Longitudinal contributions to hadronic τ decay

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A number of recent determinations of m_s using hadronic τ decay data involve inclusive analyses based on the so-called $(k,0)$ spectral weights. We show here that the OPE representations of the longitudinal contributions appearing in these analyses, which are already known to be poorly converging, have in addition an unphysical k dependence that produces a significant unphysical decrease in m_s with increasing k. We also show how, using additional sum rule constraints, the decay constants of the excited resonances in the strange scalar and pseudoscalar channels may be determined, allowing one to evaluate the longitudinal spectral contributions to the $(k,0)$ sum rules. Taking into account the very accurately known π and K pole contributions, we find that longitudinal contributions can be determined with an accuracy at the few percent level, and hence reliably subtracted, leaving an analysis for m_s involving the sum of longitudinal and transverse contributions, for which the OPE representation is much better converged.

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I. INTRODUCTION

The determination of the strange quark mass m_s has been the focus of much recent activity in both the sum rule [1–14] and lattice [15–18] communities. (A summary of the current status of both the sum rule and lattice determinations is given in Ref. [19].)

Among the sum rule approaches, those based on flavor breaking in hadronic τ decay [5–14] appear the most reliable at present. There are two reasons for this statement. First, on the experimental side, the spectral data required is known over the full kinematic range entering the relevant sum rules [20,21] and, second, on the theoretical [operater product expansion (OPE)] side, flavor-independent instanton and renormalon effects, which create potential uncertainties in analyses of the strange scalar and pseudoscalar channels [1,2,4], cancel in forming the flavor-breaking τ decay difference [7]. The τ decay sum rules are, however, not without complications. The primary complication has to do with the behavior of the OPE representation of the contributions to the inclusive decay rate of hadronic states with total spin $J=0$. In order to elaborate on this point, and to fix terminology and notation, we briefly review the relation of m_s to hadronic τ decay.

The ratio of the inclusive hadronic τ decay width through the flavor $ij=ud$ or us vector (V) or axial vector (A) weak hadronic current to the corresponding electron decay width:

$$R_{ij;V,A}\equiv \frac{\Gamma[\tau^-\to\nu_\tau \mathrm{hadrons}_{\mathrm{ij;V,A}}(\gamma)]}{\Gamma[\tau^-\to\nu_\tau e^-\overline{\nu}_e(\gamma)]}, \tag{1}$$

where (γ) indicates additional photons or lepton pairs, can be expressed in terms of weighted integrals over the spin $J=0$ (longitudinal) and $J=1$ (transverse) components of the corresponding V or A spectral functions [22], where the spectral functions are defined, as usual, by

$$\rho^{(J)}_{ij;V,A}(q^2)\equiv\frac{1}{\pi}\mathrm{Im}\,\Pi^{(J)}_{ij;V,A}(q^2). \tag{2}$$

In Eq. (2), $\Pi^{(J)}_{ij;V,A}$ are the spin J scalar components of the usual flavor ij V, A current-current correlators,

$$\begin{aligned} &i\int d^4x e^{iq\cdot x}\langle 0|T[J^\mu_{ij;V,A}(x)J^\nu_{ij;V,A}(0)^\dagger]|0\rangle,\\ &\quad\equiv(-g^{\mu\nu}q^2+q^\mu q^\nu)\Pi^{(1)}_{ij;V,A}(q^2)+q^\mu q^\nu\Pi^{(0)}_{ij;V,A}(q^2). \end{aligned} \tag{3}$$

Working with the combinations $\Pi^{(0+1)}(q^2)\equiv\Pi^{(0)}(q^2)+\Pi^{(1)}(q^2)$ and $q^2\Pi^{(0)}(q^2)$ which have no kinematic singularities, $R_{ij;V,A}$ can then be written [23,24]

$$\begin{aligned} R_{ij;V,A}&=12\pi^2 S_{EW}|V_{ij}|^2\int_0^{m_\tau^2}\frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\\ &\quad\times\left[\left(1+2\frac{s}{m_\tau^2}\right)\rho^{(0+1)}_{ij;V,A}(s)-\frac{2s}{m_\tau^2}\rho^{(0)}_{ij;V,A}(s)\right]\\ &=6\pi S_{EW}|V_{ij}|^2 i\oint_{|s|=m_\tau^2}\frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\\ &\quad\times\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(0+1)}_{ij;V,A}(s)-2\frac{s}{m_\tau^2}\Pi^{(0)}_{ij;V,A}(s)\right], \end{aligned} \tag{4}$$

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where $s=q^2=-Q^2$, $S_{EW}=1.0194$ represents the leading electroweak corrections [25], V_{ij} are the usual Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, and the second line follows from the first as a consequence of Cauchy's theorem. The second line of Eqs. (4) allows $R_{ij;V,A}$ to be evaluated using techniques based on the OPE and perturbative QCD [22–24,26,27]. The weights, $w_T(y)\equiv(1-y)^2(1+2y)$ and $w_L(y)=-2y(1-y)^2$ (with $y\equiv s/m_\tau^2$) multiplying $\Pi^{(0+1)}$ and $\Pi^{(0)}$, respectively, have (double) zeros at $s=m_\tau^2$, reflecting the fact that $s=m_\tau^2$ lies at the edge of hadronic phase space. The resulting suppression of contributions from the region of the circular contour near the timelike real axis (the region of potential breakdown of the OPE) is responsible for the high quality of the OPE representation of the inclusive hadronic rates (see Ref. [27] for a recent review).

Defining $R_{ij}\equiv R_{ij;V}+R_{ij;A}$, it is then evident that

$$\Delta R_\tau \equiv \frac{R_{ud}}{|V_{ud}|^2}-\frac{R_{us}}{|V_{us}|^2} \tag{5}$$

vanishes in the $SU(3)_F$ limit. Defining $\Delta\Pi^{(J)}\equiv\Pi^{(J)}_{ud}-\Pi^{(J)}_{us}$ and $\Delta\rho^{(J)}\equiv\rho^{(J)}_{ud}-\rho^{(J)}_{us}$, the mass-independent $D=0$ contributions, therefore, cancel by construction on the OPE side of the sum rule for ΔR_τ analogous to Eq. (4),

$$\begin{aligned}12\pi^2 S_{EW}\int_0^1 dy[w_T(y)\Delta\rho^{(0+1)}(s)+w_L(y)\Delta\rho^{(0)}(s)]\\ =6\pi i S_{EW}\oint_{|y|=1} dy[w_T(y)\Delta\Pi^{(0+1)}(s)\\ +w_L(y)\Delta\Pi^{(0)}(s)].\end{aligned} \tag{6}$$

Neglecting $m_{u,d}^2$ and $\alpha_s m_{u,d} m_s$ relative to m_s^2, the leading $D=2$ term is proportional to m_s^2 [5,24]. The integrand on the LHS of Eq. (6) is known, as a function of s, from the work of the ALEPH Collaboration [20,21]. On the OPE side, the $D=2$ contribution is known in terms of α_s once m_s is fixed; the $D=4$ contribution is known in terms of $\langle m_s\bar{s}s\rangle$; and the $D=6$ contribution is small as a result of the cancellation between the dominant $D=6$ four-quark condensate terms which occurs in the $V+A$ sum [5,24]. Alternate sum rules which also allow the spectral side to be evaluated using the measured s-dependent ud and us number distributions, without necessitating a $J=0/J=1$ separation of the experimental data, can be constructed by multiplying the integrands appearing on both sides of Eq. (6) by a common analytic factor. For the case that this factor is $(1-y)^k y^n$, the resulting sum rule is said to involve the (k,n) spectral weight, and the resulting analogue of ΔR_τ is denoted $\Delta R_\tau^{(k,n)}$. The $(k,0)$ spectral weight sum rules (with $k=0,1,2$) form the basis of a number of recent inclusive treatments of the m_s extraction problem [7–11,13,14]. We will denote the weights accompanying $\Pi^{(0+1)}$ and $\Pi^{(0)}$ in the $(k,0)$ sum rules by $w_T^{(k)}(y)=(1-y)^{2+k}(1+2y)$ and $w_L^{(k)}(y)=-2y(1-y)^{2+k}$, respectively.

