PERTURBATION THEORY IN QUANTUM ELECTRODYNAMICS

by

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CONTENTS

SUMMARY

It is possible to give an algebraic description of the quantized free electromagnetic field in which all quantization schemes are regarded as methods of obtaining representations of a quotient algebra of a certain subalgebra of the field algebra of the free electromagnetic field. In particular, the Fermi method of quantization, which was used for many early calculations, can be reformulated as such a representation, called the Fermi representation. In its original form, the Fermi method involved normalization difficulties which led to ambiguities in the calculations. The re-interpretation eliminates these ambiguities.

The Fermi representation is obtained by writing the Fock representation of the field algebra as a direct integral over the spectrum of the supplementary condition operators, and choosing a certain component representation from this decomposition. The component representations are representations of the commutator algebra of the supplementary condition operators, in which the supplementary condition operators act as scalars. They therefore determine a representation of a quotient algebra of the commutator algebra with respect to an ideal generated by the supplementary condition operators. This quotient algebra describes the physical degrees of freedom of the free electromagnetic field.

So far, our description has applied to the free field. The aim of this thesis is to show that the Fermi method can be used to derive methods for perturbation calculations in quantum electrodynamics, and to justify the Feynman rules for S-matrix calculations. Of course there remain the usual problems of interpretation of perturbation methods, since they involve representations of the free field in the infinite past and future, and inequivalent representations at other times. These difficulties are present not only in the Fermi method but in any perturbative study of quantum electrodynamics. Therefore the Fermi method will prove to be a

suitable alternative to more standard methods of quantization, such as the Gupta-Bleuler method.

Certain early papers which used the Fermi method for perturbation calculations, and to derive the S-matrix rules, point out the normalization difficulty which led to the rejection of this method in its original form. In one sense all we need to do in order to justify perturbation calculations on the basis of the Fermi method is to demonstrate that the re-interpretation eliminates this difficulty, and renders valid the treatments given in these early papers.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma, and to the best of my knowledge and belief, contains no material previously published or written by another person except where due reference is made in the text.

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CHAPTER 1

THE FREE ELECTROMAGNETIC FIELD

INTRODUCTION

The most widely used techniques of quantizing the electromagnetic field are the Gupta-Bleuler and radiation gauge methods. The basis for early perturbation calculations in quantum electrodynamics (e.g. [1],[2]) which we refer to as the Fermi method, has not been used since the development of Gupta-Bleuler quantization, because it entails normalization difficulties which other quantization methods avoid. Even in the free field case, the states which satisfy the supplementary condition correspond to vectors with infinite norm. A reformulation of the Fermi method developed by Carey, Gaffney and Hurst ([3],[4],[5]) uses a representation in which all physical states are normalizable. Our main goal is to use this formulation as a basis for perturbation calculations, in such a way that the degree of rigour is comparable with that of the more standard justifications of the Feynman rules.

We shall first consider how the normalization problem arises, and outline several methods, including that proposed in [3], of formulating the quantization in such a way that this problem does not occur. We shall also show that this approach leads to the familiar value for the effective free Hamiltonian:

$$H = \tfrac{1}{2} \int d^3x \ (E^2 + B^2) \ .$$

We shall then consider the treatment of interactions in the different schemes of quantization, reviewing the justification of perturbation calculations using radiation gauge and Gupta-Bleuler quantization, and then showing that the Fermi method can equally be used as a basis for such calculations.

QUANTIZATION OF THE FREE ELECTROMAGNETIC FIELD

Although the electromagnetic field has only two independent components, its potential is usually written as a 4-vector, $\mathcal{A}_\mu$, because this makes it possible to develop a Lorentz covariant theory. The number of independent components is reduced from four to two by requiring gauge invariance and by imposing the supplementary condition. In all the quantization schemes, the supplementary condition is in some sense a generalization of the classical requirement that

$$\partial^\mu \mathcal{A}_\mu = 0 \tag{1.1}$$

We shall use the following notation. Let $\mathcal{A}_\mu(\underline{x})$, $\mu = 0,1,2,3$, be the electromagnetic potential vector, and denote the conjugate momentum by $\Pi_\mu(\underline{x})$. According to the canonical quantization procedure, the non-vanishing commutators will be:

$$[\Pi_\mu(\underline{x}), \mathcal{A}_\nu(\underline{x}')] = i\hbar\, \delta_{\mu\nu}\, \delta(\underline{x} - \underline{x}') \ .$$

The Fourier transforms q_μ and p_μ may be defined by

$$\mathcal{A}_\mu(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, q_\mu(\underline{k})\, e^{i\underline{k}\cdot\underline{x}}$$

and

$$\Pi_\mu(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, p_\mu(\underline{k})\, e^{-i\underline{k}\cdot\underline{x}}$$

and then

$$[q_\mu(\underline{k}), p_\mu(\underline{\ell})] = i\hbar\, \delta_{\mu\nu}\, \delta(\underline{k} - \underline{\ell}) \ .$$

q_μ and p_μ may be written in terms of the photon annihilation and creation operators, a_μ and a_μ^*:

$$q_\mu(\underline{k}) = \sqrt{\frac{\hbar c}{2k}}\, \{a_\mu(\underline{k}) + a_\mu^*(-\underline{k})\},$$

$$p_j(\underline{k}) = -i \sqrt{\frac{\hbar k}{2c}}\, \{a_j(-\underline{k}) - a_j^*(\underline{k})\} - ik_j \sqrt{\frac{\hbar}{2kc}}\, \{a_o(-\underline{k}) + a_o^*(\underline{k})\} \ ,$$

$$p_o(\underline{k}) = -i \sqrt{\frac{\hbar k}{2c}}\, \{a_o(-\underline{k}) - a_o^*(\underline{k})\} + ik_j \sqrt{\frac{\hbar}{2kc}}\, \{a_j(-\underline{k}) + a_j^*(\underline{k})\} \ ,$$

which therefore satisfy

$$[a_\mu(\underline{k}),\ a^*_\nu(\underline{\ell})] = \delta_{\mu\nu}\ \delta(\underline{k} - \underline{\ell})\ . \qquad (1.2a)$$

Initially, we consider these operators acting on the Fock space whose n-particle states can be thought of as being formed by the action of n operators of the form $a^*_\mu(\underline{k})$ on the vacuum state Ω, and we adopt the usual Fock space inner product. It is convenient to define related operators

$$A_j(\underline{k}) = \sqrt{2\hbar kc}\ a_j(\underline{k})\ ,\ A_o(\underline{k}) = \sqrt{2\hbar kc}\ a^*_o(-\underline{k})\ ,$$

because these satisfy the Lorentz invariant relations

$$[A_\mu(\underline{k}),\ A^*_\nu(\underline{\ell})] = 2\hbar kc\ g_{\mu\nu}\ \delta(\underline{k} - \underline{\ell})$$

where $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, $\qquad (1.2b)$

$$[A_\mu(\underline{k}),\ A_\nu(\underline{\ell})] = [A^*_\mu(\underline{k}),\ A^*_\nu(\underline{\ell})] = 0$$

The Lagrangian is chosen to be

$$L = \tfrac{1}{2}\int d^3x\ \Big\{\Big(\frac{\partial \mathcal{A}_\mu}{\partial x^\nu} - \frac{\partial \mathcal{A}_\nu}{\partial x^\mu}\Big)\frac{\partial \mathcal{A}^\mu}{\partial x_\nu} - \Big(\frac{\partial \mathcal{A}_\mu}{\partial x_\mu}\Big)^2\Big\}$$

$$= \int d^3x\ \Big\{\tfrac{1}{2}(E^2 - B^2) - \tfrac{1}{2}\Big(\frac{\partial \mathcal{A}^\mu}{\partial x_\mu}\Big)^2\Big\}\ .$$

Then the Hamiltonian is given by

$$H = \int d^3x\ \Big\{\tfrac{1}{2}c^2\ \Pi^\mu(x)\ \Pi_\mu(x) + (\nabla \times \mathcal{A}\,)^2(x) - c\Big(\Pi^j\ \frac{\partial \mathcal{A}_o}{\partial x_j} + \Pi_o(x)\ \frac{\partial \mathcal{A}^j}{\partial x_j}\Big)\Big\}$$

or in terms of the operators (1.3),

$$H = \int d^3k\ (A^\mu(\underline{k})\ A^*_\mu(\underline{k}) + A^{\mu *}(\underline{k})\ A_\mu(\underline{k}))\ . \qquad (1.3)$$

The supplementary condition is determined by the commuting operators $\chi(x)$ and $\dot{\chi}(x)$, given by

$$\chi(x) = -\ c\Pi_o(x)\ ,$$

$$\dot{\chi}(x) = -\ c\dot{\Pi}_o(x) = c^2\ \frac{\partial \Pi_j}{\partial x_j}$$

or equivalently by the following linear combinations of Fourier transforms of these operators:

$$\chi^+(\underline{k}) = \frac{1}{c}\,\tilde{\dot{\chi}}(\underline{k}) - ik\tilde{\chi}(\underline{k}) ,$$

$$\chi^-(\underline{k}) = \frac{1}{c}\,\tilde{\dot{\chi}}(\underline{k}) + ik\tilde{\chi}(\underline{k}) .$$

In terms of A_μ or p_μ,

$$\chi^+(\underline{k}) = -\,k^\mu A_\mu(-\underline{k}),\ \chi^+(-\underline{k}) = ic\,k^\mu p_\mu(-\underline{k})$$
$$\chi^-(\underline{k}) = -\,ic\,k^\mu p_\mu(\underline{k}) = k^\mu A^*_\mu(\underline{k}) \qquad (1.4)$$

The most obvious way of attempting to render the classical supplementary condition (1.1) in a quantum theory is probably the following requirement:

> Zero is an eigenvalue of χ^+ and χ^-, and the corresponding simultaneous eigenstates are the physical states.

This is the formulation of the supplementary condition in the usual form of the Fermi method, as used by Belinfante [1], for example. In fact, however, zero is in the continuous spectrum of χ^+ and χ^-, and is not an eigenvalue, so that "states" satisfying this form of the supplementary condition cannot be constructed as normalizable vectors.

In the Fock space formalism, the state vectors are essentially described by a product of components corresponding to each kind of photon (transverse, longitudinal and scalar) at each momentum $\underline{k}$, written in terms of eigenvalues of the number operators. The vacuum state Ω is characterized by

$$a_\mu(\underline{k})\Omega = 0 \text{ for } \mu = 0,1,2,3,\ \underline{k} \in \mathbb{R}^3 \qquad (1.5)$$

and a linear vector space is spanned by the n-particle states, for all positive integers n, of the form

$$a^*_{\mu_1}(\underline{k}_1) \ldots a^*_{\mu_n}(\underline{k}_n)\Omega . \qquad (1.6)$$

For a particular momentum k, the coordinate system can be fixed such that $\underline{k} = (0,0,k)$ (where $k = |\underline{k}|$) and then the corresponding components of the supplementary condition operators are

$$\chi^{+}(\underline{k}) = \sqrt{\frac{2\hbar c}{k}}\,(a_3(\underline{k}) - a_o^*(-\underline{k}))$$
$$\chi^{-}(\underline{k}) = \sqrt{\frac{2\hbar c}{k}}\,(a_3^*(\underline{k}) - a_o(-\underline{k})) \qquad (1.7)$$

In this context, Belinfante attempted to construct the components of the product vector corresponding to longitudinal and scalar photons. For convenience, we group together those longitudinal and scalar photons which are restricted by the conditions $\chi^{\pm}(\underline{k}) = 0$ for the same value of $\underline{k}$, and regard the complete Hilbert space as a continuous tensor product [10] of spaces corresponding to each momentum $\underline{k}$. The required component of the product vector is a linear combination

$$\Sigma c_{mn}\,\chi_{m,n} \qquad (1.8)$$

of vectors

$$\chi_{m,n} = \frac{1}{\sqrt{m!n!}}\,(a_3^*(\underline{k}))^m\,(a_o^*(-\underline{k}))^n\,(\Omega)^{\underline{k}}_{0,3}$$

where $(\Omega)^{\underline{k}}_{0,3}$ satisfies

$$a_3(\underline{k})\,(\Omega)^{\underline{k}}_{0,3} = a_o(-\underline{k})\,(\Omega)^{\underline{k}}_{0,3} = 0\ .$$

Then $\chi_{m,n}$ are eigenstates of the number operators

$$N^3_{\underline{k}} = a_3^*(\underline{k})\,a_3(\underline{k})\ ,\quad N^o_{\underline{k}} = a_o^*(-\underline{k})\,a_o(-\underline{k})\ ,$$

with eigenvalues m and n respectively.

Now

$$a_3(\underline{k})\chi_{m,n} = \sqrt{m}\,\chi_{m-1,n}\ ,\quad a_o(-\underline{k})\chi_{m,n} = \sqrt{n}\,\chi_{m,n-1}$$

$$a_3^*(\underline{k})\chi_{m,n} = \sqrt{m+1}\,\chi_{m+1,n}\ ,\quad a_o^*(-\underline{k})\chi_{m,n} = \sqrt{n+1}\,\chi_{m,n+1}\ ,$$

so that the supplementary condition (1.7) leads to the recursion relations for c_{mn}:

$$\left.\begin{aligned}\sqrt{n}\, c_{mn} &= \sqrt{m}\, c_{m-1,n-1} \\ \sqrt{m}\, c_{mn} &= \sqrt{n}\, c_{m-1,n-1}\end{aligned}\right\} \; m,n \geqslant 1$$

with the solution

$$\left.\begin{aligned}c_{mn} &= 0 \quad \text{if} \quad m \neq n \\ c_{nn} &= c_{oo}\end{aligned}\right\} .$$

Thus the component (1.8) can be written

$$c_{oo} \sum_{n=o}^{\infty} \chi_{n,n} ,$$

and has norm

$$\sum_{n=o}^{\infty} |c_{nn}|^2 = |c_{oo}|^2 \sum_{n=o}^{\infty} 1$$

which is divergent. This is true for all $\underline{k}$. We conclude therefore that zero is not an eigenvalue of the supplementary condition operators, and a different formulation of the supplementary condition must be found.

In the Gupta-Bleuler method, normalization of the physical states is made possible by altering the definition of the inner product in such a way that the metric becomes indefinite. Since the field is quantized as if all four components were independent, a weaker form of the supplementary condition must be used. This condition must guarantee that all physical states have positive norm. This quantization procedure is discussed in Schweber [6] and Bogoliubov and Shirkov [11]. As before, consider a Fock representation in which the vacuum state Ω is characterized by (1.7), and a linear vector space is generated by vectors of the form (1.8), on which the usual Fock space inner product is defined by

$$(\Phi,\Psi)_F = \sum_{n=o}^{\infty} \int \frac{d^3k_1}{k_{1_o}} \cdots \int \frac{d^3k_n}{k_{n_o}} \sum_{\mu_1 \ldots \mu_n = 0}^{3} \Phi^{(n)}_{\mu_1 \ldots \mu_n}(\underline{k}_1 \ldots \underline{k}_n)\, \Psi^{(n)}_{\mu_1 \ldots \mu_n}(\underline{k}_1 \ldots \underline{k}_n)$$

where

$$\Psi^{(n)}_{\mu_1 \ldots \mu_n}(k_1 \ldots k_n) = \langle \Omega, a_{\mu_1}(k_1) \ldots a_{\mu_n}(k_n) \Psi \rangle$$

is obtained from the commutation relations (1.2a). A new bilinear form may be defined on this space:

$$(\Phi,\Psi) = (\Phi,\eta\Psi)_F$$

where η is defined by the relations:

$$a_j(\underline{k})\eta = \eta a_j(\underline{k}) \ ,$$

$$a_o(\underline{k})\eta = -\eta a_o(\underline{k}) \ ,$$

$$\eta^2 = 1 \ .$$

This amounts to changing the sign of the adjoint of a_o, and the metric obtained from the new representation is indefinite. The norm of $\chi_{n,n}$ in the appropriate component of this new metric is zero, since the contributions from timelike vectors exactly cancel the contributions from longitudinal vectors. We shall see that a similar cancellation occurs more generally.

In the indefinite metric space, a weaker form of the supplementary condition is sufficient to characterize the physical states:

$$\chi^{(+)}(\underline{k})\Psi = 0 \ , \quad \text{for all } \underline{k} \ , \tag{1.9}$$

for every physical state Ψ.

For guage invariance, observable consequences of the theory must be invariant under transformations

$$\mathcal{A}_\mu(x) \to \mathcal{A}_\mu(x) + \partial_\mu\Lambda(x) \ ,$$

where $\Box\Lambda = 0$, or equivalently

$$a_\mu(\underline{k}) \to a_\mu(\underline{k}) + k_\mu\Lambda^+(\underline{k}) \ ,$$

where

$$\Lambda(x) = \frac{i}{(2\pi)^{3/2}} \int d^3k \sqrt{\frac{\hbar c}{2k}} \left(\Lambda^+(\underline{k})\, e^{-i\underline{k}\cdot\underline{x}} + \Lambda^-(\underline{k})\, e^{i\underline{k}\cdot\underline{x}}\right)$$

and

$$\Lambda^+(\underline{k}) = \overline{\Lambda^-(\underline{k})} \ .$$

Hence the one-particle amplitude

$$\psi_\mu^{(1)}(\underline{k}) = \langle \Omega, a_\mu(\underline{k})\Psi \rangle ,$$

describes a state physically equivalent to the one described by

$$\psi_\mu^{(1)}(\underline{k}) + k_\mu \Lambda^+(\underline{k}) .$$

In parrticular, putting

$$\Lambda^+(\underline{k}) = \frac{\psi_o^{(1)}(\underline{k})}{k^o} ,$$

$$\psi_\mu^{(1)}(\underline{k}) + k_\mu \frac{\psi_o^{(1)}(\underline{k})}{k^o}$$

describes a state in which the time component vanishes. Then from (1.6) and (1.9),

$$k^j \psi_j^{(1)}(\underline{k}) = \underline{k}\cdot\underline{\psi}^{(1)}(\underline{k}) = 0 ,$$

so that the amplitude for a single longitudinal photon is also zero. A similar discussion can be carried out for states containing more than one photon. Thus there is no longer an indefinite number of unphysical photons, and the associated normalization problem will not occur in the Gupta-Bleuler theory.

