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The equivalence of Bell's inequality and the Nash inequality in a quantum game-theoretic setting

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Abstract

The interaction of competing agents is described by classical game theory. It is now well known that this can be extended to the quantum domain, where agents obey the rules of quantum mechanics. This is of emerging interest for exploring quantum foundations, quantum protocols, quantum auctions, quantum cryptography, and the dynamics of quantum cryptocurrency, for example. In this paper, we investigate two-player games in which a strategy pair can exist as a Nash equilibrium when the games obey the rules of quantum mechanics. Using a generalized Einstein-Podolsky-Rosen (EPR) setting for two-player quantum games, and considering a particular strategy pair, we identify sets of games for which the pair can exist as a Nash equilibrium only when Bell's inequality is violated. We thus determine specific games for which the Nash inequality becomes equivalent to Bell's inequality for the considered strategy pair.

I. INTRODUCTION

A quantum game [1–5] describes the strategic interaction among a set of players sharing quantum states. Players' strategic choices, or strategies [6–8], are local unitary transformations on the quantum state. The state evolves unitarily and finally the players' payoffs, or utilities, are obtained by measuring the entangled state. It turns out that under certain situations sharing of an entangled quantum state can put the players in an advantageous position and more efficient outcomes of the game can then emerge. For readers not familiar with the formalism of quantum theory [9], sharing an entangled state can be considered equivalent to the situation in which the players have (shared) access to a 'quantum system' having some intrinsically non-classical aspects. A quantum game would then involve a strategic manoeuvring of the shared quantum system in which different and perhaps more efficient outcome(s) of the game can emerge due to non-classical aspect(s) of the shared system.

Now, it is well known that non-classical, and thus apparently strange, aspects of a shared quantum system can be expressed as constraints on probabilities relevant to the shared system. Usually expressed as constraints in correlations, the famous Bell's inequality [9–14] can be re-expressed as constraints on the relevant joint probability and its marginals [15–18]. Essentially, Bell's inequality emerges as being the necessary and sufficient condition requiring a joint probability distribution to exist given a set of marginals. It is well known that Bell's inequality can be violated by a set of quantum mechanical probabilities—the probabilities that are obtained by the quantum probability rule. This turns out to be the case even though the quantum probabilities are normalized as the classical probabilities are. This is because for a set of marginal (quantum) probabilities that are obtained via the quantum probability rule, the corresponding joint probability distribution may not exist. The possibility to express truly non-classical aspects of a quantum system in only probabilistic terms [19] has led to suggestions for schemes of quantum games [26–28, 30, 33–35] that do not refer to quantum states, unitary transformations, and/or the quantum measurement.

In a classical game allowing mixed strategies, the players' strategies are convex linear combinations, with real coefficients, of their pure strategies [8]. Players' strategies in a quantum game [2, 3], however, are unitary transformations and thus belong to much larger strategy spaces. This led to the arguments [36] that quantum games can perhaps be viewed as extended classical games. In order to obtain an improved comparison between classical and quantum games, it was suggested [24] that the players' strategy sets need to be identical. This has motivated proposals [33–35] of quantum games in which players' strategies are classical, as being convex linear combinations (with real coefficients) of the classical strategies, and the quantum game emerges from the non-classical aspects of a shared probabilistic physical system—as expressed by the constraints on relevant probabilities and their marginals [15–18].

In the usual approach in the area of quantum games [5], a classical game is defined, or given, at the start and its quantum version is developed afterwards. The usual reasonable requirement being that the classical mixed-strategy game can be recovered from the quantum game. One then studies whether the quantum game offers any non-classical outcome(s). In this paper, the players' strategies in the quantum game remain classical whereas the new quantum, or non-classical, outcome(s) of the game emerge from the peculiar quantum probabilities relevant to the quantum system that two players share to play the game. In contrast to the usual approach in quantum games, in which the players' strategies are unitary transformations, here we consider a particular classical strategy pair and then enquire about the set of games for which that strategy pair can exist as a Nash equilibrium (NE) [6–8]. In particular, for a given strategy pair, we investigate whether there are such games for which that strategy pair can exist as a NE only when the corresponding Bell's inequality is violated by the quantum probabilities relevant to the shared quantum system.

We consider two-player games that can be played using the setting of generalized Einstein-Podolsky-Rosen (EPR) experiments [9, 14, 19]. As is known, in this setting a probabilistic version of Bell's inequality can be obtained [15–19]. We consider particular strategies and find the sets of games for which the strategies can exist as a NE only when Bell's inequality is violated. By identifying such games, we show that there exist strategic outcomes that can only be realized when the game is played quantum mechanically and also only when the corresponding Bell's inequality is violated.

The connection between Bell's inequality and the NE was originally reported in Ref. [20]. However, the Ref. [20] did not use an EPR setting. In the present paper, we show that the mentioned connection becomes explicitly direct by using an EPR setting in playing a quantum game.

