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Bulletin of the London Mathematical Society, 2018; OnlinePubl

Published in final form at https://doi.org/10.1112/blms.12182

Shelstad's character identity from the point of view of index theory

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November 6, 2017

Abstract

Shelstad's character identity is an equality between sums of characters of tempered representations in corresponding L-packets of two real, semisimple, linear, algebraic groups that are inner forms to each other. We reconstruct this character identity in the case of the discrete series, using index theory of elliptic operators in the framework of K-theory. Our geometric proof of the character identity is evidence that index theory can play a role in the classification of group representations via the Langlands program.

Contents

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1 Introduction

The aim of this paper is to use our previous results in [HW2] to compare representations of two connected real semisimple groups G, G' having the same Langlands dual group. We are inspired by the so-called Shelstad's character identity following from Langlands program [S1, S2]. In fact, in the local Langlands program [L1, L2, L3], every admissible representation of a real reductive Lie group is labeled by its L-parameter, which is represented by a homomorphism from the Weil group $W_{\mathbb{R}}$ of the real numbers to the Langlands dual group of G. If two real reductive groups G and G' have the same Langlands dual group, then every L-parameter ϕ of G can be identified to an L-parameter ϕ' for G'. Denote by Π_ϕ the L-packet of ϕ, i.e., the set of admissible representations of G having the same L-parameter ϕ. Shelstad's character identity states that the characters Θ_π of tempered representations π for G and G' associated to the same L-parameter satisfy the identity

$$(-1)^{\dim G/K} \sum_{\pi \in \Pi_\phi} \Theta_\pi(h) = (-1)^{\dim G'/K'} \sum_{\pi' \in \Pi_{\phi'}} \Theta_{\pi'}(h'), \qquad (1.1)$$

if h and h' are corresponding regular elements of Cartan subgroups $H < G$ and $H' < G'$, respectively.

The main result of this paper is a direct geometric proof of the character identity (1.1) in the case of discrete series representations, without referring to character formulas or the theory of Langlands program. The proof uses index theory and K-theory of C^*-algebras, and illustrates how these are related to representation theory and the Langlands program.

Acknowledgements

The authors are grateful to Kuok Fai Chao for background information about the Langlands program. The second author was supported by the Australian Research Council, through Discovery Early Career Researcher Award DE160100525.

2 Character identities and index theory

2.1 Inner forms

We recall the definition of inner forms, as discussed for example in [ABV]. Let $G_{\mathbb{C}}$ be a connected reductive complex algebraic group. A *real form* of $G_{\mathbb{C}}$ is an involutive automorphism $\sigma\colon G_{\mathbb{C}} \to G_{\mathbb{C}}$ which is antiholomorphic; i.e., $T_e\sigma(iX) = -iT_e\sigma(X)$ for all $X \in \mathfrak{g}_{\mathbb{C}}$. The involution σ is a generalised notion of complex conjugation. The *group of real points* of σ

$$G(\mathbb{R}, \sigma) = \{g \in G_{\mathbb{C}} : \sigma(g) = g\}$$

is called the *real form* of $G_{\mathbb{C}}$ associated to σ.

Definition 2.1 ([ABV, Chapter 2])**.** Two real forms $G(\mathbb{R}, \sigma)$ and $G(\mathbb{R}, \sigma')$ of $G_{\mathbb{C}}$ are said to be *inner* to each other if there is an element $g \in G_{\mathbb{C}}$ such that

$$\sigma' = C_g \circ \sigma, \tag{2.1}$$

where C_g denotes conjugation by g. The real group $G' := G(\mathbb{R}, \sigma')$ is called an *inner form* of the real group $G := G(\mathbb{R}, \sigma)$.

The set of inner forms for G can be identified bijectively to the first Galois cohomology $H^1(\Gamma, \mathrm{Inn}(G_{\mathbb{C}}))$ where Γ is the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of two elements, where the complex conjugation (the generator) takes $\eta \in \mathrm{Aut}(G_{\mathbb{C}})$ to $\sigma \circ \eta \circ \sigma^{-1} \in \mathrm{Aut}(G_{\mathbb{C}})$, and $\mathrm{Inn}(G_{\mathbb{C}})$ is the subgroup of inner automorphisms of $G_{\mathbb{C}}$ respecting the action of Γ. See [B].

Example 2.2.

- Let $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$. $G_{\mathbb{C}}$ has a compact real form $G = \mathrm{SU}(2)$ associated to $\sigma(g) = (\bar{g}^T)^{-1}$ and a split real form $G' = \mathrm{SL}(2, \mathbb{R})$ accosiated to $\sigma(g) = \bar{g}$. It can be checked that G' is the unique inner form for G.
- $\mathrm{SL}(n, \mathbb{R})$ when n odd does not have any inner form other than itself; It has an extra inner form if n is even. Note that when $n \geq 3$, $\mathrm{SL}(n, \mathbb{R})$ does not have compact inner forms in view of Theorem 2.5 below.
- $\mathrm{U}(2)$ and $\mathrm{U}(1, 1)$ are inner forms to each other.
- $\mathrm{SU}(2, 1)$ and $\mathrm{SU}(3)$ are inner forms to each other.
- $\mathrm{SO}(2p, 2q)$ where $2p + 2q = 4n$ are inner forms to each other. In particular, $\mathrm{SO}(4n)$ is the compact inner form amongst elements of this set.

If G and G' are inner forms to each other in a complex group $G_{\mathbb{C}}$, then any Cartan subgroup $H < G$ is conjugate in $G_{\mathbb{C}}$ to a Cartan subgroup $H' < G'$. See Lemma 2.1 of [L3] and [S2]. We will use this to identify elements $h \in H$ to corresponding elements $h' \in H'$.