There is an alternative method of quantization in which it is assumed from the beginning that the longitudinal and timelike components vanish:

$$\mathcal{A}^o(\underline{x}) = 0 = \underline{\nabla}\cdot\underline{\mathcal{A}}(\underline{x}) , \qquad (1.10)$$

and only the two remaining degrees of freedom are quantized. So manifest covariance is not maintained, and rather than requiring gauge invariance of all expressions, a preferred gauge, the radiation gauge, is chosen. It is then necessary to demonstrate that the results of calculations are Lorentz invariant, although quantities which appear at intermediate stages are not. Quantization in the radiation gauge is the procedure discussed in Bjorken and Drell [7]. The Lagrangian density is taken to be

$$\mathcal{L} = -\tfrac{1}{2}\left(\frac{\partial \mathcal{A}_\mu}{\partial x^\nu} - \frac{\partial \mathcal{A}_\nu}{\partial x^\mu}\right)\frac{\partial \mathcal{A}^\mu}{\partial x_\nu} = \tfrac{1}{2}(E^2 - B^2) ,$$

and the conjugate momenta are

$$\Pi^0(x) = 0 \ , \ \Pi^k(x) = E^k(x) \ .$$

The Hamiltonian is

$$H = \tfrac{1}{2} \int d^3x \ (E^2 + B^2) \ .$$

The canonical procedure indicates that the following equal-time commutators vanish:

$$0 = [\mathcal{A}^\mu(\underline{x},t), \mathcal{A}^\nu(\underline{x}',t)] = [\Pi^k(\underline{x},t), \Pi^j(\underline{x}',t)]$$
$$= [\Pi^k(\underline{x},t), \mathcal{A}^0(\underline{x},t)] \ .$$

Then $\mathcal{A}^0$ commutes with all operators and is therefore a c-number.

In the canonical procedure, the remaining equal-time commutators would be given by

$$[\Pi^i(\underline{x},t), \mathcal{A}_j(\underline{x}',t)] = -[E^i(\underline{x},t), \mathcal{A}_j(\underline{x}',t)]$$
$$= i\hbar \, \delta^i_j \, \delta(\underline{x} - \underline{x}') \ ,$$

but this contradicts Gauss' Law, since the divergence of the commutator should therefore be zero, and the divergence of the right hand side is non-zero. This inconsistency can be eliminated by choosing to replace the δ-function by the "transverse δ-function", determined by the requirement that its divergence should vanish:

$$\delta_{ij} \, \delta(\underline{x} - \underline{x}') \to \delta^{tr}_{ij}(\underline{x} - \underline{x}') = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\underline{k}\cdot(\underline{x} - \underline{x}')} \left(\delta_{ij} - \frac{k_i \, k_j}{k^2}\right) \ .$$

The commutation relations are taken to be

$$[\Pi_i(\underline{x},t), \mathcal{A}_j(\underline{x}',t)] = i\hbar \, \delta^{tr}_{ij}(\underline{x} - \underline{x}') \ . \qquad (1.11)$$

Thus $\underline{\nabla} \cdot \underline{\mathcal{A}}$ commutes with all operators, and must be a c-number.

In fact, the gauge may be chosen such that (1.10) holds. Then $\mathcal{A}^\mu(\underline{x},t)$ has only transverse components, and its plane wave expansion takes the form

$$\underline{A}(\underline{x},t) = \int \frac{d^3k}{\sqrt{2k(2\pi)^3}} \sum_{\lambda=1}^{2} \underline{\varepsilon}(\underline{k},\lambda) \{a(\underline{k},\lambda)\, e^{-ik.x} + a^*(\underline{k},\lambda)\, e^{ik.x}\} \quad (1.12)$$

where $\underline{\varepsilon}(\underline{k},\lambda)$ are transverse unit vectors satisfying

$$\underline{\varepsilon}(\underline{k},\lambda)\cdot\underline{k} = 0, \quad \underline{\varepsilon}(\underline{k},\lambda)\cdot\varepsilon(\underline{k},\lambda') = \delta_{\lambda\lambda'} ,$$

and having no time component, and a and a^* satisfy

$$[\, a(\underline{k},\lambda),\ a^*(\underline{k}',\lambda')] = \delta(\underline{k} - \underline{k}')\delta_{\lambda\lambda'} ,$$

$$[\, a(\underline{k},\lambda),\ a(\underline{k}',\lambda')] = [\, a^*(\underline{k},\lambda),\ a^*(\underline{k}',\lambda')] = 0 .$$

As a third possible approach, we consider the method suggested in [3]. The way in which the supplementary condition is introduced in the old Fermi method must be altered, but apart from this, a similar description of physical quantities is retained. We shall first develop this approach from a heuristic point of view, and then outline the rigorous formulation given in [5].

In the old formulation [1] of the supplementary condition for the Fermi methodm the values of χ^+ and χ^- are completely determined in all physical states, so that operators which do not commute with them could not correspond to observable quantities. Let $\mathcal{A}$ be the algebra generated by the photon annihilation and creation operators $a_\mu(\underline{k})$, $a_\mu^*(\underline{k})$, $\mu = 0,1,2,3$, $\underline{k} \in \mathbb{R}^3$, with a Fock representation over a Hilbert space $\mathcal{H}$. Restrict attention to those elements of $\mathcal{A}$ which commute with $\chi^{\pm}(\underline{k})$, $\underline{k} \in \mathbb{R}^3$. Let the subalgebra so defined be $\mathcal{C}$. Elements of $\mathcal{C}$ do not in general correspond to distinct physical quantities: those which differ by an operator of the form

$$\int d^3k\ (C(\underline{k})\ \chi^+(\underline{k}) + D(\underline{k})\ \chi^-(\underline{k})), \quad C(\underline{k}),\ D(\underline{k}) \in \mathcal{C} , \qquad (1.13)$$

will have the same value in all physical states. Elements of the type (1.13) form a two-sided ideal, $\mathcal{J}_0$, say, in $\mathcal{C}$. The above discussion indicates that an element of the quotient algebra

$$\mathcal{C}^L = \mathcal{C}/\mathcal{J}_0 \qquad (1.14)$$

would be a suitable description of the free electromagnetic field. This is the motivation for the reformulation of the Fermi method which is developed in [3] , [4] and [5] and which we shall now outline.

The equal-time commutation relations are equivalent to

$$[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x')] = - i\hbar c g_{\mu\nu} D(x - x') , \tag{1.15}$$

where D satisfies

$$\Box D = 0 ,$$

$$D(\underline{x},o) = 0, \left.\frac{D(\underline{x},t)}{\partial t}\right|_{t=o} = - \delta(\underline{x}) ,$$

so that

$$D(\underline{x}) = \frac{- i}{(2\pi)^3} \int d^4k \, e^{-i\underline{k}\cdot\underline{x}} \, \delta(\underline{k}^2) \, \varepsilon(k) ,$$

where

$$\varepsilon(k) = \mathrm{sgn}(k^o) .$$

If A(F) is heuristically defined as

$$A(F) = \int \mathcal{A}^\mu(x) \, f_\mu(x) \, d^3x \tag{1.16}$$

where $F = (f_\mu)$, $f_\mu \in \mathcal{S}$, then the commutation relations may be written

$$[A(F), A(F')] = i\sigma(F,F')$$

where

$$\sigma(F,F') = - \int dx \, dx' \, D(x - x') \, f^\mu(x) \, f'_\mu(x') .$$

Then the canonical commutation relations in Weyl form are

$$U(F)U(F') = e^{-\frac{1}{2}\sigma(F,F')} \, U(F + F')$$

where $U(F) = e^{iA(F)}$. σ is a degenerate quadratic form but induces a non-degenerate form on certain equivalence classes of the functions $F : F = (f_\mu)$ $F' = (f'_\mu)$ are in the same equivalence class if

$$D*f_\mu = D*f'_\mu$$
$$= \phi_\mu, \text{ say, } \mu = 0,1,2,3 .$$

ϕ satisfies

$$\Box \phi_\mu = 0, \ \mu = 0,1,2,3 \ , \tag{1.17}$$

since D does, and so we may use as labels for the equivalence classes sets of functions of the form $\phi = (\phi_\mu)$ where the ϕ_μ satisfy (1.17).

Let B be the non-degenerate symplectic form which σ induces on these equivalence classes. Then

$$B(\phi,\phi') = - \int_n dn^\mu(x) \left[\phi_\nu(x) \frac{\partial \phi^{\nu'}(x)}{\partial x^\mu} - \frac{\partial \phi_\nu(x)}{\partial x^\mu} \phi^{\nu'}(x)\right]$$

where n is a spacelike surface.

Let M_o be the space of functions

$$D*F \equiv (D*f^\mu)$$

where $f^\mu \in \mathcal{S}$, $\mu = 0,1,2,3$, and let M be the completion of M_o in the Fermi norm ([5], equation (5.3)). Then the Weyl algebra of the vector potential for the electromagnetic field is defined to be the C*-algebra of the canonical commutation relations over (M,B), $\Delta_c(M)$ (as defined in [5]). If $W(\phi_1)$ and $W(\phi_2)$ are elements of $\Delta_c(M)$ then

$$W(\phi_1)\ W(\phi_2) = e^{-\frac{i}{2}B(\phi_1,\phi_2)} W(\phi_1 + \phi_2) \ .$$

A representation of the Poincaré group can be defined on M by

$$\phi \to \phi' : \phi'_\mu(x) = \Lambda^\nu{}_\mu \ \phi_\nu(\Lambda^{-1}(x - a)) \tag{1.18a}$$

and since this leaves B invariant, the Poincaré transformations on the Weyl algebra are given by

$$W \to W' : W'(\phi) = W(\phi') \ . \tag{1.18b}$$

Let S be the subspace of M consisting of wave functions which satisfy

$$\phi_o = 0 = \underline{\nabla} \cdot \underline{\phi} \ .$$

Then the subalgebra of $\Delta_c(M)$ associated with the physical photons is $\Delta_c(S)$, the algebra generated by the elements $W(\phi)$ of $\Delta_c(M)$ for $\phi \in S$. S is not invariant under the Poincaré transformations (1.18a) so $\Delta_c(S)$ is not invariant under (1.18b). The following subspaces of M are invariant under (1.18a): the set N of elements which satisfy

$$\frac{\partial \phi_\mu}{\partial x_\mu} = 0$$

and the set T of elements of the form $\frac{\partial \Lambda}{\partial x^\mu}$ where Λ is a scalar solution of the wave equation. Denote the associated subalgebras of $\Delta_c(M)$ by $\Delta_c(N)$ and $\Delta_c(T)$. Any element (ϕ_μ) of N can be written as the sum of an element of S: $\underline{\phi}(x) - \frac{1}{\nabla^2} \nabla \underline{\nabla}\cdot\phi(x)$ and an element $\left(\frac{\partial \Lambda}{\partial x^\mu}\right)$ of T, where $\Lambda = \frac{1}{\nabla^2} \underline{\nabla}\cdot\underline{\phi}$, so that $N = S \oplus T$, and this decomposition can be used to write $\Delta_c(N)$ as the tensor product $\Delta_c(S) \otimes \Delta_c(T)$.

Let $\phi : \phi_\mu = \frac{\partial \Lambda}{\partial x^\mu}$ be an element of T. Then in similar notation to (1.16),

$$A(\phi) = \int d^4x \, \mathcal{A}^\mu(x) \left(-\frac{\partial f}{\partial x^\mu}\right) ,$$

where f is chosen such that $\Lambda = D*f$. Formal integration by parts yields

$$A(\phi) = \int d^4x \, \frac{\partial \mathcal{A}^\mu(x)}{\partial x^\mu} \, f(x) .$$

So operators $A(\phi)$ where $\phi \in T$ correspond to the supplementary condition operator $\frac{\partial \mathcal{A}^\mu}{\partial x^\mu}$.

There is a natural homomorphism π projecting $\Delta(N)$ onto $\Delta(S)$ (where $\Delta(N)$ is the algebra from which $\Delta_c(N)$ is formed by completion in the appropriate topology [5]) and which can be extended to $\Delta_c(N)$, so that there is an isomorphism between $\Delta_c(S)$ and the invariant factor algebra $\Delta_c(N)/I$, where I is the kernel of π. It can be shown [5] that $\Delta_c(T)$ is the centre of $\Delta_c(N)$. Thus elements of $\Delta_c(N)$ commute with the supplementary condition operator, and $\Delta_c(N)$ corresponds to $\mathcal{C}$, and I to $\mathcal{I}_o$ in (1.14).

The Gupta-Bleuler, radiation gauge and Fermi quantization procedures can be regarded as methods for obtaining representations for $\Delta_c(N)/I$.

The radiation gauge formulation is simply a representation of $\Delta_c(S)$, which is isomorphic to $\Delta_c(N)/I$.

The Gupta-Bleuler method is usually expressed as follows (e.g. [12]). The field algebra is represented on an indefinite metric space $\mathcal{H}$, and the physical states are equivalence classes in a subspace $\mathcal{H}'$ of $\mathcal{H}$. $\mathcal{H}'$ consists of those Φ in $\mathcal{H}$ for which

$$\left(\frac{\partial A_\mu}{\partial x_\mu}\right)^{(-)} \Phi = 0 \ ,$$

and the metric must be semidefinite on $\mathcal{H}'$:

$$< \Phi, \Phi > \geqslant 0 \ .$$

The space of physical states is

$$\mathcal{H}_{\text{phys}} = \mathcal{H}' / \mathcal{H}'' \tag{1.19}$$

where $\mathcal{H}''$ consists of the vectors of zero length:

$$\mathcal{H}'' = \{\Phi : < \Phi, \Phi > = 0\} \ .$$

Thus two different operators in the field algebra will not necessarily have distinguishable action on physical states, and this procedure can be regarded as a prescription for finding a representation of a quotient algebra of a subalgebra of the field algebra.

For the Fermi representation, a direct integral decomposition of $\mathcal{H}$, labelled by the spectrum of the supplementary condition operator, is considered. Corresponding to this decomposition of the Hilbert space, the Fock representation of $\Delta_c(M)$ restricted to $\Delta_c(N)$ may be written as a direct integral of representations of $\Delta_c(N)$ acting on each component. $\mathcal{H}$ may be written:

$$\mathcal{H} = \int_{\oplus T^\perp} d\mu(\zeta) \mathcal{H}_\zeta$$

where $\mu(\zeta)$ is defined by its characteristic function: for $\psi \in T^{\perp}$,

$$\psi \to \exp - \tfrac{1}{4}B(\psi, J_F\psi) \ .$$

Then the Fock representation acting on $\Delta_c(N)$ is given by a direct integral of representations π_ζ of $\Delta_c(N)$, where for each ζ, π_ζ acts on $\mathcal{H}_\zeta$, and for $\phi \in N$,

$$\pi_\zeta(W(\phi)) = e^{iB(\zeta,\phi)} \pi_o(W(\phi)) \ .$$

The component representation $\pi_o(\Delta_c(N))$ is distinguished because it is stable under Lorentz transformations, whereas Lorentz transformations mix the other sectors.

The direct integral π of the representations $\pi_\zeta(\Delta_c(N))$ extends to a representation of $\Delta_c(M)$ in which elements of $\Delta_c(N)$ are diagonalized. The mapping from $\Delta(N)$ to $\Delta(S)$ defined above corresponds to the homomorphism between representations: $\pi(W(\phi)) \to \pi_o(W(\phi))$, for $\phi \in N$. This is essentially a mapping from a representation of $\Delta_c(N)$ over $\mathcal{H}$ to a representation of $\Delta_c(N)/I$ over $\mathcal{H}_o$.

The different representations π_ζ have different kernels K_ζ (corresponding heuristically to ideals of $\mathcal{E}$ generated by

$$\{\chi^+(\underline{k}) - 1\ g^+(\underline{k}),\ \chi^-(\underline{k}) - 1\ g^-(\underline{k})\} \ ,$$

where the displacements $g^{\pm}$ are determined by ζ). So each representation π_ζ determines a representation of a different algebra $\Delta_c(N)/K_\zeta$, describing the observable degrees of freedom.

In general, under a Lorentz transformation $\mathcal{L}$,

$$\mathcal{H}_\zeta \to \mathcal{H}_{\mathcal{L}\zeta} \ .$$

$\mathcal{H}_o$ is invariant, and the representation π_o of $\Delta_c(N)/I$ over $\mathcal{H}_o$ is a description of the electromagnetic field in the Lorentz gauge. Transformations

$$\mathcal{A}_\mu(x) \longrightarrow \mathcal{A}_\mu(x) + \frac{\partial \Lambda}{\partial x^\mu} \ ,$$

such that

$$\Box \Lambda = 0$$

do not alter the description. Other gauge transformations correspond to the selection of a different component space.

