Positive Scalar Curvature and Callias-Type Index Theorems for Proper Actions

Hao Guo

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Contents

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Dedication

To all my family.

Abstract

This thesis by publication is a study of the equivariant index theory of Dirac operators and Callias-type operators in two distinct settings, namely on cocompact and non-cocompact manifolds with a Lie group action.

The first two chapters are a short resumé of Dirac operators and index theory and form a common introduction to the papers in the appendices.

Appendix A is joint work with my supervisors, Elder Professor Mathai Varghese and Dr. Hang Wang. For G an almost-connected Lie group acting properly and cocompactly on a manifold M, we study G-index theory of G-invariant Dirac operators. By establishing Poincaré duality for equivariant K-theory and K-homology, we are able to extend the scope of our results to include all elements of equivariant analytic K-homology, which we also show is isomorphic to equivariant geometric K-homology. Our results are applied to prove: a rigidity result for almost-complex manifolds, generalising a vanishing theorem of Hattori; an analogue of Petrie's conjecture; and Lichnerowicz-type obstructions to G-invariant Riemannian metrics on M.

Appendix B studies the much more general situation when the quotient M/G is non-compact and G is an arbitrary Lie group. I define *G-Callias-type operators* and show that they are $C^*(G)$-Fredholm by adapting analysis of Kasparov to new Hilbert $C^*(G)$-module analogues of Sobolev spaces. Questions of adjointability, regularity and essential self-adjointness are addressed in detail. The estimates on G-Callias-type operators are based on the work of Bunke [8] in the non-equivariant context. We construct explicit admissible endomorphisms for G-Callias-type operators from the K-theory of the Higson G-corona of M, a highly non-trivial group. The index theory developed here is applied to prove a general obstruction theorem for G-invariant metrics of positive scalar curvature in the non-cocompact setting.

Chapter 1

Dirac Operators

This chapter is an exposition of some well-known material on Dirac operators in the context of group actions on manifolds. The interested reader can pursue the topics further in books such as [14], [4] and [11]. The material on almost-connected groups can be found in the paper [1].

We begin by defining the Clifford algebra and the Spin and Spin^c groups, followed by the notions of Dirac and Dirac-type operators acting on sections of a Clifford module bundle.

We will recall what it means for a Lie group G to act properly on a manifold, and point out the simplifications that arise in the special case that G has only finitely many connected components. We recall the definitions of G-invariant Dirac and Dirac-type operators. We will also discuss Lichnerowicz's vanishing theorem, a prototype of index-theoretic obstructions to the existence of metrics of positive scalar curvature.

1.1 Clifford Algebras

Let $\mathbb{K}$ be a field with characteristic $\neq 2$. Let V be a vector space over $\mathbb{K}$ equipped with a non-degenerate bilinear form g. The Clifford algebra $Cl(V, g)$ associated to V and g is a unital associative $\mathbb{K}$-algebra defined as the quotient

$$Cl(V, g) := T(V)/I(V, g),$$

where $T(V)$ is the tensor algebra of V and $I(V,g)$ is the ideal of $T(V)$ generated by the set of elements

$$\{v \otimes v + g(v,v) : v \in V\}.$$

Let $\mathrm{Id} : V \to V$ be the identity map. Then $-\mathrm{Id}$ is an involutive isometry on (V,g) that induces a decomposition

$$Cl(V,g) = Cl^0(V,g) \oplus Cl^1(V,g),$$

where $Cl^0(V,g)$ and $Cl^1(V,g)$ are the $+1$ and -1-eigenspaces of $-\mathrm{Id}$. This gives $Cl(V,g)$ a $\mathbb{Z}_2$-grading, and we refer to $Cl(V,g)^0$ and $Cl(V,g)^1$ respectively as the *even* and *odd* parts of $Cl(V,g)$. As a vector space, $Cl(V,g)$ is isomorphic to Λ^*V, the real exterior algebra of V.

Define $\langle\, , \rangle$ be the usual inner product on $\mathbb{R}^n$. We shall often use the shorthand notation

$$Cl_n := Cl(\mathbb{R}^n, \langle\, , \rangle).$$

It can be shown that Cl_n is isomorphic to Cl^0_{n+1}.

Define

$$\mathbb{C}l_n := Cl_n \otimes_{\mathbb{R}} \mathbb{C}.$$

One sees that, as complex vector spaces, $\mathbb{C}l_n \cong \Lambda^*(\mathbb{C}^n)$. Further, it can be shown that

$$\mathbb{C}l_{2m} \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2^m}),$$
$$\mathbb{C}l_{2m+1} \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2^m}) \oplus \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2^m}).$$

These isomorphisms together classify the algebras $\mathbb{C}l_n$ for all $n \geq 0$.

For each $m \geq 0$, define the vector spaces

$$\Sigma_{2m} = \Sigma_{2m+1} := \mathbb{C}^{2^m}.$$

When $n = 2m$, the first isomorphism above gives a representation of $\mathbb{C}l_n$ on Σ_n. When $n = 2m+1$, the second isomorphism composed with projection onto the first factor gives a representation of $\mathbb{C}l_n$ on Σ_n. We will use these two representations to define Spin and Spin^c-Dirac operators.

Let $(e_1, \dots e_n)$ be an orthonormal basis of $\mathbb{R}^n \subseteq Cl_n \subseteq \mathbb{C}l_n$. One verifies that the element

$$\omega := i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdot \ldots \cdot e_n$$

squares to the identity operator on Σ_n and satisfies

$$v \cdot \omega = (-1)^{n-1} \omega \cdot v$$

for $v \in \mathbb{R}^n$. Thus if $n = 2m$, ω induces a $\mathbb{Z}_2$-grading on Σ_{2m}, which decomposes into a direct sum $\Sigma_{2m}^+ \oplus \Sigma_{2m}^-$ of its ± 1-eigenspaces. Elements of $\mathbb{C}l_{2m}^0$ act on Σ_{2m} preserving this decomposition, while one can verify that non-zero elements of $\mathbb{R}^n$ act as vector space isomorphisms $\Sigma_{2m}^\pm \to \Sigma_{2m}^\mp$.

1.2 The Spin and Spin^c Groups

Let $Cl_n^\times$ denote the group of units in Cl_n. There is a distinguished subgroup of $Cl_n^\times$, the *n-th Spin group*, defined as

$$\mathrm{Spin}_n := \{v_1 \cdot \ldots \cdot v_{2k} : v_j \in \mathbb{R}^n, \|v_j\| = 1, k \geq 1\}.$$

For each n, Spin_n is a compact, connected Lie group of dimension $\frac{n(n-1)}{2}$. It can be shown that for $n \geq 2$, Spin_n is the non-trivial 2-fold cover of SO_n; if $n \geq 3$, it is the universal cover of SO_n.

Restricting the representation of $\mathbb{C}l_n$ on Σ_n given above to Spin_n gives us the representation

$$\rho : \mathrm{Spin}_n \to \mathrm{Aut}(\Sigma_n),$$

referred to as the *complex spinor representation*. When n is even, ρ decomposes into a direct sum of two inequivalent irreducible representations of Spin_n; when n is odd, ρ is irreducible.

Let $U_1 \subseteq \mathbb{C}$ be the unit circle. Consider the subgroup $\mathrm{Spin}_n \times U_1 \subseteq \mathbb{C}l_n$. The *n-th Spin^c group* is defined to be the quotient

$$\mathrm{Spin}_n^c := (\mathrm{Spin}_n \times U_1) \,/\, \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ denotes the two-element subgroup

$$\{(1,1), (\epsilon, -1)\} \subseteq \mathrm{Spin}_n \times U_1,$$

where ϵ is the non-identity element in the kernel of the double covering map $\nu : \mathrm{Spin}_n \to SO_n$.

1.3 Clifford Module Bundles

Let M^n be a Riemannian manifold equipped with an orientation. Then we can write the tangent bundle TM as an associated vector bundle of its principal SO_n-bundle of oriented orthonormal frames, $P_{SO_n}(M)$:

$$TM \cong P_{SO_n}(M) \times_{\xi_n} \mathbb{R}^n,$$

defined by dividing $P_{SO_n}(M) \times \mathbb{R}^n$ by the equivalence relation

$$\xi_n := \{(p \cdot g, f) \sim (p, \rho(g)f)\},$$

where ρ is the standard representation $SO_n \to \mathrm{Aut}(\mathbb{R}^n)$. Further, since each element $g \in SO_n$ preserves the inner product on $\mathbb{R}^n$, g acts as automorphisms on the Clifford algebras Cl_n and $\mathbb{C}l_n$. That is, we have maps

$$\rho' : SO_n \to \mathrm{Aut}(Cl_n), \qquad \rho'' : SO_n \to \mathrm{Aut}(\mathbb{C}l_n).$$

Define in an analogous way to ξ_n the equivalence relations ξ_n' and ξ_n'' on $P_{SO_n}(M) \times Cl_n$ and $P_{SO_n}(M) \times \mathbb{C}l_n$ respectively. That is,

$$\xi_n' := \{(p \cdot g, f) \sim (p, \rho'(g)f)\}, \qquad \xi_n'' := \{(p \cdot g, f) \sim (p, \rho''(g)f)\}.$$

The *real and complex Clifford bundles* of TM are defined to be

$$Cl(TM) := P_{SO_n}(M) \times_{\xi_n'} Cl_n, \qquad \mathbb{C}l(TM) := P_{SO_n}(M) \times_{\xi_n''} \mathbb{C}l_n.$$

In order to define Dirac operators, we will need the notion of a *complex Clifford module bundle* over M^n. This is a Hermitian vector bundle $E \to M$ together with a bundle map called *Clifford multiplication*,

$$c : \mathbb{C}l(TM) \to \mathrm{End}(E)$$

a morphism of algebras on each fibre, with the property that for any $x \in M$, the image of $\alpha \in T_xM \subseteq \mathbb{C}l(TM)$ is skew-symmetric. If E comes with a $\mathbb{Z}_2$-grading, we require c to respect gradings on $\mathbb{C}l(TM)$ and E.

Note that in the literature, the term *Clifford bundle* may mean either the bundle $\mathbb{C}l(TM)$ (or its real analogue $Cl(TM)$) or a Clifford module bundle as we have defined it here. A Clifford module bundle is also sometimes referred to simply as a *Clifford module.*

When working with a Clifford module bundle E, we will tacitly assume that it comes with a grade-preserving Hermitian connection ∇^E that is compatible with Clifford multiplication, in the sense that given vector fields X, Y on M and a smooth section u of E, we have

$$\nabla_X^E c(Y)u = c\left(\nabla_X^{LC} Y\right) u + c(Y)\nabla_X^E u,$$

where ∇^{LC} denotes the Levi-Civita connection on TM.

1.4 Dirac Operators

Given a Clifford module bundle $E \to M^n$ with connection ∇^E, the *Dirac operator* D is defined to be the composition

$$C_c^\infty(E) \xrightarrow{\nabla^E} C_c^\infty(T^*M \otimes E) \xrightarrow{\sharp} C_c^\infty(TM \otimes E) \xrightarrow{c} C_c^\infty(E),$$

where the map $\sharp$ is given by identifying TM and T^*M using the Riemannian metric on M.

D is a first-order elliptic differential operator with initial domain the compactly supported smooth sections of E:

$$D : C_c^\infty(E) \to C_c^\infty(E).$$

One way to view D is as an unbounded, formally self-adjoint operator on $L^2(E)$. One can also take take the closure of D to get a bounded operator from the first Sobolev space $H^1(E)$ to $L^2(E)$.

Given any Hermitian vector bundle F with a Hermitian connection ∇^F, one can *twist* D by F and ∇^F to get a new Dirac operator on $E \otimes F$, as follows. Form the connection

$$\nabla^{E\otimes F} := \nabla^E \otimes 1 + 1 \otimes \nabla^F$$

on $E \otimes F$. Then the *Dirac operator* D *twisted by* F, denoted by D_F, is defined to be the composition

$$C_c^\infty(E\otimes F) \xrightarrow{\nabla^{E\otimes F}} C_c^\infty(T^*M\otimes E\otimes F)$$

$$\xrightarrow{\sharp} C_c^\infty(TM\otimes E\otimes F) \xrightarrow{c} C_c^\infty(E\otimes F),$$

where Clifford multiplication c only acts on the first factor of $E\otimes F$.

In addition to the notion of a Dirac operator associated to a Clifford module bundle, we will be concerned with two special cases of these operators on manifolds with additional geometric structures. These are Spin and Spin^c manifolds and their equivariant versions.

An oriented Riemannian manifold M^n is said to be *Spin* if there exists a principal Spin$_n$-bundle $P_{\mathrm{Spin}_n}(M)\to M$ together with a bundle map

$$\eta : P_{\mathrm{Spin}_n}(M) \to P_{SO_n}(M),$$

satisfying $\eta(pg)=\eta(p)\nu(g)$ for all $p\in P_{\mathrm{Spin}_n}(M)$ and $g\in \mathrm{Spin}_n$, where

$$\nu : \mathrm{Spin}_n \to SO_n$$

is the double covering map mentioned earlier. The existence of the bundle $P_{\mathrm{Spin}_n}(M)$ allows one to lift the structure group of TM and $\mathbb{C}l(TM)$ from SO_n to Spin$_n$, so that we have isomorphisms

$$TM \cong P_{\mathrm{Spin}_n}(M)\times_{\xi_n'''}\mathbb{R}^n, \qquad \mathbb{C}l(TM)\cong P_{\mathrm{Spin}_n}(M)\times_{\xi_n''''}\mathbb{C}l_n,$$

where the equivalence relations ξ''' and ξ'''' are defined analogously to before, using the actions of Spin$_n$ on $\mathbb{R}^n$ and $\mathbb{C}l_n$ defined by composition of the SO_n-actions with ν.

The *complex spinor bundle* $S\to M^n$ is defined to be

$$S = P_{\mathrm{Spin}_n}(M)\times_{\xi'''''}\Sigma_n,$$

where the equivalence relation ξ''''' is now defined using the the complex spin representation. Since both S and $\mathbb{C}l(TM)$ are associated bundles of $P_{\mathrm{Spin}_n}(M)$, there is a natural action

$$\mathbb{C}l(TM)\times S\to S,$$

$$[(p,v)]\times[(p,\sigma)]\mapsto[(p,v\sigma)],$$

where $p \in P_{\mathrm{Spin}_n}(M)$, $v \in \mathbb{C}l_n$ and $\sigma \in \Sigma_n$.

In order to make S into a Clifford module bundle, we need one more piece of data, namely a connection on S that is compatible with the Levi-Civita connection. The Levi-Civita connection ∇^{LC} is induced by a connection A on $P_{SO_n}(M)$. Then A is given by a collection of one-forms $\omega_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{so}_n$ satisfying certain transition rules. Composing with the inverse of the pushforward

$$\nu_* : \mathfrak{spin}_n \to \mathfrak{so}_n,$$

we get a collection of one-forms

$$\widetilde{\omega} = \nu_*^{-1}(\omega_\alpha) \in \Omega^1(U_\alpha) \otimes \mathfrak{spin}_n.$$

Composing this with the pushforward of the spin representation,

$$\rho_* : \mathfrak{spin}_n \to \mathrm{End}(\Sigma_n),$$

then gives a collection of $\mathrm{End}(S)$-valued one-forms. It can be verified that this collection defines a connection ∇^S that makes S into a Clifford module bundle. The associated Dirac operator is called the *Spin-Dirac* operator, denoted by $\partial\!\!\!/$:

$$\partial\!\!\!/ = c \circ \nabla^S,$$

where c is the representation of $\mathbb{C}l_n$ on each fibre Σ_n. Note that $\partial\!\!\!/$ is canonically induced by $P_{\mathrm{Spin}_n}(M)$ and the Levi-Civita connection on TM.

Next, an oriented Riemannian manifold M^n is said to be *Spin*^{c} if there exists a principal Spin_n^c-bundle $P_{\mathrm{Spin}_n^c}(M) \to M$ together with a bundle map

$$\eta : P_{\mathrm{Spin}_n^c}(M) \to P_{SO_n}(M),$$

satisfying $\eta(pg) = \eta(p)\nu^c(g)$ for all $p \in P_{\mathrm{Spin}_n^c}(M)$ and $g \in \mathrm{Spin}_n^c$, where ν^c is the map $\mathrm{Spin}_n^c \to SO_n$ defined by

$$\nu^c([g, u]) = \nu(g),$$

for $g \in \mathrm{Spin}_n$ and $u \in U_1$. Clearly every Spin manifold is Spin^c. The existence of the bundle $P_{\mathrm{Spin}_n^c}$ allows one to lift the structure group of TM and $\mathbb{C}l(TM)$ to Spin_n^c, so that we have isomorphisms

$$TM \cong P_{\mathrm{Spin}_n^c}(M) \times_{\xi_n''''''} \mathbb{R}^n, \qquad \mathbb{C}l(TM) \cong P_{\mathrm{Spin}_n^c}(M) \times_{\xi_n'''''''} \mathbb{C}l_n,$$

where the equivalence relations ξ'''''' and ξ''''''' are defined analogously to before.

Over any open cover $\mathcal{U} = \{U_\alpha\}$ of M, the transition functions for $P_{\mathrm{Spin}^c}(M)$ have the form $[(h_{\alpha\beta}, z_{\alpha\beta})] : U_{\alpha\beta} \to \mathrm{Spin}_n^c$, for some maps

$$h_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Spin}_n,$$

$$z_{\alpha\beta} : U_{\alpha\beta} \to U_1$$

satisfy certain transition rules. The collection $\{z_{\alpha\beta}^2\}$ specifies a complex line bundle $L \to M$, called the *determinant line bundle*. Now pick a connection on this determinant line bundle. We may assume that $\mathcal{U}$ is a good open cover - that is, each finite intersection of its members is contractible. Thus over each $U_\alpha \in \mathcal{U}$, TM is trivial and hence Spin, so we may construct the spinor bundle S_{U_α} and connection locally over U_α. For each α, twist S_{U_α} and its connection by the square root[1] of the restriction L_{U_α} and the restriction of its connection. Using a partition of unity, one can piece together the $S_{U_\alpha} \otimes L_{U_\alpha}$ and the accompanying connections to form the global spinor bundle $S \to M$ and the Spin^c-connection ∇^S. It can be verified that ∇^S makes S into a Clifford module bundle over M. The resulting Dirac operator

$$D = c \circ \nabla^S$$

is called the *Spin^c-Dirac operator*. Note that D depends on the initial choice of connection on the determinant line bundle L.

If M^n is an even-dimensional manifold with $n = 2m$, it follows from the splitting of the spinor representation $\Sigma_n = \Sigma_{2m}^+ \oplus \Sigma_{2m}^-$ that the spinor bundle S has a natural $\mathbb{Z}_2$-grading

$$S = S^+ \oplus S^-,$$

with respect to which the Dirac operator D (either Spin or Spin^c) becomes an odd-graded operator. D can be written as an off-diagonal sum of the operators

$$D^+ : S^+ \to S^-,$$

$$D^- : S^- \to S^+.$$

[1]The spin structure determines a canonical square root of the determinant line bundle.

More generally, let n be an arbitrary non-negative integer. We shall say that an operator D acting on sections of a Clifford module bundle $E \to M^n$ is a *Dirac-type operator* if the commutator of D with multiplication by a smooth f on M is equal to $c(df)$. One can show that this is equivalent to requiring the principal symbol of D^2 to be

$$\sigma_2(D^2)(x, \xi) = \|\xi\|^2, \qquad (x, \xi) \in T^*M.$$

For example, the sum of a Dirac operator D and an endomorphism Φ of E (odd-graded if E comes with a $\mathbb{Z}_2$-grading) is a Dirac-type operator.

1.5 Proper Actions

We now introduce into the picture the action of a Lie group G on the manifold M and define a notion of Dirac operators that is compatible with this action. We will always assume that G acts smoothly on M. In addition, we will always assume that the action is *proper*: that is, the inverse image of a compact set under the *action map*

$$G \times M \to M \times M,$$

$$(g, x) \mapsto (x, gx)$$

is compact. Another way of saying this is that we require that the action map be a proper map. In particular, this means that the stabiliser subgroup of any point $x \in M$ is compact.

Notice that the requirement that G act properly on M is weaker than requiring the map

$$G \times M \to M,$$

$$(g, x) \mapsto gx$$

to be proper. Indeed, the action of any Lie group G on itself by left-translations is proper, while the map $(g, x) \mapsto gx$ is proper if and only if G is compact.

In Appendix A, we will study the equivariant index theory of Dirac-type operators on manifolds on which there is a proper Lie group action. There we will assume in addition that the action is *cocompact*: that is, the space[2] of orbits M/G is compact. The case of proper, non-cocompact actions is taken up in Appendix B.

An important reference for proper G-actions is the paper [17] of Palais, which in particular proves that, given a proper action of G on a manifold M, there exists a Riemannian metric on M with respect to which the G-action is isometric. The corresponding existence of G-invariant Hermitian structures on G-equivariant vector bundles is found in Appendix B of the book [11] of Ginzburg, Guillemin, Karshon. In what follows, we shall always assume that M carries a G-invariant metric and that and any vector bundle over M carries a G-invariant Hermitian structure.

1.6 Almost-Connected Groups

In this section, suppose G is an *almost-connected* Lie group - that is, one with only finitely many connected components. It is well-known that in this case G possesses a compact subgroup K that is maximal amongst the compact subgroups with respect to inclusion. It can also be shown that K is unique up to conjugation.

By [1] Theorem A.5, there exists a subset $E \subseteq G$ such that E is diffeomorphic to $\mathbb{R}^d$ for some $d \geq 0$ and $E \times K$ is diffeomorphic to G. Thus we get a diffeomorphism

$$\mathbb{R}^d \cong G/K.$$

Now suppose G acts smoothly, but not necessarily properly, on a manifold M. Suppose there exists a G-equivariant map

$$f : M \to G/K,$$

where G acts on G/K in the obvious way, by $g \cdot hK := (gh)K$. By the remarks above, the natural map $\pi : G \to G/K$ has a global section

$$t : G/K \to G.$$

Let $N := \pi^{-1}(eK)$. Let K act on $G \times N$ by

$$k \cdot (g, n) := (gk^{-1}, k \cdot n),$$

[2] M/G may not be a manifold unless the action of G is also free.

and denote by $G \times_K S$ the orbit space of this action. Let G act on $G \times_K N$ by $g \cdot [h, n] := [gh, n]$. Then the map

$$G \times_K N \to M,$$

$$[g, n] \mapsto g \cdot n$$

is a G-equivariant diffeomorphism with inverse $M \to G \times_K N$ given by

$$x \mapsto [t \circ f(x), (t \circ f(x))^{-1} \cdot x].$$

The main theorem of [1] is that if the action of G on M is *Palais-proper*, then the map f, as defined above, exists. Recall that the action is Palais-proper if for any $x \in M$ there exists a neighbourhood $U \ni x$ such that for every $y \in M$ there exists a neighbourhood $V \ni y$ for which the set

$$\{g \in G : gV \cap U \neq \emptyset\}$$

has compact closure in G.

It follows from Abels' theorem that, given a Palais-proper action of G on M, we can find a K-invariant submanifold N of M such that $M \cong G \times_K N$. In fact, it is proved in [1] that N is unique up to G-equivariant diffeomorphism. We call N the *global K-slice* of M.

Note that although the fibre bundle $G \times_K N \to G/K \cong \mathbb{R}^d$ is trivial, it is not equivariantly trivial. That is, $G \times_K N$ is not diffeomorphic to $\mathbb{R}^d$ by a G-equivariant map.

Clearly, any proper action is Palais-proper. Thus if G acts properly on M, then M has a global K-slice N. In Appendix A, we apply Abels' theorem to equivariant index theory with applications. In particular, we shall see that the relationship between the G-manifold M and the K-manifold N translates to a close relationship between the G-equivariant index theory of general elliptic operators on M and the K-equivariant index theory of corresponding operators on the slice N.

1.7 Equivariant Dirac Operators

A map $f : X \to Y$ between G-spaces X and Y is said to be *G-equivariant* if for all $x \in X$, $f(gx) = gf(x)$.

Let $p : E \to M$ be a fibre bundle, and suppose that there is a G-action on E as well as on M. Then E is called a *G-equivariant fibre bundle* (or simply a *G-fibre bundle*) if the projection map p is G-equivariant. G acts on sections s of E by

$$gs(x) := g^{-1}s(gx), \qquad x \in M.$$

For example, an action of G on M by diffeomorphisms gives $TM \to M$ the structure of a G-equivariant vector bundle, with the G-action on TM defined by the differential. An isometric action of G on M induces an action of G on the principal bundle $P_{SO_n}(M)$ of oriented orthonormal frames and actions on $Cl(TM)$ and $\mathbb{C}l(TM)$.

We shall say that M is *G-equivariantly Spin* (or simply *G-Spin*) if in the definition of Spin manifold we also require that the principal bundle $P_{\mathrm{Spin}_n}(M)$ be equipped with a smooth G-action such that the map

$$\eta : P_{\mathrm{Spin}} \to P_{SO_n}(M)$$

is G-equivariant.

Similarly, we say that M is *G-equivariantly Spin^c* (or simply *G-Spin^c*) if the principal Spin^c-bundle $P_{\mathrm{Spin}_n^c}$ carries a smooth G-action and the map

$$\eta^c : P_{\mathrm{Spin}^c}(M) \to P_{SO_n}(M)$$

is G-equivariant.

The spinor bundle S of a G-Spin or G-Spin^c manifold is naturally a G-equivariant vector bundle. The associated Dirac operator is *G-invariant*; that is, it commutes with the action of G on sections of S. To see this, we claim that, since the group G acts by isometries, it preserves the Levi-Civita connection ∇^{LC}. More precisely, we claim that the map

$$C_c^\infty(TM) \to C_c^\infty(T^*M \otimes TM),$$

$$X \mapsto \nabla^{LC} X,$$

where X is a vector field on M, is G-equivariant with respect to the natural action of G on TM and the dual action on T^*M. Let us verify this. Suppose $g \in G$ is a fixed isometry and $\{e_1, \ldots, e_n\}$ is a local frame for TM over a neighbourhood U of a point $x \in M$. Define $\tilde{e}_i := g_* e_i$. By choosing U to be sufficiently small, we can assume that $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ is a local frame around gx. Let $\{\tilde{e}^1, \ldots, \tilde{e}^n\}$ denote the dual coframe. Then for any vector field X on M, we have

$$\begin{aligned}
g\left(\nabla^{LC} X\right)(x) &= g^{-1}\left(\nabla^{LC} X(gx)\right) \\
&= g^{-1}\left(\sum_i \nabla^{LC}_{\tilde{e}_i} X(gx) \otimes \tilde{e}^i\right) \\
&= g_*^{-1} \sum_i \nabla^{LC}_{\tilde{e}_i} X(gx) \otimes g^*(\tilde{e}^i) \\
&= g_*^{-1}\left(\sum_i g_* \left(g^* \nabla^{LC}\right)_{e_i} g_*^{-1} X(gx)\right) \otimes e^i, \\
&= \sum_i \left(g^* \nabla^{LC}\right)_{g_*^{-1}\tilde{e}_i} g_*^{-1} X(gx) \otimes e^i.
\end{aligned}$$

Now, using that the Levi-Civita connection on TM is preserved by pulling back along the isometry g, this equals

$$\begin{aligned}
&\sum_i \nabla^{LC}_{g_*^{-1}\tilde{e}_i} g_*^{-1} X(gx) \otimes e^i \\
=&\sum_i \nabla^{LC}_{e_i} g_*^{-1} X(gx) \otimes e^i \\
=&\sum_i \nabla^{LC}_{e_i} (gX)(x) \otimes e^i \\
=&\nabla^{LC}(gX)(x),
\end{aligned}$$

which proves the claim.

If M is G-Spin, ∇^{LC} is induced by a G-invariant connection on $P_{SO_n}(M)$. Hence the resulting connection

$$C_c^\infty(S) \xrightarrow{\nabla^S} C_c^\infty(T^*M \otimes S)$$

on the spinors is also G-equivariant. Moreover, one can verify that Clifford multiplication

$$c : \mathbb{C}l(TM) \to \mathrm{End}(S)$$

is a G-equivariant map, by writing $\mathbb{C}l(TM)$ and $\mathrm{End}(S)$ as associated bundles of $P_{\mathrm{Spin}_n}(M)$. Hence the G-Spin-Dirac operator

$$\partial\!\!\!/ = c \circ \nabla^S$$

is G-invariant.

Similarly, if M is a G-Spin^c manifold, one can verify that the Spin^c-Dirac operator D constructed using any G-invariant Hermitian connection on the determinant line bundle is G-invariant.

As is true for non-equivariant Dirac operators, we can define the more general notion of a *G-Clifford module bundle $E \to M$*, equipped with a G-invariant Hermitian connection ∇^E that is compatible with G-equivariant Clifford multiplication

$$c : \mathbb{C}l(TM) \to \mathrm{End}(E).$$

The G-invariant Dirac operator associated to such a bundle is defined to be $c \circ \nabla^E$.

Similarly, one can formulate the notion of a *G-invariant Dirac-type operator*. An example of such an operator is

$$B = D + \Phi,$$

where D is a G-invariant Dirac operator acting on a G-Clifford module bundle E, and Φ is a G-invariant (odd-graded if E comes with a $\mathbb{Z}_2$-grading) endomorphism of E. For well-chosen Φ, such operators can be shown to be equivariantly Fredholm (a notion we will discuss in the next chapter), even if neither M nor M/G is compact. B is then an example of a *G-Callias-type operator*, which will be object of our study in Appendix B.

1.8 Index and Positive Scalar Curvature

Let M be a compact Spin manifold with spinor bundle $S \to M$. The Spin-Dirac operator $\partial\!\!\!/$ on M is an elliptic differential operator that can be viewed as an unbounded, formally self-adjoint operator on $L^2(S)$. Any elliptic differential operator on a compact manifold is Fredholm and so has an index in $\mathbb{Z}$. Furthermore, it follows by elliptic regularity that the elements in the kernel of $\not\partial$ are smooth sections of S.[3]

If M has odd dimension, it follows that $\operatorname{index} \not\partial = 0$. [4] Therefore when M is compact, we will confine our attention to even-dimensional M. As remarked earlier, this means we can write

$$\not\partial = \not\partial^+ \oplus \not\partial^-,$$

where $\not\partial^+$ and $\not\partial^-$ are odd operators on the $\mathbb{Z}_2$-graded spinor bundle S. The object of index theory is then to study $\operatorname{index} \not\partial^+$.

The Bochner-Lichnerowicz-Weitzenbock formula (which we prefer to call simply the *Lichnerowicz formula*) for the square of $\not\partial$ states that

$$\not\partial^2 = (\nabla^S)^* \nabla^S + \frac{\kappa}{4},$$

where κ is the pointwise scalar curvature function on M associated to the Riemannian metric, and ∇^S is the connection on the spinor bundle induced by the Levi-Civita connection on TM.

Suppose that κ is uniformly positive on M. Since M is compact,

$$\inf_{x \in M} (\kappa(x)) > 0,$$

and thus $\not\partial^2$ is a strictly positive operator from the first Sobolev space $H^1(S)$ to $L^2(S)$, with bounded inverse

$$(\not\partial^2)^{-1} : L^2(S) \to H^2(S).$$

It follows that $\not\partial$ is invertible, so that it has index zero. By the Atiyah-Singer index theorem, $\operatorname{index} \not\partial$ is a topological invariant and hence independent of the choice of the Riemannian metric on M and the bundle $P_{SO_n}(M)$. Thus the index of any Spin-Dirac operator on M must vanish. Thus non-vanishing of $\operatorname{index} D$ is said to *obstruct* the existence of a Riemannian metric on M with positive scalar curvature.

[3] This also applies to Dirac operators on general Clifford module bundles.

[4] Note that this applies to any Dirac operator acting on sections of a Clifford module bundle.

This theorem is sometimes referred to as Lichnerowicz's theorem, after its originator [15], and is a prototype of vanishing theorems in index theory that provide obstructions to positive scalar curvature.

As we will see in the next chapter, the equivariant index of a G-invariant Dirac operator lives in K-theory, which makes it more challenging to compute. Still, there are natural ways to generalise Lichnerowicz's theorem to both the cocompact and non-cocompact settings, and we will explore these in the two appendices. Furthermore, there exist topological formulas for numerical *traces* of the equivariant index in terms of characteristic classes. These formulas are similar in form to the Atiyah-Singer index theorem; an example is proved in [19]. Such formulas simplify the task of proving non-vanishing of the equivariant index, which is enough to show that invariant metrics of positive scalar curvature cannot exist on M.

We will define and discuss the equivariant index in the next chapter.

Chapter 2

Fredholmness and G-Index

As mentioned in the previous chapter, a Dirac operator D acting on sections of a Clifford module bundle E over a compact manifold is an elliptic operator and hence Fredholm, meaning that it has finite-dimensional kernel and cokernel. By Atkinson's Theorem [3], one can also define Fredholmness as follows. Form the normalised Dirac operator $F_D := \frac{D}{\sqrt{D^2+1}}$, an element of the bounded operators $\mathcal{L}(L^2(E))$. Take its class $[F_D]$ in the Calkin algebra $\mathcal{L}(L^2(E))/\mathcal{K}(L^2(E))$. Then Fredholmness of D can equivalently be defined as invertibility of $[F_D]$.

This second definition of Fredholmness has the merit of being directly generalisable to the setting of Hilbert C^*-modules[1]. In particular, if A is a C^*-algebra and H is a Hilbert A-module, one can form the C^*-algebra $\mathcal{L}(H)$ of bounded adjointable operators on H and its closed two-sided ideal $\mathcal{K}(H)$ of compact operators. An operator $F \in \mathcal{L}(E)$ is then defined to be Fredholm when its class $[F] \in \mathcal{L}(H)/\mathcal{K}(H)$ is invertible. When $A = \mathbb{C}$, this gives back the usual notion of Fredholmness on Hilbert spaces.

The goal of this chapter is to prepare the reader for the study of equivariant index theory in Appendices A and B by recalling the relevant notions of Fredholmness and index used there.

We begin by defining the full and reduced group C^*-algebras associated to a Lie group G, followed by a summary of facts about the K-theory of C^*-algebras as well as Hilbert modules. For further details we refer to the

[1]The first definition can also be generalised, but not quite as neatly.

books [20], [16], and [18].

We then construct the Hilbert module $\mathcal{E}$ used in G-equivariant index theory and recall the relationship between L^2-boundedness of operators on a bundle E over a cocompact manifold M and boundedness and adjointability of the induced operator on $\mathcal{E}$, together with regularity and essential self-adjointness of G-invariant Dirac operators in this setting. Detailed proofs can be found in section 5 of [12].

We will only state the results for the Hilbert module $\mathcal{E}$ when M/G is compact. The analogous results for G-invariant Callias-type operators in the non-cocompact setting will be established in Appendix B.

2.1 Group C^*-Algebras

First we recall the definitions of the group C^*-algebras, of which there are two versions: the *maximal* (or *full*) version and the *reduced* version. Both are formed by completing $L^1(G)$, the integrable functions $G \to \mathbb{C}$, with respect to different norms, as follows.

On every locally compact group G there is a left Haar measure unique up to positive constants. Let G be a Lie group and fix a left-Haar measure $d\mu$ on G. Right-translation of $d\mu$ by a group element g^{-1} gives a new left-Haar measure $d\mu_{g^{-1}}$. We define the *modular function* $\delta : G \mapsto \mathbb{R}$ as the constant factor between these two measures. That is,

$$d\mu_{g^{-1}} = \delta(g^{-1})d\mu.$$

Often one sees the same statement written in the form $dg^{-1} = \delta(g^{-1})dg$.[2]

If $\delta \equiv 1$, G is said to be *unimodular*. In particular this means that the left and right Haar measures on G coincide.

To define the group C^*-algebras, recall that $L^1(G)$ is a Banach-*-algebra with the convolution product

$$(f * f')(g) = \int_G f(h)f'(h^{-1}g)\, dh$$

and involution

$$f^*(x) = \overline{f(x^{-1})}\delta(x),$$

[2]Another often used definition is $dg^{-1} = \delta(g)dg$.

for $f, f' \in L^1(G)$.

A *representation* χ *of a Banach-***-algebra* A on a Hilbert space H is a $*$-homomorphism $\chi : A \to \mathcal{L}(H)$, where $\mathcal{L}$ denotes the bounded operators. χ is said to be *non-degenerate* if $\chi(x)v = 0$ for all $x \in A$ implies $v = 0 \in H$.

Given a unitary representation ϕ of the group G on a Hilbert space H, one can construct a non-degenerate representation $\tilde{\phi} : L^1(G) \to \mathcal{L}(H)$ defined by

$$\tilde{\phi} : f \mapsto \int_G f(g)\phi(g)\,dg,$$

where the right-hand side is an operator defined by

$$\left\langle \left(\int_G f(g)\phi(g)\,dg \right) u, v \right\rangle := \int_G f(g)\langle \phi(g)u, v\rangle\,dg,$$

for $f \in L^1(G)$ and $u, v \in H$.

The *maximal* or *full* group C^*-algebra of G, denoted by $C^*(G)$, is the completion of $L^1(G)$ with respect to the norm

$$\|f\| = \sup \left\{ \|\phi(f)\|_{\mathcal{L}(H)} : (H, \rho) \in \hat{G} \right\},$$

where $\hat{G}$ is the unitary dual of G, consisting of all irreducible unitary representations of G. This is finite, since for any representation (H, ρ) of G,

$$\|\rho(\phi)\|_{\mathcal{L}(H)} \leq \|\phi\|_{L^1(G)}.$$

For the next definition, recall that the *left regular representation* L of G on $L^2(G)$ is given by $(L(g)f)(s) := f(g^{-1}s)$, for $g, s \in G$ and $f \in L^2(G)$. This induces an injection $\tilde{L} : L^1(G) \hookrightarrow \mathcal{L}(L^2(G))$. We define the *reduced group* C^**-algebra* as

$$C_r^*(G) := \overline{\tilde{L}(L^1(G))},$$

where the closure is taken in $\mathcal{L}(L^2(G))$.

In fact, both $C^*(G)$ and $C_r^*(G)$ can be obtained by completing $C_c(G)$, the compactly supported functions $G \to \mathbb{C}$, with respect to these norms.

Any $*$-representation of $L^1(G)$ extends to a representation of $C^*(G)$, so that in particular

$$\tilde{L} : L^1(G) \hookrightarrow \mathcal{L}(L^2(G))$$

extends to a surjective map

$$\overline{L} : C^*(G) \to C^*_r(G) \subseteq \mathcal{L}(L^2(G)).$$

When $\overline{L}$ is also injective, G is said to be *amenable*. Examples of amenable groups include compact Lie groups, abelian groups, solvable groups and finitely generated groups of subexponential growth.

When K is a compact Lie group, a more explicit description of $C^*(K)$ is available. Consider the direct sum

$$B_0(K) := \bigoplus_{(V_\pi, \pi) \in \hat{K}} \mathcal{L}(V_\pi),$$

which we define to consist of elements $(a_\pi)_{\pi \in \hat{K}}$ of the cartesian product of $\{\mathcal{L}(V_\pi) : \pi \in \hat{K}\}$ with norm vanishing at infinity: that is, for all $\epsilon > 0$, there exists a finite subset $X_\epsilon \subseteq \hat{K}$ such that for all $\pi \in \hat{K} \backslash X_\epsilon$, $\|a_\pi\|_{\mathcal{L}(V_\pi)} < \epsilon$. When equipped with the C^*-norm

$$\left\| (a_\pi)_{\pi \in \hat{K}} \right\| = \sup_{\pi \in \hat{K}} \|a_\pi\|_{\mathcal{L}(V_\pi)},$$

it can be shown that $B_0(K)$ is isomorphic to $C^*(K) \cong C^*_r(K)$.

2.2 K-Theory of C^*-Algebras

The K-theory of C^*-algebras is a noncommutative generalisation of topological K-theory. If X is a locally compact topological space, the complex topological K-theory of X, consisting of the abelian groups $K^i(X)$ for $i = 0, 1$, can be defined in terms of the K-theory of the commutative C^*-algebra $C_0(X)$:

$$K^i(X) \cong K_i(C_0(X)).$$

The K-theory of C^*-algebras will provide the receptacle for the equivariant index we define later in this chapter. In this section we recall its definition.

Let A be a C^*-algebra. For $n \in \mathbb{N}$, let $M_n(A)$ denote the algebra of $n \times n$-matrices with entries in A. Notice that

$$M_n(A) \cong A \otimes M_n(\mathbb{C})$$

as vector spaces. $M_n(A)$ is a *-algebra with the obvious operations. To put a C^*-norm on $M_n(A)$, note that we have a *-homomorphism

$$M_n(\mathbb{C}) \hookrightarrow B(\mathbb{C}^n).$$

Now pick an injective *-homomorphism

$$\phi : A \to B(H),$$

where H is a Hilbert space. Then we may view $M_n(A)$ as

$$A \otimes M_n(\mathbb{C}) \subseteq B(H) \otimes B(\mathbb{C}^n) = B(H \otimes \mathbb{C}^n),$$

from which $M_n(A)$ inherits the operator norm. It can be verified that this norm makes $M_n(A)$ a C^*-algebra. With respect to the topology induced by this norm, we can talk about homotopies between elements p and q of $M_n(A)$: paths $\gamma : [0,1] \to M_n(A)$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Write $p \sim_h q$ if p and q are homotopic. Then one verifies that $\sim_h$ is an equivalence relation on $M_n(A)$.

Let $\mathcal{P}_n(A) := \mathcal{P}(M_n(A))$ be the set of projections in $M_n(A)$ and write

$$\mathcal{P}_\infty(A) := \bigsqcup_{n=1}^{\infty} \mathcal{P}_n(A), \qquad M_\infty(A) := \bigsqcup_{n=1}^{\infty} M_n(A).$$

Define an equivalence relation $\sim_0$ on $\mathcal{P}_\infty(A)$ by declaring $p \sim_0 q$ if there exists a matrix $v \in M_\infty(A)$ such that $p = v^*v$ and $q = vv^*$. Let $\mathcal{D}(A) := \mathcal{P}_\infty(A)/\sim_0$, and denote the class of $p \in \mathcal{P}_\infty(A)$ by $[p]_\mathcal{D} \in \mathcal{D}(A)$. The operation $\oplus$ on $\mathcal{P}_\infty(A)$ defined by

$$p \oplus q := \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}.$$

descends to $\mathcal{D}(A)$ and makes it into an abelian semigroup.

If A is a unital C^*-algebra, we define the abelian group

$$K_0(A) := G(\mathcal{D}(A)),$$

where G is the Grothendieck group construction. One can show that $K_0(A)$ consists of all formal differences of classes of projections $[p]_0 - [q]_0$, where p and $q \in \mathcal{P}_n(A)$ for $n \in \mathbb{N}$.

A *-homomorphism of C^*-algebras $\phi : A \to B$ takes projections to projections and induces a map

$$\tilde{\phi} : \mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B),$$

which in turn gives a map of abelian groups

$$K_0(\phi) : K_0(A) \to K_0(B).$$

For general a C^*-algebra A, let $\tilde{A}$ denote its minimal unitisation (see [18]), so that we have a split-exact sequence of C^*-algebras

$$0 \to A \to \tilde{A} \xrightarrow{\pi} \mathbb{C} \to 0.$$

Define

$$K_0(A) := \mathrm{Ker}(K_0(\pi)).$$

One can check that if A is unital, this definition is consistent with the previous one.

