" A SECOND ORDER THEORY FOR SUPERSONIC FLOW OVER THREE-DIMENSIONAL WINGS. "

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Summary

This thesis presents a critical review of the second order theory proposed by Sugo for supersonic flow over three-dimensional wings It has been found that Sugo's theory is not exact, but is based on an approximation to a particular integral of the governing differential equation. Also, a number of important errors have been discovered. By eliminating these errors and using a more general approximate particular integral a modified theory is evolved. This is then used to find the pressure distribution over arbitrary wings in a supersonic flow.

The problem of flow over supersonic-edged delta wings is treated in detail and numerical results are obtained for one particular case of a flat plate delta wing at incidence. The results approach those of an exact theory and are clearly superior to those of Sugo for the same wing. It is thought that the modified theory will be equally satisfactory for other supersonic-edged wings and probably also for subsonic-edged wings, although the latter have not been investigated here.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of the candidate's knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text of the thesis.

I : Introduction

The problem of supersonic flow over three-dimensional wings is of fundamental importance in aerodynamics. Yet in the present state of development, the only direct theoretical method applicable to problems involving arbitrary wing shapes is the linearised small pertubation theory. Since linearised theory is useful mainly for thin wings at moderate Mach numbers, an improved general theory is obviously desirable and would be especially valuable at somewhat higher Mach numbers.

The major defects of linearised theory arise from over-simplifying the flow near influence zone boundaries and ignoring the non-linearity of the flow. The second order theory proposed by Sugo[1] can be used to improve linearised theory in both these aspects simultaneously. By considering second order contributions to the flow, some degree of non-linearity is included in both the governing differential equation and the boundary conditions; incorporation of the 'Poincare-Lighthill-Kuo' technique in the solution process enables the derivation of first order corrections to the flow field near influence zone boundaries and other shock or expansion waves, thus rendering the linearised theory uniformly valid.

The method employed is based on pertubation theory. A velocity potential is expanded in an infinite series in a small parameter of the flow, such as wing thickness or wing incidence. Curtailing the series after the second order term, the problem reduces to the solution of two linear iteration equations together with boundary conditions. As the second iteration equation is non-homogeneous, discovery of a particular integral would represent an important simplification in the theory.

In 1951, Van Dyke[2] used the pertubation method, except for 'P.L.K'. considerations, to obtain a second order theory for axially symmetric flows by the simple particular integral method. However, he could not find a particular integral suitable for the three-dimensional wing problem. Alternative methods have been used on special cases of this last problem by Clarkson[3], and Fell and Leslie[4], with some success. In 1958, Sugo[1] claimed to have found a particular integral for the problem in question. Since the direct particular integral method seemed to have practical advantages over the alternatives, Sugo's work merited investigation.

A close examination of Sugo's paper has revealed that his particular integral is only an approximation. The form of the particular integral is determined from two basic assumptions; firstly, it is assumed that the asymptotic behaviour of the flow at large distances from the wing approaches an axially symmetric form and secondly, that the particular integral satisfies the differential equation of the second order theory only in a region near the wing surface. In sections 3 and 4 several important modifications and corrections are made to Sugo's method, both in the formulation of the problem and in the derivation of an approximate particular integral. The modified theory, so derived, is used to develop a solution for the pressure distribution over a supersonic-edged delta wing in section 5. Detailed calculations are performed for a specific wing, chosen because there exists an exact theoretical result for comparison; from this comparison an empirical assessment is made. Conclusions are given in section 6.

2 : SYMBOLS.

a	: Velocity of sound
A	: $A = \pi - 2\sin^{-1}\sqrt{\dfrac{n^2 - k^2\sigma^2}{1 - k^2\sigma^2}}$
B	: $B = \sqrt{m_\omega^2 - 1}$
C	: $C = \sqrt{1-n^2}\,(1 - k^2\sigma^2)\sqrt{n^2 - k^2\sigma^2}$
C_p	: Pressure coefficient
D	: $D = \pi - 2\tan^{-1}\dfrac{\sqrt{n^2 - k^2\sigma^2}}{k\sigma\sqrt{1-n^2}}$
$h(u,y)$	: Wing Surface ordinate
k	: $k = \lvert\tan\omega\rvert$
M	: Mach number
n	: $n = k/B$
N	: $N = \dfrac{(\gamma+1)m_\omega^2}{2B^2}$ ($\doteqdot 2M$ in Sugo's notation)
q	: Velocity at a point
u,y,z	: P.L.K. coordinates
U	: Free stream velocity
x,y,z	: Cartesian Coordinates
x,r,θ	: Cylindrical Polar Coordinates
$x_1(u,y,z)$	: Quantity in P.L.K. transformation
γ	: Ratio of specific heats of a gas
ϵ	: Thickness parameter (small positive number)
λ	: Streamwise slope
μ	: Mach angle
ρ	: Local density
σ	: $\sigma = y/u$
ϕ	: Normalised pertubation velocity potential
ϕ_0	: First order " " " " " " "
ϕ_1	: Second order " " " " " " "
χ	: $\chi = \varphi_1^{(p)} - x_1\varphi_{0u}$
$\bar{\chi}$	: Approximation to χ
$\bar{\bar{\chi}}$	: Sugo's approximation to χ
ω	: Sweepback angle
Ω	: Total velocity potential

Subscripts

- s : Axially symmetric
- 0 : Quantity of linear theory
- 1 : Quantity of second order theory
- ∞ : Free stream quantity

Subscript notation is also used for partial differentiation: e.g. Ω_{xy} denotes $\frac{\partial^2 \Omega}{\partial x \partial y}$

Superscripts

- (c) : Complementary term
- (p) : Particular integral term
- * : Quantity in physical space

CHAPTER 3 : FORMULATION OF PROBLEM

3.I. Basic Flow

Consider the steady, supersonic flow of an ideal gas over an arbitrary three-dimensional thin wing. The undisturbed flow is a uniform stream of velocity U and Mach number M_∞. The flow is assumed to be adiabatic.

3.2. Wing Definition

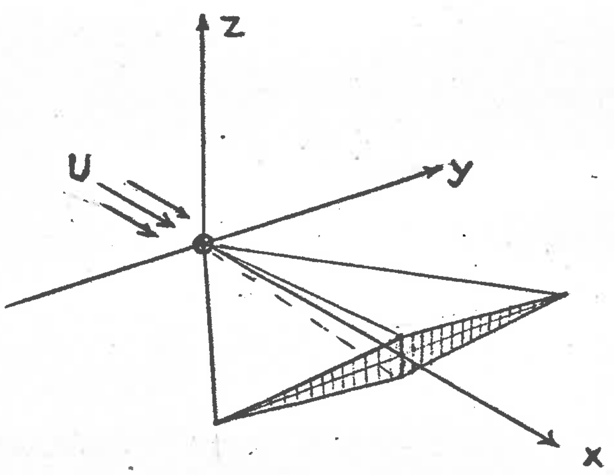
A wing may be described as a nearly planar body whose thickness in one cross-stream direction is small compared with its chord and span in the other two directions. The adjective 'thin' in this context is defined more precisely as referring to the ratio of maximum thickness of the wing to chord length; the parameter ϵ will be used throughout as a measure of this smallness e.g. the ordinates of a wing will be written as ϵ times a function of order unity. This notation will be used wherever it is necessary to distinguish between orders of magnitude of flow quantities.

3.3. Viscosity Effects

The only flow problems considered here are such that viscosity and heat conduction effects are small enough to be neglected except at bounding surfaces and shock waves; at bounding surfaces viscosity and heat conduction effects may be adequately approximated by classical boundary layer theory; shock waves arising in the flow may be considered as mathematical surfaces of discontinuity of the flow variables and are such that discontinuous entropy changes through them are $O(\epsilon^3)$, which may therefore be neglected in second order theory.

3.4. Co-ordinate System

Introduce Cartesian co-ordinates (x, y, z) with the origin at the foremost point of the wing and with axes orientated so that the undisturbed stream is parallel to the x-axis, the z-axis is perpendicular to the plane of the wing and the mean plane of the wing lies at $z=0$.



3.5. Flow Equations

The equation of motion for the total velocity potential Ω for the 3-D system described is given by (e.g. ref.5, p198)

$$(a^2-\Omega_x^2)\Omega_{xx} + (a^2-\Omega_y^2)\Omega_{yy} + (a^2-\Omega_z^2)\Omega_{zz} - 2\Omega_x\Omega_y\Omega_{xy} - 2\Omega_y\Omega_z\Omega_{yz} - 2\Omega_z\Omega_x\Omega_{xz} = 0 \quad (3.1.)$$

where a is the local speed of sound and is related to its value a_∞ in the free stream by

$$a^2 = a_\infty^2 - \frac{\gamma-1}{2}\left[(\Omega_x^2+\Omega_y^2+\Omega_z^2) - U^2\right] \quad (3.2.)$$

where γ is the ratio of specific heats of the gas.

Introducing a normalised pertubation velocity potential φ by

$$\Omega = U(x+\varphi) \quad (3.3.)$$

equation (3.1.) is reduced to

$$\begin{aligned}\beta^2\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = -m_\infty^2\Big[& \tfrac{\gamma-1}{2}(2\varphi_x + \varphi_x^2 + \varphi_y^2 + \varphi_z^2) \\ & + \varphi_{xx} + \varphi_{yy} + \varphi_{zz} + 2\varphi_x\varphi_{xx} \\ & + \varphi_x^2\varphi_{xx} + \varphi_y^2\varphi_{yy} + \varphi_z^2\varphi_{zz} \\ & + 2\varphi_y\varphi_z\varphi_{yz} + 2\varphi_z(1+\varphi_x)\varphi_{xz} \\ & + 2(1+\varphi_x)\varphi_y\varphi_{xy}\Big]\end{aligned} \quad (3.4.)$$

The small pertubation hypothesis assumes that disturbance velocities and their rates of change are of the same order as the thickness of the wing $(O(\epsilon))$. This implies that the pertubation potential introduced in equation (3.3.) is in fact $O(\epsilon)$, which is exhibited explicitly in the extended definition given in the next section.

3.6. Iteration

Following the work of Lighthill (6,7) a solution of equation (3.4.) is assumed to exist in the form

$$\begin{cases} \varphi(x,y,z) = \epsilon\,\varphi_0(u,y,z) + \epsilon^2\varphi_1(u,y,z) + \epsilon^3\varphi_2 + \cdots \\ x = u + \epsilon x_1(u,y,z) + \epsilon^2 x_2(u,y,z) + \cdots \end{cases} \quad (3.5.)$$

where each series is assumed to have a non-zero radius of convergence in regions of interest and the x_i $(i=1,2,\cdots)$ are assumed to have derivatives of all orders for $u \geq 0$. The second series of (3.5.) is introduced by the 'P.L.K.' technique.[8] This is a technique, first used in this type of problem by Lighthill[6], designed to avoid the ascending order of singularities which arises in these problems if a direct iteration procedure is employed. The functions $x_i(u,y,z)$ $(i=1,2,\cdots)$ introduced by this process are determined as part of the solution.

Substituting equation (3.5.) into equation (3.4.) and equating coefficients of like powers of ϵ, reduces the non-linear equation to an infinite sequence of linear iteration equations. Only the first two equations are retained here on the assumption that the first two terms of the series represents a good approximation to the pertubation potential.

$$O(\epsilon): \quad B^2 \varphi_{0uu} - \varphi_{0yy} - \varphi_{0zz} = 0$$
$$(B^2 = m_\infty^2 - 1) \qquad (3.6.)$$

$$O(\epsilon^2): \quad B^2\varphi_{1uu} - \varphi_{1yy} - \varphi_{1zz} = \left[-m_\infty^2\left\{(\gamma-1)m_\infty^2+2\right\}\varphi_{0u} + 2B^2 x_{1u}\right]\varphi_{0uu}$$
$$-(2m_\infty^2\varphi_{0y} + 2x_{1y})\varphi_{0uy}$$
$$-(2m_\infty^2\varphi_{0z} + 2x_{1z})\varphi_{0uz}$$
$$+\varphi_{0u}(B^2 x_{1uu} - x_{1yy} - x_{1zz})$$
$$(3.7.)$$

Equation (3.6.) is just the linear theory equation (with $x = u$ to the same order of accuracy). Equation (3.7.) is the second order equation. The non-homogeneous term involves x_1, φ_0 and their derivatives. φ_0 is the solution of equation (3.6.) which satisfies the relevant first order boundary conditions.

3.7. Boundary Conditions

3.7.I. Physical Conditions

Physical considerations suggest that the flow should satisfy the following conditions.

(i) The flow is to be tangent to the surface of the wing.

(ii) Disturbances are not propagated upstream; flow above and below wings is independent with a few exceptions (see § 3.7.3.)

(iii) Velocity boundary conditions at apex Mach cone (or envelope of Mach cones) are determined from Lighthill's analysis of conditions near a shock wave.

3.7.2. Tangency condition

Let the wings considered have upper and lower surfaces given by

$$z_u = \epsilon h^*_u(x,y) \ \ \eta(x)$$
$$z_l = \epsilon h^*_l(x,y) \ \ \eta(x)$$

where

(i) the subscripts u and l refer to upper and lower surfaces respectively.

