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Index of a Family of Lattice Dirac Operators and Its Relation to the Non-Abelian Anomaly on the Lattice

David H. Adams*

Mathematics Department and Centre for the Subatomic Structure of Matter, University of Adelaide, Adelaide, S.A. 5005, Australia
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In the continuum, a topological obstruction to the vanishing of the non-Abelian anomaly in $2n$ dimensions is given by the index of a certain Dirac operator in $2n + 2$ dimensions, or equivalently, the index of a 2-parameter family of Dirac operators in $2n$ dimensions. In this paper an analogous result is derived for chiral fermions on the lattice in the overlap formulation. This involves deriving an index theorem for a family of lattice Dirac operators satisfying the Ginsparg-Wilson relation. The index density is proportional to Lüscher's topological field in $2n + 2$ dimensions.

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The Atiyah-Singer index theorem, both for single operators [1] and families of operators [2], has been of major importance in the modern development of quantum field theory (QFT). For example, the index theorem for single operators gives topological insight into the nonvanishing of the axial anomaly [3], provides the basis for a resolution of the U(1) problem [4], and determines the dimension of the instanton moduli space (used in semiclassical investigations of Yang-Mills theory; see, e.g., [5]), while the families index theorem reveals topological obstructions to the vanishing of gauge anomalies, thereby providing constraints on allowable theories [6–8].

When attempting to get a well-defined nonperturbative formulation of QFT's (in particular, gauge theories) one successful and well-established approach has been to formulate the theory on a spacetime lattice [9]. Therefore, carrying over the index theorem, both for single operators and families, to the lattice is an interesting and important problem. For a long time this did not seem possible though, due to the fermion doubling problem and the resulting need for acceptable lattice Dirac operators to break chiral symmetry [10]. In the traditional formulations, at best only a remnant of the index theorem is retained on the lattice [11]. However, the situation has changed quite dramatically in recent years with the advent of the overlap formulation [12] and the discovery [13–15] of acceptable lattice Dirac operators satisfying the Ginsparg-Wilson (GW) relation [16]

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D \quad (a = \text{lattice spacing}). \quad (1)$$

Such operators have exactly chiral zero modes [since $D\psi = 0 \Rightarrow D(\gamma_5\psi) = (aD\gamma_5 D - \gamma_5 D)\psi = 0$], which allows one to define $\text{index } D \equiv \text{Tr}(\gamma_5|_{\text{ker}D})$ [15]. There is a “lattice index theorem” [15,17,18]

$$\text{index } D = -\frac{a}{2} \text{Tr}(\gamma_5 D) = a^4 \sum_x q(x), \quad (2)$$

where

$$q(x) = -\frac{a}{2} \text{tr}[\gamma_5 D(x, x)] \quad (3)$$

is the index density. For $SU(N)$ gauge fields on the Euclidean $2n$ -dimensional torus, $\text{index } D$ and $q(x)$ reduce to the continuum index and density in the classical continuum limit [19], at least when D is the overlap Dirac operator [13]. [Earlier results in this direction were obtained in [15,18,20]. When D is the overlap Dirac operator the right-hand side of (2) has a spectral flow interpretation which had previously been used as a definition of lattice topological charge in [12].] Furthermore, although it is not invariant under the usual chiral transformations, the fermion action $S = a^4 \sum_x \bar{\psi}(x) D \psi(x)$ exhibits an exact lattice-deformed version of chiral symmetry [17] (which was implicit in the overlap formalism): $\delta S = 0$ for $\delta\psi = \hat{\gamma}_5\psi$, $\delta\bar{\psi} = \bar{\psi}\gamma_5$ where

$$\hat{\gamma}_5 = \gamma_5(1 - aD). \quad (4)$$

An easy consequence of (1) is $\hat{\gamma}_5^2 = 1$. Furthermore, after supplementing (1) with the γ_5 -Hermiticity condition

$$D^* = \gamma_5 D \gamma_5 \quad (5)$$

we have $\hat{\gamma}_5^* = \hat{\gamma}_5$. Thus $\hat{\gamma}_5$ can be viewed as a lattice-deformed chirality matrix. The axial anomaly for the lattice-deformed chiral symmetry transformation above can be determined from the corresponding change in the fermion measure to be $\mathcal{A}(x) = -ia \text{tr}[\gamma_5 D(x, x)] = 2iq(x)$ [17]. This is completely analogous to the relation between axial anomaly and index density in the continuum [3].

