Confinement and chiral symmetry breaking via domainlike structures in the QCD vacuum

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A qualitative mechanism for the emergence of domain structured background gluon fields due to singularities in gauge field configurations is considered, and a model displaying a type of mean field approximation to the QCD partition function based on this mechanism is formulated. An estimation of the vacuum parameters (gluon condensate, topological susceptibility, string constant and quark condensate) indicates that domainlike structures lead to an area law for the Wilson loop, nonzero topological susceptibility and spontaneous breakdown of chiral symmetry. Gluon and ghost propagators in the presence of domains are calculated explicitly and their analytical properties are discussed. The Fourier transforms of the propagators are entire functions and thus describe confined dynamical fields.

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I. INTRODUCTION

Nearly every approach to the problem of the QCD ground state accepts that the vacuum is characterized by strong background gluon fields and, as results of lattice calculations suggest, by a clustered or lumpy distribution of topological charge and action density in configuration space. We shall refer to such structures interchangeably as ''clusters'' and ''domains.'' They were first observed in typical lattice gauge configurations via cooling or smearing algorithms [1,2] which incrementally suppress quantum fluctuations by locally minimizing or at least reducing the action density. For more recent work on cooling the reader is referred to [3]. On the other hand, the resulting cooled gauge fields tend to give rise to a diminished string constant indicating a loss of confinement. An alternative way of analyzing the underlying fluctuations of topological charge density is via the chirality of fermionic modes in the background of topological ''lumps,'' as originally undertaken by [4,5] and rediscussed recently in [6]. Used with lattice fermions with good chiral properties such as overlap [7,8] or domain wall [9] fermions the method indicates localization of low-lying fermionic modes with definite chirality, the very modes responsible for the chiral condensate, for example [10]. These results can be described in terms of the instanton liquid model [11] and are regarded as evidence for instantons on the lattice. While such an interpretation connects clusters of topological charge with chiral symmetry breaking, it says nothing about their relevance to confinement [12]. However, it might be significant that [8] have repeated the procedure of [6] with overlap fermions and no smearing and still observe strong localization and definite chirality of the low lying modes. This is a piece of evidence for the possibility that localization of chiral fermionic modes is due to the effective degrees of freedom responsible for both confinement and chiral symmetry breaking.

Several mechanisms of confinement have been proposed since the formulation of quantum chromodynamics. All try to realize confinement as a dual-Meissner effect, and thus rely on a condensation of singular gauge configurations such as monopoles and vortices [13,14]. In particular center vortices are considered as effective degrees of freedom relevant both to confinement and chiral symmetry breaking [14]. In general, besides the above-mentioned configurations characterized by topologically conserved charges there exist also topologically trivial domain wall singularities in gauge fixed fields [15]. The form in which singular fields occur in the gauge fixed formulation varies with the gauge choice, but their presence itself is most probably an intrinsically unavoidable feature of non-Abelian theories, universal for a large variety (if not all) of gauge fixing prescriptions. Consensus about this has been growing since the pioneering works of Gribov and Singer [16]. This suggests that the manifestation of singular gauge fields is linked to the type and dimensionality of the manifold of singularities rather than to the peculiarities of their realization within a particular gauge fixing prescription.

We take as a working hypothesis that an effect of this kind can be seen in the restrictive influence of singular gauge fields on fluctuations in the vicinity of singularities [17,18] and formulate a simplified model which allows one to study manifestations of this effect in vacuum properties and quark-gluon dynamics analytically.

The subtleties of separating fields into regular and singular parts and the behavior of regular fields at the singularities are irrelevant if one could calculate the QCD functional integral ''exactly.'' But these issues become crucial if one undertakes approximations [18]. For example, in gauge invariant quantities singularities due to ambiguities in gauge fixing should not occur. In the action such a finiteness, despite singularities in the gauge field, occurs either due to cancellations between derivative and commutator parts in the field

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strength if the singularity is topologically nontrivial (monopole or vortex) or due to finiteness of both terms separately for topologically trivial configurations (domain walls). However this cancellation of singularities in the action density can be destroyed by unconstrained fluctuations around the singular fields. Thus finiteness of the action implies specific constraints on fluctuations. In Ref. [18] an example of this is considered in detail for Polyakov gauge monopoles.

We formulate a model partition function which incorporates singularities in gauge fields effectively via their restrictive effect on fluctuations. We assume that singularities are present in general in gauge potentials, and in their vicinity one can divide an arbitrary field, $\mathcal{A}$, into singular S and regular Q parts:

$$\mathcal{A}^a_\mu(x) = S^a_\mu(x) + Q^a_\mu(x). \tag{1}$$

In order that $\mathcal{A}$ generates finite action, it must be ''close enough'' to a pure gauge configuration in the vicinity of the singularity, meaning that $[Q,S]=0$ and that the field strength for pure gauge S vanishes:

$$S_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu + ig[S_\mu, S_\nu] = 0. \tag{2}$$

This can be realized in two ways. If Eq. (2) is satisfied via a cancellation between derivative and commutator parts then the singularity in S is topologically non-trivial and non-Abelian. If the two parts separately vanish then the singularity is topologically trivial. The gauge potential S is then Abelian, namely a constant unit color vector n^a can be associated with the field S.

To be explicit, at the cost of generality, we shall take the second of these possibilities and further assume that singularities in vector potentials are concentrated on hypersurfaces ∂V_j $(j=1,\ldots,N)$ in Euclidean space of volume V, in the vicinity of which gauge fields can be divided as above into a sum of a singular pure gauge $S^{(j)}_\mu$ and regular fluctuation part $Q^{(j)}_\mu$, with a color vector n^a_j associated with $S^{(j)}$. For such fields to have finite action the fluctuations charged with respect to n_j must obey specific conditions on ∂V_j. The interiors of these regions thus constitute ''domains'' V_j. Demanding finiteness of the classical action density, one arrives at

$$\check{n}_j Q^{(j)}_\mu = 0, \quad \psi = -i\not{\eta}^j e^{i\alpha_j\gamma_5}\psi, \quad \overline{\psi} = \overline{\psi} i\not{\eta}^j e^{-i\alpha_j\gamma_5}, \tag{3}$$

for $x \in \partial V_j$, with the adjoint matrix $\check{n}_j = T^a n^a_j$ in the condition for gluons, and a bag-like boundary condition for quarks, $\eta^j_\mu(x)$ being a unit vector normal to ∂V_j.

Equations (3) indicate that gauge modes neutral with respect to n^a_j are not restricted and provide for interactions between domains. In a given domain V_j the effect of fluctuations in the rest of the system is manifested by an external gauge field $B^a_{j\mu}$ neutral with respect to n^a_j. This motivates an approximation in which domains are treated as decoupled but, simultaneously, with a compensating mean field introduced in their interiors. The model becomes analytically tractable if we consider spherical domains with fixed radius R and approximate the mean field in V_j by a covariantly constant (anti-)self-dual configuration with the field strength

$$\hat{B}^{(j)}_{\mu\nu} = \hat{n}^{(j)} B^{(j)}_{\mu\nu}, \quad \tilde{B}^{(j)}_{\mu\nu} = \pm B^{(j)}_{\mu\nu},$$

$$B^{(j)}_{\mu\nu} B^{(j)}_{\rho\nu} = B^2 \delta_{\mu\rho}, \quad \hat{n}^{(j)} = t^3 \cos\xi_j + t^8 \sin\xi_j,$$

$$\xi_j \in \{(2k+1)\pi/6\}^5_{k=0}, \tag{4}$$

where the parameter $B=\text{const}$ is the same for all domains and the constant matrix $n^a_j t^a$ belongs to the Cartan subalgebra, the generators t^a being in the fundamental representation. Note that since the mean field represents an effect of fluctuations outside the domain there is no source for this field on the boundary and therefore it should be treated as strictly homogeneous in all further calculations. The homogeneity itself appears as a simplifying approximation. Because of the uniformity (on average) of the system outside the domain, slowly varying fields should be taken into account first of all, with leading contributions to this coming from covariantly constant fields inside and on the boundary of a domain, with a dominance of (anti-)self-dual fields, since they are expected to have lower action density [19,20] than arbitrary constant fields (see also the Ginsburg-Landau type consideration in Appendix B, where the appearance of a discrete set of values ξ_j is also motivated).

The model for a partition function that we postulate and then use for calculations describes a statistical system of finite density $v^{-1}=N/V$ composed of $N\to\infty$ noninteracting spherical regions in a total Euclidean volume $V\to\infty$, each of which is characterized by a set of internal parameters with random values: the angle ω_j between chromoelectric and chromomagnetic fields, spherical angles φ_j and θ_j of the chromomagnetic field, the angle ξ_j in the color matrix $\hat{n}_j$, chirality violating angle α_j entering the fermionic boundary condition and the coordinate z_j of a domain. Clusters are characterized also by the fluctuation fields Q^j_μ, ψ^j and $\overline{\psi^j}$ satisfying boundary conditions (3), whose dynamics is driven by the QCD Lagrangian in the presence of the mean field. The propagators of fluctuation fields for a given background and boundary condition can be found analytically. Thus this partition function offers a systematic prescription for calculation of the correlation functions, based on a decomposition over fluctuations and taking the mean field into account explicitly. Such a treatment of fluctuations as perturbations of a certain background field is sensible only if the essential features of the system can be seen in the lowest orders of the decomposition (at least semi-quantitatively). In other words one has to verify whether such basic phenomena as confinement and spontaneous chiral symmetry breaking are provided by the domain-structured mean field and boundary conditions under consideration.

In the zeroth order of the expansion we shall find that the gluon condensate, topological susceptibility χ and the string constant σ for color group $SU(3)$ take the compact form

$$g^2\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)\rangle = 4B^2, \quad \chi = \frac{B^4R^4}{128\pi^2}, \quad \sigma = Bf(\pi BR^2),$$

$$f(z) = \frac{2}{3z}\left(3 - \frac{\sqrt{3}}{2z}\int_0^{2z/\sqrt{3}} \frac{dx}{x}\sin x - \frac{2\sqrt{3}}{z}\int_0^{z/\sqrt{3}} \frac{dx}{x}\sin x\right),$$

while the quark condensate density at the domain center, calculated in the lowest nonvanishing order over quark fields, reads

$$\langle\bar{\psi}\psi\rangle = -\frac{q}{2\pi^2R^3(1+q)}[2F(BR^2/2\sqrt{3}) + F(BR^2/\sqrt{3})]$$

$$F(z) = e^z - z - 1 + \frac{z^2}{4}\int_0^\infty \frac{dt e^{2t - z(\coth t - 1)/2}}{\sinh^2 t}(\coth t - 1),$$

$$q = B^2R^4/16,$$

where q is the absolute value of the topological charge associated with a single domain.

To gain numerical estimates of these quantities we fixed the mean field strength parameter B and domain radius R to fit the known value of the string tension,

$$\sqrt{B} = 947 \ \text{MeV}, \quad R^{-1} = 760 \ \text{MeV}, \tag{5}$$

which leads to the values

$$(\alpha_s/\pi)\langle F^2\rangle = 0.081 \ \text{GeV}^4, \quad \sqrt{\sigma} = 420 \ \text{MeV},$$

$$\chi = (197 \ \text{MeV})^4, \quad \langle\bar{\psi}\psi\rangle = -(228 \ \text{MeV})^3, \tag{6}$$

with domain charge $q = 0.15$ and density $v^{-1} = 42.3 \ \text{fm}^{-4}$. This estimation shows a high density of clusters and strong background fields in the system, with confinement of static charges and spontaneously broken chiral symmetry. There is no separation of scales characterizing the system, $\sqrt{B}R \approx 1$. The qualitative picture as well as numerical values obtained indicate consistency of the gross features of the model.

These results suggest that formation of clusters, predominantly (anti-)self-dual and with average size $2R \approx 0.5$ fm, can have a purely quantum origin whose explanation could require reference to the existence of obstructions in gauge fixing.

It should be noted that the physical content of the above numbers can differ from other approaches. For instance the QCD sum-rules [21] determination of the gluon condensate is not exactly comparable to ours, since in our case corrections of order $O(\alpha_s)$ and higher contain nonperturbative information via explicit dependence of quark and gluon propagators on the mean field.