(ii) $\eta(x)$ is the unit step function:

$$\eta(x) = 0 \qquad x \leq 0$$
$$\eta(x) = 1 \qquad x > 0$$

Since, in general, flows above and below are independent consider only flow above the wing and drop the subscript. For the surface

$$S^* = z - \epsilon h^*(x,y) = 0 \qquad (3.8.)$$

the tangency condition requires

$$\frac{\partial \Omega}{\partial n} = \underline{\nabla}^* \Omega \cdot \underline{\nabla}^* S^* = 0 \qquad (3.9.)$$

i.e. $(1+\varphi_x)h^*_x + \varphi_y h^*_y - \varphi_z = 0$

Transforming into P.L.K. space via equations (3.5.)

$$\epsilon h^*(x,y) = \epsilon h(u,y) + \epsilon^2 x_1(u,y,z)\, h_u(u,y)$$

and

$$\left[(1+\epsilon\varphi_{0u})(1-\epsilon x_{1u})\left(\epsilon h_u + \epsilon^2 x_1 h_{uu} + \epsilon^2 x_{1u} h_u\right)\right] + \left[\epsilon^2 \varphi_{0y} h_y\right] - \left[(1-\epsilon x_{1z})(\epsilon\varphi_{0z} + \epsilon^2\varphi_{1z})\right] = 0$$

Equating coefficients of powers of ϵ

$$O(\epsilon):\quad h_u - \varphi_{0z} = 0 \qquad (3.10.)$$

$$O(\epsilon^2):\quad \varphi_{0u}h_u - x_{1u}h_u + x_1 h_{uu} + x_{1u}h_u + \varphi_{0y}h_y + x_{1z}\varphi_{0z} - \varphi_{1z} = 0 \qquad (3.11.)$$

Assuming $\varphi_0(u,y,z)$ is a regular function of z, for z sufficiently small, these boundary conditions may be evaluated at the mean plane $z=0$.

$$O(\epsilon):\quad \varphi_{0z} = h_u \quad (z=0) \qquad (3.12.)$$

$$O(\epsilon^2):\quad \varphi_{1z} = \varphi_{0u}h_u + \varphi_{0y}h_y + x_1 h_{uu} + x_{1z}\varphi_{0z} - h\varphi_{0zz} \qquad (3.13.)$$

It should be noted that this latter boundary condition differs from that used by Sugo[1] but agrees with the form given by Clarkson[3].
The difference is significant, and in the specific calculation of § 5.2.6. it is found to be most important.

The second order boundary condition may be further simplified:

(i) The boundary condition in linear theory may be written

$$\frac{\partial z}{\partial u} = \frac{\epsilon \varphi_{oz}}{1+\epsilon \varphi_{ou}} = \epsilon h_u$$

or $\epsilon h_u (1 + \epsilon \varphi_{ou}) = \epsilon \varphi_{oz}$

Differentiate w.r.t. to z

$$\epsilon^2 h_u \varphi_{ouz} = \epsilon \varphi_{ozz}$$

which indicates that φ_{ozz} is $O(\epsilon)$ smaller than φ_{ouz}. Consequently the term $h\varphi_{ozz}$ of eq^n(3.13.) is $O(\epsilon^3)$ and may be neglected.

(ii) It will be shown later that in general $x_{1z} \equiv 0$ (see § 5.2.2.)

Hence the second order boundary condition reduces to

$$\varphi_{1z} = \varphi_{ou} h_u + \varphi_{oy} h_y + x_1 h_{uu} \qquad (3.14.)$$

3.7.3. Supersonic Flow Conditions

The conditions of § 3.7.1. (ii) require that the flow be everywhere supersonic. This is only a slightly more restrictive condition than that already imposed on the wings in this regard in § 3.3. For discussion of this feature see Ref.9 p374. Local departures, such as leading edge stagnation lines of a subsonic edge wing, can be handled by special methods; such cases will be discussed if and where they arise.

3.7.4. Influence Zone Boundary Conditions.

Boundary conditions representing flow behaviour at influence zone boundaries are obtained from Lighthill's[7] detailed analysis of conditions near a shock wave. Actually his analysis can also be used to describe behaviour at expansion boundaries as well as shock waves; in such a case it is found that the shock has zero strength and the adjoining expansion region is the dominant feature.

The correct upstream conditions are the shock wave equations applied at the shock wave location which is upstream of the envelope of Mach cones emanating from the leading edge. Linearised theory and ordinary second order theory use the simplest condition of a shock of zero strength located at the free stream Mach cone envelope. Here this boundary condition is refined by taking into account a first order correction to the influence zone boundary position. In fact this is done automatically in the solution process by the P.L.K. technique. The P.L.K. transformation, introduced initially to remove troublesome higher order singularities of the second iteration stage, achieves this effect by a co-ordinate transformation which is precisely that necessary to reposition the influence zone boundary to its first order position. Consequently P.L.K. space may be interpreted physically as a co-ordinate system relative to the adjusted influence zone boundary. Inboard flow quantities, being directly dependent on the influence zone configuration, must therefore be considered as quantities of P.L.K. space. All boundary conditions are applied at this corrected influence zone boundary position i.e. at the Mach cone envelope in P.L.K. space. The actual boundary conditions used are:

(a) First order flow quantities satisfy the zero order condition of continuity of both potential and velocity at the boundary.

(b) The second order potential is continous through the boundary. This result is immediate for an expansion boundary and follows from the continuity of tangential velocity through a shock wave in this latter case.

(c) The simple condition of continuity of second order velocity at the boundary is used here. The validity of this approximation is not obvious for a shock wave boundary in a second order theory. It may be thought that the velocity discontinuity through a shock of first order is significant. However, Wallace and Clarke[10] have shown that the contribution to the result from a consideration of this discontinuity is $O(\epsilon^{5/2})$ which is legitimately ignored in a second order theory. Continuity of velocity is obvious for a boundary dominated by an inboard expansion region.

The actual method of application of these boundary conditions is important. In this paper the P.L.K. technique is used primarily to render solutions uniformly valid within influence zones. It is then found that the regions of validity overlap between the zero order boundary and the P.L.K. (or limiting) boundary position. The actual second order boundary position is located by application of the condition of continuity of velocity there. Once the boundary has been located, the solution is uniquely determined. This interpretation differs significantly from that of Sugo in the supersonic-edged wing case.

3.8. Formulation of the Problem.

The results of the fore-going sections concerning the solution for the first and second order velocity potentials may be summarised as follows:

3.8.I. First Order Problem

(I) $B^2 \varphi_{0uu} - \varphi_{0yy} - \varphi_{0zz} = 0$

(2) $\varphi_0(u, y, z) = 0 \qquad u \leq 0$

(3) $\varphi_{0z}(u, y, 0) = h_u(u, y) \qquad u > 0$

(4) φ_0, φ_{0u} continuous at influence zone boundaries.

3.8.2. Second Order Problem

(I)
$$B^2 \varphi_{1uu} - \varphi_{1yy} - \varphi_{1zz} = \left[-M_\infty^2 \left\{(\gamma-1)M_\infty^2 + 2\right\} \varphi_{0u} + 2B^2 x_{1u}\right] \varphi_{0uu} - \left[2M_\infty^2 \varphi_{0y} + 2x_{1y}\right] \varphi_{0uy} - \left[2M_\infty^2 \varphi_{0z} + 2x_{1z}\right] \varphi_{0uz} + \varphi_{0u}\left(B^2 x_{1uu} - x_{1yy} - x_{1zz}\right)$$

where the R.H.S. is determined once the solution of eqn.(3.8.I.), and $x_1(u,y,z)$ the unknown function introduced by the P.L.K. technique, are known.

(2) Determine $x_1(u, y, z)$

(3) $\varphi_1(u, y, z) = 0 \qquad u \leq 0$

(4) $\varphi_{1z}(u, y, 0) = \varphi_{0u} h_u + \varphi_{0y} h_y + x_1 h_{uu}$

(5) φ_1, φ_{1u} continuous at influence zone boundaries.

3.9. Pressure Distribution

The aim of this work is to enable the determination of aerodynamic forces exerted over a wing. These forces may be obtained from a knowledge of the pressure distribution over the wing; once this has been determined the problem will be considered solved. The pressure distribution is most conveniently derived in the form of the pressure coefficient.

$$C_p^*(x,y,z) = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U^2} \qquad (3.15.)$$

For an isentropic flow it can be shown that

$$C_p^* = \frac{2}{\gamma M_\infty^2}\left\{\left[1 + \frac{\gamma-1}{2}M_\infty^2(1-q^2)\right]^{\frac{\gamma}{\gamma-1}} - 1\right\} \qquad (3.16.)$$

where $q^2 = (\text{velocity})^2$
$= \Omega_x^2 + \Omega_y^2 + \Omega_z^2$

Substituting from eqns. (3.3) and (3.5.) and equating coefficients of powers of ϵ :

$$O(\epsilon):\quad C_p^{(0)} = -2\varphi_{0x}$$

$$O(\epsilon^2):\quad C_p^{(1)} = -2\varphi_{1x} + 2x_{1x}\varphi_{0x} + B^2\varphi_{0x}^2 - \varphi_{0y}^2 - \varphi_{0z}^2 . \qquad (3.17.)$$

Hence $C_p = C_p^{(0)} + C_p^{(1)}$ represents the pressure coefficient to second order accuracy.

3.10. Remarks on the Problem

The only problems considered are those for which the linearised solution is known. The problem is essentially the determination of φ_{1x}, the second order contribution to the streamwise velocity component. This entails the solution of eqn.(3.7.); a linear non-homogeneous partial differential equation for which the form of the complementary function is known. If by any means a particular integral can be written down the problem is essentially solved. The complete solution consists then of the particular integral plus the complementary function; the latter adjusts the solution to fit the boundary conditions. The role of the particular integral is to transfer the non-homogeneity of the problem from the equation to the boundary conditions where it presents less difficulty. For linear partial differential equations this is always possible in principle. In fact with the aid of Green's Formula it can be shown that a particular integral of

$$B^2\varphi_{1xx} - \varphi_{1yy} - \varphi_{1zz} = -f(x,y,z)$$

is given by

$$\varphi_1^{(P)} = \frac{1}{2\pi}\iiint_\Delta \frac{f(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta}{\sqrt{(x-\xi)^2 - B^2[(y-\eta)^2 + (z-\zeta)^2]}}$$

where Δ, the region of integration, consists of that part of the fore Mach cone from the point (x,y,z) for which $f(\xi,\eta,\zeta)$ is defined. But in general this $\int^n$

is not feasible. Discovery of a direct particular solution would represent a great simplification.

An exact particular integral has not been found. Van Dyke[2] has given a partial particular integral which simplifies the triple integral given above but does not lead directly to a simple solution. Subsequently Sugo[1] claimed to have found a direct particular integral suitable for the general three-dimensional case. This particular integral and the approximations on which it is based are discussed in detail in the next section.

4: THE PARTICULAR INTEGRAL.

This section critically examines the solution proposed by Sugo[1] and then modifies it.

The second order problem has been formulated in section 3; as mentioned earlier the essential problem is the construction of a particular integral for the second order equation.

4.I. Simplification of Differential Equation.

Let $\varphi_1^{(c)}$ and $\varphi_1^{(p)}$ respectively, denote the complementary function and particular integral of equation (2.7.), which is rearranged in the following, more convenient form

$$
\begin{aligned}
B^2\varphi_{1uu} - \varphi_{1yy} - \varphi_{1zz} = \Big[-M_\infty^2 \left\{ (\gamma-1) M_\infty^2 + 2 \right\} \varphi_{0u}\varphi_{0uu} \\
- 2M_\infty^2 \varphi_{0y}\varphi_{0uy} - 2M_\infty^2 \varphi_{0z}\varphi_{0uz} \\
+ 2B^2 x_{1u}\varphi_{0uu} - 2x_{1y}\varphi_{0uy} \\
- 2x_{1z}\varphi_{0uz} + \varphi_{0u}\left(B^2 x_{1uu} - x_{1yy} - x_{1zz} \right) \quad (4.1.)
\end{aligned}
$$

In future the abbreviation P.I. will be used to denote particular integral and L will denote the linear partial differential operator

$$
L \equiv B^2 \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}
$$

Since

$$
L\left[x_1 \varphi_{0u} \right] = 2B^2 x_{1u}\varphi_{0uu} - 2x_{1y}\varphi_{0uy} - 2x_{1z}\varphi_{0uz} + \varphi_{0u}\left(B^2 x_{1uu} - x_{1yy} - x_{1zz} \right)
$$

Setting

$$
\varphi_1^{(p)} = X + x_1 \varphi_{0u} \qquad (4.2.)
$$

equation (4.I.) is reduced to

$$L[\chi] = -m_\infty^2\left[\{(\gamma-1)m_\infty^2 + 2\}\varphi_{0u}\varphi_{0uu} + 2\varphi_{0y}\varphi_{0uy} + 2\varphi_{0z}\varphi_{0uz}\right] \qquad (4.3.)$$

Following Van Dyke[2], define

$$N = \frac{(\gamma+1)m_\infty^2}{2B^2} \quad (\equiv 2m, \text{ in notation of Sugo}) \qquad (4.4.)$$

Whence equation (4.3.) becomes

$$L[\chi] = -2m_\infty^2\left[(N-1)B^2\varphi_{0u}\varphi_{0uu} + \varphi_{0y}\varphi_{0uy} + \varphi_{0z}\varphi_{0uz}\right] \qquad (4.5.)$$

This equation written in cylindrical polar co-ordinates (u,r,θ) is then

$$B^2\chi_{uu} - \chi_{rr} - \frac{1}{r}\chi_r - \frac{1}{r^2}\chi_{\theta\theta} = -2m_\infty^2\Big[(N-1)B^2\varphi_{0u}\varphi_{0uu} + \varphi_{0r}\varphi_{0ur} + \frac{1}{r^2}\varphi_{0\theta}\varphi_{0u\theta} + \frac{1}{2}\varphi_{0rr}\varphi_{0r}^2\Big] \qquad (4.6.)$$

The last term on the R.H.S. is a triple product term but is retained because of its importance in axially symmetric flow.