II. TWO-PLAYER QUANTUM GAMES USING THE EPR EXPERIMENT SETTING

The EPR setting for playing quantum games was introduced in Ref. [24] and was further investigated in Refs. [25–32]. The Refs. [26–28, 30, 33–35] investigate using the setting of generalized EPR experiments [19] for playing quantum games. This setting permits consideration of a probabilistic version of the corresponding Bell's inequality, which allows construction of quantum games without referring to the mathematical formalism of quantum mechanics including Hilbert space, unitary transformations, entangling operations, and quantum measurements [9]. The relationship between the NE and aspects of Bell's inequality have been indicated in Refs. [21–23]. The present paper's contribution consists of bringing into focus this relationship and, in particular, finding the specific games for which this relationship can be explicitly defined. Moreover, in order to achieve this the present paper uses EPR setting and the probabilistic version of Bell's inequality.

In the setting of the generalized EPR experiment, Alice and Bob are spatially separated and are unable to communicate with each other. In an individual run, both receive one half of a pair of particles originating from a common source. In the same run of the experiment, both players choose one from two given (pure) strategies. These strategies are the two directions in space along which spin or polarization measurements can be made. We denote these directions to be S_1, S_2 for Alice and S_1', S_2' for Bob. Each measurement generates $+1$ or -1 as the outcome. Experimental results are recorded for a large number of individual runs of the experiment. Payoffs are then awarded that depend on the directions the players choose over many runs (defining the players' strategies), the matrix of the game they play, and the statistics of the measurement outcomes. For instance, we denote $\Pr(+1, +1; S_1, S_1')$ as the probability of both Alice and Bob obtaining $+1$ when Alice selects the direction S_1 whereas Bob selects the direction S_1'. We write ϵ_1 for the probability $\Pr(+1, +1; S_1, S_1')$ and ϵ_8 for the probability $\Pr(-1, -1; S_1, S_2')$ and likewise one can then write down the relevant probabilities as

		Bob			
		S_1'		S_2'	
		$+1$	-1	$+1$	-1
Alice S_1	$+1$	ϵ_1	ϵ_2	ϵ_5	ϵ_6
	-1	ϵ_3	ϵ_4	ϵ_7	ϵ_8
Alice S_2	$+1$	ϵ_9	ϵ_{10}	ϵ_{13}	ϵ_{14}
	-1	ϵ_{11}	ϵ_{12}	ϵ_{15}	ϵ_{16}

. (1)

Being normalized, EPR probabilities ϵ_i satisfy the relations

$$\Sigma_{i=1}^{4}\epsilon_i = 1,\ \Sigma_{i=5}^{8}\epsilon_i = 1,\ \Sigma_{i=9}^{12}\epsilon_i = 1,\ \Sigma_{i=13}^{16}\epsilon_i = 1. \tag{2}$$

Consider in (1), for instance, the case when Alice plays her strategy S_2 and Bob plays his strategy S_1'. The two arms of the Stern-Gerlach apparatus are rotated along these two directions and the quantum measurement is performed. According to the above table, the probability that both experimental outcomes are -1 is then ϵ_{12}. Similarly, the probability that the observer 1's experimental outcome is $+1$ and observer 2's experimental outcome is -1 is given by ϵ_{10}. The other entries in (1) can similarly be explained. In the present paper, the EPR setting is enforced and that the players can only choose between two directions.

We now consider a game between two players Alice and Bob, which is defined by the real numbers a_i and b_i for $1 \leq i \leq 16$, and is given by

	Bob			
	S_1'		S_2'	
Alice S_1	(a_1, b_1)	(a_2, b_2)	(a_5, b_5)	(a_6, b_6)
	(a_3, b_3)	(a_4, b_4)	(a_7, b_7)	(a_8, b_8)
Alice S_2	(a_9, b_9)	(a_{10}, b_{10})	(a_{13}, b_{13})	(a_{14}, b_{14})
	(a_{11}, b_{11})	(a_{12}, b_{12})	(a_{15}, b_{15})	(a_{16}, b_{16})

. (3)

For this game, we now define the players' pure strategy payoff relations as

$$\Pi_{A,B}(S_1, S_1') = \Sigma_{i=1}^{4}(a_i, b_i)\epsilon_i,\ \Pi_{A,B}(S_1, S_2') = \Sigma_{i=5}^{8}(a_i, b_i)\epsilon_i,$$
$$\Pi_{A,B}(S_2, S_1') = \Sigma_{i=9}^{12}(a_i, b_i)\epsilon_i,\ \Pi_{A,B}(S_2, S_2') = \Sigma_{i=13}^{16}(a_i, b_i)\epsilon_i, \quad (4)$$

where $\Pi_A(S_1, S_2')$, for example, is Alice's payoff when she plays S_1 and Bob plays S_2'.