Example 2.3. Let $G = \mathrm{SU}(2)$ and $G' = \mathrm{SL}(2, \mathbb{R})$. Then their respective Cartan subgroups

$$T = \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} : \theta \in \mathbb{R} \right\} \qquad \text{and} \qquad T' = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

are conjugated by $\begin{bmatrix} -i & -1 \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C})$.

2.2 L-packets and character identities

Inner forms are closely related to the Langlands program. Let G be a real reductive algebraic group, and let $G^\vee$ denote the Langlands dual group of G. Consider the *L-group* ${}^L G = G^\vee \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Let $W_{\mathbb{R}}$ be the *Weil group* of real numbers, i.e., the group $\mathbb{C}^\times \cup j\mathbb{C}^\times$ subject to relations

$$j^2 = -1 \quad \text{and} \quad jz = \overline{z}j.$$

Then the irreducible admissible representations of G are parametrised by L-parameters. These are group homomorphisms

$$\phi\colon W_{\mathbb{R}} \to {}^L G$$

satisfying certain conditions. In fact, the local Langlands correspondence gives rise to a surjective map

$$f : \Pi(G) \to \Phi(G) \qquad \text{where} \tag{2.2}$$

$\Pi(G) := \{$equivalence classes of irreducible admissible representations of $G\}$;

$\Phi(G) := \{G^\vee$-conjugacy classes of L-parameters $\phi\colon W_{\mathbb{R}} \to {}^L G\}$

where $f^{-1}(\{\phi\})$ is finite for every $\phi \in \Phi(G)$. The finite set

$$\Pi_\phi := f^{-1}(\{\phi\})$$

is called the *L-packet* associated to the L-parameter ϕ. In other words, two admissible irreducible representations π_1, π_2 are said to be in the same

L-packet if and only if they have the same L-parameter. Representations having the same L-parameter are indistinguishable in the sense of Langlands.

When G and G' are inner forms to each other, they share the same L-group. Groups with the same L-group are crucial in the sense that the representation theory of these groups are closely related and can be studied globally. The character identity in Theorem 2.4 below, regarding inner forms is an instance of this philosophy. In fact, by the local Langlands correspondence, the L-packets of admissible representations for G and G' are related by their corresponding L-parameters. Hence, a character identity involving a common L-packet can be expected. For more details of Langlands program and character identity we refer to Langlands [L1, L2, L3] and Shelstad [S1, S2].

Let ϕ be an L-parameter of admissible representations of G. According to Langlands [L3], all admissible representations in the same L-packet of a tempered representation are tempered. Hence the L-parameter ϕ of a tempered representation is called tempered. Let G' be an inner form of G. Suppose G is *quasi-split*, meaning that G has a σ-invariant Borel subgroup. In that case, we have $\Phi(G') \subset \Phi(G)$. Suppose $\phi \in \Phi(G')$. We write ϕ' for ϕ, when we view it as an L-parameter of G'. Then ϕ' is tempered if ϕ is. The same line of statements remains true with "tempered' replaced by "discrete series".

For any irreducible admissible representation π, we denote its global character by Θ_π. Shelstad's character identity is stated as follows.

Theorem 2.4 (Shelstad's character identity [S1, S2])**.** *Let G and G' be connected, real reductive, linear algebraic groups, and inner forms to each other in $G_{\mathbb{C}}$. Suppose G is quasi-split. Let $K < G$ and $K' < G'$ be maximal compact subgroups and let $H < G$ and $H' < G$ be Cartan subgroups that are conjugate to each other in $G_{\mathbb{C}}$. Let h be an elliptic regular element of H and h' the corresponding element of H'. Let ϕ' be a tempered L-parameter of G', and let ϕ be the corresponding L-parameter of G. Then*

$$(-1)^{\frac{\dim G/K}{2}} \sum_{\pi \in \Pi_\phi} \Theta_\pi(h) = (-1)^{\frac{\dim G'/K'}{2}} \sum_{\pi' \in \Pi_{\phi'}} \Theta_{\pi'}(h'). \tag{2.3}$$

This is Theorem 6.3 in [S2]. In this paper, we will prove Shelstad's character identity (2.3) using K-theory and index theory when Π_ϕ and $\Pi_{\phi'}$ are L-packets of discrete series representations.

2.3 Character identities for the discrete series

We now assume in addition that the group G is semisimple, and has discrete series representations. Harish-Chandra has the following equivalent statements for existence of discrete series. See for example Theorem 22.1 in [T].

Theorem 2.5 (Harish-Chandra)**.** *The following statements are equivalent.*

1. *G has a compact Cartan subgroup $T < G$;*
2. *G has a compact inner form G';*
3. *G has discrete series, i.e., square integrable representations.*

In particular, if G has a discrete series, then it always has a compact inner form.

As a result, all but the second case in Example 2.2 admit discrete series and hence they have compact Cartan subgroups. Assume G has discrete series and T is a compact Cartan subgroup from now on.

Theorem 2.6 ([S2, Corollary 2.9])**.** *Any inner form G' of G also contains a compact Cartan subgroup.*

So an inner form G' of G also has discrete series representations.

For discrete series, we have the following property, of L-packets, which can be found for example in [Lab].

Proposition 2.7. *If G has discrete series, then two discrete series representations π_1 and π_2 are in the same L-packet if and only if they have the same infinitesimal character.*

Let $K < G$ be a maximal compact subgroup containing T. Let W_G and $W_K < W_G$ be the Weyl groups of the root systems of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, respectively. Then two discrete series representations of G have the same infinitesimal character if and only if their Harish-Chandra parameters are in the same W_G-orbit. They are equivalent if and only if they are in the same W_K-orbit, so every discrete series L-packet can be identified with W_G/W_K.