Now we define the other half of K-theory, the group K_1. Again let A be a unital C^*-algebra, and denote by $\mathcal{U}(A)$ the group of unitaries in A. Let $\mathcal{U}_n(A) := \mathcal{U}(M_n(A))$ be the group of unitaries in $M_n(A)$. Define

$$\mathcal{U}_\infty(A) := \bigsqcup_{n=1}^{\infty} \mathcal{U}_n(A)$$

and an equivalence relation $\sim_1$ on $\mathcal{U}_\infty(A)$ by declaring $u_1 \sim_1 v$ if there exists $k \geq \max(n, m)$ such that

$$\begin{bmatrix} u & 0 \\ 0 & I_{k-n} \end{bmatrix} \sim_h \begin{bmatrix} v & 0 \\ 0 & I_{k-m} \end{bmatrix},$$

where $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$. Then the quotient $\mathcal{U}_\infty(A)/\sim_1$, equipped with the multiplication operation

$$[u]_1[v]_1 := \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix},$$

is an abelian group.

For any C^*-algebra A (unital or otherwise), define

$$K_1(A) := \mathcal{U}_\infty(\tilde{A})/\sim_1 .$$

A *-homomorphism of C^*-algebras $\phi : A \to B$ takes unitaries to unitaries and induces a map

$$\tilde{\phi} : \mathcal{U}_\infty(A) \to \mathcal{U}_\infty(B).$$

In turn this induces a map of abelian groups

$$K_1(\phi) : K_1(A) \to K_1(B).$$

In summary, both K_0 and K_1 are covariant functors from the category of C^*-algebras to the category of abelian groups. They are invariant under homotopy: if ϕ and $\psi : A \to B$ are homotopic *-homomorphisms, then $K_0(\phi) = K_0(\psi)$.

Next, we list some well-known properties of the K-theory of C^*-algebras that we will need in order to define the equivariant index. Firstly, given two strongly Morita equivalent C^*-algebras A and B (see [5]), we have

$$K_i(A) \cong K_i(B)$$

where $i = 0$ or 1. For instance, if A is a separable C^*-algebra and $\mathcal{K}(H)$ is the C^*-algebra of compact operators on a separable Hilbert space H, then $K_i(A) \cong K_i(A \otimes \mathcal{K}(H))$.

Secondly, if I is a closed two-sided ideal in a C^*-algebra A, there is a natural short exact sequence

$$0 \to I \xrightarrow{i} A \xrightarrow{\pi} A/I \to 0$$

of C^*-algebras. By functoriality this induces a *six-term exact sequence* in K-theory,

$$\begin{array}{ccccc} K_1(I) & \xrightarrow{i_*} & K_1(A) & \xrightarrow{\pi_*} & K_1(A/I) \\ \delta \uparrow & & & & \downarrow \partial \\ K_0(A/I) & \xleftarrow[\pi_*]{} & K_0(A) & \xleftarrow[i_*]{} & K_0(I), \end{array}$$

where the maps ∂ and δ are called the boundary and exponential maps respectively (see [18] for their definitions).

It can be shown that, for a compact Lie group K, the K-theory of the C^*-algebra $B_0(K)$ mentioned in the previous section is isomorphic, as an abelian group, to the representation ring $R(K)$. The resulting isomorphism

$$K_0(C^*(K)) \cong K_0(C^*_r(K)) \cong R(K),$$

lets us interpret certain questions in equivariant index theory in terms of the representation theory of compact groups; we will study this connection in Appendix A. Finally, we remark that

$$K_1(C^*(K)) = K_1(C^*_r(K)) = 0.$$

2.3 Hilbert Modules

While any Dirac operator associated to a Clifford module bundle E is an unbounded operator on the Hilbert space $L^2(E)$, a G-invariant Dirac operator also defines an unbounded operator on a certain Hilbert $C^*(G)$-module $\mathcal{E}$. Fredholmness of this latter operator allows one to formulate *equivariant index theory*.

Let us begin by recalling the definition of a Hilbert module. Let A be a C^*-algebra. Then a *pre-Hilbert A-module* is a $\mathbb{C}$-vector space $\mathcal{M}$ equipped with a right-A-module structure and an A-valued inner product

$$\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to A$$

satisfying:

1. $\langle x, ya \rangle = \langle x, y \rangle a$,
2. $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$,
3. $\langle y, x \rangle = \langle x, y \rangle^*$,
4. $\langle x, x \rangle \geq 0$,
5. $\langle x, x \rangle = 0$ if and only if $x = 0$,

for all $x, y, x_1, x_2 \in \mathcal{M}$ and $a \in A$. We call a pre-Hilbert A-module $\mathcal{M}$ a *Hilbert A-module* if it is complete with respect to the norm

$$x \mapsto \|\langle x, x \rangle\|_A^{1/2}, \qquad x \in \mathcal{M}.$$

A simple example of a Hilbert A-module is A itself, with respect to the inner-product $\langle a, b \rangle = a^*b$, for $a, b \in A$. One can also consider the direct sum of countably many copies of this Hilbert A-module,

$$H_A := \bigoplus_{i \in \mathbb{N}} A.$$

An element of H_A is defined to be a sequence $(a_i)_{i \in \mathbb{N}}$ such that

$$\sum_i \langle a_i, a_i \rangle = a_i^* a_i$$

converges in A. H_A is a pre-Hilbert A-module when equipped with the inner product

$$\langle (a_i), (b_i) \rangle = \sum_{i \in \mathbb{N}} a_i^* b_i,$$

for $(a_i), (b_i) \in \oplus_{i \in \mathbb{N}} A$. In fact, one can verify that H_A is a Hilbert A-module; we call it the *standard Hilbert A-module.*

It follows from the definition above that a Hilbert $\mathbb{C}$-module is simply a complex Hilbert space. However, there are some important differences between Hilbert space theory and general Hilbert module theory. The first concerns the notion of *adjointability* of operators. Unlike in Hilbert spaces, a bounded A-linear map T from a Hilbert A-module $\mathcal{M}$ to a Hilbert A-module $\mathcal{N}$ does not in general have an adjoint - that is, there may not exist a map

$$T^* : \mathcal{N} \to \mathcal{M}$$

such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x \in \mathcal{M}$ and $y \in \mathcal{N}$. For instance, let M be a compact manifold. Let $\mathcal{N} = A = C(M)$, the continuous functions $M \to \mathbb{C}$, and let $\mathcal{M} = \{f \in A : f(x) = 0\}$ for a fixed point $x \in M$. Let $i : \mathcal{M} \hookrightarrow \mathcal{N}$ be the inclusion map. Then the adjoint of i must fix the constant function $1 \in \mathcal{N}$. However, $1 \notin \mathcal{M}$, so we conclude that i is not adjointable.

Another point of difference is that a closed submodule $\mathcal{N}$ of a Hilbert A-module $\mathcal{M}$ may not be orthogonally complementable. That is, we cannot always find another submodule $\mathcal{N}^\perp$ such that

$$\mathcal{M} = \mathcal{N} \oplus \mathcal{N}^\perp.$$

To illustrate this, consider $A = C([0,1])$ as a Hilbert module over itself. Then $C_0((0,1))$, the C^*-subalgebra of functions vanishing at infinity, is a closed submodule whose orthogonal complement is $\{0\}$.

Suppose $\mathcal{M}$ and $\mathcal{N}$ are Hilbert A-modules. We will use $\mathcal{L}(\mathcal{M}, \mathcal{N})$ to denote the bounded A-linear adjointable operators $\mathcal{M} \to \mathcal{N}$.

Given $x \in \mathcal{N}$ and $y, z \in \mathcal{M}$, consider the element $\theta_{x,y}$ of $\mathcal{L}(\mathcal{M}, \mathcal{N})$ defined by

$$\theta_{x,y}(z) := x\langle y, z\rangle_{\mathcal{M}}.$$

Evidently the adjoint of $\theta_{x,y}$ is $\theta_{y,x}$. The operators of the form $\theta_{x,y}$ are called the *rank-one* operators in $\mathcal{L}(\mathcal{M}, \mathcal{N})$. Taking the closure of the linear span of the rank-one operators gives the *compact operators* $\mathcal{K}(\mathcal{M}, \mathcal{N}) \subseteq \mathcal{L}(\mathcal{M}, \mathcal{N})$. If $\mathcal{M} = \mathcal{N}$, we will use the notation $\mathcal{L}(\mathcal{M})$ and $\mathcal{K}(\mathcal{M})$ respectively; in this case, both the bounded adjointable and compact operators form C^*-algebras.

For example, if one views A as a Hilbert module over itself, then $\mathcal{K}(A) \cong A$. One can also verify that $\mathcal{K}(H_A) \cong \mathcal{K} \otimes A$, where $\mathcal{K}$ denotes the compact operators on l^2.

2.4 A G-Sobolev Module

We will now construct a Hilbert $C^*(G)$-module $\mathcal{E}$ arising from a proper G-action on M and a G-vector bundle E. Following the terminology in Appendix B, we shall refer to $\mathcal{E}$ as the *zeroth G-Sobolev module*, although the original notion is due to Kasparov (see for example [12] section 5). In the case that M/G is compact, the functional analysis of $\mathcal{E}$ provides the foundations for G-equivariant index theory of pseudodifferential, and in particular Dirac, operators. One can think of $\mathcal{E}$ as a Hilbert $C^*(G)$-module analogue of the Hilbert space (in other words Hilbert $\mathbb{C}$-module) $L^2(E)$ for a vector bundle E.

Let M be a smooth proper G-manifold and $E \to M$ a G-equivariant Hermitian vector bundle. Suppose M and E are equipped with G-invariant metrics. Let dg be a fixed left Haar measure on M and δ be the modular function of G. Let dx denote the smooth G-invariant measure on M induced by the Riemannian metric.

The space of compactly supported smooth sections $C_c^\infty(E)$ can be given the structure of a pre-Hilbert $C_c^\infty(G)$-module, where the right $C_c^\infty(G)$-action and $C_c^\infty(G)$-valued inner product are defined by

$$(e \cdot b)(x) = \int_G g(e)(x) \cdot b(g^{-1}) \cdot \delta(g)^{-1/2}\, dg \in C_c^\infty(E),$$

$$\langle e_1, e_2 \rangle(g) = \delta(g)^{-1/2} \int_M \langle e_1(x), g(e_2)(x) \rangle_E \, dx \in C_c^\infty(G),$$

for $e, e_1, e_2 \in C_c^\infty(E), b \in C_c^\infty(G)$. As mentioned in Chapter 1, the action of G on $C_c(E)$ is defined by

$$g(e)(x) := g(e(g^{-1}x)).$$

The positivity of the above inner product in $C^*(G)$ is proved in [12] section 5. Denote by $\mathcal{E}$ the completion of the vector space $C_c^\infty(E)$ under the norm induced by $\langle\ ,\ \rangle$ above, and extend naturally the $C_c(G)$-action to a $C^*(G)$-action, and $\langle\ ,\ \rangle$ to a $C^*(G)$-valued inner product, to give $\mathcal{E}$ the structure of a Hilbert $C^*(G)$-module.

The module $\mathcal{E}$ can also be defined for the reduced group C^*-algebra of G, by completing $C_c^\infty(E)$ with respect to the $C_r^*(G)$-valued inner product in the above procedure.

Let $e, e_1, e_2 \in C_c(E)$. Then using the above definition of rank-one operators, we compute that

$$\theta_{e_1,e_2}(e)(x) = \int_M \left(\int_G \theta_{g(e_1)(x),\, g(e_2)(y)\, dg} \right) e(y)\, dy,$$

where on the right-hand side the notation θ is used for the rank-one operators on the Hilbert space $L^2(E)$. Thus any integral operator R with a G-invariant continuous kernel and proper support defines an element of $\mathcal{K}(E)$, since it can be approximated in the $\mathcal{E}$-norm by finite linear combinations of these rank-one operators.

When M/G is a compact space, it can be shown that certain G-invariant L^2-bounded operators define bounded operators on $\mathcal{E}$. Together with ellipticity of the Dirac operator, this implies that the Dirac operator (or rather, its bounded transform), is a $C^*(G)$-*Fredholm* operator: that is, it is invertible in the Calkin algebra $\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})$. Such an operator has an index in $K_i(C^*(G))$.

2.5 Boundedness and Fredholmness

We now state some key results (see [12]) that imply the G-invariant Dirac operator is bounded and Fredholm on $\mathcal{E}$, under the assumption that M/G is compact[3]. The analogous results for the G-invariant Callias-type operator in the non-cocompact case are proved in Appendix B.

Proposition. *Suppose G acts on M cocompactly and isometrically, and let E be a G-vector bundle over M. Let A be an operator on $C_c(E)$ which is $L^2(E)$-bounded, G-invariant, and has a properly supported distributional kernel. Then A defines an element of $\mathcal{L}(\mathcal{E})$ with norm $\leq$ const $\cdot \|A\|$, where $\|A\|$ is the L^2-norm of A, and the constant depends only on the supports of the operators $\mathfrak{c}A^*A + A^*A\mathfrak{c}$ and $\mathfrak{c}AA^* + AA^*\mathfrak{c}$, where $\mathfrak{c}$ is any cut-off function of our choice.*

Since the conclusion holds for all properly supported, G-invariant pseudodifferential operators, it applies in particular to the G-invariant Dirac operators we defined earlier acting on sections of a Clifford module bundle E.

The fact that a G-invariant, properly supported elliptic operator of order zero is invertible in the Calkin algebra is a consequence of the following ([12] Proposition 5.5).

Proposition. *Let M be a complete Riemannian manifold and G a locally compact group which acts on M properly and isometrically with compact quotient M/G and E a G-vector bundle over M. Form the Hilbert $C^*(G)$-module $\mathcal{E}$. If the symbol of a G-invariant properly supported operator A of*

[3]We will state them only for the Hilbert $C^*(G)$-module $\mathcal{E}$, but they apply analogously for the Hilbert $C^*_r(G)$-module $\mathcal{E}$

order 0 *is bounded at infinity in the cotangent direction by a constant* $C > 0$*, then* $A \in \mathcal{L}(\mathcal{E})$*, and the norm of* A *as an element of* $\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})$ *does not exceed* C*. If the symbol of* A *is zero at infinity, then* $A \in \mathcal{K}(\mathcal{E})$*.*

Even though the usual bounded transform of a Dirac operator,

$$F_D = \frac{D}{\sqrt{D^2+1}},$$

is not properly supported, the above result can be used to show that F_D is Fredholm. In fact, we have the following more general result (Theorem 5.8 in [12]), which also shows that D is essentially self-adjoint and regular.

Theorem. *Let* G, M, E *be as in the above proposition, let* $D : C_c^\infty(E) \to C_c^\infty(E)$ *be a formally self-adjoint* G*-invariant first-order elliptic differential operator on a vector bundle* E *over the manifold* X*. Then both operators* $D \pm i$ *have dense range as operators on* $\mathcal{E}$*, and the operators* $(D \pm i)^{-1}$ *are bounded and belong to* $\mathcal{K}(\mathcal{E})$*. The operator*

$$\frac{D}{\sqrt{D^2+1}} \in \mathcal{L}(\mathcal{E})$$

is Fredholm.

The six-term exact sequence in K-theory discussed earlier provides maps

$$\partial : K_1(\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})) \to K_0(\mathcal{K}(\mathcal{E})),$$

$$\delta : K_0(\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})) \to K_1(\mathcal{K}(\mathcal{E})).$$

We now discuss the way in which, using these maps, one obtains from the $C^*(G)$-Fredholm operator F_D, an element of $K_i(\mathcal{K}(\mathcal{E}))$, depending on whether or not $\mathcal{E}$ is $\mathbb{Z}_2$-graded. We call this element the *G-equivariant index* of F_D and denote it by $\operatorname{index}_G F_D \in K_i(\mathcal{K}(\mathcal{E}))$.

If $\mathcal{E}$ is $\mathbb{Z}_2$-graded and F_D is an odd operator with respect to this grading, then we can write

$$F_D = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}.$$

In this case, we define

$$\operatorname{index}_G F_D := \partial[P] \in K_0(\mathcal{K}(\mathcal{E})).$$

If $\mathcal{E}$ is not $\mathbb{Z}_2$-graded, set $R := \frac{1}{2}(F + 1)$. Then the class of R in $\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})$ is a projection. We define

$$\operatorname{index}_G F_D := \delta[R] \in K_1(\mathcal{K}(\mathcal{E})).$$

Now consider $\mathcal{E}$ as a Hilbert $\mathcal{K}(\mathcal{E})$-module, with $\mathcal{K}(\mathcal{E})$ acting from the left and inner product defined by

$$\langle x, y \rangle_{\mathcal{K}(\mathcal{E})} := \theta_{x,y},$$

where $\theta_{x,y}$ denotes a rank-one operator. Together with the structure $\mathcal{E}$ has as a right Hilbert $C^*(G)$-module, $\mathcal{E}$ gives a strong Morita equivalence between $\mathcal{K}(\mathcal{E})$ and $C^*(G)$.

Since K-theory of strongly Morita equivalent C^*-algebras are isomorphic, we have

$$K_i(\mathcal{K}(\mathcal{E})) \cong K_i(C^*(G)).$$

We will always interpret, by way of this isomorphism, $\operatorname{index}_G D$ as an element of $K_i(C^*(G))$.

Note that the construction applies not just to D but to any $C^*(G)$-Fredholm operator on $\mathcal{E}$. Also, if one forms the Hilbert module $\mathcal{E}$ using $C^*_r(G)$ instead of $C^*(G)$, the same procedure above yields an index in $K_i(C^*_r(G))$.

When the G-action on M is proper and cocompact, there is an equivalent way of obtaining $\operatorname{index}_G D$ starting from the G-equivariant analytic K-homology group $K_0^G(M)$, which we discuss in Appendix A. $[D]$ defines a class in $K_0^G(M)$, whose elements are in general represented by "abstract elliptic operators". One can define the index of the class $[D]$ using the descent map and intersection product in Kasparov's KK-theory. Details of this construction can be found, for example, in [12].

2.6 Preview of Results

Appendix A is a detailed study of the equivariant index theory of G-invariant Dirac operators in the cocompact setting, with a focus on almost-connected G. Our technique is to reduce questions about the G-index theory of G-invariant operators on M to the K-index theory of K-invariant operators on the K-slice N.

We will establish an equivariant version of K-theoretic Poincaré duality for both almost-connected groups and discrete groups, and this will enable us to extend the scope of our study from Dirac operators to all "abstract elliptic operators", which map the cycles in analytic K-homology. We also show that G-equivariant analytic K-homology is isomorphic to G-equivariant geometric K-homology, which we define later. As applications, we will use the G-index theory developed to give obstructions to the existence of G-invariant Riemannian metrics of positive scalar curvature on proper cocompact G-manifolds.

Appendix B studies the much more general situation when M/G is non-compact. In this setting, the standard Dirac operator D fails to be $C^*(G)$-Fredholm in general. One way to salvage the situation is to add to D a suitably chosen G-invariant potential Φ, forming an equivariant version of a Callias-type operator.

The study of such operators began in the non-equivariant setting with the work of Callias [9], who dealt with the case when M is a Euclidean space. Others [6],[2],[7],[8] then extended this to M a Riemannian manifold. More recently, the theory has been extended to certain pseudodifferential operators [13] and to Dirac operators twisted by bundles of Hilbert C^*-modules [10].

Appendix B gives the first generalisation of Callias-type operators to the G-equivariant setting. Our approach involves a combination of the techniques of Bunke [8] and Kasparov [12]. In analogy to [8], we construct the endomorphism Φ using the K-theory of an equivariant Higson corona of M, which we show is highly non-trivial. Along the way, we show that G-invariant Callias-type operators are adjointable, regular and essentially self-adjoint.

When M/G is compact, the index theory of G-Callias-type operators reduces to the index theory of G-invariant Dirac operators on G-Clifford module bundles, which explains our choice of title for this thesis.

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Appendix A

The Cocompact Setting

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Certification:	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.		
Signature		Date	May 23, 2018

Co-Author Contributions

By signing the Statement of Authorship, each author certifies that:

i. the candidate's stated contribution to the publication is accurate (as detailed above);

ii. permission is granted for the candidate in include the publication in the thesis; and

iii. the sum of all co-author contributions is equal to 100% less the candidate's stated contribution.

Name of Co-Author	Hang Wang		
Contribution to the Paper	$33\frac{1}{3}\%$		
Signature		Date	May 23, 2018

Name of Co-Author	Varghese Mathai		
Contribution to the Paper	$33\frac{1}{3}\%$		
Signature		Date	May 23, 2018

POSITIVE SCALAR CURVATURE AND POINCARÉ DUALITY FOR PROPER ACTIONS

HAO GUO, VARGHESE MATHAI, AND HANG WANG

Abstract. For G an almost-connected Lie group, we study G-equivariant index theory for proper co-compact actions with various applications, including obstructions to and existence of G-invariant Riemannian metrics of positive scalar curvature. We prove a rigidity result for almost-complex manifolds, generalising Hattori's results, and an analogue of Petrie's conjecture. When G is an almost-connected Lie group or a discrete group, we establish Poincaré duality between G-equivariant K-homology and K-theory, observing that Poincaré duality does not necessarily hold for general G.

1. Introduction

In this paper, we study G-equivariant index theory for proper co-compact actions with various applications, including obstructions to G-invariant Riemannian metrics of positive scalar curvature. As a corollary, we establish an alternate short proof of a recent result of Weiping Zhang [68] in Section 6. We also prove the existence of G-invariant Riemannian metrics of positive

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Key words and phrases. Positive scalar curvature, equivariant index theory, equivariant Poincaré duality, proper actions, almost-connected Lie groups, discrete groups, equivariant geometric K-homology, equivariant Spin^c-rigidity.

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scalar curvature under certain very general hypotheses on G and its action on the manifold (see Theorem 61).

Our G-equivariant index theory is applied to prove rigidity theorems in Section 7.1 for certain Spin^c-Dirac operators on almost-complex manifolds with a proper G-action, generalising results of Hattori [30] (see also Atiyah-Hirzebruch [3] and Hochs-Mathai [37]). In addition, we prove an analogue of Petrie's conjecture [58] in Section 7.2.

We begin in Section 2 by defining the G-equivariant geometric K-homology $K_\bullet^{\mathrm{geo},G}(X)$ for almost-connected G as a natural extension of the original definition of Baum and Douglas [10] and the definition in the compact group case [13]. It is a description of K-homology as a quotient of the equivariant bordism group over X. We extend the result in [13] for compact groups to show that it is isomorphic, via the Baum-Douglas map, to the analytic G-equivariant K-homology $K_\bullet^G(X)$ defined by Kasparov [42, 45].

Part of the technique we develop there, namely induction on analytic K-homology, is then used in Section 3 to prove one of our main theorems: equivariant Poincaré duality for almost-connected Lie groups and discrete groups. In the almost-connected case, this comes in the form of an isomorphism

$$\mathcal{PD} : K_\bullet^G(C_\tau(X)) \simeq K^\bullet_G(X), \tag{1.1}$$

where $C_\tau(X)$ is an algebra of continuous sections of the Clifford bundle. Our result extends equivariant Poincaré duality for compact groups, which is a consequence of work by Kasparov [42]. Our proof uses a result of Phillips [59] showing that there is an "induction" isomorphism of K-theory groups $K_\bullet^K(C_\tau(Y)) \simeq K_{\bullet+d}^G(C_\tau(X))$, where K is a maximal compact subgroup of G and Y is a K-slice of X. We also establish in this section versions of Poincaré duality when X is either not G-cocompact or has boundary, and relate our results to those obtained by Kasparov in [42]. In Section 3.2, we make use of previous work in [53] and [11] on equivariant K-theory and K-homology for discrete groups to prove equivariant Poincaré duality for discrete groups using the Mayer-Vietoris sequence.

The significance of our results on Poincaré duality can be seen as two-fold. First, they establish that equivariant Poincaré duality holds for a large class of *non-compact* Lie groups, and is an extension to non-compact groups the type of Poincaré duality proved in [42]. Second, the fact that the duality holds for these groups places into context the observation, made by Phillips in [60] and Lück-Oliver in [53] §5, that for an *arbitrary* Lie group G, an equivariant K-theory constructed from finite-dimensional G-vector bundles may not always be a generalised cohomology theory. This means one cannot expect Poincaré duality - as formulated in this way (for another way to formulate it in the case of non-compact group actions see Emerson and Meyer [26] and [25]) - to hold when G is an arbitrary Lie group. A concrete non-example of a group for which it does not hold is the semi-direct product $G = (S^1 \times S^1) \rtimes_\alpha \mathbb{Z}$, where $\alpha(n) = \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$, $n \in \mathbb{Z}$, which is a non-linear Lie group. As has been observed by others, in cases such as this one, Phillips' generalisation of equivariant K-theory [60] is not the same as the definition via C^*-algebras.

We observe that one general situation in which they are equivalent is when G has only finitely many connected components (see Lemma 23), and this fact guides our proof of Poincaré duality in the almost-connected group case.

In Section 4 we study the relationship of K-equivariant index theory to G-equivariant index theory, where K is again a maximal compact subgroup of G (compare [55]). The difference between our approach and, for instance, the approach in [33], is that we present a more direct proof, as well as give applications, of the fact that the following diagram relating the compact and non-compact indices commutes:

$$\begin{array}{ccc} K^K_\bullet(Y) & \xrightarrow{\text{index}_K} & K_\bullet(C^*_r(K)) \\ \simeq \downarrow \text{K-Ind} & & \simeq \downarrow \text{D-Ind} \\ K^G_{\bullet+d}(X) & \xrightarrow{\text{index}_G} & K_{\bullet+d}(C^*_r(G)). \end{array} \tag{1.2}$$

Here, the left vertical arrow is analytic induction from K to G, and the right vertical arrow is Dirac induction from K to G.

One of the consequences of this result (see Section 5) is an elegant integral trace formula for the special case when G is a connected, semisimple Lie group with finite centre, $\dim G/K$ is even and K has maximal rank. This result builds on the work of Atiyah-Schmid [7] and Lafforgue [48], but combines it with the integral formula for an L^2-index, proved by Wang in [66]. More precisely, we prove that if M is a G-Spin^c-manifold and N a K-slice, then the previous commutative diagram Then the diagram 1.2 fits into a larger commutative diagram involving the von Neumann trace on the K-theory of $C_r^*(G)$, denoted by τ_G, and a formula for the formal degree of discrete series representations in terms of the root systems of K and G (see [48]):

$$\begin{array}{ccccc}
K_0^G(M) & \xrightarrow{\text{index}_G} & K_0(C_r^*(G)) & & \\
\uparrow {\scriptstyle \text{K-Ind}} & & \uparrow {\scriptstyle \text{D-Ind}} & \searrow^{\tau_G} & \\
 & & & & \mathbb{R}. \\
 & & & \nearrow_{\Pi_K} & \\
K_0^K(N) & \xrightarrow{\text{index}_K} & R(K) & &
\end{array}$$

Here the quantity $\Pi_K([V_\mu]) := \prod_{\alpha\in\Phi^+} \frac{(\mu+\rho_c,\alpha)}{(\rho,\alpha)}$ (for details of the notation see Section 5). Using the result of Wang [66], we obtain naturally an equality of two integrals of characteristic classes, one on M and the other on the compact slice N.

Contents

2. Equivalence of Analytic and Geometric K-homologies

The goal of this section is to prove Theorem 5, which states the equivalence of two K-homology theories for a large class of groups and spaces, building on the work done in [12], [11] and [13]. Let us give a very brief introduction to these two theories.

Analytic K-homology was first studied by Atiyah [6], who was motivated by the classification of elliptic pseudo-differential operators on a locally compact topological space. Brown-Douglas-Fillmore [16] then understood it for C^*-algebras from the point of view of extension, before Kasparov [45] formulated it in the general setting of KK-theory. It is defined using analytic K-homology cycles.

Definition 1 ([45, 9])**.** Let G be a locally compact group acting properly on a Hausdorff space X. An *analytic K-homology cycle*, or *Kasparov cycle*, is a triple of the form $(\mathcal{H}, \phi, F)$, where

- $\mathcal{H}$ is a $\mathbb{Z}_2$-graded G-Hilbert space,
- $\phi : C_0(X) \to \mathcal{L}(\mathcal{H})$ is an even G-equivariant $*$-homomorphism and

- F is an odd self-adjoint bounded linear operator on $\mathcal{H}$,

where $\mathcal{L}(\mathcal{H})$ is the C^*-algebra of bounded operators on $\mathcal{H}$, such that the elements

$$\phi(a)(F^2 - 1), [\phi(a), F], [g, F]$$

belong to the $*$-subalgebra $\mathcal{K}(\mathcal{H})$ consisting of the compact operators, for any $a \in C_0(X)$ and any $g \in G$. The (even) *G-equivariant analytic K-homology of X*, denoted $K_0^G(X)$, is the abelian group generated by Kasparov cycles subject to certain equivalence relations given by homotopy. There is also an odd part, $K_1^G(X)$, whose elements are represented by cycles $(\mathcal{H}, \phi, F)$ with no imposed $\mathbb{Z}_2$-grading.

Remark 2. When A and B are G-C^*-algebras, a Kasparov (A, B)-cycle is defined similarly as in Definition 1, with the difference that $\mathcal{H}$ is now a Hilbert B-module and $C_0(X)$ is replaced by A (see [45] Definitions 2.2 and 2.3). Equivalence classes of such cycles form an abelian group $KK^G(A, B) := KK_0^G(A, B)$. There also exists an odd part $KK_1^G(A, B)$. KK-theory encompasses analytic K-homology in the sense that

$$KK_\bullet^G(C_0(X), \mathbb{C}) \simeq K_\bullet^G(X).$$

On the other hand, geometric K-homology was introduced by Baum and Douglas [10] from the perspective of Spin geometry and Dirac operators.

Definition 3 ([10, 12])**.** Let X be a G-space. A *geometric K-homology cycle* is a triple of the form (M, E, f), where

- M is a proper G-cocompact manifold with a G-equivariant Spin^c-structure,
- E is a smooth Hermitian G-equivariant vector bundle over M and
- $f : M \to X$ is a continuous G-equivariant map.

The *G-equivariant geometric K-homology*, which we shall write as $K_\bullet^{\mathrm{geo},G}(X)$, where $\bullet = 0$ or 1, is an abelian group generated by geometric cycles (M, E, f) where $\dim M \equiv 1$ or $0 \bmod 2$, subject to an equivalence relation generated by three operations: *direct sum/disjoint union*, *bordism* and *vector bundle modification.* The first relation amounts to the identification

$$(M \sqcup M, E_1 \sqcup E_2, f \sqcup f) \sim (M, E_1 \oplus E_2, f),$$

while the statement of the latter two operations are more involved. For their definitions we refer to pp. 5-6 of [11]. We will follow the definitions contained there, except that G for us is an almost-connected Lie group instead of a discrete group.

Addition in the group is given by

$$(M_1, E_1, f_1) + (M_2, E_2, f_2) = (M_1 \sqcup M_2, E_1 \sqcup E_2, f_1 \sqcup f_2),$$

with the additive inverse of (M, E, f) being given by the cycle $(-M, E, f)$, where by $-M$ we mean the G-Spin^c-manifold with the opposite G-Spin^c-structure to M (see p.5 of [11]).

Implicit in the definition of a bordism W from a G-manifold M_1 to a G-manifold M_2 is that M_1 and M_2 are contained in G-equivariant collar neighbourhoods at the ends of W. The existence of such neighbourhoods for proper actions is proved in [40] Theorem 3.5.

Example 4. Let K be a compact Lie group and Y a compact K-manifold. If Y has a K-equivariant Spin^c-structure, we can define the *fundamental class* of $K_n^{\mathrm{geo},K}(Y)$ to be

$$[Y]_K := [(Y, Y \times \mathbb{C}, \mathrm{id}_Y)] \in K_n^{\mathrm{geo},K}(Y), \qquad n = \dim Y \pmod 2.$$

Similarly, if G is an almost-connected Lie group acting properly on a G-cocompact manifold X with a G-invariant Spin^c-structure, the fundamental class of $K_n^{\mathrm{geo},G}(X)$ is given by

$$[X]_G := [(X, X \times \mathbb{C}, \mathrm{id}_X)] \in K_n^{\mathrm{geo},G}(X), \qquad n = \dim X \pmod 2.$$

One of the themes of this section of the paper is that when K is a maximal compact subgroup of G and Y sits suitably inside X as a K-invariant submanifold, there is a one-to-one correspondence between such classes, provided the homogeneous manifold G/K is also G-Spin^c.

Baum and Douglas introduced, originally in the non-equivariant setting, a natural map from geometric K-homology to analytic K-homology [10]; it extends naturally to the equivariant setting. Our version of this map, which is to say in the setting when G is an almost-connected Lie group, will be denoted by

$$BD : K_\bullet^{\mathrm{geo},G}(X) \to K_\bullet^G(X), \tag{2.1}$$

and defined as follows. Let (M, E, f) be a geometric K-homology cycle for X. Since M is G-equivariantly Spin^c by assumption, one can construct a Spin^c-*Dirac operator* on M, which acts on a spinor bundle $S_M \to M$. For reference in later sections, note that S_M is locally constructed by tensoring a Hermitian connection on the determinant line bundle L (associated to the given Spin^c-structure) with the lift of the Levi-Civita connection on TM to the local spinor bundle. Note also that if M is Spin, the Spin^c-Dirac operator can be realised as the Spin-Dirac operator twisted by a line bundle.

Let D_E be the operator D twisted by the vector bundle E. It can be viewed as an unbounded, densely-defined operator on the Hilbert space $L^2(S_M \otimes E)$. Let m be the representation of $C_0(M)$ on $L^2(S_M \otimes E)$ given by pointwise multiplication of functions on sections. Since D_E is self-adjoint, we can use functional calculus to form the L^2-bounded operator $D_E(1 + D_E^2)^{-\frac{1}{2}}$. Then the triple

$$\left(L^2(S_M \otimes E), m, D_E(1 + D_E^2)^{-\frac{1}{2}}\right)$$

is an analytic K-homology cycle defining a class in $K_0^G(M)$. Call this class $[D_E]$ (we may also denote it as a cap product $[E] \cap [D]$, depending on the context). Let $f' : C_0(X) \to C_0(M)$ be the contravariant map on algebras given by

$$(f'(g))(m) = g(f(m)), \qquad g \in C_0(X),\ m \in M. \tag{2.2}$$

Then the image of $[(M, E, f)]$ under BD is defined to be the class in $K_0^G(X)$ represented by the Kasparov cycle

$$\left(L^2(S_M \otimes E), m \circ f', D_E(1 + D_E^2)^{-\frac{1}{2}}\right).$$

We will also write this class using the pushforward notation $f_*([D_E])$. Notice that

$$f_*([D_E]) = f'^*([D_E]),$$

where $f'^* : K_0^G(M) \to K_0^G(X)$ is the covariant functorial map on analytic K-homology.

It has long been conjectured that equivariant analytic and geometric K-homologies are in general equivalent. This was proved in the case that X is a compact CW-complex without group action in [12], some twenty-five years after the conjecture was posed. Subsequently, the cases of cocompact discrete group action [11] and compact Lie group acting on a compact CW-complex [13] were confirmed. Our main aim in this section is to confirm this conjecture for the case of X a manifold admitting a proper cocompact action of an almost-connected Lie group G. More precisely, we prove:

Theorem 5. *Let G be an almost-connected Lie group acting smoothly, properly and cocompactly on a manifold X. If G/K admits a G-equivariant* Spin^c*-structure, the Baum-Douglas map relating G-equivariant analytic and geometric K-homologies is an isomorphism*

$$K_\bullet^{\mathrm{geo},G}(X) \simeq K_\bullet^G(X). \tag{2.3}$$

2.1. **Overview of the Proof.** We prove Theorem 5 in several steps. Relevant to us will will be Abels' global slice theorem, which we recall presently.

Theorem 6 (Abels [1])**.** *Let G be an almost-connected Lie group and K a maximal compact subgroup of G. Then X has a global K-slice, defined by*

$$Y = f^{-1}(eK) \subset X,$$

where $f : X \to G/K$ is a G-equivariant smooth map. Y is a K-invariant submanifold and X is diffeomorphic to the associated space

$$G \times_K Y := G \times Y/\{(gh, y) \sim (g, hy), \forall h \in K\}. \tag{2.4}$$

The associated space (2.4) is a fibre bundle over G/K with fibre the manifold Y. The G-equivariant diffeomorphism $G \times_K Y \to X$ is given by $[(g, y)] \mapsto g \cdot y$, where $[(g, y)]$ denotes the equivalence class of the pair $(g, y) \in G \times Y$ in the quotient $G \times_K Y$.

Remark 7. In the proofs that follow, we will make frequent and essential use of the fact that the slice Y in Abels' theorem is *essentially unique* in a rather strong sense, namely the K-diffeomorphism class of the slice Y depends only upon the G-diffeomorphism class of the G-action on X (for the proof of this fact see Theorem 2.2 in the paper of Abels [1] and the first paragraph of page containing it). In particular, this means that for a fixed G-action on X, the slice Y is unique up to K-diffeomorphism.

Let $d := \dim G/K$. In Section 2.2, we show carefully that that Abels' theorem can be used to write down a well-defined "induction map" on equivariant geometric K-homology,

$$i : K_\bullet^{\mathrm{geo},K}(Y) \to K_{\bullet+d}^{\mathrm{geo},G}(X),$$

and show it is an isomorphism by slightly extending Abels' theorem. In Section 2.3 we define a corresponding induction map on analytic K-homology, which we prove is an isomorphism using tools from KK-theory.

$$j : K_\bullet^{K}(Y) \to K_{\bullet+d}^{G}(X).$$

Finally, we recall the following key theorem of Baum, Oyono-Oyono, Schick and Walter, which shows that equivariant geometric and analytic K-homologies are equivalent in the case of compact group actions (note that any smooth manifold admits a CW-complex structure):

Theorem 8 ([13])**.** *Let Y be a compact CW-complex with an action of a compact Lie group K. Then the Baum-Douglas map is an isomorphism*

$$K_\bullet^{\mathrm{geo},K}(Y) \simeq K_\bullet^{K}(Y).$$

This isomorphism and the isomorphisms i and j fit into the following diagram:

$$\begin{array}{ccc} K_\bullet^{\mathrm{geo},K}(Y) & \xrightarrow{BD} & K_\bullet^{K}(Y) \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ K_{\bullet+d}^{\mathrm{geo},G}(X) & \xrightarrow{BD} & K_{\bullet+d}^{G}(X), \end{array} \tag{2.5}$$

where the bottom arrow is the G-equivariant Baum-Douglas map we defined previously. The final step is to show that this diagram commutes (Section 2.4). From this Theorem 5 follows.

Remark 9. The assumption that G/K is G-Spin^c is only used in proving Proposition 14.

2.2. **Induction on Geometric K-homology.** We define an induction map on equivariant geometric K-homology and show that, when G/K has a G-invariant Spin^c-structure, it is an isomorphism. Before doing so, we need to make an important remark about the relationship between Spin^c-structures on a G-manifold M and a given K-slice N.

Remark 10. In [36] Section 2.3, in particular Definition 2.7, a procedure called *stabilisation* is given to define a G-equivariant Spin^c-structure starting from a K-equivariant Spin^c-structure on N, together with a procedure called *destabilisation* going the other way, both based on Plymen's two-out-of-three lemma. Further, Lemma 3.9 of the same paper proves that stabilisation and destabilisation are inverses of one another. Thus there is a one-to-one correspondence between the collection of G-Spin^c-structures on M and the collection of K-Spin^c-structures on N. We note that a similar correspondence holds when M and N have boundary (see also. In the rest of this paper, whenever we speak of a "corresponding" or "compatible" G-Spin^c-structure or K-Spin^c structure given the other, we shall be doing so with this correspondence in mind.

Definition 11. Let G, X, K and Y be as before. The *geometric induction map* is given by

$$i : K^{\mathrm{geo},K}_{\bullet}(Y) \to K^{\mathrm{geo},G}_{\bullet+d}(X),$$
$$[(N, E, f)] \mapsto [(G \times_K N, G \times_K E, \tilde{f})] =: [(M, \tilde{E}, \tilde{f})].$$

Here $G \times_K N$ is equipped with the corresponding Spin^c-structure (in the sense of the remark above), while the map $f : M = G \times_K N \to G \times_K Y \cong X$ is the natural G-equivariant map determined by the K-equivariant map $f : N \to Y$ on the fibre. The vector bundle $G \times_K E$ can be defined as the pullback bundle $pr_2^* E \to G \times Y$ of E along the projection $pr_2 : G \times Y \to Y$ modulo the K-action on $G \times E$ given by $k(g, e) = (gk^{-1}, ke)$, and equipped with the pull-back Hermitian metric.

Proposition 12. *The map i is a well-defined homomorphism of abelian groups.*

Proof. One sees immediately that $(M, \tilde{E}, \tilde{f})$ satisfies the requirements of a geometric cycle. It is also easy to verify that the map i is well-defined with respect to the disjoint union/direct sum relation. To see that i is well-defined with respect to bordism, suppose that (N_1, E_1, f_1) and (N_2, E_2, f_2) are bordant cycles via a triple (S, E, f). The main point is that $G \times_K N_1$ and $-G \times_K N_2$ are bordant in the following natural way. By hypothesis, we have $\partial S = N_1 \sqcup -N_2$. Since S is a K-manifold with boundary, and K acts by self-diffeomorphisms of S, the K-action separates into two parts: an action on the boundary and an action on the interior of S. Therefore $G \times_K S$ is a G-manifold with boundary $G \times_K \partial S$ and is equal to the disjoint union of $G \times_K N_1$ and $-G \times_K N_2$. The latter two manifolds are bordant via $G \times_K S$. One then verifies that the vector bundle $G \times_K E$ restricts to $G \times_K E_1$ and $G \times_K E_2$ at the ends of the bordism, and similarly that $\tilde{f}$ restricts to $\tilde{f}_1$ and $\tilde{f}_2$.

Next, suppose that (N_1, E_1, f_1) and (N_2, E_2, f_2) are related by a vector bundle modification. In particular N_2 is the total space of the unit sphere sub-bundle of $V \oplus (N_1 \times \mathbb{R}) \to N_1$, for some G-Spin^c-vector bundle V over N_1 with fibres of even dimension $2k$. Denote the G-Spin^c-structure of V by $P_V \to N_1$. By the definition given on p. 6 of [11], we have

$$E_2 = (P_V \times_{\mathrm{Spin}^c} \beta) \otimes \pi^*(E_1),$$

where β is the Bott generator vector bundle over S^{2k} and $\pi : N_2 \to N_1$ is the projection map for the sphere bundle. Observe that $G \times_K N_2$ is just the unit sphere sub-bundle of $G \times_K V \oplus (G \times_K N_2 \times \mathbb{R})$, which is still a direct sum of an even-rank vector bundle with the trivial real one-dimensional bundle over $G \times_K N_2$. By inspection, the induced vector bundle $G \times_K E_2 = G \times_K (P_V \times_{\mathrm{Spin}^c} \beta)$ over $G \times_K N_2$ is still β when restricted over each spherical fibre of $G \times_K N_2$. It follows that the induced cycles

$(G \times_K N_1, G \times_K E_1, \tilde{f}_1)$ and $(G \times_K N_2, G \times_K E_2, \tilde{f}_2)$ are still related by a vector bundle modification, so i is also well-defined with respect to this operation.

Finally, since i preserves disjoint unions, it is a homomorphism of abelian groups. □

Having seen that it is well-defined, let us show that i is an isomorphism. First we need to establish the following variant of Abels' global slice theorem for manifolds with boundary.