Moreover, the above parameters only give a characterization of the ''bulk'' properties of the theory and say little about confinement of dynamical color modes and hadronization. Information about these aspects is contained in the quark and gluon Green's functions, in particular in their propagators. It is shown that, as expected, the Dirichlet boundary condition removes gluon zero modes, and the propagator in this problem is well defined, unlike the analogous problem in the infinite volume. Both propagators have support in the interior of the hypersphere, where they have the usual ultraviolet singularity. Thus at short distances the propagators have standard perturbative form plus power corrections. The singularity in the configuration representation of propagators is integrable and their Fourier transforms exist, so that in momentum space the propagators are entire analytical functions due to their compact support. This we regard as a manifestation of the confinement of dynamical fields.

The paper is organized as follows. In Sec. II the boundary conditions are discussed and the model partition function is defined. We consider properties of the ensemble of mean fields in Sec. III and estimate the lowest dimension gluon condensate, the topological susceptibility, the string constant and the quark condensate in the lowest nonvanishing order in fluctuation fields. Gluon and ghost propagators are calculated in Sec. IV and their analytical properties are discussed. In Sec. V we give an outline of the problems remaining to be solved and possible perspectives. The Appendixes contain some technical and illustrative material.

II. THE PARTITION FUNCTION

In this section we formulate a partition function which will be used in subsequent sections for modelling the QCD partition function in the presence of clustered background fields. It should be clear from the very beginning that we will not derive the model to be considered from the original QCD functional integral. The mathematically accurate framework for such a derivation, a self-consistent mean field approximation requiring calculation of the effective action of QCD as a functional of the mean field and characteristic functions of the domains, is yet to be formulated. The best that can be done at this stage is to identify several ingredients of the formalism required for motivating the model within QCD, postulate the model partition function, and then look for signatures of justification *a posteriori*, by means of explicit calculations.

Clustered structure of the gauge fields is introduced by the proposition that singular configurations may not be excluded *ad hoc* from the functional space of integration; rather the character of singularities should be restricted by the natural requirement that the classical action density for a given (in general, singular) configuration has to be finite.

We assume that in the vicinity of a singularity an arbitrary gluon field $\mathcal{A}$ can be divided as in Eqs. (1), (2), $\mathcal{A}^a_\mu(x) = S^a_\mu(x) + Q^a_\mu(x)$, with Q a regular field and S the singular pure gauge part,

$$\hat{S}_\mu(x) = \frac{1}{ig}[\partial_\mu U(x)]U^{-1}(x), \quad U(x) = e^{ig\hat{f}(x)}, \quad \hat{f} = f^a t^a.$$

The field strength corresponding to S^a_μ vanishes. In this paper we will consider Abelian singular configurations

$$\partial_\mu \hat{S}_\nu - \partial_\nu \hat{S}_\mu = [\hat{S}_\mu, \hat{S}_\nu] = 0.$$

This can be implemented via

$$\hat{f} = \sum_{j=1}^{N} \hat{n}_j f_j(x), \quad \hat{n}_j = \text{const}, \quad [\partial_\mu, \partial_\nu] f_j(x) = 0, \quad [\hat{n}_j, \hat{n}_k] = 0,$$

where each of the functions f_j is singular on a three-dimensional boundary ∂V_j of the (four-dimensional) region V_j, while the matrices $\hat{n}_j$ belong to the Cartan subalgebra of $SU(3)$ and can be parametrized by

$$\hat{n}_j = t_3 \cos\xi_j + t_8 \sin\xi_j, \quad 0 \leqslant \xi_j < 2\pi.$$

The boundaries of the densely packed regions V_j necessarily intersect each other and, for instance, color orientation associated with the boundary becomes ambiguous in the intersection regions. Strictly speaking, this means that the Abelian singular fields should be accompanied by topologically nontrivial vortexlike configurations, such that the three-dimensional ''domain wall'' (a topologically trivial object) should start and finish at the two-dimensional singular surfaces, corresponding to a type of dislocation. A complete picture would include the whole hierarchy of singular fields: domain walls, vortices, monopoles and instantons. It is hard to formulate a complete approach in a precise way. A qualitative discussion of this aspect of domain-like structures can be found in Ref. [22]. Even if we neglect the effects of ''dislocations'' on the boundaries ∂V_j, a self-consistent consideration is still a complicated problem. However, in this case one can get some idea about features of the required formalism by means of an artificial example—QED in the presence of the singular background fields, considered in Appendix A. The linearity of electrodynamics enables a formal definition of the free energy (effective action) as a functional of a background field and characteristic functions of clusters, and thus relegates the question about formation of clusters in a typical gauge field configuration to a competition between energy and entropy. In the case of an Abelian weakly interacting theory one hardly expects domain formation. On the contrary in non-Abelian strongly interacting theory singular fields are most probably unavoidable, but unlike QED a straightforward formulation is a difficult task.

First of all we should determine the appropriate boundary conditions for the fluctuation fields about the singular field S for finiteness of the action density. Substituting Eq. (1) into the QCD Lagrangian, we obtain

$$\begin{aligned}\mathcal{L}_{\rm QCD} = & -\frac{1}{4} Q^a_{\mu\nu} Q^a_{\mu\nu} + \bar{\psi}[i\partial\!\!\!/ - m + g\hat{\mathcal{Q}} + g\hat{\mathcal{S}}]\psi \\ & + \frac{ig}{2} Q^a_{\mu\nu}[\check{S}^{ab}_\mu Q^b_\nu - \check{S}^{ab}_\nu Q^b_\mu] - \frac{g^2}{2}[(\check{S}^2)^{bb'} Q^b_\nu Q^{b'}_\nu \\ & - (\check{S}_\mu \check{S}_\nu)^{bb'} Q^b_\mu Q^{b'}_\nu], \end{aligned} \tag{7}$$

and $Q^a_{\mu\nu}$ is the usual field strength tensor for the fluctuation field. We see from Eq. (7) that conditions on the gluon field arise

$$\check{n}_j Q_\mu = 0 \quad \text{for} \quad x \in \partial V_j, \tag{8}$$

while quark fields should satisfy the condition given in Eq. (3). Equation (8) means that the modes of the gluon field longitudinal to the color vector n^a_j are not restricted, so it is convenient to decompose gluon fluctuations inside the region V_j into transverse and longitudinal parts with respect to n^a_j:

$$\begin{aligned} & Q^a_\mu = A^{ja}_\mu + n^a_j B^j_\mu, \quad n^a_j A^{aj}_\mu \equiv 0, \\ & A^{ja}_\mu = 0 \quad \text{for} \quad x \in \partial V_j. \end{aligned} \tag{9}$$

The separation in Eq. (1) into singular and regular parts imposes certain restrictions on the gauge transformations if the original and transformed fields Q are subject to the same boundary conditions. To determine these restrictions it is sufficient to consider the infinitesimal transformation

$$\begin{aligned} & S_\mu + Q_\mu \to S_\mu + Q_\mu + \delta Q_\mu, \\ & \delta Q^a_\mu = \partial_\mu \omega^a - f^{abc}\omega^b (n^c B_\mu + A^c_\mu + S^c_\mu), \end{aligned} \tag{10}$$

from which we conclude that gauge functions should satisfy the conditions

$$\begin{aligned} & \omega^a = n^a_j \omega_j + \omega^{ja}_\perp, \quad n^a_j \omega^{ja}_\perp = 0, \\ & \partial_\mu \omega^{ja}_\perp = \omega^{ja}_\perp = 0 \quad \text{for} \quad x \in \partial V_j. \end{aligned} \tag{11}$$

The longitudinal functions ω_j need not be restricted. The condition Eq. (11) dictates that gauge fixing for the fields Q should be achieved by means of restricted gauge transformations.

The original conditions Eqs. (3) show that the interaction of quark and gluon fluctuations within the k-th region with the field fluctuations in the rest of the system can be seen as a coupling to external gauge fields which are longitudinal to the color direction n^a_k of the boundary ∂V_k. This feature motivates an approximative treatment of the partition function, in which clusters are treated as decoupled but, by way of compensation, a mean field is introduced in their interior. A self-consistent mean field approach requires calculation of the effective action as a functional of the mean field and characteristic functions of the domains. Its minima would contain information about mean field character, shape and typical domain size.

Here we assume that the effective action favors formation of clusters with typical size R and nonzero mean field. In Appendix B it is shown that with this and an arbitrary constant mean field the effective action for a domain exhibits twelve degenerate discrete minima corresponding to (anti-)self-dual configurations and six values [for $SU(3)$] of the angle ξ associated with the Weyl group. There is also a degeneracy in the orientation of the chromomagnetic field. The value $\xi_0 = \pi/6$ is specific for an ansatz with the effective action polynomial in $\mathrm{Tr}\hat{\mathcal{B}}^k$, but the period $\pi/3$ is universal. Since the volume of the domain is finite the degenerate minima do not correspond to thermodynamical phases and have to be summed in the partition function.

The partition function for the model is defined as

$$\mathcal{Z}=\mathcal{N}\lim_{V,N}\prod_{i=1}^{N}\int_V\frac{d^4z_i}{V}\int_\Sigma d\sigma_i\int_{\mathcal{F}^i_Q}\mathcal{D}Q^i\int_{\mathcal{F}^i_\psi}\mathcal{D}\psi_i\mathcal{D}\overline{\psi}_i \times\delta[D(\breve{\mathcal{B}}^{(i)})Q^{(i)}]\Delta_{\rm FP}[\breve{\mathcal{B}}^{(i)},Q^{(i)}]e^{-S^{\rm QCD}_{V_i}[Q^{(i)}+\mathcal{B}^{(i)},\psi^{(i)},\overline{\psi}^{(i)}]}, \tag{12}$$

where the thermodynamic limit assumes $V,N\to\infty$ with the density $v^{-1}=N/V$ taken finite. The fields $Q^{(i)}$, ψ_i and $\overline{\psi_i}$ are subject to boundary conditions Eq. (3), in which the original singularities are effectively encoded. Interaction between the original domains is substituted by the mean field. A background gauge condition is imposed. The Faddeev-Popov determinant should be calculated on a restricted space of functions consistent with Eq. (11), which can be implemented in the form of an integral over ghost fields $(\overline{h^a_j},h^a_j)$ subject to the boundary condition

$$\breve{n}_j h_j=0 \quad \text{for } x\in\partial V_j\,. \tag{13}$$

The integration measure $d\sigma_i$ is

$$\int_\Sigma d\sigma_i\ldots=\frac{1}{48\pi^2}\int_0^{2\pi}d\alpha_i\int_0^{2\pi}d\varphi_i\int_0^{\pi}d\theta_i\sin\theta_i \times\int_0^{\pi}d\omega_i\sum_{k=0,1}\delta(\omega_i-\pi k)\int_0^{2\pi}d\xi_i \times\sum_{l=0}^{5}\delta\left(\xi_i-(2l+1)\frac{\pi}{6}\right)\ldots. \tag{14}$$

Here φ_i and θ_i are the spherical angles of the chromomagnetic field, ω_i is the angle between the chromomagnetic and chromoelectric fields, ξ_i is the angle in the color matrix $\hat{n}_i$, α_i is the chiral angle and z_i is the center of the domain V_i with the boundary

$$(x-z_j)^2=R^2.$$

The partition function Eq. (12) describes a statistical system of density v^{-1} composed of noninteracting hyperspherical clusters, each of which is characterized by a set of internal parameters and whose internal dynamics are represented by the fluctuation fields. Correlation functions can be calculated taking the mean field into account explicitly and decomposing over the fluctuations. First of all we consider vacuum characteristics of the system to zeroth order in this expansion.

III. VACUUM PROPERTIES TO LOWEST ORDER IN FLUCTUATIONS

The above prescribed perturbative treatment of fluctuations means in particular that they cannot change vacuum properties of the system. Thus our immediate task is to test whether the mean field itself reproduces the main nonperturbative characteristics of the pure gluonic vacuum. To achieve this we have to compute vacuum expectation values of a number of basic quantities omitting integration over fluctuation fields. Thus we will calculate *n*-point connected correlators of field strength and thereby the corresponding gluon condensates, string constant and topological susceptibility.

