THE DIFFUSION OF METEOR TRAILS IN THE EARTH'S MAGNETIC FIELD

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43 Milne Street,
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5.9.63.

Dear Sir,

With reference to your letter dated August 30, 1963 I hereby give to you the authority for loan and photocopying of my thesis "The diffusion of meteor trails in the earth's magnetic field".

Yours faithfully,
(signed) J.L.A. Francey.

COPY:JJ
September 9, 1963.

Contents

Summary

A solution of the diffusion equation as applied to a meteor trail is given. Equations that show the effect of the Earth's magnetic field on the diffusion of the ionized column forming the trail are derived and solved.

The scattering of a radio wave by the trail is considered and formulae are obtained for the power received at a radio receiver on the ground in terms of the power transmitted.

The nonlinearities introduced by the space charges are considered and perturbation calculations made. In order to assess the magnitude of the effect of the space charges, a dimensional analysis is carried out. This shows that the effect is large and the equations for electrons and ions including the non linearities are then solved by numerical methods using a high speed computer.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge, no material previously published or written by another person, except where due reference is made in the text.

Signed

1. Introduction

The theory of radio reflections from meteor trails is now fairly well known. The radar equation for a column of ionized air in the atmosphere was derived by Lovell and Clegg (1948) on the assumption that the column was narrow compared with the wave-length used and that the incident wave could penetrate the column without significant modification. A very comprehensive treatment has been given by Kaiser (1953) for both low density ($< 10^{14}$ per meter) and high density trails and mention is made here of the possible effects of the Earth's magnetic field which is likely to modify the shape of the expanding ionized column.

An alternative treatment by Huxley (1952) treats the reflections as occurring not throughout the column but at the surface where the density is critical for the radio wave. In this paper also the theory of ambipolar diffusion is applied to the spreading of the meteor trail.

Weiss (1955), in discussing the measurement of diffusion coefficients from the rate of decay of meteor echoes, suggests that the Earth's magnetic field could account for some of the scatter in the experimental results obtained.

It seems that no quantitative results for the effect of the Earth's field on meteor echoes have been obtained so far. In what follows the diffusion of charged particles in a magnetic field is discussed and this is applied to the reflection of a radio wave from a meteor trail having various attitudes with respect to the Earth's field.

2. The Diffusion Equation

The equation of diffusion of either electrons or ions in zero magnetic field is given by ;-

$$\frac{\partial n}{\partial t} = D\nabla^2 n$$

where n = number density; D = diffusion coefficient.

The Greens Function for (2.1) $g(x,y,z,t;\ x_o,y_o,z_o,t_o)$ satisfies

$$-\frac{\partial g}{\partial t} + D\nabla^2 g = -\delta(x - x_o)\ \delta(y - y_o)\ \delta(z - z_o)\ \delta(t - t_o)$$

Write $g = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp -i\left[\alpha(x-x_o)+\beta(y-y_o) + \gamma(z - z_o)\right] \times$

$F(\alpha,\beta,\gamma,t)\,d\alpha d\beta d\gamma$

and since $\delta(x - x_o)\ \delta(y - y_o)\ \delta(z - z_o) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$

$\exp -i\left[\alpha(x - x_o) + \beta(y - y_o) + \gamma(z - z_o)\right] d\alpha d\beta d\gamma$

then $\frac{dF}{dt} + D(\alpha^2 + \beta^2 + \gamma^2)F = \frac{1}{(2\pi)^{\frac{3}{2}}}\ \delta(t - t_o)$

whence $F = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp t\left[-D(\alpha^2 + \beta^2 + \gamma^2)\right]$ for $t > t_o$

and $g = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\exp i\left[\alpha(x - x_o) + \beta(y - y_o) + \gamma(z - z_o)\right] \times\right.$

$\left.\exp -tD(\alpha^2+\beta^2+\gamma^2)\right\} d\alpha d\beta d\gamma$

or $g = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\alpha \exp\left[i\alpha(x - x_o) - tD\alpha^2\right]\int_{-\infty}^{\infty} d\beta \exp\left[i\beta(y - y_o)\right.$

$\left.-tD\beta^2\right] \cdot \int_{-\infty}^{\infty} d\gamma \exp\left[i\gamma(z - z_o) - tD\gamma^2\right]$

The first integral $= \int_{-\infty}^{\infty} d\alpha \exp -\left[\sqrt{tD}\,\alpha - \frac{i(x - x_o)}{2\sqrt{tD}}\right]^2 \exp\left[-\frac{(x - x_o)^2}{4tD}\right]$

$$= \int_{-\infty}^{\infty} d\phi \exp - tD\phi^2.\ \exp - \frac{(x - x_o)^2}{4tD}$$

2. Cont.

$$\text{where } \phi = \alpha - \frac{i(x-x_o)}{2tD}$$

$$= \frac{\sqrt{\pi}}{\sqrt{tD}} \quad \exp - \frac{(x - x_o)^2}{4tD}$$

$$\text{whence } g = \frac{1}{8(\pi tD)^{\frac{3}{2}}} \quad \exp -\left[(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right] / 4tD$$

and $n = No. \, g$ where No is the initial number density.

If the source No is a line rather than a point then

$$n = \int dz_o \frac{No}{8(\pi tD)^{\frac{3}{2}}} \quad \exp -\left[(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right] / 4tD$$

$$= \frac{No}{4\pi tD} \quad \exp -\left[(x - x_o)^2 + (y - y_o)^2\right] / 4tD \qquad (2.2)$$

where the line source is taken along the z_o axis.

3. Diffusion of a charged particle in a magnetic field.

A charged particle moving with velocity $\bar{v}$ in a magnetic field $\bar{B}$ will experience a force $\bar{v} \times \bar{B}$. The effect of this will be to reduce the rate of diffusion normal to the magnetic field. The reduction can be described by a "magnetic" coefficient of diffusion D_M so that for $\bar{B}$ // Z there is a flow of particles in the x-direction given by:-

$-D \frac{\partial n}{\partial x} + D_M \frac{\partial n}{\partial x} - D_M \frac{\partial n}{\partial y}$. In the y-direction the flow is given by:-

$- D \frac{\partial n}{\partial y} + D_M \frac{\partial n}{\partial y} - D_M \frac{\partial n}{\partial x}$. The flow in the z-direction is not influenced by the magnetic field so that the diffusion equation becomes:-

$$\frac{\partial n}{\partial t} = (D - D_M) \frac{\partial^2 n}{\partial x^2} + (D - D_M) \frac{\partial^2 n}{\partial y^2} + D \frac{\partial^2 n}{\partial z^2} \qquad (3.1)$$

Now write $\varepsilon = \left(\frac{D}{D - Dm}\right)^{\frac{1}{2}} x$, $\eta = \left(\frac{D}{D - Dm}\right)^{\frac{1}{2}} y$ and (3.1)

becomes $\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial \varepsilon^2} + D \frac{\partial^2 n}{\partial \eta^2} + D \frac{\partial^2 n}{\partial z^2}$ which from (2.2) has

solution for a line source //z :-

$$n = \frac{No}{4\pi t(D-D_m)} \exp - (\varepsilon^2 + \eta^2 + z^2) / 4tD$$

$$= \frac{No}{4\pi t(D-D_m)} \exp - \frac{(x^2 + y^2)}{4t(D - Dm)} \qquad (3.2)$$

for $\bar{B}$ // y the diffusion equation to be solved is :-

$\frac{\partial n}{\partial t} = (D - Dm) \frac{\partial^2 n}{\partial x^2} + D \frac{\partial^2 n}{\partial y^2} + (D - Dm) \frac{\partial^2 n}{\partial z^2}$ and in the same way as for (3.1) the solution is given by

$$n = \frac{N_o}{4\ tD^{\frac{1}{2}}(D - Dm)^{\frac{1}{2}}} \exp - \left(\frac{x^2}{4t(D - Dm)} + \frac{y^2}{4tD} \right) \qquad (3.3)$$

3. Cont.

For $\bar{B}$ in the xz plane and making an angle ϕ with the x-axis, the diffusion equation can be written

$$\frac{\partial n}{\partial t} = (D - Dm \sin \phi)\frac{\partial^2 n}{\partial x^2} + (D - Dm)\frac{\partial^2 n}{\partial y^2} + (D - Dm \cos \phi)\frac{\partial^2 n}{\partial z^2}$$

As before the solution for a line source //z is given by :-

$$n = \frac{N_o}{4\pi t(D - Dm)^{\frac{1}{2}}(D - Dm \sin \phi)^{\frac{1}{2}}} \exp -\left(\frac{x^2}{4t(D-Dm \sin \phi)} + \frac{y^2}{4t(D-Dm)}\right) \tag{3.4}$$