THE EFFECTIVE HAMILTONIAN FOR THE FREE FIELD

If a representation of $\Delta_c(N)/I$ describes the free electromagnetic field, all quantities of physical interest should be included in $\Delta_c(N)/I$. In In particular, the Poincaré transformations should be inner automorphisms of $\Delta_c(N)/I$. It is not immediately obvious that this is true, since their generators do not commute with the supplementary condition operators. In a slight abuse of notation, we shall in future refer to any - possibly unbounded - operator which commutes with the supplementary condition operators as an element of $\Delta(N)$. We shall therefore talk only of the algebraically defined $\Delta(N)$, rather than $\Delta_c(N)$ which indicates the choice of a topology.

Since the generators of Poincaré transformations are not in the commutant of $\Delta(T)$, there is no obvious rule for choosing an element of $\Delta(N)/I$ corresponding to each generator. Nonetheless, we shall find that it is possible to construct generators in $\Delta(N)/I$. This is done in [3] for the generators of homogeneous Lorentz transformations. We shall use the same type of construction to obtain the generator of time translations in $\Delta(N)/I$ from the Hamiltonian H.

For each C in $\Delta(N)$, let C^L be the equivalence class (i.e. element of $\Delta(N)/I$) to which C belongs. We wish to find an operator G in $\Delta(N)$ such that G^L can be regarded as the generator of time translations in $\Delta(N)/I$. The time evolution of any C in $\Delta(N)$ is given by

$$C(t) = e^{iHt} C\, e^{-iHt}$$

and is contained in $\Delta(N)$. If G is an element of $\Delta(N)$ such that

$$[H - G, C] \in I \quad \text{for all } C \text{ in } \Delta(N) , \tag{1.21}$$

and G commutes with H, then

$$e^{iHt} = e^{i(H-G)t}\, e^{iGt}$$

so that

$$C(t) = e^{i(H-G)t}\, (e^{iGt}\, C\, e^{-iGt})\, e^{-i(H-G)t} ,$$

and by (1.21)

$$C(t) \in (e^{iGt}\, C\, e^{-iGt}) + I ,$$

so that

$$C^L(t) \equiv (C(t))^L = (e^{iGt})^L\, C^L (e^{-iGt})^L .$$

If G is self-adjoint, $U(t) = e^{iGt}$ will be unitary, so that

$$U^*(t)U(t) = U(t)U^*(t) = 1 .$$

Thus

$$(U(t)^*)^L\, (U(t))^L = (U(t))^L\, (U(t)^*)^L = 1^L$$

and $(e^{iGt})^L$ is unitary. G^L will be a suitable generator.

We construct G as follows. Since G must be an element of $\Delta(N)$,

$$B^+(\underline{k}) \equiv [\chi^+(\underline{k}), H] = [\chi^+(\underline{k}), H - G]$$

and (1.22)

$$B^-(\underline{k}) \equiv [\chi^-(\underline{k}), H] = [\chi^-(\underline{k}), H - G] .$$

From (1.2), (1.3) and (1.4),

$$B^+(\underline{k}) = \hbar kc\, \chi^+(\underline{k}), \quad B^-(\underline{k}) = -\hbar kc\, \chi^-(\underline{k}) .$$

(1.22) will be satisfied if

$$H - G = \int d^3k\, (B^+(\underline{k})\, \xi_1(\underline{k}) + B^-(\underline{k})\, \xi_2(\underline{k})) ,$$

where

$$[\xi_1(\underline{k}), \chi^+(\underline{\ell})] = [\xi_2(\underline{k}), \chi^-(\underline{\ell})] = \delta(\underline{k} - \underline{\ell})$$

and (1.23)

$$[\xi_1(\underline{k}), \chi^-(\underline{\ell})] = [\xi_2(\underline{k}), \chi^+(\underline{\ell})] = 0 .$$

Then (1.21) will be true if $[\xi_i(\underline{k}), C] \in \Delta(N)$ for all C in $\Delta(N)$ and this follows from (1.23) by the application of the Jacobi identity to $\chi^{\pm}$, ξ_i and C.

It remains to find a family of operators $\xi_i(\underline{k})$, i=1,2, satisfying (1.23). From (1.3) and (1.4), suitable forms for ξ_i are

$$\xi_1(\underline{k}) = \alpha^{\rho}(\underline{k})\ A^*_{\rho}(-\underline{k})$$

$$\xi_2(\underline{k}) = \beta^{\rho}(\underline{k})\ A_{\rho}(\underline{k})$$

where α^{ρ} and β^{ρ} are c-numbers which may be chosen such that (1.23) holds. For example,

$$\alpha^{\rho}(\underline{k}) = \frac{-\ k^{\rho}}{4\pi\hbar k^3 c}$$

$$\beta^{\rho}(\underline{k}) = \alpha^{\rho}(\underline{k})\ .$$

Then

$$H - G = \tfrac{1}{4}\int d^3k\ \{\frac{k^i k^j}{k^2}\ (A^*_i(\underline{k})\ A_j(\underline{k}) + A_i(\underline{k})\ A^*_j(\underline{k})) + A^*_o(\underline{k})\ A_o(\underline{k}) + A_o(\underline{k})\ A^*_o(\underline{k})\}\ ,$$

so that

$$G = \tfrac{1}{4}\int d^3k\ \left(\delta^{ij} - \frac{k^i k^j}{k^2}\right)\ (A_i(\underline{k})\ A^*_j(\underline{k}) + A^*_i(\underline{k})\ A_j(\underline{k}))\ . \qquad (1.24)$$

The Fourier components of

$$\tfrac{1}{2}(E^2 + B^2) = \tfrac{1}{2}(\Pi^j\ \Pi_j + (\nabla \times \underline{A})^2)$$

are given by

$$\tfrac{1}{2}\{c^2\ p^{j*}(\underline{k})\ p_j(\underline{k}) + (\underline{k} \times \underline{q}(\underline{k}))^{j*}\ (\underline{k} \times \underline{q}(\underline{k}))_j\}$$

$$= \tfrac{1}{2}\{c^*\ p^j(-\underline{k})\ p_j(\underline{k}) + (\underline{k} \times \underline{q}(\underline{k}))^j\ (\underline{k} \times \underline{q}(\underline{k}))_j\}$$

$$= \tfrac{1}{4}\left(\delta^{ij} - \frac{k^i k^j}{k^2}\right)\ (A^*_i(\underline{k})\ A_j(\underline{k}) + A_i(\underline{k})\ A^*_j(\underline{k}))$$

$$+ \tfrac{1}{2}c^2\ \frac{k^i k^j}{k^2}\ p_i(-\underline{k})\ p_j(\underline{k})\ .$$

Since

$$\tfrac{1}{2}c^2 \frac{k^i k^j}{k^2} p_i(-\underline{k})\, p_j(\underline{k})$$

$$= -\frac{1}{8}\frac{c}{k^2}\left(\chi^+(-\underline{k}) + \chi^-(-\underline{k})\right)\left(\chi^+(\underline{k}) + \chi^-(\underline{k})\right) \varepsilon \mathcal{J}_o ,$$

$\frac{1}{2}\int (E^2 + B^2)\, d^3x$ is in the equivalence class G^L of $\Delta(N)/I$ so that G is equivalent to the usual effective Hamiltonian.

This analysis clearly demonstrates which terms of the Hamiltonian determine the time evolution of physical quantities. But we have ignored the problems associated with unbounded operators which arise in the discussion. The discussion could be given entirely in terms of the bounded operators which lie in the Manuceau C* algebras we have defined (cf [5]). We conclude that the modified Fermi method provides a valid method of deriving the usual expression for the effective Hamiltonian.

CHAPTER 2

THE INTERACTING ELECTROMAGNETIC FIELD

Perturbation methods for calculating interaction cross-sections in quantum electrodynamics can be derived on the basis of the Gupta-Bleuler or radiation gauge quantization schemes. These derivations cannot be completely rigorous, as they involve the assumption that the states of the free field and the interacting field can be accommodated in the same representation, and so, for example, an interacting state evolves from a free state in the distant past. The calculations also involve the necessity for renormalization, and the possibility that, even after renormalization, perturbation expansions may not converge. They are used because the results to which they lead agree well with experiment.

The Fermi method has previously been used as a basis for these calculations, (for example, Belinfante [1]), and the results agree with those obtained from other methods. But the divergences in the standard Fermi method which appear as non-normalizable states lead, in the interacting case, to ambiguities in the results. In the next chapter, we shall discuss interactions using the Fermi method, modified in a similar way to the formulation of the free field discussed in Chapter 1. Then the normalization difficulties and consequent ambiguities do not occur, and the only divergences remaining are those which also occur in calculations based on other methods of quantization. In this chapter, we shall consider the way in which perturbation theory can be derived from the Gupta-Bleuler and radiation gauge schemes.

Perturbation calculations in quantum electrodynamics use the first few terms of the expansion of the S-operator

$$S = 1 - \frac{i}{\hbar c}\int_{-\infty}^{\infty} H_I(t)\,dt + \left(\frac{-i}{\hbar c}\right)^2 \int_{-\infty}^{\infty} H_I(t) \int_{-\infty}^{t} H_I(t')\,dt'\,dt + \ldots$$

where H_I is the interaction Hamiltonian. S is the operator linking the initial state vector to the final state vector in a scattering process (i.e. intuitively, S can be thought of as satisfying $\Psi_I(\infty) = S\,\Psi_I(-\infty)$ where $\Psi_I(t)$ is the interaction picture state vector at time t), and the above expansion for S is obtained formally from the interaction picture equation

$$i\hbar \frac{\partial \Psi_I(t)}{\partial t} = H_I(t)\ \Psi_I(t) \ ,$$

by an iterative procedure. S may also be written as

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar c}\right)^n \int_{-\infty}^{\infty} d^4x_1 \ \ldots \int_{-\infty}^{\infty} d^4x_n \ P(\mathcal{H}_I(x_1) \ \ldots \ \mathcal{H}_I(x_n)) \ , \qquad (2.1)$$

where P orders the operators in decreasing order of their time components:

$$P(\mathcal{H}_I(x_1) \ \ldots \ \mathcal{H}_I(x_n)) = \mathcal{H}_I(x_{j1}) \ \ldots \ \mathcal{H}_I(x_{jn})$$

where

$$x^o_{j1} \geqslant \ldots \geqslant x^o_{jn} \ .$$

S-matrix elements (amplitudes of the form $(\Psi_f, S\Psi_i)$ where Ψ_i and Ψ_f are initial and final states) are evaluated on the basis of one of the quantization procedures.

PERTURBATION CALCULATIONS BASED ON THE GUPTA-BLEULER METHOD

We consider the interaction of the electromagnetic field with the electron field, whose free equations of motion are

$$(\gamma^\mu \frac{\partial}{\partial x^\mu} + m)\psi = 0$$

$$\bar{\psi}(\gamma^\mu \frac{\partial}{\partial x^\mu} - m) = 0 \ .$$

The equations describing the interacting system are

$$\{\gamma^\mu \left(\frac{\partial}{\partial x^\mu} - ie\mathcal{A}_\mu\right) + m\}\psi = 0 \ ,$$

$$\bar{\psi}\{\gamma^\mu \left(\frac{\partial}{\partial x^\mu} + ie\mathcal{A}_\mu\right) - m\} = 0 \ ,$$

$$\Box \mathcal{A}^\mu(x) = j^\mu(x)$$

where

$$j^{\mu}(x) = \frac{-ie}{2} \bar{\psi} \gamma^{\mu} \psi .$$

These correspond to the Hamiltonian H given by

$$H = H_e + H_\gamma - \frac{1}{c} \int d^3x \; j^{\mu}(x) \mathcal{A}_{\mu}(x) , \qquad (2.2)$$

where H_e is the free electron field Hamiltonian, and H_γ is the free electromagnetic field Hamiltonian. An interaction picture can be set up in which the free Hamiltonian is taken to be

$$H_o = H_e + H_\gamma$$

so that the interaction Hamiltonian is

$$H_I = - \frac{1}{c} \int j^{\mu}(x) \mathcal{A}_{\mu}(x) \; d^3x .$$

Then, since spinor operators commute with electromagnetic operators, (2.1) may be written

$$S = \sum_{n=o}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar c}\right)^n \int_{-\infty}^{\infty} d^4x_1 \ldots \int_{-\infty}^{\infty} d^4x_n \; P(j^{\mu_1}(x_1) \ldots j^{\mu_n}(x_n)) \; P(\mathcal{A}_{\mu_1}(x_1) \ldots \mathcal{A}_{\mu_n}(x_n)) . \qquad (2.3)$$

In evaluating S-matrix elements, it will be necessary to use Wick's theorem, which expresses a chronological product as a sum of normal products. The chronological product operator T is defined by

$$T(u_1(x_1) \ldots u_n(x_n)) = u_{j_1}(x_{j_1}) \ldots u_{j_n}(x_{j_n}) \; (-1)^p ,$$

where $x^o_{j_1} \geqslant \ldots \geqslant x^o_{j_n}$, $u_1, \ldots, u_n$ are operators, and p is the number of transpositions of Fermi operators involved in the transition from initial to chronological ordering. In (2.3), P can be replaced by T, since j^{μ} is quadratic in spinor field variables, so that p will always be even.

If u_1 and u_2 are any two operators in the above interacting system, then the difference between their time-ordered product and their normal ordered product $:u_1(x_1) \; u_2(x_2):$ can be expressed in terms of commutators of annihilation and creation operators, and will be a c-number. In fact,

since for the normal product the vacuum expectation value is

$$< :u_1(x_1)\ u_2(x_2): >_o = 0 \ ,$$

it follows that

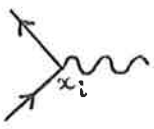
$$T(u_1(x_1)\ u_2(x_2)) = :u_1(x_1)\ u_2(x_2): + <T(u_1(x_1)\ u_2(x_2)) >_o \ .$$

From this formula, an inductive argument leads to Wick's theorem, which gives the following expression for a chronological product:

$$\begin{aligned} T(u_1(x_1) \ \ldots\ u_n(x_n)) &= :u_1(x_1) \ \ldots\ u_n(x_n): \\ &+ \sum_{\substack{\text{permutations} \\ \text{of } 1, \ \ldots\ , \ n}} \Big[< T(u_1(x_1)\ u_2(x_2)) >_o \ :u_3(x_3) \ \ldots\ u_n(x_n): + \ \ldots \\ &+ \begin{cases} <T(u_1(x_1)\ u_2(x_2)) >_o \ \ldots\ < T(u_{n-1}(x_{n-1})\ u_n(x_n)) >_o, & n \text{ even} \\ <T(u_1(x_1)\ u_2(x_2)) >_o \ \ldots\ < T(u_{n-2}(x_{n-2})\ u_{n-1}(x_{n-1})) >_o \ u_n(x_n), & n \text{ odd.} \end{cases} \Big] \end{aligned}$$

(2.4)

Thus the T-product is expressed as a sum of normal products. Feynman graphs are a diagrammatic method of listing all the arrangements in this sum for a given T-product of operators whose amplitude is to be evaluated between given initial and final states. All the normal products in (2.4) have two spinor and one electromagnetic operators with coordinates x_i, for $i=1, \ldots, n$. Therefore n vertices, labelled $x_1, \ \ldots\ , x_n$, each of which is an endpoint of two fermion lines and one photon line, will appear on the corresponding Feynman diagrams:



All the topologically distinct Feynman diagrams satisfying the requirement that this is the only type of vertex which occurs, and that the appropriate initial and final particles appear, will represent all the terms on the right hand side of (2.4). An n^{th} order perturbation calculation uses terms corresponding to diagrams with up to n vertices. An

undirected photon line with endpoints x_i and x_j corresponds to $< T(\mathcal{A}_\mu(x_i)\ \mathcal{A}_\nu(x_j)) >_o$, a directed fermion line from x_i to x_j corresponds to $<T(\bar{\psi}(x_i)\ \psi(x_j)) >_o$ or $<T(\psi(x_i)\ \bar{\psi}(x_j)) >_o$ and photon or electron field variables which appear in the normal products correspond to external lines of the appropriate particle, with one endpoint at x_i.

Now

$$< T(\mathcal{A}_\mu(x)\ \mathcal{A}_\nu(x')) >_o = < \mathcal{A}_\mu(x)\ \mathcal{A}_\nu(x')\ \theta(t - t') + \mathcal{A}_\nu(x')\ \mathcal{A}_\mu(x)\ \theta(t' - t) >_o ,$$

where

$$\theta(t) = \begin{cases} 0 , & t \geqslant 0 \\ 1 , & t > 0 \end{cases} .$$

By the definition of the vacuum, this may be rewritten

$$< T(\mathcal{A}_\mu(x)\ \mathcal{A}_\nu(x')) >_o = <\mathcal{A}_\mu^{(+)}(x)\ \mathcal{A}_\nu^{(-)}(x')\ \theta(t - t') + \mathcal{A}_\nu^{(+)}(x')\ \mathcal{A}_\mu^{(-)}(x)\ \theta(t' - t) >_o , \tag{2.5}$$

which is the amplitude for a photon to be created at x' and annihilated at x for $t > t'$, or created at x and annihilated at x' for $t' > t$. This provides a natural interpretation of the photon propagator symbol on a Feynman diagram.