In this paper, we will give a geometric proof of a special case of Theorem 2.4

Theorem 2.8. *Theorem 2.4 holds for L-packets of discrete series representations.*

This case of Theorem 2.4 can be proved by explicitly writing out Harish–Chandra's character formula for the discrete series and rearranging terms. Our proof does not involve character formulas, and will show how this case of the character identities is related to the geometry of the space G/T, and also how character identities are related to K-theory and index theory.

Example 2.9. Let $G = \mathrm{SL}(2, \mathbb{R})$, and $K = T = \mathrm{SO}(2)$. Let $\rho \in i\mathfrak{t}^*$ be the element mapping $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ to i. Let $n \in \mathbb{N}$, and set $\lambda = n\rho$. Let $\pi^G_{\pm\lambda}$ be the discrete series representation of $\mathrm{SL}(2, \mathbb{R})$ with Harish-Chandra parameter $\pm\lambda$.

We have $G^\vee = \mathrm{PGL}(2, \mathbb{C})$. The L-parameter of the two representations $\pi^G_{\pm\lambda}$, which are in the same L-packet, is the homomorphism

$$\phi_n \colon W_{\mathbb{R}} \to \mathrm{PGL}(2, \mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$

defined as follows. Let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ be the nontrivial element, i.e. complex conjugation. For $r > 0$ and $\theta \in \mathbb{R}$, set $\chi_n(re^{i\theta}) = re^{in\theta}$. Then

$$\begin{aligned} \phi_n(j) &= \left(\begin{bmatrix} 0 & (-1)^n \\ 1 & 0 \end{bmatrix}, \sigma \right) \\ \phi_n(z) &= \left(\begin{bmatrix} \chi_n(z) & 0 \\ 0 & \chi_{-n}(z) \end{bmatrix}, e \right), \end{aligned}$$

for $z \in \mathbb{C}^\times$. (See Section I.4 of [Lab].)

2.4 An equivariant index and orbital integrals

Let G be a connected, real semisimple Lie group with finite centre. Suppose G acts properly and isometrically on a Riemannian manifold M. Suppose M/G is compact. Let $E \to M$ be a $\mathbb{Z}_2$-graded, G-equivariant, Hermitian vector bundle. Let D be a G-equivariant, elliptic, self-adjoint first order differential operator on E that is odd with respect to the grading. The *reduced group C^*-algebra C^*_rG* of G is the closure in the operator norm of the algebra of all convolution operators on $L^2(G)$ by functions in $L^1(G)$. Let $K_0(C^*_rG)$ be its even K-theory. Then we have the equivariant index of D

$$\mathrm{index}_G(D) \in K_0(C^*_rG),$$

which is the image of the class defined by D in the equivariant K-homology group of M under the analytic assembly map. See [BCH] for details.

Let $g \in G$ be a semisimple element, and $Z < G$ its centraliser. The orbital integral map

$$f \mapsto \int_{G/Z} f(hgh^{-1})\, d(hZ)$$

on $C_c(G)$, defines a trace map

$$\tau_g \colon K_0(C_r^* G) \to \mathbb{C}.$$

See Section 2.1 in [HW2].

Theorem 2.1 in [HW2] is a fixed point formula for the number

$$\tau_g(\mathrm{index}_G(D)).$$

We will not use this fixed point formula, but only the following localisation or excision property of the index. This property follows directly from Theorem 2.1 in [HW2], but is in fact a step in its proof: see Proposition 4.6 in [HW2].

Proposition 2.10 (Localisation of the index)**.** *Let G', M', E' and D' be as G, M, E and D above, respectively. Suppose that G and G' are subgroups of some larger group, and that $g \in G \cap G'$. Suppose that there are g-invariant neighbourhoods U of the fixed point set M^g and U' of $(M')^g$, and a diffeomorphism $\varphi \colon U \to U'$ that commutes with g, such that*

$$\varphi^*(E'|_{U'}) = E|_U,$$

and $D'|_{U'}$ corresponds to $D|_U$ under this identification. Then

$$\tau_g(\mathrm{index}_G(D)) = \tau_g(\mathrm{index}_{G'}(D')).$$

2.5 Discrete series characters as indices

Let $K < G$ be maximal compact. Let $T < K$ be a maximal torus and suppose that T is a Cartan subgroup of G; i.e. G has a discrete series. Let

$$R := R(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}).$$

be the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Choose a positive system $R^+ \subset R$, and let ρ be half the sum of the elements of R^+. Let $\lambda \in i\mathfrak{t}^*$ be regular, and dominant with respect to R^+. Suppose $\lambda - \rho$ is integral. Let π_λ^G be the discrete series representation of G with Harish-Chandra parameter λ. The values of its character $\Theta_{\pi_\lambda^G}$ on the regular elements of T can be realised in terms of index theory.

Consider the G-manifold G/T, equipped with the G-invariant Riemannian metric defined by a K-invariant inner product on $\mathfrak{g}$. Consider the G-invariant complex structure $J^{G/T}_{R^+}$ on G/T such that, as complex vector spaces,

$$T_{eT}(G/T) = \mathfrak{g}/\mathfrak{t} \cong \bigoplus_{\alpha\in R^+} \mathfrak{g}^{\mathbb{C}}_{\alpha}. \tag{2.4}$$

Explicitly, this isomorphism is given by the inclusion $\mathfrak{g}/\mathfrak{t} \hookrightarrow (\mathfrak{g}/\mathfrak{t})^{\mathbb{C}}$ followed by projection onto the positive root spaces. Similarly, if $w \in W_G$, then we will write $J^{G/T}_{wR^+}$ for the complex structure defined as above, with R^+ replaced by wR^+.