4.2. Known Solutions.

P.I*s have been found in both the two dimensional and axially symmetric cases.

Equation (4.5.) reduces to the two-dimensional, plane case, if y dependence is removed $\left(\frac{\partial}{\partial y} \equiv 0\right)$

$$B^2\chi_{uu} - \chi_{zz} = -2M_\infty^2\left[(N-1)B^2\varphi_{ou}\varphi_{ouu} + \varphi_{oz}\varphi_{ouz}\right] \quad (4.7.)$$

P.I. $$\chi^{(p)}_{Plane} = M_\infty^2\varphi_{ou}\left[\left(1-\frac{N}{2}\right)\varphi_o + \frac{N}{2}z\,\varphi_{oz}\right] \quad (4.8.)$$

Equation (4.6.) reduces to the axially symmetric case if θ -dependence is removed $\left(\frac{\partial}{\partial\theta}\equiv 0\right)$.

$$B^2\chi_{uu} - \chi_{rr} - \frac{1}{r}\chi_r = -2M_\infty^2\left[(N-1)B^2\varphi_{ou}\varphi_{ouu} + \varphi_{or}\varphi_{our} + \frac{1}{2}\varphi_{orr}\varphi_{or}^2\right] \quad (4.9.)$$

P.I. $$\chi^{(p)}_s = M_\infty^2\varphi_{ou}\left[\varphi_o + Nr\varphi_{or}\right] - \frac{M_\infty^2}{4}r\,\varphi_{or}^3 \quad (4.10.)$$

4.3. Guiding Principles.

Sugo determines an approximate P.I. by satisfying the two conditions:

(i) The P.I. approaches the axially symmetric form asymptotically for large distances from the body (in a specified region above and below the body)

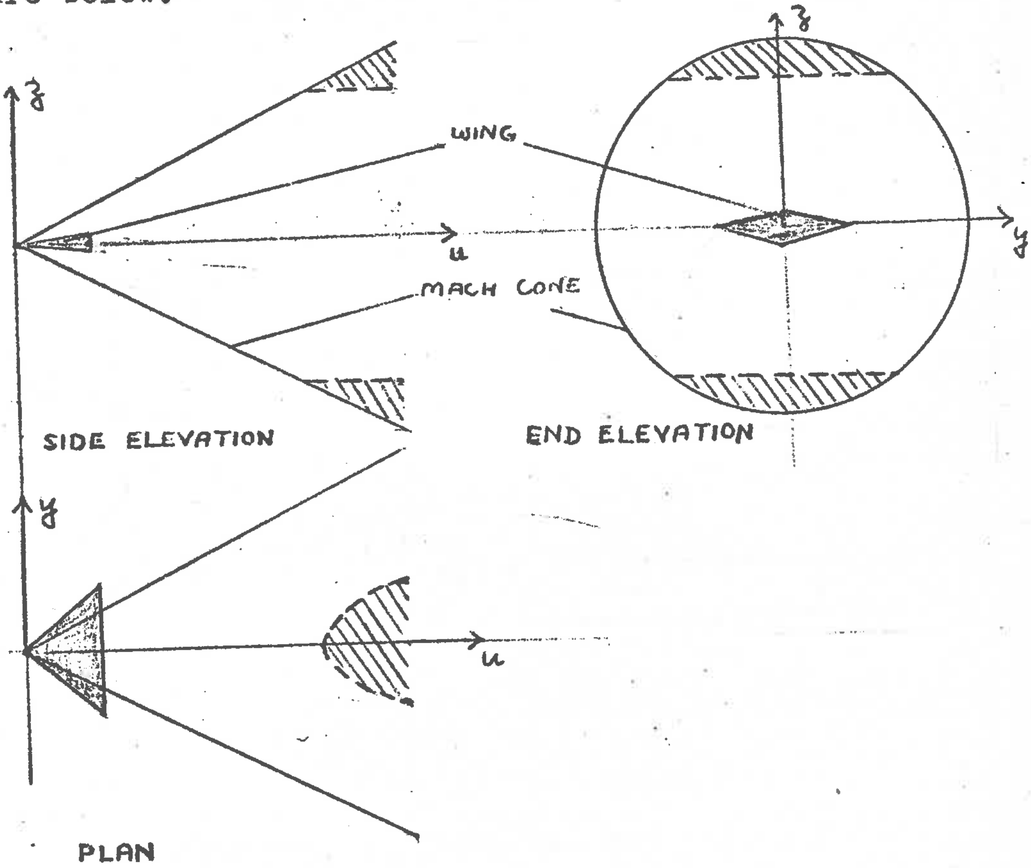
(ii) In the neighbourhood of the wing surface $(z=O(\epsilon))$ the P.I. satisfies equation (4.5.) to the order of accuracy of the equation $(O(\epsilon^2))$.

4.4. The Axial Symmetry Condition.

For large r, the dominant term of the axially symmetric P.I. is

$$M_\infty^2 Nr\varphi_{ou}\varphi_{or} = M_\infty^2 N\varphi_{ou}\left[y\varphi_{oy} + z\varphi_{oz}\right]$$

Sugo proposed that his P.I. should approach this form for the region of r large $\left(O\left(\frac{1}{\epsilon}\right)\right)$ given by z large $\left(O\left(\frac{1}{\epsilon}\right)\right)$ and y $O(1)$; to have any meaning the region considered must necessarily be inside the envelope of Mach cones. This region is indicated by the hatched portions of the figure below.



The condition then requires

$$X(u,y,z) \longrightarrow M_\infty^2 N z \, \varphi_{0u} \varphi_{0z} \qquad \left\{ \begin{matrix} z & O\left(\frac{1}{\epsilon}\right) \\ y & O(1) \end{matrix} \right\} \qquad (4.\mathrm{II})$$

As justification for this assumption, Sugo proposed that if r is large $\left(O\left(\frac{1}{\epsilon}\right)\right)$, then terms $\frac{1}{r^2}X_{\theta\theta}$ and $\frac{1}{r^2}\varphi_{0\theta}\,\varphi_{0u\theta}$ of equation (4.6.) are $O(\epsilon)$ smaller than the other terms and may be neglected; equation (4.6.) reduces to

$$B^2\varphi_{1uu} - \varphi_{1rr} - \frac{1}{r}\varphi_{1r} = -2m_\infty^2\Big[(N-1)B^2\varphi_{0u}\varphi_{0uu} + \varphi_{0r}\varphi_{0ur} + \tfrac{1}{2}\varphi_{0ur}\varphi_{0r}^2\Big]$$

This is exactly the form of the axially symmetric equation (4.9.) for which the P.I. is given by equation (4.10.) and the dominant term in the region described is given by equation (4.11.)

This argument is not valid in general. It tacitly assumes that all derivatives of φ_0 in cylindrical polar co-ordinates are of the same order. But the initial pertubation hypothesis is that the disturbance velocity components $\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$, or $\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial r}, \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$ are of the same order (basically all are $O(1)$ and actual order determined by the premultiplying power of ϵ). Whence $\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = O(1)$ giving $\frac{\partial\varphi}{\partial\theta} = r.O(1) = O(\frac{1}{\epsilon})$ for r large. This contradicts Sugo's implicit assumption.

However, since Sugo' only required the flow to be axially symmetric to $O(\epsilon^2)$ in the region of large z, it is possible that some symmetric slender wings may satisfy this condition. But this is too restrictive for a general wing theory.

4.5. Sugo's Approximate P.I.

Sugo postulates the form of his P.I. as

$$\bar{\bar{X}}^{(P)} = m_\infty^2\Big[\varphi_{0u}\Big\{\Big(1-\frac{N}{2}\Big)\varphi_0 + \frac{N}{2}y\varphi_{0y} + Nz\varphi_{0z} - Nyz\varphi_{0yz}\Big\}\Big]$$

In order to satisfy condition (4.II.), Sugo then assumes that for large z, φ_{ozz} is $O(\epsilon)$ smaller than the other derivatives of the potential. He bases this assumption on an order of magnitude investigation of flow about a slender cone; this does not inspire confidence in the assumption for wings other than those mentioned in the last paragraph of §.4.4.

Accepting this approximation for the moment, substitute into the differential equation

$$
\begin{aligned}
L[\bar{\bar{X}}^{(p)}] = -2M_\infty^2 \Big[& (N-1)B^2 \varphi_{ou}\varphi_{ouu} + \varphi_{oy}\varphi_{ouy} + \varphi_{oz}\varphi_{ouz} \Big] \\
& + \text{terms containing } \varphi_{ozz} \text{ (or derivatives)} \\
& + \text{terms multiplied by } z \\
& + \left[-N\varphi_{ouz}\varphi_{oz} + Ny\,\varphi_{ouz}\varphi_{oyz} \right]
\end{aligned}
\qquad (4.13.)
$$

In the region of the wing surface, z ~~is~~ $(O(\epsilon))$, φ_{ozz} has been shown to be $O(\epsilon^3)$, (§ 3.7.2.), and if the wing has constant slope in the stream direction φ_{ouz} vanishes on the surface. Hence the P.I. may be considered an approximation in the sense that the excess terms are $O(\epsilon^3)$ in the region near the wing surface i.e. the P.I. appears to satisfy the second order equation for the region of z sufficiently small.

4.6. Discussion.

The last three sections have summarised the approximation to the P.I. proposed by Sugo[1]. The validity of the key approximation, described in §4.3.(ii), is subsequently investigated. Nevertheless, Sugo's P.I. does not seem to be the best available in the sense of §4.3.(ii). The condition §4.3.(i), which has dubious value anyhow, forces the restriction that the wing surface must have constant slope. This restriction can be removed with no apparent loss of generality in the method. The following modification $\overline{X}^{(p)}$ to $\overline{\overline{X}}^{(p)}$ will be used henceforth.

4.7. Revised Guiding Principles.

Replace the conditions of §4.3. by the following:

(i) If any trial form satisfies the differential equation it is a P.I., independent of the methods used to derive it.

(ii) Consider three-dimensional flow as an extension of plane flow i.e. build up the P.I. from the known solution in the two-dimensional case.

(iii) In the neighbourhood of the wing surface the P.I. should satisfy equation (4.5.) to $O(\epsilon^2)$ i.e. yield an 'approximation' in the same sense as proposed by Sugo §4.3.(ii) .

4.8. Modified P.I.

The particular integral for plane flow

$$X^{(p)}_{Plane} = m_\infty^2 \varphi_{ou}\left[(1-\tfrac{N}{2})\varphi_o + \tfrac{N}{2} z \varphi_{oz}\right]$$

and the symmetry of equation (4.5.) with respect to y and z suggest the basic form for the P.I. as

$$\bar{x}^{(p)} = m_\infty^2 \varphi_{ou}\left[(1-\tfrac{N}{2})\varphi_o + \tfrac{N}{2} y \varphi_{oy} + \tfrac{N}{2} z \varphi_{oz}\right] + f(u, y, z; \varphi_o)$$

In order to satisfy condition §5.I.(iii), take

$$\bar{x}^{(p)} = m_\infty^2 \varphi_{ou}\left[(1-\tfrac{N}{2})\varphi_o + \tfrac{N}{2} y \varphi_{oy} + \tfrac{N}{2} z \varphi_{oz} - \tfrac{N}{2} y z \varphi_{oyz}\right] \qquad (4.14.)$$

Operating on this with L, yields

$$L[\bar{x}^{(p)}] = -2m_\infty^2\{(N-1)B^2\varphi_{ou}\varphi_{ouu} + \varphi_{oy}\varphi_{ouy} + \varphi_{oz}\varphi_{ouz}\}$$
$$+ \text{ terms containing } \varphi_{ozz} \text{ (or derivitives)}$$
$$+ \text{ terms multiplied by } z. \qquad (4.15.)$$

The restrictive terms requiring constant slope wings have been ~~elimated~~ eliminated (c.f.equation(4.13.)) Thus equation (4.14.) gives the form of the 'approximate' P.I., which will now be investigated in detail.

4.9. Qualitative Investigation of Approximation.

If equation (4.5.) is written in the form

$$L[x] = F[\varphi_o; u, y, z]$$

then (4.15.) gives

$$L[\bar{x}] = F[\varphi_o; u, y, z) + z[\text{terms of } O(1)] + \varphi_{ozz}.[\text{terms of } O(1)]$$

Since L is a linear operator

$$L[x-\bar{x}] = z.[\text{terms } O(1)] + \varphi_{ozz}.[\text{terms } O(1)]$$
$$= O(z) = O(\epsilon) \quad \text{as } z \to 0 \qquad (4.16)$$

The solution for the pertubation potential, and hence χ, is required only in the region near the wing surface where both ζ and $\varphi_{o\zeta\zeta}$ are $O(\epsilon)$. Then to second order accuracy, equation (4.16.) in the region of the wing surface is

$$L[\chi - \bar{\chi}] = 0$$

i.e. $\chi - \bar{\chi}$ = Terms of complementary function.

i.e. On the wing surface $\bar{\chi}^{(p)}$ represents a first approximation to $\chi^{(p)}$ in the sense that

$$\epsilon^2 \chi^{(p)} = \epsilon^2 \bar{\chi}^{(p)} + O(\epsilon^3)$$

Further investigation suggests that this argument, based rather loosely on an order of magnitude argument, may not be correct. The presence of the ζ-space variable in the differential operator and in the non-homogeneous terms presents some difficulty. The further investigations in this matter are presented in Appendix A, but it should be noted that no definite conclusion could be reached, theoretically. Therefore, in the next section, the 'approximation' is used in a specific calculation to determine whether there is any empirical justification for the method before an extensive investigation of the approximation is made.