Equation (6) [corresponding to the (0,0) spectral weight], and/or its $(k,0)$ generalizations, would allow a straightforward determination of m_s, provided the OPE representations of both the $J=0+1$ and $J=0$ contributions above were well converged at scale m_τ^2. Unfortunately, it turns out that this is not the case. The problem lies with the OPE contribution involving the product $w_L^{(k)}(y)\Pi^{(0)}(s)$. We refer to these contributions as "longitudinal" in what follows.

The source of the problem is that the perturbative series for the integrated $D=2$ longitudinal contribution [which is known to four loops, i.e., $O(\alpha_s^3)$ [28,29]] is not convergent at the scale m_τ^2 [6,7,30]. This is true whether one considers the "fixed order" (FOPT) expansion [expansion in powers of $\alpha_s(\mu^2)$ at a fixed scale, μ, e.g., $\mu=m_\tau$], or the "contour-improved" expansion (CIPT) [26,31] (in which the large logarithms are summed up by the scale choice $\mu^2=Q^2$ point-by-point over the contour). Taking the unmodified version of Eq. (6), corresponding to $k=0$ to be specific, one finds that using the central value of the ALEPH determination, $\alpha_s(m_\tau^2)=.334$, the FOPT $D=2$ series behaves as

$$\sim 1+0.99+1.24+1.59+\cdots, \tag{7}$$

while the corresponding CIPT series behaves as [21]

$$\sim 1+0.78+0.78+0.90+\cdots \tag{8}$$

[where in both cases we have normalized successive terms in the integrated series to the leading, $O(\alpha_s^0)$, term]. The integrated longitudinal $D=2$ series, truncated at $O(\alpha_s^3)$, also exhibits a very strong residual scale dependence [7]. Because of the non-convergence and strong residual scale dependence, inclusive sum rules of the type described above contain significant uncertainties associated with the presence of the longitudinal contributions. Recent inclusive analyses [7,9–11,13,14,21] proceed by taking the sum of $D=2$ $J=0+1$ and $J=0$ contributions to $O(\alpha_s^3)$. The size of the last known [$O(\alpha_s^3)$] term is taken as an estimate of the $D=2$ truncation error. In Refs. [7,10,13,14], an additional error has been included to account for the residual scale dependence of the truncated sum. The scale-dependence error is estimated by varying the scale choice in the CIPT evaluation of the $D=2$ sum according to $\mu^2=\xi^2 Q^2$, with $.75<\xi<2$, and symmetrizing the resulting variation about the central value $\xi=1$. The resulting estimated $D=2$ error is much larger than that for the remaining longitudinal terms, and hence dominates the error on the total longitudinal contribution. Taking the results of Ref. [10], which gives the detailed breakdown into individual longitudinal contributions, to be specific, and combining the quoted errors in quadrature, one finds errors of 32%, 34% and 37% for the (0,0), (1,0) and (2,0) total longitudinal contributions, respectively. These errors represent the dominant component of the total theoretical error in the inclusive analyses.

While the errors on the longitudinal contributions discussed above might appear safely conservative, we will argue in this paper that they are, in fact, almost certainly too small. We will, in addition, demonstrate that the central values of the OPE contribution to $\Delta R_\tau^{(k,0)}$ contain an unphysical dependence on k which produces a corresponding unphysical lowering of the extracted value of m_s with increasing k. Finally, we will demonstrate that it is possible to significantly reduce the uncertainties on the total longitudinal contributions, allowing one to subtract the longitudinal contributions from the experimental number distribution and work instead with sum rules for the much better behaved $0+1$ correlator difference.

The rest of the paper is organized as follows. In Sec. II we demonstrate the existence of the problem and investigate its magnitude. In Sec. III we discuss how to improve the estimate for the longitudinal contributions using sum rules for the strange scalar and pseudoscalar channels and present our numerical results. Finally, in Sec. IV we summarize our results and comment on their implications for future analyses of m_s using τ decay data.

II. PHYSICAL CONSTRAINTS ON THE LONGITUDINAL CONTRIBUTIONS TO THE SPECTRAL WEIGHT ANALYSES

If a $J=0/J=1$ spin decomposition existed for the current experimental data, one could simply subtract the longitudinal contribution from each bin of the experimental ud and us distributions, determine $\rho^{(0+1)}_{ud,us;V+A}(s)$, and use this information to analyze the $0+1$, rather than inclusive, sum rules. Unfortunately, such a decomposition does not yet exist over the whole of the kinematically allowed range. Certain general features of the longitudinal contributions to the V and A correlators, however, allow us to nonetheless obtain useful constraints.

In the chiral limit, the longitudinal spectral functions vanish except for the (massless) π and K pole terms, which contribute to $\rho^{(0)}_{ud;A}$ and $\rho^{(0)}_{us;A}$, respectively. Away from the chiral limit, $\rho^{(0)}_{ij;V}$ and $\rho^{(0)}_{ij;A}$ receive additional contributions proportional, respectively, to $(m_i-m_j)^2$ and $(m_i+m_j)^2$. For $ij=ud$, these additional contributions are numerically tiny and can be neglected. For $ij=us$, the thresholds for the nonpole contributions to $\rho^{(0)}_{us;V}$ and $\rho^{(0)}_{us;A}$ are $(m_K+m_\pi)^2\equiv s^{SS}_{th}$ and $(m_K+2m_\pi)^2\equiv s^{SPS}_{th}$, respectively. The sum of the longitudinal K and π pole contributions for the $(k,0)$ spectral weight is

$$[\Delta R^{(k,0)}_\tau]^{K+\pi}_L=48\pi^2 S_{EW}\left[\left(\frac{f_K^2}{m_\tau^2}\right)\left(\frac{m_K^2}{m_\tau^2}\right)\left(1-\frac{m_K^2}{m_\tau^2}\right)^{2+k}-\left(\frac{f_\pi^2}{m_\tau^2}\right)\left(\frac{m_\pi^2}{m_\tau^2}\right)\left(1-\frac{m_\pi^2}{m_\tau^2}\right)^{2+k}\right], \tag{9}$$

with $f_\pi=92.4$ MeV and $f_K=113.0$ MeV [32]. The π pole contribution is nearly constant with increasing k, the K pole contribution slowly decreasing with k. The remaining longitudinal contribution, which we will refer to as the "resonance contribution," is then given by

$$[\Delta R^{(k,0)}_\tau]^{resonance}_L=12\pi^2 S_{EW}\int_{y_{th}}^1 dy\, 2y(1-y)^{2+k}\times[\rho^{(0)}_{us;V}+\rho^{(0)}_{us;A}], \tag{10}$$

where $y_{th}=s^{SS}_{th}/m_\tau^2$. The V part of this contribution should be dominated by the $K_0^*(1430)$ resonance, since only the tail of the next strange scalar resonance, the $K_0^*(1950)$, lies within the kinematically allowed range, $s<m_\tau^2$. Similarly, the A contribution in Eq. (10) should be dominated by the $K(1460)$. The K and π pole longitudinal contributions are, of course, very accurately known, so it is the absence of an experimental determination of the $K_0^*(1430)$ and $K(1460)$ decay constants which prevents us from performing a reliable longitudinal subtraction.