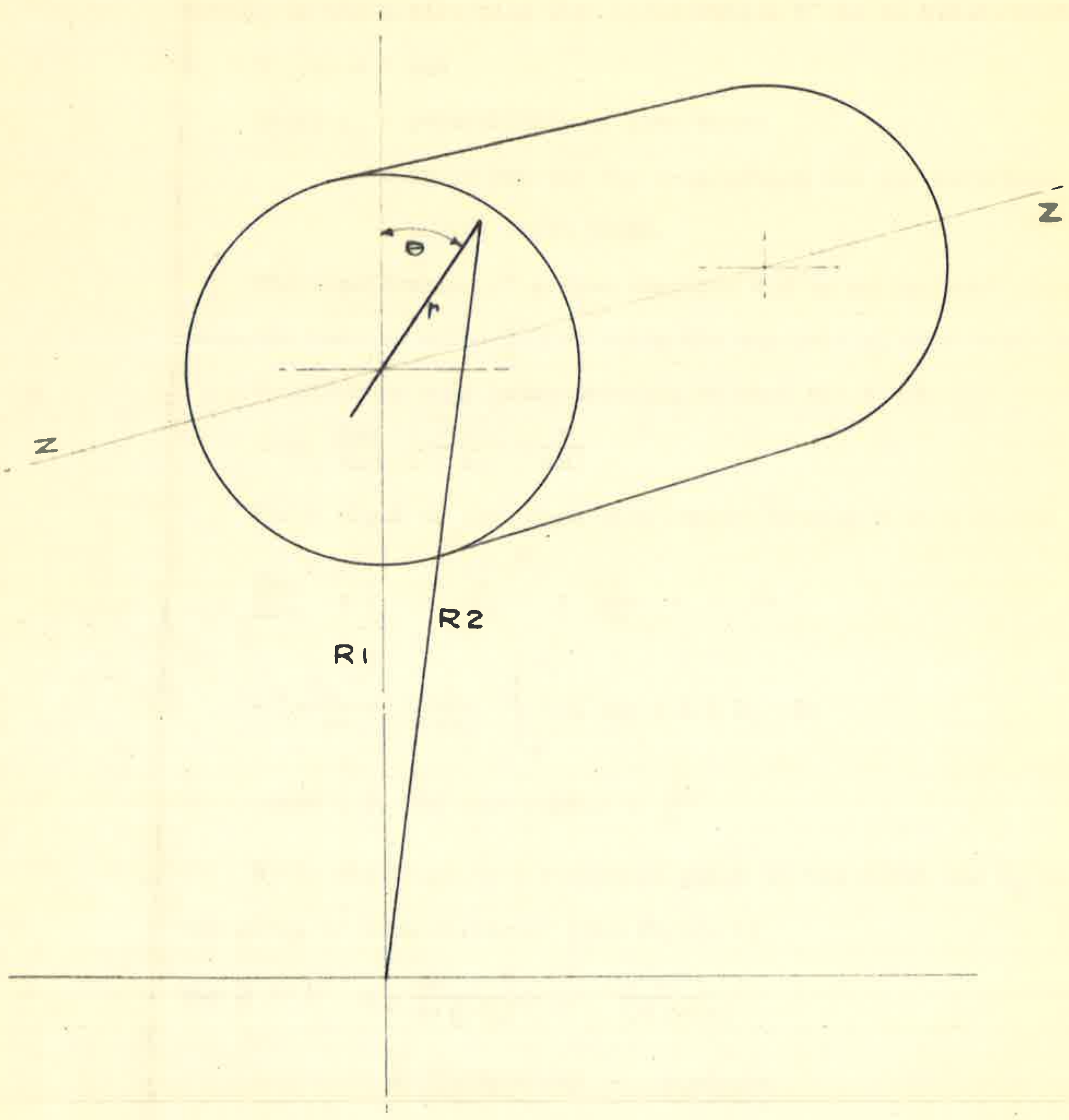

FIG 1

4. Scattering of Electromagnetic Waves by a Meteor Trail.

The scattered field strength at range R from a single electron moving in the x-direction with acceleration $\ddot{x}$ is in M.K.S. units

$\mu_o e \ddot{x} \sin \alpha / 4\pi R$

Where μ_o = permeability of free space

α = angle between the z-direction and the direction to the field point.

The acceleration of a free electron due to an incident electromagnetic wave is $-\mathrm{Einc}\,\frac{e}{m}$ (omitting the exponential time variation.)

Considering only backscattering so that $\sin \alpha = 1$

then $\frac{\mathrm{Esc}}{\mathrm{Einc}} = \frac{-\mu_o e^2}{4m} \times \frac{1}{\pi R}$

For a cloud of electrons with number density n in a region v

$$\frac{\mathrm{Esc}}{\mathrm{Einc}} = \int_v - \frac{\mu_o e^2}{4m} \times \frac{1}{\pi R} \times n \, dv$$

$$= -\frac{\mu_o e^2}{4m} \frac{1}{\pi R_1} \int_v n \exp 2 i K R_2 \, dv$$

where K is the wave number $= \frac{2\pi}{\lambda}$

R_1 is the range to a reference point in the cloud and R_2 is the range to each electron (See Figure 1)

For $\overline{\mathbf{B}} \,//\, Z$ $\quad n = \frac{No}{4\pi t(D-D_m)} \exp - \frac{(x^2+y^2)}{4t(D-Dm)}$

$$= \frac{No}{4\pi t(D-D_m)} \exp - \frac{r^2}{4t(D-Dm)}$$

4. Cont.

So $$\frac{E_{sc}}{E_{inc}} = -\frac{\mu_o e^2}{4m} \cdot \frac{1}{\pi R_1} \cdot \frac{N_o}{4\pi t(D-D_m)} \int_v dv \exp - \left[\frac{r^2}{4t(D - D_m)} + 2 i K R_2 \right] \qquad (4.1)$$

Now $$R_2^{\,2} = (R_1 + r\cos\theta)^2 + z^2 + r^2 \sin^2\theta$$

$$= R_1^{\,2} + 2r R_1 \cos\theta + r^2 + Z^2$$

$$R_2 = (R_1^{\,2} + 2rR_1 \cos\theta + r^2 + Z^2)^{\frac{1}{2}}$$

$$= R_1 \left(1 + \frac{2r\cos\theta}{R_1} + \frac{r^2 + z^2}{R_1^{\,2}}\right)^{\frac{1}{2}}$$

$$\doteq R_1 \left(1 + \frac{r\cos\theta}{R_1} + \frac{r^2 + z^2}{2R_1^{\,2}}\right)$$

$$\therefore \; R_2 - R_1 \doteq r\cos\theta + \frac{r^2 + z^2}{2R_1}$$

Replacing R_2 in the phase factor by $R_2 - R_1$ and changing to cylindrical co-ordinates the integral in 4.1 becomes

$$\int_{\infty}^{\infty} dr \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \left\{ r \exp\left[-\frac{r^2}{4t(D - D_m)} + 2 i K \left(r\cos\theta + \frac{r^2 + z^2}{2R_1} \right) \right] \right\}$$

$$= \int_{-\infty}^{\infty} dz \exp \frac{iKz^2}{R_1} \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r \exp\left[-\frac{r^2}{4t(D-D_m)} + 2 i K r\cos\theta + \frac{iKr^2}{R_1} \right] \qquad (4.2)$$

4. Cont.

The first part of (4.2) $= \int_{-\infty}^{\infty} dz \exp - \frac{K}{R_1} \left(\frac{\sqrt{2}\, Z}{1+i} \right)^2$

$$= \int_{-\infty}^{\infty} d\phi (\exp - \frac{K}{R_1} \phi^2) \cdot \frac{1+i}{\sqrt{2}} = \frac{1+i}{\sqrt{2}} \cdot \frac{\sqrt{R_1}}{\sqrt{K}} \cdot \sqrt{\pi}$$

where $\phi = \frac{\sqrt{2}z}{1+i}$, $d\phi = \frac{\sqrt{2}}{1+i} dz$

giving $\frac{(1+i)\sqrt{R_1 \lambda}}{2}$ (4.3)

The remaining part of (4.2) $= \int_0^{2\pi} d\theta \int_0^{\infty} dr \; r \; \exp$

$$\left[- \frac{r^2}{4t(D-Dm)} + 2 i K r \cos\theta + \frac{i K r^2}{R_1} \right]$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} dr \; r \; \exp - \left\{ r^2 \frac{\left[R_1 - 4t\, i\, K\, (D-Dm) \right]}{4tR_1\, (D-Dm)} - 2i K r \cos\theta \right\}$$

writing $\alpha = \frac{\left[R_1 - 4t\, i\, K\, (D-Dm) \right]}{4tR_1 (D-Dm)}$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r \exp - \alpha \left(r - \frac{i K \cos\theta}{\alpha}\right)^2 \exp - \frac{K^2 \cos^2\theta}{\alpha}$$

$$= \int_0^{2\pi} d\theta \exp - \frac{K^2 \cos^2\theta}{\alpha} \int_0^{\infty} dr\, r \exp - \alpha \left(r - \frac{i K \cos\theta}{\alpha}\right)^2$$

$$= \int_0^{2\pi} d\theta \exp - \frac{K^2\cos^2\theta}{\alpha} \left[\int_0^{\infty} d\phi \left(\phi \exp - \alpha \phi^2 + \frac{i K \cos\theta}{\alpha} \exp - \alpha\phi^2\right) \right.$$

$$\left. + \int_{\frac{-i K \cos\phi}{\alpha}}^{0} d\phi \left(\phi \exp - \alpha\phi^2 + \frac{i K \cos}{\alpha} \exp - \alpha\phi^2\right) \right] \quad (4.4)$$

4. Cont.

But $\int_0^{2\pi} d\theta \cos\theta = 0$

and the integral along the imaginary axis $\doteq 0$ since $\alpha >> K$

$$\text{So } (4.4) = \int_0^{2\pi} d\theta \; \frac{1}{2\alpha} \; \exp - \frac{K^2\cos^2\theta}{\alpha}$$

$$= \left(\frac{1}{2\alpha} \exp - \frac{K^2}{2\alpha}\right) \int_0^{2\pi} d\theta \; \exp - \frac{K^2 \cos 2\theta}{2\alpha}$$

$$= \left(\frac{1}{2\alpha} \exp - \frac{K^2}{2\alpha}\right) \int_{-\pi}^{\pi} d\chi \; \exp \frac{K^2\cos\chi}{2\alpha}$$

where $\chi = \pi - 2\theta$

$$= \left(\frac{1}{2\alpha} \exp - \frac{K^2}{2\alpha}\right) \; 2\int_0^{\pi} d\chi \; \exp \frac{K^2}{2\alpha} \cos\chi$$

$$= \left(\frac{1}{2\alpha} \exp - \frac{K^2}{2\alpha}\right) \cdot 2\pi \, I_o \left(\frac{K^2}{2\alpha}\right)$$

where Io is the modifed Bessel Function of the first kind and zero order

$$\doteq \left(\frac{1}{2\alpha} \exp - \frac{K^2}{2\alpha}\right) \cdot 2\pi \left(1 + (o)\, K^4\right)$$

$$\doteq \left\{ 4\pi \; t(D-Dm) \; \exp - \left[2t(D-Dm) \; K^2\right]\right\} \left[1 + K^4 \, t^2 \, (D-Dm)^2\right] \quad (4.5)$$

Since $\alpha \doteq \frac{1}{4t(D-Dm)}$ for large R_1

$$\therefore \quad \frac{Esc}{Einc} \doteq - \frac{\mu_o \, e^2}{4m} \cdot \frac{1}{\pi R_1} \cdot \frac{No\sqrt{R_1\,\lambda}}{4\pi t(D-Dm)\cdot 2} \cdot 4\pi t(D-Dm) \exp - \left[2t(D-Dm)K^2\right]$$

4. Cont.

$$= - \frac{\mu_o e^2}{4m} \cdot No \cdot \frac{(D-Dm)}{(D-Dm)} \cdot \frac{\sqrt{\lambda}}{2\pi \sqrt{R_1}} \cdot \exp - \left[2t(D-Dm)K^2 \right] \qquad (4.6)$$

More accurately, (4.6) should be multiplied by $1 + K^4 t^2 (D-Dm)^2$ but in practice this will differ little from unity.

