Holomorphic Immersions of Restricted Growth from Smooth Affine Algebraic Curves into the Complex Plane

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Contents

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Dedication

To my mum, dad, brother, and sister. The love, support, and inspiration that each of you provide has made me the person I am today.

Abstract

We investigate immersions of restricted growth from affine curves into the complex plane. We focus on the finite order and algebraic categories.

In the finite order case we prove a generalisation of a result due to Forstneric and Ohsawa, showing that on every affine curve there is a finite order 1-form with prescribed periods and divisor, provided we restrict the growth of the divisor at the punctures.

We also enumerate the algebraic immersions of triply punctured compact surfaces into the complex plane using the theory of dessins d'enfants and obtain an upper bound on the number of surfaces that admit such an immersion.

Notation

We use the standard notation $\mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ to denote the integers, real numbers and complex numbers respectively. To avoid confusion with the commonly used character $\mathbb{N}$ we always just use the words positive or non-negative as appropriate. We also use $\mathbb{P}^n$ to denote complex projective n-space.

We use script characters to denote sheaves over topological spaces. For example $\mathscr{E}$ is used for the smooth functions, $\mathscr{O}$ for the holomorphic functions, amd $\mathscr{M}$ for the meromorphic functions.

The divisor of a meromorphic function f is denoted $\operatorname{div}(f)$. Given a divisor D on a Riemann surface X and letting U be an open subset of X, we define the sheaf associated to D by setting

$$\mathscr{O}_D(U) = \{f \in \mathscr{M}(U) : \operatorname{div}(f|U) \geq -D\}\,.$$

To denote a sheaf of 1-forms we append a $^{(1)}$ to the script character. For example $\mathscr{O}^{(1)}$ is used to denote the holomorphic 1-forms. We use $\mathscr{E}^{0,1}$ to denote the sheaf of smooth 1-forms that can locally be written as $f d\bar{z}$ for some smooth f and a local holomorphic coordinate z.

We take neighbourhoods to be connected open sets. The topological closure of a set U is denoted using a bar over the top: $\overline{U}$. A subset $X \subset Y$ is compactly contained in Y if $\overline{X}$ is compact and $\overline{X} \subset Y$. We write this as $X \Subset Y$.

We denote the disc of radius R centred at a point $p \in \mathbb{C}$ by $D(p, R)$ and use $\mathbb{D}$ to denote the unit disc centred at 0 in $\mathbb{C}$. The support of a function f is denoted by $\operatorname{supp}(f)$. The diameter of a subset U of a metric space with metric d is denoted $\operatorname{diam}(U)$ and is defined as

$$\operatorname{diam}(U) = \sup\{d(x, y) : x, y \in U\}.$$

Chapter 1

Introduction

1.1 The setting

The idea of analytic continuation has been fundamental to the theory of Riemann surfaces since its conception. The analytic continuation of the square root function or the complex logarithm are often used as motivating examples in a first introduction to the topic. It is thus natural to ask which open Riemann surfaces can be thought of as the analytic continuation of a holomorphic function. The answer to this question is "all of them". In their landmark paper [GN67], Gunning and Narasimhan showed that every open Riemann surface admits a holomorphic immersion into $\mathbb{C}$. This theorem gives us a concrete way to view open Riemann surfaces as sitting above domains in $\mathbb{C}$ by a local homeomorphism. We can thus reconstruct any open Riemann surface simply by taking a holomorphic immersion into $\mathbb{C}$, restricting to a single coordinate chart, and then, viewing the inverse of the restricted immersion as a function in the plane, analytically continuing.

The proof presented by Gunning and Narasimhan relies on the holomorphic triviality of the tangent bundle of an open Riemann surface. This fact can be seen as an early manifestation of the Oka principle. Gunning and Narasimhan's theorem can further be viewed as the solution to the 1-dimensional case of the long standing open problem of whether every Stein manifold with trivial tangent bundle admits an immersion into complex Euclidean space of the same dimension. While this problem remains unsolved, Oka theory is known to be relevant to the higher dimensional case.

A modern development in the field of Oka theory is that of quantitative Oka theory, where one keeps track of the size and growth of the objects being constructed. Objects in the finite order category are of particular importance in this field. Functions are said to be of finite order if their growth is comparable to the exponential of an algebraic function. This category sits between the algebraic and holomorphic categories.

In a recent paper [FO13], Forstnerič and Ohsawa proved a quantitative version of Gunning and Narasimhan's immersion theorem, showing that every once-punctured compact Riemann surface admits a holomorphic immersion into $\mathbb{C}$ that is of finite order at the puncture. Chapters 3 and 4 of this thesis are devoted to filling in some omitted details in this paper and extending the main result. We employ a variety of techniques, including the Riemann–Roch theorem, holomorphic approximation and various results from geometry and topology.

Also of interest is an algebraic version of Gunning and Narasimhan's theorem. In Chapter 5 we take steps towards classifying the affine curves that admit algebraic immersions into the plane. This work still relies on results from geometry and topology, however we also employ some very different branches of mathematics including combinatorics and the theory of dessins d'enfants.

1.2 The plan of the thesis

Chapter 2

In the first two sections of Chapter 2 we introduce two techniques which we use to construct holomorphic functions with desired properties on Riemann surfaces.

The first section focusses on the Riemann–Roch formula and Serre duality. Together these theorems can be used to determine the existence of meromorphic functions with certain restrictions on their divisors. These results are fundamental to the theory of compact Riemann surfaces. Many of the original results in this thesis, especially those in Chapters 3 and 4, rely on these theorems in one way or another. However, these results do not tell us anything about functions on open Riemann surfaces in general.

In the next section we introduce terminology associated to holomorphic approximation. We then state Runge's theorem and the Mergelyan–Bishop theorem. These theorems allow us to construct holomorphic functions on open Riemann surfaces with desired local properties.

There is an interesting contrast between the flavours of mathematics used in these first two sections. In Section 2.1 we are concerned with meromorphic functions on compact Riemann surfaces. These objects have deep connections to objects in algebraic geometry which we explore later in the thesis. On the other hand, the theorems of Section 2.2 are of a hard analytic nature, their proofs requiring careful manipulation of estimates and inequalities. We will see that smooth affine algebraic curves are the natural setting in which we can weave these two strands of mathematics together.

The original proof of the Mergelyan–Bishop theorem involves highly measure theoretic techniques. Over the years different proofs have been discovered using more elementary complex analytic techniques but a modern reference is difficult to find. One such reference is [JP00]. Unfortunately there is a subtle mistake in the proof given there. This mistake was noticed and corrected by [Gar06]. We found the correction to be difficult to understand. In Appendix A we provide a complete proof using similar techniques to [JP00] and [Gar06], which we hope clarifies exactly where the mistake was made and how it can be rectified.

In Section 2.3 we briefly recount the proof of Gunning and Narasimhan's immersion theorem given in [GN67]. This theorem is the starting ingredient used to prove the main theorem of [FO13] so is essential to the generalisation that we prove in Chapter 4. We also provide a full proof for one of the preliminary lemmas in [GN67] that we use later in Section 4.2.

In Sections 2.4 and 2.5 we begin to explore the connection between compact Riemann surfaces and algebraic curves. In Section 2.4 we show how to embed a compact Riemann surface with finitely many punctures into complex Euclidean space. When paired with Chow's theorem in Section 2.5.1 we see that this embedding is in fact algebraic and the punctured surface is biholomorphic to a smooth affine algebraic curve. In Section 2.5.2 we develop the theory necessary to prove Cartan's extension theorem. This can be seen as a higher dimensional analogue of Weierstrass' interpolation theorem for Riemann surfaces.

Next we discuss the theory of finite order functions in the complex plane. In many sources this theory is developed in the context of entire functions. We give a slightly non-standard definition of finite order growth, allowing for functions defined on the complement of a compact set. We take this definition with the eventual goal of developing the theory in the setting of affine curves. The final result of this section is in the spirit of Hadamard's factorisation theorem and relates the distribution of zeros of a holomorphic function with its growth at infinity.

In the final section of Chapter 2 we discuss finite sheeted coverings of the sphere with finitely many punctures. We construct equivalent ways in which we can view such a covering in terms of complex analytic, group theoretic, and combinatorial objects. These objects provide us with a way to communicate the complicated information contained in a topological covering map in a concise and concrete manner. Moreover, they are essential to the content of Chapter 5.

Chapter 3

We begin Chapter 3 by extending the definition of finite order growth from Section 2.6 to the setting of a general Riemann surface. We then quickly specialise to the setting of affine curves (always smooth), which we view as compact Riemann surfaces with finitely many punctures. We define the sheaves of algebraic and finite order functions and 1-forms on an affine curve. With these definitions in hand we begin to prove generalisations of the results that were stated without proof in [FO13].

We first consider Proposition 2.4 of [FO13] which states that we can represent every de Rham class on a once-punctured compact Riemann surface by an algebraic 1-form. The authors attribute this fact to the general theory of coherent algebraic sheaves on affine varieties. We generalise the result to a general affine curve and give a detailed proof. The approach we take is based on a proof in [BS49] of a similar result.