Lemma 13. *Let G, K be as above, and let W be a proper G-manifold with boundary. Then there exists a global K-slice $S \subseteq W$ - that is, a K-submanifold with boundary such that we have a G-equivariant diffeomorphism $W \cong G \times_K S$ with $\partial S = S \cap \partial W$.*

Proof. First we make the claim that there exists a G-equivariant smooth map $f : W \to G/K$. We sketch its proof, mainly noting where it differs from the case proved in [1] p. 8. Let $\pi : W \to W/G$ be the natural projection. It can be shown that there exists for each $w \in W$ a G-invariant neighbourhood U_w that admits a local K-slice. Indeed, if w is in the interior of W, this follows from Proposition 2.2.2. in [57], and if $w \in \partial W$ is a boundary point, we can proceed along the lines of Palais in [57] Section 2 to obtain U_w. Note that the G-map $f_{U_w} : U_w \to G/K$ extends to a slightly larger open set containing $\overline{U}_w$, so we get smooth G-maps $f_{\overline{U}_w} : \overline{U}_w \to G/K$. Define the cover

$$\mathcal{U} := \{U_w : w \in W\}$$

of W constructed out of these U_w. Since W/G is paracompact, there is an open σ-discrete refinement of the cover $\pi(\mathcal{U}) = \{\pi(U) : U \in \mathcal{U}\}$ of $\pi(W)$. That is, there is a sequence $\mathscr{U}_n$, indexed by $n \in \mathbb{N}$, of families of open subsets of W/G such that $\bigcup_{n=1}^{\infty}(\mathscr{U}_n)$ is a cover of W/G that refines $\pi(\mathcal{U})$, and every family $\mathscr{U}_n$ is discrete (see [1] p. 6).

One can then show, as on p.8 of [1], that the set of restricted G-maps $f_{\overline{A}} := f|_{\overline{A}} : \pi^{-1}(\overline{A}) \to G/K$, with A ranging over the elements of $\mathcal{U}$, piece together to give a composite smooth G-map $f : \bigcup_{A \in \mathcal{U}} \pi^{-1}(\overline{A}) \to G/K$. The rest of the construction goes as in [1] and yields the desired map $f : W \to G/K$.

One observes, by Palais' local slice theorem [57] or otherwise, that eK is a regular value for both f and $f|_{\partial W}$. Hence $f^{-1}(eK) = S \subseteq W$ is a submanifold with boundary $\partial S = S \cap \partial W$ (for a proof of this fact see for example [54]). The map $G \times_K S \to W$ given by $[g, s] \mapsto g \cdot s$ provides a G-equivariant diffeomorphism. $\square$

Proposition 14. *Let G, X, K and Y be as before. If the homogeneous manifold G/K has a G-equivariant* Spin^c*-structure, the map i is an isomorphism.*

Proof. Let (M, E, f) be a geometric cycle for $K_\bullet^{\mathrm{geo},G}(X)$, where $\bullet = 0$ or 1. Note that M is a G-manifold with a G-equivariant Spin^c-structure. By Remark 10 one can choose a global K-slice N with a compatible K-equivariant Spin^c-structure. The restriction of E and f to the submanifold N is a preimage $[(N, E|_N, f|_N)]$ of $[(M, E, f)]$. Hence i is surjective.

To show injectivity, let $x_k \in K_0^{\mathrm{geo},K}(Y),\ k = 1, 2$, be represented by geometric cycles (N_k, E_k, f_k), such that $i(x_1) = i(x_2)$. That is, we have a relation between cycles

$$(G \times_K N_1, G \times_K E_1, \widetilde{f_1}) \sim (G \times_K N_2, G \times_K E_2, \widetilde{f_2}). \tag{2.6}$$

We show that this induces a relation

$$(N_1, E_1, f_1) \sim (N_2, E_2, f_2). \tag{2.7}$$

Indeed, if (2.6) is a relation by disjoint union/direct sum, it follows that (2.7) is also, once one remembers that the K-slice is essentially unique (see Remark 7 or Theorem 2.2 in [1]).

Now suppose (2.6) is a relation by bordism, so that $G \times_K N_1 \sqcup -G \times_K N_2$ is the boundary of another G-cocompact G-Spin^c-manifold W, which is part of a triple $(W, \tilde{E}, \tilde{f})$ with $\tilde{E}$ and $\tilde{f}$ restricting to $G \times_K E_i$ and $\tilde{f}_i$ at the ends of W. Then by Lemma 13, W has a K-slice S, with boundary $\partial S = \partial W \cap S$. Since the G-action - which by hypothesis is by self-diffeomorphisms of W - preserves the interior and boundary of W, one sees that ∂W has a K-slice ∂W_0 by the usual version of Abels' global slice theorem. Observe that,

G-equivariantly,

$$G \times_K \partial S \cong \partial W \cong G \times_K N_1 \sqcup -G \times_K N_2 \cong G \times_K (N_1 \sqcup -N_2),$$

where the first diffeomorphism follows by dimensional considerations. By uniqueness of K-slices up to K-diffeomorphism, this means that, K-equivariantly, $\partial W_0 \cong \partial S \cong N_1 \sqcup -N_2$. We now argue that S can be equipped with a K-Spin^c-structure that is compatible (in the sense of Remark 10) with the G-Spin^c-structure on W. First note that the boundary pieces $G \times_K N_1$ and $G \times_K N_2$ are contained in G-equivariant collar neighbourhoods in W (see the remark above Example 4). This allows one to form the double $\overline{W}$ of W, which is then naturally a G-Spin^c, G-cocompact manifold without boundary. Take a K-slice $\overline{S}$ of $\overline{W}$, observing that $\overline{S}$ is the double of S. By the correspondence mentioned in Remark 10, $\overline{S}$ has a K-Spin^c-structure that induces the G-Spin^c-structure on $\overline{W}$. The manifold S equipped with the restriction of this K-Spin^c-structure to S is now a K-Spin^c-bordism from N_1 to N_2. Hence N_1 and N_2 are bordant. Next, note that the vector bundle $G \times_K E$ by definition restricts to $G \times_K E_1$ and $G \times_K E_2$ on ∂W. Taking the fibre at the identity eK of the fibre bundle $G \times_K E \to G/K$ then gives a K-vector bundle whose restriction to ∂S is isomorphic to $E_1 \sqcup E_2$. One sees that, by construction, $\tilde{f}$ restricts to a K-equivariant map f on S, such that $f|_{\partial S} = f_1 \sqcup f_2$, after identifying ∂S with $N_1 \sqcup N_2$. Thus the two reduced geometric cycles (N_1, E_1, f_1) and (N_2, E_2, f_2) are still related by a bordism operation in geometric K-homology.

Finally, if (2.6) is a relation by a vector bundle modification, then there is a G-Spin^c-vector bundle V over $G \times_K N_1 =: M_1$ with even-dimensional fibres such that $G \times_K N_2 =: M_2$ is the sphere sub-bundle of $(M_1 \times \mathbb{R}) \oplus V$, while $G \times_K E_2$ is (modulo a tensor product with $\pi^*(G \times_K E_1)$, where $\pi : M_2 \to M_1$ is the projection of the sphere bundle) the bundle whose fibre over each sphere is the Bott generator vector bundle β. By Remark 10, the bundle $V|_{N_1}$ over the slice N_1 has a compatible K-equivariant Spin^c-structure. Further, the sphere sub-bundle of $(N_1 \times \mathbb{R}) \oplus (V|_{N_1})$ is precisely the restriction of M_2 to the submanifold $N_1 \subseteq M_1$; it is a K-slice of M_2. Thus after restriction, the manifolds N_1 and N_2 in (2.7) are still related by a vector bundle modification.

Let us write $G \times_K E_2$ in the form $P_{M_1} \times_{\mathrm{Spin}^c} \beta$, where P_{M_1} is the G-Spin^c-structure on M_1. By the correspondence in [36], P_{M_1} reduces to a compatible Spin^c-structure P_{N_1} on N_1, and the associated bundle reduces to $P_{N_1} \times_{\mathrm{Spin}^c} \beta$, which is still the Bott generator over each spherical fibre of N_2. One verifies also that the function $\tilde{f} \circ \pi$ restricts to $f \circ \breve{\pi}$, where $\breve{\pi}$ is the projection of the sphere bundle $N_2 \to N_1$. Thus the two geometric cycles (N_1, E_1, f_1) and (N_2, E_2, f_2) are still related by a vector bundle modification.

Hence the map i is also injective and therefore an isomorphism. □

2.3. **Induction on Analytic K-homology.** As the next step in proving the equivalence of geometric and analytic K-homologies, we now turn to the task of applying Abels' global slice theorem in the context of analytic K-homology. The goal of this subsection is to first show, using KK-theory, that there is have an isomorphism of abelian groups

$$K_\bullet^K(Y) \simeq K_{\bullet+d}^G(X),$$

before providing a natural map at the level of analytic K-cycles that realises this isomorphism. We recall the following definition (see [17] for more details). Let X be a σ-compact G-space and A, B be G-$C_0(X)$-algebras. Then the representable KK-theory $\mathcal{R}KK(X; A, B)$ is defined to be the group of equivalence classes of Kasparov (A, B)-cycles $(\mathcal{E}, \phi, T)$, defined previously, that satisfy the additional condition

$$(fa)eb = ae(fb), \qquad \forall f \in C_0(X), a \in A, b \in B, e \in \mathcal{E},$$

where the equivalence relation is identical to that defining $KK(A, B)$. The proof we give makes use of the following technical result of Kasparov.

Theorem 15 ([42, Theorem 3.4])**.** *Let X be a σ-compact space on which groups G and Γ act and assume that these actions commute. Suppose also that the Γ-action is proper and free. Then for any G-$C_0(X)$-algebras A, B, the descent map gives rise to an isomorphism*

$$\mathcal{R}KK^{G\times\Gamma}(X; A, B) \simeq \mathcal{R}KK^G(X/\Gamma; A^\Gamma, B^\Gamma).$$

Here, A^Γ is a "fixed-point subalgebra" of A under Γ, defined in [42].

We are now in a position to carry out the proof of the main result of this subsection.

Proposition 16. *Let G be an almost-connected Lie group acting properly on X and K a maximal compact subgroup. For Y a global K-slice of X, we have*

$$K_\bullet^K(Y) \simeq K_{\bullet+d}^G(X).$$

Proof. Note that for the $\mathbb{C}$-algebra $C(Y)$, the definitions of $\mathcal{R}KK$ and KK coincide:

$$K_\bullet^K(Y) \simeq \mathcal{R}KK_\bullet^K(pt; C(Y), \mathbb{C}).$$

Now a manifold is a σ-compact space, and the action of G on itself is proper and free. So from Theorem 15 we obtain (noting that $C_0(G)^G$ is precisely $\mathbb{C}$ in the sense of [42])

$$\mathcal{R}KK_\bullet^K(pt; C(Y), \mathbb{C}) \simeq \mathcal{R}KK_\bullet^{G\times K}(G; C_0(G\times Y), C_0(G)).$$

Applying Theorem 15 again to the right-hand side gives us

$$\mathcal{R}KK_\bullet^{G\times K}(G; C_0(G\times Y), C_0(G)) \simeq \mathcal{R}KK_\bullet^G(G/K; C_0(G\times_K Y), C_0(G/K)).$$

Therefore,

$$\begin{aligned} K_\bullet^K(Y) \simeq \mathcal{R}KK_\bullet^G(G/K; & C_0(G\times_K Y), C_0(G/K)) \\ &= KK_\bullet^G(C_0(G\times_K Y), C_0(G/K)), \end{aligned} \tag{2.8}$$

noting that G/K is contractible to a point. Because G is almost-connected, it is a *special manifold* in the sense of [42], so that there exist a Dirac element and a Bott element:

$$[\partial_{G/K}] \in KK_d^G(C_0(G/K), \mathbb{C}), \qquad [\beta] \in KK_d^G(\mathbb{C}, C_0(G/K)),$$

such that (denoting by $\otimes$ the Kasparov product between classes, cf. [42])

$$[\partial_{G/K}] \otimes_{\mathbb{C}} [\beta] = 1 \in KK^G(C_0(G/K), C_0(G/K)),$$
$$[\beta] \otimes_{C_0(G/K)} [\partial_{G/K}] = 1 \in KK^G(\mathbb{C}, \mathbb{C}).$$

This leads to the isomorphism

$$KK_\bullet^G(C_0(G\times_K Y), C_0(G/K)) \simeq KK_{\bullet+d}^G(C_0(G\times_K Y), \mathbb{C}).$$

Together with (2.8) we obtain $K^K_\bullet(Y) \simeq K^G_{\bullet+d}(X)$. $\square$

Now we give an explicit map for this isomorphism. From the proof of the above Proposition, we find that the image under j of a K-equivariant Kasparov cycle $[(\mathcal{H}, \phi, F)] \in K^K_\bullet(Y)$ is obtained by taking the Kasparov product of the "lifted" cycle

$$[(C_0(G \times_K \mathcal{H}), \tilde{\phi}, \tilde{F})] \in KK^G_\bullet(C_0(G \times_K Y), C_0(G/K)) \tag{2.9}$$

with the Dirac element $[\partial_{G/K}] \in KK^G_d(C_0(G/K), \mathbb{C})$. Note that $C_0(G \times_K \mathcal{H})$ is a G-Hilbert $C_0(G/K)$-module whose $C_0(G/K)$-valued inner product is given by the fibrewise inner product on $\mathcal{H}$, and $\tilde{\phi}$ is the multiplication action on $C_0(G \times_K \mathcal{H})$ and $\tilde{F}$ is a family of operators indexed by G/K and given on each fibre $\mathcal{H}$ by F.

The proof that j is an isomorphism did not make use of a G-equivariant Spin^c-structure on G/K, but for the remainder of this paper, we shall assume that such a structure on G/K exists. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G and K. There is a Lie algebra $\mathfrak{p}$ such that the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is invariant under the adjoint action of K. Our assumption of a G-equivariant Spin^c-structure on G/K means that $\mathrm{Ad} : K \to SO(\mathfrak{p})$ can be lifted to

$$\widetilde{\mathrm{Ad}} : K \to \mathrm{Spin}^c(\mathfrak{p}). \tag{2.10}$$

Remark 17. Replace G by a double cover $\widetilde{G}$ and consider the diagram

$$\begin{array}{ccc} \widetilde{K} & \xrightarrow{\widetilde{\mathrm{Ad}}} & \mathrm{Spin}^c(\mathfrak{p}) \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle \pi} \\ K & \xrightarrow{\mathrm{Ad}} & \mathrm{SO}(\mathfrak{p}), \end{array}$$

where

$$\widetilde{K} := \{(k, a) \in K \times \mathrm{Spin}^c(\mathfrak{p}); \mathrm{Ad}(k) = \pi(a)\},$$

and the maps π_K and $\widetilde{\mathrm{Ad}}$ are defined by

$$\pi_K(k, a) := k;$$
$$\widetilde{\mathrm{Ad}}(k, a) := a,$$

for $k \in K$ and $a \in \mathrm{Spin}^c(\mathfrak{p})$. Then $\widetilde{G}/\widetilde{K}$ has a G-equivariant Spin^c-structure. Indeed, for all $k \in K$,

$$\pi_K^{-1}(k) \cong \pi^{-1}(\mathrm{Ad}(k)) \cong U(1),$$

so π_K is the projection of a $U(1)$-central extension. Since G/K is contractible, $\widetilde{K}$ is the maximal compact subgroup of a $U(1)$-central extension of G.

Denote by S the K-vector space underlying $\mathfrak{p}$ in the Spin^c-representation (2.10) of K. Fix the normalising function

$$b(x) = \frac{x}{\sqrt{x^2+1}}.$$

The Dirac element can be written as

$$[\partial_{G/K}] := \left[\left((L^2(G) \otimes S)^K, m, b(\partial_{G/K})\right)\right],$$

where m is scalar multiplication of $C_0(G/K)$ on the Hilbert space $(L^2(G) \otimes S)^K$ and $\partial_{G/K}$ is the Spin^c-Dirac operator on G/K. Then the image of $[(\mathcal{H}, \phi, F)]$ under induction is

$$j[(\mathcal{H}, \phi, F)] = [(\mathcal{E}, \tilde{\phi}, \tilde{F}\sharp b(\partial_{G/K}))],$$

where $\mathcal{E}$ is the Hilbert space

$$\mathcal{E} := C_0(G \times_K \mathcal{H}) \otimes_{C_0(G/K)} (L^2(G) \otimes S)^K \simeq (L^2(G) \otimes \mathcal{H} \otimes S)^K,$$

$\tilde{\phi}$ is the representation

$$\tilde{\phi} : C_0(G \times_K Y) \to \mathcal{L}(\mathcal{E})$$

given by the obvious pointwise multiplication determined by $\phi : C(Y) \to \mathcal{L}(\mathcal{H})$, and

$$\tilde{F}\sharp b(\partial_{G/K})$$

means the Kasparov product (of operators) of $\tilde{F}$ and $b(\partial_{G/K}) = \partial_{G/K}(1 + \partial_{G/K}^2)^{-\frac{1}{2}}$. When F is also a Dirac-type operator, the Kasparov product can be explicitly written down, as follows.

Example 18. Let Y be a K-manifold with a K-equivariant Spin^c-structure and ∂_Y be the associated Spin^c-Dirac operator. Then

$$\begin{aligned}\mathcal{E} &= C_0(G \times_K L^2(Y, S_Y)) \otimes_{C_0(G/K)} (L^2(G) \otimes S)^K \\ &\simeq (L^2(G) \otimes L^2(Y, S_Y) \otimes S)^K \\ &\simeq L^2(G \times_K Y, G \times_K (S_Y \times S)) \simeq L^2(X, S_X).\end{aligned}$$

and $b(\partial_X)$ represents the Kasparov product of $b(\tilde{\partial}_Y \otimes 1)$ and $b(1 \otimes \partial_{G/K})$, namely

$$\left(\frac{\tilde{\partial}_Y}{\sqrt{\tilde{\partial}_Y^2 + 1}} \otimes 1\right) \sharp \left(1 \otimes \frac{\partial_{G/K}}{\sqrt{1 + \partial_{G/K}^2}}\right) = \frac{\partial_X}{\sqrt{1 + \partial_X^2}}.$$

Therefore

$$j[\partial_Y] = [\partial_X]. \tag{2.11}$$

This relation also holds for twisted Dirac operators (cf. the proof of Lemma 19).

2.4. **Commutativity of the Diagram.**

Lemma 19. *Diagram (2.5) commutes.*

Proof. Let x be an element of $K_\bullet^K(Y)$. From [13], there is a geometric cycle (N, E, f) representing an element of $K_\bullet^{\mathrm{geo},K}(Y)$ such that

$$x = f_*([E] \cap [D_N]).$$

By definition, the map i sends the class of geometric cycles $[(N, E, f)]$ to the class of geometric cycles

$$[(M, \tilde{E}, \tilde{f})] \in K_{\bullet+d}^{\mathrm{geo},G}(X),$$

where $M = G \times_K N$, $\tilde{E} = G \times_K E$ and $\tilde{f} : M \to X$ is the lift of $f : N \to Y$ as in Definition 11. Thus, elements in (2.5) are related as follows:

$$\begin{array}{ccc} [(N, E, f)] & \xrightarrow{BD} & f_*([E] \cap [D_N]) \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ [(M, \tilde{E}, \tilde{f})] & \xrightarrow{BD} & \tilde{f}_*([\tilde{E}] \cap [D_N]), \end{array}$$

and commutativity of (2.5) means precisely

$$j(f_*([E] \cap [D_N])) = \tilde{f}_*([\tilde{E}] \cap [D_M]).$$

Writing both sides in terms of cycles, we need to show that the Kasparov $(C_0(X), \mathbb{C})$-cycles

$$((L^2(G) \otimes L^2(S_N \otimes E) \otimes S)^K, \widetilde{m \circ f'}, b(\widetilde{D_{N,E}})\sharp b(\partial_{G/K}))$$

and

$$(L^2(S_M \otimes \tilde{E}), \tilde{m} \circ \tilde{f}', b(D_{M,\tilde{E}}))$$

give rise to the same element in $K_0^G(X)$. Here S_N stands for the spinor bundle over N and m stands for the pointwise multiplication of the algebra $C_0(N)$ of continuous functions on the Hilbert space. At this point, recall the well-known fact that the K-homology class of a Dirac operator is independent of the choice of metric. Given a K-invariant metric on N, let us pick the natural "product-type" metric on $G \times_K N$. Then we have

$$(L^2(G) \otimes L^2(S_N \otimes E) \otimes S)^K \simeq L^2(S_M \otimes \tilde{E}),$$

since $S_M = G \times_K (S_M|_N)$ and the restriction S_M to N splits into a tensor product of S_N and S. Obviously, $\tilde{m} \circ \tilde{f}'$ and $\widetilde{m \circ f'}$ both give the same representation of $C_0(X)$ on $L^2(S_M \otimes \tilde{E})$ by scalar multiplication, compatible with $\tilde{f}' : C_0(X) \to C_0(M)$. Finally, from $D_{N,E}$ to $\widetilde{D_{N,E}}$ we replace the K-equivariant Spin^c-connection ∇^E on N by the G-equivariant connection $\nabla^{\tilde{E}} = G \times_K \nabla^E$ on M. If $\pi : M = G \times_K N \to G/K$ is the projection, the Spin^c-connection $\nabla^{G/K}$ on G/K is pulled back to a G-equivariant Spin^c-connection $\pi^*\nabla^{G/K}$ on M. Let $\{e_i\}$ be a local orthonormal frame in the direction of fibres of $G \times_K N \to G/K$ and $\{f_j\}$ a local orthonormal frame of the base G/K. Then $b(\widetilde{D_{N,E}}) \sharp b(\partial_{G/K})$ looks locally like $b(x)$, where

$$x = \sum_{i=1}^{\dim N} c(e_i)\nabla^{\tilde{E}}_{e_i} + \sum_{j=1}^{d} c(f_j)\pi^*\nabla^{G/K}_{f_j}.$$

With respect to the fibration $N \to M \to G/K$, a G-invariant metric g_M on M determines a metric $g_{G/K}$ on the base G/K and a family of metrics $g_{M/(G/K)}$ on the family of manifolds $\pi : M \to G/K$ parametrised by G/K. Taking the adiabatic limit in the parameter s,

$$g_{M,s} := g_{M/(G/K)} + s^{-2}\pi^* g_{G/K}$$

approaches a product as $s \to 0$, and these metrics are G-invariant (see [14]). Since the K-homology class represented by a Dirac operator is independent of the choice of metrics, the operator $b(\widetilde{D_{N,E}}) \sharp b(\partial_{G/K})$ represents the same K-homology class as $b(D_{M,\tilde{E}})$. The lemma is then proved. □

Remark 20. Let us summarise of the consequences of Theorem 5. Let G be an almost-connected Lie group acting on a manifold X properly and cocompactly. For every element x in the analytic K-homology $K_0^G(X)$, there exists a unique class $[(M, E, f)] \in K_0^{G,geo}(X)$ such that

$$x = f_*([E] \cap [D_M]) = f_*([D_{M,E}]),$$

where $f : M \to X$. Moreover, letting K be a maximal compact subgroup, there exists a K-submanifold N such that:

- M is diffeomorphic to $G \times_K N$,
- N admits a K-equivariant Spin^c-structure compatible with the G-equivariant Spin^c-structure on M and
- $(N, E|_N, f|_N)$ is a geometric cycle for $K_d^{K,geo}(Y)$,

and there exists a unique element $(f|_N)_*([D_{N,E|_N}]) \in K_d^K(Y)$ satisfying

$$K_d^K(Y) \to K_0^G(X), \qquad (f|_N)_*([D_{N,E|_N}]) \mapsto f_*([D_{M,E}]) = x. \tag{2.12}$$

Here $Y = f(N) \subset X$ is a K-slice of X and d is the (mod 2) dimension of G/K.

Remark 21. In Theorem 4.6 in [33] and Theorem 4.5 in [34], a map

$$\text{K-Ind}_K^G\colon K_\bullet^K(Y) \to K_{\bullet+d}^G(X)$$

is constructed by a different method. In Section 6 of [33], it is shown that the K-homology class of a Spin^c-Dirac operator on Y, associated to a connection ∇^Y on the determinant line bundle of a Spin^c-structure, is mapped to the class of a Spin^c-Dirac operator on X associated to a connection ∇^X induced by ∇^Y on the determinant line bundle of the induced Spin^c-structure, by the map K-Ind$_K^G$:

$$\text{K-Ind}_K^G[\partial_Y] = [\partial_X].$$

Since we have shown that the two K-homology theories are isomorphic and that the diagram (2.5) commutes, the two induction isomorphisms i and j can be thought of as being the same map. In view of Remark 21, we shall denote both i and j by K-Ind$_K^G$, which we refer to as *induction on K-homology.* (2.12) can now be formulated in the following way:

Proposition 22. *Any $x \in K_0^G(X)$ can be represented by a G-equivariant* Spin^c*-Dirac operator on M twisted by a G-vector bundle E, where $f : M \to X$ is a continuous G-equivariant map such that*

$$x = f_*([D_{M,E}]) = \text{K-Ind}_K^G((f|_N)_*([D_{N,E|_N}])).$$

3. Poincaré Duality

3.1. **Almost-connected Groups.** We begin by remarking that Phillips' [59] generalisation of equivariant K-theory, denoted $\bar{K}_G^\bullet(X)$, which is defined using finite-dimensional equivariant vector bundles over X, is not necessarily the same as the definition via C^*-algebras, namely $K_G^\bullet(X) := K_\bullet(C_0(X) \rtimes G)$. However, in the case of almost-connected Lie groups, these groups are isomorphic, as will be argued presently.

Lemma 23. *Let G be an almost-connected Lie group acting properly and cocompactly on a smooth manifold X. Then*

$$\bar{K}_G^\bullet(X) \simeq K_G^\bullet(X) = K_\bullet(C_0(X) \rtimes G). \tag{3.1}$$

Proof. Without loss of generality, assume $\bullet = 0$. Under the hypotheses of the lemma, $C_0(X) \rtimes G$ and $C_0(Y) \rtimes K$ are strongly Morita-equivalent, by Rieffel-Green [61], where Y is a global slice given by Abels' theorem [1]. By Green-Julg [38], one has

$$K_0(C_0(Y) \rtimes K) \simeq K_K^0(Y).$$

By definition,

$$K_G^0(X) = K_0(C_0(X) \rtimes G),$$

therefore

$$K_G^0(X) \simeq K_K^0(Y).$$

But Phillips' [59] proves that $\bar{K}_G^0(X) \simeq K_K^0(Y)$, so we conclude. □

Remark 24. We do not need to specify whether or not the crossed product $C_0(X) \rtimes G$ is reduced, as the action of G on X is proper. See also Remark 39.

Remark 25. Concretely, the isomorphism (3.1) can be expressed as the composition

$$\begin{aligned} &\bar{K}_G^0(X) \to KK^G(C_0(X), C_0(X)) \\ &\to KK(C_0(X) \rtimes G, C_0(X) \rtimes G) \to K_0(C_0(X) \rtimes G), \end{aligned}$$

where the first map takes a G-equivariant vector bundle to its continuous sections, the second is the descent map (4.3) in KK-theory and the last is left multiplication via KK-product by the canonical projection (4.1) in $C_0(X) \rtimes G$. Given a G-equivariant vector bundle V over X, the images in the above sequence of maps are

$$[V] \mapsto [(\Gamma(V), 0)] \mapsto [(\Gamma(V) \rtimes G, 0)] \mapsto [p \cdot (\Gamma(V) \rtimes G)].$$

The induction on $\bar{K}^0$ is simpler than the induction on K^0. In fact, if E is a K-equivariant vector bundle over Y and p^K, p^G are the canonical projections in $C(Y) \rtimes K$, $C_0(X) \rtimes G$ respectively, then in the diagram

$$\begin{array}{ccc} \bar{K}_K^0(Y) & \longrightarrow & K_K^0(Y) \\ \downarrow & & \downarrow \\ \bar{K}_G^0(X) & \longrightarrow & K_G^0(X) \end{array} \tag{3.2}$$

we have

$$\begin{array}{ccc} [V] & \longrightarrow & [p^K \cdot (\Gamma(V) \rtimes K)] \\ \big\downarrow & & \big\downarrow \\ [G \times_K V] & \longrightarrow & [p^G \cdot (\Gamma(G \times_K V) \rtimes G)]. \end{array} \tag{3.3}$$

In particular, if V is the rank 1 trivial vector bundle over Y, then $G \times_K V$ is also a rank 1 vector bundle over X and they correspond to the canonical projections $[p^K] \in K^0_K(Y)$ and $[p^G] \in K^0_G(X)$ respectively. The induction map $K^0_K(Y) \to K^0_G(X)$ can also be understood using the isomorphism

$$K_0(C_0(X) \rtimes G) \simeq K_d(C_0(X) \rtimes K) \tag{3.4}$$

proved in [27] and the fact that X is K-homeomorphic to $Y \times \mathbb{R}^d$, where K acts on $\mathbb{R}^d$ via the diffeomorphism $\mathbb{R}^d \cong G/K$ (see Remark 5.19 in [27]). Finally, the induction maps (3.2)-(3.3) imply that both $K^0_G(X)$ and $\bar{K}^0_G(X)$ are $R(K)$-modules and that (3.1) is an isomorphism of $R(K)$-modules.

We now prove an equivariant Poincaré duality under the same hypotheses as above.

Theorem 26 (Poincaré duality: cocompact case)**.** *Let G be an almost-connected Lie group acting properly and cocompactly on a smooth manifold X. Then there are isomorphisms*

$$\mathcal{PD}_{X*} : K^G_\bullet(C_\tau(X)) \simeq K^G_\bullet(X); \tag{3.5}$$

$$\mathcal{PD}^*_X : K^\bullet_G(C_\tau(X)) \simeq K^\bullet_G(X), \tag{3.6}$$

where $C_\tau(X)$ is the algebra of continuous sections, tending to 0 at ∞, of the complex Clifford bundle associated with the tangent bundle TX of X.

Proof. We use Abels' global slice theorem to see that X is diffeomorphic to $G \times_K Y$, where K is a maximal compact subgroup of G and Y is a smooth compact manifold. Using Morita equivalence of $C_\tau(X)$ and $C_0(TX)$ [41, Theorem 2.7], the decomposition

$$TX = G \times_K [TY \oplus \mathfrak{p}], \tag{3.7}$$

where $\mathfrak{p} \oplus \mathfrak{k} = \mathfrak{g}$, and Phillips' result [60, 59] proving that induction from K to G in K-theory is an isomorphism, we obtain

$$K^K_\bullet(C_\tau(Y)) \simeq K^G_{\bullet+d}(C_\tau(X)). \tag{3.8}$$

Then, (3.8) using equivariant Poincaré duality in the compact case (see [42]),

$$\mathcal{PD} : K^K_\bullet(C_\tau(Y)) \simeq K_\bullet^K(Y),$$

together with analytic induction from K to G, which is an isomorphism

$$K^K_\bullet(Y) \simeq K^G_{\bullet+d}(X),$$

we deduce the isomorphism (3.5). The second isomorphism is proved analogously. ☐

We note that the above proof only used induction on the *analytic* version of K-homology. However, the equivalence between the two K-homologies can be used to give the following geometric interpretation of Poincaré duality.

Corollary 27. *Let G be an almost-connected Lie group acting properly and cocompactly on a manifold X. If X and G/K admit G-equivariant* Spin^c*-structures, we have*

$$\mathcal{PD}_X : K^0_G(X) \to K_a^{G,geo}(X), \quad [E] \mapsto [X] \cap [E] := [X, E \otimes \mathbb{C}, \mathrm{id}_X], \tag{3.9}$$

where E is a finite-rank G-equivariant vector bundle over X and $a = \dim X$ (mod 2).

Proof. Since X is a proper G-cocompact Spin^c-manifold, $C_\tau(X)$ is Morita-equivalent to $C_0(X)$. Thus we may write

$$\mathcal{PD}_X : K^0_G(X) \to K^G_a(X). \tag{3.10}$$

Then the claim follows by the equivalence between geometric and analytic K-homologies given in Theorem 5. ☐

Remark 28. When X is a proper and cocompact G-manifold, not necessarily Spin^c, the first isomorphism (3.5) is equivalent to

$$K^G_\bullet(X) \simeq K^G_\bullet(C_0(TX)) \simeq K_\bullet(C_0(TX) \rtimes G). \tag{3.11}$$

This isomorphism is closely related to a generalisation of the Atiyah-Singer index formula, since it is an operator-to-symbol map. In fact, recall that for a K-slice Y of X, K-invariant pseudo-differential operators represent classes in $K^K_\bullet(Y)$, while their symbols give rise to classes in $K^K_\bullet(C_0(TY))$ (cf. [5, Section 5]). Then, for a class of G-invariant pseudo-differential operators, which we shall denote by $[D_X] \in K^G_\bullet(X)$, with symbol class $[\sigma(D_X)] \in KK^G_\bullet(C_0(X), C_0(TX))$ (cf. [41]), the map

$$[D_X] \mapsto [p] \otimes_{C_0(X) \rtimes G} j^G[\sigma(D_X)]$$

realises the isomorphism (3.11). Note that using Theorem 5 and the commutative diagram

$$\begin{array}{ccc} K^G_0(M) & \longrightarrow & K^G_0(C_0(T^*M)) \\ {\scriptstyle f_*}\big\downarrow & & {\scriptstyle f_*}\big\downarrow \\ K^G_0(X) & \longrightarrow & K^G_0(C_0(T^*X)), \end{array}$$

we see that every element of $K^G_\bullet(X)$ is represented by a G-invariant Spin^c-Dirac operator.

Remark 29. When X is a proper and cocompact G-manifold, not necessarily Spin^c, the second isomorphism (3.6) maps a G-equivariant vector bundle $[E] \in K^0_G(X)$ to

$$[d_{X,E}] = [d_X] \cap [E] \in K^0_G(C_\tau(X)),$$

where $[d_X] \in K^0_G(C_\tau(X))$ is the Dirac element defined using the de Rham operator on X in [42]. This can be easily verified using induction:

$$\begin{array}{ccc} K^K_0(Y) & \longrightarrow & K^0_K(C_\tau(Y)) \\ \big\downarrow & & \big\downarrow \\ K^G_0(X) & \longrightarrow & K^0_G(C_\tau(X)), \end{array}$$

given by

$$\begin{array}{ccc} [E|_Y] & \longrightarrow & [d_Y] \cap [E|_Y] \\ \big\downarrow & & \big\downarrow \\ [E] & \longrightarrow & [d_X] \cap [E], \end{array}$$

with the help of the fact that $[d_X]$ is mapped to $[d_Y]$ under K-homology induction adapted to Clifford algebras.

Remark 30. From [59, Theorem 4.1], for any proper G-space X (not necessarily cocompact), there is an isomorphism

$$K_G^\bullet(X) \simeq K_K^\bullet(X \times \mathfrak{g}/\mathfrak{k}) \simeq K_K^{\bullet+d}(X).$$

This is the same map as (3.4). Replacing X by TX and using KK-equivalence of $C_0(TX)$ and $C_\tau(X)$ and applying Theorem 4.11 of [42] and Theorem 26, we obtain the dual version of Phillips' isomorphism for a proper *cocompact* G-space X:

$$K_\bullet^G(X) \simeq RK_{\bullet+d}^K(X) \simeq RK_\bullet^K(X \times \mathfrak{g}/\mathfrak{k}).$$

For compact manifolds $\bar{Y}$ with boundary $\partial\bar{Y} \neq \emptyset$, Poincaré duality is proved in [42]:

Theorem 31 (Poincaré duality for K-compact manifolds with boundary)**.** *Assume that $\bar{Y}$ is a smooth compact manifold with boundary $\partial\bar{Y}$ and that a compact group K acts on $\bar{Y}$ smoothly. Set $Y = \bar{Y} \setminus \partial\bar{Y}$. Then there are isomorphisms*

$$\mathcal{PD}_{Y*} : K_\bullet^K(C_\tau(Y)) \simeq K_\bullet^K(\bar{Y});$$
$$\mathcal{PD}_Y^* : K_K^\bullet(C_\tau(Y)) \simeq K_K^\bullet(\bar{Y}).$$

The proof of the following theorem is similar to that of Theorem 26 but using instead Theorem 31 and will be omitted. Observe that by Abels' global slice theorem, $\bar{X} = G \times_K \bar{Y}$, $\partial\bar{X} = G \times_K \partial\bar{Y}$, and $X = G \times_K Y$.

Theorem 32 (Poincaré duality for G-cocompact manifolds with boundary)**.** *Assume that $\bar{X}$ is a smooth G-cocompact manifold with boundary $\partial\bar{X}$ and that an almost-connected Lie group G acts on $\bar{X}$ smoothly. Set $X = \bar{X} \setminus \partial\bar{X}$. Then there are isomorphisms*

$$\mathcal{PD}_{X*} : K_\bullet^G(C_\tau(X)) \simeq K_\bullet^G(\bar{X});$$
$$\mathcal{PD}_X^* : K_G^\bullet(C_\tau(X)) \simeq K_G^\bullet(\bar{X}).$$

We now generalise Poincaré duality to the case when X is not necessarily G-cocompact.

Recall that the representable equivariant K-theory of a G-space X, as defined by Fredholm complexes in [63], denoted by $RK^{\bullet}_G(X)$, is equal to $K^G_{\bullet}(C_0(X))$ when X is G-cocompact, and is defined as the direct limit

$$RK^{\bullet}_G(X) := \lim_{Z \subset X} K^G_{\bullet}(C_0(Z))$$

over the inductive system of all cocompact G-subsets $Z \subset X$.

Theorem 32 allows us to prove Poincaré duality for non-cocompact manifolds, the main result of this section:

Theorem 33 (Poincaré duality for non-cocompact manifolds)**.** *Assume that X is a complete Riemannian manifold on which an almost-connected Lie group G acts isometrically. Then one has isomorphisms*

$$\mathcal{PD}_{X*} : K^G_{\bullet}(C_\tau(X)) \simeq RK^G_{\bullet}(X); \tag{3.12}$$

$$\mathcal{PD}^*_X : K^{\bullet}_G(C_\tau(X)) \simeq RK^{\bullet}_G(X), \tag{3.13}$$

where the right-hand side denotes the representable versions of equivariant K-homology and K-theory.

Proof. We sketch the proof here. Consider an exhaustive increasing sequence of cocompact G-manifolds with boundary $\bar{X}_j$ such that $X = \bigcup \bar{X}_j$. Theorem 32 gives us a coherent system of isomorphisms:

$$\mathcal{PD}_{j*} : K^G_{\bullet}(C_\tau(X_j)) \simeq K^G_{\bullet}(\bar{X}_j);$$
$$\mathcal{PD}^*_j : K^{\bullet}_G(C_\tau(X_j)) \simeq K^{\bullet}_G(\bar{X}_j).$$

The isomorphism $\mathcal{PD}_{X*}$ is obtained as the direct limit isomorphism of the first coherent system of isomorphisms above, and $\mathcal{PD}^*_X$ is Milnor's $\lim^1$ inverse limit of the second coherent system of isomorphisms above. ☐

Our result is related to the Poincaré duality in [42, Section 4] as follows.

Remark 34. Theorem 33 is a generalisation of Corollary 4.11 in [42]. Kasparov's first Poincaré duality ([42, Theorem 4.9]) states that for any locally compact group G,

$$RKK^G(Y \times X; A, B) \simeq RKK^G(Y; A \hat{\otimes} C_\tau(X), B),$$

where X is a complete Riemmanian manifold with an isometric G-action, Y is a σ-compact G-space and A, B are separable G-C^*-algebras. Here,

$$RKK(X; A, B) := \mathcal{R}KK(X; A(X), B(X))$$

(cf. [42, Section 2.19]). When $A = B = \mathbb{C}$ and Y is a point, this isomorphism reduces to the second isomorphism of Theorem 33.

We end this section by generalising the Atiyah-Segal completion theorem [4] using induction on K-homology and Poincaré duality.

Remark 35. Let K be a compact Lie group and A, B be G-C^*-algebras. $KK^K(A, B)$ is an $R(K)$-module. Denote by $KK^K(A, B)^\wedge$ the $I(K)$-adic completion of $KK^K(A, B)$ in the sense of [4]. In [2, Theorem 3.19], a generalisation of the Atiyah-Segal completion theorem is proved:

$$KK^K(A, B)^\wedge \simeq RKK^K(EK; A, B),$$

where $KK^K(A, B)$ is a finite $R(K)$-module. Note that choosing finite $R(K)$-modules $A = \mathbb{C}$ and $B = C(Y)$ for a compact K-manifold Y, we obtain

$$RKK^K(EK; \mathbb{C}, C(Y)) \simeq RK^0(EK \times_K Y),$$

and the Atiyah-Segal completion theorem [4] is recovered.

Now assume A, B to be G-algebras. Note that $EG = G \times_K EK$ and that there is the following induction isomorphism, from Section 3.6 of [42]:

$$RKK^K(EK; A, B) \simeq RKK^G(G \times_K EK; A, B).$$

(1) Letting $A = \mathbb{C}$ and $B = C(X)$, we have

$$KK^K_\bullet(\mathbb{C}, C_0(X)) \simeq K^\bullet_G(X)$$

as $R(K)$-modules. So its $I(K)$-adic completion is isomorphic to $RKK^G(EG; \mathbb{C}, C(X))$, and

$$K^\bullet_G(X)^\wedge \simeq RKK^G_\bullet(EG; \mathbb{C}, C_0(X)) \simeq RK^\bullet(EG \times_G X),$$

whence we recover Phillips' result [59, Theorem 5.3].

(2) Letting $A = C(X)$ and $B = \mathbb{C}$, we have from induction on K-homology that

$$KK^K_\bullet(C_0(X), \mathbb{C}) \simeq K^G_\bullet(X)$$

as $R(K)$-modules. So its $I(K)$-adic completion is isomorphic to $RKK^G(EG; C(X), \mathbb{C})$, and

$$K^G_\bullet(X)^\wedge \simeq RKK^G_\bullet(EG; C_0(X), \mathbb{C}) \simeq KK_\bullet(C_0(EG \times_G X), C_0(BG)),$$

whence we obtain a new Atiyah-Segal completion result for G-equivariant K-homology.

Note that if X is Spin^c, (1) and (2) are related by Poincaré duality (cf. [42, Theorem 4.10]).

3.2. **Discrete Groups.** Let G be a locally compact group and X a proper G-manifold where X/G is compact. As mentioned previously, the K-theory $K^*_G(X)$ fails to be a generalised cohomology in general. In [60], Phillips constructed a cohomology theory using G-vector bundles with Hilbert space fibres and shows in an example that finite-dimensional vector bundles are not enough. The example is a semidirect product of $\mathbb{Z}^2$ by the 4-torus, which is neither discrete nor linear. However, it has been verified that finite-dimensionality is enough if G is either discrete (Theorem 3.2 in [53]) or linear ([59]), and moreover that $K^*_G(X)$ is a generalised cohomology theory in these cases.

Assume G to be a discrete or linear group and X a proper G-manifold with compact quotient. Every element of K-theory $K^*_G(X)$ is represented by a compact G-equivariant vector bundle E over X [53, 59] for $* = 0$. (For $* = 1$ a K-theory element is given by G-vector bundle over the suspension of X.) Let D be the de Rham operator on X representing the Dirac element $[D]$ in the equivariant K-homology $K^*_G(C_\tau(X))$ in the sense of [42] Definition-lemma 4.2. Twisting E with the de Rham operator D gives rise to a twisted Dirac element and hence the homomorphism below:

$$\mathcal{PD} : K^*_G(X) \to K^*_G(C_\tau(X)) \qquad [E] \mapsto [D_E]. \tag{3.14}$$

Theorem 36. *Let G be a discrete group acting properly on a G manifold X with compact quotient. The Poincaré homomorphism (3.14) is an isomorphism, defining the Poincaré duality $\mathcal{PD} : K^\bullet_G(X) \simeq K^\bullet_G(C_\tau(X))$.*

Every proper compact G-manifold X is covered by G-slices of the form $G \times_H U$ where H is a compact subgroup of G and U is an H-space:

$$X = \cup_{i=1}^N G \times_{H_i} U_i.$$

We may assume that all U_is and their nonempty "intersections" are homeomorphic to $\mathbb{R}^m$ for some m. Here, U_{ij} is an nonempty intersection of U_i and U_j means that

$$[G \times_{H_i} U_i] \cap [G \times_{H_j} U_j] = G \times_{H_{ij}} U_{ij}$$

for some compact subgroup H_{ij} of G.