Note that because $\rho^{(0)}_{us;V}$ and $\rho^{(0)}_{us;A}$ are positive and the analogous ud resonance contributions are negligible, the longitudinal resonance contribution of Eq. (10) is necessarily positive and a decreasing function of k. In fact, from Eq. (10) it follows that the longitudinal $(k,0)$ resonance contributions must satisfy the inequality

$$[\Delta R^{(k+1,0)}_\tau]^{resonance}_L\leqslant\left(1-\frac{s^{SS}_{th}}{m_\tau^2}\right)[\Delta R^{(k,0)}_\tau]^{resonance}_L=0.873[\Delta R^{(k,0)}_\tau]^{resonance}_L, \tag{11}$$

where the equality would be obtained only if the entire nonpole us spectral strength lay at threshold in the scalar channel. That both the pole and resonance contributions are decreasing functions of k, of course, also means that the *total* longitudinal contribution to $\Delta R^{(k,0)}_\tau$ must be a decreasing function of k.

The fact that one expects only very small contributions from the tails of the $K_0^*(1950)$ and $K(1830)$ resonances, and that both the masses and widths of the $K_0^*(1430)$ and $K(1460)$ are similar allows us to sharpen considerably the constraint represented by Eq. (11). In the narrow width approximation (NWA), dominance by resonance contributions at $M\sim1.4$ GeV would mean that

$$[\Delta R^{(k+1,0)}_\tau]^{resonance}_L\simeq\left(1-\frac{M^2}{m_\tau^2}\right)[\Delta R^{(k,0)}_\tau]^{resonance}_L=0.38[\Delta R^{(k,0)}_\tau]^{resonance}_L. \tag{12}$$

A more refined version of this estimate is obtained by considering $K_0^*(1430)$ and $K(1460)$ Breit-Wigner forms with Particle Data Group 2000 (PDG2000) values for the masses and widths, and integrating directly over the resonance profiles. The results of this exercise are that the individual resonance contributions to $[\Delta R^{(k,0)}_\tau]_L$ for $k=0,1,2$ are in the ratios 1:0.46:0.24 for the $K_0^*(1430)$ and 1:0.43:0.20 for the $K(1460)$. We would thus expect the total resonance contributions to $[\Delta R^{(k,0)}_\tau]_L$ to satisfy

$$[\Delta R^{(1,0)}_\tau]^{resonance}_L\simeq0.44[\Delta R^{(0,0)}_\tau]^{resonance}_L$$
$$[\Delta R^{(2,0)}_\tau]^{resonance}_L\simeq0.22[\Delta R^{(0,0)}_\tau]^{resonance}_L. \tag{13}$$

Let us now consider the OPE representation of $[\Delta R^{(k,0)}_\tau]_L$. In what follows we will, for convenience, quote

TABLE I. Comparison of longitudinal and total OPE contributions to $\Delta R_\tau^{(k,0)}$ as obtained in Ref. [10]. The quoted values of $m_s(m_\tau^2)$ are those obtained in Ref. [10] using the given spectral weight $(k,0)$. This means that the difference between the total longitudinal OPE contribution and the experimental value $[\Delta R_\tau^{(k,0)}]^{exp}$ is the value of the $0+1$ OPE contribution produced by the given $m_s(m_\tau^2)$. The decrease of the longitudinal $D=2$ contributions with k is a reflection of the decrease in the extracted $m_s(m_\tau^2)$ with k; as explained in the text, for a fixed $m_s(m_\tau^2)$, the longitudinal $D=2$ values would be increasing with k. The experimental errors are those of the ALEPH collaboration.

Weight	$m_s(m_\tau^2)$(MeV)	$[\Delta R_\tau^{(k,0)}]_L^{D=2}$	$[\Delta R_\tau^{(k,0)}]_L^{OPE}$	$[\Delta R_\tau^{(k,0)}]^{exp}$
(0,0)	143	0.201 ± 0.085	0.274 ± 0.087	0.394 ± 0.137
(1,0)	121	0.150 ± 0.073	0.223 ± 0.076	0.383 ± 0.078
(2,0)	106	0.126 ± 0.072	0.199 ± 0.074	0.373 ± 0.054

the OPE results obtained in Ref. [10], since that reference provides a complete breakdown of the individual $0+1$ and longitudinal contributions (in addition to a detailed discussion of the evaluation of the various contributions, to which the interested reader is referred for details). The $D=2$ longitudinal contributions are of the form [10]

$$[\Delta R_\tau^{(k,0)}]_L^{D=2}=6\,S_{EW}(1-\epsilon_d^2)\left(\frac{m_s(m_\tau^2)}{m_\tau^2}\right)^2\Delta_{(k,0)}^L\,, \quad (14)$$

where $\epsilon_d=m_d/m_s=.053$, and $\Delta^L_{(k,0)}$, which results from the CIPT integration, depends on $\alpha_s(m_\tau^2)$. The results of Ref. [10] are

$$\Delta^L_{(0,0)}=5.1\pm2.1\pm0.5$$

$$\Delta^L_{(1,0)}=5.3\pm2.5\pm0.7$$

$$\Delta^L_{(2,0)}=5.8\pm3.2\pm0.8, \quad (15)$$

where, in each case, the first error represents the combination of the scale-dependence and truncation errors, discussed above, and the second represents the effect of the experimental uncertainty in $\alpha_s(m_\tau^2)$. The $D=4$ longitudinal contributions are, numerically, completely dominated by the term proportional to $\langle m_s\bar{s}s\rangle$. Since this term arises from the Ward identity,

$$q^4\Pi^{(0)}_{ij;V,A}(q^2)=(m_i\pm m_j)^2\Pi_{ij;S,P}(q^2)+(m_i\pm m_j)(\langle\bar{q}_iq_i\rangle\pm\langle\bar{q}_jq_j\rangle), \quad (16)$$

where $\Pi_{ij:S,P}$ are the correlators of the flavor ij scalar and pseudoscalar densities, and the plus (minus) signs on the right-hand side (RHS) correspond to the A (V) case, the Wilson coefficient of the $\langle m_s\bar{s}s\rangle$ term receives no radiative corrections. The $D=4$ contribution can therefore be evaluated rather accurately, using (i) the quark mass ratios obtained by Leutwyler [33], (ii) the Gell-Mann–Oakes–Renner (GMOR) relation, $\langle(m_u+m_d)\bar{u}u\rangle=-f_\pi^2m_\pi^2$,[1] and (iii) $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle=0.8\pm0.2$ [10]. The value turns out to be the same for the (0,0), (1,0), and (2,0) spectral weights [10]:

$$[\Delta R_\tau^{(k,0)}]_L^{D=4}=0.0726\pm0.0194. \quad (17)$$

The error in Eq. (17) is considerably smaller than that on the $D=2$ contributions. The leading four-quark $D=6$ contributions are absent from $\Pi^{(0)}_{ij;V,A}$ [24], justifying the neglect of $D=6$ contributions, while contributions of $D=8$ and higher are assumed to be negligible. The OPE representation of $[\Delta R_\tau^{(k,0)}]_L$ employed in recent inclusive analyses thus consists of the sum of $D=2$ and $D=4$ terms, with the error dominated by that on the truncated $D=2$ series.