For any integral element $\nu \in i\mathfrak{t}^*$, we have the G-equivariant line bundle

$$L^G_{\nu} := G \times_T \mathbb{C}_{\nu} \to G/T.$$

Here we write $\mathbb{C}_{\nu}$ for the vector space $\mathbb{C}$ on which T acts with weight ν. Let $\bar{\partial}_{L^G_{\lambda-\rho}}$ be the Dolbeault operator on G/T coupled to $L^G_{\lambda-\rho}$. Let $\bar{\partial}^*_{L^G_{\lambda-\rho}}$ be its formal adjoint with respect to the L^2-inner product defined by the Riemannian metric on G/T and the natural Hermitian metric on $L^G_{\lambda-\rho}$.

Proposition 2.11. *If $g \in T^{\mathrm{reg}}$, then*

$$\Theta_{\pi^G_{\lambda}}(g) = (-1)^{\dim(G/K)/2}\tau_g(\mathrm{index}_G(\bar{\partial}_{L^G_{\lambda-\rho}} + \bar{\partial}^*_{L^G_{\lambda-\rho}})).$$

For a proof, see Propositions 5.1 and 5.2 in [HW2]. This proof is based on the fact that the natural class $[\pi^G_{\lambda}] \in K_0(C^*_rG)$ (see e.g. [Laf]) equals

$$(-1)^{\dim(G/K)/2}\,\mathrm{index}_G(\bar{\partial}_{L^G_{\lambda-\rho}} + \bar{\partial}^*_{L^G_{\lambda-\rho}}).$$

3 A geometric proof

Let G be a connected, real semisimple Lie group with finite centre. Let $K < G$ be a maximal compact subgroup, and suppose a maximal torus $T < K$ is a Cartan subgroup of G. As before, Let R be the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, let $R^+ \subset R$ be a positive system, and let ρ be half the sum of the elements of R^+. Since G has a compact Cartan subgroup, it has a compact inner form G_c by Theorem 2.5. Theorem 2.8 follows from the case where $G' = G_c$, which we will assume from now on. The complexifications $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}_c$ of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_c$ of G and G_c, respectively, are equal. So if we identify T with a $G_{\mathbb{C}}$-conjugate Cartan subgroup $T_c < G_c$, then the root system of $(\mathfrak{g}^{\mathbb{C}}_c, \mathfrak{t}^{\mathbb{C}}_c)$ equals R. From now on, we will tacitly identify T and T_c with each other.

3.1 Identifying open sets

The geometric proof of Theorem 2.8 is based on suitable identifications of neighbourhoods of the fixed point sets of the actions by T on G/T and G_c/T.

Let $w \in N_{G_c}(T)$. Consider the T-invariant complex structure on $\mathfrak{g}/\mathfrak{t}$ defined by $w^{-1}R^+$ and the one on $\mathfrak{g}_c/\mathfrak{t}$ defined by R^+. Let $\mathfrak{t}^\perp \subset \mathfrak{g}$ be the orthogonal complement to $\mathfrak{t}$ in $\mathfrak{g}$ with respect to an $\mathrm{Ad}(K)$-invariant inner product. Similarly, let $\mathfrak{t}^{\perp_c} \subset \mathfrak{g}_c$ be the orthogonal complement to $\mathfrak{t}$ in $\mathfrak{g}_c$. Then we have a complex-linear isomorphism

$$\psi_w \colon \mathfrak{t}^\perp \cong \mathfrak{g}/\mathfrak{t} \cong \bigoplus_{\alpha \in R^+} \mathfrak{g}^{\mathbb{C}}_{w^{-1}\alpha} \xrightarrow{\mathrm{Ad}(w^{-1})} \bigoplus_{\alpha \in R^+} \mathfrak{g}^{\mathbb{C}}_{\alpha} \cong \mathfrak{g}_c/\mathfrak{t} \cong \mathfrak{t}^{\perp_c}. \tag{3.1}$$

Note that $\mathrm{Ad}(w)$ preserves $\mathfrak{t}^{\perp_c}$.

Lemma 3.1. *The composition* $\mathrm{Ad}(w) \circ \psi_w \colon \mathfrak{t}^\perp \to \mathfrak{t}^{\perp_c}$ *is* T*-equivariant.*

Proof. Let $X \in \mathfrak{t}^\perp$. For any choice $\tilde{R}^+$ of positive roots, let

$$p_{\tilde{R}^+} \colon \mathfrak{g}^{\mathbb{C}}/\mathfrak{t}^{\mathbb{C}} \to \bigoplus_{\alpha \in \tilde{R}^+} \mathfrak{g}^{\mathbb{C}}_{\alpha}$$

be the projection map defined by the root space decomposition. Then

$$\mathrm{Ad}(w) \circ p_{R^+} = p_{w^{-1}R^+} \circ \mathrm{Ad}(w).$$

By definition of ψ_w,

$$p_{R^+}(\psi_w(X)) = \mathrm{Ad}(w^{-1}) p_{w^{-1}R^+}(X).$$

Hence

$$p_{w^{-1}R^+}(\mathrm{Ad}(w)\psi_w(X)) = \mathrm{Ad}(w) p_{R^+}(\psi_w(X)) = p_{w^{-1}R^+}(X).$$

Since $p_{w^{-1}R^+}$ is T-equivariant, we have for all $h \in T$,

$$p_{w^{-1}R^+}(\mathrm{Ad}(h)\,\mathrm{Ad}(w)\psi_w(X)) = p_{w^{-1}R^+}(\mathrm{Ad}(h)X) = p_{w^{-1}R^+}(\mathrm{Ad}(w)\psi_w(\mathrm{Ad}(h)X)).$$