It can be seen that in the way it is defined, the game is inherently probabilistic. That is, in (3) the players' payoffs even for their pure strategies assume an underlying probability distribution as given by (1). Now, we can also define a mixed-strategy version of this game as follows. Consider Alice playing the strategy S_1 with probability p and the strategy S_2 with probability $(1-p)$ whereas Bob playing the strategy S_1' with probability q and the strategy S_2' with probability $(1-q)$. Using (3, 4) the players' mixed strategy payoff relations can then be obtained as

$$\Pi_{A,B}(p,q) = \begin{pmatrix} p \\ 1-p \end{pmatrix}^T \begin{pmatrix} \Pi_{A,B}(S_1,S_1') & \Pi_{A,B}(S_1,S_2') \\ \Pi_{A,B}(S_2,S_1') & \Pi_{A,B}(S_2,S_2') \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix}. \quad (5)$$

Assuming that the strategy pair $(p^\star, q^\star)$ is a NE, we then require

$$\Pi_A(p^\star, q^\star) - \Pi_A(p, q^\star) \geqslant 0, \quad \Pi_B(p^\star, q^\star) - \Pi_B(p^\star, q) \geqslant 0, \quad (6)$$

that takes the form

$$\Pi_A(p^\star, q^\star) - \Pi_A(p, q^\star) = (p^\star - p) \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} \Pi_A(S_1,S_1') & \Pi_A(S_1,S_2') \\ \Pi_A(S_2,S_1') & \Pi_A(S_2,S_2') \end{pmatrix} \begin{pmatrix} q^\star \\ 1-q^\star \end{pmatrix} \geq 0,$$
$$\Pi_B(p^\star, q^\star) - \Pi_B(p^\star, q) = \begin{pmatrix} p^\star \\ 1-p^\star \end{pmatrix}^T \begin{pmatrix} \Pi_B(S_1,S_1') & \Pi_B(S_1,S_2') \\ \Pi_B(S_2,S_1') & \Pi_B(S_2,S_2') \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (q^\star - q) \geq 0. \quad (7)$$

Note that the players' strategies are classical whereas the game itself is not classical as the underlying probabilities of the game are quantum mechanical as obtained from the EPR experiments. Players' payoffs are defined in terms of EPR quantum probabilities that can violate Bell's inequalities. Thus a classical game can in no way model this quantum game. This setting also circumvents the criticism of Enk and Pike [36] on quantum games. Enk and Pike noted that as the players in the quantum game have access to much larger strategy sets, the quantum game can be considered as another classical game with an extended set of classical strategies. In the considered setting, players' strategy sets are identical in both the classical and quantum games and players' payoff relations are obtained from an underlying probability distribution that is quantum mechanical.

Although this game is played using the setting of generalized EPR experiments, in which the players strategies consist of choosing between two directions, one can notice that under appropriate conditions, the game can be reduced to the usual mixed-strategy version of the standard two-player two-strategy noncooperative game. Non-cooperative games [6–8] were investigated in the early work [2, 3] on quantum games. To see this, we consider the case when

$$
\begin{aligned}
a_i(1 \leq i \leq 4) = \alpha,\ b_i(1 \leq i \leq 4) = \alpha, \\
a_i(5 \leq i \leq 8) = \beta,\ b_i(5 \leq i \leq 8) = \gamma, \\
a_i(9 \leq i \leq 12) = \gamma,\ b_i(9 \leq i \leq 12) = \beta, \\
a_i(13 \leq i \leq 16) = \delta,\ b_i(13 \leq i \leq 16) = \delta,
\end{aligned} \tag{8}
$$

and then from Eqs. (4) and (2) we obtain

$$
\begin{aligned}
\Pi_{A,B}(S_1, S_1') = \alpha \Sigma_{i=1}^{4} \epsilon_i = \alpha, \\
\Pi_A(S_1, S_2') = \beta \Sigma_{i=5}^{8} \epsilon_i, = \beta,\ \Pi_B(S_1, S_2') = \gamma \Sigma_{i=5}^{8} \epsilon_i, = \gamma, \\
\Pi_A(S_2, S_1') = \gamma \Sigma_{i=9}^{12} \epsilon_i = \gamma,\ \Pi_B(S_2, S_1') = \beta \Sigma_{i=9}^{12} \epsilon_i = \beta, \\
\Pi_{A,B}(S_2, S_2') = \delta \Sigma_{i=13}^{16} \epsilon_i = \delta.
\end{aligned} \tag{9}
$$

In view of (6), Nash inequalities for the strategy pair $(p^\star, q^\star)$ then take the form

$$
\begin{aligned}
\Pi_A(p^\star, q^\star) - \Pi_A(p, q^\star) = (p^\star - p) \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} q^\star \\ 1 - q^\star \end{pmatrix} \geq 0, \\
\Pi_B(p^\star, q^\star) - \Pi_B(p^\star, q) = \begin{pmatrix} p^\star \\ 1 - p^\star \end{pmatrix}^T \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (q^\star - q) \geq 0,
\end{aligned} \tag{10}
$$

which give us Nash inequalities for the mixed strategy $(p^\star, q^\star)$ for the following symmetric game

$$\begin{pmatrix} (\alpha,\alpha) & (\beta,\gamma) \\ (\gamma,\beta) & (\delta,\delta) \end{pmatrix}. \tag{11}$$

When $\beta < \delta < \alpha < \gamma$ this game results in the well known game of Prisoners' Dilemma. As is well known, for this game $(p^\star, q^\star) = (0,0)$ comes out as the unique NE.