It is now straightforward to see that the OPE representation just described for $[\Delta R_\tau^{(k,0)}]_L$ rather badly violates the constraints obtained above. We first observe that, from Eqs. (14) and (15), $[\Delta R_\tau^{(k,0)}]_L^{D=2}$ is a slowly increasing function of k. Since $[\Delta R_\tau^{(k,0)}]_L^{D=4}$ is constant with k, this means that the OPE representation, $[\Delta R_\tau^{(k,0)}]_L^{OPE}$ is also slowly increasing, rather than decreasing, with k.[2] Since, for values of $m_s(m_\tau^2)$ typical of those obtained in recent sum rule analyses, $m_s(m_\tau^2)\sim120$ MeV, the $D=2$ contributions are more than a factor of 2 larger than the corresponding $D=4$ contribution, this problem is likely to have important numerical consequences. The increase with k of the central values shown in Eq. (15), in fact, means that the extracted values of m_s must *necessarily* display an unphysical *decrease* with k.[3] This unphysical decrease would not, of course, be a practical (as opposed to conceptual) difficulty for the inclusive analysis if

[1]Deviations from the GMOR relation have recently been shown to be at most 6% [34]. The resulting error on the m_s analysis is completely negligible.

[2]The small $O(m_s^4)$ contributions to $[\Delta R_\tau^{(k,0)}]_L^{D=4}$, which have been neglected above, are actually also increasing with k, so the full longitudinal $D=4$ contribution is actually also (very slowly) increasing with k.

[3]It is worth noting that Ref. [14] employs a k-dependent truncation scheme, in contrast to the other inclusive analyses, which truncate at the same order for all k. The problem of the unphysical k dependence of the central values outlined above, however, remains present even for this altered scheme.

TABLE II. Longitudinal $\pi+K$ pole contributions to $[\Delta R_\tau^{(k,0)}]_L$, together with the resonance contributions implicit in the longitudinal OPE representations of Ref. [10]. The column labeled "IND" gives the latter results with each $[\Delta R_\tau^{(k,0)}]_L^{OPE}$ evaluated using the central value of the corresponding independent fit for $m_s(m_\tau^2)$. The column labeled "COMB" gives the same results, except that now the $[\Delta R_\tau^{(k,0)}]_L^{OPE}$ are all evaluated using the common value $m_s(m_\tau^2)=119$ MeV obtained in the combined analysis using the (0,0), (1,0), and (2,0) spectral weights.

Weight	$[\Delta R_\tau^{(k,0)}]_L^{K+\pi}$	$[\Delta R_\tau^{(k,0)}]_L^{resonance}$ (IND)	$[\Delta R_\tau^{(k,0)}]_L^{resonance}$ (COMB)
(0,0)	0.1204	0.154 ± 0.087	0.092 ± 0.062
(1,0)	0.1105	0.112 ± 0.076	0.107 ± 0.074
(2,0)	0.1014	0.097 ± 0.074	0.130 ± 0.092

$[\Delta R_\tau^{(k,0)}]_L^{OPE}$ represented only a small fraction of ΔR_τ. Unfortunately, this is not the case. To illustrate this point we show in Table I, for each of the three $(k,0)$ spectral weights, the numerical values of the longitudinal $D=2$ contribution and total longitudinal contribution obtained in Ref. [10], together with the 1999 ALEPH experimental values of $\Delta R_\tau^{(k,0)}$ [21]. The reader should be aware that significantly different central values of $m_s(m_\tau^2)$ were obtained for the three different spectral weights; the tabulated $D=2$ contributions correspond to these central values (also listed in the table). The quoted $D=2$ errors were obtained by combining the two theoretical errors in Eq. (15) in quadrature. We see from the table that the longitudinal contribution, in each case, represents more than half of the experimental value.

Let us attempt to quantify how large the errors associated with the unphysical k dependence of $[\Delta R_\tau^{(k,0)}]_L^{OPE}$ might be. Note that, given the accurately known values of the longitudinal π and K pole contributions, the OPE representation of $[\Delta R_\tau^{(k,0)}]_L$ should be thought of as providing an estimate for the sum of the unknown longitudinal resonance contributions. In Table II we display the values of $[\Delta R_\tau^{(k,0)}]_L^{resonance}$ implicit in the OPE results of Ref. [10], together with the values of the corresponding pole contributions, $[\Delta R_\tau^{(k,0)}]_L^{K+\pi}$. Two versions of the resonance contributions are given. In the first (labeled "IND" in the table), the values of $m_s(m_\tau^2)$ used to evaluate the three different $[\Delta R_\tau^{(k,0)}]_L^{OPE}$ are different, corresponding to the central values obtained in the relevant independent $(k,0)$ analysis of Ref. [10]. In the second (labeled "COMB" in the table), the common value, $m_s(m_\tau^2)=119$ MeV, obtained in the combined fit to all three $(k,0)$ sum rules [10], is used to compute all three of the $[\Delta R_\tau^{(k,0)}]_L^{OPE}$. The unphysical increase of $[\Delta R_\tau^{(k,0)}]_L^{OPE}$ with k, combined with the decrease of the sum of the π and K pole contributions with k, means that the nominal resonance contribution implicit in $[\Delta R_\tau^{(k,0)}]_L^{OPE}$ must be increasing with k for fixed $m_s(m_\tau^2)$. This is evident in the results of the combined analysis, but obscured by the decrease of $m_s(m_\tau^2)$ with k for the independent analysis. It is evident that both sets of results are far from satisfying the constraints given in Eqs. (13).

We are now in a position to illustrate the potential significance of the unphysical k dependence of $[\Delta R_\tau^{(k,0)}]_L^{OPE}$ on the extracted values of m_s. Let us imagine that the central OPE value provides a good approximation for one of the three $[\Delta R_\tau^{(k,0)}]_L^{resonance}$, and we use Eq. (13) to estimate the resonance contributions to the other two $(k,0)$ sum rules. We find that if we attempt to make this assumption for either $k=1$ or $k=2$ the result is a $k=0$ longitudinal contribution which exceeds the full experimental value, violating the positivity of the $0+1$ OPE representation. If we instead make the assumption for $k=0$, the resulting change in the (1,0) and (2,0) longitudinal contributions produces a shift in the extracted central $m_s(m_\tau^2)$ values from $121\to142$ MeV and $106\to133$ MeV, respectively. We stress that this exercise is for illustrative purposes only; although the consistency of the three analyses is significantly improved if one assumes that, for some reason, the $k=0$ representation is good, the resulting "extraction" of m_s is meaningless since the assumption simultaneously forces the $k=1,2$ representations to be bad, leading one to the conclusion that it is unreasonable to have assumed that the $k=0$ representation was good in the first place.