A cover $\{G \times_{H_i} U_i\}_{i=1}^N$ satisfying the above conditions is called a *good cover* of X.

As one would expect, we will apply Mayer-Vietoris sequence and the Five Lemma to prove Poincaré duality. This means that we need to extend the Poincaré duality map to non-G-cocompact manifolds. See [15] for the proof Poincaré duality for ordinary (co)-homology and [32] 11.8.11 for the K-theory analogue for compact manifolds in the nonequivariant case.

For every open proper G-manifold U, denote by $RK^\bullet_G(C_\tau(U))$ the equivariant K-homology with compact support defined by the direct limit over all G-compact submanifolds L in U with respect to the restriction map $C_\tau(U) \to C_\tau(L)$:

$$RK^\bullet_G(C_\tau(U)) = \lim_{L \subset M} K^\bullet_G(C_\tau(L)).$$

The equivariant K-theory of U, by definition, is represented by a G-vector bundle E that is trivial outside a G-compact subset $C \subset U$. Thus, the Poincaré duality map (3.14) is extended to a non G-compact manifold $U \subset M$ by

$$\text{(3.15)} \quad \mathcal{PD} : K^\bullet_G(U) \to RK^\bullet_G(C_\tau(U)) \qquad [E] \mapsto \{[D_E|_{C\cup L}]\}_{L\subset U, L/G \text{ compact}},$$

where the inductive limit is taken over G-compact submanifolds $C \cup L \subset U$.

Proof of Theorem 36. Because G is discrete, it follows from [53] that the equivariant K-theory $K^*_G(X)$ is a generalised cohomology theory. The (analytic) equivariant homology $RK^\bullet_G(C_\tau(X))$ is known to be a generalised homology theory from the work of Kasparov. Therefore for two open G-submanifolds U and V of X, we have the Mayer-Vietoris sequences for both K-theory and K-homology, with the corresponding terms related (vertically in the following diagram) by Poincaré duality maps (3.15):

$$\begin{array}{ccccc} \to K^j_G(U\cap V) & \longrightarrow & K^j_G(U)\oplus K^j_G(V) & \longrightarrow & \dots \\ \downarrow \mathcal{PD} & & \downarrow \mathcal{PD} & & \\ \to RK^j_G(C_\tau(U\cap V)) & \longrightarrow & RK^j_G(C_\tau(U))\oplus RK^j_G(C_\tau(V)) & \longrightarrow & \dots \end{array}$$

$$\begin{array}{ccc} \dots & \longrightarrow & K^j_G(U\cup V)\to \\ & & \downarrow \mathcal{PD} \\ \dots & \longrightarrow & RK^j_G(C_\tau(U\cup V))\to \end{array}$$

where $j=0,1$. This diagram commutes.

Choose a good open cover $\{G\times_{H_i} U_i\}$ for X. Using the Five Lemma and induction, it suffices to show that (3.15) is an isomorphism when U is homeomorphic to $G\times_H \mathbb{R}^m$ for some m and a finite subgroup H of G acting on $\mathbb{R}^m$: that is, show that

$$\mathcal{PD}: K^\bullet_G(G\times_H\mathbb{R}^m)\to RK^\bullet_G(C_\tau(G\times_H\mathbb{R}^m)) \tag{3.16}$$

is an isomorphism. Recall the following induction isomorphisms for discrete groups:

$$K^\bullet_H(\mathbb{R}^m)\simeq K^\bullet_G(G\times_H\mathbb{R}^m); \tag{3.17}$$

$$RK^H_\bullet(\mathbb{R}^{2m})\simeq RK^G_\bullet(G\times_H\mathbb{R}^{2m}). \tag{3.18}$$

For first isomorphism see [53] and for the second see [11]. The latter isomorphism implies

$$RK^\bullet_H(C_\tau(\mathbb{R}^m))\simeq RK^H_\bullet(T\mathbb{R}^m)\simeq RK^G_\bullet(G\times_H T\mathbb{R}^m)\simeq RK^G_\bullet(T(G\times_H\mathbb{R}^m))\simeq RK^\bullet_G(C_\tau(G\times_H\mathbb{R}^m)),$$

where the third isomorphism follows from G being discrete. Thus to prove (3.16) it only remains to show that

$$K_H^\bullet(\mathbb{R}^m) \simeq RK_H^\bullet(C_\tau(\mathbb{R}^m)).$$

But because $\mathbb{R}^m$ admits an H-Spin^c structure, we have

$$RK_\bullet^H(C_\tau(\mathbb{R}^m)) \simeq RK_H^{\bullet+m}(\mathbb{R}^m), \qquad K_{\bullet+m}^H(\mathbb{R}^m) \simeq K_H^\bullet(C_\tau(\mathbb{R}^m))$$

and so we need only to show $K_H^\bullet(C_\tau(\mathbb{R}^m)) \simeq RK_H^\bullet(\mathbb{R}^m)$. This follows from Corollary 4.11 in [42] or (4.14), which completes the proof. □

Remark 37. Our Poincaré duality between equivariant K-theory and K-homology for proper actions of an almost-connected group or a discrete group is optimal in the following sense. There are two reasons preventing us from generalising the Poincaré duality to a setting beyond discrete groups and almost connected groups. One is that equivariant K-theory given by equivariant vector bundles may not be a generalised cohomology theory and so the Mayer-Vietoris sequence cannot be applied. The other is that the induction homomorphism (3.18) on K-homology from an arbitrary compact subgroup H to G is not an isomorphism in general.

4. Index Theory of K-homology Classes

Let G be an almost-connected Lie group acting properly and cocompactly on a manifold X. Recall that elements of $K_0^G(X)$ are represented by abstract elliptic operators. In this section, we study index theory associated to each element in $K_0^G(X)$ and its relation to induction from a maximal compact subgroup.

4.1. **Higher Index.** Let G and X be as above and K a maximal compact subgroup of G. Let Y be a global K-slice of X. For a proper cocompact action there exists a *cut-off* function, a nonnegative function $c \in C_c^\infty(X)$ whose integral over every orbit is 1:

$$\int_G c(g^{-1}x)dg = 1, \qquad g \in G,\, x \in X.$$

The function c gives rise to an idempotent p in $C_0(X) \rtimes G$, satisfying

$$(p(g))(x) := \sqrt{\mu(g^{-1})c(g^{-1}x)c(x)}, \qquad g \in G,\ x \in X. \tag{4.1}$$

Here μ is the modular function on G; that is, if dg is the left Haar measure on G and $s \in G$, then μ satisfies $d(gs) = \mu(s)dg$. Note that the K-theory class $[p]$ of p in $K_0(C_0(X) \rtimes G)$ is independent of the choice of c.

Definition 38 ([44])**.** The higher index map $\mathrm{index}_G : K^G_\bullet(X) \to K_\bullet(C^*_r(G))$ is given by

$$\mathrm{index}_G(x) = [p] \otimes_{C_0(X)\rtimes G} j^G_r(x), \tag{4.2}$$

where $[p] \in K_0(C_0(X) \rtimes G)$ and j^G_r is the descent homomorphism

$$j^G_r : KK^G_\bullet(A, B) \to KK_\bullet(A \rtimes_r G, B \rtimes_r G) \tag{4.3}$$

for $A = C_0(X)$ and $B = \mathbb{C}$.

Remark 39. The reduced group C^*-algebra $C^*_r(G)$ is isomorphic to the reduced crossed product $\mathbb{C} \rtimes_r G$. The higher index map (4.2) is defined using the reduced, rather than the full, group C^*-algebra, with the idempotent $[p] \in K_0(C_0(X) \rtimes_r G)$. However, we have

$$C_0(X) \rtimes G \simeq C_0(X) \rtimes_r G$$

whenever G acts properly (cf. [42, Theorem 3.13]). Thus we shall write $C_0(X) \rtimes G$ in place of $C_0(X) \rtimes_r G$. Note also that since a compact group K acts properly on a point, we have that $C^*_r(K) \simeq C^*(K)$, whereas this does not hold for a general non-compact group G. We say that G is *amenable* when $C^*_r(G) \simeq C^*(G)$.

Remark 40. If X is a classifying space for proper actions, index_G can be used to define the *analytic assembly map* in the Baum–Connes [8, 9] and Novikov [42] conjectures. For a compact group K, $K_0(C^*_r(K))$ can be identified with the representation ring $R(K)$, while $K_1(C^*_r(K)) = 0$, and index_K is the usual equivariant index.

Remark 41. When G is an almost-connected Lie group, a classifying space for proper G-actions is G/K. Assume that G/K is Spin. For every proper cocompact G-space X, there is a continuous G-equivariant proper map $p\colon X \to G/K$ (see Theorem 6). The map p_* induced on K-homology relates the equivariant indices on X and G/K via the diagram

$$\begin{array}{ccc} K^G_\bullet(X) & \xrightarrow{\text{index}_G} & K_\bullet(C^*_r(G)). \\ {\scriptstyle p_*}\downarrow & \nearrow_{\text{index}_G}^{\cong} & \\ K^G_\bullet(G/K) & & \end{array} \tag{4.4}$$

Since the Baum–Connes conjecture is true for almost-connected groups by Theorem 1.1 in [19], the equivariant index on G/K defines an isomorphism

$$\text{index}_G : K^G_\bullet(G/K) \cong K_\bullet(C^*_r(G)). \tag{4.5}$$

4.2. **Dirac Induction.** We will assume throughout this subsection and the next that G/K is Spin. We first recall the definition of the *Dirac induction* map, which is a special case of a G-index map when $X = G/K$. First, note that we have an isomorphism

$$R(K) \simeq K^G_d(G/K), \qquad d = \dim G/K.$$

If (ρ, V) is an irreducible representation of K, its image under this isomorphism is represented by the K-homology cycle of the Spin^c-Dirac operator $D_{G/K}$ on G/K coupled with V. Taking the index of this operator then defines the *Dirac induction* of the class of (ρ, V):

$$\text{D-Ind}^G_K : R(K) \to K_d(C^*_r(G)), \qquad \text{D-Ind}^G_K([V]) := [p] \otimes_{C_0(G/K)\rtimes G} j^G_r([D^V_{G/K}]). \tag{4.6}$$

This map is an isomorphism of abelian groups by the Connes–Kasparov conjecture [20, 43, 19], proved for almost-connected groups in [19] based on important earlier results in [67, 49].

Equivalently, D-Ind^G_K can be constructed as follows. Let

$$[\partial_{G/K}] \in KK^G_d(C_0(G/K), \mathbb{C})$$

denote the Dirac element on G/K. Then, as we show below, Dirac induction $R(K) \to K_d(C_r^*(G))$ can also be defined by

(4.7)

$$\text{D-Ind}_K^G([\rho]) = [\rho] \otimes_{C_0(G/K)\rtimes G} j_r^G([\partial_{G/K}]), \quad [\rho] \in R(K) \simeq K_0(C_0(G/K) \rtimes G).$$

The last isomorphism follows from the fact that $C_r^*(K)$ and $C_0(G/K) \rtimes G$ are Morita-equivalent. To see that definitions (4.6) and (4.7) coincide, the following lemma relating $C_r^*(K)$ and $C_0(G/K) \rtimes G$ on the level of K-theory is needed.

Lemma 42. *Let V be a finitely generated projective module over $C_r^*(K)$. Then V corresponds to the projective $C_0(G/K)\rtimes G$-module $p\cdot(\Gamma(G\times_K V)\rtimes G)$ under the Morita equivalence of $C_r^*(K)$ and $C_0(G/K) \rtimes G$. That is,*

$$[V] \mapsto [p \cdot (\Gamma(G \times_K V) \rtimes G)]$$

in the isomorphism

$$K_0(C_r^*(K)) \simeq K_0(C_0(G/K) \rtimes G). \tag{4.8}$$

Proof. From [24], we know that the $C_0(G/K) \rtimes G$-module $p \cdot (C_0(G/K) \rtimes G)$ and the $C_r^*(K)$-module $\mathbb{C}$ correspond under Morita equivalence. In fact, $C_c(G)$ can be regarded as a right $C_c(K)$-module and a left $C_c(G, C_c(G/K))$-module. A cut-off function c on G/K with respect to the G-action can be lifted to an element in $C_c(G)$ satisfying

$$\langle c, c\rangle_{C_c(G,C_c(G/K))} = p, \qquad \langle c, c\rangle_{C_c(K)} = 1.$$

So the projection 1 in $C_r^*(K)$ and the projection p in $C_0(G/K)\rtimes G$ correspond under Morita equivalence. In fact, this follows from:

$$\mathcal{K}(p \cdot (C_0(G/K) \rtimes G)) \simeq p \cdot (C_0(G/K) \rtimes G) \cdot p \simeq \mathbb{C},$$

where $\mathcal{K}$ is the space of compact operators on the Hilbert $C_0(G/K) \rtimes G$-module. See details in [24, Example 5.2]. Hence, the module $1 \cdot C_r^*(K)$ corresponds to $p \cdot (C_0(G/K) \rtimes G)$ under Morita equivalence:

$$1 \cdot C_r^*(K) \sim_{M.E.} p \cdot (C_0(G/K) \rtimes G).$$

Here $\cdot$ means module multiplication given by convolution with respect to K or G. One directly calculates that $1 \cdot C_r^*(K) \simeq \mathbb{C}$, so that we have

$$\mathbb{C} \sim_{M.E.} p \cdot (C_0(G/K) \rtimes G). \tag{4.9}$$

In particular, $a \in \mathbb{C}$ is identified as an element $\tilde{a} := p \cdot f_a$ in $p \cdot (C_0(G/K) \rtimes G)$ where $(f_a(g))(x) = a$ for $g \in G$ and $x \in G/K$. This means that the Lemma is true when $V = \mathbb{C}$ is the trivial representation. The proof for general V then follows by observing that (4.8) is an isomorphism of $R(K)$-modules. In fact, (4.9) implies the following module isomorphisms:

$$p \cdot (\Gamma(G \times_K \mathbb{C}) \rtimes G) \otimes_{C_c(G, C_0(G/K))} C_c(G) \simeq \mathbb{C}; \tag{4.10}$$

$$C_c(G) \otimes_{C_c(K)} \mathbb{C} \simeq p \cdot (\Gamma(G \times_K \mathbb{C}) \rtimes G). \tag{4.11}$$

Applying the $C_r^*(K)$-module V on the left in (4.10) or on the right in (4.11), we obtain versions of the isomorphisms (4.10)-(4.11) with $\mathbb{C}$ replaced by V. Equivalently, we have:

$$V \sim_{M.E.} p \cdot (\Gamma(G \times_K V) \rtimes G).$$

Here, every $v \in V$ is identified as an element $\tilde{v} := p \cdot f_v$ in $p \cdot (\Gamma(G \times_K V) \rtimes G)$, where

$$(f_v(g))(hK) = h[(1, v)], \qquad g, h \in G,\ hK \in G/K,\ [(1, v)] \in G \times_K V.$$

This completes the proof. □

Remark 43. Under the Morita equivalence $C_r^*(K) \simeq C_0(G/K) \rtimes G$, the projection p in $C_0(G/K) \rtimes G$ corresponds the constant function 1 on K. If ρ_0 is the trivial representation of K, then under the isomorphism

$$R(K) \simeq K_0(C_r^*(K)) \simeq K_0(C_0(G/K) \rtimes G),$$

we have

$$R(K) \ni [\rho_0] \leftrightarrow [p] \in K_0(C_0(G/K) \rtimes G).$$

Remark 44. The isomorphism $K_0(C_r^*(K)) \to K_0(C_0(G/K) \rtimes G)$ given by $[V] \mapsto [p \cdot (\Gamma(G \times_K V) \rtimes G)]$ is the composition of the following maps:

$$\begin{aligned}
j :& R(K) \to KK^G(C_0(G/K), C_0(G/K)) \\
& [V] \mapsto [(\Gamma(G \times_K V), 0)]; \\
j^G :& KK^G(C_0(G/K), C_0(G/K)) \to \\
& \qquad KK(C_0(G/K) \rtimes G, C_0(G/K) \rtimes G) \\
& [(\Gamma(G \times_K V), 0)] \mapsto [(\Gamma(G \times_K V) \rtimes G, 0)]; \\
[p] \otimes_{C_0(G/K) \rtimes G} :& KK(C_0(G/K) \rtimes G, C_0(G/K) \rtimes G) \to K_0(C_0(G/K) \rtimes G) \\
& [(\Gamma(G \times_K V) \rtimes G, 0)] \mapsto [p \cdot (\Gamma(G \times_K V) \rtimes G)].
\end{aligned}$$

This description helps us prove the equivalence of definitions (4.6) and (4.7).

Lemma 45. *Let $[V] \in R(K)$. Then definitions (4.6) and (4.7) coincide:*

$$\mathrm{index}_G([D^V_{G/K}]) = [p] \otimes_{C_0(G/K) \rtimes G} j^G_r([D^V_{G/K}]) = [V] \otimes_{C_0(G/K) \rtimes G} j^G_r([\partial_{G/K}]).$$

Proof. Observe by comparing KK-cycles that we have

$$j^G_r([D^V_{G/K}]) = j^G \circ j([V]) \otimes_{C_0(G/K) \rtimes G} j^G_r([\partial_{G/K}]).$$

Thus,

$$[p] \otimes_{C_0(G/K) \rtimes G} j^G_r([D^V_{G/K}]) = [p] \otimes_{C_0(G/K)} j^G \circ j([V]) \otimes_{C_0(G/K) \rtimes G} j^G_r([\partial_{G/K}]).$$

The Lemma follows from Remark 44. ∎

4.3. **Higher Index Commutes with Induction.** The main theorem of this section is the following:

Theorem 46 (Higher index commutes with induction)**.** *The following diagram commutes:*

$$\begin{array}{ccc}
K^K_\bullet(Y) & \xrightarrow{\mathrm{index}_K} & K_\bullet(C_r^*(K)) \\
\Big\downarrow{\scriptstyle \text{K-Ind}^G_K} & & \Big\downarrow{\scriptstyle \text{D-Ind}^G_K} \\
K^G_{\bullet+d}(X) & \xrightarrow{\mathrm{index}_G} & K_{\bullet+d}(C_r^*(G)).
\end{array} \tag{4.12}$$

Proof. We only need to give the proof for $\bullet = 0$. Consider the G-equivariant maps $\lambda : Y \to pt$, $\tilde{\lambda} : X = G \times_K Y \to G/K$ and the contravariant maps on algebras

$$\lambda' : \mathbb{C} \to C_0(Y), \qquad \tilde{\lambda}' : C_0(G/K) \to C_0(X).$$

Let $(\mathcal{H}, f, F)$ be a Kasparov cycle for $K_0^K(Y)$. Then the commutativity of

$$\begin{array}{ccc} f : C_0(Y) \to \mathcal{L}(\mathcal{H}) & \longrightarrow & f \circ \lambda' \\ \downarrow & & \downarrow \\ \tilde{f} : C_0(X) \to \mathcal{L}(G \times_K \mathcal{H}) & \longrightarrow & \tilde{f} \circ \tilde{\lambda}' = \widetilde{f \circ \lambda'} \end{array}$$

implies the commutativity of

$$\begin{array}{ccc} K_0^K(Y) & \xrightarrow{\lambda_*} & K_0^K(pt) \\ \simeq\downarrow & & \simeq\downarrow \\ KK^G(C_0(X), C_0(G/K)) & \xrightarrow{\tilde{\lambda}_*} & KK^G(C_0(G/K), C_0(G/K)). \end{array} \tag{4.13}$$

Every element in $K_0^K(pt) \simeq KK^K(\mathbb{C}, \mathbb{C})$ can be represented by a K-vector space $[V]$, or a finite-dimensional representation of K. Regarding V as a $C_r^*(K)$-module, the higher index is the identity map

$$K_0^K(pt) \to K_0(C_r^*(K)), \qquad \mathrm{index}_K([V]) = [V].$$

So in the diagram

$$\begin{array}{ccc} KK^K(\mathbb{C}, \mathbb{C}) & \xrightarrow{\mathrm{index}_K} & K_0(C_r^*(K)) \\ \simeq\downarrow & & \simeq\downarrow \\ KK^G(C_0(G/K), C_0(G/K)) & \xrightarrow{\mathrm{index}_G} & K_0(C_0(G/K) \rtimes G), \end{array} \tag{4.14}$$

the elements are mapped as

$$\begin{array}{ccc} [(V, 0)] & \xrightarrow{\mathrm{index}_K} & [(V, 0)] \\ \downarrow & & \downarrow \\ [(\Gamma(G \times_K V), 0)] & \xrightarrow{\mathrm{index}_G} & [(p \cdot (\Gamma(G \times_K V) \rtimes G), 0)]. \end{array}$$

Thus, diagram (4.14) commutes if V and $p\cdot(\Gamma(G\times_K V)\rtimes G)$ are Morita-equivalent as modules; but this has been proved in Lemma 42.

Noting that the higher index map factors through the higher index map for classifying spaces (cf. (4.4)), commutativity of (4.13)-(4.14) implies that the following diagram commutes (cf. (2.8) for the left isomorphism):

$$\begin{array}{ccc} K_0^K(Y) & \xrightarrow{\text{index}_K} & K_0(C_r^*(K)) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ KK^G(C_0(X),C_0(G/K)) & \xrightarrow{\text{index}_G} & K_0(C_0(G/K)\rtimes G). \end{array} \tag{4.15}$$

Finally, note that for $x\in KK^G(C_0(X),C_0(G/K))$, we have

$$\begin{aligned} &\text{index}_G(x)\otimes_{C_0(G/K)\rtimes G} j_r^G([\partial_{G/K}]) \\ =&\big([p]\otimes_{C_0(X)\rtimes G} j^G(x)\big)\otimes_{C_0(G/K)\rtimes G} j_r^G([\partial_{G/K}]) \\ =&[p]\otimes_{C_0(X)\rtimes G}\big(j^G(x)\otimes_{C_0(G/K)\rtimes G} j_r^G([\partial_{G/K}])\big) \\ =&[p]\otimes_{C_0(X)\rtimes G} j_r^G(x\otimes_{C_0(G/K)}[\partial_{G/K}]) \\ =&\,\text{index}_G(x\otimes_{C_0(G/K)}[\partial_{G/K}]). \end{aligned}$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} KK^G(C_0(X),C_0(G/K)) & \xrightarrow{\text{index}_G} & K_0(C_0(G/K)\rtimes G)) \\ \big\downarrow{\scriptstyle\otimes_{C_0(G/K)}[\partial_{G/K}]} & & \big\downarrow{\scriptstyle\otimes_{C_0(G/K)\rtimes G} j_r^G([\partial_{G/K}])} \\ K_d^G(X) & \xrightarrow{\text{index}_G} & K_d(C_r^*(G)). \end{array} \tag{4.16}$$

Putting together (4.15) and (4.16) completes the proof. □

Remark 47. The commutative diagram in Theorem 46 is closely related to the *quantisation commutes with induction* results of [33, 34]. We acknowledge that Theorem 46 was proved in [33] by more involved diagram chasing. Here we have given a more direct proof by making use of the properties of KK-theory.

We end this subsection by stating an implication of Remark 20 and Theorem 46.

Corollary 48. *For every $x \in K_\bullet^G(X)$, there exists a geometric cycle (M, E, f) representing an element in $K_\bullet^{G,geo}(X)$ such that $x = f_*([D_{M,E}])$, and upon choosing a K-slice N of M with a compatible K-equivariant* Spin^c*-structure, we have*

$$\begin{aligned} \operatorname{index}_G x &= \operatorname{index}_G f_*([D_{M,E}]) \\ &= \text{D-Ind}_K^G(\operatorname{index}_K(f|_N)_*([D_{N,E|_N}]))) \in K_\bullet(C_r^*(G)). \end{aligned}$$

5. A Trace Theorem for Equivariant Index

We now focus on the case when G is a connected, semisimple Lie group with finite centre and K a maximal compact subgroup, such that G/K is Spin. We further assume that $d = \dim G/K$ is even and rank G = rank K, in order to allow G to have discrete series representations. Let M be a G-equivariant Spin^c-manifold with Dirac operator D_M and N a K-slice of G with Dirac operator D_N associated with a compatible (see Remark 10) K-equivariant Spin^c-structure on N. We shall use the commutativity result in Section 4 together with known representation-theoretic properties of the group $K_0(C_r^*(G))$ for such G to deduce a formula for the L^2-index of D_M in terms of the L^2-index of D_N. Since twisted Spin^c-Dirac operators exhaust $K_0^G(M)$, this allows us to relate the L^2-index of any operator representing a class in $K_0^G(M)$ to the L^2-index of K-equivariant Dirac operators in $K_0^K(N)$.

Remark 49. For simplicity, we have assumed that M has a G-equivariant Spin^c-structure. Of course, since the commutativity result of Section 4 did not require this, all of the statements in this section apply in an appropriate form to operators on any proper G-cocompact manifold X, not necessarily having a Spin^c-structure, and a compact K-slice Y.

Let $\mathcal{R}_G \subseteq \mathcal{B}(L^2(G))$ be the commutant of the right-regular representation R of G. Then $\mathcal{R}_G$ is a von Neumann algebra with a faithful, normal, semi-finite trace τ determined by

$$\tau(R(f)^* R(f)) = \int_G |f(g)|^2 \, dg,$$

for all $f \in L^2(G)$ such that $R(f)$ is a bounded operator on $L^2(G)$. Let E be a $\mathbb{Z}_2$-graded G-equivariant vector bundle over M, and let $E \cong G \times_K (E|_N)$ be a fixed choice of decomposition. Denote by $\mathcal{R}_G(E)$ the K-invariant elements of

$$\mathcal{R}_G \otimes \mathcal{B}(L^2(N, E|_N)).$$

Then $\mathcal{R}_G(E)$ is equipped with a natural trace, which we will also denote by τ, given by combining τ on $\mathcal{R}_G$ with the trace on bounded operators.

Let $D = D^+ \oplus D^-$ be an odd-graded first-order G-invariant elliptic operator acting on E and defining a class in $K_0^G(M)$. By definition, the *L^2-index* of D is (see Connes-Moscovici [22] or Wang [66])

$$L^2\text{-index}(D) := \tau(\mathrm{pr}_{\ker D^+}) - \tau(\mathrm{pr}_{\ker D^-}),$$

where pr_L denotes the projection of $L^2(M, E)$ onto a G-invariant subspace L. It was shown in [66] that the L^2-index of D can be calculated via the composition of maps

$$K_0^G(M) \xrightarrow{\mathrm{index}_G} K_0(C_r^*(G)) \xrightarrow{\tau_G} \mathbb{R},$$

where τ_G is the von Neumann trace on $C_r^*(G)$, defined first on the dense subspace $C_c(G)$ by evaluation at the identity $e \in G$:

$$\tau_G : C_c(G) \to \mathbb{C}, \qquad f \mapsto f(e),$$

and extended to $C_r^*(G)$. The same paper provides an integral formula for the L^2-index:

$$L^2\text{-index}(D) = \int_{TM} (c \circ \pi)(\hat{A}(M))^2 \cup \mathrm{ch}(\sigma_D), \tag{5.1}$$

where $\pi : TM \to M$ is the natural projection and c is a compactly supported cut-off function on M, with the property that $\int_G c(g^{-1}x)\, dg = 1$ for any $x \in M$. Since we have assumed that the manifold M is Spin^c, this integral is equal to

$$\int_M c(x) e^{\frac{1}{2} c_1(L_M)} \hat{A}(M),$$

where L_M is the line bundle defining the G-equivariant Spin^c-structure on M.

On the other hand, it is known that, for our chosen class of Lie groups G, the discrete series representations of G can be identified with elements of $K_0(C_r^*(G))$ (although not necessarily all of them) in the following way. Let (H, π) be a discrete series representation of G. Let d_H be the *formal degree* of (H, π) - equal to its mass in the Plancherel measure - and c_v be a *matrix coefficient* of the representation, given by

$$c_v : G \to \mathbb{C}, \qquad c_v(g) := \langle v, \pi(g)v \rangle,$$

for some $v \in H, ||v|| = 1$ (it is independent of the choice of v). It can be verified that $d_H c_v$ is a projection in the C^*-algebra $C_r^*(G)$ and hence defines an element in its K-theory. It is also known that the von Neumann trace τ_G (defined above) of the class $[d_H c_v] \in K_0(C_r^*(G))$ is equal to $(-1)^{d/2}$ times the formal degree (see [48]).

For a more extensive discussion we refer to [48]. It is shown there that the map taking (H, π) to $[d_H c_v] \in K_0(C_r^*(G))$ is an injection from the set of discrete series representations to a set of generating elements for $K_0(C_r^*(G))$ [48].

It is interesting to note that for connected semisimple Lie groups having discrete series, including $G = SL(2, \mathbb{R})$, the elements of $K_0(C_r^*(G))$ can be given *entirely*, in the aforesaid way, as representations from the discrete series and limits of discrete series of G. See [67]. This motivates us to understand $K_0(C_r^*(G))$, as well as G-equivariant index theory, from the point of view of representation theory of G. By comparison, the theory of representations of compact groups already plays an important role in index theory. The goal of this section is to observe an explicit representation-theoretic relationship between a G-equivariant index and its corresponding K-equivariant index, in the sense of the previous section.

The elements of $K_0(C_r^*(G))$, considered as representations of G, are in bijection with the irreducible representations of K via the Dirac induction map D-Ind (see [9]). On the other hand, there exists a formula for the formal degree of a discrete series representation of G in terms of the highest weight of a corresponding K-representation together with information about the root systems of G and K (see [7] or [49]).

To state this formula precisely, let us first fix the following notation. Let T be a common maximal torus of K and G, and choose a Weyl chamber C for the root sytem Φ of $\mathfrak{g}$, determining a set Φ^+ of positive roots. Pick the Weyl chamber for $\mathfrak{k}$ that contains C, and let Φ_c^+ be the system of compact roots. Let $\Phi_n^+ := \Phi^+ \backslash \Phi_c^+$, ρ_c and ρ_n be the half-sums of the positive-compact and non-compact roots respectively, and let $\rho := \rho_c + \rho_n$.

We can relate the formula for the formal degree proved in [48] to the commutativity result Section 4, as follows. This gives an interpretation of a result of Atiyah and Schmid (p. 25 of [7]) in terms of equivariant index theory.

Proposition 50. *Let G and K be as above, with G/K Spin. Let μ be the highest weight of an irreducible representation V_μ of K. Suppose $\mu + \rho_c$ is the Harish-Chandra parameter of a discrete series representation of (H, π) of G. Then the formal degree of (H, π) is given by*

$$d_H = (-1)^{d/2}\, \tau_G([\text{D-Ind}([V_\mu])]) = \prod_{\alpha \in \Phi^+} \frac{(\mu + \rho_c, \alpha)}{(\rho, \alpha)},$$

with both sides vanishing when the class $\text{D-Ind}[V_\mu] \in K_0(C_r^*(G))$ *is not given by a discrete series representation. We have a well-defined commutative diagram:*

$$\begin{array}{ccccc}
K_0^G(M) & \xrightarrow{\text{index}_G} & K_0(C_r^*(G)) & & \\
\uparrow{\scriptstyle \text{K-Ind}} & & \uparrow{\scriptstyle \text{D-Ind}} & \searrow^{\tau_G} & \\
 & & & & \mathbb{R}, \\
 & & & \nearrow_{\Pi_K} & \\
K_0^K(N) & \xrightarrow{\text{index}_K} & R(K) & &
\end{array}$$

where $\Pi_K([V_\mu]) := (-1)^{d/2} \prod_{\alpha \in \Phi^+} \frac{(\mu+\rho_c,\alpha)}{(\rho,\alpha)}$. *Here the Haar measure on G is normalised by*

$$\text{vol } K = \text{vol } M_1/K_1 = 1,$$

where M_1 is a maximal compact subgroup of the universal complexification $G^{\mathbb{C}}$ of G and $K_1 < G_1$ a maximal compact subgroup of a real form G_1 of $G^{\mathbb{C}}$ (see [7] for more details).

Remark 51. We have $\tau_G(\text{D-Ind}[V_\mu]) = 0$ when $\text{D-Ind}[V_\mu] \in K_0(C_r^*(G))$ is not given by a discrete series representation of G, since the Harish-Chandra parameter is then singular, hence the right-hand side of the formula above vanishes.

We can state this result in a more suggestive way that makes clear the relationship between the von Neumann traces τ_G on $K_0(C_r^*(G))$ and τ_K on $R(K) \cong K_0(C_r^*(K))$.

Proposition 52. *With the notation above, we have:*

$$\tau_G([\text{D-Ind}([V_\mu])]) = (-1)^{d/2} \left(\frac{\prod_{\alpha\in\Phi_n^+}(\mu+\rho_c,\alpha)\prod_{\alpha\in\Phi_c^+}(\rho_c,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)} \right) \tau_K([V_\mu]).$$

Proof. By the Weyl dimension formula for V_μ and the Proposition above,

$$\begin{aligned}
&\tau_G([\text{D-Ind}([V_\mu])]) = (-1)^{d/2}\, d_H \\
=&(-1)^{d/2} \prod_{\alpha\in\Phi^+} \frac{(\mu+\rho_c,\alpha)}{(\rho,\alpha)} \\
=&(-1)^{d/2} \prod_{\alpha\in\Phi_c^+} \frac{(\mu+\rho_c,\alpha)}{(\rho_c,\alpha)} \left(\frac{\prod_{\alpha\in\Phi_n^+}(\mu+\rho_c,\alpha)\prod_{\alpha\in\Phi_c^+}(\rho_c,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)} \right) \\
=&(-1)^{d/2} \dim V_\mu \left(\frac{\prod_{\alpha\in\Phi_n^+}(\mu+\rho_c,\alpha)\prod_{\alpha\in\Phi_c^+}(\rho_c,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)} \right).
\end{aligned}$$

It can be verified that the trace τ_K applied to a general element $[V] \in R(K) \cong K_0(C_r^*(K))$ returns the dimension of the representation space V, which concludes the proof. ∎

Thus τ_G and τ_K are related by a scalar factor that, up to a sign, depends only upon the highest weight μ of the representation $[V_\mu] \in R(K)$ and the chosen root systems of $\mathfrak{g}$ and $\mathfrak{k}$. In particular, as one sees by direct computation in the example $G = SL(2,\mathbb{R})$ and $K = SO(2)$, this scalar factor varies significantly depending on the weight μ (see [9] for more details on the calculation in this case), reflecting the fact that Dirac induction plays a significant role in relating the two L^2-indices for G and K.

Finally, as was mentioned in earlier, the von Neumann traces τ_G and τ_K give rise to L^2-indices when applied to the G and K-equivariant indices of operators on M and N respectively. Using Wang's formula for the L^2-index (5.1), the result above becomes an equality of integrals involving characteristic classes on the non-compact manifold M and the compact manifold N, as follows.

Corollary 53. *Let D_M and D_N be Spin^c-Dirac operators on M and N for compatible equivariant Spin^c-structures, which are defined by line bundles L_M and L_N respectively, where* $\mathrm{index}_G(D_M) \in K_0(C^*_r(G))$ *corresponds to a discrete series representation with Harish-Chandra parameter $\mu + \rho_c$. Then the L^2-indices of D_M and D_N are related by*

$$\begin{aligned}
L^2\text{-index}(D_M) &= \int_M c(x) e^{\frac{1}{2}c_1(L_M)} \hat{A}(M) \\
=&(-1)^{d/2} \left(\frac{\prod_{\alpha\in\Phi_n^+}(\mu+\rho_c,\alpha)\prod_{\alpha\in\Phi_c^+}(\rho_c,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)} \right) \int_N e^{\frac{1}{2}c_1(L_N)} \hat{A}(N) \\
=&(-1)^{d/2} \left(\frac{\prod_{\alpha\in\Phi_n^+}(\mu+\rho_c,\alpha)\prod_{\alpha\in\Phi_c^+}(\rho_c,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)} \right) L^2\text{-index}(D_N).
\end{aligned}$$

6. Positive Scalar Curvature for Proper Cocompact Actions

We now apply equivariant index theory to study obstructions to the existence of invariant metrics of positive scalar curvature on proper cocompact manifolds, before studying the existence of such metrics in the next subsection.

6.1. **Obstructions.** Suppose that G is a non-compact Lie group acting properly and cocompactly on a smooth, G-equivariantly spin, complete Riemannian manifold M. Suppose that the dimension of M is even. By the Lichnerowicz formula [52] for the square of the G-Spin-Dirac operator $\partial\!\!\!/$,

$$\partial\!\!\!/^2 = (\nabla^S)^\dagger \nabla^S + \kappa_M/4,$$

where κ_M denotes the scalar curvature of M and ∇^S denotes the connection on the spinor bundle that is induced by the Levi-Civita connection on M. Suppose now that M has pointwise-positive scalar curvature. Since M/G is compact, we have $\inf(\kappa_M) > 0$, and thus $\partial\!\!\!/^2$ is a strictly positive operator from the Sobolev space $L^{2,2}(S)$ to the Hilbert space $L^2(S)$, with a bounded inverse

$$(\partial\!\!\!/^2)^{-1} : L^2(S) \to L^{2,2}(S)$$

(see Theorem 2.11 in [29]). Let $C^*(G)$ be the maximal group C^*-algebra of G. The space $C_c^\infty(S)$ of smooth cocompactly supported sections of S has the structure of a right pre-Hilbert module over $C_c(G)$ prescribed by the formulas (for more details see [41] Section 5):

$$(s \cdot b)(x) = \int_G g(s)(x) \cdot b(g^{-1})\, dg \in C_c^\infty(S),$$
$$(s_1, s_2)(g) = \int_M (s_1(x), g(s_2)(x))\, d\mu \in C_c^\infty(G),$$

for $s, s_1, s_2 \in C_c^\infty(S)$ and $b \in C_c^\infty(G)$. Let us denote by $\mathcal{E}$ the Hilbert $C^*(G)$-module completion $\mathcal{E}$ of $C_c^\infty(S)$. Theorem 5.8 of [41] shows that the pseudodifferential operator $F = \partial\!\!\!/(\partial\!\!\!/^2 + 1)^{-\frac{1}{2}} : \Gamma(S) \to \Gamma(S)$ defines an element, which we will also call F, in the bounded adjointable operators $\mathcal{L}(\mathcal{E})$, with an index in $K_0(C^*(G))$.

The main result of this section is the following theorem on the vanishing of the equivariant index.

Theorem 54 (Vanishing Theorem 1)**.** *Let M, G, S and $\not{\partial}$ be as above, with M being even-dimensional. Suppose M admits a Riemannian metric of pointwise-positive scalar curvature. Then*

$$\mathrm{index}_G(\not{\partial}) = 0 \in K_0(C^*(G)).$$

Proof. The operator $\not{\partial} : C_c^\infty(S) \to C_c^\infty(S)$ gives a densely-defined (and a priori unbounded) operator on the Hilbert $C^*(G)$-module $\mathcal{E}$. By G-invariance, one can verify that $\not{\partial}$ is symmetric: that is, we have $C_c^\infty(S) \subseteq \mathrm{dom}(\not{\partial}^*)$, where

$$\mathrm{dom}(\not{\partial}^*) := \{y \in \mathcal{E} : \exists z \in \mathcal{E} \text{ with } \langle \not{\partial} x, y\rangle_{\mathcal{E}} = \langle x, z\rangle_{\mathcal{E}} \ \ \forall x \in C_c^\infty(S)\}.$$

Hence $\not{\partial}$ is a closable operator, and we shall write $\mathrm{dom}(\not{\partial})$ for the domain of its closure, a closed Hilbert $C^*(G)$-submodule of $\mathcal{E}$. In what follows we will be concerned with the closed operator $\not{\partial} : \mathrm{dom}(\not{\partial}) \to \mathcal{E}$. Define the *graph norm* $||\cdot||_{\mathrm{dom}(\not{\partial})}$ on $\mathrm{dom}(\not{\partial})$ to be the norm induced by the $C^*(G)$-valued inner product

$$\langle u, v\rangle_{\mathrm{dom}(\not{\partial})} := \langle u, v\rangle_{\mathcal{E}} + \langle \not{\partial} u, \not{\partial} v\rangle_{\mathcal{E}}.$$

With this norm, $\not{\partial}$ is a bounded adjointable operator $\mathrm{dom}(\not{\partial}) \to \mathcal{E}$. Since $\not{\partial}$ is a first-order elliptic differential operator on a G-cocompact manifold, Theorem 5.8 of [41] implies that both operators $\not{\partial} \pm i$ have dense range as operators on $\mathcal{E}$. Thus $\not{\partial}$ is a self-adjoint regular operator on the Hilbert $C^*(G)$-module $\mathcal{E}$ [50].

Our main task is to show that $\not{\partial}$ has an inverse $\not{\partial}^{-1} : \mathcal{E} \to \mathrm{dom}(\not{\partial})$ that is bounded and adjointable. First we show that its square $\not{\partial}^2 : \mathrm{dom}(\not{\partial}^2) \to \mathcal{E}$ is invertible in the bounded adjointable sense. Here the domain

$$\mathrm{dom}(\not{\partial}^2) := \{u \in \mathrm{dom}(\not{\partial}) : \not{\partial} u \in \mathrm{dom}(\not{\partial})\}$$

is equipped with the graph norm induced by the inner product

$$\langle u, v\rangle_{\mathrm{dom}(\not{\partial}^2)} := \langle u, v\rangle_{\mathcal{E}} + \langle \not{\partial} u, \not{\partial} v\rangle_{\mathcal{E}} + \langle \not{\partial}^2 u, \not{\partial}^2 v\rangle_{\mathcal{E}}.$$

By Proposition 1.20 of [23], $\not{\partial}^2$ is a densely-defined, closed, self-adjoint regular operator on $\mathcal{E}$. We proceed along the lines of [18] Proposition 4.9. By regularity, $\not{\partial}^2 + \mu^2$ is surjective for every positive number μ^2 (see Chapter 9 of [50]). Further, since $\not{\partial}^2$ is strictly positive, $\not{\partial}^2 + \mu^2$ is injective. By the open mapping theorem, its inverse $(\not{\partial}^2 + \mu^2)^{-1}$ is bounded. It remains to show that $(\not{\partial}^2 + \mu^2)^{-1}$ is adjointable. Write for short $B := (\not{\partial}^2 + \mu^2)^{-1}$. Note that B is self-adjoint as a bounded operator $\mathcal{E} \to \mathcal{E}$ (defined by composing B with the bounded inclusion $\mathrm{dom}(\not{\partial}^2) \hookrightarrow \mathcal{E}$), which follows from Lemma 4.1 in [50] and the estimate

$$\langle u, Bu\rangle_{\mathcal{E}} = \langle(\not{\partial} + \mu^2)Bu, Bu\rangle_{\mathcal{E}} \geq \mu^2\langle Bu, Bu\rangle_{\mathcal{E}} \geq 0.$$

Next, for any $w \in \mathcal{E}$ and $u \in \mathrm{dom}(\not{\partial}^2)$, we have

$$\begin{aligned}\langle Bu, w\rangle_{\mathrm{dom}(\not{\partial}^2)} &= \langle \not{\partial}^2 Bu, \not{\partial}^2 w\rangle_{\mathcal{E}} + \langle \not{\partial} Bu, \not{\partial} w\rangle_{\mathcal{E}} + \langle Bu, w\rangle_{\mathcal{E}} \\ &= \langle(\not{\partial}^2 + \mu^2)Bu, \not{\partial}^2 w\rangle_{\mathcal{E}} + (1-\mu^2)\langle Bu, \not{\partial}^2 w\rangle_{\mathcal{E}} + \langle u, Bw\rangle_{\mathcal{E}} \\ &= \langle u, \not{\partial}^2 w\rangle_{\mathcal{E}} + (1-\mu^2)\langle u, B\not{\partial}^2 w\rangle_{\mathcal{E}} + \langle u, Bw\rangle_{\mathcal{E}} \\ &= \langle u, (\not{\partial}^2 + (1-\mu^2)B\not{\partial}^2 + B)w\rangle_{\mathcal{E}},\end{aligned}$$

where we used symmetry of $\not{\partial}$ in $\mathcal{E}$ self-adjointness of B as shown above. This shows that $(\not{\partial}^2 + \mu^2)^{-1} \in \mathcal{L}(\mathcal{E}, \mathrm{dom}(\not{\partial}^2))$.