So $\mathrm{Ad}(w) \circ \psi_w$ is indeed T-equivariant. □

The map $\eta \colon \mathfrak{t}^\perp \to G/T$ mapping $X \in \mathfrak{t}^\perp$ to $\exp(X)T$ is T-equivariant, and a local diffeomorphism near $0 \in \mathfrak{t}^\perp$. Let $\tilde{U} \subset \mathfrak{t}^\perp$ be an $\mathrm{Ad}(T)$-invariant open neighbourhood of 0 on which this map defines a diffeomorphism onto its image $U \subset G/T$. The tangent map of η at 0 is the identification

$\mathfrak{t}^\perp \cong T_{eT}G/T$, which is complex-linear by definition. So $\eta^*(J^{G/T}_{w^{-1}R^+}|_U)$ is homotopic to $J_{w^{-1}R^+}|_{\tilde{U}}$ as T-invariant complex structures, if we choose $\tilde{U}$ small enough. So η is holomorphic up to a homotopy of T-invariant almost complex structures, which is as good as being holomorphic for index theory purposes.

Let $\eta_c\colon \mathfrak{t}^{\perp_c} \to G_c/T$ be defined by $\eta_c(Y) = \exp(Y)T$ for $Y \in \mathfrak{t}^{\perp_c}$. This is a T-equivariant map, and a local diffeomorphism near 0. Choose $\tilde{U}$ small enough, so that $\eta_c|_{\tilde{\psi}_w(\tilde{U})}$ is a diffeomorphism onto its image U_c^w, which is a neighbourhood of eT in G_c/T. Again, this diffeomorphism is holomorphic up to a T-equivariant homotopy of complex structures if we choose $\tilde{U}$ small enough. For later use, we define the map $\eta_c^w\colon \mathfrak{t}^{\perp_c} \to G_c/T$ by $\eta_c^w(Y) = \exp(Y)wT$, for $Y \in \mathfrak{t}^{\perp_c}$. This map is T-equivariant because w normalises T.

The maps ψ_w, η and η_c combine into a diffeomorphism

$$\varphi_w\colon U \xrightarrow{\eta^{-1}} \tilde{U} \xrightarrow{\psi_w} \psi_w(U) \xrightarrow{\eta_c} U_c^w,$$

holomorphic up to a homotopy of complex structures. The map ψ_w is not T-equivariant, so neither is φ_w in general.

Lemma 3.2. *The composition*

$$w \circ \varphi_w\colon U \to wU_c^w$$

is T-equivariant.

Proof. We have

$$w \circ \eta_c = \eta_c^w \circ \mathrm{Ad}(w).$$

Hence

$$w \circ \varphi_w = \eta_c^w \circ (\mathrm{Ad}(w) \circ \psi_w) \circ \eta^{-1}.$$

The maps η_c^w and η are T-equivariant, and by Lemma 3.1, so is $\mathrm{Ad}(w) \circ \psi_w$. So $w \circ \varphi_w$ is T-equivariant as well. □

3.2 Neighbourhoods of fixed point sets

Let $W_K := N_K(T)/T$ and $W_G := N_{G_c}(T)/T$. Then W_G is the Weyl group of the root system R, which explains the notation W_G. And W_K is the subgroup of W_G generated by reflections defined by compact roots. Note that

$$(G/T)^T = W_K$$

(see Lemma 6.7 in [HW1]), and

$$(G_c/T)^T = W_G.$$

Hence we have an inclusion

$$(G/T)^T \hookrightarrow (G_c/T)^T$$

and

$$(G_c/T)^T = \coprod_{[w]\in W_G/W_K} w\cdot (G/T)^T.$$

Here we fix representatives $w \in W_G$ of all classes $[w] \in W_G/W_K$ once and for all.

Consider the manifold

$$M := G/T \times W_G/W_K$$

equipped with the action by G on the first factor. Consider the G-invariant complex structure on M such that, for every $[w] \in W_G/W_K$, the restricted complex structure on $G/T \times \{[w]\}$ is $J^{G/T}_{w^{-1}R^+}$.

We have T-invariant neighbourhoods

$$V := \bigcup_{w_K\in W_K} w_K U$$

of $(G/T)^T$ and

$$V_c := \bigcup_{w\in W_G} wU_c^w$$

of $(G_c/T)^T$. We choose the set $\tilde{U}$ small enough, the sets $w_K U$ and wU_c^w are all disjoint, for $w_K \in W_K$ and $w \in W_G$. Consider the map

$$\varphi\colon V \times W_G/W_K \to V_c$$

given by

$$\varphi(w_K xT, [w]) = w_K w \varphi_w(xT), \tag{3.2}$$

for $w_K \in W_K$, $w \in W_G$ and $xT \in U$.

Lemma 3.3. *The map φ is well-defined and T-equivariant.*

Proof. Let $w_K, w'_K \in N_K(T)$ and $x, x' \in G$ be such that $w_K x T = w'_K x' T$. Fix $w \in W_G$. Since the sets $w_K U$ and $w'_K U$ are disjoint if w_K and w'_K represent different elements of W_K, there is a $t \in T$ such that $w'_K = w_K t$. There is another $t' \in T$ such that $w'_K x' = w_K x t'$. Using T-equivariance of $w\varphi_w$ and the fact that w_K normalises T, one deduces that $w'_K w \varphi_w(x'T) = w_K w \varphi_w(xT)$, and that φ is T-equivariant. □

The arguments in this section lead to the following conclusion.

Proposition 3.4. *The map φ is a T-equivariant diffeomorphism, holomorphic up to a homotopy of complex structures, from a T-invariant neighbourhood of M^T onto a T-invariant neighbourhood of $(G_c/T)^T$.*

See Figure 1 for an example of the map φ.

