains b, and vice versa. So neither $a \nleq b$ nor $b \nleq a$ in $\underset{\sim}{G}$ (otherwise, by the Stone theorem a prime filter would exist, containing one but not the other), hence $a = b$ and Q is 1 - 1.

By the proof of ad 6. in lemma 5.2.2, $Q(A \cup B) = Q(A) \cup Q(B)$ and $Q(A \cap B) = Q(A) \cap Q(B)$.

Also, $Q(a \rightarrow b) = |a \rightarrow b|$

$= \{P \in K : a \rightarrow b \in P\}$ (definition 5.2.1)

$= \{P \in K : R\ P\ |a|\ |b|\}$

$= |a| \rightarrow |b|$ in $\underset{\sim}{G}^{MG}$ (definition 5.1.5)

$= Q(a) \rightarrow Q(b)$.

And $Q(-a) = |-a| = |a|^* = -|a|$ in $\underset{\sim}{G}^{MG}$

$= -Q(a)$ as required.

So Q preserves all of the operations on G.

It remains to prove that the truth filter F of $\underset{\sim}{G}$ corresponds (under the isomorphism) to the truth filter F_1 of $\underset{\sim}{G}^{M^G}$. For all $a \in G$, $a \in F$ iff $F \in |a|$, that is, iff $F \in Q(a)$. But $F = 0$ in $\underset{\sim}{M}^G$, so we have $a \in F$ iff $0 \in Q(a)$, which holds iff $Q(a) \in F_1$ (since $F_1 = \{|a| \in \nabla : 0 \in |a|\}$), as required.

Thus the isomorphism Q is completely structure-preserving. □

Theorem 5.2.4. For all prime BB-algebras, $\underset{\sim}{G}$, a wff A is $\underset{\sim}{G}$-valid iff A is $\underset{\sim}{M}^G$-valid.

Proof By theorem 5.1.7 a wff A is $\underset{\sim}{M}^G$-valid iff A is $\underset{\sim}{G}^{M^G}$-valid.

Clearly, by the above theorem 5.2.3, A is $\underset{\sim}{G}^{M^G}$-valid iff A is $\underset{\sim}{G}$-valid, because for any interpretation I on $\underset{\sim}{G}^{M^G}$ we have a corresponding interpretation $Q^{-1} \circ I$ on $\underset{\sim}{G}$ (which verifies the same set of wffs) and for any interpretation I on $\underset{\sim}{G}$ we have a corresponding interpretation $Q \circ I$ on $\underset{\sim}{G}^{M^G}$. Hence a wff A is $\underset{\sim}{M}^G$-valid iff A is $\underset{\sim}{G}$-valid. □

Note that the algebra of ranges $\underset{\sim}{G}^M$ of a BB-model structure $\underset{\sim}{M}$ is always a prime BB-algebra, since $F = \{\alpha \in \nabla : 0 \in \alpha\}$ (definition 5.1.5), so that $\alpha \cup \beta \in F$ iff either $0 \in \alpha$ or $0 \in \beta$, that is, iff either $\alpha \in F$ or $\beta \in F$. Hence the class of prime BB-algebras exactly corresponds to the class of ranges of BB-model structures. In fact this holds for all extensions L of BB.

<u>Theorem 5.2.5.</u> BB is complete w.r.t. the class of BB-model structures.

<u>Proof</u> By theorems 3.1.4 and 4.2.7 BB is complete w.r.t. the class of prime BB-algebras. So by theorem 5.2.4. any non-theorem is refuted in some $\underset{\sim}{M}^{G}$. □

These results can be generalised to extensions L of BB.

<u>Theorem 5.2.6.</u> Lemma 5.2.2, and theorems 5.2.3 and 5.2.4 hold when BB is replaced throughout by L, where L is an extension of BB.

<u>Proof</u> That the generalisation of the lemma 5.2.2 holds is obvious, since the algebraic properties of $\underset{\sim}{G}$, corresponding to the extra rules and axioms, carry over to the extra requirements for L-model structures in the associated model structure $\underset{\sim}{M}^{G}$. Statements of the form $a \in F$ are simply replaced by statements of the form $0 \in |a|$. Theorems 5.2.3 and 5.2.4 apply because prime L-algebras are prime BB-algebras. □

Our limited extension of theorem 4.2.7 (theorem 4.2.10) prevents us from exploiting the above theorem in order to generalise our completeness result, along the lines of theorem 5.2.5. Never-the-less we state the theorem for the corresponding extensions, noting that it also holds for all of the other extensions of BB, the non-theorems of which can be falsified in <u>prime</u> algebras.

Theorem 5.2.7. Where BL is BB augmented by one or more of prefixing ($\vdash A \rightarrow B \underset{\cdot}{\rightarrow} C \rightarrow A \underset{\cdot}{\rightarrow} C \rightarrow B$), suffixing ($\vdash A \rightarrow B \underset{\cdot}{\rightarrow} B \rightarrow C \underset{\cdot}{\rightarrow} A \rightarrow C$) and contraposition ($\vdash A \rightarrow \sim B \underset{\cdot}{\rightarrow} B \rightarrow \sim A$), then BL is complete w.r.t. the class of BL-model structures.

Proof As for theorem 5.2.5, using theorems 3.2.2 and 4.2.10. □

§ 5.3. The alternative completeness theorem

We follow the method of ESL and in fact use theorem 5.2.3, applying it to the canonical model structure of BB.

Theorem 5.3.1. BB is complete w.r.t. the class of BB-model structures (mark II).

Proof Consider the Lindenbaum algebra $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \rightarrow \rangle$ where $G = \{|A| : A \text{ is a wff}\}$ ($|A| = \{B : \vdash_{BB} A \leftrightarrow B\}$) and $F = \{|A| : \vdash_{BB} A\}$ and the operators correspond to the logical connectives.[1] Since BB is prime (theorem 4.2.8) $\underset{\sim}{G}$ is prime, so applying lemma 5.2.2 we generate $\underset{\sim}{M}^G = \langle O, K, \nabla, R, * \rangle$ where $O = F$, $K = \{T : |A| \cup |B| \in T$ iff either $|A| \in T$ or $|B| \in T$; and if $|A| \in T$ and $|A| \leq |B|$ in $\underset{\sim}{G}$ then $|B| \in T\}$, $\nabla = \{\|A\|\}$ where $\|A\| = \{T \in K : |A| \in T\}$, $\|A\|^* = \big|-|A|\big| = \|\sim A\|$, and $R\, T\, \|A\|\, \|B\|$ iff $|A \rightarrow B| \in T$. Noting that $|A| \leq |B|$ in $\underset{\sim}{G}$ iff $\vdash_{BB} A \rightarrow B$, then $\underset{\sim}{M}^G$ corresponds to the canonical model structure of BB $M^c = \langle O^c, K^c, \nabla^c, R^c, *^c \rangle$ where: $O^c = \{\text{theorems of BB}\}$, $K^c = \{\text{prime BB -I. theories}\}$, $\nabla^c = \{\|A\|\}$ where $\|A\| = \{T \in K^c : A \in T\}$, $\|A\|^{*c} = \|\sim A\|$ and $R^c\, T\, \|A\|\, \|B\|$ iff $A \rightarrow B \in T$. (Where

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1. As in the proof of 3.1.4.

a BB -I. theory is a set of sentences T such that if $\vdash_{BB} A \rightarrow B$ and $A \in T$ then $B \in T$, which is closed under adjunction). The correspondence is in the sense that there is an isomorphism from $\underset{\sim}{M}^c$ to $\underset{\sim}{M}^G$, defined by $\mathcal{T} : K^c \mapsto K$ where for all $P \in K^c$, $\mathcal{T}(P) = \{|A| : A \in P\}$. This is an isomorphism because if $\vdash_{BB} A \leftrightarrow B$ and $A \in P$ then $B \in P$. Thus $\underset{\sim}{M}^c$ is a BB-model structure. Now suppose A is not a theorem of BB, then A is falsified by I on $\underset{\sim}{G}$ where $I(A) = |A|$.

So A is falsified by I' on $\underset{\sim}{M}^G$ where $I'(A, P) = T$ iff $|A| \in P$, and so by I" on $\underset{\sim}{M}^c$ where $I''(A, P) = T$ iff $A \in P$. □

Obviously, this approach can be extended to cover those $L \supseteq BB$ which are prime, just as for the approach of §5.2. Note that the use of the representation theorem 5.2.3 and $\underset{\sim}{M}^G$ is dispensible, it simply obviates the need for a direct proof of the fact that the canonical model structure is a model structure which falsifies all of the non-theorems of BB (which is an easy mechanical procedure that mimics the proof of the representation theorem - in the particular case that $\underset{\sim}{G}$ is the Lindenbaum algebra).

§ 5.4. Relational semantics for all extensions of BB

As we have seen, our relational semantics for BB is of limited applicability to extensions of BB because their adequacy hinges on whether-or-not, for each non-theorem A, a prime algebra can be constructed which falsifies A.

Following RLR, there are strategies available for catering for all extensions of BB, as in the case of B.

The two different strategies involve seizing upon different notions of what constitutes a theory of a logic, and the pivotal consideration is whether "priming" succeeds for that notion of theory-hood. By priming I mean that if some theory T doesn't contain a wff A, then there is a prime theory T' such that $T \subseteq T'$ and $A \notin T'$. We have already used the following definition of theory-hood (§5.3):

Definition 5.4.1. A set of wff T is an L-I.theory of a logic or theory L iff, whenever $A \in T$ and $\vdash_L A \rightarrow B$ then $B \in T$, and T is closed under adjunction.

This definition has the advantage that, where L is any extension of E_{fde}, the class of L-I.theories satisfies priming. The proofs in RLR[1] for the case of B and its extensions also work for BB. (Proof of Lemma 4.3 uses DR2 : $\vdash A \& B \rightarrow C$ and $\vdash B \rightarrow C \vee A \Rightarrow \vdash B \rightarrow C$, which is also a derived rule of BB and its extensions. Proof of this fact is along the lines of RLR[2] noting that only the rule form of T3.6 is needed, which follows using (R.A4) and (R.A7).) I shall prove the result using Zorns lemma, as in RLR p. 310, since this will illustrate some further points in our discussion.

Zorn's lemma : Any non-empty partially ordered set, in which every chain has an upper bound, has a maximal element.

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1. RLR, Lemma 4.3 and Lemma 4.4 (pp. 307-308).
2. RLR, p. 291.

Zorn's lemma is equivalent to the axiom of choice, which is used in the alternative priming proofs, where the required prime theory is "constructed" from the original. First we prove the following lemma.

Lemma 5.4.2. Let L be any extension of E_{fde},[1] and let T be an L-I.theory. Then for any wff A, the smallest L-I.theory which contains $T \cup \{A\}$, $Cl(T \cup \{A\})$ is the set $T_A = \{B : \vdash_L C \,\&\, A \to B$, for some $C \in T\}$.

Proof Clearly T_A is a subset of every L-I.theory which contains $T \cup \{A\}$. Thus we only need to show that T_A is a L-I.theory:

(i) Suppose that $B_1 \in T_A$ and $B_2 \in T_A$, so $\vdash_L C_1 \,\&\, A \to B_1$ and $\vdash_L C_2 \,\&\, A \to B_2$ for some $C_1, C_2 \in T$. But then by (A2), (A3), rule-transitivity and (R.A4) (page 26), we have $\vdash_L (C_1 \,\&\, C_2) \,\&\, A \to B_1 \,\&\, B_2$. Since T is closed under adjunction $B_1 \,\&\, B_2 \in T_A$, and T_A is closed under adjunction.

(ii) Suppose that $B \in T_A$ and $\vdash_L B \to C$. Then $\vdash_L D \,\&\, A \to B$ for some $D \in T$, and so by rule-transitivity $\vdash_L D \,\&\, A \to C$, thus $C \in T_A$ and T_A is closed under L-entailment. □

We are now in a position to prove the aforementioned priming result.