Having seen that there is an exact lattice index theorem for lattice Dirac operators satisfying the GW relation, and that the index and its density are related to the axial anomaly in precisely the same way as in the continuum, it is natural to ask if there is also a lattice index theorem for families of such operators such that the families index is related to gauge anomalies (or more precisely, to obstructions to the vanishing of these anomalies) in the same way

as in the continuum. In this paper we show that this is indeed the case: We derive an index theorem [(11) below] for a family of such lattice Dirac operators, parametrized by a 2-sphere in the orbit space of $SU(N)$ lattice gauge fields on the $2n$ -dimensional torus T^{2n} . This is the prototype for a more general lattice families index theorem which is currently under development [21]. We find that this index is related to an obstruction to gauge invariance in precisely the same way as in the continuum setting, where it was previously studied by Alvarez-Gaumé and Ginsparg [6]. Furthermore, the index density is found to be proportional to Lüscher's topological field $q(x, y_1, y_2)$ in $2n + 2$ dimensions [22] [given by (19) below]. This provides a natural origin for $q(x, y_1, y_2)$ in the lattice theory. (It was introduced in an *ad hoc* manner in [22].) This is of interest and potential use in connection with Lüscher's approach towards achieving gauge invariance in non-Abelian lattice chiral gauge theory: a local gauge anomaly-free formulation exists if and only if the local cohomology class represented by $q(x, y_2, y_2)$ is trivial [22].

Chiral gauge theory can be formulated on the lattice in the overlap formalism [12], which can be reformulated as a functional integral approach with lattice Dirac operator D satisfying the GW relation (1) and γ_5 -Hermiticity condition (5) [22,23]. Put a hypercubic lattice on T^{2n} , with lattice spacing a , and let C denote the finite-dimensional space of lattice spinor fields. The chiral projections $P_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$ and $\hat{P}_{\pm} = \frac{1}{2}(1 \pm \hat{\gamma}_5)$ determine decompositions $C = C_+ \oplus C_-$ and $C = \hat{C}_+ \oplus \hat{C}_-$, respectively. The (right-handed) lattice chiral determinant in this setting is

$$\det(iD_+^U) = \langle v_-, \hat{w}_+(U) \rangle, \quad (6)$$

where v_- and \hat{w}_+ are unit volume elements on C_- and \hat{C}_+ , respectively. These are unique up to phase factors; they can be written as $v_- = v_1 \wedge \cdots \wedge v_d$ and $\hat{w}_+ = \hat{w}_1 \wedge \cdots \wedge \hat{w}_d$ where v_1, \dots, v_d and $\hat{w}_1, \dots, \hat{w}_d$ are orthonormal bases for C_- and \hat{C}_+ , respectively. v_- and \hat{w}_+ are the many-body ground states in the overlap formulation [12], and correspond to the chiral fermion measures in the formulation of Refs. [22,23]. Note that $\hat{\gamma}_5 = \gamma_5(1 - aD^U)$ depends on the lattice gauge field U , so the subspace \hat{C}_+ and volume element \hat{w}_+ likewise depend on U . On the other hand, since the usual chiral decomposition $C = C_+ \oplus C_-$ does not involve U , neither does v_- . We are assuming $\dim C_{\pm} = \dim \hat{C}_{\pm} \equiv d$ (otherwise the chiral determinant vanishes). This is equivalent to assuming $\text{index} D^U = 0$, i.e., U is in the topologically trivial sector [12,22]. The space of lattice gauge fields will typically contain a subset of measure zero where D^U is not defined. In the case of the overlap Dirac operator such fields can be excluded by imposing a condition of the form $\|1 - U(p)\| < \epsilon$ on the plaquette products of U [24]. This condition is automatically satisfied close to the classical continuum limit since $1 - U(p) = a^2 F_{\mu\nu}(x) + O(a^3)$. We

will assume that the same is true for the general D that we are considering here.

Consider a circle family $\{\phi_{\theta}\}_{\theta \in S^1}$ of $SU(N)$ lattice gauge transformations. Each ϕ_{θ} is a map from the lattice sites to $SU(N)$ (assuming for simplicity that the fermion is in the fundamental representation). The action of ϕ_{θ} on U determines a circle family $S^1 = \{U^{\theta}\}_{\theta \in S^1}$ in the space \mathcal{U} of lattice gauge fields on T^{2n} . Since the modulus of (6) is gauge invariant [12], we have a map from S^1 to the unit circle in \mathbf{C} :

$$\theta \mapsto \langle v_-, \hat{w}_+(U^{\theta}) \rangle / \langle v_-, \hat{w}_+(U) \rangle. \quad (7)$$

The winding number W of this map is an obstruction to gauge invariance of the lattice chiral determinant. It was recently studied in [25] where it was shown to reduce to the continuum obstruction [6] in the classical continuum limit. In the following we will show that W is related to the index of a Dirac operator \mathcal{D} in $2n + 2$ dimensions in complete analogy with the continuum relation found in [6]. Our result is a lattice version of a special case of the families index theorem of [7] (this will be explained in detail in [21]).