A. Cereceda's analysis and the probabilitsic version of CHSH sum of correlations

A convenient solution of the system (2, 33) was reported by Cereceda in [19] and given in the Appendix A. Cereceda expressed the set of probabilities $\upsilon = \{\epsilon_2,\ \epsilon_3,\ \epsilon_6,\ \epsilon_7,\ \epsilon_{10},\ \epsilon_{11},\ \epsilon_{13},\ \epsilon_{16}\}$ in terms of the remaining set of probabilities i.e.

$$\mu = \{\epsilon_1,\ \epsilon_4,\ \epsilon_5,\ \epsilon_8,\ \epsilon_9,\ \epsilon_{12},\ \epsilon_{14},\ \epsilon_{15}\}, \tag{12}$$

and thus the elements of the set μ can be considered as independent variables.

In a particular run of the EPR experiment, the requirements of locality dictate that the outcome of $+1$ or -1 (obtained along the direction S_1 or direction S_2) is independent of whether the direction S_1' or the direction S_2' is chosen in that run. Similarly, the outcome of $+1$ or -1 (obtained along S_1' or S_2') is independent of whether the direction S_1 or the direction S_2 is chosen in that run. These locality requirements when translated in terms of the probability set ϵ_j can be expressed as Eqs. (33) in Appendix A.

Relevant to the EPR setting is the Clauser-Horne-Shimony-Holt (CHSH) form of Bell's inequality that is usually expressed in terms of the correlations $\langle S_1 S_1' \rangle$, $\langle S_1 S_2' \rangle$, $\langle S_2 S_1' \rangle$, $\langle S_2 S_2' \rangle$. Using (1) the correlation $\langle S_1 S_1' \rangle$, for instance, can be obtained as

$$\begin{aligned} \langle S_1 S_1' \rangle &= \Pr(S_1 = 1, S_1' = 1) - \Pr(S_1 = 1, S_1' = -1) \\ &- \Pr(S_1 = -1, S_1' = +1) + \Pr(S_1 = -1, S_1' = -1) \\ &= \epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4. \end{aligned} \tag{13}$$

Expressions for the correlations $\langle S_1 S_2' \rangle$, $\langle S_2 S_1' \rangle$, and $\langle S_2 S_2' \rangle$ can similarly be obtained. The CHSH sum of correlations is given as

$$\Delta = \langle S_1 S_1' \rangle + \langle S_1 S_2' \rangle + \langle S_2 S_1' \rangle - \langle S_2 S_2' \rangle, \tag{14}$$

and the CHSH inequality stating that $|\Delta| \leq 2$ holds for any theory of local hidden variables.

The set of constraints on probabilities ϵ_i that are imposed by Tsirelson's bound [37] state that the quantum prediction of the CHSH sum of correlations Δ, defined in (14), is bounded in absolute value by $2\sqrt{2}$ i.e. $|\Delta_{QM}| \leq 2\sqrt{2}$. Taking into account [19] the normalization condition (2), the quantity Δ can equivalently be expressed as

$$\Delta = 2(\epsilon_1 + \epsilon_4 + \epsilon_5 + \epsilon_8 + \epsilon_9 + \epsilon_{12} + \epsilon_{14} + \epsilon_{15} - 2). \tag{15}$$

Bell's inequality can thus be written as $0 \leq (2 - |\Delta|)$ and is violated when the discriminant $(2 - |\Delta|) < 0$. Bell's inequality is thus violated when the discriminant attains a negative value that occurs when either $\Delta > 2$ or $\Delta < -2$.

III. GAMES FOR WHICH NASH INEQUALITIES INVOLVE CHSH SUM OF CORRELATIONS

We note from (7) that Nash inequalities for the strategy pair $(p^\star, q^\star) = (1/2, 1/2)$ take the form

$$\Pi_A(1/2, 1/2) - \Pi_A(p, 1/2) =$$
$$(1/2)(1/2 - p)\left[\Pi_A(S_1, S_1') + \Pi_A(S_1, S_2') - \Pi_A(S_2, S_1') - \Pi_A(S_2, S_2')\right] \geq 0, \tag{16}$$
$$\Pi_B(1/2, 1/2) - \Pi_B(1/2, q) =$$
$$(1/2)(1/2 - q)\left[\Pi_B(S_1, S_1') - \Pi_B(S_1, S_2') + \Pi_B(S_2, S_1') - \Pi_B(S_2, S_2')\right] \geq 0, \tag{17}$$