For $\overline{B}//Y$ (4.2) becomes

$$\int_{-\infty}^{\infty} dz \exp \frac{ikz^2}{R_1} \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r \exp \left(- \frac{r^2 \cos^2\theta}{4t(D-Dm)} - \frac{r^2 \sin^2\theta}{4t D} + 2iKr\cos\theta + \frac{iKr^2}{R_1} \right)$$

of which the first part gives the result (4.3)

$$\text{The remainder} = \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r \exp \left\{ \frac{-r^2}{4t\, R_1 D(D-Dm)} \left[R_1(D-Dm)\sin^2\theta + R_1 D \cos^2\theta - 4tD(D-Dm)\, iK \right] + 2iKr\cos\theta \right\}$$

which in the same way as for (4.4) yields

$$\int_0^{2\pi} d\theta \left\{ \exp - \frac{K^2 \cos^2\theta \; 4t\, R_1 D(D-Dm)}{R_1(D-Dm)\sin^2\theta + R_1 D \cos^2\theta - 4tD(D-Dm)\, iK} \right\} \cdot$$

$$\left\{ \frac{2tD(D-Dm)R_1}{R_1(D-Dm)\sin^2\theta + R_1 D \cos^2\theta - 4tD(D-Dm)\, iK} + \frac{iK\cos\theta \quad \pi}{2} \cdot \frac{8 \left[tD(D-Dm)R_1 \right]^{\frac{3}{2}}}{\left[R_1(D-Dm)\sin^2\theta + R_1 D\cos^2\theta - 4tD(D-Dm)\, iK \right]^{\frac{3}{2}}} \right\}$$

$$\doteq \int_0^{2\pi} d\theta \left\{ \exp - \frac{4K^2 \cos^2\theta \; tD(D-Dm)}{(D-Dm)\sin^2\theta + D\cos^2\theta} \right\} \cdot$$

$$\left\{ \frac{2tD(D-Dm)\left[(D-Dm)\sin^2\theta + D\cos^2\theta \right]^{\frac{1}{2}} + 8iK\cos\theta \sqrt{\pi}\, t^{\frac{3}{2}} D^{\frac{3}{2}} (D-Dm)^{\frac{3}{2}}}{\left[(D-Dm)\sin^2\theta + \cos^2\theta \right]^{\frac{3}{2}}} \right.$$

4. Contd.

$$= \int_0^{2\pi} d\theta \left\{ \exp - \frac{4K^2 \cos^2\theta \, t D(D-Dm)}{(D-Dm)\sin^2\theta + D\cos^2\theta} \right\} \left\{ \frac{2t\, D(D-Dm)}{(D-Dm)\sin^2\theta + D\cos^2\theta} \right\} \quad (4.7)$$

$$= \int_0^{2\pi} d\theta \; 2t(D-Dm) \exp - 4(D-Dm)\, t\, K^2 \cos^2\theta \text{ if } D \gg Dm$$

$$= \left\{ 4\pi t(D-Dm) \exp - 2t(D-Dm)\, K^2 \right\} \left\{ 1 + t^2 K^4 (D-Dm)^2 + \qquad \right\}$$

$$\text{and } \frac{Esc}{Einc} = - \frac{\mu_0 e^2}{4m} \cdot No \cdot \frac{(D-Dm)^{\frac{1}{2}}}{D^{\frac{1}{2}}} \cdot \frac{\sqrt{\lambda}}{2\pi \sqrt{R_1}} \cdot \exp - 2t(D-Dm)K^2 \quad (4.8)$$

If $D \gg (D-Dm)$ then (4.7) tends to zero.

This means that the echo will vanish under these conditions.

This is probably not a very real case to consider.

For $\overline{B}$ in the XZ plane i.e. oblique to the meteor trail, (4.2) is

$$\int_{-\infty}^{\infty} dz \exp \frac{ikz^2}{R_1} \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r \exp\left\{ - \frac{r^2 \cos^2\theta}{4t(D-Dm \sin\phi)} - \frac{r^2 \sin^2\theta}{4t(D-Dm)} + \right.$$

$$\left. 2i K r \cos\theta + \frac{i K r^2}{R_1} \right\}$$

Carrying out the integrations in the same way as above the following results are obtained.

4. Cont.

$$\frac{E_{sc}}{E_{inc}} = -\frac{\mu_o e^2}{4m} \frac{N_o (D-D_m)^{\frac{1}{2}}}{(D - D_m \sin^2\phi)^{\frac{1}{2}}} \sqrt{\frac{2}{8}}\,\frac{\sqrt{\lambda}}{\pi\sqrt{R_1}} \exp - 2t(D-D_m)K^2 \qquad (4.10)$$

if $D \gg D_m$

$$\text{and } \frac{E_{sc}}{E_{inc}} = 0 \qquad (4.11)$$

if $D \gg (D-D_m)$

The scattered power $P_{sc} = P_{inc} \left(\frac{E_{sc}}{E_{inc}}\right)^2$

$$= \frac{P_t G^2}{4\pi R_1} \left(\frac{E_{sc}}{E_{inc}}\right)^2$$

where P_t = Transmitted Power, G = Transmitting Antenna Gain.

The power available at the receiver terminal is given by

$$P_R = P_{sc} \times \text{Capture Area of Receiving Antenna}$$

$$= P_{sc} \times \frac{G\lambda^2}{4\pi}$$

$$\therefore P_R = \frac{P_t G^2 \lambda^2}{16\pi^2 R_1^2} \times \left(\frac{E_{sc}}{E_{inc}}\right)^2 \qquad (4.12)$$

By inserting into (4.12) the scattering formulae previously derived, the following results are obtained :-

(a) Magnetic field parallel to axis of meteor trail

$$P_R = \left(\frac{\mu_o e^2}{4m}\right)^2 \cdot N_o^2 \cdot \frac{P_t G^2 \lambda^3}{64\pi^3 R_1^3} \exp - \frac{16\pi^2 t(D-D_m)}{\lambda^2} \qquad (4.13)$$

4. Cont.

(b) Magnetic field normal to axis of meteor trail

$$P_R = \left(\frac{\mu_o e^2}{4m}\right)^2 N_o^2 \frac{(D-Dm)}{D} \frac{Pt\, G^2 \lambda^3}{64\pi^4 R_1^3} \exp - \frac{16\pi^2 t(D-Dm)}{\lambda^2}$$

for Dm <<D (4.14)

$$P_R = 0$$

for (D-Dm)<<D (4.15)

(c) Magnetic field oblique to axis of meteor trail

$$P_R = \left(\frac{\mu_e e^2}{4m}\right)^2 N_o^2 \frac{(D-Dm)}{(D-Dm \sin \phi)} \frac{Pt\, G^2 \lambda^3}{64\pi^4 R_1^3} \exp -$$

$$\frac{16\pi^2 t(D-Dm)}{\lambda^2}$$ for Dm <<D (4.16)

$$P_R = 0$$

for (D-Dm) <<D (4.17)

For greater accuracy the formulae (4.13), (4.14) and (4.16) should be multiplied by $\left(1 + \frac{32\pi^4 t^2 (D-Dm)^2}{\lambda^4} + \frac{512\pi^8 t^4 (D-Dm)^4}{\lambda^8} + \right.$

These results may be compared with the formulae obtained neglecting the Earth's magnetic field and given by Kaiser (1953)

$$P_R = \left(\frac{\mu_o e^2}{4m}\right)^2 N_o^2 \frac{Pt\, G^2 \lambda^3}{64\pi^4 R^3} \exp - \frac{16\pi^2 tD}{\lambda^2}$$

It is apparent that ions will contribute very little to the reflected signal power.

5. The "Magnetic" Diffusion Coefficient.

A charged particle in motion in a magnetic field with velocity $\bar{v}$ attains an additional component of velocity given by $\bar{v} \times \bar{B}$. This means that an electron moving parallel to the field will be uneffected by the field while an electron moving normal to the field will tend to spiral around the field lines. It is then apparent that electrons moving with the velocity of diffusion under the action of a concentration gradient will tend to spread out more slowly in directions normal to the magnetic field than along the field.

This effect is discussed by Chapman and Cowling (1952) where it is shown that the velocity of diffusion normal to the magnetic field is reduced in the ratio $1 : \left(1 + \frac{\omega^2}{\nu^2}\right)$. Here ω = the gyro-magnetic frequency and ν = the collisional frequency. The diffusion coefficient can then be said to be reduced in this ratio so that

$$(D - D_m) = D\left(\frac{\nu^2}{\nu^2 + \omega^2}\right) \qquad (5.1)$$

The gyromagnetic frequency ω is given by $\frac{eB}{m}$ with e and m the particle charge and mass respectively and B the magnetic field strength.

The magnitude of Earth's magnetic field is taken as 0.6×10^{-4} webers/square meter at ground level in the neighbourhood of Adelaide. The field strength reduces with height by about 6 milligauss per 40 kilometers but is slightly modified by atmospheric charges so that in the region of the E-layer at 95 kilometers

5. Cont.

the magnitude is 0.586×10^{-4} webers/square meter. This height is typical for meteors. These figures for the Earth's field were kindly communicated by Mr. B. Rofe of The Weapons Research Establishment, Salisbury, South Australia whose help in this matter is gratefully acknowledged.