5: APPLICATION TO WING THEORY

5.I. Introduction

In this section a solution of a general wing-flow problem is developed using the approximate P.I. of § 4.8. For present purposes, as mentioned in § 3.9., a problem will be considered solved once the pressure coefficient distribution $C_p(x,y,z)$ is known over the wing surface.

The method employed can be summarised as follows: the P.L.K. transformation necessary to ensure convergence of the iteration process is determined(§5.2.2.); the physical problem is transformed into P.L.K. co-ordinates, where the approximate P.I. of §4.8. is used to derive an expression for the second order pressure coefficient (§§ 5.2.3.&4.); finally by inverting the P.L.K. transformation, the solution is given in physical space co-ordinates.

Only simple delta wings are considered here. The flow over these wings is conical and therefore not strictly three-dimensional. However the application of the method to such wings illustrates the essential features with less detail consideration than would be necessary for more general wings. But the most important reason for this choice is the doubtful theoretical validity of the P.I., which makes an empirical evaluat-

-ion essential; conical wings are chosen because there exist linear and exact theoretical as well as experimental results for comparison. Furthermore, as Sugo' used conical wings in his investigation, this facilitates comparison between the consequences of the original and the modified P.I.S.

Application of the method to a delta wing with supersonic leading edges, being more straightforward than for the subsonic edge case, is treated first. The theory is illustrated in the former case by reference to a flat plate at incidence. The supersonic edge case is considered in detail but time did not permit detail consideration of the subsonic edge case.

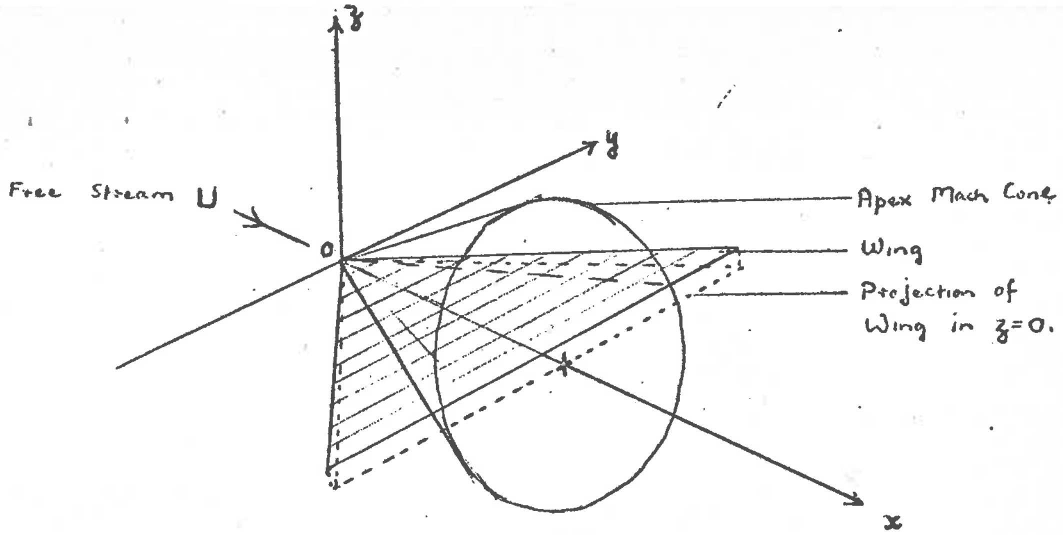
5.2. Supersonic Edged Delta Wing.

5.2.I. Wing configuration.

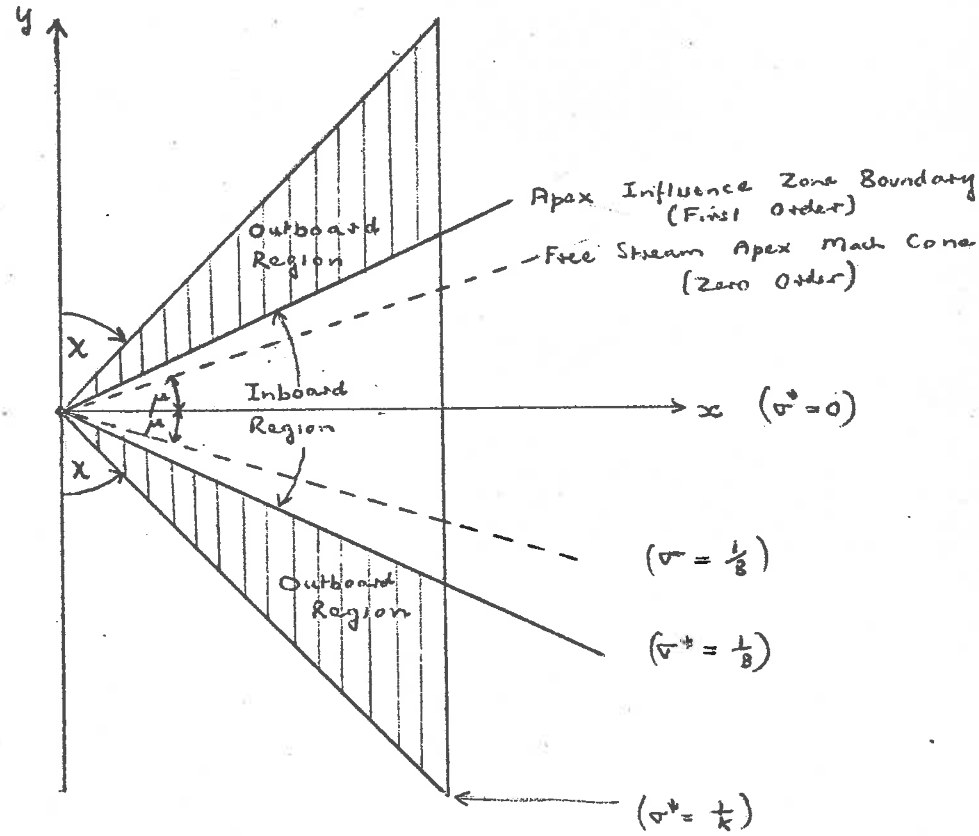
Introduce the concepts;

"Inboard region"— defined as that part of the wing contained within the apex influence zone.

"Outboard regions"— defined as those parts of the wing outside the apex influence zone.



Wing Perspective



Wing Planform

$\sigma^* = \frac{y}{x}$; $k^* = |\tan \chi|$; $B = \cot \mu$; $n^* = \frac{k^*}{B}$

Supersonic Leading Edge ; $k^* < B$; $0 < n^* < 1$

5.2.2. The P.L.K. Transformation.

The second order equation is

$$B^2 \varphi_{1uu} - \varphi_{1yy} - \varphi_{1zz} = \left[-m_\infty^2 \{(\gamma-1) m_\infty^2 + 2\} \varphi_{ou} + 2B^2 x_{1u}\right] \varphi_{ou}$$
$$- 2(m_\infty^2 \varphi_{oy} + x_{1y}) \varphi_{ouy}$$
$$- 2(m_\infty^2 \varphi_{oz} + x_{1z}) \varphi_{ouz}$$
$$+ \varphi_{ou} (B^2 x_{1uu} - x_{1yy} - x_{1zz}) \quad (5.I.)$$

where φ_o and its derivatives are supposed known from linear theory and it only remains to determine $x_1(u, y, z)$.

e.g. for a flat plate wing slope $\lambda = h_u$ the relevant linearised theory flow quantities are:

$$\left.\varphi_{ou}\right|_{z=0} = \begin{cases} \dfrac{-\lambda}{B\sqrt{1-n^2}} \\ \dfrac{-\lambda}{\pi B\sqrt{1-n^2}} \left[\pi - 2\sin^{-1}\sqrt{\dfrac{n^2-k^2\sigma^2}{1-k^2\sigma^2}}\right] & \frac{1}{B} \le \sigma \le \frac{1}{k} \end{cases} \quad [5.2.a]$$

$$\left.\varphi_{oy}\right|_{z=0} = \begin{cases} \dfrac{k\lambda}{B\sqrt{1-n^2}} \\ \dfrac{k\lambda}{\pi B\sqrt{1-n^2}} \left[\pi - 2\tan^{-1}\dfrac{\sqrt{n^2-k^2\sigma^2}}{k\sigma\sqrt{1-n^2}}\right] & \frac{1}{B} \le \sigma \le \frac{1}{k} \end{cases} \quad [5.2b]$$

$$\varphi_{ouu} = \begin{cases} 0 \\ \dfrac{2\lambda k^2\sigma^2/u}{\pi B(1-k^2\sigma^2)\sqrt{n^2-k^2\sigma^2}} \end{cases} \quad [5.2c]$$

$$\varphi_{ouy} = \begin{cases} 0 \\ \dfrac{-2\lambda k^2\sigma/u}{\pi B(1-k^2\sigma^2)\sqrt{n^2-k^2\sigma^2}} \end{cases} \quad [5.2.d]$$

$$\varphi_{oz} = h_u = \lambda \quad [5.2e]$$

Firstly, flow in the outboard region is constant and regular. Therefore outboard of the apex influence zone it is sufficient to take $x_1 \equiv 0$.

Inboard of the apex influence zone φ_{ou} and φ_{oy} are $O(\sqrt{u \pm By})$ as $u \to \pm By$ and therefore φ_{ouu}, φ_{ouy} which appear in the R.H.S. of eqn.(5.I), have square root singularities at the apex Mach cone.

c.f. eqns. (5.2a&c), (5.2 b&d) above.

The term φ_{ouz} behaves differently. If the streamwise slope λ has continuous, finite u-derivative, then φ_{ouz} (z=o) must be continuous and finite. If the slope has a discontinuity this will lead to a shock-wave or an expansion-wave at the point of discontinuity; this complication will not be considered here.

e.g. $\varphi_{ouz} = 0$ everywhere for a flat plate.

The P.L.K. technique has introduced the function $x_1(u,y,z)$ expressly to remove the singularities from the forcing term of eqn. (5.I.); x_1 is determined such that the coefficients of the singular terms vanish at the singular points (Mach cone - wing intersections) i.e. choose $x_1(u,y,z)$ such that

$$\left.\begin{aligned} 2B^2 x_{1u} - M_\infty^2\{(\gamma-1)M_\infty^2 + 2\}\varphi_{ou} &= 0 \\ x_{1y} + M_\infty^2 \varphi_{oy} &= 0 \end{aligned}\right]_{\substack{\sigma = \frac{1}{B} \\ z = 0}} \tag{5.3.}$$

and since φ_{ouz} is regular everywhere x_{1z} is arbitrary; setting $x_{1z} = 0$ simplifies $x_1(u,y,z)$ to $x_1(u,y)$.

For the flat plate wing the velocity components at the transformed Mach cone are given by eqns. (5.2.a & b). Substituting in eqn. (5.3.), gives

$$\left. x_{1u} \right|_{\substack{\sigma = \pm \frac{1}{B} \\ \zeta = 0}} = M_\infty^2 [N-1] . \left(- \frac{\lambda}{B\sqrt{1-n^2}} \right)$$

$$= \frac{-\lambda M_\infty^2 [N-1]}{B\sqrt{1-n^2}}$$

$$\left. x_{1y} \right|_{\substack{\sigma = \pm \frac{1}{B} \\ \zeta = 0}} = - \frac{M_\infty^2 \, k \lambda}{B\sqrt{1-n^2}} \qquad (5.4.)$$

It is sufficient then to take

$$x_1(u,y) = \begin{cases} 0 & k|y| > u > B|y| \\ \dfrac{-\lambda M_\infty^2 [N-1] u}{B\sqrt{1-n^2}} - \dfrac{M_\infty^2 k \lambda y}{B\sqrt{1-n^2}} & |u| < B|y| \end{cases} \qquad (5.5.)$$

For a second order theory the P.L.K. transformation is simply

$$x = u + \epsilon x_1(u,y)$$

e.g. for the flat plate

$$x = u - \frac{\lambda M_\infty^2}{B\sqrt{1-n^2}} \left[(N-1)u + ky \right]$$

or

$$x = u \left[1 - \lambda q(\sigma) \right]$$

where

$$q = \begin{cases} 0 & n \le k\sigma \le 1 \\ \dfrac{M_\infty^2}{B\sqrt{1-n^2}} \left[N - 1 + k\sigma \right] & 0 \le k\sigma \le n \end{cases} \qquad [5.6]$$

Henceforth all calculations will be carried out in P.L.K. space as apposed to physical space. Consequently, wings given in physical problems must be transformed into the corresponding wings in P.L.K. space; if quantities of physical space are denoted by an asterisk superscript, the following transformations are necessary.

$$\left.\begin{aligned} \sigma &= \sigma^{*}\left[1 - \lambda^{*} q(\sigma^{*})\right] \\ k &= k^{*}\Big/\left[1 - \lambda^{*} q(k^{*})\right] \end{aligned}\right] \quad (5.7.)$$

where $\lambda^{*} q(\sigma^{*}) \equiv \lambda q(\sigma)$ in second order theory.