We claim that $\not{\partial}^2$ is invertible. For write it as $(1 - \mu^2(\not{\partial}^2 + \mu^2)^{-1})(\not{\partial}^2 + \mu^2)$. Since $\not{\partial}^2$ is a strictly positive operator, there exists $C > 0$ such that for all $s \in \mathrm{dom}(\not{\partial}^2)$, we have $\langle \not{\partial}^2 s, s\rangle_{\mathcal{E}} \geq C\langle s, s\rangle_{\mathcal{E}}$. It follows from the Cauchy-Schwarz inequality for Hilbert modules that for any element $t \in \mathcal{E}$,

$$||\mu^2(\not{\partial}^2 + \mu^2)^{-1}t||_{\mathcal{E}} \leq \frac{\mu^2}{\mu^2 + C}||(\not{\partial}^2 + \mu^2)Bt||_{\mathcal{E}} = \frac{\mu^2}{\mu^2 + C}||t||_{\mathcal{E}}.$$

Hence $(1 - \mu^2(\not{\partial}^2 + \mu^2)^{-1})$ has an adjointable inverse given by a Neumann series. It follows that $\not{\partial}^2$ has a bounded adjointable inverse

$$(\not{\partial}^2)^{-1} = B(1 - \mu^2(\not{\partial} + \mu^2)^{-1})^{-1} : \mathcal{E} \to \mathrm{dom}(\not{\partial}^2).$$

One verifies that $\not{\partial}(\not{\partial}^2)^{-1} : \mathcal{E} \to \mathrm{dom}(\not{\partial})$ is a two-sided inverse for $\not{\partial}$. Thus $\not{\partial} + \lambda$ is invertible for all λ in a ball around $0 \in \mathbb{C}$. Thus, by the functional calculus for self-adjoint regular operators (see Theorem 1.19 of [23] for a list of its properties), we may define the operator

$$\frac{\sqrt{\not{\partial}^2+1}}{\not{\partial}} \in \mathcal{L}(\mathcal{E}),$$

which is the inverse of $F = \frac{\not{\partial}}{\sqrt{\not{\partial}^2+1}} \in \mathcal{L}(\mathcal{E})$ by the homomorphism property of the functional calculus. Recall that the G-equivariant index of $\not{\partial}$ can be computed by taking trace of the idempotent element

$$\begin{bmatrix} S_0^2 & S_0(1+S_0)Q \\ S_1 F & 1-S_1^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{Mat}_\infty(\mathcal{K}(\mathcal{E})),$$

where Q is any parametrix of F and $S_0 = 1 - QF$, $S_1 = 1 - FQ$ (see for example the first display on p. 353 of [21]). In our setting we can take $Q = F^{-1}$, hence $S_0 = S_1 = 0$. It then follows that the G-equivariant index of $\not{\partial}$ vanishes, after making the canonical identification $K_0(\mathcal{K}(\mathcal{E})) \cong K_0(C^*(G))$. □

Now, as defined in [55], integration $\int_G$ on $L^1(G)$ induces a tracial map on $C^*(G)$ and also on its K-theory. By Appendix A of this paper, we have $\int_G(\mathrm{index}_G(\not{\partial})) = \mathrm{ind}_G(\not{\partial})$ where the right-hand side is the G-invariant index of Mathai-Zhang [55]. Thus we immediately obtain as a corollary the following recent result of Zhang [68]:

Corollary 55 ([68])**.** *Let M, G and $\not{\partial}$ be as in Theorem 54, with M being even-dimensional. Then*

$$\mathrm{ind}_G(\not{\partial}) = 0,$$

where ind_G *denotes the Mathai-Zhang index.*

Using the canonical projection from $C^*(G)$ to $C^*_r(G)$, Theorem 54 also gives a vanishing result for the reduced equivariant index:

Theorem 56 (Vanishing Theorem 2)**.** *Let M, G and $\not{\partial}$ be as in Theorem 54, with M being even-dimensional. Then*

$$\mathrm{index}_G(\not{\partial}) = 0 \in K_0(C^*_r(G)).$$

Upon applying the von Neumann trace τ to this index, we conclude that

$$\mathrm{index}_{L^2}(\partial\!\!\!/) = \tau(\mathrm{index}_G(\partial\!\!\!/)) = 0,$$

where the L^2-index on the left is as defined in [66]. Let $c \in C_c^\infty(M)$ be a cut-off function, that is to say a non-negative function satisfying

$$\int_G c(g^{-1}m)\, dg = 1$$

for all $m \in M$, for a fixed left Haar measure dg on G. The *averaged $\hat{A}$-genus* of the action by G on M is then defined to be (see also [28])

$$\hat{A}_G(M) := \int_M c\hat{A}_M.$$

We have as a consequence of Vanishing Theorem 2, together with the L^2-index theorem of Wang [66], the following result proved by Fukumoto (Corollary B in [28]):

Corollary 57 ([28])**.** *Let M and G be as in Theorem 54, and assume further that G is unimodular. Then $\hat{A}_G(M) = 0$.*

6.2. **Existence.** The previous obstruction results are motivated by the following existence results. In this subsection, we suppose that G is almost-connected, while M is still a proper G-cocompact manifold. By Abels' slice theorem [1], M is G-equivariantly diffeomorphic to $G \times_K N$, where K is a maximal compact subgroup of G, N is a compact manifold with an action of K. One of our main theorems is the following.

Theorem 58. *Let G be an almost-connected Lie group and that K is a maximal compact subgroup of G. If N is a compact manifold with a K-invariant Riemannian metric of positive scalar curvature, then $M = G \times_K N$ has a G-invariant Riemannian metric of pointwise-positive scalar curvature.*

To prove it, recall the following theorem of Vilms:

Theorem 59 ([65])**.** *Let $\pi\colon M \to B$ be a fibre bundle with fibre N and structure group K. Let g_B be a Riemannian metric on B and g_N be a K-invariant Riemannian metric on N. Then there is a Riemannian metric g_M on M such that π is a Riemannian submersion with totally geodesic fibres.*

We prove an equivariant version this result, namely:

Theorem 60. *Let $\pi\colon M \to B$ be a fibre bundle with compact fibre N and structure group K. Suppose that M and B both have proper G-actions making π G-equivariant. Let g_N be a K-invariant Riemannian metric on N. Then there is a G-invariant Riemannian metric g_M on M such that π is a G-equivariant Riemannian submersion with totally geodesic fibres.*

Proof. We adapt the proof of Theorem 3.5 in [65]. Let us choose an Ehresmann connection H on M (with the horizontal lifting property), so that $TM = H \oplus V$, where V is the vertical subbundle $\ker \pi_*$. By a result of Palais [57] there exists a G-invariant metric g_B on B. Using the map $\pi_*|_H$, this induces a G-invariant metric g_H on H. Also g_N is K-invariant and so can be transferred to a metric g_V on V. Define the metric $g_M = \langle\, ,\rangle_M$ as the direct sum $g_H \oplus g_V$. One verifies that π is a Riemannian submersion. In what follows, let the Levi-Civita connections on M and N be denoted D and D^N respectively.

Note that parallel transport along a curve in B gives a K-isomorphism of fibres of M. Since the metric on N is K-invariant, the above construction of g_M means that parallel transport is furthermore an isometry of the fibres.

To show that the fibres of M are totally geodesic, it suffices to show that D and D^N act in the same way on tangent vectors to curves in any given fibre. Thus take any curve $\alpha : I \to M_b$, where M_b is the fibre over a base point $b \in B$, and parameterise it proportionally to arc-length. Let $v(0)$ be a given horizontal vector at $\alpha(0)$, and define $\sigma(t, s)$ to be the parallel translation of $\alpha(t)$ along some curve in B with initial vector $\pi_* v(0)$. Define the horizontal vector field

$$v(t) := \frac{\partial}{\partial s}\sigma(t, 0)$$

along $\alpha(t)$. For each s, let $f(s)$ denote the arc-length of the curve $\sigma(s, t)$ from $t = 0$ to 1. Since parallel transport is an isometry, the arc-length $f(s)$ is a constant with respect to s. In particular, we have $f'(0) = 0$. By the formula for the first variation of the arc-length [31], we have

$$f'(0) = \int_0^1 \frac{1}{||\alpha'(0)||} \langle D_t\alpha'(t), v(t)\rangle_M \, dt = \int_0^1 \frac{1}{f(0)} \langle D_t\alpha'(t), v(t)\rangle_M \, dt = 0,$$

whence by continuity, there must be $t_1 \in I$ such that $\langle D_t\alpha'(t_1), v(t_1)\rangle_M = 0$. Next let $\alpha_1(t) := \alpha(\frac{t}{2})$, $0 \le t \le 1$, and apply the same procedure. Since $D_t\alpha_1'(t)$ and $D_t\alpha'(t)$ are proportional, we have $\langle D_t\alpha'(t_2), v(t_2)\rangle_M = 0$ for some $0 \le t_2 \le \frac{1}{2}$. Repeating this procedure gives a sequence $t_i \to 0$ for which the quantity $\langle D_t\alpha'(t), v(t)\rangle_M$ vanishes, which implies that $\langle D_t\alpha'(0), v(0)\rangle_M = 0$.

As $v(0)$ was arbitrarily chosen in H_b, we conclude that $D_t\alpha'(0)$ is vertical and so equal to $D_t^N\alpha'(0)$. This concludes the proof that the fibres are totally geodesic. ☐

Proof of Theorem 58. Let $\kappa_{G/K}$ denote the scalar curvature of the G-invariant Riemannian metric $g_{G/K}$ on the base. Note that since G/K is a homogeneous space, $-\infty < \kappa_{G/K} < 0$ is a negative constant. Let $H \subseteq TM$ be an Ehresmann connection. Then as in the proof of Theorem 60 above, we may lift $g_{G/K}$ to a G-invariant metric g_H on H, as well as lift the K-invariant Riemannian metric g_N on N to a metric on the vertical subbundle $V \subseteq TM$. Define a G-invariant metric on M by $g_M := g_H \oplus g_V$.

Since N is compact by hypothesis, its scalar curvature κ_N satisfies $\inf\{\kappa_N\} =: \kappa_0 > 0$. Now let T and A denote the O'Neill tensors of the submersion π (their definitions can be found in [56]). By Theorem 60 above, the fibres of M are totally geodesic, so $T = 0$. Pick an orthonormal basis of horizontal vector fields $\{X_i\}$. By G-invariance, we have that for any point $p \in M$,

$$\sum_{i,j} ||A_{X_i}(X_j)||_p = \sum_{i,j} ||A_{X_i}(X_j)||_{gp}$$

for all group elements $g \in G$. This means $\sup_{p\in X}\{\sum_{i,j} ||A_{X_i}(X_j)||_p\} =: A_0 < \infty$, as M/G is compact. Now by a result of Kramer ([47] p. 596), we can relate the scalar curvatures by

$$\kappa_M(p) = \kappa_{G/K} + \kappa_N(p) - \sum_{i,j} ||A_{X_i}(X_j)||_p.$$

Upon scaling the fibre metric on N by a positive factor t, we obtain

$$\kappa_M(p) \geq \kappa_{G/K} + t^{-2}\kappa_0 - A_0 > 0 \qquad \text{whenever} \quad 0 < t < \sqrt{\frac{\kappa_0}{-\kappa_{G/K} + A_0}}.$$

Thus g_M is a G-invariant metric of positive scalar curvature on M. □

This enables us to establish the following existence theorem for PSC metrics:

Theorem 61. *Let G be an almost-connected Lie group acting properly and co-compactly on M and let K be a maximal compact subgroup of G such that the identity component of K is non-abelian. If there is a global slice such that K acts effectively on it, then M has a G-invariant Riemannian metric of positive scalar curvature.*

Proof. This theorem follows from Theorem 58 and the following theorem of Lawson and Yau:

Theorem 62 ([51])**.** *Let K be a compact Lie group whose identity component is nonabelian. If N is a compact manifold with an effective action of K, then N admits a K-invariant Riemannian metric of positive scalar curvature.*

□

7. Two Applications of the Induction Principle

The induction principle exhibited in the earlier parts of this paper allows us to generalise a few interesting results involving compact group actions. We owe inspiration to the paper of Hochs-Mathai [37], where the theorem of Atiyah and Hirzebruch is generalised to the non-compact setting.

7.1. **Hattori's Vanishing Theorem.**

In 1978, Hattori ([30], Theorem 1 and Lemma 3.1) proved the following interesting result:

Theorem 63. *Let Y be a compact, connected, almost-complex manifold of dimension greater than 2 on which S^1 acts smoothly and non-trivially, preserving the almost-complex structure. Suppose that the first Betti number of Y vanishes and that the first Chern class is*

$$c_1(Y) = k_0 x,$$

where $k_0 \in \mathbb{N}$ and $x \in H^2(Y, \mathbb{Z})$. Let L be a line bundle with $c_1(L) = kx$, for an integer k satisfying $|k| < k_0$ and $k = k_0$ (mod 2). Then

$$\operatorname{index}_{S^1}(\partial_Y^L) = 0 \quad \in R(S^1),$$

where ∂_Y^L is the equivariant Spin^c*-Dirac operator on Y twisted by L.*

Hattori's result was inspired by the vanishing theorem of Atiyah and Hirzebruch [3] for non-trivial circle actions on compact Spin-manifolds. We first mildly generalise Theorem 63 from non-trivial circle actions to non-trivial actions of compact, connected Lie groups.

Theorem 64. *Let Y be a compact, connected, almost-complex manifold of dimension greater than 2 and K a compact, connected Lie group acting smoothly and non-trivially on Y, preserving the almost-complex structure. Suppose also that the first Betti number of Y vanishes and that the first Chern class is*

$$c_1(Y) = k_0 x,$$

where $k_0 \in \mathbb{N}$ and $x \in H^2(Y, \mathbb{Z})$. Let L be a line bundle with $c_1(L) = kx$, for an integer k satisfying $|k| < k_0$ and $k = k_0$ (mod 2). Then

$$\operatorname{index}_K(\partial_Y^L) = 0 \quad \in R(K),$$

where ∂_Y^L is the equivariant Spin^c*-Dirac operator on Y twisted by L.*

Proof. For any $g \in K$, there is a maximal torus T of K containing g, whence

$$\operatorname{index}_K(\partial_Y^L)(g) = \operatorname{index}_T(\partial_Y^L)(g).$$

Since by assumption K is connected and acts on Y non-trivially, the action T of on Y given by restriction must also be non-trivial. In the special case $T = S^1 \times S^1$, this means that at most two circles in T can act trivially on

Y. Thus the circles in T that act non-trivially on Y form a dense subset of T. For each t in such a circle $S < T$, we have that

$$\mathrm{index}_T(\partial_Y^L)(t) = \mathrm{index}_S(\partial_Y^L)(t) = 0$$

by Hattori's Theorem 63. Since the character of a K-representation is continuous in K, $\mathrm{index}_T(\partial_Y^L)(t) = 0$ for all $t \in T$. This completes the argument for $T = S^1 \times S^1$.

The argument extends to the general case by induction on the dimension of T. One can show that for an l-dimensional torus T acting non-trivially on Y, no more than $l - 1$ circles can act trivially. From this it follows that, under the hypotheses of the theorem, $\mathrm{index}_K(\partial_Y^L) = 0 \in R(K)$. □

Our goal in the following subsection is to extend Theorem 64 to the non-compact setting. The result is Theorem 66, which can be stated equivalently in the form of Theorem 67. Let X be a manifold on which a connected Lie group G acts properly and isometrically. Suppose that the action is cocompact and that X has a G-equivariant Spin^c-structure. Let

$$\mathrm{index}_G(\partial_X^L) \in K_\bullet(C_r^*(G))$$

be the equivariant index of the associated Spin^c-Dirac operator.

Let $K < G$ be a maximal compact subgroup, and suppose G/K has an almost-complex structure (this is always true for a double cover of G, as pointed out in Remark 17).

Definition 65 ([37])**.** The action of G on X is *properly trivial* if all stabilisers are maximal compact subgroups of G. For a proper action, the stabilisers cannot be larger. The action is called *properly non-trivial* if it is not properly trivial.

Hattori's Theorem 64 generalises as follows.

Theorem 66. *As above, let G be a connected Lie group with a maximal compact subgroup K, and assume that G/K has an almost-complex structure. Suppose that G acts properly and cocompactly on a connected almost-complex* *manifold X and that G preserves the almost-complex structure. Suppose also that the first Betti number of X vanishes and that the first Chern class is*

$$c_1(X) = k_0 x,$$

where $k_0 \in \mathbb{N}$ and $x \in H^2(X, \mathbb{Z})$. Assume that the G-action on X is properly non-trivial. Let L be a G-equivariant line bundle with $c_1(L) = kx$, for an integer k satisfying $|k| < k_0$ and $k = k_0$ (mod 2). Then

$$\mathrm{index}_G(\partial_X^L) = 0 \quad \in K_\bullet(C_r^*(G)),$$

where ∂_X^L is the equivariant Spin^c*-Dirac operator on X twisted by L.*

Theorem 46 (*higher index commutes with induction*) allows us to deduce Theorem 66 from Hattori's Theorem 64. This is based on the fact that the Dirac induction map (4.6) relates the equivariant indices of the Spin^c-Dirac operators ∂_Y^L on Y and ∂_X^L on X, associated to the Spin^c-structures P_Y and P_X respectively, to one another. (See Corollary 48, also [36, Theorem 5.7].)

Proof of Theorem 66. Let $Y \subset X$ be as in Abels' Theorem (Theorem 6). Using the decomposition (3.7) of the tangent bundle of X, the almost-complex structures on X and on G/K (hence on $\mathfrak{p}$) give rise to an almost-complex structure on $G \times_K TY$. Since G preserves the almost-complex structure, we obtain an almost-complex structure on Y preserved by the K-action. Note that a manifold with an almost-complex structure preserved under a group action has an equivariant Spin^c-structure. By Corollary 48, we have

$$\mathrm{index}_G(\partial_X^L) = \text{D-Ind}_K^G\big(\mathrm{index}_K(\partial_Y^{L|_Y})\big). \tag{7.1}$$

Let $X_{(K)}$ be the set of points in X with stabilisers conjugate to K. By Lemma 9 of [37], the fixed point set Y^K of the action by K on Y is related to the action of G on X by

$$X_{(K)} = G \cdot Y^K \cong G/K \times Y^K.$$

The stabiliser of a point $m \in X$ is a maximal compact subgroup of G if and only if $m \in X_{(K)}$. Thus, the condition on the stabilisers of the action of G on X is equivalent to the requirement that the action of K on Y be non-trivial. Moreover, because G/K is contractible, we have

$$H^*(G \times_K Y, \mathbb{Z}) \simeq H^*(Y, \mathbb{Z}),$$

while $T(G/K)$ is trivial. Hence, $X = G \times_K Y$ and Y have the same first Betti number. In a similar fashion, we have

$$c_1(G \times_K TY) = c_1(TY), \qquad c_1(L) = c_1(G \times_K L|_Y) = c_1(L|_Y).$$

Theorem 64 now implies that, under our hypotheses,

$$\mathrm{index}_K(\partial_Y^{L|_Y}) = 0.$$

The theorem then follows from relation (7.1). □

The Theorem we have just proved has the following equivalent statement.

Theorem 67. *In the setting of Theorem 66,* $\mathrm{index}_G(\partial_X^L) \neq 0$ *if and only if there is a compact* Spin^c*-manifold* Y *with* $e^{kx/2}\hat{A}(Y) \neq 0$*, and a* G*-equivariant diffeomorphism*

$$X \simeq G/K \times Y,$$

where G *acts trivially on* Y*.*

Proof. Because the Dirac induction map D-Ind_K^G in (7.1) is an isomorphism,

$$\mathrm{index}_G(\partial_X^L) \neq 0 \quad \Leftrightarrow \quad \mathrm{index}_K(\partial_Y^{L|_Y}) \neq 0.$$

Moreover, we have an equivalence

$$\mathrm{index}_K(\partial_Y^{L|_Y}) \neq 0 \quad \Leftrightarrow \quad K \text{ acts trivially on } Y \text{ and } e^{kx}\hat{A}(Y) \neq 0,$$

which follows from Theorem 64, since if K acts trivially on Y, then $\mathrm{index}_K(\partial_Y^{L|_Y}) \in R(K)$ equals $\mathrm{index}(\partial_Y^{L|_Y}) = e^{kx}\hat{A}(Y)$ copies of the trivial representation. Since K acts trivially on Y if and only if $X = (G/K) \times Y$, the claim follows. □

Remark 68. The proofs of Theorems 66 and 67 are inspired by the proofs of Theorems 2 and 3 in [37]. Moreover (see also Remark 7 in [37]), the non-vanishing of $\mathrm{index}_G(\partial_X^L)$ in Theorems 66 and 67 can be replaced by the non-vanishing of the class $p_*[\partial_X^L]$ in (4.4) because of the Baum-Connes isomorphism (4.5).

7.2. **An Analogue of Petrie's Conjecture.** The Pontryagin class of a closed oriented manifold is not usually a homotopy invariant. However, for the Kähler manifolds $\mathbb{C}P^n$, Petrie [58] has an interesting conjecture, motivated by the question of whether or not manifolds of a given homotopy type admit non-trivial circle actions. Recall that the total Pontryagin class of $\mathbb{C}P^n$ is

$$p(\mathbb{C}P^n) = (1+x^2)^{n+1}, \qquad x \in H^2(\mathbb{C}P^n).$$

There is a natural action of S^1 on $\mathbb{C}P^n$ given by

$$(\lambda, [z_0 : \ldots : z_n]) \mapsto [\lambda^{a_0} z_0 : \ldots : \lambda^{a_n} z_n], \qquad a_i \in \mathbb{Z}.$$

Conjecture 69 (Petrie's Conjecture [58])**.** *If a closed oriented manifold Y of dimension $2n$ admits an orientation-preserving homotopy equivalence $f : Y \to \mathbb{C}P^n$ and also a non-trivial circle action, then its total Pontryagin class is given by*

$$p(Y) = f^* p(\mathbb{C}P^n).$$

It is a theorem of Hattori [30] that this conjecture holds if Y has an almost-complex structure preserved under the S^1-action and if $c_1(Y) = \pm(n+1)x$, where $x \in H^2(Y, \mathbb{Z})$ is the generator.

Recall that the $\mathcal{L}$-class is given by the multiplicative sequence of polynomials belonging to the power series $\frac{\sqrt{t}}{\tanh\sqrt{t}}$ and that the signature of a smooth compact oriented manifold Y^{4k} is equal to the $\mathcal{L}$-genus $\mathcal{L}[Y^{4k}]$. We have the following lemma, inspired by a result of Kaminker [39]:

Lemma 70. *Let Y be a compact* Spin^c*-manifold. Let $D_Y, D_{\mathbb{C}P^n}$ denote signature operators on Y and $\mathbb{C}P^n$ respectively. Then $\mathcal{L}(Y) = f^*\mathcal{L}(\mathbb{C}P^n)$ if and only if*

$$f_*[D_Y] = [D_{\mathbb{C}P^n}] \in K_\bullet^{geo}(\mathbb{C}P^n).$$

Proof. Assume without loss of generality that $\dim Y$ is even. Poincaré duality for geometric K-homology is given by taking cap product with the fundamental class $[Y]$ (cf. Example 4 and Corollary 27):

$$\mathcal{PD}_K : K^0(Y) \to K_0^{geo}(Y) \qquad [E] \to [Y] \cap [E] := [(Y, E \otimes \mathbb{C}_Y, \mathrm{id}_Y)].$$

Recall that there is a Chern character for K-homology,

$$\mathrm{ch} : K_0(Y) \to H_{\mathrm{ev}}(Y) \otimes \mathbb{Q} \qquad [(Y, E, f)] \mapsto f_*(\mathcal{PD}_H(\mathrm{ch}(E) \cup \mathrm{td}(Y))),$$

that makes the following diagram commutative:

$$\begin{array}{ccc} K^0(Y) & \xrightarrow[\simeq]{\mathrm{ch}(-)\cup\mathrm{td}(Y)} & H^{\mathrm{ev}}(Y) \otimes \mathbb{Q} \\ {\scriptstyle \mathcal{PD}_K}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \mathcal{PD}_H} \\ K_0(Y) & \xrightarrow[\mathrm{ch}(-)]{} & H_{\mathrm{ev}}(Y) \otimes \mathbb{Q}. \end{array} \tag{7.2}$$

In this diagram, $\mathrm{ch}(-) \cup \mathrm{td}(Y)$ is the Atiyah-Singer integrand and $\mathcal{PD}_H$ is the Poincaré duality map on homology and cohomology. Note that if D_Y is the signature operator on Y, then $\mathrm{ch}[\sigma(D_Y)] \cup \mathrm{td}(Y) = \mathcal{L}(Y)$. Together with the commutative diagram (7.2) we have

$$\mathcal{PD}_H(\mathcal{L}(Y)) = \mathcal{L}(Y) \cap [Y] = \mathrm{ch}(\mathcal{PD}_K([\sigma(D_Y)])).$$

The proof is then completed by naturality of the Chern character (note that $\mathcal{L}(Y) = f^*\mathcal{L}(\mathbb{C}P^n)$ if and only if the intersection products are equal, if and only if $f_*(\mathcal{L}(Y) \cap [Y]) = \mathcal{L}(\mathbb{C}P^n) \cap [\mathbb{C}P^n]$). □

Using induction on K-homology, we are able to generalise a weaker version of Hattori's result on Petrie's Conjecture, as follows.

Theorem 71 (Analogue of Petrie's Conjecture). *Let X be a connected manifold admitting an almost-complex structure and a properly non-trivial $SU(1,1)$-action preserving the almost-complex structure. If there is an $SU(1,1)$-equivariant homotopy equivalence*

$$f : X \to SU(1,1) \times_{U(1)} \mathbb{C}P^n,$$

and $c_1(X) = \pm(n+1)x$ with $x \in H^2(X, \mathbb{Z})$ being the generator, then

$$f_*[D_X] = [D_{SU(1,1)\times_{U(1)}\mathbb{C}P^n}] \in K_\bullet^{SU(1,1)}(SU(1,1) \times_{U(1)} \mathbb{C}P^n) \simeq K_\bullet^{U(1)}(\mathbb{C}P^n).$$

Proof. Let Y be a $U(1)$-slice of X. There exist $SU(1,1)$-equivariant maps

$$f : SU(1,1) \times_{U(1)} Y \to SU(1,1) \times_{U(1)} \mathbb{C}P^n,$$

$$g : SU(1,1) \times_{U(1)} \mathbb{C}P^n \to SU(1,1) \times_{U(1)} Y,$$

such that $g \circ f$ and $f \circ g$ are $SU(1,1)$-equivariantly homotopic to the respective identity maps. These maps induce $U(1)$-equivariant homotopies from $f|_Y \circ g|_{\mathbb{C}P^n}$ and $g|_{\mathbb{C}P^n} \circ f|_Y$ to the identity maps. Thus, Y is $U(1)$-homotopic to $\mathbb{C}P^n$. Hattori's result now implies that $\mathcal{L}(Y) = f^*\mathcal{L}(\mathbb{C}P^n)$. Finally, by Lemma 70 and induction on K-homology, the conclusion follows. ☐

Appendix A. Equality of Mathai-Zhang Index and Integration Trace

Let M be a smooth manifold on which a locally compact group G acts cocompactly. Let dg be a fixed left-invariant Haar measure on G and χ be the modular character of G, defined by $dg^{-1} = \chi(g)dg$. Suppose that M has a G-invariant Riemannian metric and that D is a G-invariant Dirac operator on M acting on sections of a G-equivariant Dirac bundle $E \to M$. Then D defines an element of the G-equivariant analytic K-homology $K_0^G(M)$,

$$[D] = \left[\left(L^2(E), \phi, \frac{D}{\sqrt{D^2+1}}\right)\right] \in K_0^G(M),$$

where $\phi : C_0(M) \to L^2(E)$ is the usual representation given by pointwise multiplication.

It was shown in Bunke's appendix to [55] that, for G a *unimodular* locally compact group, the Mathai-Zhang index of D (see [55] section 2 for the definitions) equals the composition

$$I : K_0^G(M) \xrightarrow{\text{index}_G} K_0(C^*(G)) \xrightarrow{\int_G} \mathbb{Z}$$

applied to $[D]$. Here index_G is the equivariant (analytic) index map and $\int_G$ is the integration trace on $K_0(C^*(G))$. More precisely, $\int_G$ is induced by the map $C^*(G) \to \mathbb{Z}$ defined on the dense subalgebra $C_c(G)$ by

$$f \mapsto \int_G f(g)\,dg.$$

The purpose of this appendix is to provide a proof, using Bunke's technique, that the Mathai-Zhang index equals the integration trace when G is an arbitrary locally compact group.

Let c be a (compactly supported) cut-off function on M, and let $L^2_c(E)^G$ be the L^2-completion of the space $\{cs : s \in C(E)^G\}$, where $C(E)^G$ is the space of G-invariant continuous sections of E. We define the Sobolev analogues $H^i_c(E)^G$, $i \geq 0$, analogously. One sees that elements of $H^i_c(E)^G$ are precisely those of the form $c\mu$, where $\mu \in H^i_{\mathrm{loc}}(E)^G$, the space of G-invariant, locally H^i sections.

Let $P^\chi_c : L^2(E) \to L^2_c(M,E)^G$ be given by

$$P^\chi_c \mu = \frac{c(x)}{\int_G \chi^{-}1(g)(c(g^{-1}x)^2)\,dg} \int_G \chi^{-1}(g)c(g^{-1}x)g\mu(g^{-1}x)\,dg.$$

By Proposition 3.1 of [64] (adapted to a left Haar measure), P^χ_c is the orthogonal projection $L^2(E) \to L^2_c(E)^G$. Now for each $t \in [0,1]$, define a linear map $P^{\chi^t}_c$ on $L^2(E) \ni \mu$ by

$$P^{\chi^t}_c \mu = \frac{c(x)}{\int_G (\chi^t)^{-1}(g)(c(g^{-1}x)^2)\,dg} \int_G (\chi^t)^{-1}(g)c(g^{-1}x)g\mu(g^{-1}x)\,dg.$$

One can verify that each $P^{\chi^t}_c$ is a projection $L^2(E) \to L^2_c(E)^G$, orthogonal when $t = 1$.

For each t, $P^{\chi^t}_c$ is a bounded operator, and the path $t \mapsto P^{\chi^t}_c$ is continuous in the strong* operator topology on $L^2(E)$. Now for each $t \in [0,1]$, define $\tilde{D}_t : H^1_c(E)^G \to L^2_c(E)^G$ by

$$c\mu \mapsto P^{\chi^t}_c D(c\mu),$$

for $\mu \in H^1_{\mathrm{loc}}(E)^G$. It follows that the path $t \mapsto \tilde{D}_t$ is continuous in the same sense and has end points

$$\tilde{D}_0 : c\mu \mapsto P^1_c D(c\mu), \qquad \tilde{D}_1 : c\mu \mapsto P^\chi_c D(c\mu).$$

A variant of the method used to prove Lemma D.1 in [55] shows that each $\tilde{D}_t$ is Fredholm.

Proposition 72. *In the above setting, the Mathai-Zhang index of D is equal to $I([D])$.*

Proof. The path $t \mapsto \tilde{D}_t$ of Fredholm operators defined above gives an equality of K-homology classes

$$[\tilde{D}_1] = [\tilde{D}_0] \in K_0^G(M),$$

where for each $t \in [0,1]$, we define the class $[\tilde{D}_t]$ in the obvious way, namely

$$[\tilde{D}_t] := \left[\left(L^2(E), \phi, \frac{\tilde{D}_t}{\sqrt{\tilde{D}_t^2 + 1}} \right) \right] \in K_0^G(M).$$

The appendix of [55] shows that

$$\text{index}(\tilde{D}_0) = I([D]),$$

where on the left is the Fredholm index of $\tilde{D}_0$. On the other hand, the Mathai-Zhang index of the Dirac operator D is equal to $\text{index}(\tilde{D}_1)$ by definition, and we conclude. □

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School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005 Australia.
E-mail address: oug.oah@gmail.com

School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005 Australia.
E-mail address: mathai.varghese@adelaide.edu.au

School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005 Australia.
E-mail address: hang.wang01@adelaide.edu.au

Appendix B

The Non-Cocompact Setting

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Authors	Hao Guo
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INDEX OF EQUIVARIANT CALLIAS-TYPE OPERATORS AND INVARIANT METRICS OF POSITIVE SCALAR CURVATURE

HAO GUO

Abstract. For a Lie group G acting properly, isometrically and non-cocompactly on a Riemannian manifold M, we study the index theory of *G-Callias-type* operators, and more broadly index theory of operators that are positive outside of a G-cocompact subset of M. We construct G-Callias-type operators using K-theory of a partial Higson corona whose K-theory is highly non-trivial. The theory is applied to obtain an obstruction to positive scalar curvature metrics in the non-cocompact setting.

1. Introduction

It is well-known that a Dirac operator D on a non-compact manifold M is not in general Fredholm, since the usual version of the Rellich lemma fails in this setting. Nevertheless, it is possible to modify D so as to make it Fredholm but still remain within the class of Dirac-type operators. One such modification is a *Callias-type operator*, which was initially studied by Callias [15] on M a Euclidean space, before being generalised to the setting of Riemannian manifolds by others [11],[5],[13],[14],[32]. A Callias-type operator may be written as $B = D + \Phi$, where Φ is an endomorphism making B invertible at infinity. As in [14], one may form the order-0 bounded transform F of B, defined formally by $F := B(B^2+f)^{-1/2}$, where f is a compactly supported function. The formal computation $F^2 - 1 = -f(B^2+f)^{-1}$ shows that F is a Fredholm operator, since multiplication by the compactly

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supported function f defines a compact operator between Sobolev spaces $H^i \to H^j$ for all $i > j$.

More generally, if M is a manifold with a proper, non-cocompact action of a Lie group G, a G-invariant Dirac operator on M need not be Fredholm in the sense of C^*-algebras. From the point of view of the equivariant index map in KK-theory, the lack of a general index for elements of the equivariant analytic K-homology $K_*^G(M)$ can be traced back to the lack of a counterpart to the canonical projection in $KK(\mathbb{C}, C_0(M) \rtimes G)$ defined by a compactly supported cut-off function when M/G is compact.

We show, by establishing G-equivariant analogues of the Sobolev spaces H^i and the Rellich lemma, that the above formal computation can be made to work in the non-cocompact setting as long as the operator being considered has positive square outside of a cocompact set. Thus we are able to establish a G-index for such operators, which includes G-invariant Callias-type operators. The situation we consider here can also be thought of, in a precise sense, as an equivariant generalisation of the situation considered by Roe in [43]; in particular, we remark that the coarse equivalence properties of the index in [43] also carry over to our setting, although we will not explore this aspect of the theory in the present work.

We begin in section 3 by constructing *G-Sobolev modules $\mathcal{E}^i$* over the maximal group C^*-algebra $C^*(G)$. This generalises Kasparov's[1] G-equivariant analogue of L^2, denoted by $\mathcal{E}$, to include Sobolev spaces of higher orders. We prove basic results on boundedness, adjointability and compactness of G-invariant properly supported operators, in particular establishing the analogue of the Rellich lemma alluded to above.

In section 4 we define the notion of *G-invertibility at infinity* - a natural generalisation of the notion of invertibility at infinity[2] to take into account a group action. We prove the following, which implies that G-invertible-at-infinity operators are equivariantly Fredholm:

Theorem 41. Suppose B is G-invertible at infinity. Let F be its bounded transform. Then $(\mathcal{E}^0, F)$ is a $C^*(G)$-Fredholm module whose index is defined

[1] See for instance [29] section 5.

[2] As defined in [14] section 1.

independently of the choice of the cocompactly supported function f used to define F.

A particular example of such an operator is given by *G-Callias-type operators*, which we define and study in section 5. We establish that these operators are essentially self-adjoint and regular, by adapting the method used in [21] to our setting. We then construct G-Callias-type operators using a partial compactification of M in the direction transverse to the G-orbits, and we show in section 6 that the K-theory of this partial compactification is highly non-trivial. In particular we have:

Theorem 77. Let M be a complete G-Riemannian manifold with M/G non-compact. Then the K-theory of the Higson G-corona of M is uncountable.

In section 7 we give an application of the theory to G-invariant metrics of positive scalar curvature on non-cocompact M. We prove:

Theorem 84. Let M be a non-cocompact G-equivariantly spin Riemannian manifold with G-spin-Dirac operator $\partial\!\!\!/_0$. If M admits a G-invariant metric with pointwise positive scalar curvature, then the G-index of every G-Callias-type operator on M vanishes. That is, for any G-admissible endomorphism Φ we have

$$\mathrm{index}_G(F) = 0 \in K_0(C^*(G)),$$

where F is the bounded transform of the G-Callias-type operator defined by Φ and $\partial\!\!\!/_0$.

This result vastly generalises two existing results on obstructions to G-invariant positive scalar curvature on proper G-spin manifolds, where G is a non-compact Lie group. The first is a recent result of Zhang (Theorem 2.2 of [47]), which was the first generalisation of Lichnerowicz' vanishing theorem to the cocompact-action case; the notion of index used there was the Mathai-Zhang index (see [37]). The second is Theorem 54 of [20], which states that the equivariant index of a G-invariant Dirac operator vanishes in the presence of G-invariant positive scalar curvature.

When M/G is compact, the equivariant index theory of G-Callias-type operators reduces to that of G-invariant Dirac operators. When M/G is non-compact, the theory we establish opens a number of new possible directions of enquiry. Among these are generalisations of Gromov and Lawson's relative index theorem [19] and Roe's partitioned index theorem [42], and a non-cocompact version of the geometric $[Q, R] = 0$ problem (see for example [23]); the latter will be considered in a forthcoming paper.

The results of this paper may be contrasted with the equivariant index theory and applications studied in [20]. Firstly, [20] deals exclusively with index theory in the cocompact case, where the $C^*(G)$-Fredholmness of the Dirac operator was already known. In addition, [20] focused almost entirely on the case of almost-connected G, where a global slice of the manifold exists, while here we let G be an arbitrary Lie group. Secondly, the estimates used in [20] to establish obstructions to positive scalar curvature relied essentially on compactness of M/G, while our estimates here work more generally.

The results can also be contrasted with those in [16], which also deals with Callias-type operators in the setting of Hilbert modules on non-compact manifolds. Whereas in [16] the relation with Hilbert module theory arises through twisting Dirac-type operators by Hilbert module bundles, the G-Sobolev modules we study here arise from the G-action. Thus one of the key tools for us is the G-invariant pseudodifferential calculus. Nevertheless, we show in subsections 5.2 and 5.3 that the technique used in [45] and [21] to establish regularity and essential self-adjointness of a Dirac operator on a non-compact, complete manifold twisted by a Hilbert module bundle also works in our setting.

Finally, other versions of G-index theory, for non-compact M/G and G, have been developed elsewhere. This includes the work of Hochs-Mathai [22],[23], Braverman [12], and Hochs-Song [26],[24],[25].

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Contents

2. Notation and Terminology

Let M be a Riemannian manifold on which a Lie group[3] G acts smoothly, properly and isometrically. In what follows, we shall call such an M a *G-Riemannian manifold.* If the orbit space M/G is compact, we shall say that the G-action is *cocompact*, or that M is a *cocompact G-manifold.* The term *non-cocompact* will mean "not necessarily cocompact", with the exception of

[3]Except in subsection 7.1, where G is almost-connected, G will always mean an arbitrary Lie group.

subsection 6.3. Except in section 7, we will suppress the Riemannian metric when mentioning M, to avoid unnecessary confusion with elements of G.

Fix a left Haar measure dg on G, with modular function $\mu\colon G \to \mathbb{R}^+$ given by $d(gs) = \mu(s)dg$. Since dg is unique up to constants, μ is well-defined independent of this choice. We shall use $d\mu$ or dx to denote the smooth G-invariant measure on M induced by the Riemannian metric.

Let $\pi\colon M \to M/G$ be the natural projection. A subset $K \subseteq M$ is called *cocompact* if $\pi(K)$ has compact closure in M/G.[4] We say that $S \subseteq M$ is *cocompactly compact* if for any cocompact $K \subseteq M$, $S \cap K$ has compact closure in M. A proper G-manifold M admits a smooth *cut-off function* $\mathfrak{c}\colon M \to [0,1]$, with the property that for all $x \in M$,

$$\int_G \mathfrak{c}(g^{-1}x)\, dg = 1.$$

Clearly, $\mathrm{supp}(\mathfrak{c})$ is cocompactly compact.

Let $\pi_1, \pi_2\colon M \times M \to M$ be the projection maps onto the first and second factors. Given an operator A on M with Schwartz kernel k_A, we say that k_A has *compact* (resp. *cocompactly compact*) support if there exists a compact (resp. cocompactly compact) subset $\tilde{K} \subseteq M$ such that $\mathrm{supp}(k_A) \subseteq \tilde{K} \times \tilde{K}$.

For A a C^*-algebra, denote its positive[5] elements by A_+. For Hilbert A-modules $\mathcal{M}$ and $\mathcal{N}$, denote the bounded adjointable and compact operators $\mathcal{M} \to \mathcal{N}$ by $\mathcal{L}(\mathcal{M},\mathcal{N})$ and $\mathcal{K}(\mathcal{M},\mathcal{N})$. If $\mathcal{M} = \mathcal{N}$, we use $\mathcal{L}(\mathcal{M})$ and $\mathcal{K}(\mathcal{M})$.

3. Sobolev Modules

We begin by defining certain Hilbert modules $\mathcal{E}^i$ over the maximal group C^*-algebra $C^*(G)$, as a generalisation of Sobolev spaces. The definition is based on Kasparov's definition of the $C^*(G)$-module $\mathcal{E}$ ([29] section 5), which is a Hilbert $C^*(G)$-module analogue of $L^2(E)$. The modules $\mathcal{E}^i$ will provide a convenient setting for the study of G-invariant differential operators in the non-cocompact setting.

Recall first the definition of $\mathcal{E}$ from [29]. Let E be a G-equivariant Hermitian vector bundle over a non-cocompact G-Riemannian manifold M. The space of compactly supported smooth sections $C_c^\infty(E)$ can be given a

[4]Thus when G is trivial, the cocompact sets are the relatively compact sets.