Remark 3.5. If one replaces the map φ_w by φ_e in (3.2), one obtains an identification $\tilde{\varphi}$ of neighbourhoods of M^T and $(G_c/T)^T$ that is simpler than the map φ we use here. However, $\tilde{\varphi}$ is not T-equivariant, which is the reason why we use the maps φ_w rather than φ_e. At the same time, using the maps φ_w means that the map φ is holomorphic up to homotopy with respect to the complex structure on M described above, whereas the map $\tilde{\varphi}$ is holomorphic up to homotopy if one uses the same complex structure $J^{G/T}_{R^+}$ on all connected components of M.

3.3 Line bundles

Consider the line bundle $L \to M$ such that for every $[w] \in W_G/W_K$,

$$L|_{G/T \times \{[w]\}} = G \times_T \mathbb{C}_{w^{-1}(\lambda-\rho)}.$$

Lemma 3.6. *There is a T-equivariant isomorphism of line bundles*

$$\varphi^*(L^{G_c}_{\lambda-\rho}|_{V_c}) \cong L|_{V \times W_G/W_K}.$$

Lemma 3.7. *Let G be any Lie group, $H < G$ a compact subgroup, and $\pi\colon H \to \mathrm{GL}(V)$ a representation of H in a finite-dimensional vector space V. Let $x \in N_G(H)$. Consider the representation $x^{-1} \cdot \pi\colon H \to \mathrm{GL}(V)$ given by $(x^{-1} \cdot \pi)(h) = \pi(x^{-1} h x)$. Write V_π and $V_{x^{-1}\cdot\pi}$ for the vector space V on which H acts via π and $x^{-1} \cdot \pi$, respectively. Fix an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{g}$. Let $\mathfrak{h}^\perp$ be the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. Define the map $\eta_x\colon \mathfrak{h}^\perp \to G/H$ by $\eta_x(X) = \exp(X) x H$, for $X \in \mathfrak{h}^\perp$. Let $\tilde{U} \subset \mathfrak{h}^\perp$ be an $\mathrm{Ad}(H)$-invariant open subset such that $\eta|_{\tilde{U}}$ is a diffeomorphism onto its image U_x. Then we have an H-equivariant isomorphism of vector bundles*

$$\eta_x^*((G \times_H V_\pi)|_{U_x}) \cong \tilde{U} \times V_{x^{-1}\cdot\pi}.$$

Proof. Note that

$$\eta_x^*((G \times_H V_\pi)|_{U_x}) = \{(X, [\exp(X)x, v]); X \in \tilde{U}, v \in V_\pi\}.$$

Define the map

$$f\colon \eta_x^*((G \times_H V_\pi)|_{U_x}) \to \tilde{U} \times V_{x^{-1}\cdot\pi}$$

by

$$f(X, [\exp(X)x, v]) = (X, v)$$

for $X \in \tilde{U}$ and $v \in V_\pi$. Note that this map is well-defined, and a vector bundle isomorphism. And for all $h \in H$,

$$\begin{aligned}
h \cdot (X, [\exp(X)x, v]) &= (\mathrm{Ad}(h)X, [h\exp(X)x, v]) \\
&= (\mathrm{Ad}(h)X, [\exp(\mathrm{Ad}(h)X)x(x^{-1}hx), v]) \\
&= (\mathrm{Ad}(h)X, [\exp(\mathrm{Ad}(h)X)x, (x^{-1}\cdot\pi)(h)v]).
\end{aligned}$$

So

$$f(h \cdot (X, [\exp(X)x, v])) = (\mathrm{Ad}(h)X, (x^{-1}\cdot\pi)(h)v) = h \cdot f(X, [\exp(X)x, v]).$$

□

Proof of Lemma 3.6. Let $w \in N_{G_c}(T)$. The claim is that we have a T-equivariant isomorphism of line bundles

$$(w \circ \varphi_w)^*(G_c \times_T \mathbb{C}_{\lambda-\rho}|_{wU_c^w}) \cong (G \times_T \mathbb{C}_{w^{-1}(\lambda-\rho)}|_U). \tag{3.3}$$

By Lemma 3.7, we have isomorphisms of T-equivariant line bundles

$$\begin{aligned}
\eta^*((G \times_T \mathbb{C}_{w^{-1}(\lambda-\rho)})|_U) &\cong \tilde{U} \times \mathbb{C}_{w^{-1}(\lambda-\rho)}; \\
(\eta_c^w)^*((G_c \times_T \mathbb{C}_{\lambda-\rho})|_{wU_c^w}) &\cong (\eta_c^w)^{-1}(wU_c^w) \times \mathbb{C}_{w^{-1}(\lambda-\rho)}.
\end{aligned} \tag{3.4}$$

Now

$$(\eta_c^w)^{-1}(wU_c^w) = (\eta_c^w)^{-1}(w\eta_c(\psi_w(\tilde{U}))) = \mathrm{Ad}(w) \circ \psi_w(\tilde{U}).$$

By Lemma 3.1, we have a T-equivariant, holomorphic diffeomorphism

$$\mathrm{Ad}(w) \circ \psi_w \colon \tilde{U} \xrightarrow{\cong} (\eta_c^w)^{-1}(wU_c^w).$$

So the two right hand sides of (3.4) are isomorphic as T-equivariant line bundles.