5.2.3. Determination of φ_{1u}

The only quantity still unknown in the expression for C_p (§ 3.9.) is the second order velocity component φ_{1u}. This is determined from a consideration of contributions from the complementary function and P.I. separately. i.e. $\varphi_{1u} = \varphi_{1u}^{(p)} + \varphi_{1u}^{(c)}$

$\varphi_{1}^{(p)}$ is given in section 4, equation (4.14.) + (4.2.);

$$\varphi_{1}^{(p)} = M_\infty^2 \varphi_{0u}\left[(1-\tfrac{N}{2})\varphi_0 + \tfrac{N}{2} y \varphi_{0y} + \tfrac{N}{2} z \varphi_{0z} - N y z \varphi_{0yz}\right] + x_1 \varphi_{0u} \quad (5.8.)$$

Differentiating

$$\left.\varphi_{1u}^{(p)}\right|_{z=0} = M_\infty^2 \varphi_{0uu}\left[(1-\tfrac{N}{2})\varphi_0 + \tfrac{N}{2} y \varphi_{0y}\right] + M_\infty^2 (1-\tfrac{N}{2})\varphi_{0u}^2 + M_\infty^2 \tfrac{N}{2} y \varphi_{0u}\varphi_{0uy} + x_{1u}\varphi_{0u} + x_1 \varphi_{0uu} \quad (5.9.)$$

Then $\varphi_{1u}^{(p)}$ is determined in terms of linear theory results ((5.2.)) and $x_1(u, y)$, ((5.5.))

It only remains to determine $\varphi_{1u}^{(c)}$. The complementary function $\varphi_{1}^{(c)}$ is given by the source type solution

$$\varphi_1^{(c)}(u,y,0) = \iint_\Delta \frac{f(\xi,\eta)\quad d\xi\, d\eta}{\sqrt{(u-\xi)^2 - B^2(y-\eta)^2}} \qquad (5.10.)$$

where

(i) $f(\xi,\eta)$ is given by (c.f. Ref. 5, section 14)

$$f(\xi,\eta) = -\frac{1}{\pi}\left.\frac{\partial\varphi_1^{(c)}}{\partial z}\right|_{z=0} \qquad (5.11.)$$

(ii) The region of integration, Δ, is that part of the wing surface (approximated by $z=0$) contained within the fore Mach cone from the point $(u,y,0)$.

From equation (3.14.)

$$\left.\frac{\partial\varphi_1}{\partial z}\right|_{z=0} = \frac{\partial\left[\varphi_1^{(c)} + \varphi_1^{(p)}\right]}{\partial z} = \varphi_{0u} h_u + \varphi_{0y} h_y + x_1 h_{uu} \qquad (5.12.)$$

whence $\frac{\partial\varphi_1^{(c)}}{\partial z}$, f and then $\varphi_{1u}^{(c)}$.

For the flat plate wing $h_u=\lambda$, $h_y=h_{uu}=0$ eqn. (5.12.) yields

$$\frac{\partial\varphi_1^{(c)}}{\partial z} = \lambda\varphi_{0u} - \varphi_{1z}^{(p)}$$

Eqn. (5.8.) gives

$$\varphi_{1z}^{(p)} = \lambda M_\infty^2 \varphi_{0u}$$

hence

$$f(u,y) = \frac{\lambda}{\pi} B^2 \varphi_{0u}$$

and

$$\varphi_1^{(c)}(u,y,0) = \iint_\Delta \frac{\frac{\lambda}{\pi} B^2 \varphi_{0u}(\xi,\eta)\ d\xi\, d\eta}{\sqrt{(u-\xi)^2 - B^2(y-\eta)^2}} \qquad [5.13]$$

where $\varphi_{0u} = \dfrac{-\lambda}{B\sqrt{1-r^2}} \qquad \frac{1}{B} \leq \sigma \leq \frac{1}{k}$

$$= -\frac{\lambda}{B\sqrt{1-r^2}}\left[1 - \frac{2}{\pi}\sin^{-1}\sqrt{\frac{r^2-k^2\sigma^2}{1-B^2\sigma^2}}\right] \quad 0 \leq \sigma < \frac{1}{B}$$

The calculation of $\varphi_{1u}^{(c)}$ from eqn. (5.I3.) is quite tedious and is relegated to Appendix B. The final result may be given in the following form:

$$\varphi_{1u}^{(c)}(\sigma, 0) = \frac{-\lambda^2}{1-r^2} \qquad \frac{1}{B} \leq \sigma \leq \frac{1}{k}$$

$$= \frac{-\lambda^2}{\pi(1-r^2)}\left[\pi - 2\sin^{-1}\sqrt{\frac{r^2-k^2\sigma^2}{1-B^2\sigma^2}}\right]$$

$$-\frac{4r\lambda^2}{\pi^2}\int_0^1 \frac{\alpha^2}{(1-\alpha^2)(1-r^2\alpha^2)}\cosh^{-1}\left|\frac{r-\alpha.k\sigma}{k\sigma-r\alpha}\right| d\alpha \qquad (5.I4.)$$

where the integral is to be interpreted as the principal value of an improper integral. Actually $\varphi_{1u}^{(c)}$ as given above was calculated by approximating to P.L.K. space by physical space. The error incurred is only $O(\epsilon^3)$ since φ_{1u} is already a second order term.

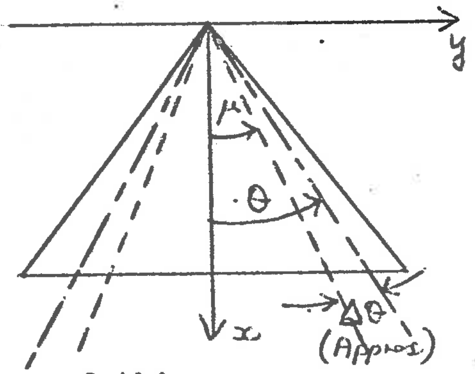
φ_{1u} is now completely determined by eqns. (5.9) and (5.I4.).

5.2.4. Validity of Solution.

Investigation of the expression for φ_{1u} just derived confirms that the P.L.K. technique has effectively removed the singularities at the free stream Mach cone and simultaneously positioned the apex influence zone boundary to its first order position. Actually the position obtained is to be interpreted as a limiting position of the influence zone boundary. Presumably the exact boundary lies between the free stream Mach cone envelope and the limit position. This concept is amplified in the discussion of the specific example in §5.2.6. It will be shown below that the boundary position obtained does correspond to the expected first order position i.e. within $O(\lambda^{*2})$; verification that singularities of the flow have been removed is not reproduced here as it is more readily seen in the similarity law form of §5.2.5.

To show that the influence zone boundary does correspond to the first order position consider the expression derived by Sheppard[11] from a consideration of linearised theory flow quantities. Defining θ as shown in the diagram below, the first order boundary location is given by (in the notation of Reference II.)

$$\tan\theta = \tan\mu + \Delta\tan\theta$$



where

$$\Delta\tan\theta = \frac{M_\infty^2 C_p^{(0)}}{2B^2}\left[k + B(N-1)\right]$$

which becomes in the notation of this paper

$$\Delta v^* \simeq \frac{\lambda^* M_\infty^2}{B^2\sqrt{1-n^{*2}}}\left[n^* + N - 1\right]$$

hence

$$v^*\Big|_{\text{Boundary}} = \frac{1}{B} + \frac{\lambda^* M_\infty^2}{B^2\sqrt{1-n^{*2}}}\left[n^* + N - 1\right]$$

From the P.L.K. technique, the adjusted boundary position is given by

$$k\sigma = n$$

which, on transformation to physical co-ordinates, yields:

$$\sigma^*_{Boundary} = \frac{1}{B} \frac{1}{[1 - \lambda^* q(\frac{1}{B})]}$$

$$\simeq \frac{1}{B} + \frac{\lambda^* M_\infty^2}{B\sqrt{1-n^{*2}}} [n + N - 1] + O(\lambda^{*2})$$

The two expressions for the boundary agree to $O(\lambda^{*2})$; i.e. the P.L.K. technique has successfully repositioned the influence zone boundary.

5.2.5. Similarity Considerations

The optimum form for the second order pressure coefficient is that which allows maximum use of dynamic similarity between flows. Van Dyke[12] has given possible similarity law forms for the second order pressure coefficient of which his equation (IIb) is the most suitable here. However van Dyke's work is based on an ordinary second order theory as opposed to an uniformly valid theory such as used here. This difference is important; it has already been mentioned in §3.7.4. but is repeated here for the sake of continuity.

An alternative way of looking at the difference between ordinary second order theory and the uniformly valid solution concerns the position of the influence zone boundary. Ordinary theory uses the simple condition of the influence zone boundary located at the free stream apex Mach cone, while an uniformly valid solution is essentially one taken relative to a corrected boundary position. Now if van Dyke's formulae are interpreted as referring to co-ordinates taken relative to the apex influence zone boundary, this is consistent with the original meaning but the generalisation to an uniformly valid solution is obvious; viz. Van Dyke's similarity law forms are applicable to the present solution provided all flow qualities are described in P.L.K. space. This is sufficient to derive most of the benifits of dynamic similarity theory except that, because the particular P.L.K. transforms may vary, similar wing -flow problems in P.L.K. space may not be similar in physical coordinates and vice-versa.

Equation (IIb) of van Dyke's paper gives the similarity law form

$$C_p(x,y,z\,;\,m,\gamma,\tau) = \frac{\tau}{\beta}P(x,\beta y,\beta z\,;\,\beta A) + \tau^2\left[p_1(\;) + \frac{m^2}{\beta^2}p_2(\;) + \frac{(\gamma+1)m^4}{\beta^4}p_3(\;)\right]$$

where p_1, p_2, p_3, are functions of the same arguments as P. Translating into the notation of this paper:

(i) $\left.\begin{cases}\epsilon \text{ (thickness)} \\ \lambda \text{ (slope)}\end{cases}\right\}$ parameter corresponds to τ

(ii) M_∞ replaces M

(iii) $B = \sqrt{M_\infty^2 - 1}$ replaces β

(iv) $n = \frac{k}{B} \propto \frac{1}{\beta A}$ (A = aspect ratio)

$\therefore g(n)$ is equivalent to $g(\beta A)$

(v) $k\sigma = \frac{ky}{x} = nB\frac{y}{x}$ $\quad = 0$

$\therefore$ F(n,$k\sigma$) is equivalent to g(x, βy, βz; βA)

(for conical flow)

The form of the similarity law is therefore

$$C_p = \frac{\lambda}{B} P(n, k\sigma) + \lambda^2 \left[p_1(\ ) + \frac{M_\infty^2}{B^2} p_2(\ ) + \frac{2N M_\infty^2}{B^2} p_3(\ ) \right] \tag{5.16.}$$

It will be shown that the expression for C_p for the flat plate wing may be put in this form. Extension to the general case should present little difficulty but is not done here.

Introduce the notation:

(i) Denote by $\overline{f}$ a function in the outboard region and by f the corresponding function inboard.

(ii) $A = \pi - 2\sin^{-1}\sqrt{\frac{n^2 - k^2\sigma^2}{1 - k^2\sigma^2}}$

$D = \pi - 2\tan^{-1}\frac{\sqrt{n^2 - k^2\sigma^2}}{k\sigma\sqrt{1 - n^2}}$

$C = \sqrt{1 - n^2}\,(1 - k^2\sigma^2)\sqrt{n^2 - k^2\sigma^2}$

The function P(n,$k\sigma$) contains the first order contribution to C_p.

viz.
$$C_p^{(0)} = -2\varphi_{0u} = \begin{cases} \dfrac{2\lambda}{B\sqrt{1 - n^2}} & \frac{1}{B} \leq \sigma \leq \frac{1}{k} \\ \dfrac{2\lambda A}{\pi B\sqrt{1 - n^2}} & 0 \leq \sigma \leq \frac{1}{B} \end{cases}$$

hence take
$$C_p^{(0)} = \frac{\lambda}{B} \begin{cases} \overline{P} & \text{Outboard} \\ P & \text{Inboard} \end{cases}$$

where
$$\bar{P} = \frac{2}{\sqrt{1-n^2}} \qquad (5.17)$$
$$P \quad \frac{2A}{\pi\sqrt{1-n^2}} \qquad (5.18)$$

The second order contribution
$$C_p^{(1)} = B^2\varphi_{ox}^2 - \varphi_{oy}^2 - \varphi_{oz}^2 + 2x_{1x}\varphi_{ox} - 2\varphi_{1x}^{(c)} - 2\varphi_{1x}^{(p)}$$
can be written in the form
$$\lambda^2\left[p_1(n,k\sigma) + \frac{M_\infty^2}{B^2}\cdot p_2(\ \ ) + \frac{2NM_\infty^2}{B^2}\, p_3(\ \ \ \ )\right]$$
where, for the flat plate wing:
$$\bar{p}_1 = \frac{2}{1-n^2} \qquad (5.19)$$
$$p_1 = -1 - \frac{n^2D^2}{\pi^2(1-n^2)} + \frac{A^2}{\pi^2(1-n^2)} + \frac{2A}{\pi(1-n^2)} + \frac{8n}{\pi^2}\int_0^1 \frac{\alpha^2}{(1-\alpha^2)(1-n^2\alpha^2)}\cosh^{-1}\left|\frac{n-\alpha.k\sigma}{k\sigma-n\alpha}\right| d\alpha \qquad (5.20)$$
$$\bar{p}_2 = \frac{-2}{1-n^2} \qquad (5.21)$$
$$p_2 = \frac{-2A^2}{\pi^2(1-n^2)} + \frac{4k^2\sigma^2}{\pi^2}\cdot\frac{1}{C}\cdot\left[\frac{A}{\pi}-1\right] + \frac{4k^3\sigma^3}{\pi}\cdot\frac{1}{C}\left[1-\frac{D}{\pi}\right] \qquad (5.22)$$
$$\bar{p}_3 = \frac{1}{2(1-n^2)} \qquad (5.23)$$
$$p_3 = \frac{A^2}{2\pi^2(1-n^2)} + \frac{2k^2\sigma^2}{\pi}\cdot\frac{1}{C}\cdot\left[1-\frac{A}{\pi}\right] \qquad (5.24)$$

Note that the apparent singularities of the second order terms p_2 and p_3 arising from the factor C which is $O(\frac{1}{\sqrt{n^2-k^2\sigma^2}})$ as $k\sigma \to \pm n$ (i.e. as $\sigma \to \pm\frac{1}{B}$) are effectively removed because of the multiplying brackets:
$$\left[1-\frac{A}{\pi}\right] = -\frac{2}{\pi}\sin^{-1}\sqrt{\frac{n^2-k^2\sigma^2}{1-k^2\sigma^2}} = O(\sqrt{n^2-k^2\sigma^2}) \text{ as } \sigma \to \pm\frac{1}{B}$$
$$\left[1-\frac{D}{\pi}\right] = \frac{2}{\pi}\tan^{-1}\frac{\sqrt{n^2-k^2\sigma^2}}{k\sigma\sqrt{1-n^2}} = O(\sqrt{n^2-k^2\sigma^2}) \text{ as } \sigma \to \pm\frac{1}{B}$$

Thus each term is in fact regular at $\sigma = \pm\frac{1}{B}$. This confirms that, for this case at least, the P.L.K. technique has rendered the solution uniformly valid.