Next we consider Proposition 2.5 of [FO13] for which the authors give no explanation. The proposition can be thought of as an algebraic version of Runge's theorem for a compact Riemann surface with a single puncture. As noted by the authors, this theorem is not actually used in the proof of the main theorem. Instead they use a stronger result: an algebraic version of the Mergelyan–Bishop theorem. We provide proofs for algebraic versions of both Runge's theorem and the Mergelyan–Bishop theorem for a general affine curve. We use the higher dimensional theory developed in Section 2.5. Just before submission of this thesis I became aware of two references that prove algebraic versions of Runge's theorem using only one-dimensional techniques. These sources are referred to in Remark 3.2.6.

In Section 3.3 we consider the finite order functions on affine curves. We start by defining the accumulation order of a divisor on an affine curve (this is our own definition). This definition is inspired by Hadamard's factorisation theorem and Theorem 2.6.9, the idea being that the growth of a finite order function at a puncture should be related to the distribution of its zeros around the puncture.

We see that this is indeed the case in Proposition 3.3.3 where we show that we can find a finite order function with a prescribed divisor on an affine curve so long as that divisor has restrictions on the accumulation order around the punctures. Moreover the order of the function at the punctures depends on the accumulation order of the divisor. This proposition generalises Proposition 2.1 of [FO13] which only allows for divisors with finite support on compact Riemann surfaces with a single puncture. We also prove a similar result for finite order 1-forms on affine curves that generalises Proposition 2.2 of [FO13].

Chapter 4

In Section 4.1 we present our generalisation of the main theorem of [FO13]. We show that on every smooth affine curve, it is possible to find an exact finite order 1-form with a prescribed divisor so long as the accumulation order of the divisor is finite at all punctures. If we take the divisor to be zero everywhere we recover the main theorem of Forstnerič and Ohsawa. While proving this we elaborate on the delicate steps that were only explained briefly in [FO13]. The proof relies on the results proved in Chapter 3 along with a generous helping of the Mergelyan–Bishop theorem.

In Section 4.2 we prove that the exact 1-form constructed in Section 4.1 can be altered so the resulting 1-form is still of finite order, has the same divisor, and has prescribed periods. This result generalises Theorem 4.1 of [FO13] which was stated without proof. It relies on the lemma from [GN67] which we prove in Section 2.3.

Chapter 5

After understanding Forstnerič and Ohsawa's theorem we might wonder if we can further restrict the growth of a holomorphic immersion of an affine curve. In Chapter 5 we consider algebraic immersions of affine curves into $\mathbb{C}$. As a consequence of the Riemann–Hurwitz formula we arrive at the following necessary condition: any affine curve that admits an algebraic immersion into $\mathbb{C}$ must be biholomorphic to a compact Riemann surface with at least three punctures. Thus there are many for which a finite order immersion is the best we can do. The three-puncture-condition is not sufficient to guarantee the existence of an algebraic immersion. For example we show that, up to biholomorphism, there is exactly one thrice punctured complex torus that admits an algebraic immersion.

The rest of the chapter can be seen as the first steps toward a classification of affine curves that admit an algebraic immersion into $\mathbb{C}$. We only consider those affine curves that are biholomorphic to a thrice punctured Riemann surface. By the material in Section 2.7 we see that an algebraic immersion of such a surface can equivalently be viewed as meromorphic function on the corresponding compact Riemann surface with exactly three critical points. We introduce the term *simple Belyi function* to refer to such functions. By representing these functions as constellations (Definition 2.7.7) and dessins d'enfants (Definition 2.7.14) we are able to show that there are only finitely many examples for each genus.

The group-theoretic representation of these objects allows us give an explicit description of each distinct algebraic immersion. The main result of the chapter is an enumeration of simple Belyi functions for low genus presented in Table 5.1. These numbers were computed with the aid of a computer. The code used to perform these computations is provided in Appendix B. Enumeration of immersions for higher genus surfaces is limited by computational power.

In the final section of the chapter we note that many of the immersions computed in the preceding section may have the same domain. Using a particular subgroup of the automorphisms of the sphere we are able to show that this is in fact the case for some of the immersions. We obtain an upper bound on the number of triply punctured surfaces that admit an algebraic immersion. This bound is also computed using a computer.

We arrive at the somewhat surprising result that there are at most two affine curves of genus two that are biholomorphic to a triply punctured compact Riemann surface and admit an algebraic immersion into $\mathbb{C}$. We are unable to determine whether these curves are distinct.

1.3 Future directions

As noted in [FO13] it is not known whether higher dimensional affine varieties admit finite order immersions into complex Euclidean space. The fact that holomorphic 1-forms are no longer automatically closed in higher dimensions poses a significant barrier that would have to be overcome if the techniques used in the one-dimensional case were to be of any help.

One might hope that the results of Chapter 4 hold for more general target spaces as well. The notion of finite order growth can be extended to maps from affine varieties to projective varieties, see [GK73]. We would then ask which affine curves admit finite order immersions into which projective curves. Of course the finite order immersions into $\mathbb{C}$ constructed in [FO13] can also be thought of as immersions into $\mathbb{P}^1$.

The enumeration of simple Belyi functions in Chapter 5 is not complete. One might hope to enumerate the simple Belyi functions of higher genus and fill in more rows of Table 5.1. We believe the runtime and memory usage of the algorithm presented in Appendix B grow factorially with the genus of the affine curve. So, while there may be many opportunities to optimise the code used, we do not expect that these will result in significant improvements in outcome unless a more efficient algorithm is developed or more advanced combinatorial theory is used.

Finally it may be possible to extend the results of Chapter 5 to a full classification of affine curves that admit algebraic immersions into $\mathbb{C}$. The next step in such a classification would most likely be to consider algebraic immersions of curves that are biholomorphic to compact Riemann surfaces with four punctures. If we assume that the immersion cannot extend as an immersion across any of the punctures, then it can be viewed as a meromorphic function on the compact surface with exactly four critical points. By the basic theory in Section 2.7 such a function must have at least three distinct critical values. Since the automorphism group of the sphere is triply transitive we can fix three of these critical values and let the fourth vary. We imagine that this would result in a moduli space of one complex dimension. This would not completely classify the algebraic immersions, however, since there would certainly be many immersions with the same critical values. It is plausible that the immersions with fixed critical values could be enumerated in a similar manner to Chapter 5.

Chapter 2

Background

2.1 The Riemann–Roch formula

The Riemann–Roch formula and Serre duality form a cornerstone for the theory of compact Riemann surfaces. They relate the vector spaces of meromorphic functions and meromorphic 1-forms with certain restrictions on their poles and zeros.

Theorem 2.1.1 (Riemann–Roch formula with Serre duality)**.** *Let D be a divisor on a compact Riemann surface X of genus g. Then $H^1(X, \mathscr{O}_D) \cong \mathscr{O}^{(1)}_{-D}(X)^*$ and*

$$\dim \mathscr{O}_D(X) - \dim \mathscr{O}^{(1)}_{-D}(X) = 1 - g + \deg D.$$

The proof of this theorem can be found in any textbook on the basic theory of Riemann surfaces, see for example [For91, Theorems 16.9 and 17.9]. The applications of this formula are of course far too numerous to account for here. Below we state two applications that we will use later on.

Theorem 2.1.2. $H^1(X, \mathscr{O}_D) = 0$ *for D with* $\deg D > 2g - 2$.

The proof of this theorem can be found in [For91, Theorem 17.16]. It relies on two facts. Firstly there is a canonical isomorphism $\mathscr{O}_{K-D} \cong \mathscr{O}^{(1)}_{-D}$ where K is a canonical divisor (that is, there is a global meromorphic 1-form with divisor K). Secondly $\dim \mathscr{O}_{K-D}(X) = 0$ when $\deg(K - D) < 0$. The result follows from Serre duality. The result below follows from the above.

Lemma 2.1.3. *Let X be a Riemann surface and $x_0 \in X$ be a point. Then there is a positive integer λ such that the divisor $D = -\lambda x_0$ is ample (that is, there is a meromorphic function whose only pole is of order λ at x_0).*

2.2 Holomorphic approximation

Holomorphic approximation is an essential tool for constructing global holomorphic functions with prescribed properties.

Definition 2.2.1. Let K be a compact subset of a Riemann surface R. We let

$$\mathscr{O}(K) = \{f|K : f \in \mathscr{O}(U) \text{ where } U \text{ is an open neighbourhood of } K\}.$$

Then we denote by $\overline{\mathscr{O}}(K)$ the closure of $\mathscr{O}(K)$ in the space of continuous functions on K with respect to the topology of uniform convergence on K. Finally we let

$$\mathscr{O}_{int}(K) = \{f : K \to \mathbb{C} : f \text{ is continuous and } f|\mathring{K} \text{ is holomorphic}\}.$$

So for any compact $K \subset R$ we have

$$\mathscr{O}(R) \xrightarrow{\text{restr}} \mathscr{O}(K) \subset \overline{\mathscr{O}}(K) \subset \mathscr{O}_{int}(K).$$

When $\mathscr{O}(R)$ is dense in $\mathscr{O}_{int}(K)$ we can approximate functions on compact sets by global holomorphic functions. Of course we cannot hope that these restriction maps should have dense image for arbitrary K, take for example a circle in $\mathbb{C}$.