Adding quantities which are zero by the definition of the vacuum, (2.5) becomes

$$\begin{aligned} < T(\mathcal{A}_\mu(x)\ \mathcal{A}_\nu(x')) >_o &= < [\mathcal{A}_\mu^{(+)}(x), \mathcal{A}_\nu^{(-)}(x')]\ \theta(t - t') \\ &\quad + [\mathcal{A}_\nu^{(+)}(x\ ), \mathcal{A}_\mu^{(-)}(x)]\ \theta(t' - t) >_o \\ &= - i\hbar c\, D_{\mu\nu}^{(+)}(x - x')\ \theta(t - t') + i\hbar c\, D_{\mu\nu}^{(-)}(x - x')\ \theta(t' - t) \\ &= (- \tfrac{1}{2}\hbar c\ (2i\, D_{\mu\nu}^{(+)}(x - x')\ \theta(t - t') - 2i\, D_{\mu\nu}^{(-)}(x - x')\ \theta(t' - t) \\ &= - \tfrac{1}{2}\hbar c\, D_{F\,\mu\nu}(x - x') \qquad \text{(defining } D_{F\,\mu\nu}\text{)} \end{aligned}$$

where

$$D_{\mu\nu} = g_{\mu\nu}\, D .$$

The positive and negative frequency parts of D can be explicitly written

$$D^{(+)}(x) = \frac{-i}{(2\pi)^3} \int e^{-ik\cdot x}\, \theta(k^o)\, \delta(k^2)\, d^4k \ ,$$

$$D^{(-)}(x) = \frac{i}{(2\pi)^3} \int e^{ik\cdot x}\, \theta(k^o)\, \delta(k^2)\, d^4k \ .$$

They may also be written with $\delta(k^2)$ replaced by $\frac{1}{k^2}$ and the k^o integration over an appropriate contour in the complex plane, or equivalently

$$D_{F_{\mu\nu}}(x) = \frac{2i}{(2\pi)^4} \int \frac{d^4k}{k^2 + i\varepsilon} e^{ik\cdot x}\, g_{\mu\nu} \ . \qquad (2.6)$$

In a similar way, it can be established that

$$< T(\psi_\alpha(x)\ \bar{\psi}_\beta(x')) >_o = -\tfrac{1}{2} S_{F_{\alpha\beta}}(x - x')$$

where

$$S_{F_{\alpha\beta}}(x - x') = \frac{-2i}{(2\pi)^4} \int \left(\frac{1}{\gamma\cdot p + m - i\varepsilon}\right)_{\alpha\beta} e^{-ip\cdot x}\, d^4p \ .$$

The expression (2.6) obtained from the expectation value (2.5) apparently involves operators and states with all four degrees of freedom independent. But the $\mathcal{A}_\mu$ could be altered by a gauge transformation, and by addition of terms which vanish in physical states by the supplementary condition. Such changes must not alter observable quantities. For example, in the propagator (2.5), the non-transverse part of $\mathcal{A}_\mu^{(-)}(x)$ may be written as a linear combination of any two independent non-transverse components. These are provided by the supplementary condition χ^+ (defined in (1.6)) and a quantity of the form $\frac{\partial \Lambda}{\partial x_\mu}$. The first of these does not contribute, because the vacuum state must satisfy the supplementary condition. The second cannot contribute to observable quantities, as it can be made to vanish by means of a gauge transformation. (That gauge invariance holds in this case is due to the fact that in calculations, (2.5) will always be multiplied by current terms, and current conservation guarantees that terms of the form

$\frac{\partial \Lambda}{\partial x^{\mu}}$ do not contribute.) Similar arguments hold for the $A_{\mu}^{(+)}$. Thus only transverse components of these operators need to be considered in (2.5). Now the expectation value in a given state of an operator with only transverse components is not affected by the presence or absence of pseudophotons in that state. The state may be replaced by one containing only transverse photons. But the indefinite metric gives the same value as the definite metric for the expectation value in purely transverse states. Thus (2.5) would be unaltered if it referred only to the vacuum for transverse photons and the expectation value relative to the definite metric.

This demonstrates how, in the Gupta-Bleuler method, the restrictions imposed by the supplementary condition and the cancellation achieved by the indefinite metric combine to prevent the unphysical photon components from contributing to observable results.

In order to determine a complete set of rules for obtaining an amplitude from a given set of Feynman diagrams, conventions must be established to account correctly for statistical factors and sign. However, we have concentrated on those aspects of the evaluation process which are different for different techniques of quantizing the electromagnetic field, and without establishing the remaining rules in detail, we now list the rules for obtaining a matrix element from an n^{th} order diagram. ([6], p.471):

1. A factor $\left(\frac{-i}{\hbar c}\right)^{n}$ for the diagram as a whole from the perturbation expansion.
2. A factor $(e\gamma^{\mu})_{\alpha\beta}$ for each vertex.
3. A factor $-\frac{1}{2}\hbar c\, D_F(x_j - x_\ell) g_{\mu\nu}$ for an internal photon line connecting the points x_j and x_ℓ.
4. A factor $(-\frac{1}{2} S_F(x_\ell - x_j))_{\alpha\beta}$ for an internal fermion line directed from x_j to x_ℓ.

5. The appropriate creation or annihilation operator $\psi_\alpha^{(\pm)}(x)$, $\bar{\psi}_\alpha^{(\pm)}(x)$, $\mathcal{A}_\mu^{(\pm)}(x)$ for each external free electron, positron or photon line leaving or arriving at x.

6. A factor (- 1) for each internal closed fermion loop.

PERTURBATION THEORY IN THE RADIATION GAUGE

For the free field in the radiation gauge, $\underline{\nabla}\cdot\underline{\mathcal{A}}$ and $\mathcal{A}_o$ are identically zero. When interactions are introduced, it is still true that they are not independent variables. The equation

$$\underline{\nabla}\cdot\underline{\mathcal{A}} = 0 \tag{2.7}$$

still holds, and $\mathcal{A}_o$ is now given by

$$\mathcal{A}_o(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \; e^{i\underline{k}\cdot\underline{x}} \frac{\tilde{j}_o(\underline{k}\; t)}{\underline{k}^2 c}$$

$$= \int d^3x' \; \frac{j_o(\underline{x}', t)}{4\pi c \; |\underline{x} - \underline{x}'|} \tag{2.8}$$

where $\tilde{j}_\mu$ is the Fourier transform of j_μ .

This is consistent with Gauss' law, since in the interacting case,

$$\underline{\nabla}\cdot\underline{E}(x) = \frac{1}{c}\, j_o(x) \; . \tag{2.9}$$

Hence $\mathcal{A}_o$ is not independent of the matter-field variables, and does not commute with them:

$$[\mathcal{A}_o(\underline{x},t) \; , \; \psi(\underline{x}',t)] = \frac{-e}{4\pi \; |\underline{x} - \underline{x}'|} \psi(x',t) \; . \tag{2.10}$$

For i = 1,2,3, $\mathcal{A}_i$ commutes with the matter-field variables. Also, for each field taken separately, the commutation relations are like the free field relations. $\underline{\mathcal{A}}$ is transverse in the radiation gauge, since (2.7) holds, and it satisfies

$$\Box \underline{\mathcal{A}} = \underline{j}^{tr} \; .$$

$\underline{j}^{tr}$ is the transverse current source, satisfying

$$\underline{\nabla}\cdot\underline{j}^{tr} = 0 \; .$$

It may be written

$$\underline{j}^{tr} = \underline{j} + \frac{\partial \underline{E}_\ell}{\partial t}$$

where $\underline{E}_\ell$ is the longitudinal part of $\underline{E}$, since $\frac{\partial \underline{E}_\ell}{\partial t}$ cancels the longitudinal component of $\underline{j}$.

The Hamiltonian may be written as

$$H = \int d^3x \, \{\bar{\psi}[\gamma_o\gamma_j \, (-i \frac{\partial}{\partial x_j} + \mathcal{A}^j) + \gamma_o m]\psi + \tfrac{1}{2} (E^2 + B^2)\}.$$

Now

$$\int d^3x \, (E^2 + B^2) = \int d^3x \, E_\ell^2 + \int d^3x \, (E_{tr}^2 + B^2) \, , \qquad (2.11)$$

where E_{tr} is the transverse component of $\underline{E}$. In the presence of interactions, we find, from (2.9),

$$\tfrac{1}{2} \int d^3x \, E_\ell^2 = \int d^3x \int d^3x' \, \frac{j_o(\underline{x},t) \, j_o(\underline{x}',t)}{8\pi c^2 \, |\underline{x} - \underline{x}'|} = \int d^3k \, \frac{\tilde{j}_o(\underline{k},t) \, \tilde{j}_o(-\underline{k},t)}{2k^2c^2} \, .$$

(2.12)

The second term in (2.11) is the free field energy for the electromagnetic field. A suitable choice for the interaction picture Hamiltonian is therefore

$$H_I(t) = \int d^3x \, \{-\frac{1}{c} j_\ell(\underline{x},t) \mathcal{A}^\ell(\underline{x},t) + j_o(\underline{x},t) \int d^3x' \, \frac{j_o(\underline{x}',t)}{8\pi c^2 \, |\underline{x} - \underline{x}'|}\} \, .$$

(2.13)

As we noted at the beginning of this chapter, the assumption that perturbation methods are a valid calculation procedure involves the assumption that an interacting state evolves from a free state in the distant past, so the free and interacting fields must be able to be accommodated in the same representation. The interaction picture makes use of this assumption, and the interaction Hamiltonian, and hence the S-matrix, are expressed in terms of the free "in-fields" - from which the interacting fields evolved. In addition to satisfying the free equations of motion, the electromagnetic in-fields have the property that they are

transverse in the radiation gauge. This property is important for S-matrix calculations, so we use notation which emphasizes the fact that in-fields are being used.

In terms of these operators, the S-matrix is

$$S = 1 + \sum_{n=1}^{\infty} \left(\frac{-i}{\hbar c}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n \, T(H_I'(t_1) \ldots H_I'(t_n)), \quad (2.14)$$

where

$$H_I'(t) = H_I^{coul}(t) + H_I^{tr}(t)$$

and

$$\int dt \, H_1^{tr}(t) = -\frac{1}{c} \int j_\mu^{in}(x) \mathcal{A}^{\mu \, in}(x) \, d^4x \, ,$$

$$\int dt \, H_1^{coul}(t) = \tfrac{1}{2} \int d^4z \, d^4z' \, \delta(z_o - z_o') \, j_\mu^{in}(z) \, \frac{n^\mu n^\nu}{4\pi \, |\underline{z} - \underline{z}'|} \, j_\nu^{in}(z') \quad (2.15)$$

where n^μ is a timelike unit vector, equal to (1,0,0,0) in the frame used for (2.13). This definition is consistent with (2.13) since $\mathcal{A}_\mu^{in}$ has only transverse components.

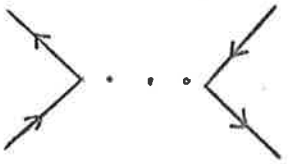
Thus the S-matrix expansion (2.14) includes all terms of the form of those in (2.3), and in addition, all those which can be obtained from an n^{th} order term in (2.3) by replacing up to n of the

$$-\frac{1}{c} \int d^4x \, j_\mu^{in}(x) \mathcal{A}^{\mu \, in}(x)$$

by

$$\int dt \, H_1^{coul}(t) \, .$$

Feynman graphs can be adapted to provide a method of listing all these terms by allowing an additional type of vertex:



to represent the Coulomb interaction between two charges. For any Feynman graph in which at least one Coulomb vertex occurs, there will be another graph, representing a term in the same sum, in which this vertex has been replaced by a propagator:

which is now taken to represent transverse photons only.

For the in-field, $\mathcal{A}^{in}$, (1.10) holds, so that the plane-wave expansion (1.12) can be used to determine the vacuum expectation value $< T(\mathcal{A}^{in}_{\mu}(x)\,\mathcal{A}^{in}_{\nu}(x')) >_0$ which corresponds to an internal photon line between x and x′:

$$< T(\mathcal{A}^{in}_{\mu}(x)\,\mathcal{A}^{in}_{\nu}(x')) > = \tfrac{1}{2}\hbar c\, D^{tr}_{F\mu\nu}(x - x')$$

$$= \int \frac{d^4k}{(2\pi)^2} \frac{e^{-i\underline{k}\cdot(\underline{x} - \underline{x}')}}{k^2 + i\varepsilon} \sum_{\lambda=1,2} \varepsilon_{\mu}(\underline{k},\lambda)\, \varepsilon_{\nu}(\underline{k},\lambda) \ .$$

A complete set of orthogonal vectors is given by $\varepsilon_{\mu}(\underline{k},1)$, $\varepsilon_{\mu}(\underline{k},2)$, n_{μ} and $\hat{k}_{\mu}$, where

$$\hat{k}_{\mu} = \frac{k^{\mu} - (k\cdot n)n^{\mu}}{\sqrt{(k\cdot n)^2 - k^2}}$$

so that

$$\sum_{\lambda=1,2} \varepsilon_{\nu}(\underline{k},\lambda)\, \varepsilon_{\mu}(\underline{k},\lambda) = - g_{\mu\nu} + n_{\mu}n_{\nu} - \hat{k}_{\nu}\hat{k}_{\mu}$$

and

$$D^{tr}_{F\mu\nu}(x - x') = D_{F\mu\nu} - \int \frac{d^4k}{(2\pi)^2} \frac{e^{-i\underline{k}\cdot(\underline{x} - \underline{x}')}}{k^2 + i\varepsilon}$$

$$\left(\frac{k^2\, n_{\nu}n_{\mu} - (k\cdot n)(k_{\nu}n_{\mu} + k_{\mu}n_{\nu}) + k_{\nu}k_{\mu}}{(k\cdot n)^2 - k^2}\right) \qquad (2.16)$$

In the S-matrix, $D^{tr}_{F\mu\nu}(x - x')$ is always multiplied by $j^{\mu}(x)\, j^{\nu}(x')$, so that, by current conservation, terms proportional to k_{μ} or k_{ν} do not contribute. Hence the second term in (2.16), multiplied by current terms, becomes

$$j_{\mu\ in}(x)\, j_{\nu\ in}(x') \int \frac{d^4k}{(2\pi)^2} \frac{e^{-i\underline{k}\cdot(\underline{x} - \underline{x}')}}{(k\cdot n)^2 - k^2} n^{\mu}n^{\nu}$$

or $$\frac{j_{\mu_{in}}(x)\, j_{\nu_{in}}(x')}{4\pi\,|\underline{x} - \underline{x}'|}\, n^{\mu}n^{\nu}\,\delta(t - t') \ .$$

This has the same form as $2H_1^{coul}(t)$. The appropriate factor corresponding just to the Coulomb interaction on a diagram, between x and x', is

$$\frac{n^{\mu}n^{\nu}\,\delta(t - t')}{8\pi\,|\underline{x} - \underline{x}'|} \ .$$

Therefore, if the statistical weight of the diagram containing the Coulomb vertex is twice that of the same diagram with this vertex replaced by a propagator, the non-covariant terms will cancel. Then diagrams containing Coulomb vertices can be ignored if D_F^{tr} is replaced by D_F in terms corresponding to the remaining diagrams, A term of order 2n in H_I^{tr} and m in H_I^{coul} will appear in the S-matrix in the term

$$\frac{1}{(2n + m)!}\,(H_I^{tr} + H_I^{coul})^{2n+m}$$

$$= \frac{1}{(2n + m)!} \sum_{k=o}^{2n+m} \frac{(2n + m)!}{k!\,(2n + m - k)!}\,(H_I^{tr})^{2n+m-k}\,(H_I^{coul})^{k}$$

(appropriately ordered.)

The required term is

$$\frac{1}{m!}\,\frac{1}{(2n)!}\,(H_1^{tr})^{2n}\,(H_1^{coul})^{m} \ .$$

There are (2n)! m! topologically identical graphs obtained by interchanging the m labels of Coulomb vertices among themselves and the 2n labels of photon-electron vertices among themselves. In addition, there are 2^m topologically equivalent graphs obtained by interchanging the pairs z and z' (in the notation of (2.15)) in H_1^{coul}. This cancels with the factor-of-2 difference between H_1^{coul} and the non-covariant part of D_F^{tr}.

Factors referring only to the electron field will not be affected by the change in quantization method. The terms corresponding to initial and final photons will be the same for both methods, since they are taken to be

free field states, and it was established in Chapter 1 that, in the Gupta-Bleuler method, these states contain only transverse photons.

It follows that the Feynman rules as stated earlier can be established on the basis of radiation gauge quantization.

CHAPTER 3

APPLICATION OF THE FERMI METHOD TO THE INTERACTING ELECTROMAGNETIC FIELD

We now return to the revised Fermi quantization procedure which was discussed for the free field in Chapter 1, and attempt to develop the quantization of the interacting field along the same lines (cf. [3]). This formulation of the interacting field can then be used as a basis for scattering calculations, and we shall show (Chapter 4) that it gives rise to the standard Feynman rules.