5.2.6. Specific Example

The solution for $C_p^*(n^*, k^* \sigma^*)$ for the supersonic edged wing (flat plate at incidence) has now been given:

Eqns. (5.16),(5.17),(5.19),(5.21),(5.23) $\rightarrow C_p$ outboard

Eqns. (5.16),(5.18),(5.20),(5.22),(5.24) $\rightarrow C_p$ inboard

and
$$\sigma^* = \frac{\sigma}{[1 - \lambda^* q(\sigma)]}$$
maps P.L.K. space into physical space, giving $C_p^*(\sigma^*)$

Consider the flat plate supersonic edge wing at incidence -4°, with 45° sweepback; on the upper or compression surface,

$$M_\infty = 3.$$
$$k^* = 1.$$
$$\lambda^* = 0.0698$$
$$\gamma = 1.40$$

The pressure distribution on this wing has already been calculated by Fowell[13], Sheppard[11] and Sugo[1] using respectively an exact theory, an improved linearised result and an approximate second order theory.

Since time did not permit the rather lengthy calculations involved in the complete inboard solution to be carried out, it was decided that an assessment of the solution could be obtained from a knowledge of

(a) C_p^* on the wing centre-line ($\sigma = 0$)

(b) C_p^* in the outboard region ($\sigma > \frac{1}{8}$)

(c) The position of the apex influence zone boundary.

The justification for this step is that all known theoretical solutions and experimental results for this class of wing are, qualitatively, of a very similar form but are distinguished quantitatively by the values at the three points (a), (b) and (c).

The values of C_p^* corresponding to (a) and (b) above are found to be

(a) C_p^* (centre-line) = 0.046

(b) C_p^* (outboard) = 0.059

The position of the apex influence zone, and the overall solution is determined by interpreting the results as follows: there exist two partial representations of the complete solution; the inboard solution, which has now been rendered uniformly valid, and the outboard solution which, in this case, represents a constant flow region. These two representations apparently overlap in the region between the zero order position of the influence zone boundary and the limiting position of this boundary as given by the inboard P.L.K. transformation. The relevant solution is extracted by applying the boundary condition of continuous velocity and pressure at the influence zone boundary i.e. the boundary position is given by the intersection of the inboard representation and the constant outboard solution. On the wing surface on the compression side of the wing the boundary is actually an expansion wave, but, as mentioned in §3.7.4., this does not affect the application of Lighthill's 'Shock-wave' analysis.

In order to assess the solution the pressure coefficient has been calculated at two further points just inboard of the boundary.

(i) At the limiting position of the inboard representation

$$\sigma = \frac{1}{B} = 0.35 \longrightarrow \sigma^* = 0.43$$

$$C_p^* = 0.066 \ (\pm .002)$$

(ii) At the zero order boundary position

$$k\sigma = k^*\sigma^* = 0.35 \longleftarrow \sigma^* = 0.35$$

$$C_p^* = 0.052 \ (\pm .003)$$

Estimating the solution from these results, together with (a) and (b) above, locates the apex influence zone boundary at

$$\sigma^*\Big|_{\text{Boundary}} = 0.42 \ (\pm .02)$$

Detailed inboard calculations are necessary to place this boundary more accurately.

The solution, and method of locating the boundary, is shown in Fig. I. together with the other theoretical results for this wing. A comparison with these other theories suggests that the present method does give a useful second order result. The solution approaches Fowell's exact solution from below much as expected for a second order theory, the chief discrepancy arising just inboard of the boundary where the present method gives a more pronounced expansion. However, the pressure gradient is very similar to that of linearised theory near the boundary and is not unreasonable.

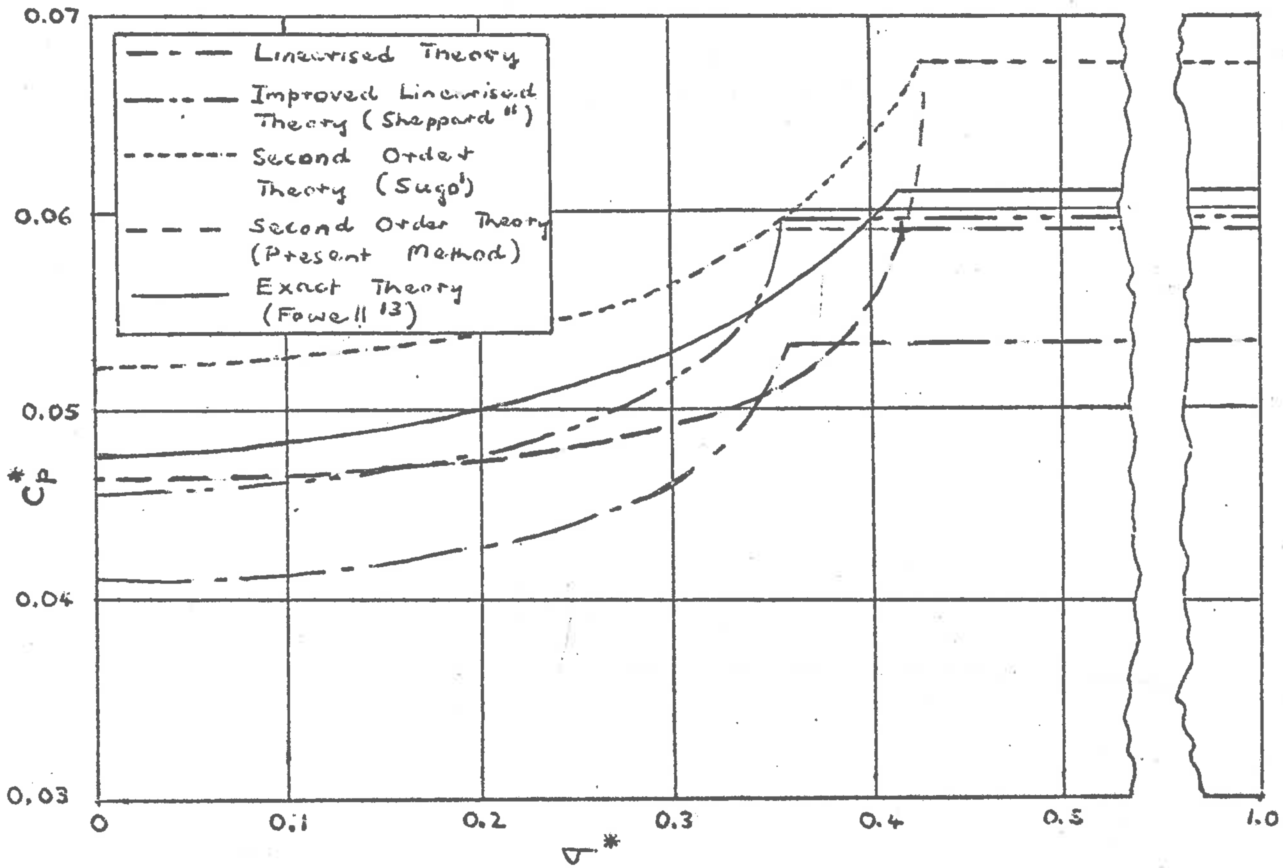


FIG. 1. PRESSURE DISTRIBUTION ON A 45° FLAT PLATE DELTA WING, $\alpha = -4°$, $M_\infty = 3$.

6: CONCLUSION

It has been shown that the second order supersonic wing theory proposed by Sugo[1] is incorrect because of errors occurring in the construction of a particular integral of the second order equation and in its application to wing problems. However, it has also been shown that a second order theory, utilising the basic approximation concept implicit in Sugo's work, has led to useful results. The only actual justification given for the approximation is empirical; the results of applying the theory to the calculation of the pressure distribution over a simple supersonic-edged, conical wing suggest that the approximation is indeed valid for this case. Furthermore there seems to be no suggestion that the approximation will be any worse for other simple wings. However, it must be emphasised that, in general, the approximation advocated will be useful only near the wing surface. Consequently, Wallace and Clarke's generalisation of their cruciform wing theory to three dimensions is in error because they assume firstly, that Sugo's particular integral is exact, and secondly that it is applicable throughout the flow region.

Application of the present method to compression surfaces of other simple supersonic-edged wings should be straight forward. Delta wings with single-wedge and double-wedge sections will require only detail modifications to boundary conditions but curved wing sections will require considerably more complex calculations; rectangular planforms should present little difficulty. Extension of the analysis to expansion surfaces may require more careful application of boundary conditions at the leading edge expansion wave and near the influence zone boundary but no other difficulties are expected.

For a subsonic-edged wing there are singularities at the influence zone boundaries and at leading edges. Singularities at the influence zone boundaries can be removed immediately, for simple wings at least, by the P.L.K. technique. Following Sugo, the leading edge singularity may be removed simultaneously by using the Karman-Tsien approximation that $\gamma = -1$ near the stagnation line, γ being the ratio of specific heats. This is a reasonable approximation in this region but the best method of incorporating it in the general solution is not clear. There is little doubt that a general result can be found which would, presumably, be as useful as the corresponding result for supersonic-edged wings.

It must be pointed out that for wings with curved leading edges the present theory would probably become intractable, because of the lengthy numerical calculations necessary. However, if an ordinary second order solution can be calculated for the flow field, then, following Wallace and Clarke,[10] such a solution need be corrected only in critical regions by using either results from the uniformly valid first order theory or a physical, shift rule ~~method~~ interpretation of the P.L.K. technique. This method may represent an important simplification in the theory but even an ordinary second order solution will involve considerable calculation. Therefore second order supersonic wing theory is unlikely to be of value in wing design unless examination of a small number of specific examples reveals means of simplifying the method without incurring significant errors. There is hope that a combined theoretical and experimental investigation of delta wings will enable a simplified theory to be developed.

ACKNOWLEDGEMENTS

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Appendix A

A.I. In section 4.9. a rather qualitative argument, based on orders of magnitude of quantities, has been used to investigate the approximation involved in the postulated form of the particular integral. It suggests that the approximations involved are legitimate for a second order theory. However, attempts to confirm this by more rigorous methods have not been successful; in fact they seem to suggest that the approximation has no mathematical justification. In this appendix some of the attempts are reproduced. None are complete. Some comments are made as to possible interpretations of the results in each case.

A.2. Consider the linear partial differential equation

$$L[\varphi] = B^2\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = F(x,y,z) \qquad \text{(A.I.)}$$

for which the complementary function $\varphi^{(c)}$ and the non-homogeneous term $F(x,y,z)$ are known. The problem is to find a particular integral. No exact particular integral could be found but there exists a function $\bar{\varphi}(x,y,z)$ s.t.

$$L[\bar{\varphi}] = G(x,y,z) \qquad \text{(A.2.)}$$

where

$$G(x,y,z) - F(x,y,z) = z R_1(x,y) + O(z^2) \qquad \text{(A.3.)}$$

This function $\bar{\varphi}(x,y,z)$ is the 'approximate' P.I. used in the solution in the main text. Does it really represent an approximation to the exact P.I. for small z $(\text{i.e. } z = O(\epsilon))$?

Define $\psi = \varphi - \bar{\varphi}$, then

$$L[\psi] = z R_1(x,y) + O(z^2) \qquad \text{(A.4.)}$$

The problem is now: is $\psi^{(p)}$ small $(O(\epsilon))$ for small z $(O(\epsilon))$?

For the wings considered, $F(x,y,z)$ (hence $R(x,y)$) may have no singularities, square root singularities or logarithmic singularities. In general it will be assumed that F is regular everywhere, i.e. either the singularities may be removed or the regions considered do not include the essential singularities.