Definition 2.2.2. Let K be a compact subset of an open Riemann surface R. The *holomorphically convex hull* of K is denoted $\hat{K}$ and is defined as the union of K with the relatively compact connected components of $R \setminus K$. We say that K is *holomorphically convex* if $K = \hat{K}$.

Theorem 2.2.3 (Runge's theorem)**.** *Let K be a holomorphically convex compact subset of an open Riemann surface R. Then the restriction map $\mathscr{O}(R) \to \mathscr{O}(K)$ has dense image with respect to the topology of uniform convergence on K.*

For a proof see [For91, Section 25]. Using Runge's theorem and the maximum principle it can be seen that for a compact subset K of an open Riemann surface R

$$\hat{K} = \left\{ x \in R : |f(x)| \le \sup_{y \in K} |f(y)| \text{ for all } f \in \mathscr{O}(R) \right\},$$

justifying the use of the term 'holomorphically convex hull'.

The following theorem is a significant strengthening of Runge's theorem.

Theorem 2.2.4 (Mergelyan–Bishop theorem). *Let K be a holomorphically convex compact subset of an open Riemann surface R. Then $\overline{\mathscr{O}}(K) = \mathscr{O}_{int}(K)$. Hence, by Runge's theorem, a function $f \in \mathscr{O}_{int}(K)$ can be uniformly approximated by a sequence (f_k) in $\mathscr{O}(R)$.*

See Appendix A for a detailed discussion of the proof.

2.3 The Gunning–Narasimhan theorem

The following theorem, proved in [GN67], is fundamental to the theory of Riemann surfaces.

Theorem 2.3.1 (Gunning–Narasimhan theorem). *Every open Riemann surface R admits a holomorphic immersion into $\mathbb{C}$.*

This theorem gives us a concrete way to think of open Riemann surfaces as sitting above domains in $\mathbb{C}$ (not necessarily as a covering space, but by a local homeomorphism). It would be difficult to improve upon the exposition given in [GN67] so we only provide a sketch below.

We begin by letting ω_0 be a nowhere zero holomorphic 1-form on R (this always exists by [For91, Corollary 26.6]). We aim to alter ω_0 so that it is exact — or equivalently alter it so that $\int_\gamma \omega_0 = 0$ for every loop γ in R — in such a way that we do not introduce any zeros.

Take an exhaustion $R_0 \Subset R_1 \Subset \cdots$ of R so that R_0 is simply connected, and so that R_k is open, has smooth boundary, and for every compact set $K \subset R_k$, $\hat{K} \subset R_k$ also, for every k. Now assume that $\omega_k \in \mathscr{O}^{(1)}(R)$ is exact on R_k, that is $\int_\gamma \omega_k = 0$ for any loop γ in R_k. We claim that there is a function $f_{k+1} \in \mathscr{O}(R)$ such that $\|f_{k+1}\|_{\overline{R}_k} < 2^{-k}$ and $\int_\gamma \omega_k \exp f_{k+1} = 0$ for all loops γ in R_{k+1}. Then $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on compact subsets of R so is holomorphic, and $\omega_0 e^f$ is exact.

The construction of the functions f_k is not trivial, indeed it forms the majority of the content of [GN67]. The following lemma shows how to alter the periods of a 1-form using continuous functions. This is an important step towards constructing the f_k. We give details of the proof as we will use it later, in Chapter 4.

Lemma 2.3.2. *Let R be a Riemann surface, $c \in \mathbb{C}$ be a fixed constant, and $\gamma : [0,1] \to R$ be a simple closed piecewise differentiable curve in R. Suppose ω is a holomorphic 1-form on R that is not identically zero and such that*

$\int_\gamma \omega \neq c$. Then there is a continuous function $g : \gamma((0,1)) \to \mathbb{C}$ with compact support such that

$$\int_\gamma e^g \omega = c \text{ and } \int_\gamma g e^g \omega \neq 0. \tag{2.1}$$

Proof. We begin by finding a step function u satisfying the conditions of (2.1) and then approximate by continuous functions. Throughout, we will use γ to denote both the map $[0,1] \to R$ and its image.

Let $I \subset \gamma(0,1)$ be the image of a closed subinterval of $(0,1)$ such that

$$\int_I \omega \neq 0.$$

This always exists since ω is not identically zero on γ. After choosing such an interval we can shrink it if necessary so that

$$\int_{\gamma \setminus I} \omega \neq c.$$

Now take $\lambda \in \mathbb{C}$ so that

$$e^\lambda = \frac{c - \int_{\gamma \setminus I} \omega}{\int_I \omega}.$$

Note that $\lambda \neq 0$ since $\int_\gamma \omega \neq c$. Then, setting $u = \lambda \chi_I$, where χ_I is the characteristic function of I, we have a discontinuous function satisfying the conditions in (2.1).

Select a uniformly bounded sequence of continuous functions $g_\nu : \gamma \to \mathbb{C}$ with

$$\operatorname{supp}(g_\nu) \subset \gamma((0,1))$$

converging uniformly to u on compact subsets of $\gamma \setminus \partial I$. Consider the complex analytic functions $\varphi, \varphi_\nu : \mathbb{C} \to \mathbb{C}$ defined by

$$\varphi(s) = \int_\gamma e^{su} \omega, \quad \varphi_\nu(s) = \int_\gamma e^{s g_\nu} \omega.$$

The functions φ_ν converge to φ uniformly on compact subsets of $\mathbb{C}$. Since $\varphi(1) = c$ and $\varphi'(1) \neq 0$ there is, by Hurwitz's theorem, a neighbourhood U of 1 such that for all sufficiently large ν there is $s_\nu \in U$ with $\varphi_\nu(s_\nu) = c$ and $\varphi'(s_\nu) \neq 0$. Take one such ν and set $g = s_\nu g_\nu$. Then g satisfies the desired properties. $\square$

2.4 Embedding affine curves into $\mathbb{C}^n$

It is well known that a compact Riemann surface X of genus g can be embedded holomorphically into $\mathbb{P}^N$ for some N. Such an embedding can be constructed by taking a divisor D on X of degree at least $2g+1$ and letting $f_0, \ldots, f_N$ form a basis for $\mathscr{O}_D(X)$. Then the map

$$F = [f_0, \ldots, f_N] : X \to \mathbb{P}^N$$

is an embedding (see [For91, Theorem 17.22] for a proof). Note that this map is well defined on all of X, even at the poles of $f_0, \ldots, f_N$. Suppose c is a pole of at least one of the $f_0, \ldots, f_N$. Then in a coordinate chart (V, z) centred around c there is a positive integer m so that for $k = 0, \ldots, N$ we can write $z^m f_k = g_k$ where g_k is holomorphic on V and for at least one k we have $g_k(c) \neq 0$. Then we set

$$F(c) = [g_0(c), \ldots, g_N(c)].$$

If we take a non-empty finite subset $C = \{c_1, \ldots, c_M\} \subset X$ we can use a similar construction to embed the punctured surface $R = X \setminus C$ into $\mathbb{C}^n$ for some n. Take the divisor $D = \lambda \sum_{j=1}^{M} c_j$ where λ is chosen large enough that $\deg D \geq 2g+1$ and so that there is a function $h \in \mathscr{M}(X)$ whose only poles are $c_1, \ldots, c_M$ and each pole has order $\lambda + 1$. The existence of such a function follows from Lemma 2.1.3. Let $f_0, \ldots, f_N$ form a basis for $\mathscr{O}_D(X)$ and $F = [f_0, \ldots, f_N]$.

Now let P be the zero set of h and consider the map

$$E = \left[f_0, \ldots, f_N, \frac{f_0}{h}, \ldots, \frac{f_N}{h}, \frac{1}{h} \right] : X \to \mathbb{P}^{2N+2}.$$

We claim E is an embedding.

Take $x \in X$. If $x \in P$, then x is a pole of $\dfrac{1}{h}$ but cannot be a pole of any of $f_0, \ldots, f_N$ (since the poles of $f_0, \ldots, f_N$ are in C) so

$$E(x) = [0, \ldots, 0, F(x), 1].$$

If $x \in C$, then

$$E(x) = [F(x), 0, \ldots, 0]$$

since $\dfrac{1}{h}$ has zeros of order $\lambda + 1$ at the points of C. Otherwise $x \notin P \cup C$, so x is not a pole of any of $h, f_0, \ldots, f_N$, and

$$E(x) = \left[F(x), \frac{1}{h(x)} F(x), \frac{1}{h(x)} \right] = [hF(x), F(x), 1].$$

Taking $y \in X$ with $x \neq y$, it follows that $E(x) \neq E(y)$ since F is injective. So E is injective.