Theorem 5.4.3. Let L be any extension of E_{fde}, then if an L-I.theory T doesn't contain a wff A, there exists an L-I.theory T' such that T' is prime, $T \subseteq T'$ and $A \notin T'$.

Proof Consider the class of L-I.theories Y such that for all $P \in Y$, $T \subseteq P$ and $A \notin P$. Y is non-empty since $T \in Y$. Every chain in Y, $P_o \subseteq P_1 \subseteq P_2 \subseteq \ldots\ldots$, has an upper bound - their union - which

1. Where L is closed under the rules of E_{fde}.

is in Y. So applying Zorn's lemma Y has a maximal element T'. We just need to check that T' is prime. Suppose that $C \vee D \in T'$ and that $C \notin T'$ and $D \notin T'$. Since T' is maximal, $A \in Cl(T' \cup \{C\})$ and $A \in Cl(T' \cup \{D\})$. Hence by lemma 5.4.2, there exists $B_1, B_2 \in T'$ such that $\vdash B_1 \& C \rightarrow A$ and $\vdash B_2 \& D \rightarrow A$. But then by (A2), (A3), rule-transitivity and (R.A7) $\vdash_L ((B_1 \& B_2) \& C) \vee ((B_1 \& B_2) \& D) \rightarrow A$ and hence by (A8) and rule-transitivity $\vdash_L (B_1 \& B_2) \& (C \vee D) \rightarrow A$.

But by our supposition $C \vee D \in T'$ and so $A \in T'$ contrary to $T' \in Y$. Hence by reductio no such wff $C \vee D$ exists, and T' is prime. □

So automatic assurance of priming ensues from the L-I.theory notion of theory-hood. However, this notion of theory-hood is somewhat adrift from that of deducibility espoused by the logic. This is most obvious for the weaker logics with rules such as rule-prefixing. So that it doesn't follow from theorem 5.4.3 that we can construct a prime algebra to falsify each non-theorem of a logic $L \supseteq E_{fde}$.

Let me expand on this point. The traditional notion of a theory is that it is a set of sentences T which is deductively closed w.r.t. the logic L in question. By deductive closure it is traditionally meant that for every finite sequence of sentences, if every one is either a thesis (in $T \cup L$) or is obtained from preceding sentences by the application of a primitive rule of L, then the final sentence in the sequence is in T. Such a definition is given in Mates 1972, p. 184. That is, a theory is traditionally a set of sentences which includes the theorems of the logic and which is closed under the rules (primitive or derived) of the logic. Clearly the L-I. theories do not correspond to this notion of theoryhood. (For example, they need not contain the theorems of L.) However, we will see that the set of sentences true on I in $\underset{\sim}{G}$, for any interpretation I on any L-algebra $\underset{\sim}{G}$, is such a theory. Thus the truth filters on the algebras reflect the traditional notion of theoryhood. It is for this reason that the above priming result for L-I. theories (theorem 5.4.3) does not ensure the required priming result for algebras (in order to obtain completeness of L w.r.t. the class of L-m.s.).

Corresponding to each interpretation I on a model $\underset{\sim}{G}$ is a "theory", namely the set of sentences which are true on I in $\underset{\sim}{G}$. However such a "theory", as well as satisfying the constraints on L-I. theories, is closed under the rules of L and contains the theorems of L. For example, in the case of BB, $T = \{A : I(A) \in F\}$, for some BB-model $\underset{\sim}{G}$ and interpretation I, satisfies : modus ponens as $a \in F$ and $a \to b \in F$ entails that $b \in F$; rule-prefixing as if $a \to b \in F$ then $c \to a \dot{\to} c \to b \in F$; (R.A4) as $a \to b \in F$ and $a \to c \in F$ entails that $a \to b \cap c \in F$; etc. . That is, T satisfies the traditional notion of theoryhood.

Conversely, if a set of sentences S contains all of the theorems of L, and S is closed under the rules of L, then we can construct an algebraic model $\underset{\sim}{G}$ and an interpretation I on $\underset{\sim}{G}$ such that $I(A) \in F$ iff $A \in S$ for all wffs A. Let me formalise this traditional notion of theoryhood before proving the above.

<u>Definition 5.4.4.</u> A set of wff T is an <u>L-D.theory</u> of a logic L iff T is closed under the primitive rules of L.

Definition 5.4.5. An L-D.theory or L-I.theory T is regular iff T contains all of the theorems of L.

Note that the traditional notion of theoryhood corresponds to regular L-D.theories.

Theorem 5.4.6. For any logic L such that BB $\subseteq$ L, if T is a regular L-D.theory then there is an L-algebra $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \rightarrow \rangle$ and an interpretation I on $\underset{\sim}{G}$ such that $I(A) \in F$ iff $A \in T$ for all wffs A.

Proof The procedure is analogous to the general completeness theorem, which uses the Lindenbaum algebra (theorem 3.1.4). Put $|A| = \{C : A \leftrightarrow C \in T\}$, $G = \{|A|\}$, $F = \{|A| : A \in T\}$, $|A| \cap |B| = |A \& B|$, $|A| \cup |B| = |A \vee B|$, $-|A| = |{\sim}A|$ and $|A| \rightarrow |B| = |A \rightarrow B|$. Proof that $\underset{\sim}{G}$ is a BB-algebra is exactly as in 3.1.4.(b) and depends on the fact that BB is contained in T, and that T is closed under the rules of BB. The extra requirements for L-algebras are satisfied analogously, in that T also contains the extra theorems of L and is closed under the extra rules of L. So $\underset{\sim}{G}$ is an L-algebra. Now simply put $I(A) = |A|$. Clearly I is an interpretation, furthermore $I(A) \in F$ iff $A \in T$. □

So we see that the class of wffs true according to some I on some $\underset{\sim}{G}$ corresponds to the class of regular L-D.theories. To prove, for a logic L $\supseteq$ BB, that in general a non-theorem A can be falsified by some I on some prime $\underset{\sim}{G}$, is, therefore, equivalent to proving that there exists a prime, regular L-D.theory which doesn't contain A. We state this result as a theorem.

Theorem 5.4.7. Let L be any extension of BB. Then the class of wffs true according to some interpretation I on some L-algebra $\underset{\sim}{G}$ corresponds to the class of regular L-D. theories. That is, every such pair (I, $\underset{\sim}{G}$) determines a unique L-D. theory T, and vice-versa.

Proof Clearly (as per the discussion prior to theorem 5.4.6) the set of wffs true on I in $\underset{\sim}{G}$ is a regular L-D. theory. Theorem 5.4.6 provides the converse. □

Since, where $\nvdash_L$ A the guarantee of the existence of a prime L-algebra which falsifies A is what we need to ensure completeness of our relational semantics of §5.3 (by using the representation theorem 5.2.3) then, in effect, what we require is the guarantee of the existence of a prime, regular L-D. theory which doesn't contain A.

So the gap in the adequacy of our relational semantics for extensions L of BB corresponds to proving whether-or-not each non-theorem of L is refuted by a prime, regular L-D. theory. Of course we cannot expect this to be true of all extensions of BB, such as BB + Av ~ A (see theorem 5.5.8). In such cases we need to modify our definition of model structures along the lines of the unreduced model structures of RLR. Note that the key to the priming proof (theorem 5.4.3) is whether the following holds:-

> If $A \in Cl\ (T \cup \{B\})$ and $A \in Cl\ (T \cup \{C\})$ then $A \in Cl\ (T \cup \{B \vee C\})$, where T is a "theory"; i.e. if T, B $\vdash$ A and T, C $\vdash$ A then T, B v C $\vdash$ A.

This is the key to any priming proof (using Zorn's lemma) since any notion of deduction $\vdash$ will satisfy the antecedent conditions of Zorn's lemma.

But, as already indicated, if the above fails to hold and priming cannot be otherwise secured, we are not lost, for completeness can be guaranteed for the following class of model structures, purely on the basis of theorem 5.4.3. These semantics, rather than depending on the properties of L-D. theories, only depend on those (in particular, priming) of L-I. theories.

Definition 5.4.8. An unreduced BB-model structure (BB-u.m.s.) is a structure $\underset{\sim}{M} = \langle O, K, \nabla, R, * \rangle$ where K is set, $O \subseteq K$[1], $\nabla \subseteq P(K)$, R is a relation on $K \times \nabla \times \nabla$ and * is a unary operation on ∇, such that:

1. $R\, x\, \alpha\, \beta$ for all $x \in O$, iff $\alpha \subseteq \beta$.
2. For all $\alpha, \beta \in \nabla$, $\{a \in K : R\, a\, \alpha\, \beta\} \in \nabla$.
3. If $R x \alpha \beta$ for all $x \in O$, then (i) if $R\, a\, \beta\, \gamma$ then $R\, a\, \alpha\, \gamma$, and (ii) if $R\, a\, \gamma\, \alpha$ then $R\, a\, \gamma\, \beta$, for all $a \in K$.
4. $\alpha^{**} = \alpha$.
5. $R\, x\, \alpha\, \beta$ for all $x \in O$ iff $R\, x\, \beta^*\, \alpha^*$ for all $x \in O$.
6. ∇ is closed under set-theoretical intersection and union.

Putting $\alpha \leqq \beta \equiv_{d.f.} R\, x\, \alpha\, \beta$ for all $x \in O$[2], then 1, 3 and 5 can be written:

1. $\alpha \leqq \beta$ iff $\alpha \subseteq \beta$.
3. If $\alpha \leqq \beta$, then .....
5. $\alpha \leqq \beta$ iff $\beta^* \leqq \alpha^*$.

Interpretation s on unreduced model structures are defined exactly as in the case of model structures (definition 5.1.2).

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1. To be contrasted with definition 5.1.1 of BB-m.s. where $O \in K$.
2. In stark contrast to the case for B, where $\alpha \leqq \beta \equiv_{d.f.} R\, x\, \alpha\, \beta$ for some $x \in O$, RLR, p.298.

Definition 5.4.9. A wff A is true on I in $\underset{\sim}{M}$, where $\underset{\sim}{M}$ is an unreduced model structure, iff $I(A, x) = T$ for all $x \in 0$. A is $\underset{\sim}{M}$-valid iff A is true on all I in $\underset{\sim}{M}$. A is BB-u.m.s. valid iff A is $\underset{\sim}{M}$-valid for all BB-u.m.s. $\underset{\sim}{M}$.

We prove that BB is sound w.r.t. the class of unreduced BB-model structures using a representation theorem, just as in the case of BB-model structures (theorem 5.1.7).

Definition 5.4.10. The algebra of ranges $\underset{\sim}{G}^M$ of an unreduced BB-model structure $\underset{\sim}{M} = \langle 0, K, \nabla, R, * \rangle$ is defined as $\underset{\sim}{G}^M = \langle \nabla, F, \cap, \cup, -, \rightarrow \rangle$ where $\cap$ and $\cup$ have their set-theoretical sense, $-\alpha = \alpha *$, $\alpha \rightarrow \beta = \{a \in K : R\ a\ \alpha\ \beta\}$ and $F = \{\alpha \in \nabla : 0 \subseteq \alpha\}$.

Clearly $\underset{\sim}{G}^M$ is well-defined: we check that it is a BB-algebra. But first, note that the difference with definition 5.1.5. is only in the specification of F.

Lemma 5.4.11. For every unreduced BB-model structure $\underset{\sim}{M} = \langle 0, K, \nabla, R, * \rangle$ the algebra of ranges $\underset{\sim}{G}^M = \langle \nabla, F, \cap, \cup, -, \rightarrow \rangle$ is a BB-algebra.

Proof ad 1 : $\langle \nabla, \cap, \cup, - \rangle$ is obviously a De Morgan lattice (as in lemma 5.1.6).

ad 2 : $\alpha \cap \beta \in F$ iff $0 \subseteq \alpha \cap \beta$, which holds iff $0 \subseteq \alpha$ and $0 \subseteq \beta$, that is, iff $\alpha \in F$ and $\beta \in F$. So F is a filter. (This property, on the usual definition of $\leq$, ensures upward closure of F.)