A. Mean field correlators

A straightforward application of Eq. (12) to the vacuum expectation value of a product of *n* field strength tensors, each of the form

$$B^a_{\mu\nu}(x)=\sum_j^N n^{(j)a}B^{(j)}_{\mu\nu}\theta(1-(x-z_j)^2/R^2),$$

gives for the connected *n*-point correlation function

$$\begin{aligned}\langle B^{a_1}_{\mu_1\nu_1}(x_1)\ldots B^{a_n}_{\mu_n\nu_n}(x_n)\rangle &=\lim_{V,N\to\infty}\sum_j^N\int_V\frac{dz_j}{V}\int d\sigma_j n^{(j)a_1}\ldots n^{(j)a_n}B^{(j)}_{\mu_1\nu_1}\ldots B^{(j)}_{\mu_n\nu_n}\\ &\times\theta(1-(x_1-z_j)^2/R^2)\ldots\theta(1-(x_n-z_j)^2/R^2)\\ &=B^n t^{a_1\ldots a_n}_{\mu_1\nu_1,\ldots,\mu_n\nu_n}\Xi_n(x_1,\ldots,x_n),\end{aligned} \tag{15}$$

where the tensor *t* is given by the integral

$$t^{a_1\ldots a_n}_{\mu_1\nu_1,\ldots,\mu_n\nu_n}=\int d\sigma_j n^{(j)a_1}\ldots n^{(j)a_n}B^{(j)}_{\mu_1\nu_1}\ldots B^{(j)}_{\mu_n\nu_n},$$

and can be calculated explicitly using the measure, Eq. (14). This tensor vanishes for odd *n*. In particular, the integral over spatial directions is defined by the generating formula

$$\frac{1}{4\pi}\int_0^{2\pi}d\varphi_j\int_0^{\pi}d\theta_j\sin\theta_j e^{iB^{(j)}_{\mu\nu}J_{\mu\nu}} =\frac{\sin\sqrt{2B^2[J_{\mu\nu}J_{\mu\nu}\pm\tilde{J}_{\mu\nu}J_{\mu\nu}]}}{\sqrt{2B^2[J_{\mu\nu}J_{\mu\nu}\pm\tilde{J}_{\mu\nu}J_{\mu\nu}]}} \tag{16}$$

where the plus and minus correspond to $B^{(j)}_{\mu\nu}$ being self-dual or anti-self-dual. The translation-invariant function

$$\Xi_n(x_1,\ldots,x_n)=\frac{1}{v}\int d^4z\,\theta(1-(x_1-z)^2/R^2)\ldots \times\theta(1-(x_n-z)^2/R^2) \tag{17}$$

can be seen as the volume of the region of overlap of *n* hyperspheres of radius *R* and centers $(x_1,\ldots,x_n)$, normalized to the volume of a single hypersphere $v=\pi^2R^4/2$,

$$\Xi_n=1, \quad \text{for } x_1=\ldots=x_n\,.$$

It is obvious from this geometrical interpretation that Ξ_n is a continuous function and vanishes if the distance between any two points $|x_i-x_j|\geqslant 2R$; correlations in the background field have finite range $2R$. The Fourier transform of Ξ_n is then an

entire analytical function and thus correlations do not have particle interpretation. It should be stressed that the statistical ensemble of background fields is not Gaussian since all connected correlators are independent of each other and cannot be reduced to the two-point correlations.

As a simplest application of the above correlators we get a gluon condensate density which to this approximation is

$$g^2\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)\rangle = 4B^2. \tag{18}$$

Note that the coupling constant is absorbed into the gauge field.

B. Topological charge and susceptibility

Another vacuum parameter which plays a significant role in the resolution of the $U_A(1)$ problem is the topological susceptibility [23–25]. To define this we consider first the topological charge density for the color group $SU(3)$,

$$Q(x) = \frac{g^2}{32\pi^2}\tilde{F}^a_{\mu\nu}(x)F^a_{\mu\nu}(x),$$

which in the mean field approximation takes the form

$$Q(x) = \frac{B^2}{8\pi^2}\sum_{j=1}^{N} \theta[1-(x-z_j)^2/R^2]\cos\omega_j, \tag{19}$$

where $\omega_j \in \{0,\pi\}$ depending on the duality of the j-th domain. We thus see that the topological charge density is constant in each domain, and the sign of this constant is uncorrelated. For a given field configuration then the topological charge is additive

$$Q = \int_V d^4x Q(x) = q(N_+ - N_-), \quad q = B^2R^4/16,$$

$$-Nq \leqslant Q \leqslant Nq$$

where q is a ''unit'' topological charge, namely the absolute value of the topological charge of a single domain, and N_+ (N_-) is the number of domains with (anti-)self-dual field, $N = N_+ + N_-$. With a fixed total number of domains N the probability of finding the topological charge Q in a given configuration is given by the distribution

$$\mathcal{P}_N(Q) = \frac{\mathcal{N}_N(Q)}{\mathcal{N}_N} = \frac{N!}{2^N(N/2-Q/2q)!(N/2+Q/2q)!},$$

where $\mathcal{N}_N(Q)$ is the number of configurations with a given charge and $\mathcal{N}_N$ is the total number of configurations. The distribution is symmetric about $Q=0$, where it has a maximum for N even. For N odd the maximum is at $Q=\pm q$. We conclude that the topological charge averaged over the ensemble of clusters vanishes.

The topological susceptibility

$$\chi = \int d^4x \langle Q(x)Q(0)\rangle \tag{20}$$

is determined by the two-point correlator of topological charge density, which in the lowest approximation reads

$$\langle Q(x)Q(y)\rangle = \frac{B^4}{64\pi^4}\Xi_2(x-y), \tag{21}$$

and we get

$$\chi = \frac{B^4R^4}{128\pi^2}. \tag{22}$$

C. Area law for the Wilson loop

In the same mean field approximation the Wilson loop is given by the integral

$$W(L) = \lim_{V,N\to\infty}\prod_{j=1}^{N}\int_V \frac{d^4z_j}{V}\int d\sigma_j \frac{1}{N_c}\mathrm{Tr} \times \exp\left\{ i\int_{S_L} d\sigma_{\mu\nu}(x)\hat{B}_{\mu\nu}(x)\right\},$$

where the measure $d\sigma_j$ corresponds to an integral over the set of parameters

$$\{z_k, \phi_k, \theta_k, \omega_k, \xi_k\}_{k=1}^N$$

of the field strength

$$\hat{B}_{\mu\nu}(x) = \sum_k \hat{n}^{(k)}B^{(k)}_{\mu\nu}\theta(1-(x-z_k)^2/R^2).$$

Note that path ordering in our case is not necessary since the matrices $\hat{n}^{(k)}$ are assumed to be in the Cartan subalgebra.

Strictly speaking the contour S_L around which the path-ordered exponential is integrated should be a rectangle whose Euclidean-time length should be taken arbitrarily large before the spatial length. It is for such a contour that one has a strict interpretation of the behavior of the exponent in terms of a static potential [26,27]. However the expectation that there be an area law is not dependent on the specific geometry of the contour. In view of the rotational properties of our approximation to the vacuum fields, it is computationally more convenient to consider a *circular* contour in the (x_3,x_4) plane of radius L with the center at the origin. If an area law is established, as will be the case, the numerical value of the resulting string constant would not be precisely that corresponding to a rectangular contour. However due to the fact that the loop must be taken large in order to extract the potential, the difference between a circle and a rectangle should not lead to radically different values of the string constant.

To illustrate the steps in the calculation while avoiding cumbersome formulas we consider here the case of color group $SU(2)$. The details of $SU(3)$ will be given in Appendix C, though the final result will be quoted below. For color $SU(2)$ we have

$$\hat{n}^{(k)} = \epsilon^k\tau_3, \quad \epsilon^k = \pm 1.$$

The thermodynamic limit $(V,N\to\infty)$ assumes that the subvolume

$$v=V/N=\pi^2R^4/2$$

is fixed. Calculation of the trace in color space leads to the result

$$\frac{1}{2}\mathrm{Tr}\exp\left\{ i\int_{S_L} d\sigma_{\mu\nu}(x)\hat{B}_{\mu\nu}(x)\right\}=\cos\left(\sum_k \epsilon^k B^{(k)}_{\mu\nu}J_{\mu\nu}(z_k)\right),$$

where we have denoted

$$J_{\mu\nu}(z_k)=\int_{S_L} d\sigma_{\mu\nu}(x)\,\theta(1-(x-z_k)^2/R^2). \tag{23}$$

Using the properties of the measure of integration over the collective coordinates one gets

$$W(L)=\lim_{V,N\to\infty}\left[\int_V\frac{d^4z_j}{V}\int d\sigma_j\frac{1}{2}(e^{iB^{(j)}_{\mu\nu}J_{\mu\nu}(z_j)}+e^{-iB^{(j)}_{\mu\nu}J_{\mu\nu}(z_j)})\right]^N.$$

We have exploited here the property that the integral over collective variables does not depend on the index j. As the contour of the Wilson loop is in the (x_3,x_4)-plane, the only nonzero components of $J_{\mu\nu}$ are

$$J_{34}=-J_{43}(z)=\int_{S_L} dx_3dx_4\,\theta(1-(x-z)^2/R^2), \tag{24}$$

and

$$B_{\mu\nu}J_{\mu\nu}(z)=2B_{43}J_{43}(z)=2E_3J_{43}(z)=2BJ_{43}(z)\cos\theta, \tag{25}$$

where θ is the angle between the chromoelectric field **E** and the third coordinate axis. Now we can calculate the integral over the spatial orientations of the vacuum field

$$\int d\sigma_j e^{iB^{(j)}_{\mu\nu}J_{\mu\nu}(z_j)}=\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta_j\sin\theta_j e^{2iBJ_{43}\cos\theta_j}=\frac{\sin 2BJ_{43}(z_j)}{2BJ_{43}(z_j)},$$

and the Wilson loop takes the form

$$W(L)=\lim_{N,V\to\infty}\left[\frac{1}{V}\int_V dz\frac{\sin 2BJ_{43}(z)}{2BJ_{43}(z)}\right]^N.$$

Calculating the integral over z we obtain finally

$$W(L)=\lim_{N\to\infty}\left[1-\frac{1}{N}U(L)\right]^N=e^{-U(L)}$$

$$U(L)=\frac{\pi^2R^2L^2}{v}\left(1-\frac{1}{2\pi BR^2}\int_0^{2\pi BR^2}\frac{dx}{x}\sin x\right)+\frac{\pi^2}{v}\left(\frac{4}{3}R^3L+\frac{1}{2}R^4\right)-\frac{\pi^2(1-\cos 2\pi BR^2)}{v(2\pi B)^2}+\frac{4\pi^2L}{v(2\pi B)^{3/2}}\int_0^{\sqrt{2\pi BR^2}}dx\sin x^2-\frac{\pi^2L^4}{v}\int_0^{R^2/L^2}ds\times\int_{(1-\sqrt{s})^2}^{(1+\sqrt{s})^2}dt\frac{\sin\{BL^2[2\varphi-\sin\varphi+s(2\psi-\sin\psi)]\}}{BL^2[2\varphi-\sin\varphi+s(2\psi-\sin\psi)]},$$

$$\cos\frac{\varphi}{2}=\frac{t-s+1}{2\sqrt{t}},\quad \cos\frac{\psi}{2}=\frac{t+s-1}{2\sqrt{st}}, \tag{26}$$

where the thermodynamic limit $(N,V\to\infty,v=V/N=\pi^2R^4/2)$ has been taken. One can check that $U(L)=0$ when $B\to 0$, as it should.

In the limit of large Wilson loop $L\gg R$ the behavior of the first four terms in $U(L)$ can be determined by inspection. For the last term a slightly more involved calculation gives the large L behavior

$$-\frac{\pi^2L^4}{v}\int_0^{R^2/L^2}ds\int_{(1-\sqrt{s})^2}^{(1+\sqrt{s})^2}dt\times\frac{\sin\{BL^2[2\varphi-\sin\varphi+s(2\psi-\sin\psi)]\}}{BL^2[2\varphi-\sin\varphi+s(2\psi-\sin\psi)]}\approx-\frac{8\pi^2LR^3}{3v}$$

with corrections coming at $\mathcal{O}(R^4)$. Thus only the first term in $U(L)$, going like L^2, displays an area dependence. The final result for the string constant for $SU(2)$ is

$$W(L)=e^{-\sigma\pi L^2+O(L)},\quad \sigma=Bf(\pi BR^2),$$

$$f(z)=\frac{2}{z}\left(1-\frac{1}{2z}\int_0^{2z}\frac{dx}{x}\sin x\right).$$

For the case of $SU(3)$, as shown in Appendix C, the function $f(z)$ turns out to be

$$f(z)=\frac{2}{3z}\left(3-\frac{\sqrt{3}}{2z}\int_0^{2z/\sqrt{3}}\frac{dx}{x}\sin x-\frac{2\sqrt{3}}{z}\int_0^{z/\sqrt{3}}\frac{dx}{x}\sin x\right). \tag{27}$$

It is positive for $z>0$ and has a maximum for $z=1.55\pi$. We choose this maximum to estimate the model parameters by fitting the string constant to the lattice result,

$$\sqrt{B}=947\ \text{MeV},\quad R^{-1}=760\ \text{MeV}, \tag{28}$$

with unit charge $q=0.15$, density $v^{-1}=42.3\ \text{fm}^{-4}$ and the ''observable'' gluonic parameters of the vacuum

$$\sqrt{\sigma}=420\ \text{MeV},\quad \chi=(197\ \text{MeV})^4,$$

$$(\alpha_s/\pi)\langle F^2\rangle=0.081\ \text{GeV}^4. \tag{29}$$

The high density ensures area law dominance already at distances $2L\approx 1.5-2$ fm.