At present there is little experimental information available on the size of the longitudinal resonance contributions. The PDG2000 compilation provides no information on $\tau\to K(1460)\nu_\tau$, and quotes only an upper bound, $B<.0005$ on the $\tau\to K_0^*(1430)\nu_\tau$ branching fraction. The latter bound corresponds to an upper bound $[\Delta R_\tau^{(0,0)}]_L^{K_0^*(1430)}<0.052$. The central longitudinal (0,0) OPE determination, if reliable, would then require a corresponding $K(1460)$ contribution *greater* than ~0.10. Taking a Breit-Wigner $K(1460)$ form with PDG2000 values for the mass and width, this corresponds to $f_{K(1460)}>100$ MeV. Such a large value is extremely unnatural given that $f_{K(1460)}/f_K\to0$ in the $SU(3)_F$ chiral limit. Not surprisingly, therefore, the lower bound $[\Delta R_\tau^{(0,0)}]_L^{K(1460)}>0.10$ turns out to be more than an order of magnitude larger than the value obtained from the sum rule analysis of the next section. As such, it is completely incompatible with the sum rules for the us pseudoscalar correlator.

III. THE EXCITED STRANGE SCALAR AND PSEUDOSCALAR RESONANCE DECAY CONSTANTS FROM SCALAR AND PSEUDOSCALAR SUM RULES

From Eqs. (16) and the Ward identities for the divergences of the flavor ij V and A currents, it follows that

$$q^4\rho_{ij;V}^{(0)}(q^2)=(m_i-m_j)^2\rho_{ij;S}(q^2)\equiv\rho_{ij;\partial V}(q^2)$$
$$q^4\rho_{ij;A}^{(0)}(q^2)=(m_i+m_j)^2\rho_{ij;P}(q^2)\equiv\rho_{ij;\partial A}(q^2), \tag{18}$$

where $\rho_{ij;S,P}(q^2)$ is the spectral function of $\Pi_{ij;S,P}(q^2)$. The contribution of the $K(1460)$ to $\rho^{(0)}_{us;A}$ on the left-hand side (LHS) is, in the NWA, $2f^2_{K(1460)}\delta(q^2-m^2_{K(1460)})$, with the usual Breit-Wigner generalization to finite width. The decay constant $f_{K(1460)}$ is defined as usual by

$$\langle 0|A^{us}_\mu|K(1460)(q)\rangle = \mathrm{i}\sqrt{2}f_{K(1460)}q_\mu . \tag{19}$$

The analogous NWA contribution of the $K(1460)$ to the RHS of Eq. (18) is $2f^2_{K(1460)}m^4_{K(1460)}\delta(q^2-m^2_{K(1460)})$. A similar relation holds between the $K^*_0(1430)$ contributions to $\rho^{(0)}_{us;V}$ and $(m_s-m_u)^2\rho_{us;S}$. It is thus possible, in principle, to determine the longitudinal resonance contributions of the preceding section indirectly, by fixing $f_{K(1460)}$ and $f_{K^*_0(1430)}$ through analyses of the us pseudoscalar and scalar correlators, respectively. In this section we will show that such a determination is, indeed, feasible.

The method involves the analysis of the correlators of the divergences of the us V and A currents, $\Pi_{us;\partial V}\equiv(m_s-m_u)^2\Pi_{us;S}$ and $\Pi_{us;\partial A}\equiv(m_s+m_u)^2\Pi_{us;P}$ using finite energy sum rules (FESR) of a type (''pinch-weighted'') known to allow an accurate reconstruction of the isovector vector spectral function using as input only the OPE, together with PDG values for the resonance masses and widths [35,36]. We will refer to these correlators as the strange scalar (SS) and strange pseudoscalar (SPS) correlators in what follows.

The basic idea of the analysis is straightforward. Analyticity leads to the general FESR relation,

$$\int_{s_{th}^{SS,SPS}}^{s_0} ds\, \rho_{us;\partial V,\partial A}(s)w(s) = \frac{-1}{2\pi i}\oint_{|s|=s_0} ds\,\Pi_{us;\partial V,\partial A}(s)w(s), \tag{20}$$

valid for any function $w(s)$ analytic in the region of the contour. The LHS is determined by the decay constants of the relevant scalar or pseudoscalar resonances, while the RHS, for large enough s_0 can be evaluated using the OPE. For the case of the analogous isovector vector correlator, it has been shown that, although the breakdown of the OPE near the timelike real axis for $s_0\sim m_\tau^2$ is not negligible [so that, for example, FESR's with $w(s)=s^k$ are *not* well satisfied for $s_0\sim m_\tau^2$ [35]], even a single zero in $w(s)$ at $s=s_0$ is enough to produce FESR's that are very well satisfied when the OPE representation for $\Pi_{ud;V}$ is employed in the analogue of the RHS of Eq. (20) [35]. A physical understanding of the origin of this behavior is provided by the arguments of Ref. [37]. As shown in Refs. [35,36], working simultaneously with FESR's based on the weight families

$$w_N(y,A)\equiv(1-y)(1+Ay)$$

$$w_D(y,A)\equiv(1-y)^2(1+Ay), \tag{21}$$

where now $y=s/s_0$, and A is a free parameter, gives enough variability in the weight profile to strongly constrain the resonance decay constants in a given channel.[4] We will denote the class of pinch-weighted FESR's as PFESR's in what follows.

The OPE representations of the SS and SPS correlators are known up to dimension $D=6$ [2,39], the $D=0$ part being determined to four-loop order $[O(\alpha_s^3)]$ [28]. It is convenient to work with the second derivative of Π with respect to Q^2, which satisfies a homogeneous renormalization group (RG) equation, allowing logarithms to be summed by the scale choice $\mu^2=Q^2$. One has, for the resulting OPE representations [2,39],

$$[\Pi''_{us;SPS/SS}(Q^2)]_{D=0}=\frac{3}{8\pi^2}\,\frac{(m_s\pm m_u)^2}{Q^2}\Big[1+\frac{11}{3}a(Q^2) + 14.1793a(Q^2)^2+77.3683a(Q^2)^3\Big]$$

$$[\Pi''_{us;SPS/SS}(Q^2)]_{D=2}=\frac{3}{4\pi^2}\,\frac{(m_s\pm m_u)^2m_s^2}{Q^4}\Big[1+\frac{28}{3}a(Q^2) + \Big(\frac{8557}{72}-\frac{77}{3}\zeta(3)\Big)a(Q^2)^2\Big]$$

[4]This is true even for correlators for which the spectral function contains significant background contributions near threshold. As an example, consider the us scalar channel. In Ref. [38], an ansatz for the corresponding spectral function has been constructed, employing the Omnes representation for the timelike scalar $K\pi$ form factor in combination with certain additional assumptions. The resulting spectral function displays a very significant background contribution near threshold (associated with the strongly attractive s-wave $I=1/2$ $K\pi$ interaction) which cannot be well represented by the tail of the $K^*_0(1430)$ resonance. If one takes as input on the OPE side of the w_N and w_D FESR's the value of m_s obtained from a FESR analysis using this spectral ansatz as input, and then, with the OPE representation so fixed, makes an incoherent-sum-of-Breit-Wigner-resonance ansatz for the spectral function, and uses matching to the OPE sides of the set of w_N and w_D FESR's above to fix the resonance decay constants, one finds that the $K_0(1430)$ peak of the spectral function is reproduced to within $\sim 2\%$, despite the fact that the near-threshold region is, of course, not well reproduced. The reason is obvious: the spectral function is small in the near-threshold region and integrals over the spectral function are sensitive dominantly to the regions where it is large, i.e., to the regions of the resonance peaks.