which hold in order for the strategy pair $(p^\star, q^\star) = (1/2, 1/2)$ to exist as a NE. Now, the presence of the terms $(1/2 - p)$ in (16) and $(1/2 - q)$ in (17) forces both expressions within the square brackets to be identically zero. Nash inequalities for the strategy pair $(p^\star, q^\star) = (1/2, 1/2)$, therefore, cannot be expressed in terms of the discriminant $(2 - |\Delta|)$. That is, the strategy pair $(p^\star, q^\star) = (1/2, 1/2)$ cannot exist as a NE when Bell's inequality is violated. We therefore consider a second example of the strategy pair $(p^\star, q^\star) = (1, 1/2)$ that allows us to establish a direct connection between Bell's inequality and Nash inequality.

Theorem 1 *For the set of games for which*

$$\Pi_A(S_2, S_1') + \Pi_A(S_2, S_2') - \Pi_A(S_1, S_1') - \Pi_A(S_1, S_2') = 2 - |\Delta| \,, \tag{18}$$

$$\Pi_B(S_1, S_1') - \Pi_B(S_1, S_2') = 0, \tag{19}$$

the strategy pair $(p^\star, q^\star) = (1, 1/2)$ *exists as a NE when Bell's inequality is violated.*

Proof. Nash inequalities (7) for the strategy pair $(p^\star, q^\star) = (1, 1/2)$ take the form:

$$\Pi_A(1, 1/2) - \Pi_A(p, 1/2) =$$

$$-(1/2)(1-p)\left[\Pi_A(S_2, S_1') + \Pi_A(S_2, S_2') - \Pi_A(S_1, S_1') - \Pi_A(S_1, S_2')\right] \geq 0, \tag{20}$$

$$\Pi_B(1, 1/2) - \Pi_A(1, q) =$$

$$\left[\Pi_B(S_1, S_1') - \Pi_B(S_1, S_2')\right](1/2 - q) \geq 0. \tag{21}$$

■

Now, the inequality (20) can hold when the term in square bracket is negative or zero. As Bell's inequality is violated when the discriminant $(2 - |\Delta|)$ is negative, therefore, for the set of games that are defined by the conditions (29, 31) the strategy pair $(p^\star, q^\star) = (1, 1/2)$ exists as a NE when Bell's inequality is violated.

For the set of games defined by the following conditions the strategy pair $(p^\star, q^\star) = (1, 1/2)$ exists as a NE when Bell's inequality is violated, for $0 \leq \Delta$,

$$\begin{aligned}
& b_1 = b_2 = b_5 = b_6 \text{ and } b_3 = b_4 = b_7 = b_8, \\
& a_2 = -a_5 + a_{12} + a_{15}, \quad a_3 = a_1 + a_4 + a_5 - a_{12} - a_{15} - 4, \\
& a_6 = a_4 + a_5 + a_8 - a_{12} - a_{15} - 4, \quad a_7 = -a_4 + a_{12} + a_{15}, \\
& a_9 = a_1 + a_4 + a_5 + a_8 - a_{12} - a_{14} - a_{15} - 4, \\
& a_{10} = a_4 + a_8 - a_{14}, \quad a_{11} = a_1 + a_5 - a_{15}, \\
& a_{13} = -a_4 - a_8 + a_{12} + a_{14} + a_{15} + 4, \quad a_{16} = a_4 + a_8 - a_{12},
\end{aligned} \tag{22}$$

and for $\Delta < 0$, the same is true for the set of games that is defined by these conditions:

$$
\begin{aligned}
&b_1 = b_2 = b_5 = b_6 \text{ and } b_3 = b_4 = b_7 = b_8, \\
&a_2 = -a_5 + a_{12} + a_{15} - 4, \quad a_3 = a_1 + a_4 + a_5 - a_{12} - a_{15} + 8, \\
&a_6 = a_4 + a_5 + a_8 - a_{12} - a_{15} + 8, \quad a_7 = -a_4 + a_{12} + a_{15} - 4, \\
&a_9 = a_1 + a_4 + a_5 + a_8 - a_{12} - a_{14} - a_{15} + 12, \\
&a_{10} = a_4 + a_8 - a_{14} + 4, \quad a_{11} = a_1 + a_5 - a_{15} + 4, \\
&a_{13} = -a_4 - a_8 + a_{12} + a_{14} + a_{15} - 8, \quad a_{16} = a_4 + a_8 - a_{12} + 4.
\end{aligned} \tag{23}
$$

Proof. Using Eqs. (4) we write Eqs. (18, 19) as,

$$
\Sigma_{i=9}^{12} a_i \epsilon_i + \Sigma_{i=13}^{16} a_i \epsilon_i - \Sigma_{i=1}^{4} a_i \epsilon_i - \Sigma_{i=5}^{8} a_i \epsilon_i = 2 - |\Delta|, \tag{24}
$$

$$
\Sigma_{i=1}^{4} b_i \epsilon_i - \Sigma_{i=5}^{8} b_i \epsilon_i = 0. \tag{25}
$$