Note that, since $\alpha \leqq \beta$ in $\underset{\sim}{M}$ iff $\alpha \subseteq \beta$ in ∇ (definition 5.4.8), we have that $\alpha \leqq \beta$ in $\underset{\sim}{M}$ iff $\alpha \leqq \beta$ in $\underset{\sim}{G}^M$ (that is, $\alpha \cap \beta = \alpha$ in $\underset{\sim}{G}^M$).

<u>ad 3</u>: $\alpha \rightarrow \beta \in F$ iff $O \subseteq \{x \in K : R\, x\, \alpha\, \beta\}$

iff $R\, x\, \alpha\, \beta$ for all $x \in O$, i.e.

iff $\alpha \leqq \beta$.

<u>ad 4</u>: If $\alpha \leqq \beta$, then $\{x \in K : R\, x\, \beta\, \gamma\} \subseteq \{x \in K : R\, x\, \alpha\, \gamma\}$ by 3(i),

i.e. $\beta \rightarrow \gamma \leqq \alpha \rightarrow \gamma$.

(The partial order on $\underset{\sim}{G}^M$ is just that induced by the subset algebra, since $\alpha \leqq \beta \equiv_{df} \alpha \cap \beta = \alpha$.)

<u>ad 5</u>: Similarly, using 3(ii) of definition 5.4.8. □

<u>Theorem 5.4.12</u>. For every unreduced BB-model structure $\underset{\sim}{M}$ and its algebra of ranges $\underset{\sim}{G}^M$, a wff A is $\underset{\sim}{M}$-valid iff A is $\underset{\sim}{G}^M$-valid.

<u>Proof</u> We show that for each interpretation I on $\underset{\sim}{M}$ there is an interpretation I' on $\underset{\sim}{G}^M$, and vice-versa, such that for every wff A, A is true on I in $\underset{\sim}{M}$ iff A is true on I' in $\underset{\sim}{G}^M$.

Suppose we have some interpretation I on $\underset{\sim}{M}$, then define I' on $\underset{\sim}{G}^M$ as: $I'(A) = |A| = \{x \in K : I(A, x) = T\}$, for all wffs of A. Just as in theorem 5.1.7 I' is well-defined, and clearly $I'(A) \in F$ iff $I(A, x) = T$ for all $x \in O$. Conversely, if we have I' on $\underset{\sim}{G}^M$, then we can define I on $\underset{\sim}{M}$ as: $I(A, x) = T$ iff $x \in I'(A)$, for all wffs A. Clearly the same strong connection holds between I and I', completing the proof of the theorem. □

Soundness quickly follows.

Theorem 5.4.13. BB is sound w.r.t. unreduced BB-model structures.

Proof Clearly it follows from theorem 5.4.12 and theorem 3.1.4 that if $\vdash_{BB}$ A then A is BB-u.m.s. valid. □

To prove that BB is complete w.r.t. the class of unreduced BB-model structures, we shall use the fact that for every non-theorem A of BB, there exists a prime, regular BB-I. theory which doesn't contain A; and that in general priming works for BB-I. theories. In our earlier discussion we saw that the completeness proof for model structures in effect relies on a priming result for L-D. theories which doesn't hold for all extensions of BB (theorem 5.5.8). But priming for L-I. theories does hold for all extensions of BB (theorem 5.4.3). Hence unlike our completeness result for BB-model structures, that for unreduced BB-model structures is generalisable to all extensions of BB.

Definition 5.4.14. The canonical unreduced BB-model structure $\underset{\sim}{M}^c = \langle O^c, K^c, \nabla^c, R^c, *^c \rangle$ is defined by:

O^c = {prime, regular BB-I.theories}; K^c = {prime BB-I.theories};

∇^c = { |A| : A is a wff} where $|A| = \{T \in K^c : A \in T\}$, that is |A| is the set of prime BB-I.theories which contain A;

for any $T \in K^c$, $R^c\ T\ |B|\ \ |C|$ iff $B \rightarrow C \in T$; and

$|A|* = |{\sim}A|$.

<u>Theorem 5.4.15.</u> The canonical unreduced BB-model structure $\underset{\sim}{M}^c$ is a BB-u.m.s.

<u>Proof</u> We check that $\underset{\sim}{M}^c$ satisfies the requirements of definition 5.4.8.

<u>ad 1</u>: To prove that $|A| \leqq |B|$ (that is, for all $T \in 0^c$, $R^c\ T\ |A|\ |B|$) iff $|A| \subseteq |B|$, we first prove that $|A| \leqq |B|$ iff $\vdash_{BB} A \rightarrow B$. We prove 'only if' by contraposition - suppose that $\nvdash_{BB} A \rightarrow B$, then by theorem 5.4.3 there exists a prime, regular BB-I.theory which doesn't contain $A \rightarrow B$, but then it is not the case that $R^c\ T\ |A|\ |B|$ for all $T \in 0^c$, i.e. $|A| \nleqq |B|$. For the converse, if $\vdash_{BB} A \rightarrow B$ then $A \rightarrow B$ is in all regular BB-I.theories, in particular all elements of 0^c, hence $|A| \leqq |B|$. It remains to prove that $\vdash_{BB} A \rightarrow B$ iff $|A| \subseteq |B|$. 'only if' follows immediately from the definition of BB-I.theories. For the converse, suppose that $|A| \subseteq |B|$. Now consider $T = \{D : \vdash_{BB} A \rightarrow D\}$. By the obvious particular case of lemma 5.4.2 T is a BB-I.theory. Suppose that $B \notin T$, then by theorem 5.4.3 there is a prime BB-I.theory T' such that $T \subseteq T'$ and $B \notin T'$. Since $\vdash_{BB} A \rightarrow A$, $A \in T'$, and T' is contrary to our supposition that $|A| \subseteq |B|$. Hence by reductio $B \in T$ and so $\vdash_{BB} A \rightarrow B$ as required.

<u>ad 2</u>: For $|A|, |B| \in \nabla^c$, $\{T \in K^c : R^c\ T\ |A|\ |B|\} = |A \rightarrow B|$, which is in ∇^c.

<u>ad 3</u> (i): Suppose that $|A| \leqq |B|$, then by the argument of ad 1 $\vdash_{BB} A \rightarrow B$. Hence $\vdash_{BB} B \rightarrow C \mathrel{\dot{\rightarrow}} A \rightarrow C$, so if $R^c\ T\ |B|\ |C|$, i.e. if $B \rightarrow C \in T$, then $A \rightarrow C \in T$, i.e. $R^c\ T\ |A|\ |C|$.

<u>ad 3</u> (ii) : Similarly.

ad 4: $|A|^{*^c *^c} = |{\sim}A|^{*^c} = |{\sim\sim}A| = |A|$ since $\vdash_{BB} A \leftrightarrow {\sim\sim}A$.

ad 5: $|A| \leq |B|$ iff $\vdash_{BB} A \rightarrow B$

iff $\vdash_{BB} \sim B \rightarrow \sim A$ (by rule-contraposition)

iff $|{\sim}B| \leq |{\sim}A|$

iff $|B|^{*^c} \leq |A|^{*^c}$.

ad 6: Suppose that $|A|, |B| \in \nabla^c$, then:

(i) $|A| \cap |B| = \{T \in K^c : A \in T \text{ and } B \in T\}$

$= |A \,\&\, B|$ by (A2), (A3) and closure of such T under BB-entailment, and their closure under adjunction.

(ii) $|A| \cup |B| = \{T \in K^c : \text{either } A \in T \text{ or } B \in T\}$

$= |A \vee B|$ by (A5), (A6) and closure under entailment, and the fact that such T are prime.

This completes the proof that $\underset{\sim}{M}^c$ is a BB-u.m.s. □

Theorem 5.4.16. BB is complete w.r.t. the class of unreduced BB-model structures.

Proof We use the canonical interpretation to falsify all non-theorems of BB.

Put $I^c(A, P) = T$ iff $P \in |A|$. I^c is obviously an interpretation on $\underset{\sim}{M}^c$:

$I^c(A \& B, P) = T$ iff $P \in |A \& B|$

iff $P \in |A|$ and $P \in |B|$ (as in ad 6(i) of theorem 5.4.15)

iff $I^c(A, P) = T$ and $I^c(B, P) = T$.

$I^c(A \vee B, P) = T$ iff $P \in |A \vee B|$

iff either $P \in |A|$ or $P \in |B|$ (as in ad 6(ii) of theorem 5.4.15)

iff either $I^c(A, P) = T$ or $I^c(B, P) = T$.

$I^c(\sim A, P) = T$ iff $P \in |\sim A|$

iff $P \in |A|^{*^c}$.

$I^c(A \rightarrow B, P) = T$ iff $P \in |A \rightarrow B|$

iff $A \rightarrow B \in P$

iff $R^c P |A| |B|$.

Furthermore, for any non theorem A of BB, by theorem 5.4.3 (or 4.2.8), there is a prime, regular BB-I.theory which doesn't contain A. So for some $P \in O^c$, $I^c(A, P) = F$ and hence A is not true on I^c in $\underset{\sim}{M}^c$. □

We now note that the definition of unreduced BB-model structures can be extended to that of unreduced L-model structures, analogously to the case of model structures. Let the logic L have an axiomatic characterisation which includes the axioms and rules of BB. Uniformly replace the sentential variables of the extra axiom and rule schemata by variables representing arbitrary members of ∇. Replace $\vdash \phi$ by $0 \subseteq \phi$ in their statements. Now replace the occurrences of connectives as follows:

$\alpha \to \beta$	becomes	$\{x \in K : R\, x\, \alpha\, \beta\}$	
$\alpha \,\&\, \beta$	becomes	$\alpha \cap \beta$	(set-theoretical
$\alpha \vee \beta$	becomes	$\alpha \cup \beta$	intersection and union)
$\sim \alpha$	becomes	α^*	

Repeat this process until all the logical connectives are removed. Finally, replace $0 \subseteq \{x \in K : R\, x\, \alpha\, \beta\}$ by $\alpha \leq \beta$.

The procedure is exactly as in the case of model structures, except that rather than $0 \in |A|$ corresponding to truth in the model, we require $0 \subseteq |A|$.

Definition 5.4.17. An unreduced L-model structure, where $BB \subseteq L$, is an unreduced BB-model structure which satisfies the conditions, according to the above recipe, corresponding to the extra axioms and rules of L.

We now extend the soundness and completeness theorems.

Theorem 5.4.18. Each $L \supseteq BB$ is sound w.r.t. the class of unreduced L-model structures.

Proof The only modification required to extend theorem 5.4.12 is to check that the algebra $\underset{\sim}{G}^M$ satisfies the extra requirements for L-algebras. But since $\alpha \in F$ iff $0 \subseteq \alpha$, the algebraic analogues of the axioms and rules hold, by the definition of L-u.m.s. . □

Definition 5.4.19. The canonical unreduced L-model structure.

Let L be any extension of BB. We define the canonical unreduced L-model structure $\underset{\sim}{M}^c = \langle 0^c, K^c, \nabla^c, R^c, *^c \rangle$ exactly as in the definition of the canonical unreduced BB-model structure (definition 5.4.14), except that BB is replaced throughout by L.

Theorem 5.4.20. For every extension L of BB, the canonical unreduced L-model structure $\underset{\sim}{M}^c$ is an unreduced L-model structure.