Taking $e = 1.6 \times 10^{-19}$ coulombs and $m = 9.1 \times 10^{-31}$ kilograms, ω for electrons $= 10^{7}$.

The collisional frequency for electrons in the atmosphere is difficult to measure directly. Deductions have been made from measurements made in pure nitrogen at low electron mean energies and an empirical formula, quoted by Crompton, Huxley and Sutton (1953), derived. A later result communicated privately by Dr. Crompton is used here, this being $\nu = 1.2 \times 10^{8} p$ where p is the atmospheric pressure in millimeters of mercury.

The atmospheric pressure expressed in millimeter of mercury is given as 1.6×10^{-4} at 100 kilometers, 0.9×10^{-3} at 95 kilometers, and 1.0×10^{-2} at 80 kilometers. Hence the electron collisional frequency $\nu = 2.0 \times 10^{4}$ at 100 kilometers, 1.0×10^{5} at 95 kilometers, and 1.2×10^{6} at 80 kilometers.

Using (5.1) $(D-Dm) = D \times 4.0 \times 10^{-6}$ at 100 kilometers.

$= D \times 1.0 \times 10^{-4}$ at 95 kilometers.

$= D \times 1.4 \times 10^{-2}$ at 80 kilometers.

Electronic diffusion is strongly effected by the Earth's magnetic field right through the meteor region. The explanation is found in consideration of the diffusion process. In the absence

5. Cont.

of external forces, particles moving with initial thermal velocities will tend, after encounters with other particles, to move from regions of high concentration to regions of lower concentration and travel in straight lines between collisions. When a magnetic field is present, the charged particles tend to spiral the field lines and in the region considered above are describing a minimum of 100 orbits between collisions. The effect of a collision will be to disturb the orbit slightly, the tendency being to increase the radius of the orbit again following the concentration gradient but moving in highly curved paths rather than straight lines.

This means that the meteor trail would tend to attain a high degree of ellipticity and the effect on the radio signal power reflected would be great. By choosing the value of $\sqrt{}$ corresponding to 95 kilometers, so that $(D-D_m) = D \times 10^{-4}$, the results of section 4 show that the magnetic field has no effect on P_R when the meteor is parallel to the field but causes a reduction of up to 10^4 times when the trail is nowmal or oblique to the field. These variations in the reflected signal are much larger than indicated by practical results. The positive ions, although ineffective as scatterers of the radio waves, are likely to exert considerable influence on the motion of the electrons and so modify these results.

6. Ambipolar Diffusion

The diffusion coefficient for electrons in the atmosphere at 95 kilometers is about 10^4 square meters per second, while that for ions is about 10, that is 1,000 times smaller than for electrons. Immediately after formation a meteor trail will start to spread out with electrons moving out more rapidly than ions. The separation of the charges will produce an electrostatic field tending to oppose the separation. This problem has been discussed by Huxley (1952) to whom the following relations (6.1) to (6.4) are due.

The equations of diffusion for electrons and ions can be written:-

$$\left.\begin{aligned} \frac{\partial n_e}{\partial t} &= D_e \nabla^2 n_e - \operatorname{div}(n_e \overline{W}_e) \\ \frac{\partial n_i}{\partial t} &= D_i \nabla^2 n_i + \operatorname{div}(n_i \overline{W}_i) \end{aligned}\right\} \qquad (6.1)$$

where n = number density

D = coefficient of diffusion

$\overline{W}$ = drift velocity in electric field $\overline{E}$

$$\operatorname{Div} \overline{E} = (n_e - n_i) \frac{e}{\varepsilon_0}$$

Subscripts e and i refer to electrons and ions respectively.

Since $\overline{W} = M\overline{E}$ where M = mobility (6.1) can be written

$$\left.\begin{aligned} M_i \frac{\partial n_e}{\partial t} &= M_i D_e \nabla^2 n_e - M_i M_e \operatorname{div}(n_e \overline{E}) \\ M_e \frac{\partial n_i}{\partial t} &= M_e D_i \nabla^2 n_i + M_e M_i \operatorname{div}(n_i \overline{E}) \end{aligned}\right\} \qquad (6.2)$$

Adding, $\frac{\partial}{\partial t}(M_i n_e + M_e n_i) = \nabla^2 (M_i D_e n_e + M_e D_i n_i) - M_i M_e \operatorname{div}(n_e - n_i)\overline{E}$

6. Cont.

Assume that $ne = ni = n$ (6.3)

and $\frac{\partial n}{\partial t} = D_A \nabla^2 n$ (6.4)

where $D_A = \frac{Mi\ De + Me\ Di}{Me + Mi}$

On the basis of Kinetic Theory it has been shown by Kaiser (1953) that $\frac{Me}{De} = \frac{e}{K\ Te}$

where e is the electronic charge, K is Boltzmann's constant, and Te is the electron temperature.

Likewise $\frac{Mi}{Di} = \frac{e}{K\ Ti}$

$$\therefore \quad \frac{Me\ Di + Mi\ De}{Me + Mi} = Di + Mi\frac{De}{Me} \qquad \text{if } Me \gg Mi$$

$$= Di + \frac{Mi\ K\ Te}{e}$$

$$= \frac{Mi\ K\ Ti}{e} + \frac{Mi\ K\ Te}{e}$$

$$= \frac{2Mi\ K\ Ti}{e} \qquad \text{if } Te = Ti$$

That is $D_A = 2Di$ (6.5)

These relations rely on the validity of (6.3), that is the assumption that ne = ni. Although this is certainly true at the instant of formation of a meteor trail it is not obvious that it should remain true. It is desirable that further investigation be made.

If it is assumed as a first approximation that the ions remain near the origin, then the equation for electrons is:-

$$\frac{\partial n}{\partial t} = (D - Dm)\frac{\partial^2 n}{\partial x^2} + (D-Dm)\frac{\partial^2 n}{\partial y^2} + D\frac{\partial^2 n}{\partial z^2} - M \operatorname{div}(n.\bar{E})$$

where $\bar{B}$ is taken to be //Z

$$= D\frac{\partial^2 n}{\partial a^2} + D\frac{\partial^2 n}{\partial b^2} + D\frac{\partial^2 n}{\partial z^2} - Mn \operatorname{div} E - M\bar{E} \operatorname{grad} n$$

where $a = \frac{\sqrt{D}}{\sqrt{D-Dm}}\, x$, $b = \frac{\sqrt{D}}{\sqrt{D-Dm}}\, y$

This can be further approximated by leaving off the last term on the right so that

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial a^2} + D\frac{\partial^2 n}{\partial b^2} + D\frac{\partial^2 n}{\partial z^2} - M e n^2 \qquad (6.6)$$

Write $\bar{n} = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} da\, db\, dz\, n \exp i\,(ka + lb + mz)$

and $$\frac{d\bar{n}}{dt} + D\,(k^2 + l^2 + m^2)\,\bar{n} = -\bar{S} \qquad (6.7)$$

where $\bar{S}$ is the Fourier Transform of $M e m^2$

The solution of the homogeneous equation corresponding to (6.7) is given by :- $\bar{n} = \bar{n}_o \exp - Dt\,(k^2 + l^2 + m^2)$

where $n = n_o$ at $t = o$

Now $\bar{S} = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M e\, n.n \exp i\,(ka + lb + mz)\, da\, db\, dz$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{n}\,(k-k', l-l', m-m')\,\bar{n}\,(k', l', m')\, dk'dl'dm'$$

6. Cont.

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{n}_o \cdot \bar{n}_o \left\{ \exp - Dt \left[(k - k')^2 + (l - l')^2 + (m - m')^2 + k'^2 + l'^2 + m'^2 \right] - \exp - Dt \,(k'^2 + l'^2 + m'^2) \right\} dk'\,dl'\,dm'$$

In this last equation account is taken of the presence at the origin of the ions so as to make $\bar{S} = o$ when $t = o$

Integrating gives $\bar{S} = \frac{1}{(4Dt)^{\frac{3}{2}}} \cdot \bar{n}_o^{\,2} \left[\left\{ \exp - \frac{Dt}{2} (k^2 + l^2 + m^2) \right\} - \sqrt{8} \right]$

$$\therefore \bar{n} = \bar{n}_o \exp - Dt\,(k^2 + l^2 + m^2) - \exp - Dt\,(k^2 + l^2 + m^2) \times$$

$$\int^{t} dt \frac{\bar{n}_o^{\,2}}{(4Dt)^{\frac{3}{2}}} \left[\exp \frac{Dt}{2} (k^2 + l^2 + m^2) - \sqrt{8} \exp Dt\,(k^2 + l^2 + m^2) \right]$$

The integral $= \frac{\bar{n}_o^{\,2}}{(4Dt)^{\frac{3}{2}}} \int^{t} dt\, t^{-\frac{3}{2}} \left(\exp \frac{At}{2} - \sqrt{8} \exp At\right)$

where $A = D(k^2 + l^2 + m^2)$

Integration by parts gives :-

$$-2t^{-\frac{1}{2}} \left(\exp \frac{At}{2} - \sqrt{8} \exp At\right) + 4t^{\frac{1}{2}} \left(\frac{A}{2} \exp \frac{At}{2} - \sqrt{8}A \exp At\right) - \frac{8}{3} t^{\frac{3}{2}} \left(\frac{A^2}{4} \exp \frac{A}{2} - \sqrt{8}A^2 \exp At\right) + \frac{16}{15} t^{\frac{5}{2}} \left(\frac{A^3}{8} \exp \frac{At}{2} - \sqrt{8}A^3 \exp At\right)$$

where higher order terms are discarded for small t.