The Hamiltonian (2.2) for the electromagnetic field interacting with the electron field can be written non-covariantly as

$$H = \int d^3x \left\{ \tfrac{1}{2}c^2 \, \Pi^\mu \Pi_\mu + \tfrac{1}{2}|\underline{\nabla} \times \underline{\mathcal{A}}|^2 - e\left(\Pi_j \frac{\partial \mathcal{A}_o}{\partial x_j} + \Pi_o \frac{\partial \mathcal{A}_j}{\partial x_j}\right) \right.$$

$$\left. - \frac{ie}{\hbar} \mathcal{A}_j \pi \gamma_o \gamma^j \psi - \frac{ie}{\hbar} \mathcal{A}_o \pi \psi - ic\pi \gamma_o \gamma^j \frac{\partial \psi}{\partial x^j} - mc\pi \gamma_o \psi \right\}$$

where π is the conjugate momentum to ψ : $\pi = i\hbar\bar{\psi}$. We wish to quantize the theory corresponding to this Hamiltonian subject to the supplementary conditions specified by χ and $\dot{\chi}$, where $\chi(x) = c\Pi_o(x)$. For the interacting case, these are equivalent to the conditions determined by

$$\begin{aligned} \chi^+(-\underline{k}) &= ic\left[k^\mu p_\mu(\underline{k}) - \frac{i}{c^2} j_o(-\underline{k}) \right] , \\ \chi^-(\underline{k}) &= -ic\left[k^\mu p_\mu(\underline{k}) + \frac{i}{c^2} j_o(-\underline{k}) \right] . \end{aligned} \tag{3.1}$$

In terms of momentum coordinates, the Hamiltonian is given by

$$H = \int d^3k \left\{ \tfrac{1}{2}c^2 p^\mu(-\underline{k}) \, p_\mu(\underline{k}) + \tfrac{1}{2}k^2\left(\delta^{j\ell} - \frac{k^j k^\ell}{k^2}\right) q_j(\underline{k}) \, q_\ell(\underline{k}) \right.$$

$$- ik_j c\left(p_o(\underline{k}) \, q^j(\underline{k}) + p^j(\underline{k}) \, q_o(\underline{k})\right) - \frac{1}{c} q^\mu(\underline{k}) \, j_\mu(-\underline{k})$$

$$\left. - ik_j c\theta(\underline{k}) \, \gamma_o \gamma^j \phi(\underline{k}) - mc\theta(\underline{k}) \gamma_o \phi(\underline{k}) \right\} ,$$

where θ and ϕ are the Fourier transforms of ψ and π respectively.

If the canonical quantization procedure is followed, and an attempt is made to impose the supplementary condition essentially by selecting the eigenspace of the supplementary condition operators with eigenvalue zero, then the following problem arises, exactly as for the free field. The so-called "physical states" selected in this way are not well-defined normalizable vectors on Fock space. However, we shall see that the following approach, discussed in Chapter 1 for the free field, may be used to introduce the supplementary condition in the interacting theory also. We define the quantum theory as a representation of the quotient algebra $\mathcal{E}^L = \mathcal{C}/\mathcal{J}$, where $\mathcal{C}$ is the algebra of operators which commute with the supplementary condition, and $\mathcal{J}$ is the ideal of elements of the form

$$\int d^3k \; (C(\underline{k})\chi^+(\underline{k}) + D(\underline{k})\chi^-(\underline{k})), \quad \text{where} \quad C(\underline{k}),\ D(\underline{k}) \in \mathcal{C} \text{ for all } \underline{k} .$$

Then the physical states are well-defined normalizable vectors in the space on which the representation of $\mathcal{E}^L$ acts, and unambiguous calculation procedures may be developed in this representation.

In terms of the Manuceau algebra notation, $\mathcal{E}$ now corresponds to an algebra including $\Delta(N)$ as well as operators involving the electron field. As it stands, the supplementary condition (3.1) depends on the amount of charge which is present. This complicates the identification of the commutator algebra, and also the description of perturbation theory calculations, where a physical system is pictured as evolving from a state in which no charge is present to interactions with charged particles, and back again. At different times during this evolution, the description of the electromagnetic part of the system will be in terms of different "observable algebras" $\Delta(N)/J$, where J may vary. Such a complication is avoided by performing a "unitary" transformation on all electromagnetic and matter field variables:

$$a \to a^S = S\, a\, \bar{S} , \quad \text{where } S = \exp\left\{\frac{1}{\hbar c}\int \frac{j_o(-k)\, k^\ell q_\ell(k)}{k^2}\, d^3k\right\} . \qquad (3.2)$$

The transformed operators have the following form:

$$
\begin{aligned}
q_\mu^S &= q_\mu \ , \qquad \mu=0,1,2,3 \ , \\
p_o^S &= p_o \ , \\
p_\ell^S(\underline{k}) &= p_\ell(\underline{k}) + \frac{ik_\ell \ j_o(-\underline{k})}{c^2 \ k^2} \ , \qquad \ell=1,2,3 \ , \\
\psi_\sigma^S(\underline{x}) &= \exp(-\alpha) \ \psi_\sigma(\underline{x}) \ , \\
\pi_\sigma^S(\underline{x}) &= \exp(\alpha) \ \pi_\sigma(\underline{x}) \ ,
\end{aligned}
\tag{3.3}
$$

where

$$
\alpha = \frac{ie}{4\pi\hbar c} \int \frac{\operatorname{div} \underline{A}(\underline{x}')}{|\underline{x} - \underline{x}'|} \, d^3x' \ .
$$

In terms of these operators, χ^+ and χ^- are given by

$$
\begin{aligned}
\chi^+(-\underline{k}) &= ick^\mu p_\mu^S(-\underline{k}) \ , \\
\chi^-(\underline{k}) &= -ick^\mu p_\mu^S(\underline{k}) \ .
\end{aligned}
\tag{3.4}
$$

Of course, due to the unbounded operator in the exponent, the transformation (3.2) is not well defined as a unitary transformation on the whole algebra. But it is a convenient shorthand for specifying an automorphism of the algebra, which, however, is not implemented, and so leads to an inequivalent representation.

The commutation relations are unaltered by the transformation, and thus the transformed spinors commute with χ^+ and χ^-, now given by (3.4). So now the introduction of charge does not alter the form of the quotient algebra, and the space on which it is represented.

In terms of the S-transformed operators, the Hamiltonian has the form

$$
\begin{aligned}
H = \int d^3k \ \{ & \tfrac{1}{2}c^2 p^{\mu S}(-\underline{k}) \ p_\mu^S(\underline{k}) + \tfrac{1}{2}k^2 \left(\delta^{j\ell} - \frac{k^j k^\ell}{k^2}\right) q_j^S(\underline{k}) \ q_\ell^S(\underline{k}) \\
& + \frac{1}{2k^2c^2} \ j_o^S(\underline{k}) \ j_o^S(-\underline{k}) + \frac{ik^j}{k^2} \ j_o^S(\underline{k}) \ p_j^S(\underline{k}) - \frac{1}{c} \ q_\ell^S(\underline{k}) \ j_m^S(-\underline{k}) \ \left(\delta^{\ell m} - \frac{k^\ell k^m}{k^2}\right) \\
& - ik^j c(p_o^S(\underline{k}) \ q_j^S(\underline{k}) + p_j^S(\underline{k}) \ q_o^S(\underline{k})) - ik^j c\theta^S(\underline{k}) \ \gamma_o \gamma_j \psi^S(\underline{k}) \\
& - mc\theta^S(\underline{k}) \ \gamma_o \psi^S(\underline{k}) \}
\end{aligned}
$$

The algebra of operators describing both fields may be written as a product

$$\mathcal{A}_e^S \times \Delta(M)^S ,$$

where $\mathcal{A}_e^S$ and $\Delta(M)^S$ are generated by the S-transformed electron and electromagnetic field operators respectively. Since the S-transformed spinors π_σ^S and ψ_σ^S commute with χ^+ and χ^-, all elements of $\mathcal{A}_e^S$ commute with the supplementary condition operators. Thus if $\Delta(N)^S$ is the sub-algebra of $\Delta(M)^S$ which commutes with the supplementary condition, the full algebra of operators commuting with the supplementary condition will be

$$\mathcal{A}_e^S \times \Delta(N)^S .$$

Also, there is an isomorphism between

$$(\mathcal{A}_e^S \times \Delta(N)^S)/I^S$$

and

$$\mathcal{A}_e^S \times (\Delta(N)^S/I^S) .$$

(Since we shall discuss only S-transformed operators, the S-superscript will be omitted.)

In order to establish that time-translation can be implemented in $\mathcal{A}_e \times \Delta(N)/I$, it is necessary to show that the Hamiltonian can be written as a sum of operators

$$H = G + K ,$$

where

$$G \in \mathcal{A}_e \times \Delta(N) \tag{3.6}$$

and

$$[K,C] \in I \quad \text{for all } C \in \mathcal{A}_e \times \Delta(N) . \tag{3.7}$$

We show that this decomposition is possible by constructing suitable operators G and K, in the same way as for the free field Hamiltonian. As in Chapter 1, we do not tackle the problems associated with the discussion

of unbounded operators, and note that a rigorous justification of our analysis should be achieved by a reformulation entirely in terms of Manuceau's C*-algebra formulation of the CCR.

From (3.4) and (3.5),

$$B^+(-\underline{k}) \equiv [\chi^+(-\underline{k}), H] = \hbar kc\, \chi^+(-\underline{k}) ,$$

and

$$B^-(\underline{k}) \equiv [\chi^-(\underline{k}), H] = -\hbar kc\, \chi^-(\underline{k}) .$$

Thus if $\xi_1(\underline{k})$ and $\xi_2(\underline{k})$, $\underline{k} \in \mathbb{R}^3$, are families of operators satisfying

$$[\chi^+(-\underline{k}), \xi_1(\underline{\ell})] = [\chi^-(\underline{k}), \xi_2(\underline{\ell})] = \delta(\underline{k} - \underline{\ell}) ,$$

$$[\chi^+(-\underline{k}), \xi_2(\underline{\ell})] = [\chi^-(\underline{k}), \xi_1(\underline{\ell})] = 0$$

then a possible choice for K will be

$$K_1 = \int d^3k\, (B^+(-\underline{k})\, \xi_1(\underline{k}) + B^-(\underline{k})\, \xi_2(\underline{k})) .$$

Suitable choices for ξ_1 and ξ_2 are

$$\xi_1(\underline{k}) = \frac{k^j q_j(-\underline{k}) - k^o q_o(-\underline{k})}{2\hbar k^2 c} ,$$

$$\xi_2(\underline{k}) = \frac{-k^j q_j(\underline{k}) + k^o q_o(\underline{k})}{2\hbar k^2 c}$$

and then

$$K_1 = \int d^3k\, ik^j c\, (p_j(\underline{k})\, q_o(\underline{k}) + p_o(\underline{k})\, q_j(\underline{k}))$$

so that, from (3.5)

$$G_1 = H - K_1 = \int d^3k \{\tfrac{1}{2}c^2 p^\mu(-\underline{k})\, p_\mu(\underline{k}) + \tfrac{1}{2}k^2 \left(\delta^{j\ell} - \frac{k^j k^\ell}{k^2}\right) q_j(\underline{k})\, q_\ell(\underline{k})$$

$$+ \frac{1}{2k^2c^2}\, j_o(+\underline{k})\, j_o(-\underline{k}) + \frac{ik^j}{k^2}\, j_o(\underline{k})\, p_j(\underline{k}) - \frac{1}{c}\, q_\ell(\underline{k})\, j_m(-\underline{k}) \left(\delta^{\ell m} - \frac{k^\ell k^m}{k^2}\right)$$

$$- ik^j c\theta(\underline{k})\, \gamma_o\gamma_j \phi(\underline{k}) - mc\theta(\underline{k})\, \gamma_o \phi(\underline{k})\} .$$

G and K as constructed do not commute, but if G is defined by

$$G = G_1 - \int d^3k \left[\tfrac{1}{2}c^2 \frac{k^\mu k^\nu}{k^2}\, p_\mu(-\underline{k})\, p_\nu(\underline{k}) + \frac{ik^j}{k^2}\, j_o(\underline{k})\, p_j(\underline{k})\right]$$

then G and K = H - G will commute. G differs from G_1 by an element of I. (We could write

$$G = G_1 - \int d^3k \{\tfrac{1}{2}\chi^+(-\underline{k})\,\chi^-(\underline{k}) - \tfrac{1}{2}\frac{c}{k^2}\,j_o(\underline{k})\,[\chi^+(\underline{k}) + \chi^-(\underline{k})]\}\ .)$$

Thus G and K still satisfy (3.6) and (3.7). The generator of time translation in $\mathcal{A}_e \times \Delta(N)/I$ is

$$G_1^L = G^L = \int d^3k \{\tfrac{1}{2}c^2 \left[\left(\delta^{j\ell} - \frac{k^j k^\ell}{k^2}\right) p_j(-\underline{k})\, p_\ell(\underline{k})\right]^L$$

$$+ \tfrac{1}{2}k^2 \left[\left(\delta^{j\ell} - \frac{k^j k^\ell}{k^2}\right) q_j(\underline{k})\, q_\ell(\underline{k})\right]^L$$

$$+ \frac{1}{2k^2c^2}\, j_o^L(\underline{k})\, j_o^L(-\underline{k}) - \frac{1}{c}\, j_m^L(-\underline{k}) \left[q_\ell(\underline{k}) \left(\delta^{\ell m} - \frac{k^\ell k^m}{k^2}\right)\right]^L$$

$$- ik^j c\theta^L(\underline{k})\, \gamma_o\gamma_j \phi^L(\underline{k}) - mc\theta^L(\underline{k})\, \gamma_o \phi^L(\underline{k})\} \ . \qquad (3.8)$$

The element G^L of $\mathcal{A}_e \times \Delta(N)/I$ such that G satisfies (3.6) is essentially unique. Suppose (G + G′) is a possible alternative. Then by (3.6), G′ commutes with $\chi^\pm$, so that it is possible to discuss the corresponding equivalence class, G'^L. Also, by (3.7), $[G',C] \in I$ for all $C \in \Delta(N)$, so that

$$[G'^L, C^L] = I = 0^L \quad \text{for all } C^L \text{ in } \mathcal{A}_e \times \Delta(N)/I \ .$$

Thus, if $\mathcal{A}_e \times \Delta(N)/I$ is irreducible, G'^L must be a multiple of the identity, 1 + I. Hence the element G^L of $\mathcal{A}_e \times \Delta(N)/I$ is determined uniquely up to an arbitrary c-number.

G^L in (3.8) has the form of the sum of the free electromagnetic Hamiltonian, the free electron field Hamiltonian, and H_I^L defined by

$$H_I^L = \int d^3k \{\frac{1}{c}\, j_m^L(-\underline{k}) \left[q_\ell(\underline{k}) \left(\delta^{\ell m} - \frac{k^\ell k^m}{k^2}\right)\right]^L + \frac{1}{2k^2c^2}\, j_o^L(\underline{k})\, j_o^L(-\underline{k})\}$$

$$= \int d^3k \{-\frac{1}{c}\, j^{mL}(-\underline{k})\, q_m^{trL}(\underline{k}) + \frac{1}{2k^2c^2}\, j_o^L(\underline{k})\, j_o^L(-\underline{k})\}$$

where

$$q_m^{tr\,L}(\underline{k}) = \left[q^{\ell}(\underline{k})\ \left(\delta_{\ell m} - \frac{k_\ell k_m}{k^2}\right)\right]^L .$$

This suggests an interaction picture in which the Hamiltonian is divided up in such a way that the interaction Hamiltonian is given by

$$H_I^L(t) = \int \left\{-\frac{1}{c}\, j^{mL}(-\underline{k},t)\ q_m^{tr\,L}(\underline{k},t) + \frac{1}{2k^2c^2}\, j_o^L(\underline{k},t)\ j_o^L(-\underline{k},t)\right\} d^3k\ , \tag{3.9}$$

where for each operator $A^L(\underline{k})$,

$$A^L(\underline{k},t) = e^{iH_o^L t/\hbar}\ A^L(\underline{k})\ e^{-iH_o^L t/\hbar}$$

and

$$H_o^L = G^L - H_I^L\ .$$

The Schrödinger equation for the interaction picture is

$$i\hbar\,\frac{\partial\Psi(t)}{\partial t} = H_I^L(t)\ \Psi(t)\ . \tag{3.10}$$

Schwinger [2], [8] did calculations in a scheme formally similar to this one (but with a supplementary condition which leads to normalization problems), maintaining explicit covariance of all expressions. We shall consider some of his calculations in Chapter 5, and in preparation for this, we now discuss his formalism, modified in the usual way to avoid the normalization problem.

The interaction Hamiltonian in $\mathcal{A}_e \times \Delta(N)/I$, (3.9), was obtained after the transformation of the full Hamiltonian to the Lorentz gauge, and taking account of the supplementary condition. It involves only transverse components of the electromagnetic field; in fact, it has the same form as the interaction Hamiltonian in the radiation gauge. If the interaction picture is adopted before transforming to the Lorentz gauge and applying the supplementary condition, the interaction Hamiltonian density corresponding to the Schrödinger Hamiltonian (2.2) is

$$\mathcal{H}_I(x) = -\frac{1}{c}\, j^{\mu}(x)\ \mathcal{A}_{\mu}(x)\ . \tag{3.11}$$

The corresponding supplementary condition operator is

$$\frac{\partial \mathcal{A}^{\mu}}{\partial x^{\mu}}(x) - \int_{\sigma} d\sigma^{\mu'} \frac{1}{c} j_{\mu}(x') D(x - x') , \qquad (3.12)$$

where x, x' lie on the arbitrary spacelike surfaces σ, σ' respectively.

Individual terms in the sum $j_{\mu}\mathcal{A}^{\mu}$ do not commute with this supplementary condition operator. To determine whether the interaction Hamiltonian itself is in Δ(N), we first write it in terms of the longitudinal, transverse and timlike components of $\mathcal{A}_{\mu}$. In a frame where the timelike component is always in the direction n = (1,0,0,0), the transverse components will be given by the Fourier transform of $q_{\ell}^{tr}(\underline{k})$, $a_{\ell}(x)$, say. Then the timelike component is $\mathcal{A}^{0}(x)$, and the longitudinal component, $(\mathcal{A}^{\ell}(x) - a^{\ell}(x))$ has the form $\frac{\partial \Lambda'(x)}{\partial x_{\ell}}$ for some scalar field Λ' satisfying $\Box \Lambda' = 0$.