Proof That $\underset{\sim}{M}^c$ is a BB-u.m.s. follows from the fact that BB $\subseteq$ L and is proved exactly as in theorem 5.4.15. $\underset{\sim}{M}^c$ also satisfies the extra requirements for L-u.m.s. (definition 5.4.17) because every axiom A occurs in each element of 0^c (so that $0^c \subseteq |A|$); and if the antecedent(s) of a rule occurs in every element of 0^c, then so does the consequent (since by priming (theorem 5.4.3), the former occurs iff that antecedent(s) is a theorem of L). □

Theorem 5.4.21. For every extension L of BB, L is complete w.r.t. the class of unreduced L-model structures.

Proof Exactly as in theorem 5.4.16, by theorem 5.4.3 the canonical interpretation on $\underset{\sim}{M}^c$ falsifies every non-theorem of L. □

So the unreduced model structures are adequate for all extensions of BB. Where the (stronger) model structures of §5.1 fail to be adequate, we can fall back onto the unreduced model structures in order to characterise the corresponding logic.

§ 5.5 Theories and model structures

In section §5.4 we saw that the impediment to universal applicability of L-model structures as characterisations of extensions L of BB corresponds to the fact that priming doesn't hold, in general, for L-D.theories. And, that the universal applicability of unreduced L-model structures is ensured by the fact that priming does hold for L-I.theories. In this section we continue to examine the relationships between these two notions of theoryhood and the two types of relational semantics.

First, it is instructive to prove the completeness result for unreduced model structures using an alternative strategy, that of ALG II (and theorem 5.2.5), where an appropriate representation theorem is used.

Definition 5.5.1. Let $\underset{\sim}{G} = \langle G, F, \cap, \cup, -, \rightarrow \rangle$ be any BB-algebra. The associated unreduced model structure, $\underset{\sim}{M}^G$, of $\underset{\sim}{G}$ is defined as follows: $O = \{P\colon\ P \text{ is a prime filter in } \underset{\sim}{G} \text{ and } F \subseteq P\}$; $K = \{\text{prime filters in } \underset{\sim}{G}\}$; denoting $\{P \in K : a \in P\}$ by $|a|$ for all $a \in G$, $\nabla = \{|a| : a \in G\}$; $|a|^* = |-a|$; and $R\,P\,|a|\,|b|$ iff $a \rightarrow b \in P$.

Clearly $\underset{\sim}{M}^G$ is well-defined, we check that it is indeed an unreduced BB-model structure. Note that, contrary to definition 5.2.1, there is no restriction to prime algebras.

Lemma 5.5.2. For every BB-algebra $\underset{\sim}{G}$, the associated unreduced model structure $\underset{\sim}{M}^G$ is a BB-u.m.s.

Proof ad 1. : $|a| \leqq |b|$ iff $|a| \subseteq |b|$. Now $R\,P\,|a|\,|b|$ for all $P \in O$ iff every prime filter which contains F also contains $a \to b$. Since $\underset{\sim}{G}$ is a distributive lattice the latter holds iff $a \to b \in F$ (using priming once more) (Grätzer 1978, p.64), which holds iff $a \leqq b$ in $\underset{\sim}{G}$. But $a \leqq b$ iff $\{P \in K : a \in P\} \subseteq \{P \in K : b \in P\}$, once again by the prime filter theorem. So we have $|a| \leqq |b|$ iff $|a| \subseteq |b|$ as required.

ad 2. : For all $|a|, |b| \in \nabla$, $\{P \in K : R\,P\,|a|\,|b|\} \in \nabla$ because the set is just $|a \to b|$.

ad 3.(i) : Suffixing. Suppose that $|a| \leqq |b|$, then as in the proof of 1., $a \leqq b$ in $\underset{\sim}{G}$, and so $b \to c \leqq a \to c$ in $\underset{\sim}{G}$, for all $c \in G$. Hence any filter containing $b \to c$ also contains $a \to c$, so if $R\,P\,|b|\,|c|$ then $R\,P\,|a|\,|c|$, as required.

ad 3.(ii) : Prefixing. Similarly.

ad 4. : Obviously $|a|^{**} = |a|$.

ad 5. : $|a| \leqq |b|$ iff $|b|^* \leqq |a|^*$. Clearly follows from rule-contraposition in $\underset{\sim}{G}$, and the fact that $|a| \leqq |b|$ iff $a \leqq b$ in $\underset{\sim}{G}$.

ad 6. : ∇ is closed under set-theoretical intersection and union. Proof is exactly as in lemma 5.2.2. □

As in the case of prime algebras and their associated model structures (definition 5.2.1), we show that a wff A is $\underset{\sim}{M}^G$-valid iff A is $\underset{\sim}{G}$-valid by proving that $\underset{\sim}{G}$ can be regarded as the algebra of ranges of $\underset{\sim}{M}^G$, that is, $\underset{\sim}{G} \equiv \underset{\sim}{G}^{M^G}$.

Theorem 5.2.3. Let $\underset{\sim}{G}$ be any BB-algebra. Then there is a structure-preserving isomorphism from $\underset{\sim}{G}$ to $\underset{\sim}{G}^{M^G}$, the algebra of ranges of $\underset{\sim}{M}^G$.

Proof Consider the function $Q : G \mapsto \nabla$ defined by $Q(a) = |a|$. Recall that $\underset{\sim}{G}^{M^G} = \langle \nabla, F_1, \cap, \cup, -, \rightarrow \rangle$ where ∇ is that of $\underset{\sim}{M}^G$, $\cap$ and $\cup$ have their set-theoretical sense, $F_1 = \{|a| \in \nabla : 0 \subseteq |a|\}$, $-|a| = |a|*$ and $|a| \rightarrow |b| = \{P \in K : R\,P\,|a|\,|b|\}$ (definition 5.4.10). We check that Q is an isomorphism. Clearly it is onto, by the definition of ∇. That it is 1-1 is exactly as for theorem 5.2.3. That it preserves the operations is also exactly as for theorem 5.2.3. Finally, we show that the truth filter F of $\underset{\sim}{G}$ is the inverse-image of F_1 under Q. For all $a \in \underset{\sim}{G}$, $a \in F$ iff for every $P \in 0$, $a \in P$ (by the prime filter theorem). Thus, $a \in F$ iff $P \in |a|$ for every $P \in 0$, that is, iff $0 \subseteq |a|$, which holds iff $|a| \in F_1$. Thus $a \in F$ iff $Q(a) \in F_1$. □

Theorem 5.5.4. For all BB-algebras $\underset{\sim}{G}$, a wff a is $\underset{\sim}{G}$-valid iff A is $\underset{\sim}{M}^G$-valid.

Proof The theorem follows from the fact that $\underset{\sim}{G} \equiv \underset{\sim}{G}^{M^G}$ and the fact that a wff A is $\underset{\sim}{M}^G$-valid iff A is $\underset{\sim}{G}^{M^G}$-valid (theorem 5.4.12), exactly as for theorem 5.2.4. □

This obviously provides us with an alternative completeness proof along the lines of theorem 5.2.5.

Theorem 5.5.5. BB is complete w.r.t. the class of unreduced BB-model structures.

Proof The theorem follows immediately from theorem 5.5.4 above and theorem 3.1.4. □

These results can be extended to all logics L which contain BB, just as before (c.f. theorems 5.2.6 and 5.2.7). However, in this case the extension is perfectly general, for there is no requirement that $\underset{\sim}{G}$ be prime in definition 5.5.1. So it is not dependent on completeness results w.r.t. prime algebras.

Theorem 5.5.6. Lemma 5.5.2, and theorems 5.5.3, 5.5.4 and 5.5.5 hold when BB is replaced throughout by L, where L is an extension of BB.

Proof That the generalisation of the lemma 5.5.2 holds is obvious from the definition of unreduced L-model structures (definition 5.4.17). Statements of the form $a \in F$ are simply replaced by statements of the form $0 \subseteq |a|$. Theorems 5.5.3 and 5.5.4 apply because L-algebras are in particular BB-algebras. The generalisation of theorem 5.5.5 then follows using theorem 3.2.2. □

Note that theorems 5.5.5 and 5.5.6 can easily be modified to provide soundness of the logics w.r.t. the unreduced model structures as well. For given any unreduced model structure $\underset{\sim}{M}$ we have the algebra of ranges $\underset{\sim}{G}^{M}$ with the corresponding $\underset{\sim}{M}^{G^M}$ such that a wff A is $\underset{\sim}{M}$-valid iff A is $\underset{\sim}{G}^{M}$-valid (theorem 5.4.12), which holds iff A is $\underset{\sim}{M}^{G^M}$-valid (theorem 5.5.4). So $\underset{\sim}{M}$-validity corresponds to $\underset{\sim}{M}^{G^M}$-validity and soundness follows using theorem 5.5.4.

The above completeness proof shows that the move from L-model structures to unreduced L-model structures is a move from adequately characterising only the *prime* L-algebras to adequately characterising *all* L-algebras, and hence L. This is highlighted by the fact that the class of sets of wff true on interpretations I on L-model structures $\underset{\sim}{M}$ is identical with the class of *prime* regular L-D. theories (and hence to the class of sets of wff true on interpretations I on *prime* L-algebras $\underset{\sim}{G}$, by theorem 5.4.7), whereas the class of sets of wff true on interpretations I on *unreduced* L-model structures $\underset{\sim}{M}$ is identical with the class of *all* regular L-D. theories (and hence to the class of sets of wff true on interpretations I on L-algebras $\underset{\sim}{G}$). We now prove these facts.

Theorem 5.5.7. Let the logic L be any extension of BB. Then the class of prime, regular L-D. theories is identical with the class of sets of wff true on interpretations I in L-model structures $\underset{\sim}{M}$. Alternatively, every such pair (I, $\underset{\sim}{M}$) determines a unique prime, regular L-D. theory, and vice-versa.

Proof Given a prime, regular L-D. theory T, there is a prime L-algebra $\underset{\sim}{G}$ and an interpretation I on $\underset{\sim}{G}$ which verifies all and only the wffs in T (by theorem 5.4.7). But applying theorem 5.2.3 we have an I' on the L-m.s. $\underset{\sim}{M}^{G}$ which verifies exactly T. [Alternatively, one can construct the required T-canonical L-m.s. $\underset{\sim}{M} = \langle 0, K, \nabla, R, * \rangle$ directly (c.f. theorem 5.3.1), putting : $0 = T$; $K = \{$prime L-I. theories$\}$; $\nabla = \{|A| : A$ is a wff$\}$ where $|A| = \{P \in K : A \in P\}$; $R\,x\,|A|\ |B|$ iff $A \rightarrow B \in x$; and $|A|^* = |{\sim}A|$. And use the canonical interpretation $I(A,P) = T$ iff $P \in |A|$.]

Conversely, suppose we have an L-m.s. $\underset{\sim}{M} = \langle 0, K, \nabla, R, * \rangle$ and an interpretation I on $\underset{\sim}{M}$: $T = \{A : I(A, 0) = T\}$ (definition 5.1.4) is regular because L-m.s. verify the theorems of L (theorem 5.1.11). Furthermore, T is closed under the rules of L because corresponding to each, $\vdash A \Rightarrow \vdash B$, we have that $0 \in \alpha \Rightarrow 0 \in \beta$ where α and β are the model-theoretic analogues of A and B respectively (as per definition 5.1.9). But this ensures that if $0 \in \{x \in K : I(A, x) = T\}$ then $0 \in \{x \in K : I(B,x) = T\}$, that is, if $A \in T$ then $B \in T$. Finally, T is prime since $I(A \vee B, 0) = T$ iffeither $I(A, 0) = T$ or $I(B,0) = T$ (definition 5.1.2). Thus T is a prime, regular L-D. theory. □

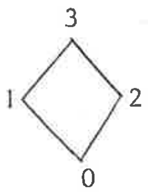
Before proving the second fact, that the set of wffs true on I in L-u.m.s. $\underset{\sim}{M}$ is identical with the set of regular L-D.theories, we prove the promised incompleteness result. Namely, that some extensions L of BB cannot be characterised by the L-model structures. We do so by displaying a particular non-theorem of a particular such L which is a member of every prime, regular L-D.theory. That is to say, we display a counter-example which shows that priming (w.r.t. the L-D.theories) fails for L. Hence, by the above theorem 5.5.7, that particular non-theorem is L-m.s. valid. This highlights the need to utilise the unreduced model structures.