6. Cont.

Hence $\bar{n} = \bar{n}_o \exp - Dt\,(k^2 + l^2 + m^2) + \dfrac{\bar{n}_o^{\,2}}{4D^{\frac{3}{2}}t^{\frac{1}{2}}} \Big\{ \Big[\exp - \dfrac{Dt}{2}$

$(k^2 + l^2 + m^2)\Big] \;\; -\sqrt{8}\Big\}$

$- \dfrac{\bar{n}_o^{\,2}}{2D^{\frac{1}{2}}} (k^2 + l^2 + m^2)\, t^{\frac{1}{2}} \Big\{\Big[\frac{1}{2} \exp - \dfrac{Dt}{2} (k^2 + l^2 + m^2)\Big] \;\; -\sqrt{8}\Big\}$

$+ \dfrac{\bar{n}_o^{\,2}}{3} (k^2 + l^2 + m^2)^2\, t^{\frac{3}{2}} D^{\frac{1}{2}} \Big\{ \Big[\frac{1}{4} \exp - \dfrac{Dt}{2} (k^2 + l^2 + m^2)\Big] -\sqrt{8}\Big\}$

$- 2\, \dfrac{\bar{n}_o^{\,2}}{15} (k^2 + l^2 + m^2)^3\, t^{\frac{5}{2}} D^{\frac{3}{2}} \Big\{\Big[\frac{1}{8} \exp - \dfrac{Dt}{2} (k^2 + l^2 + m^2)\Big] -\sqrt{8}\Big\}$

Inverting this gives :-

$$n = \frac{n_o}{8(\pi tD)^{\frac{3}{2}}} \exp - (a^2 + b^2 + z^2)/\,4tD$$

$$+ \frac{M\,e\,n_o^{\,2}}{8\pi^{\frac{3}{2}} D^3\, t^2} \exp - (a^2 + b^2 + z^2)/\,2tD$$

$$+ \frac{3M\,e\,n_o^{\,2}}{2^{\frac{7}{2}} \pi^{\frac{3}{2}} D^3\, t^2} \exp - (a^2 + b^2 + z^2)/\,2tD$$

$$- \frac{M\,e n_o^{\,2}}{2^{\frac{7}{2}} \pi^{\frac{3}{2}} D^4 t^3} \exp - (a^2 + b^2\, z^2)/\,2tD$$

$$+ \frac{5M.e\,n_o^{\,2}}{2^{\frac{3}{2}} \pi^{\frac{3}{2}} D^3 t^2} \exp - (a^2 + b^2 + z^2)/\,2tD$$

$$- \frac{10M\,e\,n_o^{\,2}(a^2 + b^2 + z^2)}{3.\,2^{\frac{3}{2}}\,\pi^{\frac{3}{2}}\,D^4 t^3} \exp - (a^2 + b^2 + z^2)/2tD$$

6. cont.

$$+ \frac{M e n_o^2 (a^2 + b^2 + z^2)^2}{3 . 2^{3/2} \pi^{3/2} D^5 t^4} \quad \exp - (a^2 + b^2 + z^2)/2tD$$

Collecting terms and integrating with respect to z to represent a line source :-

$$n = \frac{N_o}{4\pi t(D-Dm)} \exp - \frac{(x^2 + y^2)}{4t(D-Dm)} + \frac{2 Me N_o^2}{\pi t^{3/2}(D-Dm)^{5/2}} \exp - \frac{(x^2 + y^2)}{2t(D-Dm)}$$

$$- \frac{3 Me N_o^2 (x^2 + y^2)}{2\pi t^{5/2}(D-Dm)^{7/2}} \exp - \frac{(x^2 + y^2)}{2t(D-Dm)}$$

$$+ \frac{Me N_o^2}{6\pi t^{7/2}} \cdot \frac{(x^2 + y^2)^2}{(D-Dm)^{9/2}} \exp - \frac{(x^2 + y^2)}{2t(D-Dm)} \qquad (6.8)$$

It is obvious from this very approximate calculation that the electric field built up by separation of the charges will have a considerable effect on the electron distribution. It will be difficult to estimate the magnitude of the effect from this calculation with any degree of accuracy and better methods are required.

7. Dimensional Analysis

The differential equations for electrons and ions are :-

$$\frac{\partial n_e}{\partial t} = D_e \nabla^2 n_e - M_e \operatorname{div} (n_e \bar{E})$$

$$\frac{\partial n_i}{\partial t} = D_i \nabla^2 n_i + M_i \operatorname{div} (n_i \bar{E})$$

writing $N = \dfrac{M_i n_e + M_e n_i}{M_e + M_i}$

and $n = n_e - n_i$

so that $n_e = N + \dfrac{M_e n}{M_e + M_i}$

and $n_i = N - \dfrac{M_i n}{M_e + M_i}$

Then $\dfrac{\partial}{\partial t} \left(N + \dfrac{M_e n}{M_e + M_i}\right) = D_e \nabla^2 \left(N + \dfrac{M_e n}{M_e + M_i}\right) -$

$M_e \operatorname{div} \left(N + \dfrac{M_e n}{M_e + M_i}\right) . \bar{E}$

and $\dfrac{\partial}{\partial t} \left(N - \dfrac{M_i n}{M_e + M_i}\right) = D_i \nabla^2 \left(N - \dfrac{M_i n}{M_e + M_i}\right) +$

$M_i \operatorname{div} \left(N - \dfrac{M_i n}{M_e + M_i}\right) . \bar{E}$

$\therefore (M_i + M_e) \dfrac{\partial N}{\partial t} = (M_i D_e + M_e D_i) \nabla^2 N + (D_e - D_i)$

$\dfrac{M_i M_e}{M_e + M_i} \nabla^2 n - M_e M_i \operatorname{div} (n\bar{E})$

7. Cont.

$$\text{or } \frac{\partial N}{\partial t} = \left(\frac{M_i D_e + M_e D_i}{M_e + M_i}\right) \nabla^2 N + \frac{(D_e - D_i) M_e M_i}{(M_e + M_i)^2} \nabla^2 n$$

$$- \frac{M_e M_i}{M_e + M_i} \operatorname{div} (n \overline{E}) \qquad (7.1)$$

$$\text{and } \frac{\partial n}{\partial t} = \left(\frac{D_e M_e + D_i M_i}{M_e + M_i}\right) \nabla^2 n + (D_e - D_i) \nabla^2 N - (M_e + M_i)$$

$$\operatorname{div} (N \overline{E}) - (M_e - M_i) \operatorname{div} (n \overline{E}) \qquad (7.2)$$

In order to estimate the size of the various terms in these equations it is useful to express the parameters in dimensionless form. The dimensions of the parameters in terms of length (L), time (T), mass (M) and charge (Q) are :-

Diffusion Coefficient D_e or D_i $= L^2 T^{-1}$

Mobility M_e or M_i $= M^{-1} T Q$

Charge e $= Q$

Electric field strength E $= M L T^{-2} Q^{-1}$

Dielectric Constant ε_o $= M^{-1} L^{-3} T^2 Q^2$

The last parameter ε_o is included since $\operatorname{div} \overline{E} = \frac{n e}{\varepsilon_o}$

7. Cont.

In order to express E in dimensionless form write:-

$$L^{2a}\ T^{-a}\ M^{-b}\ T^{b}\ Q^{b}\ Q^{c}\ M^{-d}\ L^{-3d}\ T^{2d}\ Q^{2d}$$

L: $2a - 3d = 1$, $a = \frac{3}{2}d + \frac{1}{2}$, $a = 2$

M: $-b - d = 1$, $b = -d - 1$, $b = -2$

T: $-a + b + 2d = -2$, $-\frac{3}{2}d - \frac{1}{2} - d - 1 + 2d = -2$, $d = 1$

Q: $b + c + 2d = -1$, $c = -1$

Hence a dimensionless form of E $= \dfrac{M_e^2 \cdot e}{D_e^2 \cdot \varepsilon_o} \cdot E = \mathcal{E}$ say.

It looks as if the equations (7.1) and(7.2) could be given in terms of the diffusion coefficients, mobility and a length (a). By writing:-

L: $2a - 3d = 1$, $a = \frac{3}{2}d + \frac{1}{2}$, $a = -1$

M: $-b - d = 0$, $b = -d$, $b = 1$

T: $-a + b + 2d = 0$, $-\frac{1}{2} - \frac{3}{2}d - d + 2d = 0$, $d = -1$

Q: $b + c + 2d = 0$, $c = 1$

it is seen that $\dfrac{M_e \cdot e}{\varepsilon_o D_e}$ has the dimension of length

and $\alpha = \dfrac{M_e\ e}{a\ \varepsilon_o\ D_e}$ is dimensionless

also $\beta = \dfrac{D_i}{D_e}$ and $\gamma = \dfrac{M_i}{M_e}$ are dimensionless.

hence $E = \dfrac{D_e^2\ \varepsilon_o}{M_e^2\ e} \cdot \mathcal{E} = \dfrac{D_e\ \mathcal{E}}{M_e\ a\ \alpha}$

7. Cont.

∴ (7.1) can be written

$$\frac{\partial N}{\partial t} = \frac{M_e D_e \left(\frac{M_i}{M_e} + \frac{D_i}{D_e}\right)}{M_e \left(1 + \frac{M_i}{M_e}\right)} \nabla^2 N + \frac{D_e M_e^2 \left(1 - \frac{D_i}{D_e}\right) \frac{M_i}{M_e}}{M_e^2 \left(1 + \frac{M_i}{M_e}\right)^2} \cdot \nabla^2 n$$

$$- \frac{M_e^2 \left(\frac{M_i}{M_e}\right)}{M_e \left(1 + \frac{M_i}{M_e}\right)} \operatorname{div} n \overline{E}$$

$$= \frac{D_e (\gamma + \beta)}{1 + \gamma} \cdot \nabla^2 N + \frac{D_e \gamma (1 - \beta)}{(1 + \gamma)^2} \cdot \nabla^2 n - \frac{M_e \gamma}{1 + \gamma} \operatorname{div}$$

$$\left(\frac{n D_e \mathcal{E}}{M_e a \alpha} \right)$$

Now write $\tau = \frac{t D_e}{a^2}$, $\zeta = \frac{x}{a}$ and change the variables in the derivatives so that

$$\frac{\partial N}{\partial \tau} = \frac{\gamma + \beta}{1 + \gamma} \cdot \nabla_\zeta^2 N + \frac{\gamma(1 - \beta)}{(1 + \gamma)^2} \cdot \nabla_\zeta^2 n - \frac{\gamma}{\alpha (1 + \gamma)} \operatorname{div}_\zeta$$

$$(n \mathcal{E}) \qquad (7.3)$$

In the same way

$$\frac{\partial n}{\partial \tau} = \frac{1 + \beta \gamma}{1 + \gamma} \cdot \nabla_\zeta^2 n + (1 - \beta) \nabla_\zeta^2 N - \frac{1 + \gamma}{\alpha} \operatorname{div}_\zeta (N \mathcal{E})$$

$$- \frac{1 - \gamma}{\alpha} \operatorname{div}_\zeta (n \mathcal{E}) \qquad (7.4)$$