Also, E is an immersion. For $k = 0, \ldots, 2N+2$ let (U_k, w_k) be the coordinate chart on $\mathbb{P}^{2N+2}$ defined by $U_k = \{[z_0, \ldots, z_{2N+2}] : z_k \neq 0\}$ and

$$w_k[z_0, \ldots, z_{2N+2}] = \frac{1}{z_k}\left(z_0, \ldots, \hat{z_k}, \ldots, z_{2N+2}\right).$$

Then for $x \in X \setminus (C \cup P)$ we have

$$w_{2N+2} \circ E(x) = (hf_0(x), \ldots, hf_N(x), f_0(x), \ldots, f_N(x)).$$

For $x \in C$, let m be the maximum order of the poles of $f_0, \ldots, f_N$. Then in a coordinate neighbourhood (V, z) centred on x for $k = 0, \ldots, N$ we can write $z^m f_k(z) = g_k(z)$ where $g_j(x) \neq 0$ for some j. Then

$$w_j \circ E(x) = \frac{1}{g_j(x)}(g_0(x), \ldots, \hat{g}_j(x), \ldots, g_N(x), 0, \ldots, 0).$$

Finally for $x \in P$

$$w_{2N+2} \circ E(x) = (0, \ldots, 0, f_0(x), \ldots, f_N(x)).$$

In all of these cases we see that $dE(x)$ can only be zero when $dF(x)$ is also. Since F is an immersion this is never the case, therefore E is an immersion.

Let $H = \left\{[z_0, \ldots, z_{2N+1}, 0] \in \mathbb{P}^{2N+2}\right\}$. Then E embeds X into $\mathbb{P}^{2N+2}$ in such a way that $E^{-1}(H) = C$. Thus post-composing with $w_{2N+2} : U_{2N+2} \to \mathbb{C}^{2N+2}$ results in an embedding of $R = X \setminus C$ into $\mathbb{C}^{2N+1}$. So we have the following theorem.

Theorem 2.4.1. *Let X be a compact Riemann surface, $C \subset X$ be a non-empty finite subset, and $R = X \setminus C$. Then R can be embedded holomorphically into $\mathbb{C}^n$ for some n, such that the components of this embedding are meromorphic on X.*

We recall the theorem of Narasimhan [Nar60] that every open Riemann surface can be holomorphically embedded into $\mathbb{C}^3$.

2.5 Higher dimensional theory

There are two main results from the theory of several complex variables which we employ in this thesis: Chow's theorem and Cartan's extension theorem. We state these theorems and outline their proofs in the subsections below.

2.5.1 Chow's theorem

Let U be an open subset of a complex manifold X. Then $A \subset U$ is said to be an *analytic subvariety* of U if A is closed in U and if for every $a \in A$ there is a neighbourhood V and holomorphic functions $f_1, \ldots, f_m$ on V such that

$$A \cap V = \{z \in V : f_1(z) = \cdots = f_m(z) = 0\}.$$

Similarly an *algebraic subvariety* of $X = \mathbb{C}^n$ or $\mathbb{P}^n$ is a set of the form

$$\{z \in X : p_1(z) = \cdots = p_m(z) = 0\}$$

where $p_1, \ldots, p_m$ are polynomials in the case of $\mathbb{C}^n$, or homogeneous polynomials in the case of $\mathbb{P}^n$.

Theorem 2.5.1 (Chow's theorem)**.** *Every analytic subvariety of $\mathbb{P}^n$ is algebraic.*

This theorem was originally proved by Chow in [Cho49]. We present a simplified proof from [RS53] relying on the following theorem which we state without proof.

Theorem 2.5.2 (Remmert–Stein extension theorem)**.** *Let X be a complex manifold, Z be an analytic subvariety of X of dimension d and Y be an analytic subvariety of $X \setminus Z$ of dimension greater than d. Then the closure $\overline{Y}$ of Y in X is an analytic subvariety of X.*

Note that the requirement on the dimension of Y is necessary. For example take $X = \mathbb{C}^2$, $Z = \{(0, z_2) : z_2 \in \mathbb{C}\}$, and

$$Y = \left\{ \left(\frac{1}{n}, 0\right) \in \mathbb{C}^2 : n = 1, 2, \ldots \right\}.$$

Then Y is an analytic subvariety of $X \setminus Z$ since it is discrete, however $\overline{Y}$ is not an analytic subvariety of X since any holomorphic function that is zero on Y must be zero on the set $\{(z_1, 0) : z_1 \in \mathbb{C}\}$.

Proof of Chow's theorem. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the map sending a point $(z_0, \ldots, z_n)$ to its equivalence class $[z_0, \ldots, z_n]$ in $\mathbb{P}^n$. The fibres of π are complex lines through the origin in $\mathbb{C}^{n+1}$. Let X be an analytic subvariety of $\mathbb{P}^n$ and let $Y = \pi^{-1}(X)$. So Y is a cone and an analytic subvariety of $\mathbb{C}^{n+1} \setminus \{0\}$ with dimension greater than 0. Then by Theorem 2.5.2, $\overline{Y} = Y \cup \{0\}$ is an analytic subvariety of $\mathbb{C}^{n+1}$. Thus there is a neighbourhood U of 0 and functions $f_1, \ldots, f_m \in \mathscr{O}(U)$ such that $\overline{Y} \cap U = \{z \in U : f_1(z) = \cdots = f_m(z) = 0\}$.

Now for $j = 1, \ldots, m$ we expand f_j in a Taylor series about 0. So $f_j = \sum_{k=0}^{\infty} f_{jk}$ where f_{jk} is a homogeneous polynomial of degree k. Since Y is a cone, for any $z \in Y$ there is some $\varepsilon > 0$ such that for all $t \in \mathbb{C}$ with $|t| < \varepsilon$ we have $tz \in U \cap \overline{Y}$. So for all j

$$f_j(tz) = \sum_{k=0}^{\infty} t^k f_{jk}(z) = 0.$$

It follows that $f_{jk}(z) = 0$ for all $z \in Y$ and all j and k. So

$$Y \subset \{z \in \mathbb{C}^{n+1} \setminus \{0\} : f_{jk}(z) = 0 \text{ for all } j \text{ and } k\}.$$

Conversely if $z \in \mathbb{C}^{n+1} \setminus \{0\}$ and $f_{jk}(z) = 0$ for all j and k then for t sufficiently close to 0, $tz \in U$ and $f_{jk}(tz) = t^k f_{jk}(z) = 0$ for all j and k. Thus $tz \in Y$ and hence $z \in Y$.

So X is the common zero locus of the homogeneous polynomials f_{jk}. By Hilbert's basis theorem X is then an algebraic subvariety. □

Remark 2.5.3. Let X be a compact Riemann surface, C a finite subset and $R = X \setminus C$. Consider the embedding $E : X \to \mathbb{P}^n$ constructed in Section 2.4 that had the property that $E|R \to \mathbb{C}^n$ was an embedding. The image of E is, by the implicit function theorem, an analytic subvariety of $\mathbb{P}^n$. By Chow's theorem it is also an algebraic subvariety. So we see that X is biholomorphic to a projective algebraic curve. Similarly R is biholomorphic to a smooth affine algebraic curve.

In fact, with more work we can say even more. Projective algebraic curves along with algebraic maps between them form a category. It can be seen that this category is equivalent to the category of compact Riemann surfaces and holomorphic maps.

2.5.2 Cartan's extension theorem

Definition 2.5.4 (Coherent sheaves)**.** Suppose $\mathscr{S}$ is a sheaf of abelian groups on $\mathbb{C}^n$ with an $\mathscr{O}$-module structure, that is, for every open $U \subset \mathbb{C}^n$ there is a scalar multiplication map

$$\mathscr{O}(U) \times \mathscr{S}(U) \to \mathscr{S}(U)$$

commuting with restrictions, making $\mathscr{S}(U)$ into an $\mathscr{O}(U)$-module. Then $\mathscr{S}$ is called a *sheaf of $\mathscr{O}$-modules*. We often simply say that $\mathscr{S}$ is an $\mathscr{O}$-module over $\mathbb{C}^n$.

Let $\mathscr{S}$ be an $\mathscr{O}$-module on $\mathbb{C}^n$. Then $\mathscr{S}$ is *locally finitely generated* if every point $x \in \mathbb{C}^n$ has a neighbourhood U with sections $f_1, \dots, f_k \in \mathscr{S}(U)$ such that for all $y \in U$

$$\mathscr{S}_y = (f_1)_y \mathscr{O}_y + \cdots + (f_k)_y \mathscr{O}_y$$

where $\mathscr{S}_y$ denotes the stalk of $\mathscr{S}$ at y and $(f_j)_y$ denotes the germ of f_j at y.

Given an open subset $U \subset \mathbb{C}^n$ and $f_1, \dots, f_k \in \mathscr{S}(U)$ we define the *sheaf of relations between* $f_1, \dots, f_k$, denoted $\mathscr{R}(f_1, \dots, f_k)$, by letting

$$\mathscr{R}(f_1, \dots, f_k)(V) = \{(\varphi_1, \dots, \varphi_k) \in \mathscr{O}^k(V) : \varphi_1 f_1 + \cdots + \varphi_k f_k = 0\}$$

where $V \subset U$ is open.

Finally, we say $\mathscr{S}$ is *coherent* if it is locally finitely generated and if for all open sets $U \subset X$ and $f_1, \dots, f_k \in \mathscr{S}(U)$ the sheaf of relations $\mathscr{R}(f_1, \dots, f_k)$ is locally finitely generated as a sheaf on U.

The notion of coherence makes sense in the context of sheaves of $\mathscr{O}$-modules on arbitrary complex manifolds. For our purposes however, we only require coherent sheaves on $\mathbb{C}^n$.