<u>Theorem 5.5.8.</u> Where $L = BB + Av \sim A$ (and $L = BB + Av \sim A + (A4) + (A7) = B + Av \sim A$), there is a non-theorem of L which is a member of every prime, regular L-D. theory.

<u>Proof</u> The following is not a theorem of L :

$$Av \sim (Av \sim A \rightarrow A \,\&\, \sim A)$$

(Neither $\sim (Av \sim A \rightarrow A \,\&\, \sim A)$ nor $Av \sim A \rightarrow A \,\&\, \sim A$ is a theorem of L.) The following L-algebra falsifies $Av \sim (Av \sim A \rightarrow A \,\&\, \sim A)$.



→	0	1	2	*3
0	3	3	3	3
1	1	3	1	3
2	1	1	3	3
*3	1	1	1	3

x	0	1	2	3
~x	3	2	1	0

$F = \{3\}$

Putting I(A) = 2 : $2 \cup -(2 \cup -2 \rightarrow 2 \cap -2)$

$= 2 \cup -(3 \rightarrow 0)$

$= 2 \cup -1$

$= 2 \notin F$

Note that in fact (A4) and (A7) are verified, so the above is a $B + Av \sim A$ - algebra, and hence the above wff is not a theorem of $B + Av \sim A$ either.

We show that no regular, prime L-D. theory T exists which doesn't contain $\phi = p \vee \sim (p \vee \sim p \rightarrow p \,\&\, \sim p)$.

For suppose that $\phi \notin T$ where T is a regular, prime L-D. theory, then since $\vdash_L \phi \vee \sim \phi$ and T is prime $\sim \phi \in T$. So $\sim p \;\&\; (p \vee \sim p \rightarrow p \;\&\; \sim p) \in T$.

(T contains E_{fde} and is closed under modus ponens so that De Morgan transformation and double negation is available.)

Now by (A2) and (A3), the fact that T is regular, and closure under modus ponens, $\sim p \in T$ and $p \vee \sim p \rightarrow p \;\&\; \sim p \in T$. By $\vdash_L \sim p \rightarrow p \vee \sim p$, $\vdash_L p \;\&\; \sim p \rightarrow p$, regularity, rule-suffixing (say) and modus ponens, $p \in T$. But $\vdash_L p \rightarrow \phi$ and so by closure under modus ponens (as T is regular), we have $\phi \in T$ contrary to our supposition. By reductio the claim is proved. □

Thus for L = BB + Av ~ A and L = B + Av ~ A there are non-theorems of L which are valid in every prime L-algebra, and L is not complete w.r.t. the class of L-model structures. The same holds for any other extension L of BB where some non-theorem cannot be rejected by a prime, regular L-D.theory. In such cases our only recourse are the unreduced L-model structures.

Theorem 5.5.9. Let the logic L be any extension of BB. Then the class of regular L-D. theories is identical with the class of sets of wff true on interpretations I in unreduced L-model structures $\underset{\sim}{M}$. Alternatively, every such pair (I, $\underset{\sim}{M}$) determines a unique regular L-D. theory, and vice-versa.

<u>Proof</u> The proof is similar to that of theorem 5.5.7. Given a regular L-D.theory T, there is an L-algebra $\underset{\sim}{G}$ and an interpretation I on $\underset{\sim}{G}$ which verifies all and only the wffs in T (by theorem 5.4.7). And applying theorem 5.5.4 we have an I' on the L-u.m.s. $\underset{\sim}{M}^G$ which verifies exactly T.

[Alternatively, one can construct the required T-canonical L-u.m.s. $\underset{\sim}{M} = \langle O, K, \nabla, R, * \rangle$ directly, putting: $O = \{$prime, regular T-I.theories$\}$; $K = \{$prime T-I.theories$\}$; $\nabla = \{|A| : A$ is a wff$\}$ where $|A| = \{P \in K : A \in P\}$; $R\ P\ |B|\ |C|$ iff $B \to C \in P$; and $|A|^* = |{\sim}A|$ (definition 5.4.14). Since T is an appropriate sort of extension of BB, we can use the T-canonical interpretation I of theorem 5.4.21, and by the theorem I verifies exactly T in $\underset{\sim}{M}$.]

Conversely, suppose we have an L-u.m.s. $\underset{\sim}{M} = \langle O, K, \nabla, R, * \rangle$ and an interpretation I on $\underset{\sim}{M}$. $U = \{A : I(A,x) = T$ for all $x \in O\}$ (definition 5.4.9) is regular because L-u.m.s. verify the theorems of L (theorem 5.4.18). Furthermore, U is closed under the rules of L because their model-theoretic analogues (as per definition 5.4.17) hold in $\underset{\sim}{M}$. So U is a regular L-D.theory. □

Note, this displays some of the structure of the class of such (I, $\underset{\sim}{M}$). The proper subset consisting of (I', $\underset{\sim}{M}$') where I' is the canonical interpretation on a canonical (w.r.t. a regular L-D.theory, i.e. that defined in theorem 5.5.9) L-u.m.s. $\underset{\sim}{M}$, really is canonical for the class of such (I, $\underset{\sim}{M}$). For the rest are redundant from this perspective.

The above theorem states that there is a function

$$F : \{(I, \underset{\sim}{M})\} \mapsto \{\text{regular L-D.theories}\}$$

and a function $G : \{\text{regular L-D.theories}\} \mapsto \{(I', \underset{\sim}{M}')\}$, so we have that $G \circ F : \{(I, \underset{\sim}{M})\} \mapsto \{(I', \underset{\sim}{M}')\}$ is a function. Thus every $(I, \underset{\sim}{M})$ has a unique (canonical) representative in the class of $(I', \underset{\sim}{M}')$, each $(I', \underset{\sim}{M}')$ representing its inverse image under $G \circ F$.

Obviously the same comment applies to the set of $(I, \underset{\sim}{M})$ where $\underset{\sim}{M}$ is an L-model structure, and the set of <u>prime</u>, regular L-D.theories. By theorem 5.5.7 we have the functions

$$F : \{(I, \underset{\sim}{M})\} \mapsto \{\text{prime, regular L-D.theories}\},$$

$$G : \{\text{prime, regular L-D.theories}\} \mapsto \{(I', \underset{\sim}{M}')\}, \text{ and so}$$

$$G \circ F : \{(I, M)\} \mapsto \{(I', \underset{\sim}{M}')\}.$$

Where the $(I', \underset{\sim}{M}')$ represent the canonical interpretations on canonical model structures (determined by prime, regular L-D.theories) (c.f. the proof of theorem 5.5.7).

Of course, given the correspondence between L-algebras and regular L-D.theories (theorem 5.4.7), this is another perspective on theorems 5.5.4 and 5.2.4. The first giving an L-u.m.s. characterisation of L-algebras, and the second giving an L-m.s. characterisation of <u>prime</u> L-algebras.

This illustrates the key role played by the L-I.theories in the move from L-model structures to unreduced L-model structures. The class of interpretations I on L-model structures $\underset{\sim}{M}$ has a proper subset of canonical representatives $(I', \underset{\sim}{M}')$ which deliver every possible set of wffs true on any I in any $\underset{\sim}{M}$ (identical with the set of all prime, regular

L-D.theories). The same holds for unreduced L-model structures (the (I', $\underset{\sim}{M}$') corresponding to the set of all regular L-D.theories). The canonical representatives are constructed using prime L.I.theories. Since priming does not, in general, hold for L-D. theories (theorem 5.5.8) the L-m.s. canonical representatives (and hence the L-m.s.) do not adequately characterise L, in general. It is the fact that priming does hold for L-I.theories (theorem 5.4.3) which ensures that the L-u.m.s. canonical representatives (and hence the L-u.m.s.) do adequately characterise L.

To provide relational semantics for all extensions of BB we have exploited the weaker notion of theoryhood, that of L-I.theories, and the corresponding unreduced model structures.

Let us review the various inter-relationships between the two notions of theoryhood, and the two types of relational semantics.

- The class of prime, regular L-D.theories is:
  - (i) equal to the class of sets of wff true on interpretations I on prime L-algebras $\underset{\sim}{G}$ (theorem 5.4.7);
  - (ii) equal to the class of sets of wff true on interpretations I on L-model structures $\underset{\sim}{M}$ (theorem 5.5.7);
  - (iii) isomorphic with the class of interpretations I on algebras of ranges of L-m.s. $\underset{\sim}{M}$ (theorem 5.2.3. and (i));
  - (iv) isomorphic with the class of canonical interpretations I' on canonical L-m.s. $\underset{\sim}{M}$'.

- The class of regular L-D.theories is:

  (i) equal to the class of sets of wff true on interpretations I on L-algebras $\underset{\sim}{G}$;

  (ii) equal to the class of sets of wff true on interpretations I on unreduced L-model structures $\underset{\sim}{M}$ (theorem 5.5.9);

  (iii) isomorphic with the class of interpretations I on algebras of ranges of L-u.m.s. $\underset{\sim}{M}$ (theorem 5.5.3 and (i));

  (iv) isomorphic with the class of canonical interpretations I' on canonical L-u.m.s. $\underset{\sim}{M}'$.

The following diagram may be helpful. The I' and $\underset{\sim}{M}'$ denote the canonical structures mentioned in the proofs of theorems 5.5.7 and 5.5.9.

(I', L-u.m.s. $\underset{\sim}{M}'$)

$\Updownarrow$

(I,$\underset{\sim}{G}$) ≡ <u>regular L-D.theories</u> ≡ (I, L-u.m.s. $\underset{\sim}{M}$)

| | |

(I, prime $\underset{\sim}{G}$) ≡ <u>prime, regular L-D.theories</u> ≡ (I, L-m.s. $\underset{\sim}{M}$)

$\Updownarrow$

(I', L-m.s. $\underset{\sim}{M}'$)

The top layer adequately characterises extensions L of BB, whereas the bottom layer does not, in general.

It is clear that the traditional notion of theoryhood (corresponding to the regular L-D.theories) provides an important explanatory and unifying tool for analysing the relational semantics of extensions of BB. Further study of the relationships between the two notions of theories (and others, such as that akin to L-I.theoryhood, namely closure under provable material (or other) implication - see page 83) is clearly warranted.

I suggest that one can regard the regular L-D.theories as doing as the logic L does, and the (regular) L-I.theories as doing as the logic says it does. This means regarding the formulation of L as providing a universal paradigm of good reasoning, as well as looking to the provable entailments in L. For example, the set of theorems of Ackermann's system Π' is identical with the set of theorems of E (RLR, p.290), but Π' is formulated as E plus disjunctive syllogism. The class of Π'-I.theories is equal to the class of E-I.theories. That is, the notion of L-I.theoryhood does not distinguish between the two logics (both say they do the same thing). Clearly the class of Π'-D.theories is very different from the class of E-D.theories, however. The formulation of Π' espouses a different paradigm of reasoning.

On this account, the desire for deducibility relations expressed by one's primitive rules to be "supported" by a corresponding provable entailment (the reason given by Belnap for dropping disjunctive syllogism in formulating E, Belnap 1959), is a desire for the logic L to do as it says it does. (And it is difficult to see why the above requirement ought apply to the rules, and yet stop short of requiring the regular L-D.theories to be identical with the regular L-I.theories.)

Since the L-I.theories do as L <u>says</u> it does, it is not surprising that L is characterised by unreduced L-model structures (constructed using the L-I.theories and dependent only on their properties). Our considerations suggest that an appropriate strategy is the following:-

(i) If priming works for the L-D.theories of a logic L (or even for just the regular L-D.theories), then L can be characterised by the class of L-model structures.