[5]This will always mean "positive semi-definite".

pre-Hilbert $C_c^\infty(G)$-module structure, with right $C_c^\infty(G)$-action and $C_c^\infty(G)$-valued inner product given by

$$(e\cdot b)(x) = \int_G g(e)(x)\cdot b(g^{-1})\cdot \mu(g)^{-1/2}\,dg \in C_c^\infty(E),$$

$$\langle e_1, e_2\rangle(g) = \mu(g)^{-1/2}\int_M \langle e_1(x), g(e_2)(x)\rangle_E\,d\mu \in C_c^\infty(G),$$

for $e, e_1, e_2 \in C_c^\infty(E)$ and $b \in C_c^\infty(G)$. Here G acts on $C_c(E)$ by

$$g(e)(x) \coloneqq g(e(g^{-1}x)).$$

Let $\mathcal{E}$ be the completion of $C_c^\infty(E)$ under the norm induced by the above inner product, and extend naturally the $C_c(G)$-action to a $C^*(G)$-action on $\mathcal{E}$ and the inner product to a $C^*(G)$-valued inner product. This gives $\mathcal{E}$ the structure of a Hilbert $C^*(G)$-module.

We generalise this definition as follows.

Definition 1. Let M be a non-cocompact G-Riemannian manifold and E a Hermitian G-vector bundle over M. Let B be a G-invariant, formally self-adjoint Dirac-type operator on E with initial domain $C_c^\infty(E)$. For each integer $i \geq 0$, let $C_c^{\infty,i}(E)$ be the pre-Hilbert $C_c^\infty(G)$-module with right $C_c^\infty(G)$-action and $C_c^\infty(G)$-valued inner product given by

$$(e\cdot b)(x) = \int_G g(e)(x)\cdot b(g^{-1})\mu(g)^{-1/2}\,dg \in C_c^\infty(E),$$

$$\langle e_1, e_2\rangle_i(g) = \mu(g)^{-1/2}\sum_{k=0}^{i}\int_M \langle B^k e_1(x), B^k g(e_2)(x)\rangle_E\,d\mu \in C_c^\infty(G),$$

for $e, e_1, e_2 \in C_c^\infty(E)$ and $b \in C_c^\infty(G)$, and where we set B^0 equal to the identity operator. Positivity of these inner products is proved in Lemma 2. Denote by $\mathcal{E}^i(E)$, or simply $\mathcal{E}^i$, the vector space completion of $C_c^{\infty,i}(E)$ with respect to the norm induced by $\langle\ ,\ \rangle_i$, and extend naturally the $C_c(G)$-action to a $C^*(G)$-action and $\langle\ ,\ \rangle_i$ to a $C^*(G)$-valued inner product, to give $\mathcal{E}^i$ the structure of a Hilbert $C^*(G)$-module. Let $\|\cdot\|_i$ denote the associated norm. We call $\mathcal{E}^i$ the *i-th G-Sobolev module with respect to B*.

Lemma 2. *For each $i \geq 0$, the $C_c(G)$-valued inner product $\langle\ ,\ \rangle_i$ on the pre-Hilbert $C_c(G)$-module $C_c^{\infty,i}(E)$ is positive in $C^*(G)$.*

Proof. For any $u \in C_c^{\infty,i}(E)$ we can write

$$\langle u, u\rangle_i = \sum_{k=0}^{i} \langle B^k u, B^k u\rangle_0 \in C_c(G) \subseteq C^*(G).$$

It can be shown as in [29] section 5 by embedding into a larger Hilbert module that each of the summands is in $C^*(G)_+$. Although the proof in [29] deals with the case of M/G compact, the embedding used there works equally well for the non-cocompact case, noting that the support of a cut-off function $\mathfrak{c}$ is always cocompactly compact. Now the result follows from the fact that a sum of finitely many positive elements in a C^*-algebra is positive. □

Definition 3. A *non-cocompact* (resp. *cocompact*) *G-triple* (G, M, E) consists of a Lie group G, a non-cocompact (resp. *cocompact*) proper G-manifold M and a Hermitian G-vector bundle $E \to M$, together with a G-invariant Dirac-type operator B on E and the collection $\{\mathcal{E}^i\}$ of G-Sobolev modules formed using B. If $E = E^+ \oplus E^-$ is $\mathbb{Z}_2$-graded and $B = B^- \oplus B^+$ is an odd operator, we shall call (G, M, E) a $\mathbb{Z}_2$-graded G-triple.

Given a G-triple with operator B, let $\overline{B^i}$ denote the operator whose graph is the closure of the graph of B^i. $\mathrm{dom}(\overline{B^i})$ equipped with the graph norm is then isomorphic to $\mathcal{E}^i$. Thus $\overline{B^i}$ is a bounded operator $\mathcal{E}^i \to \mathcal{E}^0$ for all $i \geq 0$, and Proposition 7 implies that $\overline{B^i}$ is adjointable. Further, the results of subsections 5.2 and 5.3 imply that B is regular and essentially self-adjoint. Therefore, except in those subsections, we will not make the notational distinction between B^i and $\overline{B^i}$.

3.1. **Boundedness and Adjointability.** We now establish basic boundedness and adjointability results for G-Sobolev modules. When M/G is compact, Kasparov ([29] Theorem 5.4) proved that an $L^2(E)$-bounded, G-invariant operator on $C_c(E)$ with properly supported Schwartz kernel defines a bounded adjointable operator on $\mathcal{E} = \mathcal{E}^0$. The same method of proof yields the following.

Proposition 4. *Let (G, M, E) be a cocompact G-triple. Let A be an operator on $C_c^\infty(E)$ that is G-invariant and bounded when viewed as an operator $H^i(E) \to H^j(E)$. If A has properly supported Schwartz kernel, then A defines an element of $\mathcal{L}(\mathcal{E}^i, \mathcal{E}^j)$.*

Remark 5. The action of G on $C_c^\infty(E)$ extends to a unitary action on each $H^i(E)$, so we may speak of G-invariant operators $H^i(E) \to H^j(E)$.

The proof of the above proposition uses the following lemma.

Lemma 6. *Let (G, M, E) be a cocompact G-triple. Let T be a bounded positive operator on $H^i(E)$ with compactly supported Schwartz kernel. Then $\langle e, \int_G s(T)\, ds\rangle_{\mathcal{E}^i} \in C^*(G)_+$.*

Proof. The proof is similar to [29] Lemma 5.3, so we only give a sketch. T has a unique positive square root $T^{1/2} \colon H^i(E) \to H^i(E)$, so that

$$\langle s(e), T(s(e)\rangle_{H^i(E)} = \langle T^{1/2}(s(e)), T^{1/2}(s(e))\rangle_{H^i(E)}$$

for $e \in C_c(E)$ and $s \in G$. The function $G \to H^i(E)$ given by $s \mapsto T^{1/2}(s(e))$ has compact support in G. Thus for any unitary representation of G on a Hilbert space H and any $h \in H$, the integral

$$v := \int_G \mu^{-1/2}(s) T^{1/2}(s(e)) \otimes s(h)\, ds$$

is a well-defined element of $H^i(E) \otimes H$ with norm equal to

$$\begin{aligned}\int_G \int_G & \mu^{-1/2}(t)\mu^{-1/2}(s)\langle T^{1/2}s(e), T^{1/2}(t(e))\rangle_{H^i(E)} \cdot \langle s(h), t(h)\rangle_H \, ds\, dt \\ &= \int_G \left\langle e, \left(\int_G s(T)\, ds\right)(e)\right\rangle_{\mathcal{E}^i}(t) \cdot \langle h, t(h)\rangle_H \, dt.\end{aligned}$$

Thus $\langle e, (\int_G s(T)\, ds)(e)\rangle_{\mathcal{E}^i}$ is a positive operator on H for all unitary representations of G, where we let $f \in C_c(G)$ act on H by

$$f(h) := \int_G f(g) g(h)\, dg.$$

It follows that $\langle e, \int_G s(T)\, ds\rangle_{\mathcal{E}^i}$ is a positive element of $C^*(G)$. □

Proof. (of Proposition) Let $\mathfrak{c}$ be a cut-off function on M. The operator $A_1 := (\mathfrak{c}A^*A + A^*A\mathfrak{c})/2$ is bounded on $H^i(E)$ by $\|A\|^2\,\|\mathfrak{c}\|$, where $\|A\|$ is the norm of $A : H^i(E) \to H^j(E)$. Note that A_1 is self-adjoint with compactly supported Schwartz kernel. Let $\mathfrak{c}_1$ be a non-negative, compactly supported function on M, identically 1 on the support of the kernel of A_1. Then the operator $A_2 := \mathfrak{c}_1^2\,\|A\|^2\,\|\mathfrak{c}\| - A_1$ is positive, bounded and has Schwartz kernel with compact support. Now the G-invariant function $x \mapsto \int_G g(\mathfrak{c}_1^2)(x)\, dg$ is bounded on M by some constant M_0. Since $\int_G g(A_1)\,dg = A^*A$, the previous lemma applied to A_2 shows that for any $e \in C_c^\infty(E)$,

$$\left\langle e, \left(\int_G g(A_2)\,dg\right) e \right\rangle_{\mathcal{E}^i} = \left(\int_G \mathfrak{c}_1^2 \|A\|^2 \|\mathfrak{c}\|\; dg\right) \langle e, e\rangle_{\mathcal{E}^i} - \langle e, A_1(e)\rangle_{\mathcal{E}^i}$$

is positive in $C^*(G)$. Hence $\langle e, A^*A(e)\rangle_{\mathcal{E}^i} \leq M_0 \|A\|^2 \|\mathfrak{c}\| \langle e, e\rangle_{\mathcal{E}^i} \in C^*(G)$, and A extends to an operator on all of $\mathcal{E}^i$. The left-hand side above equals $\langle A(e), A(e)\rangle_{\mathcal{E}}$, hence A is bounded by a constant depending only on the choice of $\mathfrak{c}$. By the same reasoning, $A^*\colon H^j(E) \to H^i(E)$ defines a bounded operator $\mathcal{E}^j(E) \to \mathcal{E}^i(E)$ that one checks is the adjoint of A. □

Proposition 7. *Let (G, M, E) be a non-cocompact G-triple with B as above. For $j \geq 0$, B defines an element of $\mathcal{L}(\mathcal{E}^{j+1}, \mathcal{E}^j)$.*

Proof. Boundedness is clear. Now since G acts on M properly, there exists a countable, locally finite open covering $\mathcal{U}$ of M by G-stable open subsets U_k, $k \in \mathbb{N}$, such that each U_k is cocompact. By [39] (see also [40] Theorem 5.2.5) one can find a G-invariant partition of unity $\{\rho_k\}$ subordinate to $\mathcal{U}$. Now one can form the modules $\mathcal{E}^i$ by first forming analogous local modules $\mathcal{E}^i_{U_k}$ on U_k, where U_k is considered as an open G-submanifold of a cocompact G-manifold without boundary, namely the double $\overline{U_k}^+$ of the cocompact G-manifold-with-boundary $\overline{U_k}$. (This can be done since there exists a G-equivariant collar neighbourhood of $\partial\overline{U_k}$ inside $\overline{U_k}$, by Theorem 3.5 of [28].) One can then use $\{\rho_k\}$ to form the inner product on $\mathcal{E}^i$. For example, in the case of $\mathcal{E}^0$, we have

$$\begin{aligned}\langle s, t\rangle_{\mathcal{E}^0}(g) = \langle \mu(g)^{-1/2} s, gt\rangle_{L^2(E)} &= \mu(g)^{-1/2} \sum_k \langle \sqrt{\rho_k} s, \sqrt{\rho_k} gt\rangle_{L^2\left(E|_{U_k}\right)} \\ &= \mu(g)^{-1/2} \sum_k \langle \sqrt{\rho_k} s, g\sqrt{\rho_k} t\rangle_{L^2\left(E|_{U_k}\right)} \\ &= \sum_k \langle \sqrt{\rho_k} s, \sqrt{\rho_k} t\rangle_{\mathcal{E}^0_{U_k}}(g),\end{aligned}$$

for $s, t \in C_c^\infty(E)$ and $g \in G$, where we have used G-invariance of ρ_k. By Proposition 7, the operator B restricted to sections supported on each neighbourhood U_k is in $\mathcal{L}(\mathcal{E}^{j+1}_{U_k}, \mathcal{E}^j_{U_k})$. By [29] Theorem 5.8, the local inverse $((B^2+1)_{U_k})^{-1}\colon \mathcal{E}^l_{U_k} \to \mathcal{E}^{l+2}_{U_k}$ exists for all $l \geq 0$. One can verify that

$B_{U_k}((B^2+1)_{U_k})^{-1}$ is the adjoint of B_{U_k}, and that

$$\sum_k B_{U_k}((B^2+1)_{U_k})^{-1}\rho_k : \mathcal{E}^j \to \mathcal{E}^{j+1},$$

which is well-defined by local finiteness of $\mathcal{U}$, is the adjoint of B. □

Corollary 8. *Let (G, M, E) be a non-cocompact G-triple with B as above. For $j \geq 0$, B^i defines an element of $\mathcal{L}(\mathcal{E}^{j+i}, \mathcal{E}^j)$.*

Lemma 9. *Let (G, M, E) be a non-cocompact G-triple and A a G-invariant operator on $C_c^\infty(E)$ that is bounded $H^i(E) \to H^j(E)$ for some i and j. Then the adjoint $A^*\colon H^j(E) \to H^i(E)$ is also G-invariant.*

Proof. For $s_1, s_2 \in C_c^\infty(E)$ we have

$$\begin{aligned}\langle s_1, g(A^* s_2)\rangle_{H^i(E)} &= \langle g^{-1}(s_1), A^* s_2\rangle_{H^i(E)}\\ &= \langle g^{-1}(A s_1), s_2\rangle_{H^0(E)}\\ &= \langle s_1, A^* g(s_2)\rangle_{H^i(E)}.\end{aligned}$$

□

Proposition 10. *Let (G, M, E) be a non-cocompact G-triple. Then multiplication by a G-invariant function $f\colon M \to \mathbb{C}$ for which $\|f\|_\infty < \infty$ defines an element of $\mathcal{L}(\mathcal{E}^i, \mathcal{E}^0)$ for all $i \geq 0$.*

Proof. Boundedness follows from

$$\|\langle fe, fe\rangle_{\mathcal{E}^0}\|_{C^*(G)} \leq C^2 \|\langle e, e\rangle_{\mathcal{E}^0}\|_{C^*(G)} \leq C^2 \|\langle e, e\rangle_{\mathcal{E}^i}\|_{C^*(G)}.$$

Now let $f^*\colon H^0(E) \to H^i(E)$ be the adjoint of $f\colon H^i(E) \to H^0(E)$. Since $f\colon H^i(E) \to H^0(E)$ is bounded and G-invariant, the lemma above implies that f^* is G-invariant. For $e_1, e_2 \in C_c^\infty(E)$ and $g \in G$, we have

$$\begin{aligned}\langle fe_1, e_2\rangle_{\mathcal{E}^0}(g) &= \mu(g)^{-1/2}\langle fe_1, g(e_2)\rangle_{H^0(E)}\\ &= \mu(g)^{-1/2}\langle e_1, g(f^* e_2)\rangle_{H^i(E)}\\ &= \langle e_1, f^* e_2\rangle_{\mathcal{E}^i}(g).\end{aligned}$$

Hence f^* is the adjoint for $f\colon \mathcal{E}^i \to \mathcal{E}^0$ and therefore bounded [34]. □

3.2. **A Rellich-type Lemma.** Recall the following analogue of the Rellich lemma for M a non-compact manifold.

Lemma 11. *Let $f\colon M \to \mathbb{C}$ be a compactly supported function. Then multiplication by f is a compact operator $H^s \to H^t$ if $s > t$.*

We generalise this to the setting of the $\mathcal{E}^i$. Let us begin by considering the torus T^n. For $m \in \mathbb{Z}^n$, consider the function $\phi_m\colon T^n \to \mathbb{C}$, $x \mapsto e^{2\pi i\langle m,x\rangle}$, where $x = (x_1, \ldots, x_n)$ with $0 \leq x_k \leq 2\pi$ for each k. Then for any s, an orthogonal Hilbert basis for $H^s(T^n)$ is given by $\{\phi_m\colon m \in \mathbb{Z}^n\}$. Let $k_s \coloneqq (1 + \sum_{i=1}^n m_i^2)^{-s}$. Then $\{k_s\phi_m : m \in \mathbb{Z}^n\}$ is an orthonormal Hilbert basis for $H^s(T^n)$. If $s \geq t$, the inclusion $i : H^s \to H^t$ can be written as the formal sum of rank-one operators on Sobolev spaces $i = \sum_{m\in\mathbb{Z}^n} \theta_{k_t\phi_m,k_s\phi_m}$, converging if and only if $s > t$. An analogous statement holds on a general compact manifold instead of T^n.

Now suppose M is a non-cocompact G-manifold. Let f be a G-invariant, cocompactly supported function on M and $\mathfrak{c}$ a cut-off function. By Lemma 11, multiplication by $\mathfrak{c}f$ is a compact operator $H^s(M) \to H^t(M)$.

A rank-one element in the sense of Hilbert modules of $\mathcal{K}(\mathcal{E}^s, \mathcal{E}^t)$ is a certain G-average of a rank-one operator on Sobolev spaces $\theta_{e_1,e_2}\colon H^s(M) \to H^t(M)$, where e_1 and e_2 are compactly supported sections (see [36]). Hence the operator

$$\int_G g \left(\sum_m \theta_{\mathfrak{c}fk_s\phi_m,\, \mathfrak{c}fk_{s+1}\phi_m} \right) dg$$

is compact $\mathcal{E}^s \to \mathcal{E}^t$. By the above remarks, this is equal to:

$$\begin{aligned} &\int_G \sum_m \theta_{g(\mathfrak{c}fk_s\phi_m),\, g(\mathfrak{c}fk_{s+1}\phi_m)}\, dg \\ &= \int_G g(\mathfrak{c}f)\, dg \\ &= f \in \mathcal{K}(\mathcal{E}^s, \mathcal{E}^t). \end{aligned}$$

This proves the following Rellich-type lemma for G-Sobolev modules.

Theorem 12. *Let f be a cocompactly supported G-invariant function. Then multiplication by f is an element of $\mathcal{K}(\mathcal{E}^s, \mathcal{E}^t)$ for $s > t$.*

4. G-invertibility at Infinity

We now introduce a notion of invertibility at infinity [14] appropriate to the G-equivariant setting; see [14] for the definition without a group action. Let (G, M, E) be a $\mathbb{Z}_2$-graded non-cocompact G-triple. It follows from our previous results that $B^2 \in \mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)$. Further, we have:

Lemma 13. *Let $f \colon M \to \mathbb{C}$ be a continuous G-invariant function for which $\|f\|_\infty < \infty$. Then $B^2 + f \in \mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)$.*

Proof. This follows from Proposition 7. □

Recall that an unbounded operator A on a Hilbert module $\mathcal{M}$ is said to be *regular* if its graph is orthogonally complementable in $\mathcal{M} \oplus \mathcal{M}$. It can be shown that A is both regular and self-adjoint if and only if $\exists \mu \in i\mathbb{R}$ such that both $A \pm \mu \colon \mathcal{M} \to \mathcal{M}$ have dense range [18].

Definition 14. Let (G, M, E) be a $\mathbb{Z}_2$-graded G-triple equipped with a regular, self-adjoint operator B. Then $B \colon \mathcal{E}^0 \to \mathcal{E}^0$ is said to be *G-invertible at infinity* if there exists a non-negative, G-invariant, cocompactly supported function f on M such that $B^2 + f \in \mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)$ has an inverse $(B^2 + f)^{-1}$ in $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$.

Remark 15. When the acting group G is trivial, we will use the term *invertible at infinity*; this is consistent with the usage in [14].

To prove that operators that are G-invertible at infinity have an index, we adapt Bunke's estimates in [14]. The difference in our approach is that the Hilbert $C^*(G)$-module structure of $\mathcal{E}^i$ arises from the G-action. Still, most of the estimates in [14] carry over to our setting. Hence we will only sketch the proofs where the details can be found in [14].

Lemma 16. *Let B be an operator that is G-invertible at infinity. Then*

$$d := \inf_{\psi \in \mathcal{E}^2, \|\psi\|_{\mathcal{E}^0} = 1} \left(\|B\psi\|_{\mathcal{E}^0}^2 + \left\| \sqrt{f}\psi \right\|_{\mathcal{E}^0}^2 \right) > 0.$$

Proof. Let $\psi \in \mathcal{E}^2$ with $\|\psi\|_{\mathcal{E}^0} = 1$. Let $R := (B^2 + f)^{-1}$ for a cocompactly supported function f. Then

$$\begin{aligned} 1 &= \|\langle \psi, \psi \rangle_{\mathcal{E}^0}\| \\ &= \left\| \langle (B^2 + f) R\psi, \psi \rangle_{\mathcal{E}^0} \right\| \\ &\leq \|B\|_{\mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)} \|R\|_{\mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)} \|B\psi\|_{\mathcal{E}^0} + \sup_{x \in M} |\sqrt{f(x)}| \, \|R\|_{\mathcal{L}(\mathcal{E}^0)} \left\| \sqrt{f}\psi \right\|_{\mathcal{E}^0}. \end{aligned}$$

Since the inclusion $\mathcal{E}^2 \hookrightarrow \mathcal{E}^0$ is bounded, this expression is bounded by

$$\begin{aligned} &\|B\|_{\mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)} \|R\|_{\mathcal{L}(\mathcal{E}^0)} \|B\psi\|_{\mathcal{E}^0} + \sup_{x \in M} |\sqrt{f(x)}| \, \|R\|_{\mathcal{L}(\mathcal{E}^0)} \left\| \sqrt{f}\psi \right\|_{\mathcal{E}^0} \\ &\leq C \left(\|B\psi\|_{\mathcal{E}^0}^2 + \left\| \sqrt{f}\psi \right\|_{\mathcal{E}^0}^2 \right)^{1/2} \end{aligned}$$

for some constants C' and C. Thus $d \geq \frac{1}{C^2} > 0$. □

Corollary 17. *Let B and d be as above and $\lambda \in \mathbb{R}$. Then*

$$\left\| (B^2 + f + \lambda^2)\psi \right\|_{\mathcal{E}^0} \geq (d + \lambda^2) \|\psi\|_{\mathcal{E}^0} \qquad \forall \psi \in \mathcal{E}^2.$$

Proof. By the previous lemma, we see that

$$\langle B\psi, B\psi \rangle_{\mathcal{E}^0} + \langle \sqrt{f}\psi, \sqrt{f}\psi \rangle_{\mathcal{E}^0} = \langle (B^2 + f)\psi, \psi \rangle_{\mathcal{E}^0} \geq d$$

for all $\psi \in \mathcal{E}^2$ with $\|\psi\|_{\mathcal{E}^0} = 1$. This means that for arbitrary $\psi \in \mathcal{E}^2$,

$$\left\| (B^2 + f + \lambda^2)\psi \right\|_{\mathcal{E}^0} \|\psi\|_{\mathcal{E}^0} \geq \langle (B^2 + f + \lambda^2)\psi, \psi \rangle_{\mathcal{E}^0} \geq (d + \lambda^2) \|\psi\|_{\mathcal{E}^0}^2.$$

□

We now turn to estimates involving the resolvent, which we denote by

$$R(\lambda) \coloneqq (B^2 + f + \lambda^2)^{-1} : \mathcal{E}^0 \to \mathcal{E}^2$$

whenever it exists.

Lemma 18. *Suppose $B \colon \mathcal{E}^1 \to \mathcal{E}^0$ is G-invertible at infinity. Then*
(a) for all $\lambda \geq 0$, $R(\lambda) \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$ exists, and

$$\|R(\lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq (d + \lambda^2)^{-1};$$

(b) there exists C such that for all $\lambda \geq 0$,

$$\left\| B^2 R(\lambda) \right\|_{\mathcal{L}(\mathcal{E}^0)} \leq C.$$

Proof. Assume that (a) is true for all $0 \leq \lambda \leq \Lambda$. That this is true for $\Lambda = 0$ follows from G-invertible at infinity and the corollary above. Indeed, since the inclusion $\mathcal{E}^2 \hookrightarrow \mathcal{E}^0$ is bounded adjointable, $R(\lambda) \in \mathcal{L}(\mathcal{E}^0)$ for such λ. To get the estimate, notice that for any $\phi \in \mathcal{E}^0$, the above corollary with $\lambda = 0$ applied to the element $R(0)\phi \in \mathcal{E}^2$ gives

$$\|R(0)\phi\|_{\mathcal{E}^0} \leq \frac{1}{d} \left\|(B^2 + f)R(0)\phi\right\|_{\mathcal{E}^0} = \frac{1}{d} \|\phi\|_{\mathcal{E}^0},$$

which proves $\|R(\lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq (d + \lambda^2)^{-1}$ for all λ in this range. With this in hand, we can show existence of $R(\lambda)$ for λ in the range $|\lambda^2 - \Lambda^2| < d + \Lambda^2$. For such λ it is true that

$$\left\|(\Lambda^2 - \lambda^2)R(\Lambda)\right\|_{\mathcal{L}(\mathcal{E}^0)} \leq |\Lambda^2 - \lambda^2| \, \|R(\Lambda)\|_{\mathcal{L}(\mathcal{E}^0)} < (d + \Lambda^2) \, \|R(\Lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq 1,$$

where the final inequality follows from

$$\|R(\Lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq \frac{1}{d + \Lambda^2}.$$

Thus the Neumann series $\sum_{i=0}^{\infty}(\Lambda^2 - \lambda^2)^i R(\Lambda)^i$ converges and defines an element of $\mathcal{L}(\mathcal{E}^0)$ with adjoint

$$\sum_{i=0}^{\infty}(\Lambda^2 - \lambda^2)^i (R(\Lambda)^*)^i.$$

Now we claim that the operator

$$R(\Lambda)\left(\sum_{i=0}^{\infty}(\Lambda^2 - \lambda^2)^i R(\Lambda)^i\right) : \mathcal{E}^0 \to \mathcal{E}^2,$$

interpreted as the composition

$$\mathcal{E}^0 \xrightarrow{\sum_{i=0}^{\infty}(\Lambda^2-\lambda^2)^i R(\Lambda)^i} \mathcal{E}^0 \xrightarrow{R(\Lambda)} \mathcal{E}^2,$$

is the inverse of $B^2 + f + \lambda^2$. To verify this, write for short $T := B^2 + f + \lambda^2$ and $S := B^2 + f + \Lambda^2$. Then we have the relations

$$1 - TS^{-1} = (\Lambda^2 - \lambda^2)R(\Lambda)$$

$$T = (1 - (1 - TS^{-1}))S,$$

which means

$$R(\lambda) = T^{-1} = S^{-1}\left(\sum_{i=0}^{\infty}(1 - TS^{-1})^i\right) = R(\Lambda)\left(\sum_{i=0}^{\infty}(\Lambda^2 - \lambda^2)^i R(\Lambda)^i\right).$$

Thus for all λ such that $|\lambda^2 - \Lambda^2| < d + \Lambda^2$, $R(\lambda) \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$ exists. We can now apply the above corollary to $R(\lambda)\phi \in \mathcal{E}^2$ for any $\phi \in \mathcal{E}^0$, which yields the desired estimate for λ in this interval:

$$\|R(\lambda)\phi\|_{\mathcal{E}^0} \leq \frac{1}{d+\lambda^2}\left\|(B^2 + f + \lambda^2)R(\lambda)\phi\right\|_{\mathcal{E}^0} = \frac{1}{d+\lambda^2}\|\phi\|_{\mathcal{E}^0}.$$

Iterating this argument countably many times, we exhaust the positive part of $\mathbb{R}$ and get (a). (b) follows from (a) by the triangle inequality applied to

$$B^2(B^2+f+\lambda^2)^{-1} = (B^2+f+\lambda^2)(B^2+f+\lambda^2)^{-1} - (f+\lambda^2)(B^2+f+\lambda^2)^{-1},$$

yielding

$$\left\|B^2 R(\lambda)\phi\right\|_{\mathcal{E}^0} \leq \|\phi\|_{\mathcal{E}^0} + \left\|(f+\lambda^2)R(\lambda)\phi\right\|_{\mathcal{E}^0} \leq C\|\phi\|_{\mathcal{E}^0}, \qquad \forall \phi \in \mathcal{E}^0.$$

□

Remark 19. The above argument in fact gives a stronger existence result for $R(\lambda)$. In particular, $R(\lambda)$ exists whenever $\mathrm{Im}(\lambda) < \sqrt{d}$.

We would like to form the operator $R(0)^{\frac{1}{2}}$ via functional calculus on $R(\lambda) \in \mathcal{L}(\mathcal{E}^0)$, in order to define a bounded version of B. This is analogous to forming the normalised Dirac operator $\frac{D}{\sqrt{D^2+1}}$. First we must verify that $R(\lambda)$ is a self-adjoint element of the C^*-algebra $\mathcal{L}(\mathcal{E}^0)$.

Lemma 20. *$R(\lambda)$ is self-adjoint with respect to the $\mathcal{E}^0$-inner product.*

Proof. For all $\phi \in \mathcal{E}^2$ and $\psi \in \mathcal{E}^0$, we have

$$\begin{aligned}\langle R(\lambda)(B^2 + f + \lambda^2)\phi, \psi\rangle_{\mathcal{E}^0} &- \langle (B^2 + f + \lambda^2)\phi, R(\lambda)\psi\rangle_{\mathcal{E}^0} \\ &= \langle \phi, \psi\rangle_{\mathcal{E}^0} - \langle \phi, (B^2 + f + \lambda^2)R(\lambda)\psi\rangle_{\mathcal{E}^0} \\ &= 0.\end{aligned}$$

But any $\rho \in \mathcal{E}^0$ can be written as $(D^2 + f + \lambda^2)\phi$ for some $\phi \in \mathcal{E}^2$, so for any $\rho, \psi \in \mathcal{E}^0$,

$$\langle (R(\lambda)\rho, \psi\rangle_{\mathcal{E}^0} = \langle \rho, R(\lambda)\psi\rangle_{\mathcal{E}^0}.$$

□

Remark 21. In fact, by the Neumann series expansion of the adjoint mentioned in the proof of Lemma 18, self-adjointness of $R(\lambda) \in \mathcal{L}(\mathcal{E}^0)$ for any $\lambda > 0$ follows once we have verified that $R(0)$ is self-adjoint.

Lemma 22. *We have* $BR(\lambda) \in \mathcal{L}(\mathcal{E}^0)$ *and*

$$\|BR(\lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq C(d + \lambda^2)^{-1/2}$$

for some $C < \infty$ *independent of* $\lambda \geq 0$*, with adjoint*

$$(BR(\lambda))^* = BR(\lambda) + R(\lambda)c(df)R(\lambda),$$

where c *denotes Clifford multiplication.*

Proof. There exists $C < \infty$ such that for all $\phi \in \mathcal{E}^0$ with $\|\phi\|_0 = 1$,

$$\begin{aligned} \|BR(\lambda)\phi\|_{\mathcal{E}^0}^2 &= \|\langle BR(\lambda)\phi, BR(\lambda)\phi\rangle\|_{\mathcal{E}^0} \\ &\leq \|R(\lambda)\phi\|_{\mathcal{E}^0} \left\|B^2 R(\lambda)\phi\right\|_{\mathcal{E}^0} \\ &\leq C^2(B + \lambda)^{-1}, \end{aligned}$$

where we have made use of Lemma 18. Taking square roots gives the desired estimate. For the second statement, note that $R(\lambda)^* = R(\lambda)$ by Lemma 20, so that for all $\psi, \phi \in \mathcal{E}^0$,

$$\begin{aligned} \langle \psi, BR(\lambda)\phi\rangle_{\mathcal{E}^0} &= \langle (B^2 + f + \lambda^2)R(\lambda)\psi, BR(\lambda)\phi\rangle_{\mathcal{E}^0} \\ &= \langle B^2 R(\lambda)\psi, BR(\lambda)\phi\rangle_0 + \langle (f + \lambda^2)R(\lambda)\psi, BR(\lambda)\phi\rangle_{\mathcal{E}^0} \\ &= \langle BR(\lambda)\psi, \phi\rangle_{\mathcal{E}^0} - \langle R(\lambda)\psi, \left[B, (f + \lambda^2)R(\lambda)\right] R(\lambda)\phi\rangle_0 \\ &= \langle (BR(\lambda) + R(\lambda)c(df)R(\lambda))\psi, \phi\rangle_{\mathcal{E}^0}. \end{aligned}$$

□

Lemma 23. *The commutator of* B *and* $R(\lambda)$ *acts on* $\phi \in \mathcal{E}^1$ *by*

$$[B, R(\lambda)]\,\phi = -R(\lambda)c(df)R(\lambda)\phi.$$

Proof. Let $\psi, \phi \in \mathcal{E}^1$. Then using the previous lemma we have

$$\begin{aligned} \langle \psi, R(\lambda)B\phi\rangle_{\mathcal{E}^0} &= \langle BR(\lambda)\psi, \phi\rangle_{\mathcal{E}^0} \\ &= \langle \psi, (BR(\lambda) + R(\lambda)c(df)R(\lambda))\phi\rangle_{\mathcal{E}^0}. \end{aligned}$$

Density of $\mathcal{E}^1 \subseteq \mathcal{E}^0$ and non-degeneracy of $\langle\, ,\, \rangle_{\mathcal{E}^0}$ imply the result. □

Lemma 24. *The operator*

$$(R(0)+\kappa)^{-1} \in \mathcal{L}(\mathcal{E}^0)$$

exists for all $\kappa \in (-\infty, \frac{1}{d}) \cup (0, \infty)$.

Proof. By the Neumann series expansion in Lemma 18, we see that there exists a sufficiently small $\mu > 0$ such that $B^2 + f + \mu i$ and $B^2 + f - \mu i$ are both invertible. By Proposition 4.1 of [27], this means that B^2+f is a regular self-adjoint operator with spectrum contained in $(-\infty, d]$. By the continuous functional calculus for regular self-adjoint operators (Theorem 1.19 in [18]), the spectrum of $R(0) \in \mathcal{L}(\mathcal{E}^0)$ is a subset of $\left[0, \frac{1}{d}\right]$. In particular, this means $(R(0)+\kappa)^{-1} \in \mathcal{L}(\mathcal{E}^0)$ exists for all $\kappa \in (-\infty, -\frac{1}{d}) \cup (0, \infty)$. □

The next lemma is clear.

Lemma 25. *For* $\lambda > 0$ *we have*

$$R(\lambda) = \frac{1}{\lambda^2} R(0) \left(R(0) + \frac{1}{\lambda^2} \right)^{-1}.$$

Lemma 26. *Suppose* (G, M, E) *is a* G*-triple equipped with an operator* B *that is* G*-invertible at infinity. Then for any* $\psi \in \mathcal{E}^1$*, the integral*

$$\frac{2}{\pi} \int_0^\infty B R(\lambda) \psi \, d\lambda$$

converges in $\mathcal{E}^0$ *and defines a bounded operator* $\mathcal{E}^1 \to \mathcal{E}^0$*, where elements of* $\mathcal{E}^1$ *are given the* $\mathcal{E}^0$*-norm. This operator extends to an element of* $F \in \mathcal{L}(\mathcal{E}^0)$.

Proof. We proceed as in [14] Lemma 1.8, with D replaced by B, H^i replaced by $\mathcal{E}^i$ and $B(H^0)$ replaced by $\mathcal{L}(\mathcal{E}^0)$. By the previous lemma, $\frac{2}{\pi}\int_0^\infty R(\lambda)\, d\lambda = R(0)^{1/2}$, the right-hand side being defined by functional calculus in $\mathcal{L}(\mathcal{E}^0)$. One sees that $R(0)^{1/2}B$ extends by continuity to an element $L \in \mathcal{L}(\mathcal{E}^0)$ such that for $\psi \in \mathcal{E}^1$, $F\psi = L\psi - \frac{2}{\pi}\int_0^\infty R(\lambda)c(df)R(\lambda)\psi\, d\lambda$. The continuous extension of this operator defines $F \in \mathcal{L}(\mathcal{E}^0)$. □

Remark 27. $BR(\lambda)$ takes $\psi \in \mathcal{E}^{1,+}$ to an element of $\mathcal{E}^{0,-}$, hence the operator $\int_0^\infty BR(\lambda)\, d\lambda$ is odd.[6] It follows that F is an odd operator on $\mathcal{E}^0$.

[6]The integral performed in $\mathcal{L}(\mathcal{E}^1, \mathcal{E}^0)$.

Proposition 28. *The above definition of F is equivalent to*

$$F\phi := \frac{2}{\pi}\int_0^\infty BR(\lambda)\phi\, d\lambda \qquad \forall \phi \in \mathcal{E}^0.$$

Proof. Let $\phi = \lim_{n\to\infty}\phi_n$. Then for all ψ in the dense submodule $\mathcal{E}^1$,

$$\begin{aligned}\langle \psi, F\phi\rangle_{\mathcal{E}^0} &= \lim_{n\to\infty}\left\langle \psi, \frac{2}{\pi}\int_0^\infty BR(\lambda)\phi_n\, d\lambda\right\rangle_{\mathcal{E}^0} \\ &= \frac{2}{\pi}\int_0^\infty \langle R(\lambda)B\psi, \lim_{n\to\infty}\phi_n\rangle_{\mathcal{E}^0}\, d\lambda \\ &= \left\langle \psi, \frac{2}{\pi}\int_0^\infty BR(\lambda)\phi\, d\lambda\right\rangle_{\mathcal{E}^0}.\end{aligned}$$

□

Lemma 29. $F - F^* \in \mathcal{K}(\mathcal{E}^0)$.

Proof. Note that

$$(F - F^*)\psi = \frac{2}{\pi}\int_0^\infty R(\lambda)c(df)R(\lambda)\psi\, d\lambda.$$

The integrand is compact by the Rellich Lemma and the integral converges by

$$\|R(\lambda)c(df)R(\lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq C(d+\lambda^2)^{-2}.$$

□

Lemma 30. $[f, R(\lambda)] = R(\lambda)(Bc(df) + c(df)B)R(\lambda) \in \mathcal{L}(\mathcal{E}^0)$.

Proof. For an element $\phi = (B^2 + f + \lambda^2)\psi$, we have

$$\begin{aligned}&(fR(\lambda) - R(\lambda)f)(B^2 + f + \lambda^2)\psi \\ =&f\psi - R(\lambda)(B^2 f - Bc(df) - c(df)B + f^2 + f\lambda^2)\psi \\ =&f\psi - R(\lambda)((B^2 + f + \lambda^2)f - Bc(df) + c(df)B)\psi.\end{aligned}$$

It follows that

$$[f, R(\lambda)](D^2 + f + \lambda^2)\psi = R(\lambda)(Bc(df) + c(df)B)R(\lambda)\phi.$$

One then uses that $R(\lambda)$, B and $c(df)$ are bounded and adjointable. □

Theorem 31. *Let B be G-invertible at infinity. Then $F^2 \sim 1$ modulo $\mathcal{K}(\mathcal{E}^0)$.*

Proof. Note that by [14] Lemma 1.8, $R(0)^{1/2}B$ extends by continuity to an operator $L \in \mathcal{L}(\mathcal{E}^0)$ and that F differs from L by a compact operator. Furthermore, $F - F^* \in \mathcal{K}(\mathcal{E}^0)$ by the previous lemma. Thus it is sufficient to show that $LL^* - 1 \in \mathcal{K}(\mathcal{E}^0)$. For $\psi \in \mathcal{E}^2$, we have

$$\begin{aligned}(LL^* - 1)\psi &= R(0)^{1/2}B^2RR(0)^{1/2}(B^2 + f)\psi - \psi \\ &= R(0)(B^2 + f)\psi - \psi - R(0)^{1/2}fR(0)R(0)^{1/2}(B^2 + f)\psi \\ &= -R(0)^{1/2}fR(0)^{1/2}\psi.\end{aligned}$$

By Lemma 30, $fR(0)^{1/2}$ differs from $R(0)^{1/2}f$ by a compact operator:

$$\begin{aligned}fR(0)^{1/2} &= \frac{2}{\pi}\int_0^\infty (R(\lambda)f + R(\lambda)(Bc(df) + c(df)B)R(\lambda))\,d\lambda \\ &= R(0)^{1/2}f + \frac{2}{\pi}\int_0^\infty R(\lambda)(Bc(df) + c(df)B)R(\lambda)\,d\lambda.\end{aligned}$$

The last integral is compact, hence $-R(0)^{1/2}fR(0)^{1/2} \in \mathcal{K}(\mathcal{E}^0)$. This operator agrees with $LL^* - 1 \in \mathcal{L}(\mathcal{E}^0)$ on a dense submodule and thus on all of $\mathcal{E}^0$. It follows that $LL^* - 1$ is compact. □

Thus we arrive at our first main result.

Theorem 32. *Let (G, M, E) be a $\mathbb{Z}_2$-graded G-triple equipped with an odd operator B that is G-invertible at infinity. Then the bounded transform of B, $F \in \mathcal{L}(\mathcal{E}^0)$, is $C^*(G)$-Fredholm with an index in $K_0(C^*(G))$.*

We shall write $\mathrm{index}_G(F)$ for the $C^*(G)$-index of F.

4.1. A Simplified Definition of F.

We now show that in fact

$$F = \frac{2}{\pi}\int_0^\infty BR(\lambda)\,d\lambda = BR(0)^{1/2}.$$

This will simplify certain calculations, for instance in section 7. The argument uses facts about regular operators and Bochner integration.