Below we have two examples. The first is one of the few easy, non-trivial examples of a coherent sheaf. The second demonstrates that the condition on the sheaf of relations is not automatically satisfied for a locally finitely generated $\mathscr{O}$-module.

Example 2.5.5. $\mathscr{O}$ is a coherent $\mathscr{O}$-module over $\mathbb{C}$.

Proof. Firstly $\mathscr{O}$ is locally finitely generated by the constant section 1. Suppose $U \subset \mathbb{C}$ is open and $f_1, \dots, f_k \in \mathscr{O}(U)$ are not identically zero. We need to show that $\mathscr{R}(f_1, \dots, f_k)$ is locally finitely generated. For each $c \in U$ let $n_1, \dots, n_k$ be the smallest numbers such that there are functions g_j holomorphic in a neighbourhood of c with $g_j(c) \neq 0$ and so that $f_j(z) = g_j(z)(z-c)^{n_j}$. Suppose that n_1 is the smallest of the n_j. Then for any $\varphi_2, \dots, \varphi_k \in \mathscr{O}(U)$ let

$$\varphi_1(z) = -\frac{\varphi_2(z) g_2(z)(z-c)^{n_2} + \cdots + \varphi_k(z) g_k(z)(z-c)^{n_k}}{g_1(z)(z-c)^{n_1}}.$$

Then $(\varphi_1, \dots, \varphi_k) \in \mathscr{R}(f_1, \dots, f_k)$. This shows that $\varphi_2, \dots, \varphi_k$ are free and φ_1 is completely determined, thus $\mathscr{R}(f_1, \dots, f_k) \cong \mathscr{O}^{k-1}$. So $\mathscr{R}(f_1, \dots, f_k)$ is locally finitely generated. $\square$

Example 2.5.6. Define a sheaf $\mathscr{S}$ over $\mathbb{C}$ as follows. Let $V \subset U \subset \mathbb{C}$ be open sets and set

$$\mathscr{S}(U) = \begin{cases} \mathscr{O}(U) & \text{if } 0 \in U, \\ 0 & \text{otherwise,} \end{cases}$$

and define restriction maps $t_V^U : \mathscr{S}(U) \to \mathscr{S}(V)$ by

$$t_V^U(f) = \begin{cases} f|V & \text{if } 0 \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathscr{S}$ is a locally finitely generated $\mathscr{O}$-module but is not coherent.

Proof. The stalk of $\mathscr{S}$ at $z \in \mathbb{C}$ is trivial except when $z = 0$. When $z = 0$ the stalk is $\mathscr{O}_0$ so clearly $\mathscr{S}$ is locally finitely generated.

Now take the constant function $1 \in \mathscr{S}(\mathbb{C})$. Then for an open set $U \subset \mathbb{C}$ we have $t_U^{\mathbb{C}}(1) = 1$ if $0 \in U$ or 0 otherwise so

$$\mathscr{R}(1)(U) = \begin{cases} 0 & \text{if } 0 \in U, \\ \mathscr{O}(U) & \text{otherwise,} \end{cases}$$

and we see that $\mathscr{R}(1)$ is not locally finitely generated at 0. □

Finding other coherent sheaves turns out to be difficult.

Theorem 2.5.7 (Oka). *$\mathscr{O}$ is a coherent $\mathscr{O}$-module over $\mathbb{C}^n$ for all n.*

The proof of this theorem is much more difficult than in the 1-dimensional case. We refer to [GR84, Chapter 2] for a detailed discussion.

The following lemma follows immediately from the definition of coherence.

Lemma 2.5.8. *Let $\mathscr{S}$ be a coherent $\mathscr{O}$-module over $\mathbb{C}^n$ and let $\mathscr{T}$ be a submodule. Then $\mathscr{T}$ is coherent if and only if it is locally finitely generated.*

Example 2.5.9 (Ideal sheaf)**.** Let M be a closed complex d-dimensional submanifold of $\mathbb{C}^n$ with $d < n$ and define the *ideal sheaf* $\mathscr{I}^M$ to be the submodule of $\mathscr{O}$ over $\mathbb{C}^n$ comprised of functions that vanish on M. Then $\mathscr{I}^M$ is coherent.

Proof. Since $\mathscr{I}^M$ is a submodule of the coherent module $\mathscr{O}$ we only need to show that it is locally finitely generated. For $x \notin M$ there is a neighbourhood U of x such that $\mathscr{I}^M(U) = \mathscr{O}(U)$ so $\mathscr{I}^M$ is locally finitely generated and hence coherent outside of M. Take $x \in M$. Then there is a coordinate chart $(U, (z_1, \ldots, z_n))$ around x so that

$$M \cap U = \{(z_1, \ldots, z_d, 0, \ldots, 0) \in U : z_1, \ldots, z_d \in \mathbb{C}\}.$$

Since $x \in M$, $x = (x_1, \dots, x_d, 0, \dots, 0)$. Now for any neighbourhood V of x any $f \in \mathscr{I}^M(V)$ has a power series

$$f(z) = \sum_{i_1,\dots,i_n=0}^{\infty} c_{i_1,\dots,i_n}(z_1 - x_1)^{i_1} \cdots (z_d - x_d)^{i_d} z_{d+1}^{i_{d+1}} \cdots z_n^{i_n}$$

with $c_{i_1,\dots,i_n} = 0$ whenever $i_{d+1} = \cdots = i_n = 0$. So $\mathscr{I}_x^M$ is generated by the germs of the functions $z_{d+1}, \dots, z_n$. Clearly the germs of these functions also generate the stalk $\mathscr{I}_y^M$ for any $y \in M \cap U$.

Now take $y \in U \setminus M$. We claim that the stalk $\mathscr{I}_y^M$ is generated by the germ of the function z_n. To see this we only need to note that on a sufficiently small neighbourhood V of y, z_n is invertible and thus $1 \in (z_n)_y \mathscr{O}_y$ and therefore $(z_n)_y \mathscr{O}_y = \mathscr{O}_y = \mathscr{I}_y^M$. □

Theorem 2.5.10 (Cartan's theorem B)**.** *Let $\mathscr{S}$ be a coherent $\mathscr{O}$-module over $\mathbb{C}^n$. Then $H^q(\mathbb{C}^n, \mathscr{S}) = 0$ for all $q \geq 1$.*

In fact this theorem holds for coherent sheaves on a particular class of complex manifolds known as the Stein manifolds, however, for our purposes $\mathbb{C}^n$ is sufficient.

The proof of this theorem is too long for us to recount here so we give a very brief sketch below. We first show that given a coherent $\mathscr{O}$-module $\mathscr{S}$ on a neighbourhood of a bounded polydisc X there is an exact sequence

$$0 \to \mathscr{O}^{n_k} \to \cdots \to \mathscr{O}^{n_0} \to \mathscr{S} \to 0$$

of $\mathscr{O}$-modules on X. We then show that any coherent $\mathscr{O}$-module that fits into such a sequence has the property that $H^q(X, \mathscr{S}) = 0$ for $q \geq 1$. Then by exhausting $\mathbb{C}^n$ by polydiscs and using Runge approximation we are able to split any q-cocycle thus showing that $H^q(\mathbb{C}^n, \mathscr{S}) = 0$ for $q \geq 1$. To prove the theorem for an arbitrary Stein manifold requires more work still, we refer to [Hör90, Theorem 7.4.3].

From this theorem we have the following application.

Theorem 2.5.11 (Cartan's extension theorem)**.** *Every holomorphic function defined on a closed complex submanifold M of $\mathbb{C}^n$ extends to a holomorphic function on all of $\mathbb{C}^n$.*

Proof. Define an $\mathscr{O}$-module $\mathscr{O}_M$ over $\mathbb{C}^n$ by letting

$$\mathscr{O}_M(U) = \begin{cases} 0 & \text{if } U \cap M = \emptyset, \\ \mathscr{O}(U \cap M) & \text{otherwise,} \end{cases}$$

and using the obvious restriction maps. Then we have a short exact sequence of $\mathscr{O}$-modules

$$0 \to \mathscr{I}^M \to \mathscr{O} \to \mathscr{O}_M \to 0,$$

where the map $\mathscr{O} \to \mathscr{O}_M$ is given by restriction. This induces a long exact sequence in cohomology

$$0 \to \mathscr{I}^M(\mathbb{C}^n) \to \mathscr{O}(\mathbb{C}^n) \to \mathscr{O}_M(\mathbb{C}^n) \to H^1(\mathbb{C}^n, \mathscr{I}^M) \to \cdots .$$

By Theorem 2.5.10 $H^1(\mathbb{C}^n, \mathscr{I}^M) = 0$ so the map $\mathscr{O}(\mathbb{C}^n) \to \mathscr{O}_M(\mathbb{C}^n) = \mathscr{O}(M)$ is surjective. Thus every holomorphic function on M is the restriction of a holomorphic function on $\mathbb{C}^n$. $\square$

2.6 Functions of finite order

Throughout this thesis we will be interested in holomorphic functions defined on a punctured Riemann surface with restricted growth at the punctures. The simplest case we could consider is the complex plane, that is, the Riemann sphere $\mathbb{P}^1$ with ∞ removed. The theory of these functions is well developed.