7. Cont.

γ and β reflect the electron - ion mass ratio and can each be taken to be 10^{-3}.

$$\alpha = \frac{M_e\, e}{a\, \varepsilon_o\, D_e} = \frac{10^8 \times 10^{-19}}{a \times 10^{-11} \times 10^4} = \frac{1}{a \times 10^4}$$

A typical length for the system could be the length corresponding to the duration of the radio echo from a meteor trail. A duration of the order of 0.1 second is quite common, so in terms of dimensions

$$L = \sqrt{T\, D_e} = \text{about } 30$$

that is $a = 30$

$$\therefore \quad \alpha = 3 \times 10^{-6}$$

and
$$\frac{\partial N}{\partial \tau} = 2 \times 10^{-3} \nabla_\zeta^2 N + 10^{-3} \nabla_\zeta^2 n - 300 \operatorname{div}_\zeta (n \mathcal{E}) \qquad (7.5)$$

$$\frac{\partial n}{\partial \tau} = \nabla_\zeta^2 n + \nabla_\zeta^2 N - 3 \times 10^5 \operatorname{div}_\zeta (n \mathcal{E}) - 3 \times 10^5 \operatorname{div}_\zeta (N \mathcal{E}) \qquad (7.6)$$

If the terms involving $\mathcal{E}$ are regarded as a perturbation then the equations to be solved are:-

$$\frac{\partial N}{\partial \tau} = 2A \nabla^2 N + A \nabla^2 n \qquad (7.7)$$

$$\frac{\partial n}{\partial \tau} = \nabla^2 n + \nabla^2 N \qquad (7.8)$$

in which $A = 10^{-3}$

7. Cont.

Writing $N' = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} d^3 k\, N \exp i\, \bar{k}.\, \xi$

and $n' = \frac{1}{2\pi^{\frac{3}{2}}} \int_{-\infty}^{\infty} d^3 k\, n \exp i\, \bar{k}.\xi$

then $\frac{dN'}{d\tau} = -\, 2A\bar{k}^2 N' - A\bar{k}^2\, n'$

$\frac{dn'}{d\tau} = -\, \bar{k}^2\, n' - \bar{k}^2\, N'$

Now let $y_1\,(p) = \int_0^{\infty} d\tau\, N'\,(\tau) \exp - p\tau$

$y_2\,(p) = \int_0^{\infty} d\tau\, n'\,(\tau) \exp - p\tau$

and $N' = N'_o$ for $\tau = o$

$n' = o$ for $\tau = o$

Then $-\, N'_o + p\, y_1 + 2\, A\bar{k}^2\, y_1 + A\bar{k}^2\, y_2 = 0$

$p\, y_2 + \bar{k}^2\, y_2 + \bar{k}^2\, y_1 = 0$

$y_2 = -\, \frac{\bar{k}^2\, y_1}{p + \bar{k}^2}$

$-\, N'_o + p\, y_1 + 2A\bar{k}^2\, y_1 - \frac{A\bar{k}^4\, y_1}{p + \bar{k}^2} = 0$

$$y_1 = \frac{N'_o}{p + 2A\bar{k}^2 - \frac{A\bar{k}^4}{p + \bar{k}^2}} = \frac{N'_o\, (p + \bar{k}^2)}{p^2 + p\bar{k}^2\, (2A + 1) + A\bar{k}^4} \qquad (7.9)$$

7. Cont.

and hence $y_2 = \dfrac{-N_o' \bar{k}^2}{p^2 + p\bar{k}^2 (2A + 1) + A\bar{k}^4}$ (7.10)

(7.9) and (7.10) have poles at $p = \bar{k}^2 \left[-(1 + 2A) \pm \sqrt{1 + 4A^2} \right] /2$

That is $p_o = -\bar{k}^2 (1 + 2A - \sqrt{1 + 4A^2})/2$

$p_1 = -\bar{k}^2 (1 + 2A + \sqrt{1 + 4A^2})/2$

So $y_1 = \dfrac{B}{p-p_o} + \dfrac{C}{p-p_1}$ and $B(p-p_1) + C (p-p_o) = p + \bar{k}^2$

$$B = \frac{p_o + \bar{k}^2}{p_o - p_1}, \quad C = \frac{p_1 + \bar{k}^2}{p_1 - p_o}$$

$y_2 = \dfrac{B'}{p-p_o} + \dfrac{C'}{p-p_1}$ and $B' (p-p_1) + C' (p-p_o) = - \bar{k}^2$

$$B' = \frac{-\bar{k}^2}{p_o-p_1}, \quad C' = \frac{-\bar{k}^2}{p_1 - p_o}$$

Resulting in $N' = \dfrac{N_o' (-1 + 2A + \sqrt{1 + 4A^2})}{2 \sqrt{1 + 4A^2}} \exp - \left[\dfrac{\bar{k}^2}{2} \tau (1 + 2A + \sqrt{1 + 4A^2}) \right]$

$- \dfrac{N_o' (-1 + 2A - \sqrt{1 + 4A^2})}{2 \sqrt{1 + 4A^2}} \exp - \left[\dfrac{\bar{k}^2}{2} \tau(1 + 2A - \sqrt{1 + 4A^2}) \right]$

and $n' = \dfrac{N_o'}{\sqrt{1 + 4A^2}} \exp - \left[\dfrac{\bar{k}^2}{2} \tau (1 + 2A + \sqrt{1 + 4A^2}) \right]$

7. Cont.

$$- \frac{N_o'}{\sqrt{1 + 4A^2}} \exp - \left[\frac{\bar{k}^2}{2} \tau \left(1 + 2A - \sqrt{1 + 4A^2}\right) \right]$$

And to a reasonable approximation:-

$$N' = A N_o' \exp - \bar{k}^2 \tau (1 + A) - (A - 1) N_o' \exp - \bar{k}^2 \tau A \qquad (7.11)$$

$$n' = N_o' \exp - \bar{k}^2 \tau (1 + A) - N_o' \exp - \bar{k}^2 \tau A \qquad (7.12)$$

For a line source //Z (7.12) gives for n

$$n = \frac{N_o}{8\pi\tau \; (1 + A)} \exp - \frac{\zeta^2}{4\tau(1 + A)} - \frac{N_o}{8\pi\tau A} \exp - \frac{\zeta^2}{4 \tau A} \qquad (7.13)$$

Now $\nabla^2 \phi = \frac{e\, n}{\varepsilon_o}$

and a dimensionless ϕ, say $\bar{\phi} = \phi \frac{M_e^2 e}{a D_e^2 \varepsilon_o}$

$$\therefore \; \frac{a D_e^2 \varepsilon_o}{M_e^2 e} \cdot \frac{1}{a^2} \nabla_\zeta^2 \bar{\phi} = \frac{e . n}{\varepsilon_o}$$

and $\nabla_\zeta^2 \bar{\phi} = \alpha^2 a^3 n$

$= 3 \times 10^{-7} n$

So with cylindrical symmetry :-

$$\frac{1}{\zeta} \frac{d}{d\zeta} \left(\zeta \frac{d\bar{\phi}}{d\zeta}\right) = 3 \times 10^{-7} n$$

7. Cont.

$$\therefore \frac{d\phi}{d\zeta} = \frac{3 \times 10^{-7}}{\zeta} \left(\int^{\zeta} d\zeta' \, \zeta' \, n(\zeta') \right)$$

$$= \frac{3 \times 10^{-7}}{\zeta} \left(\frac{N_o}{4\pi} \exp - \frac{\zeta^2}{4\tau(1+A)} - \frac{N_o}{4\pi} \exp - \frac{\zeta^2}{4\tau A} \right) + \frac{\lambda}{\zeta}$$

Continuity at the origin gives

$$\frac{d\phi}{d\zeta} = \frac{3 \times 10^{-7}}{\zeta} \left(\frac{N_o}{4\pi} \exp - \frac{\zeta^2}{4\tau(1+A)} - \frac{N_o}{4\pi} \exp - \frac{\zeta^2}{4\tau A} \right)$$

but $\mathcal{E} = - \dfrac{d\phi}{d\zeta}$

$$= \frac{3 \times 10^{-7}}{\zeta} \left(\frac{N_o}{4\pi} \exp - \frac{\zeta^2}{4\tau A} - \frac{N_o}{4\pi} \exp - \frac{\zeta^2}{4\tau(1+A)} \right)$$

$$\therefore \quad \zeta n \mathcal{E} = 3 \times 10^{-7} \left[\frac{N_o^2}{32\pi^2 \tau(1+A)} \exp - \frac{\zeta^2}{2\tau(1+A)} - \frac{N_o^2}{32\pi^2 \tau(1+A)} \exp - \frac{\zeta^2(1+2A)}{4A\tau} - \frac{N_o^2}{32\pi^2 \tau A} \exp - \frac{\zeta^2 (1+2A)}{4A\tau} + \frac{N_o^2}{32\pi^2 \tau A} \exp - \frac{\zeta^2}{2\tau A} \right]$$

Hence $\operatorname{div}_\zeta (n\mathcal{E}) \doteqdot 3 \times 10^{-7} \dfrac{N_o^2}{32\pi^2 \tau^2} \exp - \dfrac{\zeta^2}{2\tau}$ (7.14) where

the terms with A in the denominator of the exponent are omitted.