The result for the gluon condensate is larger than the most recent estimate within QCD sum rules [28]. As already mentioned, our value is not directly comparable to sum-rule results due to differences in the content of $O(\alpha_s)$ corrections. What appears to be important is that all these quantities are nonzero, and their values can be fit to the expected numbers simultaneously.

Obviously, if B goes to zero then the string constant vanishes. This underscores the role of the gluon condensate in the confinement of static charges. On the other hand we can also see that if the number of domains is fixed and the thermodynamic limit is defined as $V,R\to\infty,N=\text{const}<\infty$, namely if the clusters are macroscopically large, then $W(L)=1$, which indicates the absence of a linear potential between static (infinitely heavy) charges in a purely homogeneous field. However this does not mean that heavy quarks ($m_Q^2\gg B$) are not confined if domains are macroscopically large. As is shown in Ref. [29], the nonrelativistic potential is quadratic in the distance between heavy quarks with the coefficient proportional to m_Q^{-1}.

Since we have integrated over background fields exactly the role of a finite range of correlation functions is hidden in the above calculation. In order to see this role explicitly one would need to decompose the integrand into an infinite series and integrate term by term. At this step all correlation functions of the background field up to infinite order would be manifest. The arguments of Refs. [30] about the crucial importance of a fast decay of correlators for confinement of static charges would be seen to apply here via this representation.

A comment on the values of the parameters R and B appearing in our estimation is in order. We observe that there is no separation of the two scales characterizing the vacuum. The average strength of vacuum fields B and the average domain size R, are comparable to each other, $\sqrt{B}R\approx 1$. Neither large domains nor stochasticity of background fields are seen here which *a posteriori* justifies the mean field averaging prescription in the partition function. This prescription corresponds to a system less ordered than, for instance, a spin glass. Nor does the partition function represent a heterophase mixture [31], since the condition for quasiequilibrium is not satisfied: one may not think of these clusters as droplets of different thermodynamic phases as they are too small and too transient compared to the basic scale of interactions determined in this picture by the gluon condensate value. The mean field in the clusters is singled out not due to a hierarchy of scales, but due to certain specific properties: the (anti-)self-duality, and the Abelian character if dislocations at the boundaries are neglected. Homogeneity of the background field appears as an approximation.

D. Quark condensate density at domain center

A complete consideration of the fermionic eigen-problem for the background field and boundary conditions under consideration will be given in a separate work. However to complete the picture of vacuum properties in the model, we estimate here the quark condensate density at the domain center.

A complete calculation of the quark condensate in the lowest nonvanishing order over fluctuations requires solution of the equations

$$(i\not{D}-m)S(x,y)=-\delta(x,y), \tag{30}$$

$$i\not{\eta}(x)e^{i\alpha\gamma_5}S(x,y)=-S(x,y),\quad (x-z)^2=R^2, \tag{31}$$

$$S(x,y)i\not{\eta}(y)e^{-i\alpha\gamma_5}=S(x,y),\quad (y-z)^2=R^2, \tag{32}$$

where $\eta_\mu(x)=(x-z)_\mu/|x-z|$, and D_μ is the covariant derivative in the fundamental representation,

$$D_\mu=\partial_\mu-i\hat{B}_\mu=\partial_\mu+\frac{i}{2}\hat{n}B_{\mu\nu}x_\nu.$$

Substituting

$$S=(i\not{D}+m)[P_\pm\mathcal{H}_0+P_\mp O_+\mathcal{H}_{+1}+P_\mp O_-\mathcal{H}_{-1}], \tag{33}$$

into Eq. (30) where

$$O_\pm=N_+\Sigma_\pm+N_-\Sigma_\mp,$$

$$N_\pm=\frac{1}{2}(1\pm\hat{n}/|\hat{n}|),$$

$$\Sigma_\pm=\frac{1}{2}(1\pm\vec{\Sigma}\vec{B}/B),$$

and $\hat{B}=|\hat{n}|B$, shows that the scalar functions $\mathcal{H}_\zeta$, with $\zeta=0,\pm 1$, should satisfy the equations:

$$(-D^2+m^2+2\zeta\hat{B})\mathcal{H}_\zeta=\delta(x,y). \tag{34}$$

We note that if solutions vanishing at infinity were sought, then the Green function $\mathcal{H}_{-1}$ would be divergent in the massless limit due to the contribution of zero modes of the Dirac operator in the presence of the (anti-)self-dual homogeneous field. The present bag-like boundary conditions remove zero eigen-values from the spectrum, and the massless limit is regular. Due to averaging over self- and anti-self-dual configurations and all possible values of angle α in the partition function, chiral symmetry is not broken explicitly. However, as we show below, a nonzero quark condensate arises in the massless limit due to an interplay of random distribution of the domains with self- and anti-self-dual field and the boundary conditions with the random value of the chirality violating angle α.

In order to avoid cumbersome calculations and expose the role of the former zero modes in a transparent way we turn to the particular choice $y=z=0$ and calculate the value of the quark condensate at the center of the domain. In this case the

functions $\mathcal{H}_\zeta$ can depend only on x_μ, $B_{\mu\nu}x_\nu$ and $\eta_\mu = x_\mu/\sqrt{x^2}$, and hence are functions of x^2 only, and the general solutions for scalar Green's functions take the form

$$\mathcal{H}_\zeta = \Delta(x^2|\mu_\zeta) + C_\zeta \Phi(x^2|\mu_\zeta),$$

where $\mu_\zeta = m^2/2B + \zeta$, and

$$\Phi(x^2|\mu) = e^{-Bx^2/4} M(1+\mu, 2, Bx^2/2).$$

Here $\Delta(x^2|\mu)$ is the vanishing at infinity scalar propagator with mass $2B\mu$ in the homogeneous (anti-)self-dual field and Φ is a solution to the homogeneous equation regular at $x^2 = 0$, expressed in terms of the confluent hypergeometric function. The constants C_ζ can be used to fit the boundary condition. Terms with $\mathcal{H}_0$ and $\mathcal{H}_{+1}$ are regular in the massless limit and cannot contribute to the trace of the quark propagator. Thus we concentrate on the term $\mathcal{H}_{-1}$. Using identities

$$\gamma_\mu B_{\mu\rho} x_\rho P_\mp \Sigma_+ = iB\not{x} P_\mp \Sigma_+ ,$$

$$\gamma_\mu B_{\mu\rho} x_\rho P_\mp \Sigma_- = -iB\not{x} P_\mp \Sigma_- ,$$

one can show that the boundary condition is satisfied if on the boundary

$$2e^{\mp i\alpha} m\mathcal{H}_{-1} = -2\mathcal{H}'_{-1} - \hat{B}R^2 \mathcal{H}_{-1},$$

which implies that in the massless limit C_{-1} takes the form

$$C_{-1} = -\frac{\hat{B}^2}{4\pi^2 m^2} + \frac{e^{\pm i\alpha}}{2\pi^2 R^3 m} F(\hat{B}R^2/2) + O(1),$$

$$F(z) = e^z - z - 1 + \frac{z^2}{4}\int_0^\infty \frac{dt e^{2t - z(\coth t - 1)/2}}{\sinh^2 t}(\coth t - 1).$$

Moreover the singular terms cancel in $\not{D}\mathcal{H}_{-1}$ and

$$\lim_{m\to 0} m\mathcal{H}_{-1}(x,0) = \frac{e^{\pm i\alpha}}{2\pi^2 R^3} F(\hat{B}R^2/2) e^{-\hat{B}x^2/4}, \qquad (35)$$

and thus,

$$\mathrm{Tr}\, S(0,0) = \frac{e^{\pm i\alpha}}{2\pi^2 R^3} \sum_{|\hat{n}|} F(\hat{B}R^2/2). \qquad (36)$$

It should be noted that the part of the propagator responsible for the nonzero trace, Eq. (35), is proportional to the zero mode of the Dirac operator

$$i\not{D} P_\mp O_- e^{-\hat{B}x^2/4} \equiv 0.$$

Now we have to average Eq. (36) over domain configurations taking into account the quark determinant. According to [32] the α-dependence of the quark determinant is

$$\exp\left\{\frac{i\alpha}{32\pi^2}\int dx \tilde{B}_{\mu\nu} B_{\mu\nu} \theta(1 - x^2/R^2)\right\} = \exp\{\pm iq\alpha\}$$

where q is the topological charge associated with a domain, and thus a θ-term is generated effectively by the quark determinant. After averaging over α we get a finite value for the condensate at the center of the domain

$$\langle \bar{\psi}\psi \rangle = -\frac{q}{2\pi^2 R^3 (1+q)} \sum_{|\hat{n}|} F(\hat{B}R^2/2).$$

Numerically this is equal to $\langle \bar{\psi}\psi \rangle = -(228\ \mathrm{MeV})^3$ for B and R fixed by the string constant as in Eq. (28).

IV. PROPAGATORS IN THE PRESENCE OF DOMAINS

In order to study in more detail the influence of domain structure and the mean field on the properties of the dynamical quarks and gluons we have to find their propagators. They can be analytically calculated by reduction to the scalar problem, essentially that of a four-dimensional harmonic oscillator with total angular momentum coupled to the external field. The general solution is given by decomposition over hyperspherical harmonics. In the following section we present the exact solution for the scalar propagator, though with most derivations relegated to Appendixes D and E. With the scalar result we derive propagators for ghost and gluon fluctuations in an external (anti-)self-dual field with Dirichlet boundary conditions imposed on the fluctuations on a hyperspherical surface.

A. Scalar propagator

The problem to be solved is given by the scalar Green's function equation

$$[(\partial_\mu - iB_\mu)^2 - \mathcal{M}^2] G(x,x'|\mu) = -\delta^{(4)}(x - x'),$$

$$B_\mu = -\frac{1}{2} B_{\mu\nu} x_\nu, \quad B_{\mu\nu} B_{\mu\rho} = B^2 \delta_{\nu\rho}, \qquad (37)$$

with the homogeneous Dirichlet boundary condition

$$G(x,x'|\mu)_{x^2 = R^2} = G(x,x'|\mu)_{x'^2 = R^2} = 0,$$

where R is the radius of a hypersphere centered at the origin and $\mu = \mathcal{M}^2/2B$.

We present first the solution to the corresponding eigenvalue problem:

$$-[(\partial_\mu - iB_\mu)^2 - \mathcal{M}^2]\psi_\lambda = \lambda \psi_\lambda .$$

One may choose $B_{\mu\nu}$ such that

$$B_{34} = E, \quad B_{12} = B, \quad -B \leqslant E \leqslant B.$$

A representation of the eigenfunctions in terms of a complete orthonormalized set of eigenfunctions of the four-dimensional Laplace operator is achieved in the following hyperspherical coordinate system (see for example, [33])

$$x_1 = r \sin\eta \cos\phi$$

$$x_2 = r \sin\eta \sin\phi$$

$$x_3 = r\cos\eta\cos\chi$$

$$x_4 = r\cos\eta\sin\chi. \tag{38}$$

The angular eigenfunctions are

$$C_{km_1m_2}(\eta,\phi,\chi) = (-1)^{|m_1+m_2|}(2\pi)^{-1}\Theta_k^{m_1-m_2,m_1+m_2}(\eta) \times \exp i[(m_1-m_2)\chi+(m_1+m_2)\phi]$$

$$\Theta_k^{k-r-s,s-r}(\eta) = \sqrt{2(k+1)(k-r)!(k-s)!r!s!} \times \sum_{n=0}^{r} \frac{(-1)^{r-n}\cos^{k-r-s+2n}\eta\sin^{r+s-2n}\eta}{(k-r-s+n)!n!(r-n)!(s-n)!},$$

$$(r,s=0,1,\ldots,k),$$

where k, m_1, m_2 are respectively the orbital angular momentum and the two azimuthal quantum numbers, relevant for a four-dimensional hyperspherical symmetry. That the $C_{km_1m_2}$ are eigenfunctions with the said eigenvalues is proven in Appendix D.