Definition 2.6.1. Suppose U is the complement of a compact set in $\mathbb{C}$. A holomorphic function $f : U \to \mathbb{C}$ is said to be of *finite order* if there are non-negative numbers A, B, R, and ν such that

$$|f(z)| \leq A \exp(B|z|^\nu) \tag{2.2}$$

for all z with $|z| \geq R$. The *order* μ of f is then defined as the infimum of numbers $\nu \geq 0$ such that there are non-negative numbers A, B and R such that (2.2) holds.

We now establish some equivalent definitions that will later prove useful.

Lemma 2.6.2. *A holomorphic function f defined on the complement of a compact set in $\mathbb{C}$ has order μ if and only if for every $\varepsilon > 0$ there is $R > 0$ such that*

$$|f(z)| \leq \exp|z|^{(\mu+\varepsilon)}$$

for all z with $|z| > R$ and this property is not satisfied for any smaller μ.

Proof. The reverse direction is immediate from the definition. To prove the forward direction we notice that for $x \in \mathbb{R}$ and $\varepsilon, A, B, \nu > 0$

$$\lim_{x\to\infty} \frac{A \exp Bx^\nu}{\exp x^{\nu+\varepsilon}} = 0.$$

It follows that there is $R > 0$ such that for all $|z| > R$

$$A \exp B|z|^{\nu} < \exp |z|^{\nu+\epsilon}.$$

Hence, if f has order μ, we have

$$|f(z)| \leq \exp |z|^{\mu+\varepsilon}$$

for all $|z|$ sufficiently large. It is clear that the above inequality could not hold for any smaller μ. $\square$

Lemma 2.6.3. *A holomorphic function f has order μ if and only if*

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \mu \tag{2.3}$$

where $M(r) = \max\{|f(re^{i\theta})| : 0 \leq \theta \leq 2\pi\}$.

Proof. Suppose f satisfies (2.3) and take $\varepsilon > 0$. Then there is $R > 1$ such that

$$\frac{\log\log M(r)}{\log r} \leq \mu + \varepsilon$$

for all $r > R$. Then, for such r,

$$M(r) \leq \exp(r^{\mu+\varepsilon}).$$

This is true for all $\epsilon > 0$, so the order of f is at most μ. It is clear that this is not satisfied for any smaller μ, so f has order μ.

Now suppose f has order μ. Then for any $\varepsilon > 0$, there is $R > 0$ such that

$$|f(z)| \leq \exp |z|^{\mu+\varepsilon}$$

for all $|z| > R$. Then for any $\varepsilon > 0$,

$$M(r) \leq \exp r^{\mu+\varepsilon}$$

for all r sufficiently large. The result follows by taking logarithms. $\square$

The order of an entire function can also be found using the coefficients of the function's Taylor series.

Proposition 2.6.4. *Suppose f is an entire function of order μ and has Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then*

$$\mu = \limsup_{k\to\infty} \frac{k \log k}{\log(1/|a_k|)}$$

where we take the quotient on the right to be 0 if $a_k = 0$.

Proof. We follow the proof given in [Boa54]. Let

$$\eta = \limsup_{k\to\infty} \frac{k\log k}{\log(1/|a_k|)}.$$

First we show that $\mu \geq \eta$. Since μ is non-negative this is true if $\eta = 0$. Suppose $0 < \eta \leq \infty$ and notice the following elementary fact for all $r > 0$:

$$|a_k| = \left|\frac{f^{(n)}(0)}{k!}\right| \leq \frac{1}{2\pi}\int_{|z|=r}\left|\frac{f(z)}{z^{k+1}}\right||dz| \leq \frac{M(r)}{r^k}. \tag{2.4}$$

Take $0 < \varepsilon < \eta$ and let $A = \eta - \varepsilon$ if $\eta < \infty$ and $A = \varepsilon$ if $\eta = \infty$. We think of ε being small if η is finite and large if $\eta = \infty$ so that A is "close" to η. Then for infinitely many k,

$$k\log k \geq A\log\frac{1}{|a_k|}.$$

Then

$$\log|a_k| \geq \frac{-k\log k}{A}$$

so taking logarithms in (2.4) gives for all r

$$\log M(r) \geq k\log r + \log|a_k| \geq k\log r - \frac{k\log k}{A}$$

for infinitely many k. For such k we can take $r = (ek)^{\frac{1}{A}}$ and then

$$\log M(r) \geq \frac{k}{A} = \frac{r^A}{eA}.$$

So

$$\frac{\log\log M(r)}{\log(r)} \geq A - \frac{\log(eA)}{\log r}$$

holds for such r. The second term in the right hand side of the above inequality approaches zero as r increases and since A is independent of r we see that

$$\mu = \limsup_{r\to\infty}\frac{\log\log M(r)}{\log r} \geq A = \begin{cases}\eta - \varepsilon & \text{if } \eta < \infty,\\ \varepsilon & \text{if } \eta = \infty.\end{cases}$$

Since ε was arbitrary we see that $\eta \leq \mu$.

Next we show that $\mu \leq \eta$. Of course if $\eta = \infty$ there is nothing to prove so assume otherwise. Take $\varepsilon > 0$. Then for k large enough that $\log|a_k| < 0$ and so that

$$0 \leq \frac{k\log k}{\log(1/|a_k|)} \leq \eta + \varepsilon$$

we have

$$|a_k| \leq k^{-k/(\eta+\varepsilon)}.$$

Then

$$M(r) \leq \sum_{k=0}^{\infty} |a_k| r^k \leq \sum_{k=0}^{\infty} k^{-k/(\eta+\varepsilon)} r^k.$$

We split the last sum into two parts and bound them separately. Let

$$S_1(r) = \sum_{k<(2r)^{\eta+\varepsilon}} k^{-k/(\eta+\varepsilon)} r^k \quad \text{and} \quad S_2(r) = \sum_{k\geq(2r)^{\eta+\varepsilon}} k^{-k/(\eta+\varepsilon)} r^k.$$

Then we have

$$S_2(r) = \sum_{k\geq(2r)^{\eta+\varepsilon}} (rk^{-\frac{1}{\eta+\varepsilon}})^k \leq \sum_{k\geq(2r)^{\eta+\varepsilon}} \left(\frac{1}{2}\right)^k < 1.$$

And for S_1:

$$\begin{aligned} S_1(r) &= \sum_{k<(2r)^{\eta+\varepsilon}} k^{-k/(\eta+\varepsilon)} r^k \\ &\leq r^{(2r)^{\eta+\varepsilon}} \sum_{k<(2r)^{\eta+\varepsilon}} k^{-k/(\eta+\varepsilon)} \\ &\leq r^{(2r)^{\eta+\varepsilon}} \sum_{k=0}^{\infty} k^{-k/(\eta+\varepsilon)}. \end{aligned}$$

The last series converges, so $S_1(r)$ has growth comparable to $r^{(2r)^{\eta+\varepsilon}}$. So when r is sufficiently large,

$$M(r) \leq S_1(r) + S_2(r) \leq r^{r^{\eta+2\varepsilon}},$$

so

$$\mu = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \leq \limsup_{r\to\infty}\ \eta + 2\varepsilon + \frac{\log\log r}{\log r} = \eta + 2\varepsilon.$$

Since ε is arbitrary, $\mu \leq \eta$. $\square$

This proposition allow us to calculate some explicit examples.

Example 2.6.5. 1. Every polynomial has order 0.

2. Let f be a polynomial of degree n. Then $\exp f$ has order n.

3. For a complex number q with $|q| < 1$, the function $f : \mathbb{C} \to \mathbb{C}$ defined by
$$f(z) = \sum_{k=0}^{\infty} q^{k^2} z^k$$
has order zero.

Given two functions of finite order we can construct new finite order functions.

Lemma 2.6.6. *Suppose f and g are holomorphic functions defined on the complement of compact sets with order μ and ν respectively. Then*

1. *f' has order μ.*
2. *$af + bg$ has order $\eta \le \max(\mu, \nu)$, where $a, b \in \mathbb{C}$.*
3. *fg has order $\eta \le \max(\mu, \nu)$.*
4. *if f has no zeros then $1/f$ has order μ.*

Proof. The first property follows from Proposition 2.6.4. For the second property we only need to note that if $\alpha < \beta$ then $\exp |z|^\alpha < \exp |z|^\beta$ for $|z| > 1$. Then for every $\epsilon > 0$ there is $R > 0$ such that
$$\begin{aligned} |af(z) + bg(z)| &\le a \exp |z|^{\mu + \frac{\varepsilon}{2}} + b \exp |z|^{\nu + \frac{\varepsilon}{2}} \\ &\le (a + b) \exp |z|^{\max(\mu,\nu)+\varepsilon} \end{aligned}$$
for $|z| > R$. Since this is true for all $\varepsilon > 0$, $af + bg$ has order at most $\max(\mu, \nu)$.

For the third property, take $\varepsilon > 0$. For $|z|$ sufficiently large
$$|f(z)| \le \exp |z|^{\mu+\varepsilon} \text{ and } |g(z)| \le \exp |z|^{\nu+\varepsilon}.$$
Then
$$|fg(z)| \le \exp(|z|^{\mu+\varepsilon} + |z|^{\nu+\varepsilon})$$
and since ε is arbitrary and for sufficiently large z, $|z|^\alpha + |z|^\beta \le C|z|^{\max(\alpha,\beta)}$ for some constant C that depends on α and β, we get the result.