Choose a disk family $\mathcal{B}^2 = \{U^{(\theta,t)}\}_{(\theta,t) \in B^2}$ in \mathcal{U} with boundary S^1 , i.e., with $U^{(\theta,1)} = U^{\theta}$. (Such a family might not exist in general due to the restrictions on \mathcal{U} needed to ensure that D is well defined. However its existence is guaranteed close to the classical continuum limit: we can take the lattice transcript of the continuum family of [6].) This determines a family of lattice Dirac operators $D^{(\theta,t)} = D^{U^{(\theta,t)}}$. Setting $\hat{\Gamma}^1 = \sigma_1 \otimes \hat{\gamma}_5$ (where $\hat{\gamma}_5 = \hat{\gamma}_5^{(\theta,t)}$) and $\hat{\Gamma}^2 = \sigma_2 \otimes 1$, we define the Dirac operator in $2n + 2$ dimensions in the lattice setting to be

$$\mathcal{D} = \hat{\Gamma}^{\alpha} i(\partial_{\alpha} + A_{\alpha}) \quad \alpha = 1, 2. \quad (8)$$

The derivatives are with respect to the continuous Cartesian coordinates (y_1, y_2) on B^2 and we have introduced a continuum $SU(N)$ gauge field $A = A_{\alpha} dy_{\alpha}$ on B^2 with $A_{\alpha}(y_1, y_2, x)$ a function of lattice site x as well as (y_1, y_2) . \mathcal{D} extends in a natural way to an elliptic 1st order differential operator on the vector fields with values in a vector bundle over the closed manifold $S^2 = B^2 \cup_{S^1} \tilde{B}^2$ as follows. The fiber of the vector bundle is $\mathbf{C}^2 \otimes C$ (i.e., the representation space of the Pauli matrices tensored with the finite-dimensional vector space of lattice spinor fields on T^{2n}) and the transition function at the common boundary S^1 of B^2 and \tilde{B}^2 is $1 \otimes \Phi^{-1}$ where $\Phi(\theta) = \phi_{\theta}$. A vector field in this vector bundle consists of a function $\Psi(\theta, t)$ on B^2 together with a function $\tilde{\Psi}(\theta, s)$ on \tilde{B}^2 , both taking values in $\mathbf{C}^2 \otimes C$, and related at the common boundary S^1 by

$$\tilde{\Psi}(\theta, 1) = \Phi(\theta)^{-1} \cdot \Psi(\theta, 1) \equiv 1 \otimes \phi_{\theta}^{-1} \cdot \Psi(\theta, 1). \quad (9)$$

\mathcal{D} is defined on Ψ by (8), and is defined to act on $\tilde{\Psi}$ as $\hat{\Gamma}_U^\alpha i(\partial_\alpha + \tilde{A}_\alpha)$ where $\hat{\Gamma}_U^1 = \sigma_1 \otimes \hat{\gamma}_5^U$ and $\hat{\Gamma}_U^2 = \Gamma^2$. The gauge-covariance of \mathcal{D} implies that $D^{(\theta,1)} = D^{U^\theta} = \phi_\theta \circ D^U \circ \phi_\theta^{-1}$ and $\hat{\gamma}_5^{(\theta,1)} = \phi_\theta \circ \hat{\gamma}_5^U \circ \phi_\theta^{-1}$. Using these it is easily checked that \mathcal{D} respects the relation (9), and is therefore a well-defined operator on the vector fields in the above vector bundle over S^2 , provided the gauge field $\tilde{A} = \tilde{A}_\alpha dy_\alpha$ on \tilde{B}^2 is related to the field A on B^2 at the common boundary S^1 by

$$\tilde{A}(\theta, 1, x) = \phi_\theta(x)^{-1}A(\theta, 1, x)\phi_\theta(x) + \phi_\theta(x)^{-1}d_\theta\phi_\theta(x). \quad (10)$$

[For example, we can take $A \equiv 0$ and $\tilde{A}(\theta, s, x) = s\phi_\theta(x)^{-1}d_\theta\phi_\theta(x)$ in terms of polar coordinates (θ, s) on \tilde{B}^2 .]