$$[\Pi''_{us;SPS/SS}(Q^2)]_{D=4}=\frac{(m_s\pm m_u)^2}{Q^6}\Bigg[\mp 2\left(1+\frac{23}{3}a(Q^2)\right)\times\langle m_s\overline{u}u\rangle-\frac{1}{9}\left(1+\frac{121}{18}a(Q^2)\right)I_G+\left(1+\frac{64}{9}a(Q^2)\right)I_s-\frac{3m_s(Q^2)^4}{7\pi^2}\left(\frac{1}{a(Q^2)}+\frac{155}{24}\right)\Bigg]$$

$$[\Pi''_{us;SPS/SS}(Q^2)]_{D=6}=\frac{(m_s\pm m_u)^2}{Q^8}\Bigg(\pm 3[\langle m_i g\overline{q_j}\sigma\cdot Gq_j+m_j g\overline{q_i}\sigma\cdot Gq_i\rangle]-\frac{32}{9}\pi^2 a\rho_{VSA}[\langle\overline{q_i}q_i\rangle^2+\langle\overline{q_j}q_j\rangle^2-9\langle\overline{q_i}q_i\rangle\langle\overline{q_j}q_j\rangle]\Bigg), \tag{22}$$

where I_G and I_s are the RG invariant versions of the gluon and strange quark condensate, as defined in Ref. [29], ρ_{VSA} describes the deviation of the four-quark condensates from their vacuum saturation values, and the upper (lower) sign corresponds throughout to the SPS (SS) case.

One should bear in mind that, on the theoretical side of SS (SPS) sum rules, the contribution of direct instantons to the SS (SPS) correlator is not contained in the OPE representation. Such effects are known to play a potentially important role in scalar and pseudoscalar channels [40–43], particularly at lower scales ~ 1 GeV.[5] As a result, one must include an estimate of direct instanton contributions, in addition to OPE contributions, in the theoretical representation of the SS (SPS) correlator. A convenient, and phenomenologically constrained, model for making such an estimate is the instanton liquid model (ILM) [44].[6] ILM contributions to the theoretical side of polynomial-weighted SPS PFESR's can be obtained from the result [45]

$$\frac{-1}{2\pi i}\oint_{|s|=s_0}ds\, s^k[\Pi_{us;P}(s)]_{ILM}=\frac{-3[m_s+m_u]^2\eta_{us}}{4\pi}\int_0^{s_0}ds\, s^{k+1}J_1(\rho_I\sqrt{s})Y_1(\rho_I\sqrt{s}), \tag{23}$$

where $\rho_I\simeq(1/0.6\ \ \mathrm{GeV})$ is the average instanton size (a parameter of the ILM), η_{us} is an $SU(3)$-breaking factor whose value in the ILM is ~ 0.6 [42], and the result is relevant to scales $\sim 1\ \ \mathrm{GeV}^2$. The corresponding result for the SS channel is obtained by the replacement $(m_s+m_u)^2\to-(m_s-m_u)^2$.

It is important to remember that, for a given scale, the ILM contribution to a typical scalar or pseudoscalar FESR is much larger than that to the corresponding Borel sum rule (BSR).[7] At the scales we will be employing, ILM contributions to the SPS and SS BSR's are, in fact, quite small, while those to our PFESR's are still non-negligible. Consistency between PFESR and BSR analyses thus represents a non-trivial constraint on the reliability of the ILM representation of instanton contributions [46]. In Ref. [46], this consistency check was implemented as follows. First, the families of PFESR's noted above were used to make a simultaneous determination of m_i+m_j and the resonance decay constants relevant to the flavor ij pseudoscalar channel, the values obtained for m_i+m_j and the resonance decay constants being sensitive to whether or not the OPE was supplemented with ILM contributions. The resulting PFESR-generated values for the decay constants were then used as input to a BSR analysis of the same correlator, an alternate determination of m_i+m_j being the output of this analysis. The PFESR and BSR determinations of m_i+m_j should then be consistent. The only non-trivial sensitivity to ILM contributions in the BSR analysis is that associated with the input PFESR values for the resonance decay constants. Consistency of the two determinations was found only when ILM contributions were included on the theoretical sides of the PFESR's [46].

In what follows, therefore, we will employ the ILM to estimate direct instanton effects, and determine the strange scalar and pseudoscalar resonance decay constants in a combined w_N, w_D PFESR analysis. Compatibility of the PFESR and BSR quark mass determinations will be imposed as an additional consistency requirement.[8] The Borel transform of the ILM contribution to the SPS correlator required for this consistency check is given by

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[5]The Borel transform of the OPE representation of the correlator of the flavor ij pseudoscalar density, for example, displays the wrong dependence on the Borel mass, M, in the chiral limit: while the $exp(-s/M^2)$-weighted spectral integral must become independent of M in this limit, the Borel transformed OPE representation displays a strong dependence on M.

[6]The incorrect M dependence of the theoretical side of the Borel transformed pseudoscalar correlator sum rule is cured once the OPE representation is supplemented with ILM contributions [41,42].

[7]The reason is straightforward: ILM contributions to the scalar and pseudoscalar correlators, $\Pi_{ij;S,P}(Q^2)$, are proportional to $Q^2[K_{-1}(\rho_I\sqrt{Q^2})]^2$. The modulus of the MacDonald function $K_{-1}(\rho_I\sqrt{Q^2})$, on the circle $Q^2=|Q^2|e^{i\theta}$, is typically much larger for non-zero θ than it is for the spacelike point, $\theta=0$. The integral around the circle $|s|=s_0$ present on the theoretical side of a FESR thus samples regions of the complex Q^2 plane where the ILM contributions are enhanced.

[8]Errors associated with uncertainties in the input resonance masses and widths and the input values of parameters appearing on the theoretical sides of the sum rules occur for both the PFESR and BSR analyses and are strongly correlated. The BSR analysis has additional errors associated with the use of the ''continuum'' approximation for the high-s part of the spectral integral and the uncertainty in the choice of the ''continuum threshold'' parameter. For the SPS case, these were estimated to produce an uncertainty of $\sim 9\%$ in m_s+m_u [46]. We have employed this same estimate for the additional BSR uncertainty in our combined PFESR/BSR SS channel analysis.

$$\frac{3\rho_I^2(m_s+m_u)^2M^6}{8\pi^2}[K_0(\rho_I^2M^2/2)+K_1(\rho_I^2M^2/2)]. \quad (24)$$

That for the SS correlator is obtained by the replacement $(m_s+m_u)^2\to-(m_s-m_u)^2$. Expressions for the Borel transform of the OPE representations are well known, and can be found in Refs. [2,29,47].