In the same way (7.9) gives for a line source //Z:-

$$N = \frac{A N_o}{8\pi\tau(1+A)} \exp - \frac{\zeta^2}{4\tau(1+A)} - \frac{(A-1)N_o}{8\pi\tau A} \exp - \frac{\zeta^2}{4 \tau A}$$

7. Cont.

$$\therefore \frac{\partial N}{\partial \tau} = \frac{N_o A \zeta^2}{32\pi \tau^3 (1+A)^2} \exp - \frac{\zeta^2}{4\tau(1+A)} - \frac{A N_o}{8\pi \tau^2 (1+A)}$$

$$\exp - \frac{\zeta^2}{4\tau(1+A)}$$

$$- \frac{N_o (A-1) \zeta^2}{32\pi \tau^3 A^2} \exp - \frac{\zeta^2}{4 \tau A} + \frac{(A-1) N_o}{8\pi \tau^2 A} \exp - \frac{\zeta^2}{4 \tau A}$$

$$\doteqdot A \left[\frac{N_o \zeta^2}{32\pi \tau^2} - \frac{N_o}{8\pi \tau^2} \right] \exp - \frac{\zeta^2}{4\tau} \qquad (7.15)$$

Comparing the sizes of (7.14) and (7.15) it is seen that the perturbation becomes as large as the linear terms in (7.5) at a small value of $\tau \doteqdot \frac{1}{80}$.

This means that the electric field associated with charge separation very soon reaches a large size and will tend to dominate the motion of the particles. It is desirable to attempt a more exact solution of the equations.

8. Numerical Solution

The diffusion equations for electrons and ions can be written in dimensionless form :-

$$\frac{\partial n_e}{\partial \tau} = \nabla^2 n_e - \frac{1}{\alpha} \operatorname{div} (n_e \mathcal{E})$$

$$\frac{\partial n_i}{\partial \tau} = \beta \nabla^2 n_i + \frac{\Upsilon}{\alpha} \operatorname{div} (n_i \mathcal{E})$$

In cylindrical co-ordinates these become

$$\frac{\partial n_e}{\partial \tau} = \frac{1}{\zeta} \frac{\partial}{\partial \zeta} \left(\zeta \frac{\partial n_e}{\partial \zeta}\right) + \frac{\alpha^2 a^3}{\alpha} n_e (n_e - n_i) - \frac{1}{\alpha} \mathcal{E} \cdot \frac{\partial n_e}{\partial \zeta}$$

$$\frac{\partial n_i}{\partial \tau} = \frac{\beta}{\zeta} \frac{\partial}{\partial \zeta} \left(\zeta \frac{\partial n_i}{\partial \zeta}\right) + \frac{\Upsilon \alpha^2 a^3}{\alpha} n_i (n_e - n_i) + \frac{\Upsilon}{\alpha} \mathcal{E} \cdot \frac{n_i}{\zeta}$$

As shown in section 7, the electric field $\mathcal{E}$ is given by

$$\mathcal{E} = -\frac{d \phi}{d \zeta} = -\alpha^2 a^3 \cdot \frac{1}{\zeta} \int^{\zeta} \zeta' (n_e - n_i)_{\zeta'} \, d\zeta'$$

$$\therefore \frac{\partial n_e}{\partial \tau} = \frac{\partial^2 n_e}{\partial \zeta^2} + \frac{1}{\zeta} \frac{\partial n_e}{\partial \zeta} - \frac{n_e (n_e - n_i)}{11} + \frac{1}{11 \zeta} \left(\frac{\partial n_e}{\partial \zeta}\right) \left[\int^{\zeta} \zeta' (n_e - n_i)_{\zeta'} \, d\zeta' \right] \qquad (8.1)$$

$$10^3 \frac{\partial n_i}{\partial \tau} = \frac{\partial^2 n_i}{\partial \zeta^2} + \frac{1}{\zeta} \frac{\partial n_i}{\partial \zeta} + \frac{n_i (n_e - n_i)}{11} - \frac{1}{11 \zeta} \left(\frac{\partial n_i}{\partial \zeta}\right) \left[\int^{\zeta} \zeta' (n_e - n_i)_{\zeta'} \, d\zeta' \right] \qquad (8.2)$$

8. Cont.

These equations which are of the parabolic type can be solved by numerical methods provided a high speed electronic computer is available.

If the space (ζ) variables are replaced by finite differences, there results a system of ordinary differential equations since all the variables become functions of τ only, and one differential equation appears at each interval. So in the equation for electrons replace n_e (ζ, τ) by A (jh, τ) where h is a suitable step length along the ζ axis and j runs from 0 to 20. Strictly j should be taken to a value such that at the end point A (jh, τ) remains constant for all τ thus imposing a boundary condition. However 20 is taken here as the upper limit in order to keep the calculation within practical limits even using a high speed computer. To simplify the notation A (jh, τ) is now called A (J) it being remembered that A (J) is a function of τ only.

In the same way n_i (ζ, τ) becomes B (J).

The equations to solved are now :-

$$\frac{dA(J)}{d\tau} = \left[A(J+1) - 2A(J) + A(J-1)\right] / h^2 + \frac{1}{Jh}\left[A(J+1) - A(J-1)\right] /2h$$

$$- \frac{1}{11} \cdot A(J)\left[A(J) - B(J)\right] + \frac{1}{11Jh} \cdot \frac{\left[A(J+1) - A(J-1)\right]}{2h} \times SI(J) \tag{8.3}$$

$$\frac{dB(J)}{d\tau} = \left[B(J+1) - 2B(J) + B(J+1)\right] / h^2 + \frac{1}{Jh} \cdot \left[B(J+1) - B(J-1)\right] /2h$$

$$+ \frac{1}{11} \cdot B(J)\left[A(J) - B(J)\right] - \frac{1}{11Jh} \cdot \frac{\left[B(J+1) - B(J-1)\right]}{2h} \times SI(J) \tag{8.4}$$

8. Cont.

In these equations J can take on the values 1 to 19. SI(J) is a number calculated for each value of J and is the result of evaluating the integrals on the right of equations (8.1) and (8.2) according to Simpson's rule.

In order to obtain starting values for the calculation it can be assumed that the initial distributions for electrons and ions are gaussian. This is thought to be true in practice.

Let the distribution be of the form :-

$$c \exp - \frac{r^2}{2\sigma^2}$$

There are then a total number in the group given by

$$c \int_0^R dr \; r \exp - \frac{r^2}{2\sigma^2}$$

$$= c \int_0^{\frac{R}{\sqrt{2}\sigma}} (\exp -\rho^2) \; \sqrt{2}\,\sigma\rho \; . \; \sqrt{2}\,\sigma \, d\rho$$

$$= c\,\sigma^2 \left[- \exp - \frac{R^2}{2\,\sigma^2} + 1 \right]$$

choosing $c\,\sigma^2 = N_o$

and $\exp - \frac{R^2}{2\,\sigma^2} = \frac{1}{2}$ so that R becomes the distance from the origin at which the distribution has dropped to half the size at the origin :-

Then $\frac{R^2}{2\,\sigma^2} = \log 2.$

8. Cont.

$$\frac{1}{2\sigma^2} = \frac{\log 2}{R^2}$$

and the initial distribution has the form

$$\frac{2 N_o \log 2}{R^2} \exp - \frac{r^2 \log 2}{R^2}$$

$$\doteq \frac{2 N_o}{R^2} \exp - \frac{r^2}{R^2}$$

The distance R is taken to be about 2 mean free paths which at 95 kilometers altitude represents $\frac{1}{2}$ metre or $\frac{1}{60}$ in dimensionless (ζ) form.

The step length h for the finite difference approximation can now be chosen so as to have 3 steps inside the region extending out to R. That is h is taken as $\frac{1}{180}$.

The initial distribution is then calculated by the computer in the form :-

$$A(J) = B(J) = 2 N_o \times 60^2 \times \exp - (60^2 \times J^2 h^2)$$

$$= 2 N_o \times 60^2 \times \exp - \frac{J^2}{9} \qquad J = 0, 20$$

The functions SI(J) can now be calculated in the form

$$SI(1) = \frac{h}{3} . h (A(1) - B(1))$$

$$SI(L) + SI(L-1) + \frac{h}{3} . Lh. (A(L) - B(L)) + 3.(L-1)h. (A(L-1) - B(L-1))$$

$L = 2, 4, 6 \ldots\ldots\ldots 18$

$$SI(M) = SI(M-1) + \frac{h}{3} . Mh\ (A(M) - B(M)) + 1.\ (M-1)h.\ (A(M-1)$$

$$-B(M-1))$$

$M = 3,5,7 \ldots\ldots\ldots\ldots\ldots 19.$

This form of calculation is well suited to a high speed computer.

The initial values of the derivatives DA(J) and DB(J) are now calculated by the computer. Since every term on the right of (8.3) and (8.4) is now known this is a straightforward calculation.

Using the information now stored in the computer the difference equations can now be integrated by means of the Runge-Kutta process. The fourth order process which is very accurate is used. In this method, four estimates of the slope are made over a suitable τ step-length and new values of the dependent variables A and B are obtained. This is carried out for each equation in turn and the whole process repeated for as many steps as desired. After each integration step the computer can be made to print out the new values of A and B.

Owing to the non-linear terms in the equations (8.3) and (8.4) it is difficult to predict the best step-length to be used for the integrations and trial and error methods have to be used. If the step-length is chosen too small it requires an unnecessarily large number of integrations to show the behaviour of the system of particles. If the step-length is chosen too large, inaccuracies associated with the finite difference approximation can become very large and the process can become unstable.