More generally, in a frame of reference specified by the timelike vector n_{μ} (which may depend on x), the field $\mathcal{A}_{\mu}$ written as a sum of timelike, longitudinal and transverse components becomes

$$\mathcal{A}_{\mu}(x) = n_{\mu}n_{\nu}\frac{\partial \Lambda}{\partial x_{\nu}}(x) - \left(\frac{\partial}{\partial x^{\mu}} + n_{\mu}n_{\nu}\frac{\partial}{\partial x_{\nu}}\right)\Lambda'(x) + a_{\mu}(x) , \qquad (3.13)$$

where

$$\Box \Lambda = \Box \Lambda' = \Box a_{\mu} = 0 \quad , \quad \text{and} \quad \frac{\partial a_{\mu}(x)}{\partial x_{\mu}} = 0 = n_{\mu}a^{\mu}(x) .$$

Then the Hamiltonian density becomes

$$\mathcal{H}_{I}(x) = j^{\mu}(x)\, n_{\mu}n_{\nu}\frac{\partial}{\partial x_{\nu}}(\Lambda(x) - \Lambda'(x)) - j^{\mu}(x)\frac{\partial \Lambda'(x)}{\partial x^{\mu}} + j^{\mu}(x)\, a_{\mu}(x) , \qquad (3.14)$$

and the supplementary condition operator (3.12) is

$$n_{\mu}n_{\nu}\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x_{\nu}}(\Lambda(x) - \Lambda'(x)) - \int_{\sigma'} d\sigma'\, n^{\nu}\, j_{\nu}(x')\, D(x - x')$$

or

$$(\Lambda(x) - \Lambda'(x)) - \int_{\sigma'} d\sigma'\, n^{\nu}\, j_{\nu}(x')\, \mathcal{D}(x - x') , \qquad (3.15)$$

where $\mathcal{D}$ is determined by

$$\left.\begin{aligned} \Box^2 \mathcal{D}(x) &= 0 \ , \\ \left(n_\mu \frac{\partial}{\partial x_\mu}\right)^2 \mathcal{D}(x) &= D(x) \end{aligned}\right\} \tag{3.16}$$

Then the commutation relations (1.15) are equivalent to

$$\begin{aligned} [\mathcal{A}_\mu(x), \mathcal{A}_\nu(x')] &= - i\hbar c\, g_{\mu\nu} D(x - x') + i\hbar c \left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right. \\ &\quad \left. + \left(n_\mu \frac{\partial}{\partial x^\nu} + n_\nu \frac{\partial}{\partial x^\mu}\right) n^\lambda \frac{\partial}{\partial x^\lambda}\right] \mathcal{D}(x - x') \ , \end{aligned} \tag{3.17}$$

$$[\Lambda(x), \Lambda(x')] = - [\Lambda'(x), \Lambda'(x')] = - i\hbar c\, \mathcal{D}(x - x') \ ,$$

$$[\Lambda(x), \Lambda'(x')] = [\Lambda(x), \mathcal{A}_\mu(x')] = [\Lambda'(x), \mathcal{A}_\mu(x')] = 0$$

The canonical transformation (3.2) makes the supplementary condition independent of the electron field, and it is then easy to tell by inspection which operators are in $\Delta(N)$. In the present notation, the transformation (3.2) is given by

$$A(x) \to e^{-i\Sigma(\sigma)}\, A(x)\, e^{i\Sigma(\sigma)}$$

for any A in $\mathcal{A}$, where σ is a spacelike surface containing x, with normal $n_\mu(x)$, say, and

$$\Sigma(\sigma) = \frac{1}{\hbar c} \int_\sigma \frac{1}{c} j^\mu(x)\, \Lambda'(x)\, d\sigma_\mu \ . \tag{3.18}$$

Under this transformation, the supplementary condition operator simplifies to

$$\Lambda(x) - \Lambda'(x) \tag{3.19}$$

while $\mathcal{H}_I(x)$ becomes

$$\begin{aligned} \mathcal{H}_{I,\Sigma}(x) &= - \frac{1}{c} j_\mu(x) \left(\mathcal{A}_\mu(x) + n_\mu n_\nu \frac{\partial}{\partial x_\nu} (\Lambda(x) - \Lambda'(x))\right) \\ &\quad + \frac{1}{c^2} \int_\sigma d\sigma'_\lambda\, j^\lambda(x')\, j^\mu(x) \left(\tfrac{1}{2} \frac{\partial \mathcal{D}(x - x')}{\partial x^\mu} + n_\mu n_\nu \frac{\partial \mathcal{D}(x - x')}{\partial x_\nu}\right) . \end{aligned} \tag{3.20}$$

Clearly, $\mathcal{H}_{I,\Sigma}(x)$ commutes with the operator (3.19). It is therefore an element of $\mathcal{A}_e \times \Delta(N)$, and belongs to one of the equivalence classes in $\mathcal{A}_e \times \Delta(N)/I$. This equivalence class may be written

$$\mathcal{H}^L_{I,\Sigma}(x) = -\frac{1}{c} j^L_\mu(x) \mathcal{A}^{\mu\,L}(x) + \frac{1}{c^2}\int_\sigma d\sigma'_\lambda \, j^{\lambda\,L}(x') \, j^{\mu\,L}(x) \left(\tfrac{1}{2} \frac{\partial \mathcal{D}(x - x')}{\partial x^\mu} + n_\mu n_\nu \frac{\partial \mathcal{D}(x - x')}{\partial x_\nu}\right) . \tag{3.21}$$

The interacting electromagnetic field is described as a representation of $\mathcal{A}_e \times \Delta(N)/I$ on a Hilbert space of states, and the equation of motion in this representation corresponding to the interaction Hamiltonian (3.21) is

$$i\hbar \frac{\delta \Psi[\sigma]}{\delta \sigma(x)} = H^L_{I,\Sigma}(x) \; \Psi[\sigma] . \tag{3.22}$$

It is not necessary to impose the supplementary condition in conjunction with this equation. It has already been accounted for in defining the algebra $\mathcal{A}_e \times \Delta(N)/I$, to which the Hamiltonian $\mathcal{H}^L_{I,\Sigma}(x)$ belongs. For a given calculation, all the necessary transformations and rearrangements of the equation of motion may now be carried out wholly within the representation of the quotient algebra, starting from equation (3.22).

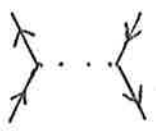
In the next chapter, we shall derive the Feynman rules for scattering calculations in this formlism, and in the following chapter, we shall consider other calculations in the quotient algebra.

CHAPTER 4

EVALUATION OF S-MATRIX ELEMENTS IN THE FERMI METHOD

In view of the correspondence between calculations involving operators in the quotient algebra $\mathcal{A}_e \times \Delta(N)/I$ and those carried out in terms of operators in the full field algebra $\mathcal{A} = \mathcal{A}_e \times \Delta(M)$, it seems reasonable to expect that the formulation of the Fermi method as a representation of the quotient algebra will lead to the standard rules for the evaluation of S-matrix elements. In this chapter, we shall demonstrate that the Feynman rules for scattering calculations can in fact be justified on the basis of the revised Fermi method. The discussion of scattering calculations based on the Gupta-Bleuler method in Chapter 2 showed that these rules follow most directly from a formulation in which the components of the field are not separated into physical and unphysical parts. However, we have also seen that quantization in the radiation gauge leads to the same results. The form of the interaction Hamiltonian in the revised Fermi method (3.9) is similar to that found in the radiation gauge (2.13): they both consist of a Coulomb term plus the inner product of the current with the transverse electromagnetic field.

Thus it may be expected that the terms in the S-matrix for the revised Fermi method will correspond to the same Feynman diagrams as those used for the radiation gauge calculations. These diagrams involve two kinds of vertices: those representing Coulomb interactions,



and the electron-photon vertex

In order to perform perturbation calculations in the quotient algebra formulation of the Fermi method, we assume that the S-matrix S^L is an element of $\mathcal{A}_e \times \Delta(N)/I$, and apply the usual iterative procedure to the interaction equation (3.10) or (3.22) in order to obtain a perturbation expansion in $\mathcal{A}_e \times \Delta(N)/I$:

$$S^L = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar c}\right)^n \int_{-\infty}^{\infty} dt_1 \ \ldots \ \int_{-\infty}^{\infty} dt_n \ P(H_1^L(t_1) \ \ldots \ H_1^L(t_n)) \qquad (4.1)$$

where $H_1^L(t)$ is given by (3.9), or (3.21), with all operators replaced by their equivalence classes in $\mathcal{A}_e \times \Delta(N)/I$. As anticipated in the above discussion, the terms of S^L correspond to those in the expansion (2.14) for S, obtained in the radiation gauge. We wish to conclude that, as in the radiation gauge formulation, the S-matrix element for given initial and final states has a term corresponding to each Feynman diagram with the appropriate external states, and involving the two types of vertex mentioned above. This will certainly follow if Wick's theorem for time-ordered products holds in the present formulation. Wick's theorem is established by an inductive argument from the identity

$$T(\phi_1(x) \ \phi_2(x)) = :\phi_1(x) \ \phi_2(x): + < T(\phi_1(x) \ \phi_2(x)) >_{vac} , \qquad (4.2)$$

(where ϕ_i may represent any electromagnetic or electron field operator occurring in the S-matrix expansion (4.1). Thus the only electromagnetic operators which need to be considered are the transverse components.) To check that the identity (4.2) holds, we must verify first that the form of the commutation relations in $\mathcal{A}_e \times \Delta(N)/I$ implies that different orderings of the product of operators $\phi_1(x_1) \ \phi_2(x_2)$ differ by a c-number, so that

$$T(\phi_1(x_1) \ \phi_2(x_2)) = :\phi_1(x_1) \ \phi_2(x_2): + \text{c-number} , \qquad (4.3)$$

and secondly that a normal-ordered product has zero vacuum expectation value.

Since the mapping from $\mathcal{A}_e \times \Delta(N)$ to $\mathcal{A}_e \times \Delta(N)/I$ is a homomorphism, we deduce from the commutation and anticommutation relations of operators in $\mathcal{A}_e \times \Delta(N)$ that

$$\{\psi^L(x_1),\ \bar{\psi}^L(x_2)\} = \{\psi(x_1),\ \bar{\psi}(x_2)\}^L = \delta(x_1 - x_2)$$

and, by (3.17), that

$$\left[a_\mu^L(x_1),\ a_\nu^L(x_2)\right] = -\,i\hbar c\, g_{\mu\nu}\, D(x_1 - x_2)$$
$$+\,i\hbar c \left[\frac{\partial}{\partial x_1^\mu}\,\frac{\partial}{\partial x_1^\nu} + \left(n_\mu \frac{\partial}{\partial x_1^\nu} + n_\nu \frac{\partial}{\partial x_1^\mu}\right) n^\lambda \frac{\partial}{\partial x_1^\lambda}\right] \mathcal{D}(x_1 - x_2)\ . \tag{4.4}$$

Since only the transverse components of the electromagnetic field appear in the S-matrix (4.1), these commutation relations are sufficient to establish (4.3). In order to establish (4.2), it remains to discuss the definition of the vacuum in $\mathcal{A}_e \times \Delta(N)/I$, and to show that normal-ordered products have zero vacuum expectation value. The usual characterization of the vacuum Ω for the electron field can be translated into the present formulation:

$$\psi^{(+)L}(x)\Omega = \bar{\psi}^{(+)L}(x)\Omega = 0\ . \tag{4.5}$$

(The relationships between some different conventions for positive and negative frequency components are given by Bogoliubov and Shirkov [8] p.653. We follow Schwinger's convention, in which (4.5) holds and in a normal-ordered product, all negative frequency components stand to the left of all positive frequency components.) Hence any normal-ordered product of electron field operators in $\mathcal{A}_e$ will have zero vacuum expectation value. Positive frequency components of the transverse electromagnetic field should also satisfy

$$a_\mu^{(+)L}(x)\Omega = 0\ , \tag{4.6}$$

and since the only physical free photons are transverse modes, the definition

of the vacuum should not make reference to other modes. We now check that (4.6) is a Lorentz invariant requirement, i.e. it is independent of the vector n^μ introduced in (3.13). Consider an alternative definition of the transverse components of a field:

$$\bar{a}_\mu(x) = a_\mu(x) - \bar{n}_\mu \bar{n}_\nu \frac{\partial}{\partial x_\nu} \Lambda(x) + \left(\frac{\partial}{\partial x^\mu} + \bar{n}_\mu \bar{n}_\nu \frac{\partial}{\partial x_\nu}\right) \Lambda'(x) ,$$

where $\bar{n}_\mu$ is related to n_μ by a proper Lorentz transformation. Then

$$\bar{a}_\mu(x) = a_\mu(x) + (n_\mu n_\nu - \bar{n}_\mu \bar{n}_\nu) \frac{\partial}{\partial x_\nu} (\Lambda(x) - \Lambda'(x)) ,$$

so that $\bar{a}_\mu$ and a_μ differ by an element of the ideal generated by the supplementary condition operator and hence

$$\bar{a}_\mu^L(x) = a_\mu^L(x) ,$$

and in particular

$$\bar{a}_\mu^{(+)L}(x) = a_\mu^{(+)L}(x) .$$

Thus the definition of the vacuum (4.5), (4.6) is Lorentz invariant. It also guarantees that a normal-ordering of any term in the S-matrix (4.1) will have zero vacuum expectation value. This fact, together with (4.3) leads to the identity (4.2) from which Wick's theorem follows (cf. [7], § 17.4). To perform a perturbation calculation, Wick's theorem may be used to expand the S-matrix (4.1) as a sum of terms consisting of a normal-ordered product multiplied by 2-point vacuum expectation values, and appropriate matrix elements may be calculated. Alternatively, one can write down all the topologically distinct Feynman diagrams, involving both kinds of vertex, with the appropriate external lines. Then each diagram corresponds to a term in the expansion of the S-matrix element, under the following correspondence:

$< T(a^{\mu L}(x_i)\ a^{\nu L}(x_j)) >$ corresponding to an internal photon line between x_i and x_j;

$< T(\bar{\psi}^L_\alpha(x_i)\ \psi^L_\beta(x_j)) >$ corresponding to an internal fermion line from x_i to x_j;

the appropriate creation or annihilation operator $\psi^{(\pm)\,L}_\alpha(x)$, $\bar{\psi}^{(\pm)\,L}_\alpha(x)$, $a^{(\pm)\,L}_\mu(x)$ for each external free electron, positron or photon line leaving or arriving at x;

a factor $(e\gamma^\mu)_{\alpha\beta}$ for each photon-electron vertex, and for each end of every Coulomb interaction vertex.

A Coulomb interaction vertex, with the photon lines adjoining it corresponds to the Coulomb term in the Hamiltonian density. Thus those terms of the S-matrix which correspond to a diagram containing that vertex include a factor

$$\int d^4x \int_\sigma n^\mu\, d\sigma'\, \frac{1}{c^2}\, j^\nu(x)\, j^\mu(x') \left(n_\nu n^\lambda \frac{\partial}{\partial x^\lambda} + \tfrac{1}{2} \frac{\partial}{\partial x^\nu}\right) \mathcal{D}(x - x') \quad . \quad (4.7)$$

(choosing for each x a surface σ with constant normal n^μ so that $d\sigma'_\mu = n_\mu\, d\sigma'$).

The terms corresponding to diagrams in which the vertex has been replaced by a photon propagator contain a factor

$$\int d^4x \int d^4x' \frac{1}{c^2}\, j^\nu(x)\, j^\mu(x') < T(a^L_\mu(x)\, a^L_\nu(x')) >_o \ ,$$

corresponding to the propagator and adjoining electron lines.

To find an expression for the term $< T(a^L_\mu(x)\, a^L_\nu(x')) >_o$ in terms of the commutator function D, we use the commutation relations (4.4) for the a^L_μ:

$$< T(a^L_\mu(x)\, a^L_\nu(x')) >_o = < \left[a^{(+)\,L}_\mu(x), a^{(-)\,L}_\nu(x')\right] > \theta(x^o - x^{o'}) + < a^{(+)}_\nu(x'), a^{(-)}_\mu(x) > \theta(x^{o'} - x^o)$$

$$= - \tfrac{1}{2}\hbar c \, g_{\mu\nu} \, D_F(x - x') + \tfrac{1}{2}\hbar c \left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + \left(n_\mu \frac{\partial}{\partial x^\nu} + n_\nu \frac{\partial}{\partial x^\mu}\right) n^\lambda \frac{\partial}{\partial x^\lambda}\right] \left(n^\rho \frac{\partial}{\partial x^\rho}\right)^{-2} D_F(x - x') \, , \tag{4.8}$$

where D_F is the usual propagator function,

$$D_F(x) = 2i(D^{(+)}(x) \, \theta(x^o) - D^{(-)}(x) \, \theta(- x^o)) \, . \tag{4.9}$$

The first term in (4.8) is the propagator function corresponding to a photon line in the standard Feynman rules. We shall show that the second term exactly cancels the Coulomb term (4.7). Then as in the radiation gauge calculation, the correct results can be obtained by omitting diagrams involving Coulomb vertices, and writing down the propagator function $\frac{1}{2}\hbar c \, g_{\mu\nu} \, D_F(x - x')$ corresponding to a photon line from x to x'. The statistical weighting of diagrams will be the same as in the radiation gauge calculations discussed in Chapter 2, and hence the required cancellations will occur if the S-matrix term obtained from the second term in (4.8) is equal to minus twice that from the Coulomb term (4.7).