We saw in the proof of Lemma 3.2 that

$$w \circ \varphi_w \circ \eta = \eta_c^w \circ (\mathrm{Ad}(w) \circ \psi_w).$$

So the pullback of the left hand side of (3.3) along η equals

$$\begin{aligned}(\mathrm{Ad}(w)\circ\psi_w)^*(\eta_c^w)^*((G_c\times_T\mathbb{C}_{\lambda-\rho}))|_{wU_c^w}) &= (\mathrm{Ad}(w)\circ\psi_w)^*\big((\eta_c^w)^{-1}(wU_c^w)\times\mathbb{C}_{w^{-1}(\lambda-\rho)}\big)\\ &= (\eta_c^w\circ\mathrm{Ad}(w)\circ\psi_w)^{-1}(wU_c^w)\times\mathbb{C}_{w^{-1}(\lambda-\rho)}.\\ &= (w\circ\eta_c\circ\psi_w)^{-1}(wU_c^w)\times\mathbb{C}_{w^{-1}(\lambda-\rho)}\\ &= \tilde{U}\times\mathbb{C}_{w^{-1}(\lambda-\rho)}.\end{aligned}$$

The latter line bundle is the pullback of the right hand side of (3.3) along η. □

3.4 Proof of Theorem 2.8

Since G_c is compact, the claim is that if ϕ is the L-parameter of π_λ^G, and ϕ' is the corresponding L-parameter for G_c,

$$(-1)^{\dim(G/K)/2}\sum_{\pi\in\Pi_\phi}\Theta_\pi|_{T^{\mathrm{reg}}} = \sum_{\pi'\in\Pi_{\phi'}}\Theta_{\pi'}|_{T^{\mathrm{reg}}}. \tag{3.5}$$

Let $g\in T$ be an element whose powers are dense in T. Let $\bar\partial_L$ be the Dolbeault operator on M coupled to L. Propositions 2.10 and 3.4 and Lemma 3.6 imply that

$$\begin{aligned}\tau_g(\mathrm{index}_{G_c}(\bar\partial_{L^{G_c}_{\lambda-\rho}}+\bar\partial^*_{L^{G_c}_{\lambda-\rho}})) &= \tau_g(\mathrm{index}_G(\bar\partial_L+\bar\partial_L^*))\\ &= \sum_{[w]\in W_G/W_K}\tau_g(\mathrm{index}_G(\bar\partial_{L^G_{w^{-1}(\lambda-\rho)}}+\bar\partial^*_{L^G_{w^{-1}(\lambda-\rho)}})).\end{aligned} \tag{3.6}$$

For any regular element $\nu\in i\mathfrak{t}^*$, let ρ_ν be half the sum of the roots in R with positive inner products with ν. Then for any $w\in W_G$, $\rho_{w\lambda}=w\rho$. So the right hand side of (3.6) equals

$$\sum_{[w]\in W_G/W_K}\tau_g(\mathrm{index}(\bar\partial_{L^G_{w^{-1}\lambda-\rho_{w^{-1}\lambda}}}+\bar\partial^*_{L^G_{w^{-1}\lambda-\rho_{w^{-1}\lambda}}})).$$

By Proposition 2.11, this equals

$$(-1)^{\dim(G/K)/2}\sum_{[w]\in W_G/W_K}\Theta_{\pi^G_{w^{-1}\lambda}}(g).$$

As noted below Proposition 2.7, we have

$$\Pi_\phi = \{\pi^G_{w\lambda}; w\in W_G\}/W_K,$$

and similarly,

$$\Pi_{\phi'} = \{\pi_\lambda^{G_c}\}.$$

Note that $\pi_\lambda^{G_c}$ is the irreducible representation of G_c with infinitesimal character λ, so with highest weight $\lambda-\rho$. The Borel–Weil–Bott theorem therefore implies that the right hand side of (3.5), evaluated at g, equals

$$\Theta_{\pi_\lambda^{G_c}}(g) = \tau_g(\mathrm{index}_{G_c}(\bar\partial_{L^{G_c}_{\lambda-\rho}} + \bar\partial^*_{L^{G_c}_{\lambda-\rho}})),$$

which by the above considerations equals the left hand side of (3.5), evaluated at g. We have assumed that the powers of g are dense in T, but as such elements are dense in T^{reg}, Theorem 2.8 follows in the case where $G' = G_c$ since both sides of (3.5) are analytic on T^{reg}. As noted at the start of Section 3, that case implies the general case, so the theorem has been proved.

3.5 Example: $G = \mathrm{SL}(2, \mathbb{R})$

Let $G = \mathrm{SL}(2,\mathbb{R})$, and $G' = G_c = \mathrm{SU}(2)$. Note that G is quasi-split. Take $K = T = \mathrm{SO}(2) \cong \mathrm{U}(1) \hookrightarrow \mathrm{SU}(2)$. Let $\rho \in i\mathfrak{t}^*$ be the element mapping $X := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathfrak{t}$ to i. Let $n \in \mathbb{N}$, and set $\lambda = n\rho$. Let $\pi^G_{\pm\lambda}$ be the discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ with Harish-Chandra parameter $\pm\lambda$. Let ϕ be the L-parameter of π^G_λ; see Example 2.9. Then $\Pi_\phi = \{\pi^G_\lambda, \pi^G_{-\lambda}\}$. Let ϕ' be the corresponding L-parameter for G_c. Then $\Pi_{\phi'} = \{\pi^{G_c}_\lambda\}$, where $\pi^{G_c}_\lambda$ is the irreducible representation of $\mathrm{SU}(2)$ with infinitesimal character λ, so with highest weight $\lambda - \rho$.