A.3. Suppose the function ψ introduced above may be expanded in the form

$$\psi(x,y,z) = \psi_0(x,y) + z\psi_1(x,y) + z^2\psi_2(x,y) + \cdots \qquad \text{(A.5.)}$$

and the R.H.S. of (A.4.) may be taken as

$$R(x,y,z) = zR_1(x,y) + z^2 R_2(x,y) + \ldots \qquad \text{(A.6.)}$$

Substituting these expansions in (A.4.) and collecting coefficients of powers of z yields

$$
\begin{aligned}
&B^2 \psi_{0xx} - \psi_{0yy} - \psi_2 = 0 \\
&B^2 \psi_{1xx} - \psi_{1yy} - 3!\,\psi_3 = R_1 \\
&B^2 \psi_{2xx} - \psi_{2yy} - \tfrac{4!}{2!}\,\psi_4 = R_2 \\
&B^2 \psi_{3xx} - \psi_{3yy} - \tfrac{5!}{3!}\,\psi_5 = R_3
\end{aligned}
\tag{A.7.}
$$

If $\bar{\varphi}$ is a suitable approximation to φ, then $\bar{\varphi}(x,y,0) = \varphi(x,y,0)$ which requires $\psi_0(x,y) \equiv 0$

(A.7.) gives n equations for $n+2$ unknowns.

Two more equations are needed.

The boundary condition eqn (2.I.4) gives

$$
\begin{aligned}
&\frac{\partial \varphi}{\partial z} = \frac{\partial \bar{\varphi}}{\partial z} \qquad (z=0) \\
&\text{ie}\quad \frac{\partial \psi}{\partial z} = 0 \qquad (z=0)
\end{aligned}
\tag{A.8.}
$$

But $\dfrac{\partial \psi}{\partial z} = \psi_1 + 2z\psi_2 + O(z^2)$

Hence (A.8.) gives $\psi_1 = 0$

Substituting back into (A.7.) yields $\psi_3 \left(= \frac{R_1}{3!}\right)$ whence $\psi_5, \psi_7, \ldots\ldots\ldots$ However the method breaks down because there does not exist another boundary condition on the surface $z=0$.

A.4. As mentioned earlier the Green's Formula method is not particularly useful in general because of the difficulty of evaluating the triple integral. However it would appear that equation (A.4.) is amenable to such a method, especially if the region of integration is chosen to exclude singularities of $R(x,y)$ e.g. inboard of the apex influence zone of a supersonic edge delta wing.

The particular integral of (A.4.) is then

$$\psi^{(p)}(x,y,z) = \iiint_{\Delta} \frac{\zeta R_1(\xi,\eta) + O(\zeta^2) \quad d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2 - B^2[(y-\eta)^2 + (z-\zeta)^2]}}$$

Is this expression $O(z)$, z small ?

After extensive investigation and integration over possible surfaces, the details of which are not included, it was found there was no reason whatsoever for suggesting that $\psi^{(p)} = O(z)$ (in fact the residue terms suggested the approximation has little meaning in this sense.)

A.5. Since there exists an exact P.I. for the two dimensional problem, this affords a means of checking the proposed approximation. In particular consider plane flow over a curved boundary. The particular integral for a second order equation for this problem has been given by Van Dyke[2] pp 494-7. In the following investigation, results derived by Van Dyke[2] are quoted (in the notation of this paper) without proof.

The second order equation for plane flow

$$B^2\varphi_{1xx} - \varphi_{1zz} = -2M_\infty^2\left[(N-1)B^2\varphi_{0x}\varphi_{0xx} + \varphi_{0x}\varphi_{0xz}\right]$$

has an exact P.I.

$$\varphi_{1\,\text{Plane}}^{(p)} = -m_\omega^2 \frac{N}{2} z\, \varphi_{0x}\, \varphi_{0z}$$

Consider flow past a curved wall whose equation is

$$z = \epsilon\, g(x)$$

The solution of the first order problem is then

$$\varphi_0 = \frac{1}{B}\, g(x - Bz)$$

Substituting in the second order equation given above, yields

$$B^2\varphi_{1xx} - \varphi_{1zz} = -2m_\omega^2 N g'(x-Bz)\, g''(x-Bz)$$

with P.I.

$$\varphi_1^{(p)} = \frac{m_\omega^2 N}{2B} z\left[g'(x-Bz)\right]^2$$

Now following the approximation adopted in this paper, neglect terms of $O(\epsilon^3)$; the second order equation becomes

$$B^2\varphi_{1xx} - \varphi_{1zz} = -2m_\omega^2 N g'(x)\, g''(x)$$

It must be shown that any 'approximate' P.I. which satisfies this last equation differs from the exact P.I. only by terms of $O(\epsilon)$ or complementary function terms.

Consider two specific examples.

(i) Flow past a sharp corner

$$g(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$$

Then

$$\varphi_1^{(p)} = \frac{m_\omega^2 N z}{2B}$$

But the approximate form of the second order equation is

$$B^2\varphi_{1xx} - \varphi_{1zz} = 0$$

Hence

$$\bar{\varphi}_1^{(p)} = 0$$

and the result

$$\varphi_1^{(p)} - \bar{\varphi}_1^{(p)} = O(z)$$

is almost trival for this case.

(ii) Flow past a parabolic bend

$$g(x) = \begin{cases} 0 & x < 0 \\ \tfrac{1}{2}x^2 & x \geq 0 \end{cases}$$

Then

$$\varphi_1^{(p)} = \frac{m_\infty^2 N z}{2B}(x - Bz)^2$$

The approximate equation is

$$B^2\varphi_{1xx} - \varphi_{1zz} = -2m_\infty^2 N x$$

An obvious P.I. of this equation is

$$\bar{\varphi}_1^{(p)} = -\frac{2m_\infty^2}{6B^2} N x^3$$

Hence

$$\varphi_1^{(p)} - \bar{\varphi}_1^{(p)} = \frac{2m_\infty^2}{6B^2} N x^3$$

But this difference is not significant because it can easily be written as a combination of complementary function terms plus terms of $O(z)$

e.g. $x^3 = (x - Bz)^3 + 3Bx^2z - 3B^2xz^2 + B^3z^3$

$= \text{C.F.} + O(z)$

$$\therefore \varphi_1^{(p)} - \bar{\varphi}_1^{(p)} = \text{C.F.} + O(z)$$

It has been found that the approximation is valid in all 2-D examples tested. However, as there is a significant difference between the 3-D and 2-D wave equations, the results need not generalise. Attempts to construct counter examples all failed.

Appendix B. Calculation of the contribution to the streamwise velocity component from the second order complementary function term.

It has been shown in § 5.2.3. that the second order complementary function is given by

$$\varphi_1^{(c)}(u,y,0) = \iint_{\Delta} \frac{\frac{\lambda}{\pi}\beta^2 \varphi_{ou}(\xi,\eta,0)\, d\xi\, d\eta}{\sqrt{(u-\xi)^2 - \beta^2(y-\eta)^2}} \quad \text{(B.I.)}$$

Where the integration extends over the region of the wing included in the fore mach cone from the point $P(u,y,0)$. There is a difficulty here, in that the integration includes inboard and outboard regions for which the P.L.K. coordinates differ. However, since φ_1 is already $O(\epsilon^2)$ the difference between these two spaces gives a contribution $O(\epsilon^3)$ which is not significant. The following calculations are best considered in physical space coordinates (equivalent to P.L.K. form outboard) For convenience asterisks are not included and $u(= x + O(\epsilon))$ notation is also retained.

In the evaluation of this integral points outboard, (denote by $\bar{P}$) and inboard points(P), must be considered separately.

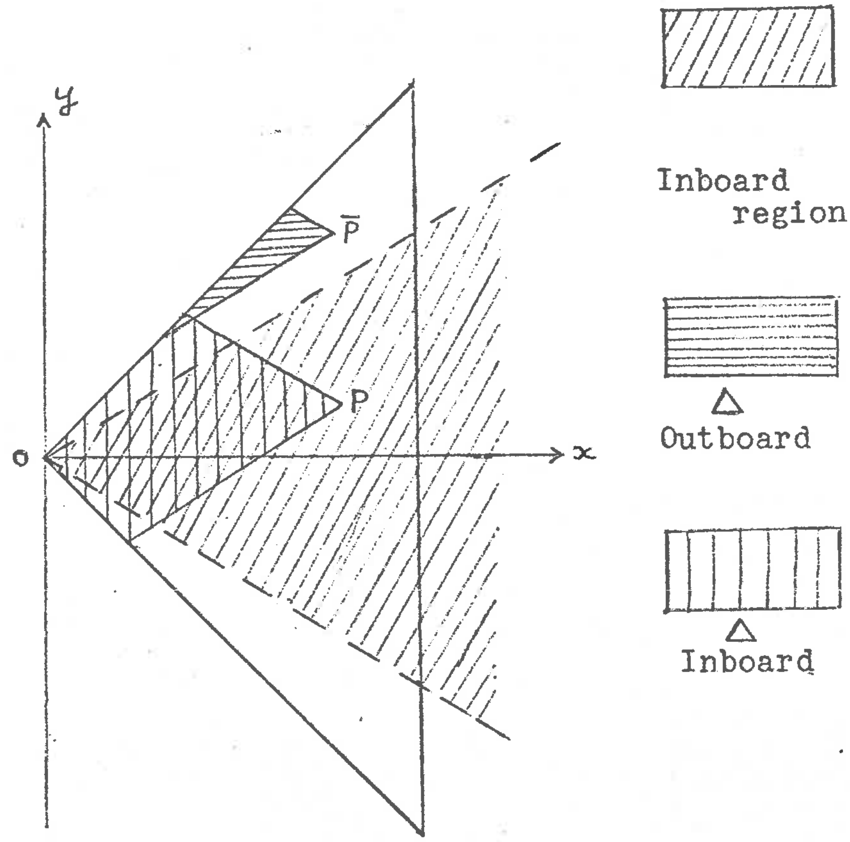

Fig. 1b

For a point $\bar{P}$ in the outboard region of this wing, flow is equivalent to that over an inclined rectangular wing of infinite aspect ratio and slope,

$$\frac{\beta\lambda^2}{\sqrt{1-n^2}}$$

hence

$$\varphi^{(c)}_{1u}\Big|_{\bar{P}} = \frac{-\lambda^2}{1-n^2}$$

(B.2.)

For the representative inboard point (P)

$$\varphi_1^{(c)}(u,y,0) = \frac{B^2\lambda}{\pi}\left(-\frac{\lambda}{\pi B\sqrt{1-n^2}}\right)\iint_\Delta \frac{\left[\pi - 2\sin^{-1}\sqrt{\frac{n^2-k^2h^2}{1-k^2h^2}}\right]}{\sqrt{(u-\xi)^2 - B^2(y-\eta)^2}}\,d\xi\,d\eta$$

$$\left(h = \frac{\eta}{\xi}\right)$$

$$= -\frac{B\lambda^2}{\pi\sqrt{1-n^2}}\iint_\Delta \frac{1}{\sqrt{(u-\xi)^2 - B^2(y-\eta)^2}}\,d\xi\,d\eta$$

$$+\frac{2B\lambda^2}{\pi^2\sqrt{1-n^2}}\iint_\Delta \frac{\sin^{-1}\sqrt{\frac{n^2-k^2h^2}{1-k^2h^2}}}{\sqrt{(u-\xi)^2 - B^2(y-\eta)^2}}\,d\xi\,d\eta$$

(B.3.)

The first term of (B.3.) gives a contribution to $\varphi_{1u}^{(c)}$

$$\varphi_{1u}^{(c)}\Big|_P^{(1)} = \frac{-\lambda^2}{\pi(1-n^2)}\left[\pi - 2\sin^{-1}\sqrt{\frac{n^2-k^2\sigma^2}{1-k^2\sigma^2}}\right]$$

(B.4.)

It only remains to determine the contribution to $\varphi_{1u}^{(c)}$ from the second term of (B.3.)

viz.

$$\varphi_1^{(c)}\Big|_P^{(2)} = \frac{2B\lambda^2}{\pi^2\sqrt{1-n^2}}\iint_{\Delta\cap\Gamma} \frac{\sin^{-1}\sqrt{\frac{n^2-k^2h^2}{1-k^2h^2}}}{\sqrt{(u-\xi)^2 - B^2(y-\eta)^2}}\,d\xi\,d\eta$$

(B.5.)

where the region of integration is now confined to the region of the wing contained within the intersection of the fore Mach cone (Δ) from P and the apex Mach cone (Γ) because the integral is defined as zero outboard.

Instead of evaluating (B.5.) directly for the symmetrical delta wing it is simpler to build up the result by considering the half wing shown in Fig.2b below.

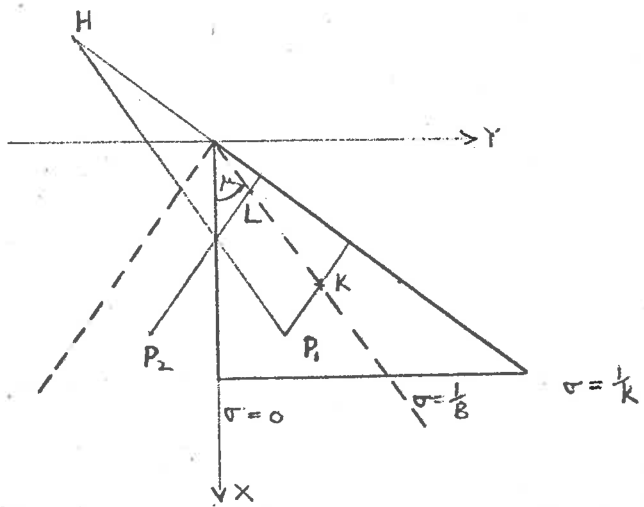

Fig.2b.

Denote by P_1 an inboard point on the half wing (i.e. $y > 0$) and P_2 an inboard point outside the wing (i.e. $y < 0$).