(ii) If it is not the case that every non-theorem of L has a prime, regular L-D.theory which rejects it, or if proof of the contrary is wanting, then we can characterise L by the class of unreduced L-model structures.

The need to move from (i) to (ii) in the case of some of the weaker relevant logics - both extensions of BB and of B - signals the fact that our semantics, and the standard semantics for B and its extensions, provides a characterisation of entailment (provable implication) rather than of deducibility as espoused by the formulation of the logic.

For such logics the gap between the two is rather large. Deducibility espoused by the formulation of a logic L is captured by the regular L-D.theories, whilst provable implication corresponds to the L-I.theories. Such considerations ought incline one to view "strong completeness" theorems for the relational semantics with a grain of salt. For example RLR Theorem 4.10(I)[1]:-

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1. RLR, p.315.

"Where $\mho$ is a set of wff not L-derivable from regular set S of wff, there is an L-model which verifies every member of S and falsifies every member of $\mho$."

In this context an L-model is an unreduced relational model structure plus a particular interpretation on it. L-derivability is closely tied to L-I.theoryhood i.e. provable implication: X is L-derivable from Y iff there exists wffs $A_1,\ldots\ldots,A_m$ in Y and $B_1,\ldots\ldots,B_n$ in X such that $\vdash_L A_1 \& \ldots\ldots \& A_m \rightarrow B_1 \vee \ldots\ldots \vee B_n$ [1]. The result also carries over to our unreduced L-model structures.

It seems rather odd that, for the weaker logics (with rules other than modus ponens and adjunction) it could be that $\mho \subseteq Cl\ (S)$ w.r.t. L-D.theories, i.e. every L-D.theory which contains S also contains $\mho$, whilst the antecedent conditions of "strong completeness" are satisfied. So according to the notion of deducibility characterised by the formulation of L, $\mho$ may be deducible from S, and yet an (unreduced) L-model verifies S and falsifies $\mho$. For these logics logical dependence viz-a-viz the traditional notion of deducibility is not captured by the relational semantics. A further manifestation of the split between I.theoryhood and D.theoryhood (and the long haul towards a deduction theorem!).

These relevant (and otherwise intensional) logics open up a rich horizon of possibilities apropos notions of theoryhood.

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1. RLR, p. 307.

§ 5.6. Comparison with the relational semantics for B and its extensions and connexive logics

Our relational semantics for BB and its extensions are somewhat unusual, in that the ternary relation R is on $K \times \nabla \times \nabla$ and the binary relation * is on $\nabla \times \nabla$. The relations are not defined just on the class of "worlds", but include the possible ranges of wffs (as in ESL). This is not the case for the standard semantics for B and its extensions, where it is not necessary to utilise P(K). Let us examine why we need the different approach.

A B-m.s. $\underset{\sim}{M} = \langle O, K, R, * \rangle$ has R defined on $K \times K \times K$ and * defined on $K \times K$.

In the case of B, we have, for all a, b, c, d $\in$ K:

(1) If $a \leqq b$ and R b c d, then R a c d[1].

And the following requirements on interpretations:

(2) I(A → B, a) = T iff, for every b, c $\in$ K, if both R a b c and I(A, b) = T, then I(B, c) = T;

(3) I(A & B, a) = T iff I(A, a) = T and I(B, a) = T.

These three requirements swiftly ensure the valididy of (A4) A → B & A → C $\dot{\rightarrow}$ A → B & C, as follows.

We just need to check that whenever $x \leqq y$ for x, y $\in$ K, then if I(A → B & A → C, x) = T then I (A → B & C, y) = T, for all interpretations I on all model structures $\underset{\sim}{M}$.

(Where ROxyholds iff $x \leqq y$, then the above is equivalent to the requirement that (A4) is true at O.)

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1. RLR, p.298.

So suppose that $x \leqq y$ and $I(A \rightarrow B \& A \rightarrow C, x) = T$. By (3) $I(A \rightarrow B, x) = T$ and $I(A \rightarrow C, x) = T$. Now by (1), if $R\,y\,z\,w$ then $R\,x\,z\,w$. Consider the value of $A \rightarrow B \& C$ at y. If $R\,y\,z\,w$ and $I(A,z) = T$, since $R\,x\,z\,w$, by (2) $I(B,w) = T$ and $I(C,w) = T$, so by (3) $I(B \& C, w) = T$, and hence by (2) $I(A \rightarrow B \& C, y) = T$. Since the argument is perfectly general for model structures, (1), (2) and (3) suffice to ensure the validity of (A4).

Since (A4) is not in general a theorem of the extensions of BB, this shows that the standard approach is inadequate for our purposes.

One might attempt to contain the relations over worlds by loosening up (1), (2) or (3). (1) ensures that the class of valid wff is closed under rule-suffixing, which we also need to be the case. In fact, retaining the evaluation rule (2), if (1) doesn't hold then we can construct model structures where rule-suffixing fails to hold. (2) is a natural evaluation rule for $\rightarrow$, and it is difficult to see how it might be modified yet still deliver rule-suffixing, along with some modification of (1). The remaining alternative is to modify (3), the evaluation rule for conjunction.

Connexive semantics provide a possible strategy for modifying (3) and I will briefly indicate why that strategy is unsuccessful.

Three of the important theorems affirmed by modern connexive logics are $(A \rightarrow B) \rightarrow \sim (A \rightarrow \sim B)$ (Boethius), $\sim (A \rightarrow \sim A)$ (Aristotle) and $\sim((A \rightarrow B) \& (A \rightarrow \sim B))$ (Strawson) (Mortenson 198??). Note that the addition of any one of these to a logic which has a theorem of the form $A \rightarrow \sim A$ results in a negation-inconsistent set of theorems (assuming that $\vdash A \rightarrow A$ and adjunction hold). Furthermore, $A \& \sim A \rightarrow \sim (A \& \sim A)$

is a Tautological Entailment, so the addition of one of the above theses to any extension of E_{fde} (which includes all of the standard relevant logics, BB and its extensions, and classical logic) results in inconsistency. Thus, the E_{fde} theorems $A \& \sim A \rightarrow A$, and $A \& \sim A \rightarrow \sim A$ are rejected by connexive logics. A typical motivation for this rejection is that $\sim A$ cancels out A when they are conjoined, so that the content of $A \& \sim A$ is less than that of A and of that of $\sim A$ (Routley 1979). So conjunction becomes an intensional connective. The semantics are similar to those for B and its extensions, with the addition of a further ternary relation S on the set of worlds K, as well as the generation relation G on wff x K. The modifications in the specification of an interpretation I on a model are:

$I(A \& B, a) = T$ iff there exist $b, c \in K$ s.t. $S\, b\, c\, a$ and $I(A, b) = T$ and $I(B, c) = T$;

If $A\, G\, b$ then $I(A, b) = T$.

The latter clause is a somewhat undesirable feature of a semantics because it involves a relation between propositions and worlds, which is a part-determinant of which propositions are true at which worlds. Although G is only needed to validate the particularly connexive theses, so that we can drop it in the case of BB, it is necessary to impose connections of a similar sort and of a worse order.

Routley provides the modelling conditions for all of BB except (A2), (A3), (A5), (A6), (R.A4) and (R.A7) (Routley 1979). To obtain the first four axioms, we need to add that:

† If $S\, a\, b\, c$ then $a \leqq c$ and $b \leqq c$.

The remaining rules are the stumbling block. To validate (R.A4) without also validating (A4), we need to impose an incestuous relationship between I and S. Now $A \rightarrow B$ and $A \rightarrow C$ are true on I in $\underset{\sim}{M} = \langle 0, K, R, S, * \rangle$ only if for all $x \in K$, $I(A, x) = T$ entails that $I(B, x) = T$ and $I(C, x) = T$. If this is the case, to validate (R.A4) we need to require that, for all $x \in K$, $I(A, x) = T$ entails that $I(B \& C, x) = T$. (Using $|A| = \{x \in K : I(A, x) = T\}$ we need: If $|A| \subseteq |B|$ and $|A| \subseteq |C|$, then $|A| \subseteq |B \& C|$.) But for this to hold we need: There exists $m, n \in K$ such that $S\ m\ n\ x$ and $I(B, m) = T$ and $I(C, n) = T$. It is impossible to ensure that such will be the case, in a manner not sensitive to the particular I, without also validating (A4). For suppose that, for some $a \in K$, it is not the case that $S\ a\ a\ a$. Then we can augment any assignment I using three <u>new</u> propositional variables p_1, p_2, p_3 putting $I(p_i, a) = T$ and otherwise $I(p_i, x) = F$ for $x \in K$ except where $a \leqq x$. Clearly $I(p_1, x) = I(p_2, x) = I(p_3, x)$ for all $x \in K$. This ensures that $p_1 \rightarrow p_2$ and $p_1 \rightarrow p_3$ are true on I in $\underset{\sim}{M}$. For $p_1 \rightarrow p_2 \& p_3$ to be true, we need in particular that $I(p_2 \& p_3, a) = T$ since $I(p_1, a) = T$. So we require that for some $m, n \in K$, $S\ m\ n\ a$ and $I(p_2, m) = T$ and $I(p_3, n) = T$, and by †, $m \leqq a$ and $n \leqq a$. But there are no such $m \leqq a$ and $n \leqq a$, other than $m = n = a$, since by our assignment $I(p_i, x) = F$ for all $x < a$. Hence we require $S\ a\ a\ a$. Thus to ensure that (R.A4) is validated for all interpretations I, we need to specify that $S\ x\ x\ x$ holds for all $x \in K$. This renders S redundant, for then $I(A \& B, x) = T$ iff $I(A, x) = T$ and $I(B, x) = T$. So we are back to our original assignment rule for conjunction (3) and so (A4) is also validated. So to validate (R.A4), but not (A4), we need to use the following I-sensitive condition:

If $|A| \subseteq |B|$ and $|A| \subseteq |C|$, then for all $x \in K$, if $x \in |A|$ then there exist $m, n \in K$ s.t. $S\, m\, n\, x$ and $m \in |B|$ and $n \in |C|$ (where $|A| = \{x \in K : I(A, x) = T\}$).

This negates the point of utilising a B-type relational semantics rather than those I have presented in §5.1 and §5.4, for it is clear that the ranges of wff determined by the interpretation I (corresponding to a subset of the power set of K) are coming heavily into play. Even if such a strategy of modifying (3) were successful, it ought to be regarded very suspiciously, for the ensuing relational semantics would misleadingly suggest that conjunction in BB is intensional rather than extensional, which it clearly is not (as I have already argued). The role of conjunction in augmented variable sharing, the algebraic semantics and the relational semantics (and their De Morganality), show that conjunction in BB and its extensions is extensional.

Of course, for those BL $\supseteq$ BB such that BL = L, and so BL is an extension of B (for example L = R), the standard semantics for B (and its extensions) apply. So in these cases we can use the standard semantics rather than our specialised semantics which cater for extensions of BB. For example, since BR = R, the standard relational semantics for R obviously also adequately characterise BR.

CHAPTER 6

The mistake in ALG II and a partial remedy

§ 6.0. Introduction

As mentioned in §5.0, there is a mistake in ALG II. The strategy in ALG II for the proof of completeness of B and its extensions w.r.t. the relational model structures involves showing how to generate them from prime algebras (groupoids) in a manner which preserves validity in the model, and using the algebraic completeness result. An essential ingredient is what we may call the Priming Lemma: That any non-theorem of L can be falsified in a prime L-groupoid. This strategy is amply illustrated in §5.2 where we utilise it for our own purposes. However, as we found, it breaks down for some of the weaker extensions of BB, because the above essential ingredient fails to hold (theorem 5.5.8).

In ALG II there is a general proof that for a large class of extensions L of B, non-theorems of L can be falsified in prime L-groupoids.[1] So it might be thought that our algebraic semantics, or the class of BL logics (with base logic BB), are in this respect inadequate compared to the standard relevant logics (with base logic B). The purpose of this chapter is to point out that such is not the case. The difficulties with priming do not stop at B (as theorem 5.5.8 shows).