The eigenfunctions for the complete problem are a product of radial and angular parts,

$$\psi(x) = f(r)C_{km_1m_2}(\eta,\phi,\chi).$$

The radial equation has a solution expressed in terms of the confluent hypergeometric function, in the notation of [34],

$$f(r) = (Br^2/2)^{k/2}e^{-Br^2/4} \times M\left(\frac{k}{2}+1-m_{2,1}+\frac{\mathcal{M}^2-\lambda}{2B},k+2;Br^2/2\right), \tag{39}$$

where the function regular at $r=0$ is chosen for normalizability. Here $m_{2,1}$ should be put equal to m_2 for the self-dual field, and m_1 for the anti-self-dual field. It is convenient to denote

$$n_{2,1} = \frac{k}{2}+1-m_{2,1}. \tag{40}$$

Another independent solution (not normalizable in our problem), which is regular at infinity and singular at the origin, would be obtained by replacing the function M with the function U. Imposition of the Dirichlet condition at $r=R$ forces eigenvalues λ to take discrete values defined by the zeros of $M(a,b,z)$ as a function of a at fixed b and z. The eigenvalues λ are strictly positive. As we will see below, the case of $\mathcal{M}^2=-2B$ will be met in the problem for the gluon propagator. In this case the lowest eigenvalue λ_0 is defined by $(k=m_1=m_2=0)$

$$M(-\lambda_0,2;BR^2/2)=0,$$

and is a positive function of BR^2, as can be checked.

The propagator can be found by the standard method of decomposition over hyperspherical harmonics, see for instance [35], exploiting the above hyperspherical representation. The derivation is given in Appendix E but it consists of essentially two steps. First we exploit completeness of the angular eigenfunctions in order to reduce the four dimensional problem to a one-dimensional Sturm-Liouville problem with a known exact solution for the radial dependence. Second we add a solution to the homogeneous equation with a coefficient selected to implement the above boundary condition.

Using the solution to the eigenvalue equation with $\lambda=0$, one gets independent solutions for the homogeneous equation. In the notation of [34], the two solutions (respectively regular at the origin and at infinity) are

$$R_1(r|k,n_{2,1},\mu) = r^k e^{-Br^2/4}M(n_{2,1}+\mu,k+2;Br^2/2)$$

$$R_2(r|k,n_{2,1},\mu) = r^k e^{-Br^2/4}U(n_{2,1}+\mu,k+2;Br^2/2). \tag{41}$$

The Sturm-Liouville equation is satisfied by

$$X_{kn_{2,1}}(r,r'|\mu) = B\frac{\Gamma(n_{2,1}+\mu)}{4\Gamma(k+2)}R_{kn_{2,1}}(r,r'|\mu) \tag{42}$$

with

$$R_{kn_{2,1}}(r,r'|\mu) = \begin{cases} R_1(r|k,n_{2,1},\mu)R_2(r'|k,n_{2,1},\mu), & r<r' \\ R_1(r'|k,n_{2,1},\mu)R_2(r|k,n_{2,1},\mu), & r>r'. \end{cases} \tag{43}$$

In terms of these quantities, the Green's function is

$$G(x,x'|\mu) = \frac{B}{4}\sum_{k,m_1,m_2}\frac{\Gamma(n_{2,1}+\mu)}{\Gamma(k+2)}C_{km_1m_2}(\eta',\phi',\chi') \times C_{km_1m_2}(\eta,\phi,\chi) \times \left[R_{kn_{2,1}}(r,r'|\mu) - \frac{U(n_{2,1}+\mu,k+2;BR^2/2)}{M(n_{2,1}+\mu,k+2;BR^2/2)} \times R_1(r|k,n_{2,1},\mu)R_1(r'|k,n_{2,1},\mu)\right]. \tag{44}$$

The first term inside square brackets guarantees G to be a Green's function through the solution to the Sturm-Liouville equation. The coefficient in the second term is determined by the boundary condition. The only singularity in G is the usual ultraviolet one at $x=x'$. For more details we refer the reader to Appendix D.

B. Ghost and gluon propagators

We work in the background gauge

$$D_\mu^{ab}A_\mu^b = 0,$$

and use conventions for the adjoint representation of color $SU(3)$:

$$\check{D}_\mu=\partial_\mu-i\check{n}\check{B}_\mu,\quad \check{n}=T^a n^a,\quad T^a_{bc}=-if^{abc}.$$
$$\check{n}=T^3\cos\xi_k+T^8\sin\xi_k,\quad \xi_k=(2k+1)\pi/6,\quad k=0,1,\ldots,5. \tag{45}$$
$$\check{B}_{\mu\nu}=\check{n}B_{\mu\nu},\quad \check{B}=\sqrt{\check{n}^2}B.$$

The color matrix $\check{n}$ can be diagonalized by the unitary transformation

$$n=U\check{n}U^\dagger=\mathrm{diag}(\zeta_1,-\zeta_1,0,\zeta_2,-\zeta_2,\zeta_3,-\zeta_3,0)$$
$$\zeta_1=\sin\xi,\quad \zeta_2=(\sin\xi+\sqrt{3}\cos\xi)/2,$$
$$\zeta_3=(-\sin\xi+\sqrt{3}\cos\xi)/2,$$

where $U=\mathrm{diag}(W,0,W,W,0)$ with

$$W=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}. \tag{46}$$

For the values of the angle ξ_k as in Eq. (45) the ζ_i take values from the set $(\pm1,\pm1/2)$. Namely, for $k=0,1,2,3,4,5$ respectively, the three ζ_i are $(1/2,1,1/2)$, $(1,1/2,-1/2)$, $(1/2,-1/2,-1)$, $(-1,-1/2,1/2)$, $(-1/2,1/2,1)$. The diagonalized covariant derivative takes the form

$$D_\mu=U\check{D}_\mu U^\dagger=\partial_\mu-inB_\mu. \tag{47}$$

In the Feynman gauge, diagonalized equations for the ghost and gluon propagators take the form

$$-D^2G(x,x')=\delta(x-x'),$$
$$(-D^2\delta_{\mu\nu}+2inB_{\mu\nu})G_{\nu\rho}(x,x')=\delta^{\mu\rho}\delta(x-x'), \tag{48}$$

with the diagonalized boundary conditions

$$nG_{\nu\rho}(x,x')=0,\ nG(x,x')=0,\ \text{for } x^2=R^2 \text{ or } x'^2=R^2,$$

where n and the propagators are diagonal matrices in color indices. As evident above, the matrix n has two zero eigenvalues and the corresponding gluon components are not restricted by the above boundary condition so the equations for these modes are simply the free ones. These modes are thus not confined in the model under consideration.

The scalar equation for the ghost propagator has been solved in the previous section, where one should simply replace $B\to nB$ and put $\mathcal{M}=0$. The equation for the gluon propagator and the boundary condition can be further diagonalized with respect to Lorentz indices and is thus reduced to four scalar equations, each of which has a well defined solution since, as discussed above, dangerous zero modes, so called chromons [19,20], do not satisfy Dirichlet conditions and do not contribute to the Green's function. The original propagators are restored by the inverse transformations.

In the case of the gluon propagator one can avoid the second diagonalization by looking for a solution of the form

$$G_{\nu\rho}(x,x')=(D^2\delta_{\nu\rho}+2inB_{\nu\rho})\Delta(x,x'). \tag{49}$$

Substitution of Eq. (49) into the original Green's function equation gives

$$[-(D^2)^2+4n^2B^2]\Delta(x,x')=\delta(x-x'),$$

with the solution given formally by

$$\Delta(x,x')=\frac{1}{-(D^2)^2+4n^2B^2}\,\delta(x-x')$$
$$=\frac{1}{4|n|B}\left(\frac{1}{-D^2+2|n|B}-\frac{1}{-D^2-2|n|B}\right)\times\delta(x-x'). \tag{50}$$

The two terms above are nothing but scalar propagators with "mass term" $\mathcal{M}^2=\pm2|n|B$. Substituting Eq. (50) into Eq. (49) and using notation of the previous subsection for the scalar propagator with $B\to|n|B$ and $\mu=\pm1$ one gets after simple manipulations

$$G_{\mu\nu}(x,x')=\frac{1}{2}\delta_{\mu\nu}[G(x,x'|1)+G(x,x'|-1)]+\frac{inB_{\mu\nu}}{2|n|B}[G(x,x'|1)-G(x,x'|-1)], \tag{51}$$

where only nonzero elements of the diagonal matrix n are involved. With this representation it is clear that the boundary condition for the gluon propagator is satisfied if the scalar Green's functions $G(x,x'|\pm1)$ are subject to the homogeneous Dirichlet condition independently of each other. An explicit form is obtained via Eq. (44) by substitution $B\to|n|B$ and $\mu=\pm1$. This can be done straightforwardly for the terms with $k-m_{2,1}>0$ in the expansion over hyperspherical harmonics, as is obvious from the integral representations of the confluent hypergeometric functions [34] $(b>a>0)$

$$M(a,b,z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 dte^{zt}t^{a-1}(1-t)^{b-a-1},$$
$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty dte^{-zt}t^{a-1}(1+t)^{b-a-1}, \tag{52}$$

with $a=1+\mu+k/2-m_{2,1}$ and $b=k+2$, in our case. Special comment is required for the terms with $a=k/2-m_{2,1}=0$ in the decomposition of $G(x,x'|-1)$.

Using the representations Eqs. (52) it immediately follows that

$$\lim_{a\to0}M(a,b,z)=1+O(a),\quad \lim_{a\to0}U(a,b,z)=1+O(a).$$

With this and Eqs. (41)–(44) one can be convinced that the singularity in the gamma-function in Eq. (44) at $a=n_{2,1}-1=k/2-m_{2,1}=0$ is cancelled by the contribution coming from the expression in the square brackets. Thus we conclude that

the gluon propagator exists. This confirms the absence of the zero modes under the imposed Dirichlet boundary conditions.

C. A comment on analytical properties

The scalar propagator Eq. (44) which determines the analytic properties of the off-diagonal components of the ghost and gluon propagators has compact support in the hyperspherical region of radius R in Euclidean space-time with the usual ultraviolet integrable singularity at $x'=x$. Thus the Fourier transform of the propagator averaged over domain position, given by the integral

$$\tilde{G}(p^2)=\int_{V_R} d^4x e^{ipx} G(x),$$
$$G(x-y)=v^{-1}\int_V dz G(x-z,y-z), \tag{53}$$

leads to a $\tilde{G}(p^2)$ which is an entire analytical function in the complex p^2 plane. Entire propagators are typical for nonlocal field theories and have been interpreted as confinement of dynamical charged fields [19,36,37]. Thus the presence of domains maintains confinement of off-diagonal gluons and ghosts.

An instructive example is given by a toy calculation which illustrates the qualitative behavior of the Fourier transform of propagators with compact support in a finite region of R^4. We calculate the Fourier transform of the function

$$D(x)=\frac{\theta(1-x^2/R^2)}{4\pi^2 x^2}.$$

The calculation proceeds via the following steps (where $p=|p|$)

$$\begin{aligned}\tilde{D}(p)&=\frac{1}{\pi}\int_{-1}^{1} dt\sqrt{1-t^2}\int_0^R dr r e^{iprt}\\ &=\frac{2}{\pi}\int_0^1 dt\sqrt{1-t^2}\int_0^R dr r\cos(prt)\\ &=\frac{1}{p}\int_0^R dr J_1(pr)=\frac{1}{p^2}\int_0^{Rp} dx J_1(x)\\ &=\frac{2}{p^2}\sum_{k=0}^{\infty} J_{2k+2}(Rp)\\ &=\frac{1-J_0(Rp)}{p^2},\end{aligned} \tag{54}$$

where we have used the identity

$$J_0(z)+2\sum_{k=1}^{\infty} J_{2k}(z)=1.$$

This propagator is an entire function with the properties

$$\tilde{D}(0)=R^2/4,\ \tilde{D}(ip)=\frac{I_0(Rp)-1}{p^2},$$
$$\lim_{p^2\to\infty}\tilde{D}(p)=\frac{1}{p^2}\left[1-\sqrt{\frac{2}{\pi Rp}}\cos(Rp-\pi/4)\right], \tag{55}$$
$$\lim_{p^2\to\infty}\tilde{D}(ip)=\frac{e^{Rp}}{p^2\sqrt{2\pi Rp}},$$

which indicate the standard p^{-2} behavior for asymptotically large Euclidean momenta, and an exponential rising in the physical region (large energy). We intend to consider elsewhere detailed analytic properties of the Green's functions in the present approach (including that for fermions).