The space of vector fields in the above vector bundle over S^2 is denoted by \mathcal{V} in the following. The chirality operator $\Gamma_5 = \sigma_3 \otimes 1$ determines a chiral decomposition $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$. The ellipticity of \mathcal{D} follows easily from the facts that $\sigma_1 \otimes \hat{\gamma}_5$ anticommutes with $\sigma_2 \otimes 1$ and $(\sigma_1 \otimes \hat{\gamma}_5)^2 = (\sigma_2 \otimes 1)^2 = 1 \otimes 1$. Also, \mathcal{D} is formally self-adjoint with respect to the natural inner product in \mathcal{V} since $\sigma_1 \otimes \hat{\gamma}_5$ and $\sigma_2 \otimes 1$ are self-adjoint on $\mathbf{C}^2 \otimes C$. \mathcal{D} anticommutes with $\Gamma_5 = \sigma_3 \otimes 1$ and therefore has a chiral decomposition $\begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}$ and $\text{index } \mathcal{D} = \dim \ker \mathcal{D}_+ - \dim \ker \mathcal{D}_-$. The following formula for the index is derived below.

Theorem:

$$\text{index } \mathcal{D} = -\frac{1}{2\pi i} \left(\int_{B^2} \text{Tr}(\hat{P}_+ d\hat{P}_+ d\hat{P}_+) + \frac{1}{2} \int_{S^1} \text{Tr}(\phi_\theta^{-1} d_\theta \phi_\theta \hat{\gamma}_5^U) \right). \quad (11)$$

Here \hat{P}_+ is to be viewed as a function on the space \mathcal{U} of lattice gauge fields whose values are operators on C (i.e., finite-dimensional matrices), and d is the exterior derivative on \mathcal{U} . Thus the first integrand is a 2-form on \mathcal{U} and can be integrated over the disc B^2 in \mathcal{U} to get a C number. The second integrand is a 1-form on the boundary S^1 of B^2 , with $\hat{\gamma}_5^U = \hat{\gamma}_5^{(0,1)}$ constant.

By Eq. (3.11) of [25] the obstruction (winding number) W associated with the map (7) equals the right-hand side of (11) without the minus sign. It follows that

$$W = -\text{index } \mathcal{D}. \quad (12)$$

This is the promised lattice analog of the result of [6]. It follows from (12) and the result of [25] that index \mathcal{D} reduces in the classical continuum limit to minus the degree of the map $\Phi: S^1 \times T^{2n} \rightarrow \text{SU}(N)$, $\Phi(\theta, x) = \phi_\theta(x)$, which is precisely the index of the continuum Dirac operator in $2n + 2$ dimensions [6]. The 2-form in the first term in the right-hand side of (11) has appeared previously in the overlap formalism in [26], where it was interpreted as a form of Berry's curvature. [The Berry phase is associated with the state $w_+(U)$ in (6).] It is interesting to note that a version of this 2-form also arises in the context of the quantized Hall effect [27]. The second term in (11) arises in [25] as the integral of the covariant gauge anomaly (and vanishes in the special case where $U = 1$).

The index formula (11), together with (19) for the index density, and the relation (12) are the main results of this paper. The proof is as follows. We start from the formula

$$\text{index } \mathcal{D} = \text{Tr}(\Gamma_5 e^{-\tau \mathcal{D}^2}) \quad \forall \tau > 0. \quad (13)$$

This can be evaluated in the $\tau \rightarrow 0$ limit by familiar techniques as in [6]: It is seen to be the sum of a contribution from the B^2 part of $S^2 = B^2 \cup_{S^1} \tilde{B}^2$, given by

$$\int_{B^2} d^2y a^4 \sum_x q_{\mathcal{D}}(x, y_1, y_2), \quad (14)$$

where

$$q_{\mathcal{D}}(x, y_1, y_2) = \lim_{\tau \rightarrow 0} \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \text{tr}[\Gamma_5 e^{-ik \cdot y} (e^{-\tau \mathcal{D}^2}) e^{ik \cdot y}](x, x), \quad (15)$$

and an analogous contribution from the \tilde{B}^2 part. In (15) the trace is over spinor and flavor indices; $\mathcal{O}(x, y)$ denotes the kernel function of an operator \mathcal{O} on scalar lattice fields. From (8) we calculate