In the first section we show that the essential Priming Lemma for extensions of B fails (thus Theorem 7 of ALG II is false)[2]. In the second section we indicate where the proof of the Priming Lemma in ALG II goes awry, by providing a counter-example to an essential lemma

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1. ALG II, Theorem 7.
2. Hence the proof of the adequacy of the relational semantics in ALG II (Theorem 10) is invalid.

for the case of $\underline{\underline{B}}$. ($\underline{\underline{B}}$ is defined in the next section.) Note that this shows that the proof fails for cases even where the Priming Lemma <u>does</u> hold. Finally, we indicate how to resurrect a proof of the Priming Lemma for $\underline{\underline{B}}$ and a couple of its extensions, similar to our proofs for BB (theorems 4.2.8 and 4.2.11).

§ 6.1. <u>The Priming Lemma fails</u>

We show that Theorem 7 of ALG II is erroneous, the theorem states:

"Where L is any one of the extensions of $\underline{\underline{B}}$, or its parts introduced, A is a theorem of L iff A is valid in every prime L-groupoid."[1]

$\underline{\underline{B}}$ is just B with fusion ° and the sentential constant t added to the language, plus the following extra axioms and rule:

(A10) $\vdash t$; (A11) $\vdash t \mathrel{\underset{\cdot}{\to}} A \to A$;

(R6) $\vdash (A \circ B) \to C \iff \vdash A \mathrel{\underset{\cdot}{\to}} B \to C$ (residuation).

($\underline{\underline{B}}$ is a conservative extension of B.)

Theorem 5.5.8 states that a non-theorem of B + A v ~ A is a member of every prime, regular B + A v ~ A - D.theory. By the correspondence between regular L-D.theories and the classes of wffs true on particular interpretations in particular L-algebras, just as in the case of BB + A v ~ A, it follows that there exists a non-theorem of B + A v ~ A which is valid in every prime B + A v ~ A - algebra (theorem 5.4.7). We show that the same holds for $\underline{\underline{B}}$ + A v ~ A and $\underline{\underline{B}}$ + A v ~ A - groupoids.

1. ALG II, page 26.

<u>Definition 6.1.1.</u> A <u>De Morgan groupoid</u> is a structure

$\underset{\sim}{G} = \langle G, 1, \cap, \cup, -, \rightarrow, \circ \rangle$ such that:

(i) $\langle G, \cap, \cup, - \rangle$ is a De Morgan lattice;

(ii) $\circ$ is a binary operation on G and $1 \circ a = a$ for all $a \in G$;

(iii) $a \circ b \leqq c$ in $\underset{\sim}{G}$ iff $a \leqq b \rightarrow c$, for all a, b, c $\in$ G

(where as usual $x \leqq y =_{df} x \cap y = x$).

<u>Definition 6.1.2.</u> An <u>interpretation</u> I on a De Morgan groupoid $\underset{\sim}{G}$ is defined exactly as for BB-algebras (definition 3.1.2) plus:

$I(A \circ B) = I(A) \circ I(B)$ and $I(t) = 1$.

Validity in De Morgan groupoids, etc., are defined exactly as in definition 3.1.3.

$\underline{\underline{B}}$ is sound and complete w.r.t. the class of De Morgan groupoids. (Theorem 2 of ALG II).

<u>Definition 6.1.3.</u> A <u>$\underline{\underline{B}}$ + A v ~ A - groupoid</u> is a De Morgan groupoid which also satisfies: $1 \leqq a \cup -a$ for all $a \in G$.

<u>Theorem 6.1.4.</u> There is a non-theorem of $\underline{\underline{B}}$ + A v ~ A which is valid in every prime $\underline{\underline{B}}$ + A v ~ A - groupoid.

Proof Now the truth filters of the L-groupoids are closed under the algebraic analogues of the axioms and rules of L.[1] So the set of wffs true on an interpretation in an L-groupoid is a regular L-D.theory (c.f. theorem 5.4.7).

Note that p v ~ (p v ~ p → p & ~ p) is not a theorem of $\underline{\underline{B}}$ + A v ~ A. We can use the appropriate conservative extension result or simply augment the L-algebra of theorem 5.5.8 by the following definition of fusion,

∘	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	1	2	3
3	0	1	2	3

and put 1 = 3 (the groupoid "strongest truth"). This gives a $\underline{\underline{B}}$ + A v ~ A - groupoid which falsifies the above wff.

Now if some prime $\underline{\underline{B}}$ + A v ~ A - groupoid falsifies p v ~ (p v ~ p → p & ~ p) then we have a prime, regular $\underline{\underline{B}}$ + A v ~ A - D.theory which doesn't contain p v ~ (p v ~ p → p & ~ p). But the proper subset of wffs of such a theory, which don't contain ∘ or t, is a prime, regular B + A v ~ A - D.theory (since the rules and axioms of B are a subset of those for $\underline{\underline{B}}$) which doesn't contain p v ~ (p v ~ p → p & ~p), contrary to theorem 5.5.8.

Hence by reductio no such prime $\underline{\underline{B}}$ + A v ~ A - groupoid exists. □

1. See the soundness proof in ALG II.

Noting that $\underline{\underline{B}}$ + A v ~ A is amongst the class of extensions of $\underline{\underline{B}}$ introduced in ALG II, the above result refutes Theorem 7 of ALG II.

§ 6.2. Where the Priming Lemma goes awry

The proof of Theorem 7 in ALG II fails in the particular case that L = $\underline{\underline{B}}$. The proof uses a lemma which states that given an L-groupoid $\underset{\sim}{G}$ and a filter F in $\underset{\sim}{G}$ such that 1 ∈ F, a structure called a quotient L-groupoid $\frac{\underset{\sim}{G}}{F}$ which "factors out" F, is an L-groupoid.[1] $\frac{\underset{\sim}{G}}{F}$ simply identifies elements which F says are logically (entailment-wise) equivalent: $|a| = \{b \in G : (a \rightarrow b) \cap (b \rightarrow a) \in F\}$.
And operations are defined representation-wise, e.g. $|a| \cap |b| = |a \cap b|$. However this fails to deliver an L-groupoid. The problem is once again that gap between traditional deducibility and provable entailment, the L-D.theories and L-I.theories. Factoring out w.r.t. equivalence according to entailment will be successful where the underlying structures correspond to L-I.theories, but will fail where the underlying structures correspond to L-D.theories. So the above sort of factoring process works in the case of the relational model structures, which characterise provable implication rather than traditional deducibility, and fails in the case of algebras (in general), which characterise traditional deducibility (c.f. §5.5).

I shall simply specify the De Morgan groupoid ($\underline{\underline{B}}$-groupoid) and filter F, which provides a counter-example.

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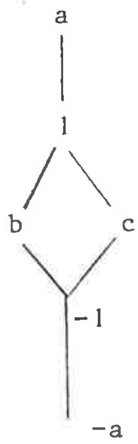
1. ALG II p. 28.



x	-a	-1	b	c	1	a
-x	a	1	b	c	-1	-a

→	-a	-1	b	c	1	a
-a	a	a	a	a	a	a
-1	-a	a	a	a	a	a
b	-a	b	a	b	a	a
c	-a	-a	-a	a	a	a
1	-a	-a	-a	b	a	a
a	-a	-a	-a	-a	-a	a

∘	-a	-1	b	c	1	a
-a	-a	-a	-a	-a	-a	-a
-1	-a	-1	-1	c	c	a
b	-a	-1	-1	c	c	a
c	-a	-1	b	c	1	a
1	-a	-1	b	c	1	a
a	-a	-1	b	c	1	a

$\cap$ and $\cup$ are specified by the Hasse diagram in the usual way. For the prime filter $F = \{b, 1, a\}$ $\frac{\underset{\sim}{G}}{F}$ is not a De Morgan groupoid. Residuation fails because we have $|1| \circ |b| = |b| \leq |c|$ (since $b \rightarrow c \in F$) and $|1| \nleq |b| \rightarrow |c|$.

§ 6.3. A valid proof of the Priming Lemma for limited extensions of B

In §4.2 we proved that BB is both prime and negation-consistent by showing that given any BB-model $\underset{\sim}{G}$ and interpretation I on it, there is a prime and negation-consistent BB-model $\underset{\sim}{G}'$ and interpretation I' on it such that all wffs false on I in $\underset{\sim}{G}$ are also false on I' in $\underset{\sim}{G}'$ (theorem 4.2.7). It immediately follows that every nontheorem of BB can be falsified in a prime BB-algebra, by the algebraic completeness result (theorem 3.1.4) - one applies the priming process to the Lindenbaum algebra. In fact the procedure of theorem 4.2.7 can be extended to De Morgan groupoids, so that it is possible to use this strategy to prove that $\underline{\underline{B}}$ can be characterised by the class of prime De Morgan groupoids (and similarly the extensions of $\underline{\underline{B}}$ corresponding to those of BB for which the process works - theorem 4.2.11). Which is Theorem 7 of ALG II in the particular case that L is replaced by $\underline{\underline{B}}$.

Let us first note that the result holds for B-algebras.

Theorem 6.3.1. The class of BB-algebras which satisfy (A4) and (A7) (i.e. the B-algebras) satisfy theorem 4.2.7.

Proof We simply need to prove that where $\underset{\sim}{G}$ satisfies

(i) $(a \to b) \cap (a \to c) \leqq a \to b \cap c$ and

(ii) $(a \to c) \cap (b \to c) \leqq a \cup b \to c$, for all a, b, c $\in$ G

then the $\underset{\sim}{G}'$ of theorem 4.2.7 satisfies

(i') $(a^{\alpha} \to' b^{\beta}) \cap' (a^{\alpha} \to' c^{\gamma}) \leqq' a^{\alpha} \to' (b^{\beta} \cap' c^{\gamma})$ and

(ii') $(a^{\alpha} \to' c^{\gamma}) \cap' (b^{\beta} \to' c^{\gamma}) \leqq' (a^{\alpha} \cup' b^{\beta}) \to' c^{\gamma}$

for all a, b, c $\in$ G and $\alpha, \beta, \gamma \in S$ (definition 4.2.1).

Suppose that $\underset{\sim}{G}$ satisfies (i). Now by definition 4.2.2

$$(a^{\alpha} \rightarrow' b^{\beta}) \cap' (a^{\alpha} \rightarrow' c^{\gamma}) = (a \rightarrow b)^{x} \cap' (a \rightarrow c)^{y} \quad \text{where } x, y \in S - \{v\}$$

$$= (a \rightarrow b \cap a \rightarrow c)^{x \cap y}$$

Where $x \cap y = \wedge$ iff $a \rightarrow b$, $a \rightarrow c \in F$ and $\alpha \leqq \beta$ and $\alpha \leqq \gamma$.

$$a^{\alpha} \rightarrow' (b^{\beta} \cap' c^{\gamma}) = a^{\alpha} \rightarrow' (b \cap c)^{\beta \cap \gamma}$$

$$= (a \rightarrow b \cap c)^{z} \quad \text{where } z \in S - \{v\} \text{ and}$$

$z = \wedge$ iff $a \rightarrow b \cap c \in F$ and $\alpha \leqq \beta \cap \gamma$. Thus $z = \wedge$ iff $x \cap y = \wedge$.

Hence by (i), and the fact that $m^{\wedge} \leqq' n^{\wedge}$ if $m \leqq n$ in $\underset{\sim}{G}$, (i') holds.

The proof that if $\underset{\sim}{G}$ satisfies (ii) then $\underset{\sim}{G}'$ satisfies (ii') is similar. □

So we have the Priming Lemma for B, and an independent proof of:

<u>Theorem 6.3.2.</u> B is both prime and negation-consistent.