V. CONCLUSIONS AND OPEN PROBLEMS

The idea of domains in the vacuum is not a new one and various hints and attempts at implementation of such an idea can be found [19,38–40]. These approaches assume explicitly or implicitly that the boundaries of domains are populated by the (chromo)electric and/or (chromo)magnetic ''charges and/or currents'' which produce nonzero field strength inside domains. Thus the source for the mean field inside is assumed to be present on the boundary. Specific configurations suitable in principle for a description of such domains are known (see for instance [38,41–44]). In this picture domains are assumed to be stable and in this sense are somewhat similar to the usual domains in ferromagnets.

The model presented in this work differs cardinally from this picture. The central idea that enables us to introduce and consider domains is the observation made in Refs. [17,18] that the presence of singular pure gauge background fields imposes specific conditions on quark, ghost and gluon fluctuations. The boundaries correspond to the locations of singularities in the pure gauge vector potentials which by themselves do not generate any field strength. Such boundaries make their presence felt only via their impact on quantum fluctuations. The mean field inside domains appears as a collective effect of quantum fluctuations, which themselves remain subject to certain boundary conditions. The domains are not stable in this picture, but describe a specific class of field fluctuations in the system. Within this model all the fundamental features of the QCD vacuum—gluon condensation, topological susceptibility, confinement of static and dynamical charges and a non-zero quark condensate—emerge in a transparent and simple way.

So far we have discussed this mechanism in a purely qualitative manner and the relationship of the model with real QCD has to be clarified. It should be recalled that our motivation skipped over two points, both requiring more formal justification. In the first step we prescribed a particular way of dealing with singular pure gauges and thus the QCD functional integral incorporated densely packed interacting domains. In the second step we replaced this integral by a model partition function describing decoupled hyperspheri-

cal domains or clusters. The role of the mean field inside domains is to compensate effectively for the decoupling. The problem of verification of both steps remains open. In particular, a possible relationship between this kind of domain formation via singular pure gauges and the Gribov problem has yet to be understood.

Concerning phenomenological applications, a complete solution to the fermionic eigenvalue problem would immediately enable clarification of the connection between the picture of spontaneous chiral symmetry breaking in this model and the Banks-Casher relation [10].

With quark and gluon propagators in the mean field detailed applications to hadron physics are accessible. The meson spectrum, for example, can be computed via a bosonization procedure as applied in [37,45] or via Bethe-Salpeter equations. Entire quark and gluon propagators are expected to give rise to the Regge character of the spectrum of relativistic bound states [45,46].

In this context the $U_A(1)$ problem can be also addressed. Preliminary estimations show that due to nonzero topological susceptibility the pseudoscalar correlators in the isovector and isoscalar channels are different in the massless limit and strong splitting between the masses of the η' and π mesons is expected. Alternately, the anomalous Ward identity of [23] could be studied order by order in the decomposition over fluctuation fields. The fact that for the pure glue theory a reasonable value for the topological susceptibility is obtained simultaneous with a non-zero quark condensate is encouraging in this respect.

It would be tempting to look for the present picture in lattice simulations. However, domains of constant field can only be taken seriously in a statistical sense, so one should compare results not configuration by configuration (say, after moderate cooling) but for correlators and condensates calculated within the model and on the lattice, where a full statistical ensemble has been taken into account.

Returning to gluonic fluctuations, the picture of dynamical confinement remains incomplete. Diagonal or ''neutral'' gluons remain freely propagating modes in the pre-mean field framework. Intuitively it is clear that this problem is ultimately related to the topological triviality of the class of singular field we have considered here. Incorporating a wider hierarchy of singular fields can resolve this problem.

Finally, a set of open problems relate to the general properties of quantum field theory with domain-like structures and Dirichlet boundary conditions on fluctuation fields. Entire propagators, which appear as a result, indicate that the theory is nonlocal. The ramifications of non-locality need to be investigated, particularly in light of recent work by Efimov [47]. Although the choice of boundary conditions is not expected to generally influence short-distance singularities in Green's functions, the explicit structure of ultraviolet divergencies and the question of renormalizability of a quantum field model with Dirichlet boundary conditions imposed on fields in regions of space should be investigated explicitly.

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APPENDIX A: QUANTUM ELECTRODYNAMICS

The purpose of the following is to illustrate how the effective action as a functional of the mean field and characteristic functions can be defined formally in the Abelian case. Consider QED

$$L=-\frac{1}{4}Q_{\mu\nu}Q_{\mu\nu}+\bar{\psi}(i\not{\partial}-m-e\not{Q})\psi-e\bar{\psi}\not{S}\psi \qquad \text{(A1)}$$

in the presence of an external pure gauge singular field of the form

$$S_\mu=\sum_j^N \partial_\mu f_j(x),$$

where the functions f_j have topologically trivial singularities on hypersurfaces ∂V_j and are assumed to be not Fourier transformable. Gauge transformations which would remove such a pure gauge field are then not defined. Here $Q_{\mu\nu}=\partial_\mu Q_\nu-\partial_\nu Q_\mu$ is the field strength for the photon fields. Thus S appears only in the interaction term coupling to the fermion field. The fluctuation fields Q, ψ and $\bar{\psi}$ are assumed to be regular differentiable functions everywhere in Euclidean space. It should be stressed that unlike non-Abelian theory there is no internal necessity for considering singular fields in electrodynamics, and the example below is artificial in this sense.

The field S in the vicinity of the j-th singular surface can be represented as

$$S_\mu\sim\eta^j_\mu(\eta^j_\nu\partial_\nu)f_j(x), \qquad \text{(A2)}$$

where η^j_μ is a unit vector normal to the surface ∂V_j. Finiteness of the action density thus requires that

$$\bar{\psi}(x)\not{\eta}^j(x)\psi(x)=0,\ \ x\in\partial V_j. \qquad \text{(A3)}$$

This condition is satisfied if, for x on the boundary

$$\psi=-i\not{\eta}^j e^{i\alpha_j\gamma_5}\psi,\ \ \bar{\psi}=\bar{\psi}i\not{\eta}^j e^{-i\alpha_j\gamma_5}, \qquad \text{(A4)}$$

which is the well-known bag-like boundary condition [32]. Note that we are working in Euclidean space-time and the fields ψ and $\bar{\psi}$ are independent variables. The angle α_j is

arbitrary and need not be the same for different j. It should be stressed that the boundary condition violates chiral symmetry. No conditions on the photon fluctuation field Q arise since it is decoupled from S. Now we can write down the functional integral straightforwardly

$$\mathcal{Z}[S]=\int \mathcal{D}Q\,\delta(\partial Q)\int_{\mathcal{F}_S}\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{-\mathcal{S}[Q+S,\psi,\overline{\psi}]}, \quad \text{(A5)}$$

where now the space $\mathcal{F}_S$ contains only those fields which satisfy the boundary conditions Eq. (A4). We stress that the field S in Eq. (A5) is considered as a fixed background field. Gauge fixing for the field Q can be achieved by regular gauge transformations. We see from Eq. (A5) that due to the presence of the singular field S, the integral over fermionic fluctuations is separated into integrations over fields inside subregions V_j bounded by the surfaces ∂V_j where the background field is singular, and these fluctuations are subject to the boundary conditions (A4).

Now we define a procedure for averaging over singular configurations. This is done by identifying the set of different singular configurations with a set of characteristic functions dividing Euclidean space into subregions, whose boundaries coincide with the singular surfaces of a given singular field

$$\{S\}\leftrightarrow\{\chi_1,\ldots,\chi_N\},\ \sum_j^N\int_V d^4x\chi_j^\kappa(x)=V,$$

where the requirement of conservation of the total volume is imposed. Integration over S is defined as an averaging over an ensemble of characteristic functions and angles α_j coming through the fermionic boundary conditions,

$$\int \mathcal{D}S\mathcal{Z}[S]\leftrightarrow\prod_j^N\int\mathcal{D}\chi_j d\alpha_j\delta\left(1-V^{-1}\sum_k^N\int d^4x\chi_k(x)\right)\times\mathcal{Z}[\chi_1,\ldots,\chi_N|\alpha_1,\ldots,\alpha_N], \quad \text{(A6)}$$

$$\mathcal{Z}[\chi|\alpha]=\prod_j^N\int\mathcal{D}Q\,\delta(\partial Q)\int_{\mathcal{F}_j(\alpha_j)}\mathcal{D}\psi_j\mathcal{D}\overline{\psi}_j\times\exp\left\{-\int_{V_j}d^4x\left[\frac{1}{4}Q_{\mu\nu}^2-\overline{\psi}_j(i\partial\!\!\!/-m-e\mathcal{Q}\!\!\!/)\psi_j\right]\right\}. \quad \text{(A7)}$$

In this representation translation invariance as well as chiral symmetry (for $m=0$) are restored because of the averaging over all α_j and characteristic functions χ_j.

Let us integrate out the photon field and factorize the part of the fermionic integral corresponding to the k-th region. We obtain

$$\begin{aligned}\mathcal{Z}_k[\chi|\alpha]=&\int_{\mathcal{F}_k(\alpha_k)}\mathcal{D}\psi_k\mathcal{D}\overline{\psi}_k\prod_{j\neq k}^N\int_{\mathcal{F}_j(\alpha_j)}\mathcal{D}\psi_j\mathcal{D}\overline{\psi}_j\\&\times\exp\left\{\int_{V_k}d^4x\overline{\psi}_k(i\partial\!\!\!/-m)\psi_k+\frac{e^2}{2}\int_{V_k}d^4xd^4yJ_k(x)D(x-y)J_k(y)\right\}\\&\times\exp\left\{\int_{V_j}d^4x\overline{\psi}_j(i\partial\!\!\!/-m)\psi_j+e^2\sum_{j\neq k}\int_{V_k}d^4x\int_{V_j}d^4yJ_k(x)D(x-y)J_j(y)+\frac{e^2}{2}\sum_{j,j'\neq k}\int_{V_j}d^4x\int_{V_j'}d^4yJ_j(x)D(x-y)J_{j'}(y)\right\},\end{aligned}$$

where $J_j(x)$ denotes the electromagnetic current and $D(x-y)$ is the standard photon propagator. Inserting "unity" represented as

$$1=\prod_{x\in V_k}\int\mathcal{D}B^k\delta\left[B^k-e\sum_{j\neq k}\int_{V_j}d^4yD(x-y)J_j(y)\right],$$

we arrive at the representation

$$\begin{aligned}\mathcal{Z}[\chi|\alpha]=&\int\mathcal{D}B^k\exp\{-\mathcal{S}_{\rm eff}[B^k|\chi,\alpha]\}\\&\times\int_{\mathcal{F}_k(\alpha_k)}\mathcal{D}\psi_k\mathcal{D}\overline{\psi}_k\exp\left\{\int_{V_k}d^4x\overline{\psi}_k(i\partial\!\!\!/-m+eB\!\!\!/^k)\psi_k+\frac{e^2}{2}\int_{V_k}d^4xd^4yJ_k(x)D(x-y)J_k(y)\right\}, \quad \text{(A8)}\end{aligned}$$

$$\begin{aligned}&e^{-\mathcal{S}_{\rm eff}[B^k|\chi,\alpha]}\\&=\prod_{j\neq k}^N\int_{\mathcal{F}_j(\alpha_j)}\mathcal{D}\psi_j\mathcal{D}\overline{\psi}_j\times\delta\left[B^k-e\sum_{j\neq k}\int_{V_j}d^4yD(x-y)J_j(y)\right]\\&\times\exp\left\{\int_{V_j}d^4x\overline{\psi}_j(i\partial\!\!\!/-m)\psi_j+\frac{e^2}{2}\sum_{j,j'\neq k}\int_{V_j}d^4x\int_{V_j'}d^4yJ_j(x)D(x-y)J_{j'}(y)\right\}.\end{aligned} \quad \text{(A9)}$$

In this representation the partition function $\mathcal{Z}[\chi|\alpha]$ is defined by the fluctuations of the fermion field in an arbitrarily chosen subregion V_k in the presence of the electromagnetic field B_μ^k, the dynamics of which are governed by the effective action $S_{\rm eff}$.