First note that $R(0)^{1/2}$ has range equal to $\mathcal{E}^1$ (see [34]). Moreover, using the functional calculus for regular operators (see [33] section 7, [18] Theorem 1.19 and [34] Chapter 10), we deduce that $R(0)^{1/2} \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$. Using these facts, we have that:

Proposition 33. *Let $F \in \mathcal{L}(\mathcal{E}^0)$ be defined as in the previous subsection. Then $F = BR(0)^{1/2}$.*

Proof. Recall that bounded linear maps commute with Bochner integration (see [3] Lemma 11.45). Since F is the continuous extension of

$$\mathcal{E}^1 \ni \psi \mapsto \frac{2}{\pi} \int_0^\infty BR(\lambda)\psi \, d\lambda \in \mathcal{E}^0,$$

F acts on a general element $\phi = \lim_{n\to\infty} \psi_n \in \mathcal{E}^0$ by

$$F\phi = \lim_{n\to\infty} \frac{2}{\pi} \int_0^\infty BR(\lambda)\psi_n \, d\lambda.$$

Since $R(0)^{1/2}$ is bounded $\mathcal{E}^0 \to \mathcal{E}^1$,

$$\begin{aligned} BR(0)^{1/2}\phi &= BR(0)^{1/2} \lim_{n\to\infty} \psi_n \\ &= B \lim_{n\to\infty} R(0)^{1/2}\psi_n, \end{aligned}$$

where the second limit is taken in $\mathcal{E}^1$. This is equal to

$$\begin{aligned} &\lim_{n\to\infty} BR(0)^{1/2}\psi_n \\ &= \lim_{n\to\infty} \frac{2B}{\pi} \left(\int_0^\infty R(\lambda) \, d\lambda \right)_0 \psi_n \\ &= \lim_{n\to\infty} \frac{2B}{\pi} \left(\int_0^\infty R(\lambda) \, d\lambda \right)_1 \psi_n, \end{aligned}$$

where the subscripts $_0$ and $_1$ indicate integration performed in $\mathcal{L}(\mathcal{E}^0)$ and $\mathcal{L}(\mathcal{E}^1)$ respectively (see the remark below). This equals

$$\lim_{n\to\infty} \frac{2B}{\pi} \int_0^\infty R(\lambda)\psi_n \, d\lambda,$$

since pairing with ψ_n is a bounded linear map $\mathcal{L}(\mathcal{E}^1) \to \mathcal{E}^1$. Since the bounded operator $B\colon \mathcal{E}^1 \to \mathcal{E}^0$ commutes with integration in $\mathcal{E}^1$, this equals

$$BR(0)^{1/2}\phi = \frac{2}{\pi} \lim_{n\to\infty} \int_0^\infty BR(\lambda)\psi_n \, d\lambda = F\phi.$$

□

Remark 34. The square roots of $R(0) \in \mathcal{L}(\mathcal{E}^0)$ and $R(0) \in \mathcal{L}(\mathcal{E}^1)$ agree on their common domain $\mathcal{E}^1$. In other words,

$$\left(\int_0^\infty R(\lambda)\, d\lambda\right)_0 \psi = \left(\int_0^\infty R(\lambda)\, d\lambda\right)_1 \psi,$$

for all $\psi \in \mathcal{E}^1$. To see this, note that the inclusion $i : \mathcal{E}^1 \to \mathcal{E}^0$ commutes with integration in $\mathcal{L}(\mathcal{E}^1)$:

$$i\int_0^\infty R(\lambda)\psi_n\, d\lambda = \int_0^\infty iR(\lambda)\psi_n\, d\lambda = \int_0^\infty R(\lambda)\psi_n\, d\lambda.$$

This gives another proof of:

Corollary 35. *$F = BR(0)^{1/2} \colon \mathcal{E}^0 \to \mathcal{E}^0$ is an odd operator.*

Proof. The functional calculus of an even operator is even, hence $R(0)^{1/2}$ is even. F is the composition of the odd operator B with $R(0)^{1/2}$. ☐

Remark 36. To make sense of the Bochner integrals of $R(\lambda)$ used above in the context of the non-separable Banach spaces $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$ and $\mathcal{L}(\mathcal{E}^1)$, it is necessary for the integrand to be a strongly measurable function of $\lambda \in [0, \infty)$. By Pettis' measurability theorem ([41] Theorem 1.1) and the fact that $\lambda \mapsto R(\lambda)$ is continuous, it suffices to show that the image $R(\lambda)$ for all λ is contained in a closed separable subspace of the target Banach space. One sees that this indeed the case from the expansion of $R(\lambda)$ as a Neumann series over countably many intervals, as shown in the following (we state it only for $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$, but the case of $\mathcal{L}(\mathcal{E}^1)$ is similar):

Lemma 37. *The image of the map*

$$[0, \infty) \to \mathcal{L}(\mathcal{E}^0, \mathcal{E}^1), \quad \lambda \mapsto R(\lambda)$$

is contained in a closed separable subspace of $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$.

Proof. Define a sequence $(\Lambda_k)_{k\in\mathbb{N}}$ by $\Lambda_k := \sqrt{\frac{(k-1)d}{2}}$, with d defined as in Lemma 16. From the proof of Lemma 18 one sees that, for $k \geq 1$ and each $\lambda \in [a_k, a_{k+1}] =: J_k$,

$$R(\lambda) = R(\Lambda_k)\left(\sum_{i=0}^\infty (\Lambda_k^2 - \lambda^2)^i R(\Lambda_k)^i\right).$$

Thus for λ belonging to each of the countably many intervals J_k, $k \in \mathbb{N}$, the resolvent $R(\lambda)$ belongs to the closure of the span of $\mathcal{A}_k := \{R(\Lambda_k)^i : i \geq 0\}$. It follows that the closure of the span of the countable union

$$\bigcup_{k \in \mathbb{N}} \mathcal{A}_k \subseteq \mathcal{L}(\mathcal{E}^0, \mathcal{E}^1),$$

which is again separable, contains $R(\lambda)$ for all $\lambda \in [0, \infty)$. □

4.2. G**-invertible Operators as** KK**-elements.** We now study G-invertible-at-infinity operators and their indices via KK-theory. We begin by introducing some notation.

Let $C_b^G(M)$ denote the algebra of bounded, continuous G-invariant functions on M. Let $C_g^{\infty,G}(M) \subset C_b^G(M)$ denote the subalgebra of smooth functions such that for all $\epsilon > 0$, there exists a cocompact subset $M_0 \subseteq M$ such that $\|df\|_{T^*M} < \epsilon$ on $M \backslash M_0$. Let $C_0^G(M)$ be the algebra of G-invariant functions f on M such that for all $\epsilon > 0$ there exists a cocompact subset $M_1 \subseteq M$ such that $|f| < \epsilon$ on $M \backslash M_1$. Thus $C_b^{\infty,G}(M)$ consists of those bounded smooth G-invariant functions f for which $\|df\|_{T^*M} \in C_0^G(M)$.

Remark 38. In [14], the notation $C_g(M)$ is used for the closure in $\|\cdot\|_\infty$ of the smooth functions f on M for which $\|df\|_{T^*M} \in C_0(M)$.

Definition 39. Let the algebra $C_g^G(M)$ be the completion of $C_g^{\infty,G}(M)$ in $C_b^G(M)$ with respect to $\|\cdot\|_\infty$. Define

$$C(\partial_h^G(\overline{M})) = \frac{C_g^G(M)}{C_0^G(M)}.$$

The maximal ideal space $\overline{M}^G$ of $C_g^G(M)$ is called the *Higson G-compactification* of M. The maximal ideal space $\partial_h^G \overline{M}$ of $C(\partial_h^G \overline{M})$ is called the *Higson G-corona* of M.

When G is trivial, the Higson G-compactification and the Higson G-corona give back the Higson compactification and Higson corona of M considered in [14][7]. $C_g^G(M)$, which is still commutative and unital for general G, corresponds to the algebra of continuous functions with values in $\mathbb{C}$ on some *compact* space, which therefore cannot carry a proper G-action if G is not compact. Indeed, we have:

[7]In [14] the notation $\partial_h M$ is used instead of $\partial_h \overline{M}$

Lemma 40. $\overline{M}^G$ *is a compactification of* M/G.

Proof. $C_g^G(M)$ contains $C_0^G(M) \cong C_0(M/G)$, which is the closure of the space of G-invariant cocompact functions on M under $\|\cdot\|_\infty$. Thus $C_g^G(M)$ separates points of M/G from closed subsets and contains the constant functions. Thus $\overline{M}^G$ is a compactification of M/G. □

Theorem 41. *Let* (G, M, E) *be a* G*-triple equipped with an operator* B *that is* G*-invertible at infinity. Let* F *be the bounded transform of* B *defined using a cocompactly supported function* f*, as in Lemma 26. Then* $(\mathcal{E}^0, F)$ *is a Kasparov module over the pair of* C^**-algebras* $(\mathbb{C}, C^*(G))$*. The class* $[\mathcal{E}^0, F] \in KK(\mathbb{C}, C^*(G))$ *is independent of the choice of* f*.*

Proof. The only thing left to prove is that $[\mathcal{E}^0, F]$ is independent of the choice of f.

Next, let f_1 be another smooth G-invariant, cocompactly supported function on M for which $(B^2 + f_1)^{-1} \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$. Define the bounded transforms F and F_1 of B using f and f_1 respectively. We have

$$R(\lambda) - R_1(\lambda) = R(\lambda)(f_1 - f)R_1(\lambda).$$

Thus $BR(\lambda) - BR_1(\lambda)$ can be written as the following composition of bounded adjointable operators:

$$\mathcal{E}^0 \xrightarrow{R_1(\lambda)} \mathcal{E}^2 \xrightarrow{f_1 - f} \mathcal{E}^0 \xrightarrow{BR(\lambda)} \mathcal{E}^0,$$

where the middle arrow is a compact operator by Theorem 12. It follows that $BR(\lambda) - BR_1(\lambda) \in \mathcal{K}(\mathcal{E}^0)$. We also have that

$$\|BR(\lambda) - BR_1(\lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq C(d + \lambda^2)^{-3/2}.$$

From this estimate and the dominated convergence theorem it follows that for any $\mathcal{E}^0 \ni \phi = \lim_{n\to\infty} \psi_n$,

$$\begin{aligned}(F - F_1)\phi &= \frac{2}{\pi} \lim_{n\to\infty} \int_0^\infty (BR(\lambda) - BR_1(\lambda))\psi_n \, d\lambda \\ &= \frac{2}{\pi} \int_0^\infty (BR(\lambda) - BR_1(\lambda))\phi \, d\lambda.\end{aligned}$$

□

The image of $[\mathcal{E}^0, F]$ under the isomorphism

$$KK(\mathbb{C}, C^*(G)) \cong K_0(C^*(G))$$

coincides with $\mathrm{index}_G(F)$ defined after Theorem 32, hence we will denote this map also by index_G.

Remark 42. With F as above, one can in fact show that $[F, h] \in \mathcal{K}(\mathcal{E}^0)$ for all h in $C_g^G(M)$. Indeed, it suffices to establish this for all $h \in C_g^{\infty,G}(M)$. One can, for example, proceed as in [14] Lemma 1.12, replacing $C_g^\infty(M)$ with $C_g^{\infty,G}(M)$. However, a shorter proof is as follows. First note that $hR(0)^{1/2}$ and $R(0)^{1/2}h$ differ by the operator

$$\frac{2}{\pi}\int_0^\infty R(\lambda)(Bc(dh) + c(dh)B)R(\lambda)\, d\lambda.$$

Since $h \in C_g^{\infty,G}(M)$, $\|dh\|_{T^*M} \in C_0^{\infty,G}(M)$. Thus dh can be approximated by cocompactly supported endomorphisms, for which the integral converges absolutely. On the other hand, hB and Bh differ by $c(dh)$, which is again a limit of cocompactly supported endomorphisms. Thus $hBR(0)^{1/2} - BR(0)^{1/2}h$ is compact.

Remark 43. Despite the above remark, we have not shown that $(\mathcal{E}^0, F)$ gives a cycle in $KK(C_g^G(M), C^*(G))$, since the non-separability of $C_g^G(M)$ makes it unclear whether this KK-theory forms a group. Note that even if it does, the equivariant index map we get would still not quite be an analogue of the Baum-Connes assembly map

$$KK^G(C_0(M), \mathbb{C}) \to K_*(C^*(G))$$

defined using the descent homomorphism and pairing with a canonical projection when M/G is compact [29]. Indeed, to define an analogue of this map using $C_g^G(M)$ one we would need to pair with a projection in a suitable crossed product algebra adapted to G-invertible-at-infinity operators on non-cocompact manifolds.

4.3. **A Simple Example.** Let G be a discrete group and $\tilde{M} := G \times M$. Then the action of G on $\tilde{M}$ given by $g \cdot (h, x) := (gh, x)$ is proper and free. Let B be an operator on a bundle $E \to M$ that is invertible at infinity. B lifts in an obvious way (see [6]) to an operator $\tilde{B}$ on the lifted bundle

$\tilde{E} \to \tilde{M}$ that is G-invertible at infinity. Denote the normalisations of B and $\tilde{B}$ by $F = BR(0)^{1/2}$ and $\tilde{F} = \tilde{B}\tilde{R}(0)^{1/2}$ respectively.

Following [4] Theorem 3.8 or 3.12, one can construct a parametrix Q for the Fredholm operator B so that the operators

$$BQ - 1 =: S_0, \qquad QB - 1 =: S_1$$

are of trace class. By [4] Lemma 3.1,

$$\begin{aligned} \operatorname{index} B &= \text{trace } S_1 - \text{trace } S_0 \\ &= \int_M k_{S_1}(x,x)\,dx - \int_M k_{S_0}(x,x)\,dx. \end{aligned}$$

The parametrix Q lifts to a G-invariant parametrix $\tilde{Q}$ for $\tilde{B}$, so that

$$\tilde{B}\tilde{Q} - 1 := \tilde{S}_0, \qquad \tilde{Q}\tilde{B} - 1 := \tilde{S}_1$$

are lifts of S_0 and S_1. Pick a cut-off function $\mathfrak{c}_0$ on G, which has a natural G-action given by group multiplication. Then $\mathfrak{c}_0$ extends in the obvious way to a cut-off function $\mathfrak{c}$ on $\tilde{M} = G \times M$, defined by $\mathfrak{c}(g,m) := \mathfrak{c}_0(g)$, and the quantity

$$\text{trace}_G(\tilde{S}_i) := \int_{\tilde{M}} \mathfrak{c}(\tilde{x}) k_{\tilde{S}_i}(\tilde{x},\tilde{x})\,d\tilde{x} = \int_G \mathfrak{c}_0(g) \left(\int_M k_{S_i}(x,x)\,dx \right) dg = \text{trace } S_i$$

makes sense. In fact, it can be shown (similarly to [46]) that $\text{trace}_G(\tilde{S}_i)$ is independent of the choice of $\mathfrak{c}_0$ and that it can be expressed in terms of a homomorphism τ_G of abelian groups

$$\tau_G \colon \langle \operatorname{index}_G \tilde{F} \rangle \to \mathbb{R},$$

where the domain is the subgroup of $K_0(C^*(G))$ generated by the element $\operatorname{index}_G \tilde{F}$. Moreover,

$$\begin{aligned} \tau_G(\operatorname{index}_G \tilde{F}) &= \text{trace}_G(\tilde{S}_1) - \text{trace}_G(\tilde{S}_0) \\ &= \text{trace } S_1 - \text{trace } S_0 \\ &= \operatorname{index} B. \end{aligned}$$

It follows that if $\text{trace}_G(\tilde{S}_1) - \text{trace}_G(\tilde{S}_0) \neq 0$ then $\operatorname{index}_G \tilde{F} \neq 0$ in $K_0(C^*(G))$. This gives an easy way of seeing that the map index_G we have defined in this section is non-trivial for non-cocompact manifolds. For a concrete example, let N be the Kummer hyperplane in $\mathbb{CP}^3$ defined by

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0.$$

Then N is spin with $\hat{A}(N) = 2$. Denote the spin-Dirac operator on N by $\partial\!\!\!/_N$, and note that the spin structure on N extends to a spin structure on the cylinder $M = \mathbb{R} \times N$, with associated spin-Dirac operator $\partial\!\!\!/_M$ acting on the spinor bundle $S \to M$. Consider a smooth monotonic function $\psi\colon M \to [0,1]$ such that $\psi|_{(-\infty,-1]} \equiv 1$ and $\psi|_{[1,\infty)} \equiv -1$. The Callias-type operator[8] $B := D + \Phi$, where

$$D := \begin{bmatrix} 0 & \partial\!\!\!/_M \\ \partial\!\!\!/_M & 0 \end{bmatrix}, \qquad \Phi := i \otimes \begin{bmatrix} 0 & -\psi \\ \psi & 0 \end{bmatrix},$$

acts on the $\mathbb{Z}_2$-graded bundle $S \otimes (\mathbb{C} \oplus \mathbb{C}^{\mathrm{op}})$, where $^{\mathrm{op}}$ means the opposite grading, and is invertible at infinity. Let F_B denote the bounded transform of B. Then by [14] Theorem 2.9 (see also [5] Theorems 1.5 and 2.1),

$$\operatorname{index} F_B = \operatorname{index} \partial\!\!\!/_N = \hat{A}(N) = 2.$$

Lifting B to an operator $\tilde{B}$ on $\tilde{M} := G \times M$ as above gives, in the language of the next section, a G-Callias-type operator. Letting $F_{\tilde{B}}$ be its bounded transform, one sees that

$$\operatorname{index}_G F_{\tilde{B}} \neq 0 \in K_0(C^*(G)).$$

5. G-Callias-type Operators

We now define, for G a general Lie group, *G-invariant Callias-type operators* and prove that they are G-invertible at infinity. This notion generalises the Callias-type operators studied in [14] section 2 to the equivariant setting. Throughout this section, except when indicated otherwise, M will be a non-cocompact G-Riemannian manifold.

Definition 44. Let $E \to M$ be a $\mathbb{Z}_2$-graded G-Clifford bundle with Dirac operator D. An odd-graded, G-invariant endomorphism $\Phi \in C^1(M, \operatorname{End} E)$ is called *G-admissible for D* (or simply *G-admissible*) if

(a) $\Phi D + D\Phi$ is a bounded, order-0 bundle endomorphism;

[8] For the definition of Callias-type operators see the next section or [14] for the non-equivariant case.

(b) Φ is self-adjoint with respect to the inner product on E;
(c) there exists a cocompact subset $K \subseteq M$ and $C > 0$ such that

$$\Phi D + D\Phi + \Phi^2 \geq C \text{ on } M\backslash K.$$

Definition 45. Let E and D be as in the previous definition. A *G-invariant Callias-type operator* (or simply a *G-Callias-type operator*) is an operator of the form

$$B := D + \Phi,$$

where the endomorphism Φ is G-admissible for D.

We postpone the explicit construction of G-admissible endomorphisms until section 6. Our first task is to show that a G-Callias-type operator is G-invertible at infinity and hence $C^*(G)$-Fredholm.

5.1. **Positivity of B^2+f.** Let $B = D+\Phi$ be a G-Callias-type operator. We first show that there exists a cocompactly supported, G-invariant function f such that $B^2 + f$ is a positive unbounded operator with respect to the $C^*(G)$-valued inner product on $\mathcal{E}^0$.

Lemma 46. *Let $B = D + \Phi$ be a G-Callias-type operator on $E \to M$. Then there exist a G-invariant, cocompactly supported function f and a constant $C > 0$ such that for all $s \in H^2(E)$,*

$$\langle (B^2 + f)s, s\rangle_{H^0} \geq C\langle s, s\rangle_{H^0}.$$

Proof. Let $\pi\colon M \to M/G$ be the projection. The G-bundle E descends to a topological vector bundle $\check{E}$ over M/G, while the G-invariant bundle map $\Phi D + D\Phi + \Phi^2$ descends to a continuous bundle map χ on $\check{E}$. Let K be the cocompact in Definition 44 (c). Then over K, $\Phi D + D\Phi + \Phi^2$ is bounded below by the same constant as for χ, namely

$$\inf_{x\in\pi(K)}\left(\inf_{v\in\check{E}_x}\left(\frac{\langle \chi v, v\rangle}{\|v\|^2}\right)\right) \geq \inf_{x\in\pi(K)}(-\|\chi\|).$$

Adding a sufficient large, compactly supported function $s\colon M/G \to [0,\infty)$ to $\Phi D + D\Phi + \Phi^2$ makes it positive on K. The result now follows from the fact that D^2 is non-negative and that on $M\backslash K$ we have $\Phi D + D\Phi + \Phi^2 \geq C$, where C is the constant in Definition 44 (c). ∎

The above lemma and Corollary 8 imply that, given a G-Callias-type operator B, $B^2 + f$ defines an element of $\mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)$. We now show that this operator is positive in the sense of $C^*(G)$. The proof is inspired by Kasparov's proof of [29] Lemma 5.3.

Proposition 47. *Let (G, M, E) be a $\mathbb{Z}_2$-graded non-cocompact G-triple and B a G-Callias-type operator on E. Then there exists a G-invariant cocompactly supported function f and a constant $C > 0$ such that for all $s \in \mathcal{E}^2$,*

$$\langle (B^2 + f)s, s\rangle_{\mathcal{E}^0} \geq C\langle s, s\rangle_{\mathcal{E}^0}.$$

Proof. Let $B = D+\Phi$ and $\mathfrak{c}$ be a cut-off function on M. Since $\langle D^2 e, e\rangle_{\mathcal{E}^0} \geq 0$ for all $e \in C_c^\infty(E)$, it suffices to show that there exist f and $C > 0$ such that

$$\langle (B^2 + f - D^2)e, e\rangle_{\mathcal{E}^0} = \langle (D\Phi + \Phi D + \Phi^2 + f)e, e\rangle_{\mathcal{E}^0} \geq C\langle e, e\rangle_{\mathcal{E}^0},$$

which is equivalent to

$$\langle (D\Phi + \Phi D + \Phi^2 + f - C)e, e\rangle_{\mathcal{E}^0} \in C^*(G)_+.$$

Choose f as in the previous lemma, so that $D\Phi + \Phi D + \Phi^2 + f - C$ is a bounded positive operator on $L^2(E)$. Since G acts by unitaries on $L^2(E)$, and conjugating by unitaries preserves the functional calculus, $D\Phi + \Phi D + \Phi^2 + f - C$ has a bounded, G-invariant positive square root Q. Now the operator $Q\mathfrak{c}Q$ has cocompactly compactly supported Schwartz kernel

$$k_{Q\mathfrak{c}Q}(x, y) = \int_M k_Q(x, z)\mathfrak{c}(z)k_Q(z, y)\, d\mu(z),$$

since $\mathfrak{c}$ has cocompactly compact support. Let $\tilde{K}$ be a cocompactly compact subset of M containing $\operatorname{supp}(k_{Q\mathfrak{c}Q})$. Define $a\colon G \to \mathbb{R}$ by

$$g \mapsto \langle Q\mathfrak{c}Qg(e), g(e)\rangle_{H^0(E)} = \int_M \left\langle \int_M k_{Q\mathfrak{c}Q}(x, y)g(e)(y)\, d\mu(y), g(e)(x)\right\rangle_E d\mu(x).$$

Then $\operatorname{supp}(a)$ is contained in

$$\begin{aligned} &\{g \in G \,|\, \operatorname{supp}(g(e)) \cap \tilde{K} \neq \emptyset\} \\ &= \{g \in G \,|\, \operatorname{supp}(g(e)) \cap \operatorname{supp}(g(e)) \cap \tilde{K} \neq \emptyset\} \\ &\subseteq \{g \in G \,|\, \operatorname{supp}(g(e)) \cap G \cdot \operatorname{supp}(e) \cap \tilde{K} \neq \emptyset\}. \end{aligned}$$

Since the set $G \cdot \operatorname{supp}(e) \cap \tilde{K}$ is compact, $\operatorname{supp}(a)$ is a compact subset of G, by properness of the G-action. Thus the map $G \mapsto H^0(E)$, $g \mapsto \sqrt{\mathfrak{c}}Q(g(e))$

has compact support in G. It follows that for any unitary representation of G on a Hilbert space $(H, (\ ,\ )_H)$ and $h \in H$,

$$v := \int_G \mu^{-1/2}(g)\sqrt{\mathfrak{c}}Q(g(e)) \otimes g(h)\, dg$$

is a well-defined vector in $H^0(E) \otimes H$. Its norm $\|v\|_{H^0(E)\otimes H}$ is equal to

$$\begin{aligned}
&\int_G \int_G \mu^{-1/2}(g')\mu^{-1/2}(g)\langle \sqrt{\mathfrak{c}}Qg(e), \sqrt{\mathfrak{c}}Q(g'(e))\rangle_{H^0} \cdot (g(h), g'(h))_H\, dg\, dg' \\
&= \int_G \int_G \mu^{-1/2}(g')\mu^{-1/2}(g)\langle Q\mathfrak{c}Qg(e), g'(e)\rangle_{H^0} \cdot (g(h), g'(h))_H\, dg\, dg' \\
&= \int_G \int_G \mu^{-1/2}(g)\mu^{-1/2}(g')\langle Q\mathfrak{c}Qg(e), g'(e)\rangle_{H^0} \cdot (g(h), g'(h))_H\, dg'\, dg,
\end{aligned}$$

by Fubini's theorem. Since $\langle\ ,\ \rangle_{H^0(E)}$ and $(\ ,\ )_H$ are G-invariant[9], this equals

$$\int_G \int_G \mu^{-1/2}(g)\mu^{-1/2}(g')\langle g'^{-1}(Q\mathfrak{c}Q)g(e), e\rangle_{H^0} \cdot (g'^{-1}g(h), h)_H\, dg'\, dg.$$

$$= \int_G \int_G \mu^{-1/2}(g)\mu^{-1/2}(g')\langle g^{-1}(Q\mathfrak{c}Q)g'^{-1}g(e), e\rangle_{H^0} \cdot (g'^{-1}g(h), h)_H\, dg'\, dg.$$

Substituting $g' \mapsto gg'$ and following simliar computations to the proof of [29] Lemma 5.3, one sees that this is equal to

$$\int_G \left\langle \left(\int_G g(Q\mathfrak{c}Q)\, dg \right)(e), e \right\rangle_{\mathcal{E}^0} (g') \cdot (g'(h), h)_H\, dg'.$$

Thus $\langle (\int_G g(Q\mathfrak{c}Q)\, dg)(e), e\rangle_{\mathcal{E}^0}$ is a positive operator on H for all unitary representations of G, where we let $f \in C_c(G)$ act on H by

$$f \cdot h := \int_G f(g)g(h)\, dg.$$

It follows that the element

$$\langle (D\Phi + \Phi D + \Phi^2 + f - C)e, e\rangle_{\mathcal{E}^0} = \langle Q^2 e, e\rangle_{\mathcal{E}^0} = \left\langle \left(\int_G g(Q\mathfrak{c}Q)\, dg \right)(e), e \right\rangle_{\mathcal{E}^0}$$

[9] G acts unitarily on H by hypothesis and unitarily on $H^0(E)$ by Remark 5.

is in $C^*(G)_+$, where we have used the fact that, since Q is G-invariant,

$$\int_G g(Q\mathfrak{c}Q)\,dg = Q^2.$$

□

Next, we show that B has a regular self-adjoint extension. In fact, we show that B is essentially self-adjoint with regular closure.

5.2. **Essential Self-adjointness and Regularity of** B**.** The notation in this subsection and the next will distinguish between the formally self-adjoint operator B and its closure $\overline{B}$.

Proposition 48. *Let (G, M, E) be a non-cocompact $\mathbb{Z}_2$-graded G-triple with M complete. Let B be a G-Callias-type operator on E. Then B is essentially self-adjoint in $\langle\,,\,\rangle_{\mathcal{E}^0}$, and $\overline{B}$ is regular.*

Proof. When (G, M, E) is cocompact with M complete, the regularity and self-adjointness of $\overline{B}$ was established in [29] Theorem 5.8. Now suppose M is complete and non-cocompact. There exists a family $\{a_\epsilon \colon \epsilon > 0\}$ of compactly supported smooth functions taking values in $[0, 1]$ satisfying

$$\bigcup_{\epsilon>0}\{a_\epsilon^{-1}(1)\} = M, \qquad \sup_{x\in M}\|da_\epsilon(x)\| \leq \epsilon, \qquad a_{\epsilon_1}^{-1}(1) \subseteq a_{\epsilon_2}^{-1}(1) \text{ if } \epsilon_2 \leq \epsilon_1.$$

Let $s \in C_c^\infty(E)$. For a given $\epsilon > 0$, choose a_ϵ so that $a_\epsilon \equiv 1$ on $\mathrm{supp}(s)$. Since there exists an exhaustion of M by cocompact, G-stable open subsets $\{U_k \colon k \in \mathbb{N}\}$, we can pick i large enough so that U_i contains $\mathrm{supp}(a_\epsilon)$. Now form the double M' of the closure $\overline{U_i}$, using a G-equivariant collar neighbourhood (which exists by [28] Theorem 3.5), and extend the action of G to M' naturally. Then M' is a cocompact G-Riemannian manifold without boundary, and all G-structures on U_i naturally extend to M' also.

In particular, this gives a G-Callias-type operator B', acting on $E' \to M'$. Since M' is cocompact, the closure $\overline{B'}$ is regular and self-adjoint. Now use $\overline{B'}$ to form G-Sobolev modules $\{\mathcal{E}'^{,i}\}$ on M', following section 3. Because $\overline{B'} + i \colon \mathcal{E}'^{,1} \to \mathcal{E}'^{,0}$ is onto, we can find $e \in \mathcal{E}'^{,1}$ for which $(\overline{B'} + i)e = s$. We have $\langle s, s\rangle_{\mathcal{E}^{,0}} \geq \langle e, e\rangle_{\mathcal{E}^{,0}} \in C^*(G)_+$ and

$$(\overline{B}+i)(a_\epsilon e) = (\overline{B'}+i)(a_\epsilon x) = [\overline{B'}, a_\epsilon]e + a_\epsilon(\overline{B'}+i)e = c(\nabla a_\epsilon)e + a_\epsilon(\overline{B'}+i)e.$$

Note that $c(\nabla a_\epsilon)$ is a bounded operator on $\mathcal{E}^0$, but since a_ϵ was not assumed to be G-invariant, it could fail to be adjointable. Nevertheless, boundedness is enough, since it enables us to conclude

$$\left\|(\overline{B}+i)(a_\epsilon e)\right\|_{\mathcal{E}^0} = \left\|c(\nabla a_\epsilon)e + a_\epsilon(\overline{B'}+i)e\right\|_{\mathcal{E}^0} \leq (1+\epsilon)\left\|s\right\|_{\mathcal{E}^0}.$$

Taking a sequence $\epsilon_i \to 0$, this implies that

$$(\overline{B}+i)(a_{\epsilon_i} e) \to a_{\epsilon_i}(\overline{B'}+i)e = a_{\epsilon_i} s = s,$$

thus $\overline{B}+i$ has dense range. □

5.3. **Non-complete Cocompact Manifolds.** In this subsection we show that a G-invariant Dirac-type operator B on any cocompact G-manifold M is regular and self-adjoint, without the assumption of completeness. The reader who is interested in G-Callias-type operators may skip this subsection without loss of continuity.

First recall that the *formal adjoint* $P^\natural$ of a pseudodifferential operator P of order m is the pseudodifferential operator obtained by taking the adjoint of the symbol of P.

Lemma 49. *Let P be a G-invariant pseudodifferential operator of order k. Then $P^\natural$ is also a G-invariant pseudodifferential operator of order k.*

Let us recall the following boundedness result for elliptic operators. (More generally, the next two results hold for any G-invariant properly supported operator with symbol bounded at infinity in the cotangent direction by some $C>0$ in the sense of [29] Theorem 3.2.)

Proposition 50. *Let (G, M, E) be a cocompact G-triple, and let P be a G-invariant, properly supported elliptic pseudodifferential operator of order $k \leq 0$ on E. Then P extends to a bounded operator $P\colon H^0(E) \to H^{-k}(E)$.*

Proof. The case $k=0$ is proven in [29] Lemma 5.2. For $k<0$ we have

$$\langle Pu, Pu\rangle_{H^{-k}(E)} = \sum_{i=0}^{-k}\langle B^i Pu, B^i Pu\rangle_{H^0(E)}.$$

Since each operator B^iP, $0 \leq i \leq k$ is G-invariant, properly supported and of order ≤ 0, the $k=0$ case implies that for each i, there exists $C_i > 0$

such that for all $u \in H^{k-i}(E)$, we have $\langle B^i Pu, B^i Pu\rangle_{H^0(E)} \leq C_i \|u\|^2_{H^0(E)}$. It follows that

$$\|Pu\|^2_{H^{-k}(E)} = \sum_{i=0}^{-k} \left\|B^i Pu\right\|^2_{H^0(E)} \leq \left(\sum_{i=0}^{-k} C_i\right) \|u\|^2_{H^0(E)}.$$

□

Corollary 51. *A G-invariant, properly supported elliptic pseudodifferential operator P of order $k \leq 0$ defines a bounded adjointable operator $\mathcal{E}^0 \to \mathcal{E}^{-k}$.*

Proof. P is a G-invariant, properly supported operator on $C_c^\infty(E)$ that extends to a bounded operator $H^0(E) \to H^{-k}(E)$ by the above proposition. By Proposition 4, P defines a bounded adjointable operator $\mathcal{E}^0 \to \mathcal{E}^{-k}$. □

Proposition 52. *Let P be a G-invariant, properly supported elliptic pseudodifferential operator of order $k \leq 0$. Then P and $P^\natural$ define elements of $\mathcal{L}(\mathcal{E}^0)$ and $P^* = P^\natural \in \mathcal{L}(\mathcal{E}^0)$.*

Proof. By the corollary above, P^* and $P^\natural$ define bounded adjointable operators on $\mathcal{E}^0$. Let k_P and $k_{P^\natural}$ denote the Schwartz kernels of P and $P^\natural$. By hypothesis, k_P and $k_{P^\natural}$ are G-invariant. It follows from the definition of the formal adjoint that for all $x, y \in M$, $k_{P^\natural}(x,y) = k_P(x,y)^*$, where * means the adjoint of an endomorphism. It follows from the definition in section 3 of the $C_c(G)$-valued inner product $\langle\ ,\ \rangle_0$ on $C_c^{\infty,0}(E)$ that for $g \in G$,

$$\begin{aligned}\langle u, P^*v\rangle_0(g) &= \langle Pu, v\rangle_0(g)\\ &= \int_M \left\langle \int_M k_P(x,y)u(y)\,dy, (gv)(x)\right\rangle_E d\mu\\ &= \int_M \left\langle u(x), \int_M k_{P^\natural}(x,y)((gv)(y))\,dy\right\rangle_E d\mu.\end{aligned}$$

By G-invariance of $k_{P^\natural}$, this is equal to

$$\int_M \left\langle u(x), \int_M k_{gP^\natural}(x,y)v(y)\,dy\right\rangle_E d\mu = \langle u, P^\natural v\rangle_0(g).$$

Since P^* and $P^\natural$ agree on a dense subset, they must agree on $\mathcal{E}^0$. □

Since B is G-invariant, properly supported and elliptic of order 1, it has a G-invariant properly supported parametrix Q of order -1, such that

$$R \coloneqq 1 - BQ, \qquad T \coloneqq 1 - QB$$

are G-invariant smoothing operators. By Proposition 4, the closures $\overline{B}$ and $\overline{Q}$ belong to $\mathcal{L}(\mathcal{E}^1, \mathcal{E}^0)$ and $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$ respectively. We also have $\overline{T} \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$ and $\overline{BT}, \overline{BQ} \in \mathcal{L}(\mathcal{E}^0)$. These properties of the G-invariant pseudodifferential calculus now enable us to proceed as in [21]:

Lemma 53. *Let (G, M, E) be a cocompact G-triple, and let B be a G-Callias-type operator on E. Let Q, R and T be as above. Then $\overline{B} \circ \overline{Q} = \overline{BQ}$ and $\overline{B} \circ \overline{T} = \overline{BT}$.*

Proof. First we show that $\overline{BQ} = \overline{B} \circ \overline{Q}$. By the observations we have just made, $\mathrm{dom}(\overline{B} \circ \overline{Q}) = \mathrm{dom}(\overline{BQ}) = \mathcal{E}^0$, hence it suffices to show that the two operators agree on $\mathcal{E}^0$. Thus let $x \in \mathcal{E}^0$. Taking the initial domains of B, Q and $B \circ Q$ to be $C_c^\infty(E)$, this means, by definition, that there exists a sequence x_n in $C_c^\infty(E)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} B \circ Q(x_n) = \overline{BQ}x$, with both limits taken in the $\mathcal{E}^0$-norm. Since $\overline{Q} \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$,

$$\overline{Q}x = \overline{Q} \lim_{n\to\infty} x_n = \lim_{n\to\infty} \overline{Q} x_n = \lim_{n\to\infty} Q x_n$$

in the $\mathcal{E}^1$-norm. Since $\overline{B} \in \mathcal{L}(\mathcal{E}^1, \mathcal{E}^0)$ and $Qx_n \in C_c^\infty(E)$, this implies

$$\overline{B} \circ \overline{Q}(x) = \overline{B} \lim_{n\to\infty} Q x_n = \lim_{n\to\infty} \overline{B} \circ Q(x_n) = \lim_{n\to\infty} B \circ Q(x_n) = \overline{BQ}x.$$

The same argument with Q replaced by T shows that $\overline{B} \circ \overline{T} = \overline{BT}$. □

Lemma 54. *Let (G, M, E) be a cocompact G-triple, B be a G-Callias-type operator on E and Q, R and T be as above. Then $\mathcal{E}^1 = \mathrm{im}(\overline{Q}) + \mathrm{im}(\overline{T})$.*

Proof. That $\mathrm{im}(\overline{Q}) + \mathrm{im}(\overline{T}) \subseteq \mathcal{E}^1$ is clear. To show $\mathcal{E}^1 \subseteq \mathrm{im}(\overline{Q}) + \mathrm{im}(\overline{T})$, let $x \in \mathcal{E}^1$. Then there exists a sequence $x_n \in C_c^\infty(E)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Bx_n = \overline{B}x$, in $\mathcal{E}^0$. Now $QBx_n = x_n - Tx_n$, hence

$$\overline{Q} \circ \overline{B}(x) = \overline{Q} \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} QBx_n = \lim_{x\to\infty} (x_n - Tx_n) = x - \overline{T}x.$$

This shows that $x \in \mathrm{im}(\overline{Q}) + \mathrm{im}(\overline{T})$. □

Theorem 55. *Let (G, M, E) be a cocompact G-triple, with B be a G-Callias-type operator on E. Then B is essentially self-adjoint.*

Proof. Since B is symmetric in $\langle\, , \,\rangle_{\mathcal{E}^0}$, we have $\overline{B} \subseteq B^*$. For the other inclusion, note that the relation $BQ = 1 - R$ implies $(BQ)^* = 1 - R^*$, while we also have $Q^*B^* \subseteq (BQ)^*$. Now $Q^* = \overline{Q^\natural}$ by Lemma 49 (which applies because Q has order ≤ 0). Since $Q^\natural$ is G-invariant and properly supported, the domain of Q^* is $\mathcal{E}^0$, and we have $\mathrm{dom}(B^*) = \mathrm{dom}(Q^*B^*)$. If $x \in \mathrm{dom}(B^*)$, $Q^*B^* \subseteq (BQ)^* = 1 - R^*$, so $Q^*(B^*x) + R^*x = x$. Hence

$$\mathrm{dom}(B^*) \subseteq \mathrm{im}(Q^*) + \mathrm{im}(R^*) = \mathrm{im}(\overline{Q^\natural}) + \mathrm{im}(\overline{R^\natural}) \subseteq \mathrm{dom}(\overline{B}) = \mathcal{E}^1,$$

where the middle two relations follow from the fact that $Q^\natural$ and $R^\natural$ are G-invariant, properly supported elliptic operators of order ≤ -1. □

Theorem 56. *Let (G, M, E) be a cocompact G-triple, and let B be a G-Callias-type operator on E. Then $\overline{B}$ is regular.*

Proof. It suffices to show that the graph $G(\overline{B})$ of $\overline{B}$, which is a closed submodule of $\mathcal{E}^0 \oplus \mathcal{E}^0$, has an orthogonal complement. By a standard criterion (see [34] p. 238) it is enough to show that $G(\overline{B})$ is the image of some bounded adjointable map $U \colon \mathcal{E}^0 \oplus \mathcal{E}^0 \to \mathcal{E}^0 \oplus \mathcal{E}^0$. Define

$$U \colon \mathcal{E}^0 \oplus \mathcal{E}^0 \to \mathcal{E}^0 \oplus \mathcal{E}^0,$$
$$(x, y) \mapsto (\overline{Q}x + \overline{T}y, \overline{BQ}x + \overline{BT}y).$$

By the first lemma above, $\overline{BQ}x + \overline{BT}y = \overline{B} \circ \overline{Q}(x) + \overline{B} \circ \overline{T}(y)$. Together with the second lemma, this means $G(\overline{B}) = \mathrm{im}(U)$. As the graph of a closed operator, $\mathrm{im}(U) = G(\overline{B})$ is closed and hence orthogonally complementable by [34] Theorem 3.2. □

This completes the proof of essential self-adjointness of B and regularity of $\overline{B}$ for a general cocompact manifold. The results then extend to the non-cocompact setting as in the proof of Proposition 48.

5.4. ***G*-invertibility at Infinity of *B*.** In this subsection we establish the following key result:

Theorem 57. *Let (G, M, E) be a non-cocompact $\mathbb{Z}_2$-graded G-triple with $B = D + \Phi$ a G-Callias-type operator. Then B is G-invertible at infinity.*

First, we have:

Proposition 58. *For $0 \neq \mu \in \mathbb{R}$, $B^2 + \mu^2 \colon \mathcal{E}^2 \to \mathcal{E}^0$ is bijective, with an inverse in $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$.*

Proof. By part (e) of [16] 3.3, surjectivity follows from the fact that B has regular closure, which was shown in the previous subsection. Injectivity follows from the estimate that for all $v \in \mathcal{E}^2$,

$$\left\|(B^2+\mu^2)v\right\|_{\mathcal{E}^0} \geq \mu^2 \left\|v\right\|_{\mathcal{E}^0}.$$

By the open mapping theorem, the inverse $(B^2+\mu^2)^{-1}$ is bounded. Adjointability is not an issue, as it can be seen from general theory that a bounded inverse of an invertible bounded adjointable operator T between Hilbert A-modules $\mathcal{M}$, $\mathcal{N}$ is must be adjointable, as follows. It follows from [10] Theorem II.7.2.9 that $\operatorname{ran}(T^*) = \ker(T)^{\perp} = \{0\} = \mathcal{M}$ and $\ker(T^*) = \operatorname{ran}(T)^{\perp} = F^{\perp} = \{0\}$. Hence T^* has an inverse satisfying

$$\langle T^{-1}y, x\rangle_{\mathcal{M}} = \langle T^{-1}y, T^*(T^*)^{-1}x\rangle_{\mathcal{M}} = \langle y, (T^*)^{-1}x\rangle_{\mathcal{N}}.$$

Thus the inverse of T^{-1} is $(T^*)^{-1}$. □

Remark 59. The explicit formula

$$((B^2+\mu^2)^{-1})^* = B^2 + (1-\mu^2)(B^2+\mu)^{-1}B^2 + (B^2+\mu)^{-1}$$

can be proven as in [16] Proposition 4.9.

Theorem 60. *Let (G, M, E) be a non-cocompact $\mathbb{Z}_2$-graded G-triple and $B = D + \Phi$ a G-Callias-type operator on E. Then there exists a non-negative, G-invariant, cocompactly supported function f on M such that $B^2 + f$ is invertible at infinity.*

Proof. The analogue of the argument in [16] subsection 4.10 also works in our situation, having established the results in the previous sections. Thus we only sketch the argument, highlighting the differences. First one shows that for $\mathbb{R} \ni \mu \neq 0$ and $f\colon M \to \mathbb{R}$ a G-invariant uniformly bounded smooth function with $\mu^2 > \|f\|_\infty$, the operator $B^2+\mu^2+f$ has an inverse in $\mathcal{L}(\mathcal{E}^0) \cap \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$. This inverse is given by the Neumann series

$$(B^2+\mu^2+f)^{-1} = (B^2+\mu^2)^{-1}\sum_{k=0}^{\infty}(-1)^k\left\{f(B^2+\mu^2)^{-1}\right\}^k.$$

By Proposition 47, we can pick a G-invariant cocompactly supported function f and a constant $C > 0$ such that for all $s \in \mathcal{E}^2$,

$$\langle (B^2+f)s, s\rangle_{\mathcal{E}^0} \geq C\langle s, s\rangle_{\mathcal{E}^0}.$$

Pick $\mu \neq 0$ such that $\mu^2 > \|f\|_\infty$. We may write

$$B^2 + f = (1 - \mu^2(B^2 + \mu^2 + f)^{-1})(B^2 + \mu^2 + f).$$

The Cauchy-Schwartz inequality then yields

$$\left\|\mu^2(B^2 + \mu^2 + f)^{-1}\right\|_{\mathcal{L}(\mathcal{E}^0)} \leq \frac{\mu^2}{\mu^2 + C} < 1,$$

so that the Neumann series

$$(B^2 + f)^{-1} = (B^2 + \mu^2 + f)^{-1} \sum_{k=0}^{\infty} \left\{\mu^2(B^2 + \mu^2 + f)^{-1}\right\}^k$$

converges in norm and gives an element of $\mathcal{L}(\mathcal{E}^0) \cap \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$. □

6. The Endomorphism Φ

We now construct G-admissible endomorphisms Φ using K-theory of the Higson G-corona defined in section 4. This is an equivariant version of the construction in [14].

We also establish that the K-theory of the Higson G-corona is highly non-trivial, which provides some motivation to study of index theory of G-Callias-type operators.