Equation (B.5.) yields for a point P_1

$$\varphi_1^{(c)} = \frac{2B\lambda^2}{\pi^2\sqrt{1-n^2}}\Bigg[\int_0^y d\eta \int_{B\eta}^{u-B(y-\eta)} d\xi \frac{\sin^{-1}\sqrt{\frac{n^2\xi^2-k^2\eta^2}{\xi^2-k^2\eta^2}}}{\sqrt{(u-\xi)^2-B^2(y-\eta)^2}}$$

$$+\int_y^{y_K} d\eta \int_{B\eta}^{u-B(\eta-y)} d\xi \frac{\sin^{-1}\sqrt{\frac{n^2\xi^2-k^2\eta^2}{\xi^2-k^2\eta^2}}}{\sqrt{(u-\xi)^2-B^2(y-\eta)^2}}$$

(B.6.)

where

$$y_K = \frac{u+By}{2B}$$

Integrate by parts w.r.t. to ξ ; this serves to change the integral from one with an (integrable) singularity at the upper limit to one with the singularity at the lower limit (which is independent of u) thereby simplifying differentiation w.r. to u.

$$\varphi_1^{(c)} = \frac{2B\lambda^2}{\pi^2\sqrt{1-n^2}}\left[\int_0^y d\eta \int_{B\eta}^{u-B(y-\eta)} d\xi \frac{k^2\eta^2\sqrt{1-n^2}}{\sqrt{n^2\xi^2-k^2\eta^2}\,(\xi^2-k^2\eta^2)} \cosh^{-1}\left|\frac{u-\xi}{B(y-\eta)}\right| \right.$$

$$\left. + \int_y^{y_k} d\eta \int_{B\eta}^{u-B(\eta-y)} d\xi \frac{k^2\eta^2\sqrt{1-n^2}}{\sqrt{n^2\xi^2-k^2\eta^2}\,(\xi^2-k^2\eta^2)} \cosh^{-1}\left|\frac{u-\xi}{B(y-\eta)}\right|\right]$$

Differentiate w.r.to u

$$\varphi_{1u}^{(c)} = \frac{2B\lambda^2}{\pi^2\sqrt{1-n^2}}\left[\int_0^y d\eta \int_{B\eta}^{u-B(y-\eta)} d\xi \frac{\sqrt{1-n^2}\cdot k^2\eta^2}{\sqrt{n^2\xi^2-k^2\eta^2}\,(\xi^2-k^2\eta^2)} \frac{1}{\sqrt{(u-\xi)^2-B^2(y-\eta)^2}}\right.$$

$$+ \cancel{\int_0^y d\eta \cosh^{-1}(1) \times F(\xi,\eta)}$$

$$+ \int_y^{y_k} d\eta \int_{B\eta}^{u-B(\eta-y)} d\xi \frac{\sqrt{1-n^2}\, k^2\eta^2}{\sqrt{n^2\xi^2-k^2\eta^2}\,(\xi^2-k^2\eta^2)\sqrt{(u-\xi)^2-B^2(y-\eta)^2}}$$

$$+ \cancel{\int_y^{y_k} d\eta \cosh^{-1}(1) \times F(\xi,\eta)}$$

$$\left. + \cancel{\frac{1}{2B}\int_{\frac{u+By}{2}}^{\frac{u+By}{2}} d\xi\; G(\xi,\eta)}\right]$$

$\therefore$

$$\varphi_{1u}^{(c)} = \frac{2B\lambda^2}{\pi^2\sqrt{1-n^2}}\left[\int_0^y d\eta \int_{B\eta}^{u-B(y-\eta)} d\xi \frac{\sqrt{1-n^2}\, k^2\eta^2}{\sqrt{n^2\xi^2-k^2\eta^2}\,(\xi^2-k^2\eta^2)\sqrt{(u-\xi)^2-B^2(y-\eta)^2}}\right.$$

$$+ \int_y^{\frac{u+By}{2B}} d\eta \int_{B\eta}^{u-B(\eta-y)} d\xi \frac{\sqrt{1-n^2}\, k^2\eta^2}{\sqrt{n^2\xi^2-k^2\eta^2}\,(\xi^2-k^2\eta^2)\sqrt{(u-\xi)^2-B^2(y-\eta)^2}}$$

(B.7.)

Since the flow is in fact conical, this last expression may be further simplified by introducing the conical variable $h = \frac{\eta}{\xi}$ and transforming from (ξ,η) to (ξ,h) variables.

Then

$$\varphi_{1u}^{(c)}(\sigma,0) = \frac{2B\lambda^2}{\pi^2\sqrt{1-n^2}}\left[\int_0^{\sigma} dh \int_0^{\xi_1} d\xi \frac{\sqrt{1-n^2}\,.\,k^2h^2}{\sqrt{(u-\xi)^2 - B^2(y-h\xi)^2}\,(1-k^2h^2)\sqrt{n^2-k^2h^2}}\right.$$

$$\left.+\int_{\sigma}^{\frac{1}{B}} dh \int_0^{\xi_1} d\xi \frac{\sqrt{1-n^2}\,.\,k^2h^2}{(1-k^2h^2)\sqrt{n^2-k^2h^2}\,\sqrt{(u-\xi)^2 - B^2(y-h\xi)^2}}\right]$$

(B.8.)

where ξ_1, is the last value of ξ (in (ξ,h) co-ords) which gives a contribution to the flow at P.

Performing the ξ-integration and simplifying

$$\varphi_{1u}^{(c)}\Big|_{P_1}^{(2)} = \frac{-2B\lambda^2}{\pi^2\sqrt{1-n^2}}\int_0^{\frac{1}{B}} dh \frac{\sqrt{1-n^2}\;k^2h^2}{\sqrt{n^2-k^2h^2}\,(1-k^2h^2)\sqrt{1-B^2h^2}}\,\cosh^{-1}\left|\frac{1-B^2\sigma h}{B(\sigma-h)}\right|$$

(B.9.)

But the wing also induces a flow for points P_2 off the wing surface but inside the apex influence zone. From (B.5.) and Fig.2b

$$\varphi_1^{(c)}\Big|_{P_2}^{(2)} = \frac{2B\lambda^2}{\pi^2\sqrt{1-n^2}}\int_0^{y_L} d\eta \int_{B\eta}^{u-B(\eta-y)} d\xi \frac{\sin^{-1}\sqrt{\dfrac{n^2\xi^2-k^2\eta^2}{\xi^2-k^2\eta^2}}}{\sqrt{(u-\xi)^2 - B^2(y-\eta)^2}}$$

where $$y_L = \frac{u+By}{2B}$$

Carrying through the analysis as for P_1 , the following result is obtained.

$$\varphi_{1u}^{(c)}\Big|_{P_2}^{(2)} = \frac{-2B\lambda^2}{\pi^2\sqrt{1-n^2}}\int_0^{\frac{1}{B}} dh \frac{\sqrt{1-n^2}\;k^2h^2}{\sqrt{n^2-k^2h^2}\,(1-k^2h^2)\sqrt{1-B^2h^2}}\,\cosh^{-1}\left|\frac{1-B^2\sigma h}{B(\sigma-h)}\right|$$

(B.10.)

i.e. $$\varphi_{1u}^{(c)}\Big|_{P_1}^{(2)} = \varphi_{1u}^{(c)}\Big|_{P_2}^{(2)}$$

Superposing two such half wings, the positive one and its negative counterpart, reconstructs the original wing for which then

$$\varphi_{1u}^{(c)}\Big|_P^{(2)} = \frac{-4B\lambda^2}{\pi^2}\int_0^{\frac{1}{B}} \frac{k^2h^2}{\sqrt{n^2-k^2h^2}\,(1-k^2h^2)\sqrt{1-B^2h^2}}\; \cosh^{-1}\left|\frac{1-B^2h\sigma}{B(\sigma-h)}\right| dh \qquad \text{(B.II.)}$$

This may be further simplified by the change of variables $\alpha = Bh$.

$$\varphi_{1u}^{(c)}\Big|_P^{(2)} = -\frac{4n\lambda^2}{\pi^2}\int_0^1 \frac{\alpha^2}{(1-n^2\alpha^2)(1-\alpha^2)}\; \cosh^{-1}\left|\frac{n-k\sigma.\alpha}{k\sigma-n\alpha}\right| d\alpha \qquad \text{(B.I2.)}$$

On inspection, the integral of (B.I2.) has apparent singularities of the form $O\left(\frac{1}{x-x_0}\right)$ as $x \rightarrow x_0$ at $\alpha = 1$ and $\alpha = B\sigma$. ($\alpha = \frac{1}{n}$ is outside range of integration.) The apparent divergence of the integral at $\alpha = B\sigma$ is removed by taking the principal value. As the integration could not be performed analytically, a numerical evaluation is necessary.

In order to determine the degree of refinement necessary to obtain a suitable approximation to the integral by numerical methods, the centre-line integral ($\sigma = 0$) is first evaluated for varying subdivision steps.

On the wing centre-line ($\sigma = 0$) the integral contributing to $\varphi_{1u}^{(c)}$ (equ.(B.I2.)) simplifies to

$$\varphi_{1u}^{(c)}\Big|_P^{(2)} = -\frac{4n\lambda^2}{\pi^2}\int_0^1 \frac{\alpha^2}{(1-n^2\alpha^2)(1-\alpha^2)} \cosh^{-1}\left(\frac{1}{\alpha}\right) d\alpha$$

(B.I3.)

where

(i) $\cosh^{-1}\left(\frac{1}{\alpha}\right) = O(\log\alpha)$ as $\alpha \to 0$

Hence
$\alpha^2 \cosh^{-1}\left(\frac{1}{\alpha}\right) \to 0$ as $\alpha \to 0$

i.e. The integral vanishes as $\alpha \to 0$ instead of giving the singularity expected from the general form.

(ii) It can be shown that

$$\frac{\cosh^{-1}\frac{1}{\alpha}}{1-\alpha} = O\left(\frac{1}{\sqrt{1-\alpha}}\right)$$

as $\alpha \to 1$

In fact $\frac{\cosh^{-1}\frac{1}{\alpha}}{1-\alpha} < \frac{1.2}{\sqrt{1-\alpha}}$; $.99 < \alpha < 1$

(iii) Since the integral of (B.I3.) (denote by I) is multiplied by

$$\frac{4n\lambda^2}{\pi^2} \simeq \frac{\lambda^2}{10}$$

the integral itself need only be evaluated such that

$$|\text{Error in } I| < 10.\lambda^2 . |I| = O(\lambda^4)$$

(i.e. contribution to total error is not significant)

A crude numerical integration shows $I \simeq 1$

Therefore it is sufficient to evaluate I s.t.

$$|\text{Error}| < .05$$

Truncating the integral at $\alpha = .9999$ and using inequality of (ii) shows

$$\left| \text{Error} \,|_{\text{Truncation}} \right| < .03$$

The numerical integration of

$$I = \int_{0.0000}^{.9999} \frac{\alpha^2}{(1-\alpha^2)(1-n^2\alpha^2)} \cosh^{-1}\left(\frac{1}{\alpha}\right) d\alpha$$

$$= \int_{0.00}^{0.99} \text{"} \, d\alpha + \int_{.9900}^{.9999} \text{"} \, d\alpha$$

using intervals of length .01 in the first integral and length .0001 in the second yields the value

$$I = 0.97 \pm .05$$

which is within the desired accuracy.

This value is used in the calculation of C_p on the centre-line

A more general method had to be evolved for calculation of the integral off the centre-line. This was derived by modification of a method for evaluating singular integrals given by Roper[14].

The calculation proceeds as follows:

Set

$$I = I_1 + I_2 + I_3$$

Where

$$I_1 = \int_0^{\beta\sigma - \delta} G(\alpha) \cosh^{-1} \left| \frac{n - k\sigma\alpha}{k\sigma - n\alpha} \right| d\alpha$$

$$I_2 = \delta .\, G(B\sigma + \delta)\, \cosh^{-1} \left| \frac{n - \frac{k^2\sigma^2}{n} - k\sigma .\, \delta}{n\delta} \right|$$

$$+\, \delta .\, G(B\sigma - \delta)\, \cosh^{-1} \left| \frac{n - \frac{k^2\sigma^2}{n} + k\sigma .\, \delta}{n\delta} \right|$$

$$+ \int_{B\sigma-\delta}^{B\sigma+\delta} \frac{2\alpha\,[1 - n^2\alpha^4]}{(1-\alpha^2)^2(1-n^2\alpha^2)^2}\,(B\sigma - \alpha)\, \cosh^{-1} \left| \frac{n - k\sigma .\, \alpha}{k\sigma - n\alpha} \right| d\alpha$$

$$+ \int_{B\sigma-\delta}^{B\sigma+\delta} G(\alpha)\, \frac{n^2 - k^2\sigma^2 \quad d\alpha}{n\sqrt{(n - k\sigma .\, \alpha)^2 - (k\sigma - n\alpha)^2}}$$

$$I_3 = \int_{B\sigma-\delta}^{.9999} G(\alpha)\, \cosh^{-1} \left| \frac{n - k\sigma .\, \alpha}{k\sigma - n\alpha} \right| d\alpha$$

Where

(i) $G(\alpha) = \dfrac{\alpha^2}{(1-\alpha^2)(1-n^2\alpha^2)}$

(ii) δ is chosen such that the first integral term of I_2 may be neglected to the order of accuracy of the calculation.

(iii) All integrals, except the one mentioned in (ii), are now non-singular and integrable.

(iv) The behaviour of I_3 as $\alpha \to 1$ corresponds to that of the centre-line integral and is truncated in the same manner.

Since each of the integrals into which I has been analysed can now be integrated by standard numerical methods, the essential difficulties have been overcome.

Time did not permit these calculations to be carried through.

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