For the final property, note that $f(1/z)$ is a function on a punctured neighbourhood of 0. Hence there is a number m and a holomorphic function h such that $f(1/z) = z^m \exp h(z)$. Now for a complex number A and a holomorphic function F, $\exp F$ has the same order as $\exp AF$. So letting $k(z) = h(1/z)$, since $f(z) = z^{-m} \exp k(z)$ has order μ so too does $1/f = z^m \exp -k(z)$. $\square$

It is not true that the reciprocal of a general finite order function is of finite order, indeed the reciprocal is not even holomorphic in general. We might hope that if we were to alter our definition of finite order functions to include *meromorphic* functions that satisfy (2.2) then the reciprocal would be of finite order. Unfortunately this is still not the case. We will see shortly that many functions with finite order have zeros accumulating at ∞. The reciprocal of such a function would have poles accumulating at ∞ and so could not satisfy the equivalent definition of finite order given in Lemma 2.6.3.

There is an alternative definition of the order of a meromorphic function in terms of the Nevanlinna characteristic function. Under this definition it can be shown that the reciprocal of a finite order function is of finite order [CY01, Proposition 1.5.2]. To develop the background needed for this definition would take us too far afield.

Using Jensen's formula we can relate the order of a function to the distribution of its zeros.

Theorem 2.6.7 (Jensen's formula)**.** *Let U be an open set in $\mathbb{C}$ containing $D(0,R)$. Suppose $f \in \mathscr{O}(U)$ has no zeros on the circle of radius R and $f(0) \neq 0$. Denote the zeros of f by (a_n) repeated according to multiplicity. Then*

$$\log|f(0)| + \sum_{|a_n|<R} \log \frac{R}{|a_n|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta.$$

Note that if $f(0) = 0$ there is some k such that $f(z) = z^k g(z)$ where $g(0) \neq 0$; then Jensen's formula can be applied to g. The proof of this theorem can be found in almost any textbook on basic complex analysis; we recommend [SS03, page 135].

Corollary 2.6.8. *Suppose f is an entire function of order μ with infinitely many zeros. Let $(a_n)_{n=1}^{\infty}$ be a listing of the zeros of f repeated according to multiplicity. Then the sum*

$$\sum_{n=1}^{\infty} |a_n|^{-(\mu+\varepsilon)} < \infty$$

converges for every $\varepsilon > 0$.

Proof. Let $\mathfrak{n}(r)$ denote the number of zeros of f inside the disc of radius r centred on the origin. Viewed as a distribution, the derivative of $\mathfrak{n}$ is an infinite linear combination of delta functions. Then by integration by parts

$$\sum_{|a_n|<R} |a_n|^{-\alpha} = \int_0^R \frac{d\mathfrak{n}(r)}{r^\alpha} = \frac{\mathfrak{n}(R)}{R^\alpha} + \alpha\int_0^R \frac{\mathfrak{n}(r)}{r^{\alpha+1}}dr. \tag{2.5}$$

We will show that in the limit as $R \to \infty$ the right hand side of (2.5) is finite for appropriately chosen α. First note that

$$\sum_{|a_n|<R} \log \frac{R}{|a_n|} = \sum_{|a_n|<R} \int_{|a_n|}^{R} \frac{dr}{r} = \int_0^R \frac{\mathfrak{n}(r)}{r} dr.$$

Then

$$\mathfrak{n}(R) \log 2 = \mathfrak{n}(R) \int_R^{2R} \frac{dr}{r}$$

and since $\mathfrak{n}$ is non-decreasing

$$\mathfrak{n}(R) \int_R^{2R} \frac{dr}{r} \leq \int_R^{2R} \frac{\mathfrak{n}(r)}{r} dr.$$

Multiplication by a rational function does not change the growth of f. Thus we can replace f by $Az^{-k}f$ where $A \in \mathbb{C}$ and k a non-negative integer are chosen so that $f(0) = 1$. Then, by Jensen's formula, for every $R > 0$ with $R \neq |a_n|$ for every n we have

$$\int_0^{2R} \frac{\mathfrak{n}(r)}{r} dr = \sum_{|a_n|<2R} \log \frac{2R}{|a_n|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(2Re^{i\theta})| d\theta.$$

Then

$$\mathfrak{n}(R) \log 2 \leq \int_0^{2R} \frac{\mathfrak{n}(r)}{r} dr \leq \log M(2R).$$

Since f has order μ, for every $\varepsilon > 0$

$$M(2R) \leq \exp R^{\mu+\varepsilon}$$

holds for R sufficiently large. Thus for R sufficiently large and satisfying $R \neq |a_n|$ for every n, we have

$$\mathfrak{n}(R) \leq \frac{2^{\mu+\varepsilon}}{\log 2} R^{\mu+\varepsilon}.$$

Since $\mathfrak{n}$ is increasing this is true for all R sufficiently large.

Finally, if we take $\alpha > \mu + \varepsilon$ the right hand side of (2.5) is finite in the limit as $R \to \infty$. Since ε is arbitrary $\sum_{n=0}^{\infty} |a_n|^{-(\mu+\varepsilon)} < \infty$ for all $\varepsilon > 0$. □

Theorem 2.6.9. *For a positive integer k, let*

$$E_k(z) = (1 - z) \exp\left(z + \frac{z}{2} + \cdots + \frac{z^k}{k}\right).$$

Suppose $(a_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{C}$ such that $a_n \to \infty$ and so that there is a number $\mu > 0$ for which

$$\sum_{n=1}^{\infty} |a_n|^{-(\mu+\varepsilon)} < \infty$$

holds for all $\varepsilon > 0$ but not for any $\varepsilon < 0$. Let k be the integer such that $k \le \mu < k+1$. Then the function

$$f(z) = \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

is entire, has order μ and has zeros at the points specified by the sequence (a_n) and nowhere else.

Proof. We refer to [SS03, section 5.4] for a proof that f is entire. To show that it has order μ first note that by Corollary 2.6.8 the order of f is at least μ. We now show that the order is at most μ. For every $\varepsilon > 0$ there is $R > 0$ such that for all $|z| > R$,

$$|E_k(z)| = \left|(1-z)\exp\left(1 + \cdots + \frac{z^k}{k}\right)\right| \le \exp|z^{\mu+\varepsilon}|.$$

Then for $|z| > R$,

$$\begin{aligned} |f(z)| &= \left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)\right| \\ &\le \exp\left(|z|^{\mu+\varepsilon}\sum_{n=1}^{\infty}|a_n|^{-(\mu+\varepsilon)}\right) \\ &\le \exp\left(c|z|^{\mu+\varepsilon}\right) \end{aligned}$$

where $\sum_{n=1}^{\infty}|a_n|^{-(\mu+\varepsilon)} = c < \infty$. Since this is true for all $\varepsilon > 0$ the order of f cannot be more than μ and hence f has order μ. $\square$

2.7 Ramified coverings of the sphere and dessins d'enfants

2.7.1 Coverings of the sphere with finitely many punctures

Riemann surfaces and meromorphic functions

The following construction can be found in [For91, Theorem 8.4].

Construction 2.7.1. Given a non-constant meromorphic function f on a compact Riemann surface X we can construct a finite-sheeted covering map of the sphere with finitely many punctures (we simply remove the critical values of f from $\mathbb{P}^1$ and their preimages from X). The number of sheets of the resulting covering map is then equal to the degree of f as a map.

Conversely, letting $Y = \mathbb{P}^1 \setminus \{c_1, \dots, c_k\}$, given an n-sheeted covering $p : X \to Y$ with X being path connected we can recover a meromorphic function on a compact Riemann surface. Firstly X can be given a complex structure by pulling back the complex structure on Y using p, so p can be thought of as a holomorphic map from the non-compact Riemann surface X into Y.

Now for $i = 1, \dots, k$, around each c_i there is a neighbourhood U_i such that, letting $V_i = p^{-1}(U_i \setminus \{c_i\})$, $p|V_i$ is a covering of $U_i \setminus \{c_i\}$. Now let J_i be the number of connected components of V_i and denote these components by W_{ij}, where $j = 1, \dots, J_i$. If $c_i = \infty$ define

$$p_{ij} = \frac{1}{p|W_{ij}},$$

otherwise set $p_{ij} = p|W_{ij} - c_i$. Then p_{ij} is a finite-sheeted holomorphic covering of a punctured neighbourhood of 0, thus in suitable coordinates $p_{ij} = z^m$ for some integer $1 \leq m \leq n$. Then $(p_{ij})^{\frac{1}{m}}$ is single-valued and a biholomorphism from W_{ij} to a punctured neighbourhood of $0 \in \mathbb{C}$. We now "fill in" the puncture in the following way. Let P_{ij} be an abstract point and extend $(p_{ij})^{\frac{1}{m}}$ to the set $W_{ij} \cup \{P_{ij}\}$ by setting $(p_{ij})^{\frac{1}{m}}(P_{ij}) = 0$. We then equip $W_{ij} \cup \{P_{ij}\}$ with a topology by saying a subset A is open if and only if $(p_{ij})^{\frac{1}{m}}(A)$ is open in $\mathbb{C}$. In this way $(p_{ij})^{\frac{1}{m}}$ extends to a biholomorphism from $W_{ij} \cup \{P_{ij}\}$ to a neighbourhood of 0 (so $(p_{ij})^{\frac{1}{m}}$ is a holomorphic coordinate chart around P_{ij}). And so, in precisely the same way, we see that $p|W_{ij}$ extends to $W_{ij} \cup \{P_{ij}\}$ as a branched cover of U_i whose only branch point is P_{ij}.