From the second part of (4.8), including adjoining current terms, we obtain

$$\int d^4x \int d^4x' \, \frac{1}{c^2} \, j^\nu(x') \, j^\mu(x) \, \tfrac{1}{2}\hbar c \left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + \left(n_\mu \frac{\partial}{\partial x^\nu} + n_\nu \frac{\partial}{\partial x^\mu}\right) n^\lambda \frac{\partial}{\partial x^\lambda}\right] \left(n^\rho \frac{\partial}{\partial x^\rho}\right)^{-2} D_F(x - x')$$

$$= i\hbar c \int d^4x \int d^4x' \, \frac{1}{c^2} \, j^\nu(x') \, j^\mu(x) \left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + \left(n_\mu \frac{\partial}{\partial x^\nu} + n_\nu \frac{\partial}{\partial x^\mu}\right) n^\lambda \frac{\partial}{\partial x^\lambda}\right] \left(n^\rho \frac{\partial}{\partial x^\rho}\right)^{-2} (D^{(+)}(x - x') \, \theta(x^o - x^{o'})$$

$$- D^{(-)}(x - x') \, \theta(x^{o'} - x^o)) \qquad \text{(by (4.9))}$$

$$= i\hbar c \int d^4x \left[\int_{-\infty}^{\sigma(\tau)} d^4x' \, \frac{1}{c^2} \, j^\nu(x') \, j^\mu(x) \left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right.\right.$$

$$+\left(n_\mu \frac{\partial}{\partial x^\nu} + n_\nu \frac{\partial}{\partial x^\mu}\right) n^\lambda \frac{\partial}{\partial x^\lambda}\Big] \left(n^\rho \frac{\partial}{\partial x^\rho}\right)^{-2} D^{(+)}(x - x')$$

$$- \int_{\sigma(\tau)}^{\infty} d^4x' \frac{1}{c^2} j^\nu(x') j^\mu(x) \Big[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + \left(n_\mu \frac{\partial}{\partial x^\nu}\right.$$

$$\left. + n_\nu \frac{\partial}{\partial x^\mu}\right) n^\lambda \frac{\partial}{\partial x^\lambda}\Big] \left(n^\rho \frac{\partial}{\partial x^\rho}\right)^{-2} D^{(-)}(x - x')\Big]$$

(where $\tau = n_\mu x^\mu$, and $n_\mu x'^\mu = \tau$ for $x' \in \sigma(\tau)$)

$$= i\hbar c \int d^4x \int_{\sigma(\tau)} d\sigma' \frac{1}{c^2} j^\nu(x') j^\mu(x) \Big[n_\mu \frac{\partial}{\partial x^\nu}$$

$$+ 2n_\mu n_\nu n^\lambda \frac{\partial}{\partial x^\lambda}\Big] \left(n^\rho \frac{\partial}{\partial x^\rho}\right)^{-2} D(x - x')$$

(integrating by parts and using current conservation)

$$= 2\, i\hbar c \int d^4x \int_{\sigma(\tau)} n_\mu \, d\sigma' \frac{1}{c^2} j^\nu(x') j^\mu(x) \left(n_\nu n^\lambda \frac{\partial}{\partial x^\lambda}\right.$$

$$\left. + \tfrac{1}{2} \frac{\partial}{\partial x^\nu}\right) \mathcal{D}(x - x')$$

$$= -2\left(\frac{\hbar c}{i}\right) \times \text{Coulomb term}$$

where the Coulomb term is given by (4.7).

If the Coulomb factor is part of the expansion of H^{n+1} in the S-matrix, the corresponding propagator will be part of the H^{n+1} term. Thus there will be an extra factor of $\frac{i}{\hbar c}$ multiplying the term containing the propagator.

We conclude that the correct results are obtained for scattering calculations if only the diagrams involving photon-electron vertices are used, and the usual propagator function

$$-\tfrac{1}{2}\hbar c \, D_F(x_j - x_i) g_{\mu\nu}$$

is assumed to correspond to an internal photon line between x_i and x_j.

Hence calculations of S-matrix elements in the quotient algebra formulation of the Fermi method lead to the Feynman rules for scattering

calculations. *The modified Fermi method thus provides a possible basis for the justification of the usual perturbation methods of calculation in quantum electrodynamics.* The justification of these rules on the basis of the Fermi method was the subject of a paper by Coester and Jauch [9]. They used Schwinger's formalism to derive the usual rules for perturbation calculations and noted (footnote p.153) that the question of the normalizability of physical states was unresolved. *Our formulation of the calculations resolves this problem and thereby explains why Schwinger* [8] *and Coester and Jauch* [9] *obtain the correct results.* As a further example, in the following chapter we shall develop a reinterpretation of Schwinger's self-energy calculation [8] in terms of a representation of the quotient algebra $\mathcal{A}_e \times \Delta(N)/I$.

CHAPTER 5

THE SELF ENERGY OF THE ELECTRON

In addition to S-matrix calculations, there are other observable consequences of the theory, which should be calculable on the basis of any satisfactory quantization procedure. In particular, the S-matrix contains only amplitudes calculated between states in the infinite past and future. As an example of another calculation, we shall show that the self energy of the electron can be calculated to second order by methods formally similar to those of Belinfante [1] and Schwinger [8].

To follow the approach of Belinfante, we again consider the photon annihilation and creation operators $a_\mu(\underline{k})$, $a^*_\mu(\underline{k})$, as acting on the $\underline{k}^{th}$ component of a continuous tensor product space, indexed by the momentum $\underline{k}$.

The interaction Hamiltonian is given by

$$H_I = \int d^3x \; j^\mu(x) \; \mathcal{A}_\mu(x) \; .$$

Since the interaction Hamiltonian belongs to $\mathcal{C}_e \times \Delta(N)$, there is a unique element, H_I^L, of $\mathcal{C}_e \times \Delta(N)/I$ to which it corresponds. This correspondence must be used when evaluating the matrix elements contributing to the self energy:

$$(\Psi_o, \; H_I \; \Psi_i) \; (\Psi_i, \; H_I \; \Psi_o)$$

where Ψ_o is the incoming and outgoing electron, and Ψ_i ranges over the intermediate states of processes which contribute to the self energy.

Belinfante calculated the self energy of an electron at rest, to second order. He listed nine different kinds of electron-photon interactions which contribute to this value. Only one of these processes involves transverse photons: the emission and reabsorption of a transverse photon of momentum $\underline{k}$. In addition, there is the same process for longitudinal and for timelike photons; also either of these nonphysical photons can be first absorbed and then re-emitted, as for a short time they

can be present in the vacuum of the photon field. There is also the possibility of the absorption of a timelike and a longitudinal photon of the same momentum, since their energy contributions will cancel, leaving the electron at rest; similarly for the emission of both. All these pairs of processes occurring in either order give all the possible self energy terms to second order.

Belinfante found that the process involving transverse photons, integrated over all possible momenta, gave the usual logarithmically divergent expression for the self energy of the electron. The contribution from the remaining processes was ambiguous, due to the infinite sums involved. Depending at what point in the calculation the infinite normalization (described in Chapter 1) was isolated and set equal to 1, the contribution from unphysical photons could be quadratically divergent or zero.

If we regard the calculation as taking place in a representation of the quotient algebra, then the operators in H_I corresponding to longitudinal and timelike photons are multiples of the supplementary condition $(a_3(\underline{k}) - a_o^*(-\underline{k}))$ or $(a_3^*(\underline{k}) - a_o(\underline{k}))$, and so are in the zero equivalence class of $\Delta(N)/I$, and do not contribute to the calculation. We are left with the logarithmic divergence, which can be regularized in the usual way.

Like those of Belinfante, Schwinger's equations also involve a formulation of the supplementary condition which leads to physical states which are not well-defined, as they are not normalizable. But again, we shall find that the equations of motion as he writes them can be regarded as pertaining to the formulation of the interacting electromagnetic field as a representation of the quotient algebra $\mathcal{A}_e \times \Delta(N)/I$ of $\mathcal{A}_e \times \Delta(N)$ with respect to the ideal generated by the supplementary condition operators. With this reinterpretation, the normalization problem does not arise.

The Gupta-Bleuler method has the advantage that all perturbation calculations can be done using explicitly covariant expressions involving the electromagnetic potential $\mathcal{A}_\mu$ acting on the indefinite metric space $\mathcal{H}$, and without explicit application of the supplementary condition. Then the result can be projected onto the physical space $\mathcal{H}_{phys}$, as defined in (1.19). We shall see how Schwinger's methods of calculation indicate that equally convenient expressions involving the potential $\mathcal{A}_\mu$ may be used in the Fermi method. At any stage in the calculation, the corresponding operators in the quotient algebra may be found. Thus the covariant form of the Hamiltonian in terms of the unconstrained 4-vector potential acts on the direct integral space $\mathcal{H}$ defined in (1.20). This is the Fock space for the vector potential. The quotient algebra is represented on one of the components $\mathcal{H}_\zeta$ in the central decomposition of $\mathcal{H}$. An operator gauge transformation of the form (3.2) puts the Hamiltonian in a form in which the corresponding quotient algebra operator is easily deduced.

For perturbation calculations in any method of quantization, the ease with which the transition is made between the field algebra description and the quotient algebra description is due to the following property. *The interaction picture Hamiltonian (3.9) commutes with the supplementary condition operators.* This means that, contrary to the case of the full Hamiltonian (3.5), no subtractions are necessary in order to find an internal generator for time translations.

Schwinger does a calculation of the electron self energy using two alternative formulations of the Hamiltonian operator. One involves only the transverse components of the electromagnetic potential, as we saw in Chapter 3, while the other involves the full electromagnetic potential, and an explicit supplementary condition. We shall interpret the first version of the calculation as taking place in the physical algebra

$\mathcal{A}_e \times \Delta(N)/I$. The existence of the second version demonstrates that calculations can be done in the convenient form involving the potential $\mathcal{A}^\mu$, and a prescription can be given for interpreting them in terms of the physical algebra.

First we consider the calculation taking place in the quotient algebra. In Chapter 3, we wrote down the quotient algebra equation of motion in a non-covariant radiation gauge form (3.10), and later in covariant form, (3.22). We shall now verify that the equation (3.22) reduces to (3.10) in the appropriate frame.

If we choose the flat spacelike surface σ with normal $n^\mu = (1,0,0,0)$, and perform the integration over the surface, (3.11) becomes

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = \int d^3x \{ -\frac{1}{c} j^L_\ell(x)\, \mathcal{A}^{\ell L}(x) + \frac{1}{c} \int_{t'=t} d^3x' \tfrac{1}{2} \Big(\frac{\partial \mathcal{D}(x - x')}{\partial x_\ell} j^L_\ell(x) - \frac{\partial \mathcal{D}(x - x')}{\partial x^o} j^{oL}(x)\ j^{oL}(x') \Big\} \Psi(t) \,. \tag{5.1}$$

The first term on the right hand side is

$$\int d^3x \left(-\frac{1}{c} j^L_\ell(x)\, \mathcal{A}^{\ell L}(x) \right) = \int d^3k \left(-\frac{1}{c} \tilde{j}^L_\ell(-\underline{k},t)\, q^{\ell tr L}(\underline{k},t) \right) ,$$

which has the same form as the first term in the Hamiltonian (3.9). The second term,

$$J(t) = \frac{1}{2c^2} \int d^3x \int_{t'=t} d^3x' \Big\{ \frac{\partial \mathcal{D}(x - x')}{\partial x^\ell} j^{\ell L}(x) - \frac{\partial \mathcal{D}(x - x')}{\partial x^o} j^{oL}(x) \Big\} j^{oL}(x')$$

may also be written in terms of Fourier transforms. D(x) can be written

$$D(x) = -\frac{1}{(2\pi)^3} \int d^3k\ e^{i\underline{k}\cdot\underline{x}} \frac{\sin |\underline{k}|\ x^o}{|\underline{k}|} ,$$

so that

$$\tilde{D}(\underline{k},t) = -\frac{1}{(2\pi)^{3/2}} \frac{\sin |\underline{k}|\ ct}{|\underline{k}|} .$$

Now by (3.16), with $n^\mu = (1,0,0,0)$,

$$\nabla^2 \mathcal{D}(x) = - D(x) .$$

Hence

$$\tilde{\mathcal{D}}(\underline{k},t) = - \frac{1}{(2\pi)^{3/2}} \frac{\sin |\underline{k}|\, ct}{|\underline{k}|^3} ,$$

and it follows that

$$\int_{t'=t} d^3x' \frac{\partial \mathcal{D}(x - x')}{\partial x_\ell} j^L(x')$$

$$= - \int d^3k \, (- ik_\ell) \, e^{i\underline{k}\cdot\underline{x}} \frac{1}{(2\pi)^{3/2}} \frac{\sin |\underline{k}|\, c(t - t')}{|\underline{k}|^3} j^{\ell L}(\underline{k},t) ,$$

which is zero, since $\sin |\underline{k}|\, c(t - t')$ is zero on the surface $t = t'$.

The second term in J(t),

$$- \frac{1}{2c^2} \int d^3x \int_{t=t'} d^3x' \frac{\partial \mathcal{D}(x - x')}{\partial x^o} j^{oL}(x) \, j^{oL}(x')$$

$$= - \frac{1}{2c^2} \int d^3x \int d^3x' \, j^{oL}(x) \, j^{oL}(x') \frac{1}{c} \frac{\partial}{\partial t} \left[\frac{1}{(2\pi)^{3/2}} \right.$$

$$\left. \int_{t'=t} d^3k \, e^{i\underline{k}\cdot(\underline{x} - \underline{x}')} \tilde{\mathcal{D}}(\underline{k},t - t') \right]$$

$$= \frac{1}{2c^3} \int_{t'=t} d^3k \int \frac{d^3x}{(2\pi)^{3/2}} e^{i\underline{k}\cdot\underline{x}} j^{oL}(x)$$

$$\int \frac{d^3x'}{(2\pi)^{3/2}} e^{-i\underline{k}\cdot\underline{x}'} j^{oL}(x') \, |\underline{k}| \, c \frac{\cos |\underline{k}|\, c(t - t')}{|\underline{k}|^3}$$

$$= \int_{t'=t} d^3k \frac{1}{2 \, |\underline{k}|^2 \, c^2} \tilde{j}^{oL}(\underline{k},t) \, \tilde{j}^{oL}(- \underline{k},t) .$$

Thus (3.22) is equivalent to

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = \int d^3k \left\{ \frac{\tilde{j}^{oL}(\underline{k},t) \, \tilde{j}^{L}_{o}(- \underline{k},t)}{2 \, |\underline{k}|^2 \, c^2} - \frac{1}{c} j^{\ell L}(- \underline{k},t) \, q_\ell^{trL}(\underline{k},t) \right\} \Psi(t) , \tag{5.2}$$

which is just the equation of motion (3.10) with Hamiltonian (3.9).

Starting from this equation of motion, we shall now follow the steps of Schwinger's self energy calculation [8]. As first order processes do not contribute to the self energy, we seek a transformation in the observable algebra to eliminate first order terms in (3.10):

$$\Psi(t) \to e^{-iT^L} \Psi(t) \ , \tag{5.3}$$

for some T^L in $\mathcal{A}_e \times \Delta(N)/I$. Under a transformation of this form, (3.10) becomes

$$i\hbar \frac{\partial}{\partial t} (e^{-iT^L} \Psi(t)) = e^{iT^L} H_1^L(t) \, e^{-iT^L} \Psi(t) \ ,$$

or

$$i\hbar \frac{\partial \Psi(t)}{\partial t} + i\hbar \, e^{iT^L} \frac{\partial e^{-iT^L}}{\partial t} \Psi(t) = e^{iT^L} H_1^L(t) \, e^{-iT^L} \Psi(t) \ ,$$

i.e.

$$i\hbar \frac{\partial \Psi(t)}{\partial t} + \left\{\hbar \frac{\partial T^L}{\partial t} + \frac{i\hbar}{2} \left[T^L, \frac{\partial T^L}{\partial t}\right] + \ldots\right\} \Psi(t)$$

$$= \{H_1^L(t) + i[T^L, H_1^L(t)] + \ldots\} \Psi(t) \ .$$

The first order terms are contained in

$$\left(\hbar \frac{\partial T^L}{\partial t} - H_1^L(t)\right) \ ,$$

and will be eliminated if we choose T^L such that

$$\hbar \frac{\partial T^L}{\partial t} = \int d^3k \, (-\tfrac{1}{c}) \, q_m^{tr^L}(\underline{k},t) \, j^{m^L}(-\underline{k},t) = H_1^L(t) - J(t) \ .$$

A suitable choice for T^L is

$$T^L = \frac{1}{\hbar} \int_{-\infty}^{t} (H_1^L(t) - J(t)) \, dt \ .$$

Then, to second order,

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = \{J^L(t) + \tfrac{i}{2}[T^L, H_1^L(t) + J^L(t)]\} \Psi(t)$$

$$= \int d^3k \left\{\frac{j_o^L(\underline{k},t) \, j_o^L(-\underline{k},t)}{2k^2 c^2} - \frac{i}{4\hbar c} \int d^3\ell \int dt' \, \varepsilon(t - t') \left[j_m^L(-\underline{k},t)\right.\right.$$

$$q^{tr\ m}{}^{L}(\underline{k},t),\ j_n^L(-\underline{\ell},t)\ q^{tr\ n}{}^{L}(\underline{\ell},t)\Big]\Big\}\ \Psi(t)\ . \qquad (5.4)$$

(5.4) can be written, to second order,

$$i\hbar\frac{\partial\Psi_1(t)}{\partial t} = \int d^3k\ \Big[\frac{j_o^L(\underline{k},t)\ j_o^L(-\underline{k},t)}{2\ k^2\ c^2} - \frac{1}{4c}\{j_m^L(-\underline{k},t),\ (\delta_T q^{tr\ m})^{L}(\underline{k},t)\}$$

$$- \frac{1}{4c}\{q_\ell^{tr}{}^{L}(\underline{k},t),\ (\delta_T j^\ell)^L(-\underline{k},t)\}\Big]\ \Psi(t)\ , \qquad (5.5)$$

where $(\delta_T j_\ell)^L(\underline{k},t) = i[T^L, j_\ell(\underline{k},t)]$

and $(\delta_T q_\ell^{tr})^{L}(\underline{k},t) = i\,[T^L,(q_\ell^{tr})^{L}(\underline{k},t)]$.