Now, using Eq. (12), the left sides of Eq. (24) can then be expressed in terms of the probabilities from the set μ as follows:

$$
\begin{aligned}
&\Sigma_{i=9}^{12} a_i \epsilon_i + \Sigma_{i=13}^{16} a_i \epsilon_i - \Sigma_{i=1}^{4} a_i \epsilon_i - \Sigma_{i=5}^{8} a_i \epsilon_i = \\
&(\epsilon_1/2)(-2a_1 + a_2 + a_3 - a_6 + a_7 - a_{10} + a_{11} - a_{13} + a_{16}) + \\
&(\epsilon_4/2)(-2a_4 + a_2 + a_3 + a_6 - a_7 + a_{10} - a_{11} + a_{13} - a_{16}) + \\
&(\epsilon_5/2)(-2a_5 - a_2 + a_3 + a_6 + a_7 + a_{10} - a_{11} + a_{13} - a_{16}) + \\
&(\epsilon_8/2)(-2a_8 + a_2 - a_3 + a_6 + a_7 - a_{10} + a_{11} - a_{13} + a_{16}) + \\
&(\epsilon_9/2)(2a_9 + a_2 - a_3 + a_6 - a_7 - a_{10} - a_{11} + a_{13} - a_{16}) + \\
&(\epsilon_{12}/2)(2a_{12} - a_2 + a_3 - a_6 + a_7 - a_{10} - a_{11} - a_{13} + a_{16}) + \\
&(\epsilon_{14}/2)(2a_{14} - a_2 + a_3 - a_6 + a_7 + a_{10} - a_{11} - a_{13} - a_{16}) + \\
&(\epsilon_{15}/2)(2a_{15} + a_2 - a_3 + a_6 - a_7 - a_{10} + a_{11} - a_{13} - a_{16}) + \\
&(1/2)(-a_2 - a_3 - a_6 - a_7 + a_{10} + a_{11} + a_{13} + a_{16}) = 2 - |\Delta|.
\end{aligned} \tag{26}
$$

Similarly, the left side of Eq. (25) now takes the form:

$$\Sigma_{i=1}^{4} b_i \epsilon_i - \Sigma_{i=5}^{8} b_i \epsilon_i =$$
$$(\epsilon_1/2)(2b_1 - b_2 - b_3 - b_6 + b_7) + (\epsilon_4/2)(-b_2 - b_3 + 2b_4 + b_6 - b_7) +$$
$$(\epsilon_5/2)(b_2 - b_3 - 2b_5 + b_6 + b_7) + (\epsilon_8/2)(-b_2 + b_3 + b_6 + b_7 - 2b_8) +$$
$$(\epsilon_9/2)(-b_2 + b_3 + b_6 - b_7) + (\epsilon_{12}/2)(b_2 - b_3 - b_6 + b_7) +$$
$$(\epsilon_{14}/2)(b_2 - b_3 - b_6 + b_7) + (\epsilon_{15}/2)(-b_2 + b_3 + b_6 - b_7) +$$
$$(1/2)(b_2 + b_3 - b_6 - b_7) = 0. \tag{27}$$

As the probabilities in the set μ are considered independent, comparing the two sides of Eq. (27) leads us to obtain:

$$b_1 = b_2 = b_5 = b_6 \text{ and } b_3 = b_4 = b_7 = b_8. \tag{28}$$

Consider now the right side of Eq. (26). As $\Delta = 2(\epsilon_1 + \epsilon_4 + \epsilon_5 + \epsilon_8 + \epsilon_9 + \epsilon_{12} + \epsilon_{14} + \epsilon_{15} - 2)$, the discriminant $(2 - |\Delta|)$ can be negative for the following two cases: a) For case $0 \leq \Delta$, we have $2 - |\Delta| = 2 - \Delta$. The rank of the system (26) is 7. We take $a_1, a_4, a_5, a_8, a_{12}, a_{14}, a_{15}$ as independently chosen constants and compare the coefficients of the independent probabilities in the set μ on the two sides of Eq. (26). This gives the set of conditions (22). That is, for the set of games defined by the conditions (28, 22) the strategy pair $(p^\star, q^\star) = (1, 1/2)$ exists as a NE when the Bell's inequality is violated. For case $\Delta < 0$, we have $|\Delta| = -\Delta$ and $2 - |\Delta| = 2 + \Delta$. Following the steps from the last case, we obtain the set of conditions (23). As before, for the set of games that are defined by the conditions (28, 23), the strategy pair $(p^\star, q^\star) = (1, 1/2)$ exists as a NE when the Bell's inequality is violated. ■