$$\begin{aligned} -\mathcal{D}^2 &= \partial^2 + (2A_1 + \hat{\gamma}_5 \nabla_1 \hat{\gamma}_5 - i\sigma_3 \nabla_2 \hat{\gamma}_5) \partial_1 \\ &\quad + 2A_2 \partial_2 + \nabla_\alpha A_\alpha \\ &\quad + i\sigma_3 \hat{\gamma}_5 F_{12} + (\hat{\gamma}_5 \nabla_1 \hat{\gamma}_5 - i\sigma_3 \nabla_2 \hat{\gamma}_5) A_1, \end{aligned} \quad (16)$$

where $\nabla_\alpha \hat{\gamma}_5 \equiv \partial_\alpha \hat{\gamma}_5 + [A_\alpha, \hat{\gamma}_5]$ (as in [22]); for notational simplicity we have omitted the \otimes symbol. After substituting this in (14) and making a change of variables $k_j \rightarrow \tau^{-1/2} k_j$ we find

$$\begin{aligned} q_{\mathcal{D}}(x, y_1, y_2) &= \lim_{\tau \rightarrow 0} \frac{1}{\tau} \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \text{tr}(\sigma_3 \exp\{-k^2 + \sqrt{\tau}(2A_1 + \hat{\gamma}_5 \nabla_1 \hat{\gamma}_5 - i\sigma_3 \nabla_2 \hat{\gamma}_5) ik_1 + \sqrt{\tau}(2A_2) ik_2 \\ &\quad + \tau[\nabla_\alpha A_\alpha + \hat{\gamma}_5 \nabla_1 \hat{\gamma}_5 + i\sigma_3(\hat{\gamma}_5 F_{12} - \nabla_2 \hat{\gamma}_5 A_1)]\})(x, x). \end{aligned} \quad (17)$$

This can be calculated by expanding the integrand in powers of $\sqrt{\tau}$. Terms with odd powers of σ_3 give a vanishing contribution, as do terms with k dependence of the form $e^{-k^2} k_1^p k_2^q$ where either p or q is odd. The remaining terms are

$$\tau i e^{-k^2} \text{tr}[(2A_1 \nabla_2 \hat{\gamma}_5 + \hat{\gamma}_5 \nabla_1 \hat{\gamma}_5 \nabla_2 \hat{\gamma}_5) k_1^2 + \hat{\gamma}_5 F_{12} - \nabla_2 \hat{\gamma}_5 A_1](x, x) + O(\tau^{3/2}), \quad (18)$$

where we have used the fact that $\hat{\gamma}_5 \nabla_\alpha \hat{\gamma}_5 = -\nabla_\alpha \hat{\gamma}_5 \hat{\gamma}_5$ (an easy consequence of $\hat{\gamma}_5^2 = 1$). Evaluating the integral in (17) with this integrand, we find that the contributions from the $2A_1 \nabla_2 \hat{\gamma}_5 k_1^2$ and $-\nabla_2 \hat{\gamma}_5 A_1$ terms in $\text{tr}(\dots)$ in (18) cancel, resulting in

$$q_{\mathcal{D}}(x, y_1, y_2) = \frac{-1}{2\pi i} \left(\frac{1}{4} \text{tr}(\hat{\gamma}_5 \nabla_1 \hat{\gamma}_5 \nabla_2 \hat{\gamma}_5)(x, x) + \frac{1}{2} \text{tr}(\hat{\gamma}_5 F_{12})(x, x) \right). \quad (19)$$

Modulo the numerical factor $-1/2\pi$, this coincides with Lüscher's topological field Eq. (9.8) of [22]. Summing over the lattice sites gives (cf. Appendix B of [22])

$$a^4 \sum_x q_{\mathcal{D}}(x, y_1, y_2) = \frac{-1}{2\pi i} \text{Tr} \left(\frac{1}{4} (\hat{\gamma}_5 \partial_1 \hat{\gamma}_5 \partial_2 \hat{\gamma}_5) - \frac{1}{2} \partial_1 (A_2 \hat{\gamma}_5) + \frac{1}{2} \partial_2 (A_1 \hat{\gamma}_5) \right). \quad (20)$$

The contribution to (14) from the first term in (20) gives the first term in the index formula (11). The contribution to (14) from the remaining terms in (20) reduces to $-\frac{1}{2} \int_{S^1} \text{Tr}[A(\theta, 1) \hat{\gamma}_5^{(\theta, 1)}]$ in polar coordinates. The analogous contribution to index \mathcal{D} from the \tilde{B}^2 part is only $+\frac{1}{2} \int_{S^1} \text{Tr}[\tilde{A}(\theta, 1) \hat{\gamma}_5^U]$ (since $\hat{\gamma}_5^U$ is constant). Adding these and using (10) we get the second term in (11).

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*Email address: dadams@maths.adelaide.edu.au

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