$(\delta_T j_\ell)^L$ is, to second order, the change in j_ℓ^L caused by the transformation which eliminates first order interaction terms. It is the current induced by the electromagnetic field. Similarly, $(\delta_T q_\ell^{tr})^{L}$ is, to first order, the electromagnetic field induced by the current as a result of the first order coupling. (It is multiplied by the current wherever it occurs, and so only needs to be taken to first order in a second order calculation.) In covariant notation, a second order equation corresponding to (3.22) is

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)} = \Big[-\frac{1}{4c}\{\{j^L(x),\ (\delta_T\mathcal{A}^\mu)^L(x)\} + \{(\delta_T j_\mu)^L(x),\mathcal{A}^{\mu\ L}(x)\}\Big)$$

$$-\frac{1}{c^2}\int_\sigma\Big(\tfrac{1}{2}\frac{\partial\mathcal{D}(x-x')}{\partial x^\mu} + n_\mu n_\nu\frac{\partial\mathcal{D}(x-x')}{\partial x_\nu}\ j^{\mu\ L}(x)\ j_\lambda^L(x')\ d\sigma^{\lambda'}\Big]\ \Psi[\sigma]\ , \qquad (5.6)$$

and

$$T[\sigma] = -\frac{1}{2\hbar c^2}\int_{-\infty}^{\infty} j_\mu^L(x')\ \mathcal{A}^{\mu\ L}(x')\ \varepsilon[\sigma,\sigma']\ d\omega'\ .$$

The self energy of the electron can be calculated from this second order equation. The interaction Hamiltonian density is written as

$$\mathcal{H}_1^L(x) = \mathcal{H}_{0,0}^{\ L}(x) + \mathcal{H}_{1,0}^{\ L}(x) + \mathcal{H}_{2,0}^{\ L}(x) + \mathcal{H}_{1,1}^{\ L}(x)$$

where $\mathcal{H}_{i,j}^{\ L}$ describes transitions involving i electrons and j photons (these are the only terms which contribute to second order). We shall find the

$\mathcal{H}^{L}_{1,0}(x)$ term by selecting the appropriate terms from a Wick ordering of Hamiltonian.

The terms involving the electromagnetic field can be written

$$- \frac{1}{4c} \left(\{ j^{L}_{\mu}(x),\ (\delta_T \mathcal{A}^{\mu})^{L}(x) \} + \{ (\delta_T j_{\mu})^{L}(x),\ \mathcal{A}^{\mu\,L}(x) \} \right)$$

$$= \frac{-i}{8\hbar c^3} \int_{-\infty}^{\infty} \epsilon[\sigma,\sigma']\, d\omega' \left(\{ j^{L}_{\mu}(x), j^{\nu\,L}(x') \} [\mathcal{A}^{L}_{\nu}(x'),\ \mathcal{A}^{\mu\,L}(x)] \right.$$

$$\left. + [\, j^{\nu\,L}(x'),\ j^{L}_{\mu}(x)]\ \{ \mathcal{A}^{L}_{\nu}(x'),\ \mathcal{A}^{\mu\,L}(x) \} \right) . \qquad (5.7)$$

We require the vacuum expectation value of the electromagnetic factors, and the value of the current terms in the 1-particle state. Using the definition of the vacuum (4.6), we find

$$[\mathcal{A}^{L}_{\nu}(x'),\ \mathcal{A}^{\mu\,L}(x)] = + i\hbar c\, g^{\mu}{}_{\nu}\, D(x - x')$$

$$- i\hbar c \left[\frac{\partial}{\partial x_{\mu}} \frac{\partial}{\partial x^{\nu}} + \left(n^{\mu} \frac{\partial}{\partial x^{\nu}} + n_{\nu} \frac{\partial}{\partial x_{\mu}} \right) n^{\lambda} \frac{\partial}{\partial x^{\lambda}} \right] \mathcal{D}(x - x')$$

$$= < [\mathcal{A}^{L}_{\nu}(x'),\ \mathcal{A}^{\mu\,L}(x)\ ] >_{o} , \qquad (5.8)$$

while

$$< \{ \mathcal{A}^{L}_{\nu}(x'),\ \mathcal{A}^{\mu\,L}(x) \} >_{o} = - \hbar c\, g^{\mu}{}_{\nu}\, D^{(1)}(x - x')$$

$$+ \hbar c \left[\frac{\partial}{\partial x_{\mu}} \frac{\partial}{\partial x^{\nu}} + \left(n^{\mu} \frac{\partial}{\partial x^{\nu}} + n_{\nu} \frac{\partial}{\partial x_{\mu}} \right) n^{\lambda} \frac{\partial}{\partial x^{\lambda}} \right] \mathcal{D}^{(1)}(x - x') ,$$

where

$$D^{(1)}(x) = i\, (D^{(+)}(x) - D^{(-)}(x)) . \qquad (5.9)$$

In general, a Wick ordering of a product of current terms is given by

$$j_{\mu}(x) j_{\nu}(x') = :\bar{\psi}_{\alpha}(x) \gamma_{\mu}^{\alpha\beta} \psi_{\beta}(x): :\bar{\psi}_{\nu}(x') \gamma_{\nu}^{\gamma\delta} \psi_{\delta}(x'):$$

$$= \gamma_{\mu}^{\alpha\beta} \gamma_{\nu}^{\gamma\delta} \{ < \bar{\psi}_{\alpha}(x) \psi_{\delta}(x') > :\psi_{\beta}(x) \bar{\psi}_{\gamma}(x'): + < \psi_{\beta}(x) \bar{\psi}_{\gamma}(x') > :\bar{\psi}_{\alpha}(x) \psi_{\delta}(x'):$$

$$+ :\bar{\psi}_{\alpha}(x) \psi_{\beta}(x) \bar{\psi}_{\gamma}(x') \psi_{\delta}(x'): + < \bar{\psi}_{\alpha}(x) \psi_{\delta}(x') > < \psi_{\beta}(x) \bar{\psi}_{\gamma}(x') > \} , \qquad (5.10)$$

and the first two terms in this decomposition contribute to the expectation value in a 1-particle state. Thus the contribution to the 1-particle expectation value from $[j^L_\nu(x'), j^L_\mu(x)]$ is

$$\gamma_\mu^{\alpha\beta}\gamma_\nu^{\gamma\delta}\big(:\psi_\beta(x)\bar\psi_\gamma(x'):<[\bar\psi_\alpha(x),\psi_\delta(x')]> + :\bar\psi_\alpha(x)\psi_\delta(x'):<[\psi_\beta(x),\bar\psi_\gamma(x')]>\,, \tag{5.11}$$

and the contribution from $\{j^L_\nu(x'), j^L_\mu(x)\}$ is

$$\gamma_\mu^{\alpha\beta}\gamma_\nu^{\gamma\delta}\big(:\psi_\beta(x)\bar\psi_\gamma(x'):<\{\bar\psi_\alpha(x),\psi_\delta(x')\}> + :\bar\psi_\alpha(x)\psi_\delta(x'):<\{\psi_\beta(x),\bar\psi_\gamma(x')\}>\,. \tag{5.12}$$

So the contribution of the operator (5.7) to the self energy term of the Hamiltonian density is $T_1(x) + T_2(x)$, where

$$T_1(x) = \frac{-i}{8\hbar c^3}\int_{-\infty}^{\infty}\varepsilon[\sigma,\sigma']\,d\omega'\,\gamma_\mu^{\alpha\beta}\gamma_\nu^{\gamma\delta}\Big[\big(:\psi_\beta(x)\bar\psi_\gamma(x'):<\{\bar\psi_\alpha(x),\psi_\delta(x')\}> + :\bar\psi_\alpha(x)\psi_\delta(x'):<\{\psi_\beta(x),\bar\psi_\gamma(x')\}>\big)\, i\hbar c\, g_{\mu\nu}\, D(x-x')$$

$$+\big(:\psi_\beta(x)\bar\psi_\gamma(x'):<[\bar\psi_\alpha(x),\psi_\delta(x')]> + :\bar\psi_\alpha(x)\psi_\delta(x'):<[\psi_\beta(x),\bar\psi_\gamma(x')]>\big)\,(-\hbar c\, g_{\mu\nu}\, D^{(1)}(x-x'))\,, \tag{5.13}$$

and

$$T_2(x) = \frac{1}{8c^2}\int_{-\infty}^{\infty}\varepsilon[\sigma\,\sigma']\,d\omega'\,\gamma_\mu^{\alpha\beta}\gamma_\nu^{\gamma\delta}\Big[\big(:\psi_\beta(x)\bar\psi_\gamma(x'):<\{\bar\psi_\alpha(x),\psi_\delta(x')\}> + :\bar\psi_\alpha(x)\psi_\delta(x'):<\{\psi_\beta(x),\bar\psi_\gamma(x)\}>\big)\,\big(\partial^\mu\partial^\nu + (n^\mu\partial^\nu + n^\nu\partial^\mu)\,n^\lambda\frac{\partial}{\partial x^\lambda}\big)\mathcal{D}(x-x')$$

$$+\big(:\psi_\beta(x)\bar\psi_\gamma(x'):<[\bar\psi_\alpha(x),\psi_\delta(x')]> + :\bar\psi_\alpha(x)\psi_\delta(x'):<[\psi_\beta(x),\bar\psi_\gamma(x')]>\big)\,i\,\big(\partial^\mu\partial^\nu + (n^\nu\partial^\mu + n^\nu\partial^\mu)\,n^\lambda\frac{\partial}{\partial x^\lambda}\big)\,\mathcal{D}^{(1)}(x-x')\Big]\,. \tag{5.14}$$

We shall show that $T_2(x)$ exactly cancels with the contribution of the Coulomb term to the electron self energy.

We have

$$T_2(x) = \frac{1}{8c^2}\, 2 \int_{-\infty}^{\sigma} d\omega'\, \gamma_\mu^{\alpha\beta}\gamma_\nu^{\gamma\delta}\Big[\big(:\psi_\beta(x)\bar\psi_\gamma(x'): < 2\bar\psi_\alpha(x)\psi_\delta(x') >$$

$$+ :\bar\psi_\alpha(x)\psi_\delta(x'): < 2\psi_\beta(x)\psi_\gamma(x') >\big)$$

$$\times (\partial^\mu\partial^\nu + (n^\mu\partial^\nu + n^\nu\partial^\mu)\, n^\lambda\partial_\lambda)\, \mathcal{D}^{(+)}(x - x')$$

$$+ (:\psi_\beta(x)\bar\psi_\gamma(x'): < 2\psi_\delta(x')\bar\psi_\alpha(x) > + :\bar\psi_\alpha(x)\psi_\delta(x'):$$

$$< 2\bar\psi_\gamma(x')\psi_\beta(x) >)\ (\partial^\mu\partial^\nu + (n^\mu\partial^\nu + n^\nu\partial^\mu)\, n^\lambda\partial_\lambda)\, \mathcal{D}^{(-)}(x - x')\Big]$$

$$= \frac{1}{c^2}\int_\sigma d\sigma'\, \gamma_\mu^{\alpha\beta}\gamma_\nu^{\gamma\delta}\ (:\psi_\beta(x)\bar\psi_\gamma(x') < \bar\psi_\alpha(x)\psi_\delta(x') >$$

$$+ :\bar\psi_\alpha(x)\psi_\delta(x') < \psi_\beta(x)\bar\psi_\nu(x') >)\ n^\mu\ \Big(\tfrac{1}{2}\frac{\partial}{\partial x_\nu} + n^\nu n^\lambda\frac{\partial}{\partial x^\lambda}\Big)\, \mathcal{D}(x - x')$$

and using (5.10), the 1-particle component of the Coulomb term is

$$-\frac{1}{c^2}\int_\sigma \Big(\tfrac{1}{2}\frac{\partial\mathcal{D}(x - x')}{\partial x_\mu} + n^\mu n_\lambda\frac{\partial\mathcal{D}(x - x')}{\partial x_\lambda}\Big)\gamma_\mu^{\alpha\beta}\gamma_\nu^{\gamma\delta}\,[< \bar\psi_\alpha(x)\psi_\delta(x') >$$

$$:\psi_\beta(x)\bar\psi_\gamma(x'): + < \psi_\beta(x)\bar\psi_\gamma(x') > :\bar\psi_\alpha(x)\psi_\delta(x'):]\quad n^\nu\, d\sigma'\ .$$

Thus these terms cancel, and the self energy term of the interaction Hamiltonian density is $T_1(x)$, defined in equation (5.13).

We have chosen a radiation gauge type of representation of the algebra $\mathcal{A}_e \times \Delta(N)/I$, so that there are separate contributions to interaction calculations from Coulomb interactions and electron-photon interactions. But we always find a cancellation between the Coulomb term and non-covariant parts of the transverse photon propagator, which lead to a much simpler final result. Such a result would have been obtained from the full 4-dimensional photon field interacting with the electron field with no Coulomb term; that is, if the interaction Hamiltonian $-\frac{1}{c}\, j_\mu(x)\, \mathcal{A}^\mu(x)$ is

used, and the Σ-transformation (3.18) is not applied. However, there is a problem in trying to do a calculation with the full 4-component potential because of the difficulty of defining states which satisfy the supplementary condition. Schwinger points out that if this interaction Hamiltonian is used, and it is regarded as acting on the Fock space for the vector potential (our direct integral space $\mathcal{H}$) then it is always possible at any stage to re-interpret the equations obtained as determining corresponding equations in terms of transverse potentials. The same procedure allows re-interpretation in our quotient algebra formulation as used in the calculation above. The correspondence is achieved by applying the operator gauge transformation (3.2), or equivalently

$$A(x) \to e^{-i\Sigma[\sigma]}\ A(x)\ e^{i\Sigma[\sigma]}\ , \tag{5.16}$$

with Σ as defined in (3.18). This makes the supplementary condition independent of the matter field, and it is then easy to project the interaction Hamiltonian so obtained onto the appropriate element of its direct integral decomposition (e.g. the operator (3.20) gives rise in this way to the quotient algebra operator (3.21).) All matter field operators remain in the same form, and multiples of the supplementary condition operator, now in terms of components of the electromagnetic potential only, are set equal to zero. Because the interaction Hamiltonian always commutes with the supplementary condition, this is an unambiguous procedure. Thus after the transformation (5.16), the representation $\mathcal{H}_0$ is selected. This choice of gauge gives rise to a representation of $\mathcal{A}_e \chi\, \Delta(N)/I$ which we call the Fermi representation.

6. CONCLUSIONS

The algebraic formulation given in [3] , [4] and [5] for the free electromagnetic field seems to be an important departure from other descriptions of the quantum theory, in the following sense. Elements of the field algebra are no longer represented as operators which act on physical states in the quantum formulation; instead, the quantum theory is defined to be a representation of a quotient algebra of a subalgebra of the field algebra. But in fact this description is not at odds with standard formulations: both the Gupta-Bleuler and radiation gauge quantizations can be interpreted as prescriptions for finding representations of the quotient algebra.

The indefinite metric formulation of Gupta and Bleuler can accommodate quantization in many different gauges, including choices which preserve Lorentz covariance and invariance under Lorentz gauge transformations. On the other hand, the radiation gauge formulation is a particular choice of gauge, in which Lorentz covariance has been given up, but a representation on a space with definite metric is possible.

Gauges which differ by a non-Lorentz gauge transformation will lead in general to inequivalent representations. We might take Segal's approach of regarding the quantum theory as being given by an abstract C*-algebra (in this case, $\Delta_c(N)/I$). In order to do calculations we must find a way of deciding which representations will be physical.

In the Fermi method, a representation of $\Delta_c(N)/I$ is found by selecting components in a direct integral decomposition (1.20) of a representation of $\Delta_c(N)$ with respect to the supplementary condition operators. Therefore the representation of the quotient algebra acts on a component Hilbert space $\mathcal{H}_\zeta$ in the direct integral. The choice of Hilbert space is a matter of convenience and corresponds to a choice of gauge. A choice can be made which preserves Lorentz gauge invariance; namely $\mathcal{H}_0$ in the free theory.

In quantum electrodynamics, where interaction with an electron field is introduced, we have described a relationship between the methods of quantization which is completely similar to that which exists in the free case.

As is well-known, the radiation gauge and Gupta-Bleuler quantizations give identical results for perturbation calculations. It would seem reasonable to regard as physical any quantization procedure which also agreed with these well-accepted calculational results.

These results were also achieved in earlier calculations, such as those of Schwinger and Belinfante, which however contained ambiguities, due basically to the problem of interpreting the supplementary condition in the quantum theory. We have now seen that these calculations can be interpreted as taking place in the Fermi representation of the quotient algebra, obtained as an element of the direct decomposition of the commutator algebra, $\mathcal{A}_e \times \Delta(N)$ of the supplementary condition operators. Furthermore, explicitly covariant calculations can be performed in the representation of $\mathcal{A}_e \times \Delta(N)$, with the understanding that the physical information is contained in the appropriate representation of the quotient algebra.

With this re-interpretation of the early calculations, the ambiguities are eliminated, and the Fermi representation is established as another representation of the abstract algebra, which must be regarded as physical because it yields the correct results for the standard calculations of quantum electrodynamics.

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