IV. AN EXAMPLE

As a specific example, and in view of Eqs. (28), we assign the value of 1 arbitrarily to b_1, b_2, b_5, b_6 and also the same value to b_3, b_4, b_7, b_8. Also, as Eq. (25) does not involve the constants b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16} we also assign the value of 1 to them. Likewise, we assign the value of 1 to the independently chosen constants a_1, a_4, a_5, a_8, a_{12}, a_{14}, a_{15}. With reference to Eqs. (22, 23) we obtain the following two games,

$$
\begin{array}{cc|cc|cc}
 & & \multicolumn{4}{c}{\text{Bob}} \\
 & & \multicolumn{2}{c}{S_1'} & \multicolumn{2}{c}{S_2'} \\
 & S_1 & (1,1) & (1,1) & (1,1) & (-3,1) \\
 & & (-3,1) & (1,1) & (1,1) & (1,1) \\
\text{Alice} & & & & & \\ \hline
 & & (-3,1) & (1,1) & (5,1) & (1,1) \\
 & S_2 & (1,1) & (1,1) & (1,1) & (1,1)
\end{array}, \tag{29}
$$

for which we consider the strategy pair $(p^{\star}, q^{\star}) = (1, 1/2)$. We use Eqs. (26) under the assumption $0 \leq \Delta$, where Δ is defined by Eq. (15), to obtain Nash inequalities for the game (29) as

$$
\begin{aligned}
\Pi_A(1, 1/2) - \Pi_A(p, 1/2) &= -(1/2)(1-p)\left[2 - \Delta\right] \geq 0, \\
\Pi_B(1, 1/2) - \Pi_A(1, q) &= 0.
\end{aligned} \tag{30}
$$

As $0 \leq (1-p) \leq 1$, for this game, the strategy pair $(p^{\star}, q^{\star}) = (1, 1/2)$ exists as a NE when $2 < \Delta$. The converse is also true in that when $2 < \Delta$ the strategy pair $(p^{\star}, q^{\star}) = (1, 1/2)$ becomes a NE. That is, for the considered game and the strategy pair, Nash and Bell's inequalities become equivalent.

$$
\begin{array}{cc|cc|cc}
 & & \multicolumn{4}{c}{\text{Bob}} \\
 & & \multicolumn{2}{c}{S_1'} & \multicolumn{2}{c}{S_2'} \\
 & S_1 & (1,1) & (-3,1) & (1,1) & (9,1) \\
 & & (9,1) & (1,1) & (-3,1) & (1,1) \\
\text{Alice} & & & & & \\ \hline
 & & (13,1) & (5,1) & (-7,1) & (1,1) \\
 & S_2 & (5,1) & (1,1) & (1,1) & (5,1)
\end{array}. \tag{31}
$$

Similarly, now considering the game (31) for the same strategy pair, we obtain Nash inequalities for the strategy pair $(p^{\star}, q^{\star}) = (1, 1/2)$ as follows and with the assumption that $\Delta < 0$,

$$\Pi_A(1,1/2) - \Pi_A(p,1/2) = -(1/2)(1-p)\,[2+\Delta] \geq 0,$$
$$\Pi_B(1,1/2) - \Pi_A(1,q) = 0. \tag{32}$$

For this game, the strategy pair $(p^\star, q^\star) = (1, 1/2)$ exists as a NE when $\Delta < -2$. The converse is also true in that when $\Delta < -2$ the strategy pair $(p^\star, q^\star) = (1, 1/2)$ becomes a NE. That is, for the considered game and the strategy pair, Nash and Bell's inequalities becomes equivalent. The strategy pair $(p^\star, q^\star) = (1, 1/2)$ therefore exists as a NE in both the games (29, 31) when Bell's inequality is violated.

As (29, 31) are especially-designed games, their classical versions do not have an existing name. The behavior of these games changes from their quantum pay-off versions in that the particular mixed strategy pair $(p^\star, q^\star) = (1, 1/2)$ can exist only in the quantum versions of these games.

V. DISCUSSION

In the quantum games considered in this paper, the players' strategies are classical consisting of convex linear combinations—with real coefficients—of their pure classical strategies, whereas the players' payoffs are obtained directly from the set of quantum probabilities that underlie the playing of the game. We consider the probabilistic form of Bell's inequality that can be violated by the set of quantum probabilities. We then show that there exist such games in which a classical pair of strategies can be a NE only when the underlying probabilities of the game are truly quantum mechanical in that they violate Bell's inequality. In the usual approach to quantum games, a game is given, or known, and pairs of quantum strategies, consisting of unitary transformations, are determined that constitute a NE. In a striking contrast this usual approach, a classical strategy pair is considered as given whereas the set of games is then determine for which that classical strategy pair becomes a NE only when Bell's inequality is violated.

Some of the earliest criticisms [36] of quantum games questioned whether such games are genuinely quantum mechanical. It was suggested that the violation of Bell's inequality can decidedly determine whether a quantum game is genuinely quantum or not. Although deriving Bell's inequality does not require quantum theory, its violation is a well established feature that is achieveable only in the truely quantum mechanical regime.