<u>Proof</u> Apply the procedure of theorems 4.2.7 and 6.3.1 to the Lindenbaum algebra of B, just as in the proof of theorem 4.2.8. □

The procedure only needs to be slightly augmented to cater for De Morgan groupoids. I will simply state the extra requirements and note that the required De Morgan groupoid properties ensue.

Where $\underset{\sim}{G}$ is a De Morgan groupoid, then for $\underset{\sim}{G}'$ (of definition 4.2.2) we also need to define the new fusion operator o'. We do so as follows:

$$a^{\wedge} \text{ o}' \; b^{\beta} = \begin{cases} (a \text{ o } b)^{\beta} & \text{if it exists,} \\ a \text{ o } b & \text{otherwise.} \end{cases}$$

$$\text{For } \alpha \neq \wedge \quad a^{\alpha} \text{ o}' \; b^{\beta} = \begin{cases} (a \text{ o } b)^{\vee} & \text{if it exists ,} \\ a \text{ o } b & \text{otherwise.} \end{cases}$$

The "strongest truth" of $\underset{\sim}{G}'$ is $1^{\wedge}$ where 1 is that of $\underset{\sim}{G}$.

Whence we arrive at the required result:

Theorem 6.3.3. A is a theorem of $\underline{\underline{B}}$ iff A is valid in every prime $\underline{\underline{B}}$-groupoid (De Morgan groupoid).

Obviously the corresponding extensions of theorem 4.2.7 also apply, i.e. the result holds where one or more of prefixing, suffixing and contraposition are added to $\underline{\underline{B}}$ (theorem 4.2.11).

CHAPTER 7

Extensions and open problems

§ 7.0. Introduction

In this chapter we present some open problems, ruminations and extensions.

§ 7.1. B-ing and predicate logics

We have confined our considerations to propositional logics, however nothing prohibits extending our L-Hierarchy process to predicate logics. We could add instantiation $(x)\ Ax \rightarrow At$ and generalisation $\vdash A \rightarrow B \implies \vdash A \rightarrow (x)\ B$ (where x is not free in A) to the intensional base, and construct the corresponding L-Hierarchy (definition 2.3.1). Theorem 2.4.1 still applies and so the correspondence of the L-Hierarchy and axiomatic formulation is preserved.

Since R = BR then the corresponding predicate logics are also equivalent, RQ = BRQ. Thus RQ has an L-Hierarchy. This alternative characterisation of RQ might provide a useful tool for research into RQ and its theories (e.g. arithmetic, $R^{\#}$).

§ 7.2. Undecidability

Alasdair Urquhart's proof that E and R (and a host of other extensions of T) are undecidable (Urquhart 198?) depends on their particular extensional/intensional interplay. When this is weakened, the proof fails to go through. Hence the decidability or otherwise of BT and BE and other relatively strong extensions of BB remains an open question.

§ 7.3. Ortho - BL

Bob Meyer showed that Ortho-R (R minus distribution) is decidable by displaying its equivalence to a suitable Gentzen consecution calculus (Meyer 1966). This calculus is obtained by adding the following rules to the Gentzenisation of $R_{\overset{\sim}{\to}}$ ($LR_{\overset{\sim}{\to}}$) :

$$(\vdash \&) \quad \frac{\alpha \vdash A, \beta \qquad \alpha \vdash B, \beta}{\alpha \vdash A \,\&\, B, \beta} \qquad\qquad (\& \vdash) \quad \frac{\alpha, A \vdash \beta}{\alpha, A \,\&\, B \vdash \beta}$$

$$(\vdash \mathrm{v}) \quad \frac{\alpha \vdash A, \beta}{\alpha \vdash A \mathrm{\ v\ } B, \beta} \qquad\qquad (\mathrm{v} \vdash) \quad \frac{\alpha, A \vdash \beta \qquad \alpha, B \vdash \beta}{\alpha, A \mathrm{\ v\ } B \vdash \beta}$$

where A and B are wffs and α and β are sequences of wffs. Clearly ($\vdash$ &) and (v $\vdash$) correspond to the rules (R.A4) and (R.A7). We can only derive (A4) and (A7) when the interplay between the intensional fragment and these rules is strong enough. Thus Meyer's result for Ortho-R turns on the fact that R = BR.

We can similarly apply this process in the case of E. We add the above rules to a Gentzenisation $LE_{\overset{\sim}{\to}}$ of $E_{\overset{\sim}{\to}}$. But keeping in mind the above considerations we ought not expect the resulting calculus to be equivalent to Ortho-E, since BE $\subsetneq$ E.

Conjecture $LE_{\overset{\sim}{\to}}$ augmented with the above & and v introduction rules is equivalent to Ortho-BE. In general, for an intensional sequence calculus $L_{\overset{\sim}{\to}}$ which is an extension of $BB_{\overset{\sim}{\to}}$ (= $B_{\overset{\sim}{\to}}$), adding the above rules gives Ortho-(BB + $L_{\overset{\sim}{\to}}$).

If such were the case, then as for Ortho-R we could prove that Ortho-BE is decidable.

§ 7.4. BMo and modal logics

The 8-valued logic BM_o (definition 3.3.4) merits further investigation. It is almost modal in the following sense.

The system $S1^o$ is formulated (Feys 1965, p. 43):

Axiom schemata: (1) $A \,\&\, B \rightarrow A$; (2) $A \,\&\, B \rightarrow B \,\&\, A$;

(3) $(A \,\&\, B) \,\&\, C \rightarrow A \,\&\, (B \,\&\, C)$; (4) $A \rightarrow A \,\&\, A$;

(5) $(A \rightarrow B) \,\&\, (B \rightarrow C) \underset{\cdot}{\rightarrow} A \rightarrow C$.

Rules: Modus ponens, adjunction and replacement of strict equivalents.

Definitions: (6) '$A \rightarrow B$' for '$\sim \Diamond\ (A \,\&\, \sim B)$' ;

and v, $\supset$, $\leftrightarrow$ and $\Box$ as usual.

Now (6) says that $A \rightarrow B$ = df $\Box\ (A \supset B)$ and it follows from (6) that $\Box A =_{df} \sim A \rightarrow A$. So we can regard (6) as incorporating both:

(6A) $A \rightarrow B =_{df} A \,\&\, {\sim}B \rightarrow \sim A \vee B$; and

(6B) $\Box A =_{df} \sim A \rightarrow A$.

Fact: BM_o verifies (1) - (5), and hence the logic given by (1) - (5) and (6B).

Further fact: BM_o verifies $A \rightarrow \Diamond A$, $\Diamond (A \,\&\, B) \rightarrow \Diamond A$, $A \rightarrow B \underset{\cdot}{\rightarrow} \Diamond A \rightarrow \Diamond B$ and $\Box A \rightarrow \Box \Box A$.

Thus BM_o verifies what we might call $S1^o$-(6A), S1-(6A), $S2^o$-(6A),S2-(6A S3-(6A) and S4-(6A). While $\Diamond \Diamond A$ is falsified, all its values are either 7 or 6 in BM_o . We can add 6 to the truth filter of BM_o (so replacing $F = \{3, 7\}$ by $F' = \{2, 3, 6, 7\}$) and we still have a BE-algebra which falsifies (A4) $A \to B \;\&\; A \to C \underset{\cdot}{\to} A \to B \;\&\; C$ and (A7) $A \to C \;\&\; B \to C \underset{\cdot}{\to} A \vee B \to C$. So on this slight modification, BM_o also verifies S6-(6A) and S7-(6A).

Unfortunately (6A) does too much intensional work in the above formulation of $S1^o$, as BM_o does not verify $S3\overset{\sim}{\to}$ (Hacking 1963). The stumbling block is $A \to B \underset{\cdot}{\to} A \to \sim B \underset{\cdot}{\to} A \to C$, however this is verified where A is strict, that is, A is an implication. Similarly $A \to B \underset{\cdot}{\to} C \underset{\cdot}{\to} A \to B$, for $S4\overset{\sim}{\to}$, fails to hold, except when C is strict.

So BM_o is almost modal in the sense that it verifies many modal theses of the standard modal logics. For example, if all the variables in an $S4\overset{\sim}{\to}$ theorem are replaced by implications, then the resulting wff is BM_o-valid. Thus it is clear that the L/BL distinction remains a real distinction for many "almost modal" systems - keeping in mind that (A4) and (A7) are invalid in BM_o.

Of course, the perfectly general L-Hierarchy process (definition 2.3.1) can be applied directly to the intensional components of these modal logics. It would be interesting to see how much of the modal nature of the logics remained. For example, $S4 = E + A \to B \underset{\cdot}{\to} C \underset{\cdot}{\to} A \to B$, so BS4 exists (definition 2.5.2) and is equivalent to the L-Hierarchy obtained using $L\overset{\sim}{\to} = S4\overset{\sim}{\to}$ and adjunction. It seems to me likely that $\nvdash_{BS4}$(A4) and $\nvdash_{BS4}$(A7), and that all theorems of S4 which don't contain either & or v are theorems of BS4 (so that the reduction of modalities for S4 also holds for BS4).

§ 7.5. Exploiting the L-Hierarchy

Unfortunately, having proved the equivalence of many systems to an alternative L-Hierarchy formulation (for example, all of the BL logics), I have not greatly exploited that alternative formulation in considering the properties of these logics. The L-Hierarchy provides strong intuitive insights as to the extensional/intensional interplay in such logics and ought, it seems, have much fruit to bear. The following questions come to mind: Can we prove that BB is prime using the corresponding L-Hierarchy; are all intensional extensions of BB (i.e. BB + $L_{\tilde{\rightarrow}}$) prime? The L-Hierarchy might also provide clues for more efficient decision procedures for those appropriate extensions of FDE (definition 2.1.11) which are decidable (and also for proofs of decidability).

CHAPTER 8

Conclusion

We have seen that Anderson and Belnap's notion of variable sharing has a natural generalisation which can be used to formulate logics of arbitrary degree. Furthermore, the extensional/intensional interplay of such logics is perspicuous and strongly intuitively motivated.

The L-Hierarchy process is very general. It can be applied to any set of purely extensional and purely intensional schemata. However, the process represents one fixed analytic perspective, which can be summed up in the following dictum:

> Separate the process which gives rise to entailments purely on the basis of extensional & and v from those entailments and theorems which have other independent motivation, and use it as the core of the extensional/intensional interplay.

This is obviously the key to FDE (definition 2.1.11), WE (definition 2.1.12) and the L-Hierarchy (definition 2.3.1). From this perspective the claim that classical propositional calculus merely depicts the behaviour of extensional connectives, and so, properly understood, its theses are valid, is somewhat misguided. For the purely extensional and-gate, or-gate behaviour of & and v is captured by (2) - (6) of FDE (and WE). To obtain the classical propositional calculus we need to move out of the realm of what & and v signify, and give particular answers to the philosophical questions of what constitutes valid deduction and the nature of truth in all logically (!) possible "worlds". For as I have already noted in §2.1, we need to add $A \& \sim A \rightarrow B$ and $B \rightarrow A v \sim A$ for all atoms B and propositional variables A, to (2) - (6), in order to obtain classical propositional logic (when $\rightarrow$ is replaced by $\supset$).

The L-Hierarchy provides a sound bed-rock upon which to build one's logic, which involves the bare presuppositions incorporated in purely extensional & and v. It provides the most natural way of moving up from the Tautological Entailments to logics of arbitrary degree.

We have also seen that from this perspective, the standard minimal (full degree) relevant logic B is too strong, and have put forward BB as an appropriate base logic. The properties of BB, and the fully developed algebraic and relational semantics for all extensions of BB (including, in particular, the BL logics) displayed in chapters 3 and 5, show that BB can perfectly adequately function in such a role.

Finally, the notion of L-D.theory introduced in §5.4, as well as the standard one of L-I.theory, plays a crucial analytical role in understanding the relational semantics forBB and its extensions. There are clearly many philosophical and logical questions about notions of theoryhood for such relevant (and other intensional) logics which ought to be addressed.

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