We use the following input values on the OPE+ILM side of our sum rules: $\rho_I=1/(0.6\ \text{GeV})$ [42,44], $\alpha_s(m_\tau^2)=0.334\pm.022$ [20,48], $\langle\alpha_s G^2\rangle=(0.07\pm0.01)\ \text{GeV}^4$ [49], $(m_u+m_d)\langle\bar{u}u\rangle=-f_\pi^2m_\pi^2$ (the GMOR relation), $0.7<\langle\bar{s}s\rangle/\langle\bar{u}u\rangle\equiv r_c<1$ [2,29]; $\langle g\bar{q}\sigma Fq\rangle=(0.8\pm0.2\ \text{GeV}^2)\langle\bar{q}q\rangle$ [50] and $0<\rho_{VSA}<10$. The $D=0$, 2 and 4 OPE integrals are evaluated using the contour-improvement prescription [26,31], since this is known to improve convergence and reduce residual scale dependence [26]. The running coupling and running mass required in this procedure are obtained using the 4-loop-truncated versions of the β [51] and γ [52] functions, with the value of $\alpha_s(m_\tau^2)$ given above as input.

The analysis of the SPS channel has already been performed in Ref. [46], to which the interested reader is referred for a detailed discussion. The results of that analysis are

$$m_s(2\ \text{GeV})=100\pm12\ \text{MeV} \quad (25)$$

$$f_{K(1460)}=21.4\pm2.8\ \text{MeV} \quad (26)$$

$$0<f_{K(1830)}<8.9\ \text{MeV}, \quad (27)$$

where the errors have been obtained by combining the "theory" and "method" errors of Ref. [46] in quadrature.[9] The lack of a strong constraint on $f_{K(1830)}$ is a result of the smallness of the $K(1830)$ contribution to the various PFESR spectral integrals. Since only the tail of the $K(1830)$ contributes to hadronic τ decay, and the endpoint region is strongly suppressed by the kinematic weight factor, this uncertainty plays a negligible role for our purposes. Two further points should be stressed: first, the value of m_s obtained from the SPS PFESR/BSR analysis is consistent with that obtained from recent analyses based on hadronic τ decay data and, second, even if one completely neglects ILM contributions (ignoring the resulting inconsistency between PFESR and BSR mass determinations), one obtains a value $f_{K(1460)}=22.9\pm2.7$ MeV compatible with that given in Eq. (26) within errors.[10] For the purposes of determining the strange pseudoscalar longitudinal contributions in hadronic τ decay, the result of Eq. (26) thus appears very robust.

A simultaneous PFESR determination of m_s+m_u and the excited resonance decay constants in the SPS channel is possible only because one part of the spectral function (the K pole contribution) is well determined experimentally. Unfortunately the experimental spectral constraints are considerably weaker in the SS channel.

The SS spectral function should be dominated by contributions from $K\pi$ intermediate states up to and including the $K_0^*(1430)$ region, since the $K_0^*(1430)$ displays essentially no inelasticity. Unitarity and the Omnes representation of the timelike scalar $K\pi$ form factor allow one to represent the $K\pi$ component of the spectral function in terms of K_{l3} data and $K\pi$ phases [2,38]. There are, however, ambiguities in this representation. In the literature, it has been assumed that a possible polynomial prefactor is absent from the Omnes representation and, in addition, that the corresponding asymptotic behavior of the $K\pi$ phase required by quark counting rules has already been reached at the upper edge of the currently accessible experimental range, $s\simeq2.9\ \text{GeV}^2$. The spectral ansatz which results from these assumptions [38] serves as the basis for a number of recent sum rule of analyses of the SS channel [1] and corresponds (reading from Fig. 2 of Ref. [38]) to the constraint

$$26.2\ \text{MeV}<f_{K_0^*(1430)}<31.0\ \text{MeV}. \quad (28)$$

The corresponding value for m_s is, averaging the errors quoted in Refs. [1(d)] and [2(b)],

$$m_s(2\ \text{GeV})=115\pm15\ \text{MeV}. \quad (29)$$

It turns out, however, that quite sizable deviations from the asymptotic value of the phase are allowed in the region of the $K_0^*(1950)$ without violating the known ChPT constraint on the slope of the form factor at $s=0$. These can, in turn, produce non-trivial deviations of the spectral function from that obtained in Ref. [38], even in the region of the $K_0^*(1430)$. There are thus potentially significant uncertainties not yet reflected in the range of values for $f_{K_0^*(1430)}$ given in Eq. (28).

Without fully constrained experimental values for the SS spectral function, the PFESR and/or BSR analyses allow us to determine $f_{K_0^*(1430)}$ and $f_{K_0^*(1950)}$ only after an input value for $(m_s-m_u)^2$ has been provided on the OPE+ILM side of the sum rules. The reason is that, at the scales employed in our analyses, those terms in the OPE proportional to m_s^4 are numerically tiny, so the OPE representation is, to a very good approximation, proportional to $(m_s-m_u)^2$. Thus, once one finds an optimized spectral ansatz for a particular value of m_s-m_u, say $m_s-m_u\equiv m_0$, an equally-well-optimized ansatz for any other value, $m_s-m_u\equiv m_1$, can be obtained simply by rescaling the fitted decay constants by m_1/m_0. The PFESR analysis thus allows only a determination of the ratios $f_i/(m_s-m_u)$.

In our SS PFESR analysis, our spectral ansatz consists of an incoherent sum of $K_0^*(1430)$ and $K_0^*(1950)$ Breit-Wigner

[9]The method errors refer to changes in the output produced by varying the s_0 and A ranges used in the PFESR analysis, or by performing w_N or w_D family analyses separately, rather than a combined analysis. A breakdown of the contributions to the combined error may be found in Ref. [46].

[10]The biggest impact of neglecting ILM contributions is on $f_{K(1830)}$, which becomes 14.5 ± 1.4 MeV. The PFESR value of $m_s(2\ \text{GeV})$ is also altered, the central value becoming 116 MeV, but it is difficult to assign meaningful errors to this number since the PFESR and BSR determinations are not consistent in this case.

resonance forms, with PDG2000 values of the resonance masses and widths. We employ the same PFESR analysis window as used in our earlier study of the SPS channel, namely $3.0\ \mathrm{GeV}^2 \leqslant s_0 \leqslant 4.0\ \mathrm{GeV}^2$ and $0 \leqslant A \leqslant 4$. The different A values correspond to weights with significantly different relative weightings between the first and second resonance regions, and hence are useful in tightening constraints on the resonance decay constants. As noted above, as a self-consistency check on the ILM representation of direct instanton effects, we also require consistency between the value of $m_s - m_u$ used as input to the PFESR analysis and that obtained as output from the corresponding BSR analysis, in which PFESR values of the resonance decay constants are used. The results of this determination are

$$f_{K_0^*(1430)} = [22.5 \pm 2.1]\left(\frac{m_s(2\ \mathrm{GeV})}{100\ \mathrm{MeV}}\right) \tag{30}$$

$$f_{K_0^*(1950)} = [17.6 \pm 2.0]\left(\frac{m_s(2\ \mathrm{GeV})}{100\ \mathrm{MeV}}\right), \tag{31}$$

where we have combined all sources of error in quadrature. If we consider the range of m_s values given in Eq. (29), the corresponding range of $f_{K_0^*(1430)}$ allowed by Eq. (30) is

$$20.4\ \mathrm{MeV} < f_{K_0^*(1430)} < 32.0\ \mathrm{MeV}, \tag{32}$$

which turns out to be in good agreement with that given by Eq. (28).