Fix $g \in T$ with dense powers. Now $G_c/T = S^2$, and $(G_c/T)^g$ consists of the north and south poles. And G/T is the hyperbolic plane $\{x^2 + y^2 - z^2 = n\}$, and $(G/T)^g$ is the single point $eT = (0, 0, n)$. We have $W_K = \{e\}$ and $W_G = \mathbb{Z}_2$, where the nontrivial element of W_G is represented by[1] $w := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in N_{\mathrm{SU}(2)}(\mathrm{U}(1))$. This element maps the north pole of $G_c/T = S^2$ to the south pole. So we see that indeed

$$(G_c/T)^g = (G/T)^g \cup w(G/T)^g,$$

The map φ identifying neighbourhoods of these fixed point sets in G_c/T and $G/T \times W_G/W_K$, respectively, is depicted in Figure 1.

[1]Note that as matrices, we have $w = X$. We use different letters because w is viewed as a group element, whereas X is viewed as a Lie algebra element.

The Atiyah–Segal–Singer or Atiyah–Bott fixed point formula implies that, if $g = \exp(tX)$,

$$\mathrm{index}_{\mathrm{SU}(2)}(\bar{\partial}_{L^{\mathrm{SU}(2)}_{\lambda-\rho}} + \bar{\partial}^*_{L^{\mathrm{SU}(2)}_{\lambda-\rho}})(g) = \frac{e^{\lambda}(g) - e^{-\lambda}(g)}{e^{\rho}(g) - e^{-\rho}(g)} = \frac{\sin(nt)}{\sin(t)}. \tag{3.7}$$

Theorem 2.1 in [HW2] implies that

$$\tau_g(\mathrm{index}_{\mathrm{SL}(2,\mathbb{R})}(\bar{\partial}_{L^{\mathrm{SL}(2,\mathbb{R})}_{\pm(\lambda-\rho)}} + \bar{\partial}^*_{L^{\mathrm{SL}(2,\mathbb{R})}_{\pm(\lambda-\rho)}})) = \frac{e^{\pm\lambda}(g)}{\pm(e^{\rho}(g) - e^{-\rho}(g))} = \pm\frac{e^{\pm int}}{2i\sin(t)}.$$

Let ϕ be the L-parameter of π^G_λ, and let ϕ' be the corresponding L-parameter for SU(2). Then we conclude that

$$\begin{aligned}(-1)^{\dim(\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2))/2} \sum_{\pi\in\Pi_\phi} \Theta_\pi(g) = \\ \tau_g(\mathrm{index}_{\mathrm{SL}(2,\mathbb{R})}(\bar{\partial}_{L^{\mathrm{SL}(2,\mathbb{R})}_{\lambda-\rho}} + \bar{\partial}^*_{L^{\mathrm{SL}(2,\mathbb{R})}_{\lambda-\rho}})) + \tau_g(\mathrm{index}_{\mathrm{SL}(2,\mathbb{R})}(\bar{\partial}_{L^{\mathrm{SL}(2,\mathbb{R})}_{-(\lambda-\rho)}} + \bar{\partial}^*_{L^{\mathrm{SL}(2,\mathbb{R})}_{-(\lambda-\rho)}})) = \\ \mathrm{index}_{\mathrm{SU}(2)}(\bar{\partial}_{L^{\mathrm{SU}(2)}_{\lambda-\rho}} + \bar{\partial}^*_{L^{\mathrm{SU}(2)}_{\lambda-\rho}})(g) = \\ (-1)^{\dim(\mathrm{SU}(2)/\mathrm{SU}(2))/2} \sum_{\pi'\in\Pi_{\phi'}} \Theta_{\pi'}(g).\end{aligned}$$

In this example, we see from the fixed point formulas used, and Weyl's and Harish-Chandra's character formulas, that the respective indices evaluated at g equal the characters of the corresponding representations. And the character identity follows directly from the explicit expressions for these characters. But note that in the proof of Theorem 2.8, these fixed point formulas and character formulas were not used.

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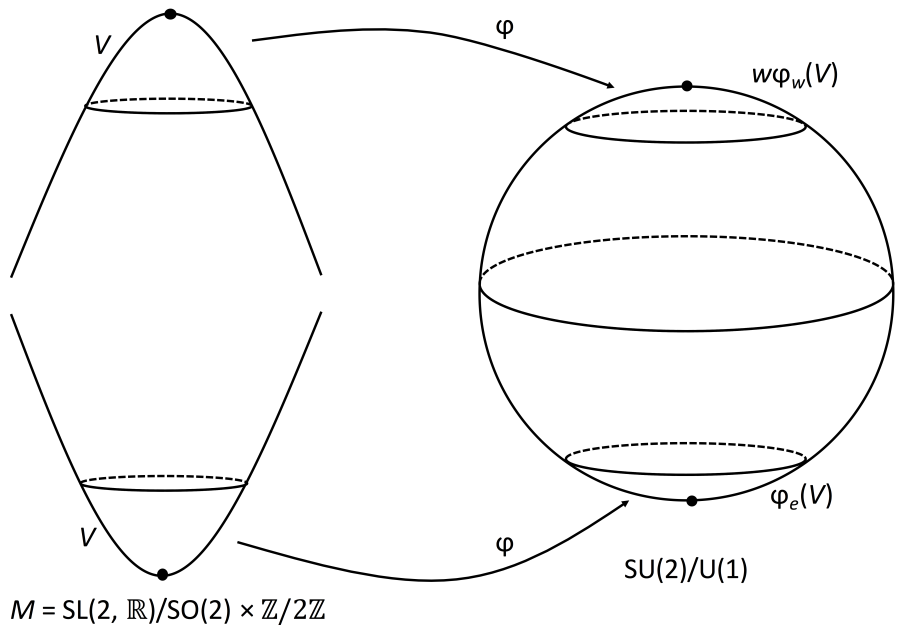

Figure 1: The map φ for $G = \mathrm{SL}(2, \mathbb{R})$

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