As is seen from Eq. (A8), this effective action has appeared as an integral (or collective effect) of field fluctuations in the rest of the infinite system, outside the k-th domain. This action functionally depends on a division provided by a particular set of characteristic functions.

Physically different situations would arise depending on the properties of the effective action. This becomes obvious if we write down the partition function averaged over an ensemble of the characteristic functions and boundary conditions (angles α_j),

$$\mathcal{Z}=\prod_j^N \int \mathcal{D}\chi_j d\alpha_j \delta\left(1-V^{-1}\sum_i^N \int d^4x \chi_i(x)\right)$$
$$\times \int \mathcal{D}B^k \exp\{-\mathcal{S}_{\rm eff}[B^k|\chi,\alpha]\}$$
$$\times \int_{\mathcal{F}_k(\alpha_k)} \mathcal{D}\psi_k \mathcal{D}\bar{\psi}_k \exp\left\{\int_{V_k} d^4x \bar{\psi}_k (i\partial\!\!\!/ - m + eB\!\!\!/^k)\psi_k \right.$$
$$\left. + \frac{e^2}{2}\int_{V_k} d^4x d^4y J_k(x) D(x-y) J_k(y)\right\}. \qquad \text{(A10)}$$

Two qualitatively different pictures are possible. If the functional $S_{\rm eff}$ has an absolute minimum at $B_k=0$ and infinitely small volumes of all regions V_j $(j\neq k)$, then we recover the standard QED partition function in the infinite volume. If, however, the minimum is at nonzero mean field and some nonvanishing averaged size of subregions is supportable, we depart from standard electrodynamics. In principle the effective action can be calculated (at least within perturbation theory). As mentioned in the Introduction, there is no reason to expect that the second scenario is realized in electrodynamics.

APPENDIX B: EFFECTIVE POTENTIAL FOR THE CONSTANT FIELD

Consider a covariantly constant Abelian field with the field strength parametrized as

$$B^a_{\mu\nu}=n^a B_{\mu\nu},\ \ n^a T^a = T^3\cos\xi + T^8 \sin\xi,$$

$$E_i = B_{4i},\ \ H_i=\frac{1}{2}\epsilon_{ijk}B_{jk},\ \ \mathbf{E}^2+\mathbf{H}^2=2B^2,$$

$$\mathbf{EH}=|\mathbf{E}||\mathbf{H}|\cos\omega,\ \ (\mathbf{EH})^2=\mathbf{H}^2(2B^2-\mathbf{H}^2)\cos^2\omega,$$

and let the gauge invariant effective potential be given by the series

$$U_{\rm eff}(B,\omega,\xi)=\sum_{k=1}^{\infty} A_k {\rm Tr}\,\check{B}^{2k},$$

$$(\check{B}^{2k})_{\mu\nu}=\check{n}^{2k}B_{\mu\alpha_1}\ldots B_{\alpha_{k-1}\nu}, \qquad \text{(B1)}$$

with A_k constants. One can show that if $U_{\rm eff}$ is bounded from below and has a nontrivial minimum as a function of parameter B then there is a set of twelve discrete minima corresponding to an (anti-)self-dual field and six values of the angle ξ.

The odd powers of $\check{n}$ and B do not appear in the potential, since this would mean violation of Weyl symmetry and parity respectively. The Weyl group is a discrete subgroup of global $SU(3)$ and in this case can be seen as the group of permutations of the eigenvalues of the matrix $\check{n}$. Such permutations can be arranged by a shift of the angle parametrizing the Abelian field configuration, $\xi\to\xi+\pi n/3$. In other words, the effective potential is periodic in ξ with a period $\pi/3$, the angle of $SU(3)$. It is also periodic in ω with the period π due to invariance under parity. This can be checked using formulas

$${\rm Tr}\,\check{n}^2=3,\ \ {\rm Tr}\,\check{n}^4=(9/4),\quad {\rm Tr}\,\check{n}^6=(3/16)[10+\cos(6\xi)],$$

$${\rm Tr}\,B^2=-2(\mathbf{E}^2+\mathbf{H}^2)=-4B^2,$$

$${\rm Tr}\,B^4=2[(\mathbf{E}^2+\mathbf{H}^2)^2-2(\mathbf{EH})^2]=8\left[B^4-\frac{1}{2}(\mathbf{EH})^2\right],$$

$${\rm Tr}\,B^6=-2(\mathbf{E}^2+\mathbf{H}^2)[(\mathbf{E}^2+\mathbf{H}^2)^2-3(\mathbf{EH})^2]$$
$$=-16B^2\left[B^4-\frac{3}{4}(\mathbf{EH})^2\right].$$

A nontrivial dependence on ξ appears for $k\geqslant 3$. Higher terms depend on ξ via functions $\cos 6l\xi$ $(l\geqslant 1)$. Taking into acount the first three terms in the decomposition and calculating the traces as above one gets

$$U_{\rm eff}=-C_1B^2+\frac{C_2}{\Lambda^4}\left[B^4-\frac{1}{2}(\mathbf{EH})^2\right]+\frac{C_3}{\Lambda^8}B^2(10+\cos 6\xi)$$
$$\times\left[B^4-\frac{3}{4}(\mathbf{EH})^2\right].$$

The coefficients C_2 and C_3 are assumed to be positive to provide for the boundedness of the potential from below, and Λ is a scale. The sign of the constant C_1 is of particular importance. If C_1 is negative the minimum is trivial $B=0$. For C_1 positive, the potential has a minimum at nonzero B. Using the identity

$$(\mathbf{EH})^2=\mathbf{H}^2(2B^2-\mathbf{H}^2)\cos^2\omega,$$

it is easy to check that there are degenerate absolute minima corresponding to field configurations with the parameters

$$\mathbf{H}^2=\mathcal{B}^2,\ \ \omega_n=\pi n\ \ (n=0,1),$$

$$\xi_k=(2k+1)\pi/6\ \ (k=0,1,\ldots,5),$$

$$\mathcal{B}^2=2\Lambda^4(\sqrt{C_2^2+3C_1C_3}-C_2)/3C_3>0.$$

These twelve discrete degenerate minima correspond to self-dual and anti-self-dual field configurations and six values of angle ξ, and there is a continuous degeneracy relating to

orientations of the chromomagnetic field **H**. With the simplest polynomial form for $U_{\rm eff}$ as above we have $\xi_0=\pi/6$. This value depends on the form of the effective potential, however the period $\pi/3$ related to the Weyl symmetry is universal.

APPENDIX C: THE WILSON LOOP FOR $SU(3)$

For SU(3) the eigenvalues of the color matrix

$$\hat{n}_j = t^3\cos\xi_j + t^8\sin\xi_j$$

take values from the set

$$\frac{1}{\sqrt{3}},\quad -\frac{1}{\sqrt{3}},\quad \frac{1}{2\sqrt{3}},\quad -\frac{1}{2\sqrt{3}}$$

for all different values of the vacuum angle

$$\xi_j \in [(2l+1)\pi/3]_{l=0,\ldots,5}.$$

This leads to the following result for the trace averaged over the vacuum angle

$$W(L)=\lim_{V,N\to\infty}\left[\int_V \frac{d^4z_j}{V}\int d\sigma_j \frac{1}{6}\left(e^{(i/\sqrt{3})B^j_{\mu\nu}J_{\mu\nu}(z_j)}+e^{-(i/\sqrt{3})B^j_{\mu\nu}J_{\mu\nu}(z_j)}+2e^{(i/2\sqrt{3})B^j_{\mu\nu}J_{\mu\nu}(z_j)}+2e^{-(i/2\sqrt{3})B^j_{\mu\nu}J_{\mu\nu}(z_j)}\right)\right]^N.$$

Then the Wilson loop takes the form

$$W(L)=\lim_{N\to\infty}\left[1-\frac{1}{N}U(L)\right]^N=e^{-U(L)}$$

$$U(L)=\frac{\pi^2R^2L^2}{3v}\left(3-\frac{\sqrt{3}}{2\pi BR^2}\int_0^{2\pi BR^2/\sqrt{3}}\frac{dx}{x}\sin x-\frac{2\sqrt{3}}{\pi BR^2}\int_0^{\pi BR^2/\sqrt{3}}\frac{dx}{x}\sin x\right)+\frac{\pi^2}{v}\left(\frac{4}{3}R^3L+\frac{1}{2}R^4\right)$$
$$-\frac{\pi^2(1-\cos 2\pi BR^2/\sqrt{3})}{v(2\pi B/\sqrt{3})^2}-\frac{2\pi^2(1-\cos\pi BR^2/\sqrt{3})}{v(\pi B/\sqrt{3})^2}$$
$$+\frac{4\pi^2L}{v}\left(\frac{1}{(2\pi B/\sqrt{3})^{3/2}}\int_0^{\sqrt{2\pi BR^2/\sqrt{3}}}dx\frac{\sin x}{x}+\frac{2}{(\pi B/\sqrt{3})^{3/2}}\int_0^{\sqrt{\pi BR^2/\sqrt{3}}}dx\frac{\sin x}{x}\right)$$
$$-\frac{\pi^2L^4}{v}\int_0^{R^2/L^2}ds\int_{(1-\sqrt{s})^2}^{(1+\sqrt{s})^2}dt\left(\sqrt{3}\frac{\sin\{BL^2[2\varphi-\sin\varphi+s(2\psi-\sin\psi)]/\sqrt{3}\}}{BL^2[2\varphi-\sin\varphi+s(2\psi-\sin\psi)]}\right.$$
$$\left.+4\sqrt{3}\frac{\sin\{BL^2[2\varphi-\sin\varphi+s(2\psi-\sin\psi)]/2\sqrt{3}\}}{BL^2[2\varphi-\sin\varphi+s(2\psi-\sin\psi)]}\right).$$

This leads to the string constant as given through the function as in Eq. (27).

APPENDIX D: SCALAR FIELD EIGENVALUE PROBLEM

We introduce the $O(4)$ generators

$$L_i = i\epsilon_{ijk}x_j\partial_k,$$

$$M_i = i(x_4\partial_i - x_i\partial_4),$$

respectively for spatial rotations and Euclidean ''boosts.'' These satisfy the usual commutation relations.

Now in the scalar field eigenvalue problem we encounter the structure

$$(\partial_\mu - iB_\mu)^2 = \partial^2 - 2iB_\mu\partial_\mu - B_\mu B_\mu.$$

Given that

$$B_\mu\partial_\mu = -\frac{1}{2}B_{\mu\nu}x_\nu\partial_\mu = \frac{1}{2i}(EM_3 - BL_3)$$

we see that it is better to go over to the $O(3)\times O(3)$ generators,

$$\mathbf{K}_1 = \frac{1}{2}(\mathbf{L}+\mathbf{M})$$

$$\mathbf{K}_2 = \frac{1}{2}(\mathbf{L}-\mathbf{M}).$$

So with the field self-dual/anti-self-dual, $E=\pm B$ we obtain,

$$B_\mu \partial_\mu = \begin{cases} iBK_{2z}, & E=B, \\ iBK_{1z}, & E=-B. \end{cases}$$

Then the four dimensional Laplace operator can be written

$$\partial^2 = \frac{1}{r^3}\partial_r(r^3\partial_r) - \frac{4}{r^2}\mathbf{K}_1^2.$$

The usual considerations show then that the eigenvalues of the complete set of mutually commuting operators $\mathbf{K}_1^2 = \mathbf{K}_2^2$, K_{1z} and K_{2z} are

$$\frac{k}{2}\left(\frac{k}{2}+1\right) \quad (k=0,1,\ldots), \quad m_1, m_2 = \frac{k}{2}, \frac{k}{2}-1, \ldots, -\frac{k}{2}. \tag{D1}$$

Eigenfunctions corresponding to these values can be found such that

$$\begin{aligned} K_{1z}C_{km_1m_2} &= m_1 C_{km_1m_2}, \\ K_{2z}C_{km_1m_2} &= m_2 C_{km_1m_2}, \\ \mathbf{K}_1^2 C_{km_1m_2} &= \frac{k}{2}\left(\frac{k}{2}+1\right) C_{km_1m_2}. \end{aligned} \tag{D2}$$

In the hyperspherical coordinates

$$\begin{aligned} x_1 &= r\sin\eta\cos\phi \\ x_2 &= r\sin\eta\sin\phi \\ x_3 &= r\cos\eta\cos\chi \\ x_4 &= r\cos\eta\sin\chi \end{aligned}$$