With the values above for the decay constants of the SS and SPS resonances it is straightforward to compute the expected $\tau \to K_0^*(1430)\nu_\tau$ branching fraction, and also the values of the resonance contributions to $[\Delta R_\tau^{(0,0)}]_L$. Taking $83\ \mathrm{MeV} < m_s(2\ \mathrm{GeV}) < 130\ \mathrm{MeV}$ [19], the result of Eq. (30) corresponds to

$$0.00003 < B[\tau \to K_0^*(1430)\nu_\tau] < 0.00011, \tag{33}$$

and hence satisfies the constraint given by the PDG2000 upper bound. As expected on kinematic grounds, the $K(1830)$ and $K_0^*(1950)$ contributions to $[\Delta R_\tau^{(0,0)}]_L$ are negligible.[11] The corresponding $K(1460)$ and $K_0^*(1430)$ contributions, which follow from Eqs. (26) and (30), are[12]

$$[\Delta R_\tau^{(0,0)}]_L^{K(1460)} = 0.0052 \pm 0.0014 \tag{34}$$

and

$$[\Delta R_\tau^{(0,0)}]_L^{K_0^*(1430)} = [0.0059 \pm 0.0011]\left[\frac{m_s(2\ \mathrm{GeV})}{100\ \mathrm{MeV}}\right]^2. \tag{35}$$

The sum of the SS and SPS resonance contributions is thus $\sim 10\%$ of the $K+\pi$ pole contribution. This level of suppression of resonance relative to pole contributions is in fact quite natural, and represents a combination of chiral and kinematic effects. With $y_K \equiv m_K^2/m_\tau^s$ and $y_{res} \simeq (1.4\ \mathrm{GeV}^2)/m_\tau^2$, one has $w_L^{(0)}(y_K) = .13$ and $w_L^{(0)}(y_{res}) = .16$. Thus, although $w_L^{(0)}(y_K)$, which is proportional to m_K^2, and hence of $O(m_s)$ in the chiral counting, is formally suppressed by one power of m_s relative to $w_L^{(0)}(y_{res})$, the factor $(1-y)^2$ in $w_L^{(0)}(y)$ produces a kinematic suppression of $w_L^{(0)}(y_{res})$ which largely undoes this effect. As a result, one expects resonance contributions to $[\Delta R_\tau^{(0,0)}]_L$ to be suppressed, relative to the K contribution, by the ratio $[f_{K_0^*(1430),K(1460)}/f_K]^2$, which has an m_s^2 chiral suppression, ~ 0.1. Naive chiral counting would have produced instead a less strong suppression, of order m_s, ~ 0.3.

As noted above, the result of Eq. (34) is more than an order of magnitude smaller than the lower bound implied by the combination of the PDG2000 upper bound on the $K_0^*(1430)$ branching fraction and the assumption that the OPE representation of the longitudinal (0,0) spectral weight contribution is reliable. To satisfy the lower bound, one would require a value of $f_{K(1460)}$ a factor of $\sim \sqrt{20}$ larger than that given above. Such a large value, however, leads to an exceptionally poor "optimized" PFESR OPE/spectral match. The value of m_s corresponding to this "optimized" match, moreover, produces a $(0+1)$ OPE contribution that already exceeds the experimental value for $\Delta R_\tau^{(0,0)}$, and hence violates the positivity of the longitudinal SS and SPS contributions.

IV. SUMMARY AND DISCUSSION

We have shown that determinations of m_s based on inclusive $(k,0)$ spectral weight analyses of flavor breaking in hadronic τ decay have an unavoidable, unphysical dependence on k, and that the impact of this unphysical behavior on the extracted values for m_s is numerically significant. The problem has been shown to result from the unphysical behavior with respect to k of the OPE representation of the longitudinal contributions to the ud and us correlators, when truncated at $D=6$. If truncation at $D=6$ is justified, then the problem lies with the $D=2$ part of the OPE representation (whose integrated contour-improved series, in any case, already displays rather bad non-converging behavior). Part of the problem, however, may lie in the neglect of higher dimension contributions, particularly since contributions unsuppressed by additional factors of α_s, and having dimensions up to $D=8$, 10 and 12, are in principle present for the (0,0), (1,0) and (2,0) analyses, respectively. Since nothing is known phenomenologically about the values of $D=8$ and

[11] The $K_0^*(1950)$ contribution is a factor of ~ 20 smaller than the $K_0^*(1430)$ contribution, the $K(1830)$ contribution a factor of ~ 60 smaller than the $K(1460)$ contribution.

[12] For definiteness, we have computed these contributions using the value $|V_{us}| = 0.2196 \pm 0.0023$ obtained from the analysis of K_{e3} decay data. The results, of course, scale as $1/|V_{us}|^2$.

higher condensates, the only way to investigate this question is to consider spectral weight (or other PFESR) analyses with $s_0 \neq m_\tau^2$, and try to use the s_0 dependence to separate contributions of different dimension.[13] To work with $s_0 \neq m_\tau^2$, however, one must necessarily perform a non-inclusive analysis, since different kinematic factors, both of which are specific to $s_0 = m_\tau^2$, are associated with the $0+1$ and longitudinal contributions to the experimental number distribution. In the region of the spectrum where the separation into $0+1$ and longitudinal contributions is not straightforward (the excited SS and SPS resonance region) we have provided determinations of the SS and SPS resonance decay constants accurate to $\sim 10\%$. This allows one to evaluate the resonance part of the longitudinal contribution with an accuracy of $\sim 20\%$. Even for the (0,0) analysis, such an uncertainty corresponds to only a $\sim 2\%$ uncertainty on the total longitudinal subtraction; for weights that more strongly suppress the excited resonance region, the corresponding uncertainty is even smaller.[14]

We conclude with a comment on the implications of our results for future τ decay determinations of m_s. It is our opinion that the bad behavior of the OPE representation of the longitudinal contributions precludes the reliable use of an inclusive analysis, and forces us to make a theoretical evaluation of the longitudinal resonance contributions to the spectrum. As a result, the previous dis-incentive to studying the s_0 dependence of any particular PFESR (the non-inclusivity of such an analysis) is no longer in play. Since the $D = 2,4,6$ terms in the OPE representation of $\Delta\Pi^{(0+1)}$ are well behaved, a study of the s_0 dependence then becomes crucial either to demonstrating explicitly that higher dimension contributions can indeed be safely neglected or to constraining their magnitude if they cannot. Because of the very strong correlations between spectral integrals corresponding to different s_0, but fixed weight, $w(s/s_0)$, truncated OPE representations which either miss, or pass obliquely through the experimental error band for the s_0-dependent spectral integral results, will both signal the presence of such neglected higher D terms. Our determinations of the SS and SPS resonance decay constants make it possible for the s_0 dependence of the $0+1$ sum rules to be studied in a straightforward manner, and we believe that such a study should be part of all future investigations.

[13] Contributions of dimension $D = 2N$ scale with s_0 as $1/s_0^{N-1}$ up to logarithms.

[14] Although the large amount of new τ data that will be generated by the B factory experiments will eventually dramatically change the experimental situation, at present experimental errors on the $V+A$ us number distribution are quite large ($\sim 20-30\,\%$) beyond the K^* region. As a result, reduced errors on m_s, at least at present, require the use of weights that fall off with s in this region more strongly than does the (0,0) transverse weight. The uncertainties on our determinations of the decay constants thus play a negligible role in current analyses.

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