The correct step length is related to the value chosen for N_o since this governs the size of the non-linear terms. For $N_o = 10^3$, so

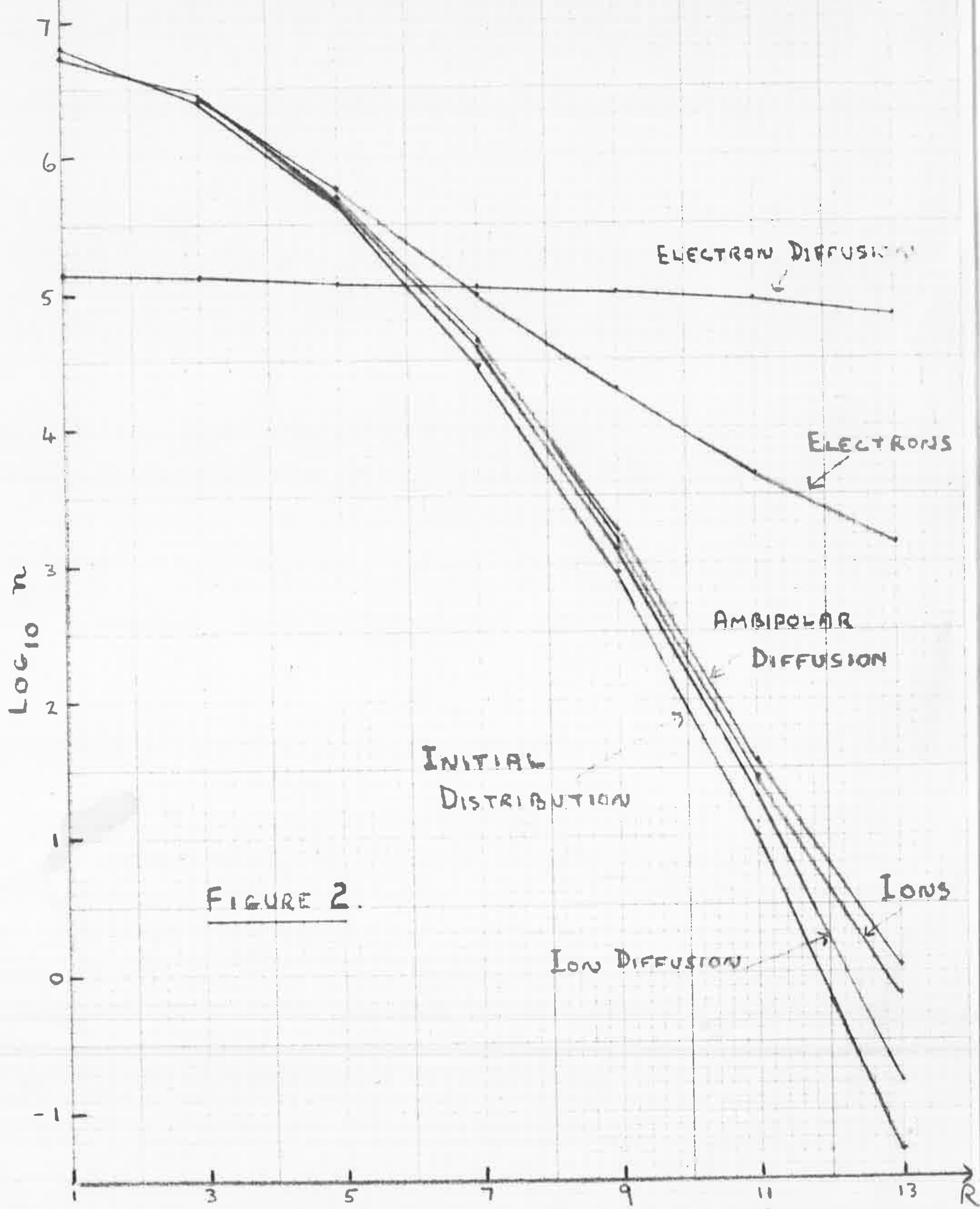

FIGURE 2.

DISTRIBUTION AT $\tau = 3 \times 10^{-3}$

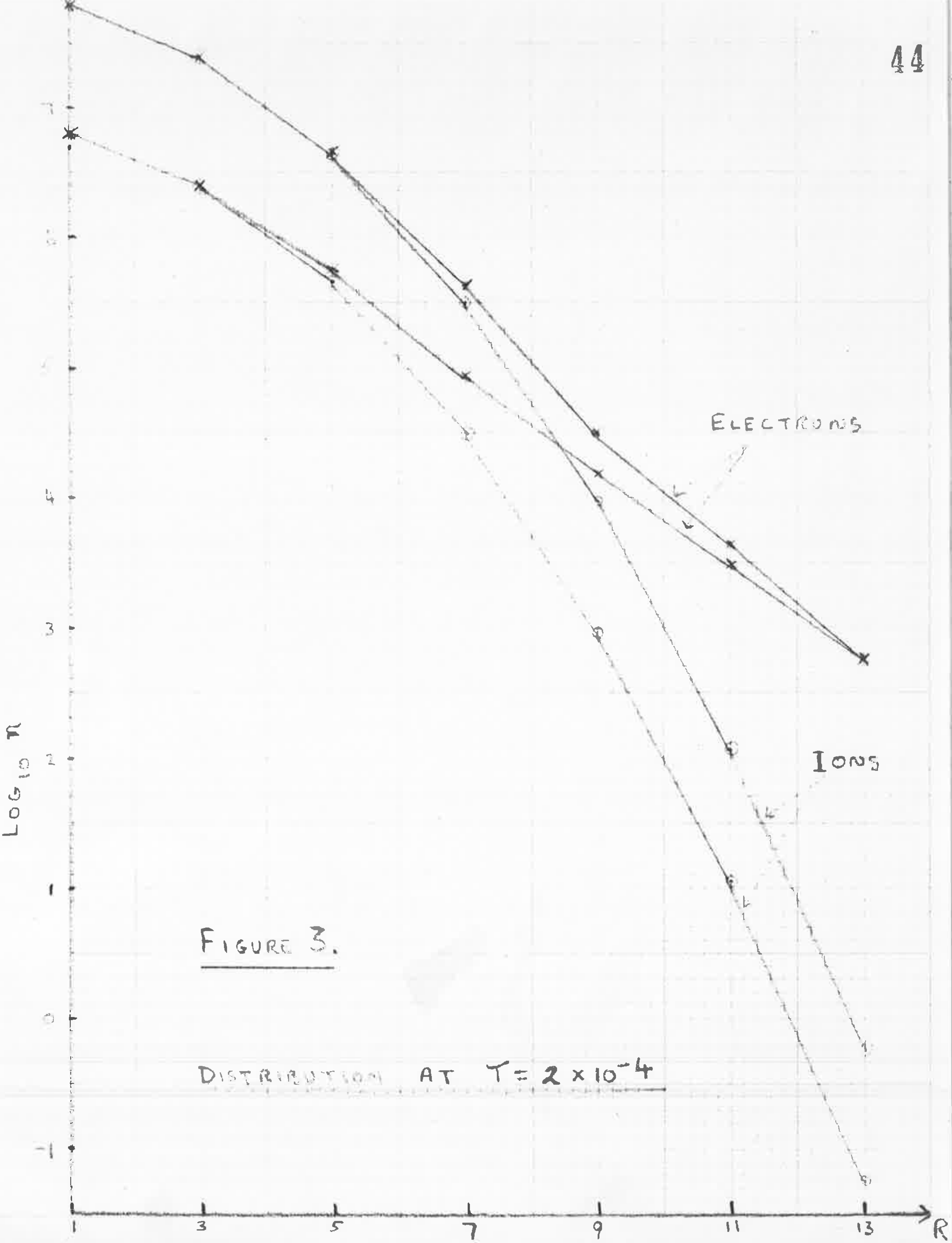

Figure 3.

Effect of Increase in N_0 by a Factor of 10

8. Cont.

that A and B are about 10^7 at the origin, the step-length for $\tau = 10^{-6}$. For $N_o = 10^{10}$ and A and B about 10^{14} the step-length required is 10^{-13}. As it happens this latter case would require an impossibly large number of integration steps to show how the system behaves since it turns out that the spreading out of particles is controlled by the comparatively slowly moving ions.

With $N_o = 10^3$ and a τ step-length of 10^{-6} the integrations have been carried out over 3,000 steps and the resulting values of A and B plotted against ζ. As a check the linear equations for electron diffusion and ion diffusion each separately have been integrated in exactly the same way as described above. The linear equation for Ambipolar diffusion has also been integrated in the same way.

The results shown as Figure 2 indicate that for densities greater than about 10^5 per cubic meter the electrons and ions move out together at the Ambipolar diffusion rate. Below this figure the charges tend to separate indicating that the electrostatic forces are no longer powerful enough to prevent separation due to differential diffusion.

It can be inferred that for an initial distribution of about 10^{14}, which is the practical case for low density meteor-trails, the charges will spread out at the Ambipolar diffusion rate for the duration of a radio echo. This inference has been tested by increasing N_o by a factor of 10 and carrying out the integrations as before with a shorterstep-length. The results are shown on Figure 3 from which

8. Cont.

it can be seen that the effect of the increase is to keep the electron and ion distributions more nearly the same further away from the origin.

9. Conclusion

The results of section 8 indicate that the assumption (6.3), that electrons and ions diffuse at the same rate, is valid. In the absence of a magnetic field or in directions parallel to the field the coefficient of diffusion $D_A = 2D_i$. In directions normal to the magnetic field, as shown in section 5, the diffusion coefficient for electrons is reduced and at 95 kilometers $D_e \ll D_i$ if the ions are assumed not to be effected by the magnetic field. It can be concluded that in this case $D_A = D_i$.

In the notation of section 3 :-

$$(D - Dm) = D_i$$

$$D = 2D_i$$

The results of section 4 now show that the Earth's magnetic field has no effect on the reflected signal power when the meteor trail is parallel to the field direction. When the trail is normal to the field there is a reduction in the reflected power by a factor of 2. This is the worst case for signal reduction. The oblique case will be between factors of 1 and 2 depending on the angle which the trail makes with the field.

The reductions should be observable and it is thought that experimental data at present being analysed at Adelaide may give some confirmation.

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