$4\mathbf{K}_1^2$ takes the form,

$$4\mathbf{K}_1^2 = -\frac{\partial^2}{\partial\eta^2} - \mathrm{cosec}^2\eta\frac{\partial^2}{\partial\phi^2} - \sec^2\eta\frac{\partial^2}{\partial\chi^2} + (\tan\eta - \cot\eta)\frac{\partial}{\partial\eta}.$$

Thus

$$\begin{aligned} 4\mathbf{K}_1^2 C_{km_1m_2}(\eta,\phi,\chi) = \exp i[(m_1-m_2)\chi + (m_1+m_2)\phi] \\ \times\left[-\frac{\partial^2}{\partial\eta^2} + (m_1+m_2)^2\mathrm{cosec}^2\eta \right. \\ + (m_1-m_2)^2\sec^2\eta + (\tan\eta \\ \left. - \cot\eta)\frac{\partial}{\partial\eta}\right]\Theta_k^{m_1-m_2,m_1+m_2}(\eta). \end{aligned}$$

Up to normalization, the angular eigenfunctions Θ can be better written in terms of the hypergeometric function,

$$\begin{aligned} &\Theta_k^{m_1-m_2,m_1+m_2}(\eta) \\ &\quad \propto \cos^{m_1-m_2}(\eta)\sin^{k-m_1+m_2}(\eta)\,{}_2\mathrm{F}_1 \\ &\quad\quad \times\left(-\frac{k}{2}+m_1, -\frac{k}{2}-m_2;\ m_1-m_2+1;\ -\cot^2\eta\right). \end{aligned}$$

For compactness of notation we denote

$$h(\eta) = \cos^{m_1-m_2}(\eta)\sin^{k-m_1+m_2}(\eta),$$

$$u(\eta) = {}_2\mathrm{F}_1\left(-\frac{k}{2}+m_1, -\frac{k}{2}-m_2;\ m_1-m_2+1;\ -\cot^2\eta\right).$$

Then, after some tedious calculation, one gets

$$\begin{aligned} &4\mathbf{K}_1^2 C_{km_1m_2}(\eta,\phi,\chi) \\ &\quad = \exp i[(m_1-m_2)\chi + (m_1+m_2)\phi]h(\eta) \\ &\quad\quad \times\left[-\frac{d^2u}{d\eta} + \cot\eta\frac{du}{d\eta}\left(\frac{2(m_1-m_2)+1}{\cot^2\eta}\right.\right. \\ &\quad\quad \left.-[2(k-m_1+m_2)+1]\right) + u[(2m_1-k)(2m_2+k) \\ &\quad\quad \left.\times\cot^2\eta + 2k(m_1-m_2+1) + 4m_1m_2]\right]. \end{aligned}$$

Rewriting in terms of the variable $z=-\cot^2\eta$ one eventually brings this to the form

$$\begin{aligned} &4\mathbf{K}_1^2 C_{km_1m_2}(\eta,\phi,\chi) \\ &\quad = \exp i[(m_1-m_2)\chi + (m_1+m_2)\phi]4h(\eta)(1-z) \\ &\quad\quad \times\left[z(1-z)\frac{d^2u}{dz^2} + \frac{du}{dz}[(m_1-m_2+1)\right. \\ &\quad\quad -(1-k+m_1-m_2)z] - \left(\frac{k}{2}-m_1\right)\left(\frac{k}{2}+m_2\right)u \\ &\quad\quad \left.+\frac{k}{2}\left(\frac{k}{2}+1\right)\frac{u}{1-z}\right]. \end{aligned} \tag{D3}$$

But u satisfies the hypergeometric equation. Thus the first three sets of terms in the square brackets of Eq. (D3) vanish, leaving

$$\begin{aligned} 4\mathbf{K}_1^2 C_{km_1m_2}(\eta,\phi,\chi) = 4\frac{k}{2}\left(\frac{k}{2}+1\right) \\ \times e^{i[(m_1-m_2)\chi+(m_1+m_2)\phi]}h(\eta)u(\eta), \end{aligned}$$

namely, Eq. (D2), which completes the proof.

Putting all this together we arrive at the following representation for the square of the covariant derivative

$$(\partial_\mu - iB_\mu)^2 = \frac{1}{r^3}\partial_r(r^3\partial_r) - \frac{4}{r^2}\mathbf{K}_1^2 + 2BK_{2z} - \frac{1}{4}B^2r^2, \quad E = B$$

$$(\partial_\mu - iB_\mu)^2 = \frac{1}{r^3}\partial_r(r^3\partial_r) - \frac{4}{r^2}\mathbf{K}_1^2 + 2BK_{1z} - \frac{1}{4}B^2r^2,$$

$$E = -B.$$

Using the eigenfunctions $C_{km_1m_2}$ we can reduce the original eigenvalue problem to that corresponding to a radial operator

$$-\left[\frac{1}{r^3}\partial_r(r^3\partial_r) - \frac{k(k+2)}{r^2} + 2m_{2,1}B - \frac{B^2r^2}{4} - \mathcal{M}^2\right]f(r) = \lambda f(r),$$

where $m_{2,1}$ is defined in the main text.

So the complete eigenfunctions will be a product of radial and angular parts,

$$\psi(x) = f(r)C_{km_1m_2}(\eta,\phi,\chi).$$

The radial function can be solved, as described in the main body of the paper, leading to the solution given in Eq. (39).

A comment is necessary at this point on the half-integer values of the azimuthal quantum numbers. The angular eigenfunctions depend only on the sum and differences of ms, which will be whole integers. Also the combination

$$n_{2,1} \equiv \frac{k}{2} - m_{2,1} + 1 = \frac{k}{2} + \frac{k}{2} + 1, \frac{k}{2} + \frac{k}{2}, \ldots, \frac{k}{2} - \frac{k}{2} + 1 = k+1, k, \ldots, 1.$$

This combination, and thus the eigenvalue spectrum, will always involve integral values.

APPENDIX E: DERIVATION OF THE SCALAR FIELD PROPAGATOR

We represent the delta function in the hyperspherical coordinates,

$$\delta^{(4)}(x-x') = \frac{\delta(r-r')\delta(\eta-\eta')\delta(\phi-\phi')\delta(\chi-\chi')}{r^3 \sin\eta\cos\eta},$$

where the primed variables correspond to the hyperspherical coordinates of x'. We use the ansatz,

$$G(x,x'|\mu) = \sum_{k,m_1,m_2} V_{km_1m_2}(\eta',\phi',\chi') \times C_{km_1m_2}(\eta,\phi,\chi)X_{km_1m_2}(r,r'|\mu).$$

Inserting this into the original Green's function equation, exploiting the delta-function representation and the fact that the functions C are eigenfunctions of the covariant derivative squared operator, yields

$$\sum_{k,m_1,m_2} V_{km_1m_2}(\eta',\phi',\chi')C_{km_1m_2}(\eta,\phi,\chi) \times \left[\frac{1}{r^3}\partial_r(r^3\partial_r) - \frac{k(k+2)}{r^2} + 2m_{2,1}B - \frac{B^2r^2}{4} - \mathcal{M}^2\right] \times X_{km_1m_2}(r,r'|\mu), = -\frac{\delta(r-r')\delta(\eta-\eta')\delta(\phi-\phi')\delta(\chi-\chi')}{r^3\sin\eta\cos\eta}.$$

We can now separate the angular from the radial dependence in two equations,

$$\sum_{k,m_1,m_2} V_{km_1m_2}(\eta',\phi',\chi')C_{km_1m_2}(\eta,\phi,\chi) = \frac{\delta(\eta-\eta')\delta(\phi-\phi')\delta(\chi-\chi')}{\sin\eta\cos\eta}, \tag{E1}$$

$$\left[\frac{1}{r^3}\partial_r(r^3\partial_r) - \frac{k(k+2)}{r^2} + 2m_{2,1}B - \frac{B^2r^2}{4} - \mathcal{M}^2\right] \times X_{km_1m_2}(r,r'|\mu) = -\frac{\delta(r-r')}{r^3}. \tag{E2}$$

Equation (E1) we recognize as the completeness relation for the angular eigenfunctions, thus $V=C$. Using Eq. (E2), we read off that X does not depend on both quantum numbers m_2, m_1 but on one of them, depending on whether the field was self-dual or anti-self-dual. In fact we shall indicate this dependence on $m_{2,1}$ via the quantum number $n_{2,1}$ so that $X = X_{kn_{2,1}}(r,r'|\mu)$.

We can solve the radial problem by solving for the radial Green's function in the infinite volume and then adding a solution to the homogeneous equation with an arbitrary coefficient. The coefficient is fixed by imposing the finite boundary condition at $r=R$.

Using our solution to the eigenvalue equation we can easily extract homogeneous solutions, namely to the equation:

$$\left[\frac{1}{r^3}\partial_r(r^3\partial_r) - \frac{k(k+2)}{r^2} + 2m_{2,1}B - \frac{B^2r^2}{4} - \mathcal{M}^2\right]R(r) = 0.$$

In the notation of [34], the two solutions (respectively regular at infinity and the origin) are

$$R_1(r|k,n_{2,1},\mu) = r^k e^{-Br^2/4}M(n_{2,1}+\mu, k+2; Br^2/2)$$
$$R_2(r|k,n_{2,1},\mu) = r^k e^{-Br^2/4}U(n_{2,1}+\mu, k+2; Br^2/2) \tag{E3}$$

with $n_{2,1}$ defined in Eq. (40).

We next solve for the Green's function in the $R=\infty$ case by recasting the problem in the form of a Sturm-Liouville equation,

$$[\partial_r p(r)\partial_r + q(r)]X(r,r') = -\delta(r-r'),$$

so that

$$p(r)=r^3$$

$$q(r)=-\left[\frac{B^2r^2}{4}+\frac{k(k+2)}{r^2}-2Bm_{2,1}+\mathcal{M}^2\right]r^3.$$

The Sturm-Liouville equation is known to have the solution,

$$X(r,r')=-\frac{1}{p(r')w(r')}R(r,r')$$

with $w(r)$ the Wronskian of the homogeneous solutions,

$$w=R_1R_2'-R_2R_1'$$

and

$$R(r,r')=\begin{cases}R_1(r)R_2(r'), & r<r',\\ R_1(r')R_2(r), & r>r'.\end{cases}$$

The Wronskian for the two solutions is evaluated to be

$$w(r)=-\frac{\Gamma(k+2)\sqrt{2B}}{\Gamma(n_{2,1}+\mu)(Br^2/2)^{3/2}}$$

so that

$$p(r)w(r)=\frac{-4\Gamma(k+2)}{B\Gamma(n_{2,1}+\mu)}.$$

We now construct the full Green's function and impose the boundary condition at finite R:

$$G(x,x'|\mu)=\sum_{k,m_1,m_2}C_{km_1m_2}(\eta',\phi',\chi')C_{km_1m_2}(\eta,\phi,\chi)\times[X_{kn_{2,1}}(r,r'|\mu)+A_{kn_{2,1}}(R|B,\mu)\times R_1(r|k,n_{2,1},\mu)R_1(r'|k,n_{2,1},\mu)].$$

Imposing $G(R,r'|\mu)=0$ with $r'<r=R$ we have

$$A_{kn_{2,1}}(R|\mu)=-\frac{X_{kn_{2,1}}(R,r'|\mu)}{R_1(R|k,n_{2,1},\mu)R_1(r'|k,n_{2,1},\mu)}$$

$$=+\frac{R_{kn_{2,1}}(R,r')}{w(R)p(R)R_1(R|k,n_{2,1},\mu)R_1(r'|k,n_{2,1},\mu)}$$

$$=+\frac{R_1(r'|k,n_{2,1},\mu)R_2(R|k,n_{2,1},\mu)}{w(R)p(R)R_1(R|k,n_{2,1},\mu)R_1(r'|k,n_{2,1},\mu)}$$

$$=\frac{R_2(R|k,n_{2,1},\mu)}{w(R)p(R)R_1(R|k,n_{2,1},\mu)}.$$

Thus finally,

$$A_{kn_{2,1}}(R|\mu)=-B\frac{\Gamma(n_{2,1}+\mu)}{4\Gamma(k+2)}\frac{U(n_{2,1}+\mu,k+2;BR^2/2)}{M(n_{2,1}+\mu,k+2;BR^2/2)}. \tag{E4}$$

It is straightforward to see that imposing $G(r,R|\mu)=0$, $r<R$ will give the same result for A.

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