As the players' strategies even in the quantum game are restricted to the classical ones, and the players' payoff relations are obtained from the set of underlying quantum mechanical probabilities, our approach is not susceptible to the Enk and Pike type argument [36]—stating that a quantum game with quantum mechanical strategies can be considered equivalent to another extended classical game. In our approach this criticism is avoided as the players' strategies in both the classical and quantum games remain identical.

The generalized EPR setting used in this paper assumes that the repeated runs of the EPR experiment are performed in order to obtain the expected values of quantum mechanical observables. In particular, it is not the case that the measurement outcomes of an individual run lead to the players revising their strategic moves in the next run in view of their payoffs in the previous run, as is the case in repeated games. A study of repeated quantum games using generalized EPR experiments is an open question for future work.

Note that although the players' strategies are classical, quantum mechanics is central to the setting of the considered quantum game. Players' payoff relations have underlying quantum probability distributions. The physical system that is used to play this game is the standard EPR type apparatus involving Stern-Gerlach type measurements. Local unitary transformations are used as the players' strategies in the standard schemes to play quantum games whereas classical strategies, akin to rotating the arms of an EPR apparatus, are the players' strategies in this present paper [24–32]. We identify sets of games in which, for a considered classical mixed strategy, the Nash inequality becomes equivalent to Bell's inequality.

The results of this paper can be extended to multi-player games. This would involve consideration of the N-partite Bell's inequality [38]—a situation in which use of geometric algebra has been shown to offer a tractable setting for the analysis of N-partite interactions [32]. Also, consideration of two-player games with multi strategies will involve Bell's inequalities with many observables [39].

Appendix A

When translated in terms of the probability set ϵ_j, the locality requirements can be expressed as

$$
\begin{aligned}
\epsilon_1 + \epsilon_2 &= \epsilon_5 + \epsilon_6, \quad \epsilon_1 + \epsilon_3 = \epsilon_9 + \epsilon_{11}, \\
\epsilon_9 + \epsilon_{10} &= \epsilon_{13} + \epsilon_{14}, \quad \epsilon_5 + \epsilon_7 = \epsilon_{13} + \epsilon_{15}, \\
\epsilon_3 + \epsilon_4 &= \epsilon_7 + \epsilon_8, \quad \epsilon_{11} + \epsilon_{12} = \epsilon_{15} + \epsilon_{16}, \\
\epsilon_2 + \epsilon_4 &= \epsilon_{10} + \epsilon_{12}, \quad \epsilon_6 + \epsilon_8 = \epsilon_{14} + \epsilon_{16}.
\end{aligned} \tag{33}
$$

Cereceda [19] reports a convenient solution of the system (2, 33) for which the set of probabilities $\upsilon = \{\epsilon_2,\ \epsilon_3,\ \epsilon_6,\ \epsilon_7,\ \epsilon_{10},\ \epsilon_{11},\ \epsilon_{13},\ \epsilon_{16}\}$ is expressed in terms of the remaining set of probabilities i.e.

$$
\begin{aligned}
\epsilon_2 &= (1 - \epsilon_1 - \epsilon_4 + \epsilon_5 - \epsilon_8 - \epsilon_9 + \epsilon_{12} + \epsilon_{14} - \epsilon_{15})/2, \\
\epsilon_3 &= (1 - \epsilon_1 - \epsilon_4 - \epsilon_5 + \epsilon_8 + \epsilon_9 - \epsilon_{12} - \epsilon_{14} + \epsilon_{15})/2, \\
\epsilon_6 &= (1 + \epsilon_1 - \epsilon_4 - \epsilon_5 - \epsilon_8 - \epsilon_9 + \epsilon_{12} + \epsilon_{14} - \epsilon_{15})/2, \\
\epsilon_7 &= (1 - \epsilon_1 + \epsilon_4 - \epsilon_5 - \epsilon_8 + \epsilon_9 - \epsilon_{12} - \epsilon_{14} + \epsilon_{15})/2, \\
\epsilon_{10} &= (1 - \epsilon_1 + \epsilon_4 + \epsilon_5 - \epsilon_8 - \epsilon_9 - \epsilon_{12} + \epsilon_{14} - \epsilon_{15})/2, \\
\epsilon_{11} &= (1 + \epsilon_1 - \epsilon_4 - \epsilon_5 + \epsilon_8 - \epsilon_9 - \epsilon_{12} - \epsilon_{14} + \epsilon_{15})/2, \\
\epsilon_{13} &= (1 - \epsilon_1 + \epsilon_4 + \epsilon_5 - \epsilon_8 + \epsilon_9 - \epsilon_{12} - \epsilon_{14} - \epsilon_{15})/2, \\
\epsilon_{16} &= (1 + \epsilon_1 - \epsilon_4 - \epsilon_5 + \epsilon_8 - \epsilon_9 + \epsilon_{12} - \epsilon_{14} - \epsilon_{15})/2.
\end{aligned} \tag{34}
$$

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