6.1. **The Construction.**

We first show how to construct a G-admissible endomorphism Φ starting from an element of K-theory (even or odd, depending on the dimension of M).

Suppose first that M is an odd-dimensional complete non-cocompact G-Riemannian manifold. Let $E_0 \to M$ be a complex Clifford bundle with an ungraded Dirac operator D_{E_0}. Let P be the entry-wise lift of a projection

$$P_0 \in \mathrm{Mat}_l(C(\partial_h^G(\overline{M})))$$

to an element of $\mathrm{Mat}_l(C_g^G(M))$, for some $l \geq 0$. Form the $\mathbb{Z}_2$-graded Clifford bundle $E := E_0 \otimes (\mathbb{C}^l \oplus (\mathbb{C}^l)^{op})$, with the associated Dirac operator

$$D := \begin{bmatrix} 0 & D_{E_0} \otimes d^l \\ D_{E_0} \otimes d^l & 0 \end{bmatrix},$$

where d^l is the trivial connection on $\mathbb{C}^l \to M$. Define the endomorphism

$$\Phi := i \otimes \begin{bmatrix} 0 & 1 - 2P \\ 2P - 1 & 0 \end{bmatrix} \in C^1(M, \mathrm{End}(E)).$$

Proposition 61. *Φ is a G-admissible endomorphism.*

Proof. Clearly Φ is odd-graded and self-adjoint. Let ∇^{E_0} denote the Clifford connection on E_0. Then the twisted connection on $E_0 \otimes \mathbb{C}^l$ is $\nabla^{E_0} \otimes 1 + 1 \otimes d^l$. Since $1 - 2P$ acts only on the factor $\mathbb{C}^l$, it commutes with $\nabla^{E_0} \otimes 1$. Also,

$$[1 \otimes d^l, 1 \otimes (1 - 2P)] = d^{l^2} P,$$

where d^{l^2} is the trivial connection on $\mathrm{End}(\mathbb{C}^l)$. Thus we have

$$D\Phi + \Phi D = 2i \otimes \begin{bmatrix} -c(d^{l^2} P) & 0 \\ 0 & c(d^{l^2} P) \end{bmatrix}.$$

By definition of $C_g^{\infty,G}(M)$, $c(d^{l^2} P) \to 0$ and $\Phi^2 \to 1$ at infinity in the direction transverse to the G-orbits. Thus there exists a cocompact subset $K \subseteq M$ such that on $M \backslash K$, $D\Phi + \Phi D + \Phi^2 \geq c$ for some constant $c > 0$. $\square$

By Theorem 57, the operator $B \coloneqq D + \Phi$ is G-invertible at infinity. Thus B has an index in $K_0(C^*(G))$ by Theorem 32. Let F be the bounded transform of B. By the same reasoning as in [14] section 2, $\mathrm{index}_G(F)$ depends only on the class of projection P_0 in $K_0(C(\partial_h^G(\overline{M}))$.

Note that the endomorphism Φ arising from the zero element of $K_0(C(\partial_h^G(\overline{M})))$ gives rise to the invertible operator

$$B = \begin{bmatrix} 0 & D + i \\ D - i & 0 \end{bmatrix},$$

for which $\mathrm{index}_G(BR(0)^{1/2}) = 0 \in K_0(C^*(G))$.

Suppose instead that M is an even-dimensional, complete non-cocompact G-Riemannian manifold. Let $E_0 \to M$ be a $\mathbb{Z}_2$-graded Clifford bundle with grading z. An element $[U_0] \in K_1(C(\partial_h^G(\overline{M})))$ is represented by a unitary matrix $U_0 \in \mathrm{Mat}_l(C(\partial_h^G(\overline{M})))$ for some $l \geq 0$. Let U be a lift of U_0 such that $U^*U - 1 \in \mathrm{Mat}_l(C_0^G(M))$ and $U \pmod{\mathrm{Mat}_l(C_0^G(M))}$ represents $[U_0]$. Form the $\mathbb{Z}_2$-graded bundle $E \coloneqq E_0 \otimes (\mathbb{C}^l \oplus (\mathbb{C}^{l,op}))$ with Dirac operator D formed from the connection on E_0 twisted by d^l. Let

$$\Phi \coloneqq z \otimes \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix} \in C^1(M, \mathrm{End}(E)),$$

and define $B = D + \Phi$.

Proposition 62. *Φ is a G-admissible endomorphism.*

Proof. Clearly Φ is odd-graded and self-adjoint. Similarly to the calculation in the previous proposition, one has

$$D\Phi + \Phi D = z \begin{bmatrix} -c(d^{l^2}U) & 0 \\ 0 & c(d^{l^2}U) \end{bmatrix}.$$

By definition of $C_g^{\infty,G}(M)$, $c(d^{l^2}(U)) \to 0$ and $\Phi^2 \to 1$ at infinity in the direction transverse to the G-orbits. It follows that there is some cocompact subset $K \subseteq M$ such that on $M\backslash K$ $D\Phi + \Phi D + \Phi^2 \geq c > 0$ for some c. $\square$

As in the odd-dimensional case, one can verify that the $\mathrm{index}_G(F)$, where F is the bounded transform of B, depends only on $[U_0] \in K_1(C(\partial_h^G(\overline{M})))$.

For $\dim M \equiv i \pmod 2$, $i = 0, 1$, the map

$$K_i(C(\partial_h^G(\overline{M}))) \to K_0(C^*(G)),$$

$$[R] \mapsto \mathrm{index}_G(F_R),$$

is a homomorphism of abelian groups. Here R denotes a projection or unitary matrix representative of K_0 or K_1, depending on $\dim M$, and F_R denotes the bounded transform of $B = D + \Phi_R$, where Φ_R is formed using R.

6.2. **Interpretation as a KK-product.** Let F be the bounded transform of a G-Callias-type operator $B = D + \Phi$. In this subsection we interpret the cycle $[\mathcal{E}^0, F]$ in terms of a KK-pairing, similar to those constructed in [14] and [32].

Suppose first that M is even-dimensional, with a Dirac operator D_0 acting on a $\mathbb{Z}_2$-graded G-Clifford bundle $E_0 \to M$. Given a G-admissible Φ of the kind constructed in the previous subsection, one can form the bundle E and the operator D. Let $B = D + \Phi$. Using the procedure in section 3, form the G-Sobolev modules $\mathcal{E}^i_{D_0}$ and $\mathcal{E}^i$ associated to D_0 and B respectively. Note that $\mathcal{E}^0 = \mathcal{E}^0_{D_0} \otimes \mathbb{C}^2$.

Let $R_{D_0}(0) \coloneqq (D_0^2 + 1)^{-1}$. Then by the same kind of analysis as in section 4, one knows that $R_{D_0}(0)^{1/2}$ is a bounded adjointable operator $\mathcal{E}^0_{D_0} \to \mathcal{E}^1_{D_0}$. Define its bounded transform $F' \coloneqq D_0 R_{D_0}(0)^{1/2}$.

Proposition 63. *The pair* $(F', \mathcal{E}^0_{D_0})$ *defines a cycle*

$$[D_0] \coloneqq [F', \mathcal{E}^0_{D_0}] \in KK(C_0^G(M), C^*(G)).$$

Proof. Each $a \in C_0^G(M)$ defines a bounded adjointable operator $\mathcal{E}^0_{D_0} \to \mathcal{E}^0_{D_0}$. Let $C_c^G(M) \subseteq C_b(M)$ be the subring of G-invariant, cocompactly supported functions on M, with closure $C_0^G(M)$ (see subsection 4.2). It suffices to show that for all $a \in C_c^G(M)$, we have

$$a((F')^2 - 1) \in \mathcal{K}(\mathcal{E}^0_{D_0}), \qquad F'a - aF' \in \mathcal{K}(\mathcal{E}^0_{D_0}).$$

The second relation follows from the same sort of argument used to prove Theorem 41, applied to F' instead of F. By functional calculus of the regular operator D_0 one has

$$(F')^2 - 1 = -R_{D_0}(0) \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2).$$

Hence by Theorem 12, $a((F')^2 - 1))$ is a compact operator $\mathcal{E}^0_{D_0} \to \mathcal{E}^0_{D_0}$. □

From here, our construction is similar that in [14] 2.3.3 and 2.4.3 for the non-equivariant case, so we will be rather brief. The differences are: instead of the algebra $C_g(M)$ used in [14], we use $\mathbb{C}$; instead of $C_0(M)$ we use $C_0^G(M)$; and the $\mathbb{C}$-Fredholm cycle $[h, F_2]$ used there is replaced by the $C^*(G)$-Fredholm cycle $[D] = [\mathcal{E}^0_{D_0}, F']$.

Define the $\mathbb{Z}_2$-graded Hilbert $C_0^G(M)$-module

$$L := C_0^G(M) \otimes \mathbb{C}^l \otimes \mathbb{C}^2,$$

and consider the cycle $[\Phi] := [L, \Phi] \in KK(\mathbb{C}, C_0^G(M))$. We have:

Proposition 64. *Let E, E_0, D, D_0, B be as before, with $\dim M$ even. Then*

$$[\mathcal{E}^0, F] = [\Phi] \otimes_{C_0^G(M)} [D_0] \in KK(\mathbb{C}, C^*(G)).$$

Next we consider the case of odd-dimensional M with the initial Dirac operator D_0 being ungraded.

Let $C^{1,0}$ be the Clifford algebra generated by a single element X satisfying the relation $X^2 = -1$. Let z be the grading of $C^{1,0}$, and form the operator

$$D = D_0 \otimes zX,$$

which defines a class $[D] \in KK(C_0^G(M) \otimes C^{1,0}, C^*(G))$. Consider

$$L := C_0^G(M) \otimes \mathbb{C}^l \otimes C^{1,0}$$

as a $\mathbb{Z}_2$-graded Hilbert $C_0^G(M) \otimes C^{1,0}$-module. Write

$$\Phi = i(1 - 2P) \otimes X,$$

which defines a bounded adjointable operator on L. One verifies as in [14] 2.3.3 that the pair (L, Φ) defines a Kasparov module

$$[\Phi] := [L, \Phi] \in KK(\mathbb{C}, C_0^G(M) \otimes C^{1,0}).$$

Let $\tau_{C^{1,0}}$ be the isomorphism

$$KK(\mathbb{C}, C^*(G)) \xrightarrow{\sim} KK(\mathbb{C} \otimes C^{1,0}, C^*(G) \otimes C^{1,0})$$

defined in [9] 17.8.5. Then we have:

Proposition 65. *Let E, D, B be as before, with $\dim M$ odd. Then*

$$[\mathcal{E}^0, F] = \tau_{C^{1,0}}^{-1}([\Phi] \otimes_{C_0(M)} [D]) \in KK(\mathbb{C}, C^*(G)).$$

6.3. **The G-corona.** We now prove that the K-theory of the Higson G-corona is highly non-trivial. In this section we will use "non-cocompact" to mean not cocompact, as opposed to not necessarily cocompact.

Recall the definition of the Higson G-compactification $\overline{M}^G$ from section 4. We will refer to this as the "smooth" definition, in contrast to the more frequently seen definition of the Higson compactification $\overline{X}^d$ of a metric space (X, d), in the non-equivariant setting. We are motivated by the following result:

Proposition 66. [30] *Let (X, d) be a non-compact connected metric space with proper metric d. Then $\check{H}^1(\overline{X}^d)$ contains a subgroup isomorphic to the additive reals $(\mathbb{R}, +)$.*

Corollary 67. *Let (X, d) be as in the above proposition. Then $K^1(\overline{X}^d)$ is uncountable.*

Proof. The Chern character gives an isomorphism

$$\mathrm{ch}\colon K^1(\overline{X}^d) \otimes \mathbb{Q} \xrightarrow{\sim} \check{H}^{\mathrm{odd}}(\overline{X}^d, \mathbb{Q}) \cong \check{H}^{\mathrm{odd}}(\overline{X}^d) \otimes \mathbb{Q}.$$

By the previous proposition, $\check{H}^1(X)^d \otimes \mathbb{Q}$, and hence $K^1(\overline{X}^d) \otimes \mathbb{Q}$, is uncountable. Since tensoring with $\mathbb{Q}$ does not alter countability, it follows that $K^1(\overline{X}^d)$ is uncountable. □

From the isomorphism $C_b(M/G) \cong C_b^G(M)$ and the observation that $C_g^G(M)$ is a closed subring of $C_b^G(M)$ containing the constant functions and also $C_0^G(M) \cong C_0(M/G)$, one sees that $C_g^G(M)$ determines a compactification of M/G.

Definition 68. Let M and G be as above. The compactification corresponding to $C_g^G(M)$ is the *Higson G-compactification of M*, denoted by $\overline{M}^G$.

$\overline{M}^G$ is *partial* compactification of M in the direction transverse to the G-orbits. It is the unique compactification of M/G such that a function $f\colon M/G \to \mathbb{C}$ extends continuously to $\overline{M}^G$ if and only if $f \in C_g^G(M)$.

One can also characterise $\overline{M}^G$ using functions with values in a compact submanifold $Y \subseteq \mathbb{R}^N$ for some N.[10] Recall that $f \in C_g^{\infty,G}(M)$ if f is bounded, smooth and G-invariant, and $\|df\|_{T^*M} \in C_0^G(M)$.

Definition 69. Let M be a non-cocompact G-Riemannian manifold and Y a compact submanifold of $\mathbb{R}^N$ for some $N \geq 0$. Let $\pi_i\colon \mathbb{R}^N \to \mathbb{R}$ be the projection map onto the i-th coordinate. Suppose $f\colon M \to Y$ is a G-invariant function. Then we say that $f \in C_g^G(M, Y)$ if $\pi_i \circ f \in C_g^G(M)$ for each $1 \leq i \leq N$.

Proposition 70. *Let M be a complete non-cocompact G-Riemannian manifold. Let $Y \subseteq \mathbb{R}^N$ be a compact submanifold for some $N \geq 0$ and $f\colon M/G \to Y$ continuous. Then f has a continuous extension to the Higson G-compactification $\overline{M}^G$ if and only if $f \in C_g^G(M, Y)$. Further, $\overline{M}^G$ is the unique such compactification of M/G.*

Proof. Let $f \in C_g^G(M, Y)$. Without loss of generality, since Y is compact, we may take Y to be a submanifold of $[0,1]^N$. For each j, let $\pi_j\colon [0,1]^N \to [0,1]$ be the projection onto the j-th coordinate. Then $\pi_j \circ f \in C_g^G(M)$ and hence can be extended to a continuous function $\overline{\pi_j \circ f}$ on $\overline{M}^G$. The function

$$\begin{aligned} \overline{f}\colon \overline{M}^G &\to [0,1]^N, \\ x &\mapsto \left(\overline{\pi_1 \circ f}(x), \ldots, \overline{\pi_N \circ f}(x)\right) \end{aligned}$$

is a continuous extension of f to $\overline{M}^G$, taking values in $Y \subseteq [0,1]^N$.

On the other hand, suppose $f\colon M \to Y$ is a continuous map not in $C_g^G(M, Y)$. Then there exists some j for which $\pi_j \circ f \notin C_g^G(M)$. Thus $\pi_j \circ f$ does not extend continuously to $\overline{M}^G$. It follows that f does not extend continuously to $\overline{M}^G$, for otherwise $\pi_j \circ f$ would also.

[10]Compare [30] Proposition 1.

To show uniqueness, suppose $\mathcal{C}$ is a compactification of M/G with the above properties. Then the set of continuous function $M/G \to \mathbb{C}$ that extend to $\mathcal{C}$ is precisely the closed subring $C_g^G(M) \subseteq C_b^G(M)$. Taking the maximal ideals of $C_g^G(M)$ with the convex-hull topology then recovers $\overline{M}^G$. $\square$

Now we apply this proposition to $Y = S^1 \subseteq \mathbb{R}^2$. The first Čech cohomology $\check{H}^1(\overline{M}^G)$ can be identified with the group $[\overline{M}^G, S^1]$ of homotopy classes of maps $\overline{M}^G \to S^1$, with a certain group operation derived from pointwise multiplication of functions. Let $e\colon \mathbb{R} \to S^1$ be the covering map $x \mapsto e^{2\pi i x} \in S^1 \subseteq \mathbb{C}$. Let the algebra $C_g^{G,u}(M)$ be defined by the same conditions as $C_g^G(M)$ except that the functions need not be bounded.

For clarity, write $C_g^{G,b}(M) := C_g^G(M)$, and let $C_g^{G,b}(M,\mathbb{R})$ and $C_g^{G,u}(M,\mathbb{R})$ be the real-valued functions in $C_g^{G,b}(M)$ and $C_g^{G,u}(M)$.

Suppose $f \in C_g^{G,u}(M,\mathbb{R})$. Then it is easy to see that $e \circ f \in C_g^G(M,S^1)$, and so by the above proposition, f has an extension $\overline{e \circ f}\colon \overline{M}^G \to S^1$. Define the map $a\colon C_g^{G,u}(M) \to \check{H}^1(\overline{M}^G)$ by

$$f \mapsto [\overline{e \circ f}] \in [\overline{M}^G, S^1] \cong \check{H}^1(\overline{M}^G).$$

Proposition 71. *There is an exact sequence of abelian groups*

$$0 \to C_g^{G,b}(M,\mathbb{R}) \hookrightarrow C_g^{G,u}(M,\mathbb{R}) \xrightarrow{a} \check{H}^1(\overline{M}^G).$$

Proof. Clearly the sequence is exact at $C_g^{G,b}(M,\mathbb{R})$. To show that $C_g^{G,b}(M,\mathbb{R}) \subseteq \ker(a)$, let $f \in C_g^{G,b}(M,\mathbb{R})$. Then f has an extension $\overline{f}\colon \overline{M}^G \to \mathbb{R}$. Since $\overline{f}$ is a lift for $\overline{e \circ f}$, the latter map is null-homotopic. Next, to show that $\ker(a) \subseteq C_g^{G,b}(M,\mathbb{R})$, suppose $f \in C_g^{G,u}(M,\mathbb{R})$ and $a(f) = 0 \in [\overline{M}^G, S^1]$. This implies that $a(f) = [\overline{e \circ g}]$ is null-homotopic and hence $\overline{e \circ f}$ has a lift $q\colon \overline{M}^G \to \mathbb{R}$. Since both $q|_{M/G}$ and f are lifts for $\overline{e \circ f}|_{M/G}$, they must be related by a deck transformation. Since q is bounded, f must be bounded, and $f \in C_g^{G,b}(M,\mathbb{R})$. Hence $\ker(a) = C_g^{G,b}(M,\mathbb{R})$. $\square$

Corollary 72. *$\check{H}^1(\overline{M}^G)$ contains a subgroup isomorphic to $C_g^{G,u}(M,\mathbb{R})/C_g^{G,b}(M,\mathbb{R})$.*

We now show that $C_g^{G,u}(M,\mathbb{R})/C_g^{G,b}(M,\mathbb{R})$ is uncountable. Since the action of G on M is proper and isometric, M/G has a natural distance function

$d_{M/G}$ given by the minimal geodesic distance between orbits (see [2] p. 74). $d_{M/G}$ lifts to a function d_M on M that is well-defined on pairs of orbits. For a fixed point $x_0 \in M$, the function

$$d_{x_0} \colon M \to \mathbb{R}, \qquad x \mapsto d_M(x_0, x)$$

is G-invariant and 1-Lipschitz. We now show that d_{x_0} can be approximated by a smooth G-invariant Lipschitz function.

Proposition 73. *For any $\epsilon > 0$, there is a smooth G-invariant function $d_{x_0,\epsilon}$ on M such that for any $x \in M$,*

$$|d_{x_0,\epsilon}(x) - d_{x_0}(x)| < \epsilon, \qquad \|d(d_{x_0,\epsilon})\|_{T^*M} \leq 1 + \epsilon.$$

Proof. Since M is a complete Riemannian manifold, it is a proper metric space and hence separable. By [7], for any function $\delta : M \to (0, \infty)$ and $r > 0$, one can construct a smooth approximation f_{x_0} such that for all $x \in M$,

$$|f_{x_0}(x) - d_{x_0}(x)| < \delta(x), \qquad \|df_{x_0}(x)\|_{T^*M} < 1 + r.$$

Since the G-action is proper, we can find a cut-off function $\mathfrak{c}$ on M. Let $\tilde{f}_{x_0}$ be the G-average of f_{x_0}, defined by

$$\tilde{f}_{x_0}(x) := \int_G \mathfrak{c}(g^{-1}x) f_{x_0}(g^{-1}x)\, dg.$$

Clearly $\tilde{f}_{x_0}$ is G-invariant. We now argue that $\tilde{f}_{x_0}$ is Lipschitz. Observe that $\left\|d\tilde{f}_{x_0}\right\|_{T^*M}$ is equal to

$$\begin{aligned}
&\left\|\int_G \left(d\mathfrak{c}(g^{-1}x)\right) f_{x_0}(g^{-1}(x)) + \mathfrak{c}(g^{-1}x) df_{x_0}(g^{-1}x)\, dg\right\|_{T^*M} \\
\leq &\left\|\int_G \left(d\mathfrak{c}(g^{-1}x)\right) f_{x_0}(g^{-1}(x))\, dg\right\|_{T^*M} + \int_G \mathfrak{c}(g^{-1}x) \left\|df_{x_0}(g^{-1}x)\right\|_{T^*M} dg \\
= &\left\|\int_G d\mathfrak{c}(g^{-1}x) \left(d(g^{-1}(x)) + \delta_x\right) dg\right\|_{T^*M} + \int_G \mathfrak{c}(g^{-1}x) \left\|df_{x_0}(g^{-1}x)\right\|_{T^*M} dg,
\end{aligned}$$

where $\delta_x := f_{x_0}(g^{-1}(x)) - d(g^{-1}(x))$. Suppose we choose $\delta_x \leq \delta$ uniformly for some constant $\delta > 0$. Then the above expression is bounded by

$$\left\|\int_G \left(d\mathfrak{c}(g^{-1}x)\right) d(g^{-1}(x))\, dg\right\|_{T^*M} + \delta \int_G \left\|d\mathfrak{c}(g^{-1}x)\right\|_{T^*M} dg$$

$$+ \int_G \mathfrak{c}(g^{-1}x) \left\| df_{x_0}(g^{-1}x) \right\|_{T^*M} dg.$$

The first summand vanishes because d is G-invariant, since for any G-invariant function $l\colon M \to \mathbb{R}$, we have

$$\int_G d\mathfrak{c}(g^{-1}x) l(g^{-1}x)\, dg = l(x) \int_G d\mathfrak{c}(g^{-1}x)\, dg = l(x)d(1) = 0.$$

For a fixed $x \in M$, let $\operatorname{supp} \mathfrak{c}(g^{-1}x)$ be the support in G of the function $g \mapsto \mathfrak{c}(g^{-1}x)$. Let U be a G-stable, cocompact subset of M, and let δ_U be an upper-bound for δ on U. Then for $x \in U$, we have

$$\delta_U \int_G \left\| d\mathfrak{c}(g^{-1}x) \right\|_{T^*M} dg \leq \delta_U \left(\sup_{x \in U} \| d\mathfrak{c}(x) \| \right) \left(\sup_{x \in U} \left\{ \operatorname{vol} \left(\operatorname{supp} \mathfrak{c}(g^{-1}x) \right) \right\} \right).$$

The function $x \mapsto d\mathfrak{c}(x)$ has cocompactly compact support and hence is bounded above on U. The function $x \mapsto \operatorname{vol}(\operatorname{supp} \mathfrak{c}(g^{-1}x))$ is G-invariant and so descends to a compact set in M/G, whence it is also bounded above on U. The above product is bounded by $\delta_U C_U$, where the constant $C_U > 0$ depends only on the cocompact set U. The third summand above is bounded by $1 + r$, given our choice of f_{x_0}. Thus the whole expression is strictly less than $1 + \delta_U C_U + r$. By picking $r < \frac{\epsilon}{2}$ and δ_U such that $\delta_U C_U < \frac{\epsilon}{2}$, this expression can be made to be less than $1 + \epsilon$.

By further choosing $\delta_U \leq \epsilon$, we have that for $x \in U$,

$$\left| \tilde{f}_{x_0}(x) - d_{x_0}(x) \right| = \int_G \mathfrak{c}(g^{-1}x) \left| f_{x_0}(g^{-1}x) - d_{x_0}(g^{-1}x) \right| dg \leq \delta_U \leq \epsilon.$$

To obtain the estimate on all of M, let $\mathcal{U} = \{U_i\colon i \in \mathbb{N}\}$ be a locally finite, countable open cover of M by G-stable cocompact sets. There exists a smooth partition of unity subordinate to $\mathcal{U}$ consisting of G-invariant functions ψ_{U_i} [39]. Then $\delta(x) \coloneqq \sum_{i=1}^{\infty} \psi_i(x) \delta_{U_i}(x)$ is a well-defined smooth function $M \to \mathbb{R}$, so we may choose the approximation f_{x_0} so that for all $x \in M$, $|f_{x_0} - d_{x_0}| < \delta(x)$. For each $x \in M$, let $C_x \coloneqq \max\{C_i\colon x \in U_i,\ i \in \mathbb{N}\}$. Then it holds that for all $x \in M$,

$$\delta(x) C_x < \frac{\epsilon}{2}.$$

By our previous calculations this implies that for all $x \in M$,

$$\left| \tilde{f}_{x_0}(x) - d_{x_0}(x) \right| \leq \epsilon, \qquad \left\| d\tilde{f}_{x_0} \right\|_{T^*M} \leq 1 + \epsilon.$$

Finally, the desired function is $d_{x_0,\epsilon} \coloneqq \tilde{f}_{x_0}$. □

Remark 74. This argument applies more generally to produce a G-invariant smooth approximation $\tilde{f}$ starting from a G-invariant Lipschitz function f.

Proposition 75. *Let M be a non-cocompact G-Riemannian manifold. Then $\check{H}^1(\overline{M}^G)$ contains a subgroup isomorphic to $(\mathbb{R}, +)$.*

Proof. We follow the idea in [30]. Fix $x_0 \in M$. Define $d_M, d_{M/G}, d_{x_0}$ and $d_{x_0,\epsilon}$ as in the proof of the previous proposition, so that for $y \in M$,

$$|d_{x_0,\epsilon}(y) - d_{x_0}(y)| < \epsilon, \qquad \|d(d_{x_0,\epsilon})(y)\|_{T^*M} < 2.$$

For each $r \in \mathbb{R}$, consider the function $\rho_r \in C_g^G(M)$ defined by

$$\rho_r \colon M \to \mathbb{R}, \qquad y \mapsto r \ln d_{x_0,\epsilon}(y).$$

If $r \neq s$, $\rho_r - \rho_s$ is unbounded. Hence the subgroup

$$\{[\rho_r] \colon r \in \mathbb{R}\} \subseteq \frac{C_g^{G,u}(M, \mathbb{R})}{C_g^{G,b}(M, \mathbb{R})}$$

is uncountable; therefore $C_g^{G,u}(M, \mathbb{R})/C_g^{G,b}(M, \mathbb{R})$ has rank at least $2^{\aleph_0}$. Now (M, d_M) is a proper non-compact metric space and hence separable. It follows that $(M/G, d_{M/G})$ is also separable. Thus the rank of $C_g^{G,u}(M, \mathbb{R})$, and hence that of $C_g^{G,u}(M, \mathbb{R})/C_g^{G,b}(M, \mathbb{R})$, is at most $2^{\aleph_0}$. Since $C_g^{G,u}(M, \mathbb{R})/C_g^{G,b}(M, \mathbb{R})$ is an abelian, divisible and torsion-free group, it is isomorphic to $(\mathbb{R}, +)$. □

Corollary 76. *Let M be a complete non-cocompact G-Riemannian manifold. Then both $K_1(C_g^G(M))$ and $K_1(C(\partial_h^G(\overline{M})))$ are uncountable.*

Proof. The first statement follows from the Chern character isomorphism

$$\mathrm{ch} \colon K^1(\overline{M}^G) \otimes \mathbb{Q} \xrightarrow{\sim} \check{H}^1(\overline{M}^G, \mathbb{Q}) \cong \check{H}^1(\overline{M}^G) \otimes \mathbb{Q}$$

together with the above proposition, noting that $K_1(C_g^G(M)) \cong K^1(\overline{M}^G)$. The second statement follows from the first by the Five Lemma. □

This shows that when M is a complete non-cocompact G-Riemannian manifold, the group $K_1(C(\partial_h^G(\overline{M})))$ is uncountable. On the other hand $K_0(C_g^G(M))) \cong K^0(\overline{M}^G)$ contains a copy of $\mathbb{Z}$, since $\overline{M}^G$ is a compact space, so that $K_0(C(\partial_h^G(\overline{M}))) \cong K^0(\partial_h^G(\overline{M}))$ is also infinite. In summary, we have:

Theorem 77. *Let M be a complete non-cocompact G-Riemannian manifold. Then the K-theory of the Higson G-corona of M is uncountable.*

7. Invariant Metrics of Positive Scalar Curvature

7.1. **An Existence Result.** In this subsection, let G be an almost-connected Lie group acting properly on manifold M. By Abels' the global slice theorem [1], M has a K-slice N. Recall the following result [20].

Theorem 78. *Let $\pi\colon M \to B$ be a fibre bundle with compact fibre N and structure group K. Suppose that M and B both have proper G-actions making π G-equivariant. Let g_N be a K-invariant Riemannian metric on N. Then there is a G-invariant Riemannian metric g_M on M such that π is a G-equivariant Riemannian submersion with totally geodesic fibres.*

This result enables us, by the same method as in [20], to prove:

Proposition 79. *Suppose G is an almost-connected Lie group acting properly on M, and let K be a maximal compact subgroup of G and N a global K-slice of M, so that we have $M \cong G \times_K N$. Then if N admits a K-invariant positive scalar curvature metric, M admits a G-invariant positive scalar curvature metric.*

Proof. Let $\kappa_{G/K}$ denote the scalar curvature of the G-invariant Riemannian metric $g_{G/K}$ on the base. Note that since G/K is a homogeneous space, $-\infty < \kappa_{G/K} < 0$ is a negative constant. Let $H \subseteq TM$ be an Ehresmann connection. Then as in the proof of Theorem 78 above, we may lift $g_{G/K}$ to a G-invariant metric g_H on H, as well as lift the K-invariant Riemannian metric g_N on N to a metric on the vertical subbundle $V \subseteq TM$. Define a G-invariant metric on M by $g_M \coloneqq g_H \oplus g_V$.

Since N is compact by hypothesis, its scalar curvature κ_N satisfies $\inf\{\kappa_N\} =: \kappa_0 > 0$. Now let T and A denote the O'Neill tensors of the submersion π (their definitions can be found in [38]). By Theorem 78 above, the fibres of M are totally geodesic, so $T = 0$. Pick an orthonormal basis of horizontal vector fields $\{X_i\}$. By G-invariance, we have that at any point $p \in M$,

$$\sum_{i,j} ||A_{X_i}(X_j)||_p = \sum_{i,j} ||A_{X_i}(X_j)||_{h\cdot p}$$

for all group elements $h \in G$. This means $\sup_{p\in X}\{\sum_{i,j}||A_{X_i}(X_j)||_p\} =: A_0 < \infty$, as M/G is compact. Now by a result of Kramer ([31] p. 596), we can relate the scalar curvatures by

$$\kappa_M(p) = \kappa_{G/K} + \kappa_N(p) - \sum_{i,j} ||A_{X_i}(X_j)||_p.$$

Upon scaling the fibre metric on N by a positive factor t, we obtain

$$\kappa_M(p) \geq \kappa_{G/K} + t^{-2}\kappa_0 - A_0 > 0 \qquad \text{whenever} \quad 0 < t < \sqrt{\frac{\kappa_0}{-\kappa_{G/K} + A_0}}.$$

Thus g_M is a G-invariant metric of positive scalar curvature on M. □

By a straightforward generalisation of a result in [35], if K is non-abelian and K acts freely on a non-compact manifold N with bounded sectional curvature, then N admits a K-invariant metric of positive scalar curvature. Thus we obtain the following existence result.

Theorem 80. *Let G be an almost-connected Lie group acting properly on M, K a maximal compact subgroup of G and N a global K-slice of M with bounded sectional curvature. Then if K is non-abelian and acts freely on N, M has a G-invariant Riemannian metric of positive scalar curvature.*

The next subsection establishes a vanishing theorem for the index of G-Callias-type operators in the presence of G-invariant positive scalar curvature.

7.2. A Non-cocompact Lichnerowicz-type Obstruction.

Let G be a Lie group and M a G-equivariantly spin, non-cocompact Riemannian manifold. Let $S \to M$ denote the spinor bundle and $\partial\!\!\!/_0$ the G-spin-Dirac operator. Let Φ be a G-admissible endomorphism constructed from a projection or unitary in $K_i(C(\partial_h^G(\overline{M})))$, and form the G-Callias-type operator $B = \partial\!\!\!/ + \Phi$, where $\partial\!\!\!/$ is a $\mathbb{Z}_2$-graded version of $\partial\!\!\!/_0$ acting on the $\mathbb{Z}_2$-graded bundle E constructed from $S = E_0$ (in the notation of section 5).

Form the $\mathbb{Z}_2$-graded G-Sobolev modules $\mathcal{E}^i = \mathcal{E}^i(E)$ as in section 3. Then by Proposition 7, B extends to a bounded adjointable operator $\mathcal{E}^1 \to \mathcal{E}^0$. For $\lambda \in \mathbb{R}$, let $R(\lambda) = (B^2 + f + \lambda^2)^{-1}$, which exists by Lemma 18. Normalising B gives rise to the operator $F := BR(0)^{1/2} \in \mathcal{L}(\mathcal{E}^0)$, by subsection 4.1. Writing F in this way lets us to check the following basic fact.

Lemma 81. *Suppose a G-Callias-type operator B has a bounded inverse $B^{-1}\colon \mathcal{E}^0 \to \mathcal{E}^1$. Then F is invertible in $\mathcal{L}(\mathcal{E}^0)$.*

Proof. By hypothesis B has an inverse B^{-1}, while

$$R(0)^{1/2} = \left((B^2+f)^{-1}\right)^{1/2} = \left((B^2+f)^{1/2}\right)^{-1}$$

is an isomorphism $\mathcal{E}^0 \to \mathcal{E}^1$. Thus a right-inverse for F is given by

$$F^{-1} := (R(0)^{1/2})^{-1}B^{-1} = \left(\sqrt{B^2+f+\lambda^2}\right)B^{-1}\colon \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^0.$$

One can check that this is also a left-inverse for F. □

It follows that if B is invertible then $\operatorname{index}_G F = 0 \in K_0(C^*(G))$. One can see this for example by using the explicit idempotent

$$\begin{bmatrix} S_0^2 & S_0(1+S_0)Q \\ S_1F & 1-S_1^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \operatorname{Mat}_\infty(\mathcal{K}(\mathcal{E}^0)),$$

for $\operatorname{index}_G(F)$, where Q is a parametrix for F and $S_0 = 1-QF$, $S_1 = 1-FQ$ ([17] p. 353); by the above lemma we may take $Q = F^{-1}$, and $S_0 = S_1 = 0$.

Denote the G-invariant Riemannian metric on M by g, which is not to be confused with elements of the group G. Suppose that the scalar curvature κ^g associated to g is everywhere positive. By Lichnerowicz's formula,

$$B^2 = \partial\!\!\!/^2 + \partial\!\!\!/\Phi + \Phi\partial\!\!\!/ + \Phi^2 = \nabla^*\nabla + \frac{\kappa^g}{4} + \partial\!\!\!/\Phi + \Phi\partial\!\!\!/ + \Phi^2,$$

where ∇ is the connection Laplacian on E.

We now show that g can be scaled by an appropriate constant to make B invertible (compare [8] section 2).

For $r > 0$, define the metric rg on M by $rg(v,w) := r \cdot g(v,w)$, and let D^{rg} be the associated Dirac operator. The G-admissible endomophism Φ is still G-admissible for D^{rg}, so the operator $B^{rg} := D^{rg} + \Phi$ is again of G-Callias-type.

Proposition 82. *There exists a positive constant r for which the operator B^{rg} defined above is strictly positive.*

Proof. The operator $\nabla^*\nabla$ is positive, and Φ is pointwise self-adjoint by construction. The endomorphism $\partial\!\!\!/\Phi + \Phi\partial\!\!\!/$ is bounded away from 0 outside a cocompact subset $K \subseteq M$. It is equal to Clifford multiplication by a one-form dR, where R is a projection if $\dim M$ is odd and a unitary if $\dim M$ is even (see subsection 6.1), and is bounded below by a constant C_Φ depending on Φ. Now if $\frac{\kappa^g}{4} > -C_\Phi + \epsilon$ everywhere on K for some $\epsilon > 0$, then B^2 is strictly positive. Otherwise, scale the metric g by a positive number r, so that $\kappa^{rg} = \frac{\kappa^g}{r^2}$, and $c^{rg} = \frac{c}{r}$, where c means Clifford multiplication by a one-form. The latter implies that $C_\Phi^{rg} = \frac{C_\Phi}{r}$, since, for a fixed Φ, the constant C_Φ scales with the metric in the same way as c. As K is cocompact, κ^g is bounded below by some $\kappa_0 > 0$ on K. Thus we can find $r > 0$ such that

$$\frac{\kappa^{rg}}{4} > -C_\Phi^{rg} + \epsilon.$$

It follows that $(B^{rg})^2$ is a strictly positive operator. □

Proposition 83. *Let B be a G-Callias-type operator such that B^2 is strictly positive. Then $B\colon \mathcal{E}^1 \to \mathcal{E}^0$ is invertible.*

Proof. It suffices to show that $B^2\colon \mathcal{E}^2 \to \mathcal{E}^0$ is invertible, from which the conclusion follows since $B(B^2)^{-1}\colon \mathcal{E}^0 \to \mathcal{E}^1$ is a two-sided inverse of B. First we show that $B^2+\mu$ is invertible. By regularity of B and hence B^2, $B^2+\mu^2$ is surjective for every positive number μ^2 (see Chapter 9 of [34]). Further, since B^2 is strictly positive, $B^2+\mu^2$ is injective. By the open mapping theorem, its inverse $(B^2+\mu^2)^{-1}$ is bounded. It remains to show that $(B^2+\mu^2)^{-1}$ is adjointable. Write for short $T := (B^2+\mu^2)^{-1}$. Then T is self-adjoint as a bounded operator $\mathcal{E}^0 \to \mathcal{E}^0$ (defined by composing B with the bounded inclusion $\mathcal{E}^2 \hookrightarrow \mathcal{E}^0$), which follows from Lemma 4.1 in [34] and the estimate

$$\langle u, Tu\rangle_{\mathcal{E}^0} = \langle (B+\mu^2)Tu, Tu\rangle_{\mathcal{E}^0} \geq \mu^2 \langle Tu, Tu\rangle_{\mathcal{E}^0} \geq 0.$$

Next, for any $w \in \mathcal{E}^0$ and $u \in \mathcal{E}^2$, we have

$$\begin{aligned}
\langle Tu, w\rangle_{\mathcal{E}^2} &= \langle B^2Tu, B^2w\rangle_{\mathcal{E}^0} + \langle BTu, Bw\rangle_{\mathcal{E}^0} + \langle Tu, w\rangle_{\mathcal{E}^0} \\
&= \langle (B^2+\mu^2)Tu, B^2w\rangle_{\mathcal{E}^0} + (1-\mu^2)\langle Tu, B^2w\rangle_{\mathcal{E}^0} + \langle u, Tw\rangle_{\mathcal{E}^0} \\
&= \langle u, B^2w\rangle_{\mathcal{E}^0} + (1-\mu^2)\langle u, TB^2w\rangle_{\mathcal{E}^0} + \langle u, Tw\rangle_{\mathcal{E}^0} \\
&= \langle u, (B^2 + (1-\mu^2)TB^2 + B)w\rangle_{\mathcal{E}^0},
\end{aligned}$$

where we used symmetry of B in $\mathcal{E}^0$ self-adjointness of T as shown above. This shows that $(B^2+\mu^2)^{-1} \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$.

Now note that one can write B^2 as

$$(1-\mu^2(B^2+\mu^2)^{-1})(B^2+\mu^2).$$

Since B^2 is a strictly positive operator, there exists $C > 0$ such that for all $s \in \mathcal{E}^2$, we have $\langle B^2 s, s\rangle_{\mathcal{E}^0} \geq C\langle s, s\rangle_{\mathcal{E}^0}$. It follows from the Cauchy-Schwarz inequality for Hilbert modules that for any $\psi \in \mathcal{E}^0$,

$$||\mu^2(B^2+\mu^2)^{-1}\psi||_{\mathcal{E}^0} \leq \frac{\mu^2}{\mu^2+C}||(B^2+\mu^2)T\psi||_{\mathcal{E}^0} = \frac{\mu^2}{\mu^2+C}||\psi||_{\mathcal{E}^0}.$$

Hence $(1-\mu^2(B^2+\mu^2)^{-1})$ has an adjointable inverse given by a Neumann series. It follows that B^2 has a bounded adjointable two-sided inverse, given by

$$(B^2)^{-1} = T(1-\mu^2(B+\mu^2)^{-1})^{-1} \colon \mathcal{E}^0 \to \mathcal{E}^2.$$

By the first sentence of the proof, the conclusion follows. □

This leads us to the main result of this section.

Theorem 84. *Let M be a non-cocompact G-spin Riemannian manifold with Dirac operator $\partial\!\!\!/_0$. Suppose M admits a G-invariant positive scalar curvature metric. Let D be the $\mathbb{Z}_2$-graded Dirac operator formed from $D_0 = \partial\!\!\!/_0$ as in section 5, and let $B = D + \Phi$ for a G-admissible Φ. Then*

$$\operatorname{index}_G F = 0 \in K_0(C^*(G)),$$

where F is the bounded transform of B.

Proof. Since B is G-invertible at infinity, it follows from the results in section 4 that $(\mathcal{E}^0, F)$ defines a class

$$[B] \coloneqq [\mathcal{E}^0, F] \in KK(\mathbb{C}, C^*(G)).$$

This class is independent of the choice of metric on M (see [44]). In particular, let F and F^{rg} be the normalised Callias-type operators associated to the metrics g and rg respectively, for some $r > 0$. Since $[\mathcal{E}^0, F]$ and $[\mathcal{E}^0, F^{rg}]$ are related by an element of $\mathcal{K}(\mathcal{E}^0)$, $\operatorname{index}_G F = \operatorname{index}_G F^{rg}$. By Propositions 82 and 83, we can find an r such that $F^{rg} = B^{rg}R^{rg}(0)^{1/2}$ is invertible. □

Remark 85. Scaling the metric g does not change the class $[\mathcal{E}^0_D, D] \in KK(C^G_0(M), C^*(G))$ (see subsection 6.2). Further, scaling the metric by a constant as we did above does not alter the space $C^G_g(M)$. In particular, if Φ is G-admissible for D then it is also G-admissible for D^{rg}. Thus $B^{rg} = D^{rg} + \Phi$ is still of G-Callias-type and $[\mathcal{E}^0, F^{rg}]$ can be computed using the KK-product given in Propositions 64 and 65 (note that this may not happen when the metric g is scaled by an arbitrary function f, since Φ may no longer be G-admissible). Thus when $\kappa^{rg} > 0$ everywhere on M, we have

$$\begin{aligned}\operatorname{index}_G F &= [\Phi] \otimes_{C_0(M)} [D] \\ &= [\Phi] \otimes_{C_0(M)} [D^{rg}] \\ &= \operatorname{index}_G F^{rg} \\ &= 0 \in K_0(C^*(G)).\end{aligned}$$

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