Thus

$$\overline{X} = X \cup \bigcup_{i=1}^{k} \bigcup_{j=1}^{J_i} \{P_{ij}\}$$

is a compact Riemann surface and p extends to a meromorphic function on $\overline{X}$ whose branch points lie in the set $\bigcup_{i=1}^{k} \bigcup_{j=1}^{J_i} \{P_{ij}\}$.

Definition 2.7.2. We say two topological covering maps $p : X \to Y$ and $q : Z \to Y$ are *equivalent* if there is a homeomorphism $h : X \to Z$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "p"] & & Z \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

In a similar way we say that pairs (X, f) and (Z, g) where X and Z are Riemann surfaces, and f and g are meromorphic functions are *equivalent* if there is a biholomorphism $h : X \to Z$ such that $f = g \circ h$.

Theorem 2.7.3. *Let $Y = \mathbb{P}^1 \setminus C$ where C is a non-empty finite subset of $\mathbb{P}^1$. Suppose $p : X \to Y$ and $q : Z \to Y$ are finite sheeted covering maps. By Construction 2.7.1 these coverings give rise to Riemann surface–meromorphic function pairs $\left(\overline{X}, \hat{p}\right)$ and $\left(\overline{Z}, \hat{q}\right)$ respectively. Then p and q are equivalent as covering maps if and only if $\left(\overline{X}, \hat{p}\right)$ and $\left(\overline{Z}, \hat{q}\right)$ are equivalent as Riemann surface–meromorphic function pairs.*

We refer to [For91, Theorem 8.5] for a proof.

Group-theoretic representations of coverings

Finite-sheeted coverings of the punctured sphere also have equivalent formulations in terms of group-theoretic objects. Let $Y = \mathbb{P}^1 \setminus C$ where $C = \{c_1, \ldots, c_k\}$, $k \geq 1$, is a finite subset of $\mathbb{P}^1$ and take a base point $y_0 \in Y$. Then $\pi_1(Y, y_0)$ is isomorphic to the free group with $k-1$ generators. In order to preserve the symmetry between the points in C we usually view $\pi_1(Y, y_0)$ as the group with presentation

$$\langle \gamma_1, \ldots, \gamma_k | \gamma_1 \cdots \gamma_k \rangle$$

and think of the generator γ_i as a loop based at y_0 that is freely homotopic to a small circle around c_i. Suppose $p : X \to Y$ is an n-sheeted covering map with X being path connected. Let $E = p^{-1}(y_0) = \{e_1, \ldots, e_n\}$ denote the fibre over y_0. Using the map p we can define an action of $\pi_1(Y, y_0)$ on E.

Construction 2.7.4. For each point $e_i \in E$ and each generator γ_j of $\pi_1(Y, y_0)$ there is a lifting by p of γ_j to a path $\tilde{\gamma}_{ij}$ in X with starting point e_i. We define $\gamma_j(e_i)$ to be the endpoint of $\tilde{\gamma}_{ij}$. This action clearly respects the group operation of $\pi_1(Y, y_0)$. We call the action thus defined on E the *monodromy action of $\pi_1(Y, y_0)$ associated to the map p.*

Lemma 2.7.5. *The monodromy action associated to a finite-sheeted covering map $p : X \to Y$ is transitive on the fibre $E = p^{-1}(y_0)$.*

Proof. This follows from the fact that X is path-connected. The image of any path joining $e_1, e_2 \in E$ under p is a loop γ in Y. Thus the class $[\gamma] \in \pi_1(Y, y_0)$ has $\gamma(e_1) = e_2$. □

If we identify each point in E with a unique number between 1 and n, then the monodromy action associated to a covering map induces a group homomorphism $h : \pi_1(Y, y_0) \to S_n$ where the action of $h(\gamma)$ on $\{1, \ldots, n\}$ mirrors the action of γ on E. Of course h depends on how we label the points in the set E so h is only unique up to conjugation by an element of S_n.

Later we will see that a covering map is determined by its associated monodromy action. For this reason it is useful to decouple the two ideas.

Definition 2.7.6 (Monodromy representation)**.** A *monodromy representation* of $\pi_1(Y, y_0)$ is a group homomorphism $h : \pi_1(Y, y_0) \to S_n$ for some n such that the image acts transitively on $\{1, \ldots, n\}$. We say that two monodromy representations h_1 and h_2 are *equivalent* if there is an element $\rho \in S_n$ such that $h_1 = \rho^{-1} h_2 \rho$. Clearly the monodromy representations of equivalent covering maps are equivalent.

Now let us fix a generating set $\{\gamma_1, \ldots, \gamma_k\}$ for $\pi_1(Y, y_0)$ subject to the relation $\gamma_1 \cdots \gamma_k = \mathrm{id}$. Then a given monodromy representation of $\pi_1(Y, y_0)$ is completely determined by the images of $\gamma_1, \ldots, \gamma_k$. So a monodromy representation of $Y = \mathbb{P}^1 \setminus C$ determines and is determined by a sequence of k permutations $[\sigma_1, \ldots, \sigma_k]$ such that

- $\sigma_1 \cdots \sigma_k = \mathrm{id}$,
- the group generated by $\sigma_1, \ldots, \sigma_k$ acts transitively on the set $\{1, \ldots, n\}$.

Definition 2.7.7 (Constellation [LZ04, Definition 1.1.1])**.** A finite sequence $[\sigma_1, \ldots, \sigma_k]$ of permutations in S_n for some n satisfying the above properties is known as a *constellation* of length k. The constellation determined by an n-sheeted covering map of Y is only unique up to conjugation by an element of S_n so we will say that constellations $[\sigma_1, \ldots, \sigma_k]$ and $[\tau_1, \ldots, \tau_k]$ are *equivalent* if there is $\rho \in S_n$ such that $\sigma_j = \rho^{-1} \tau_j \rho$ for all j.

Given a constellation $[\sigma_1, \ldots, \sigma_k]$ one constructs a monodromy representation of $\pi_1(Y, y_0)$ where $Y = \mathbb{P}^1 \setminus \{c_1, \ldots, c_k\}$ simply by mapping generators $\gamma_1, \ldots, \gamma_k$ of $\pi_1(Y, y_0)$ to $\sigma_1, \ldots, \sigma_k$. It is clear that equivalent constellations give rise to equivalent monodromy representations and vice versa. Note that there is no canonical choice of a generating set $\{\gamma_1, \ldots, \gamma_k\}$ and different choices of this set may give rise to non-equivalent constellations.

Now given a monodromy representation $h : \pi_1(Y, y_0) \to S_n$ and $i \in \{1, \ldots, n\}$ consider the stabiliser

$$\mathrm{Stab}(i) = \{\gamma \in \pi_1(Y, y_0) : h(\gamma)(i) = i\}.$$

Since the image of h acts transitively, for any $j \in \{1, \ldots, n\}$ there is $\gamma \in \pi_1(Y, y_0)$ such that $\gamma(i) = j$. Thus $\mathrm{Stab}(j) = \gamma^{-1}\,\mathrm{Stab}(i)\gamma$ so, up to conjugation, this subgroup is unique.

Lemma 2.7.8. *Given a monodromy representation $h : \pi_1(Y, y_0) \to S_n$ and $i \in \{1, \ldots, n\}$, $M = \mathrm{Stab}(i)$ has index n in $\pi_1(Y, y_0)$.*

Proof. Take $\gamma_1, \gamma_2 \in \pi_1(Y, y_0)$. Then $M\gamma_1 = M\gamma_2$ if and only if $\gamma_1\gamma_2^{-1} \in M$. This is the case only if $\gamma_1(i) = \gamma_2(i)$. Thus the cosets of M are in bijection to the elements of $\{1, \ldots, n\}$ so M has index n. $\square$

Definition 2.7.9 (Monodromy subgroup)**.** A subgroup of $\pi_1(Y, y_0)$ with finite index is called a *monodromy subgroup* of $\pi_1(Y, y_0)$. Two monodromy subgroups M_1 and M_2 are said to be *equivalent* if they are conjugate.

Construction 2.7.10. Given a monodromy subgroup $M < \pi_1(Y, y_0)$ of index n we can construct a topological space X and an n-sheeted covering map $p : X \to Y$.

Consider the set of oriented paths in Y that begin at y_0. We endow this space with a topology in the following way: take an oriented path α with starting point y_0 and end point x, and an open, connected, and simply connected neighbourhood U of x. Let $[U, \alpha]$ be the set of curves β with end point lying in U and such that there is a curve u lying entirely within U such that β is homotopic to $\alpha \cdot u$ (this construction gives rise to the universal covering of Y, see [For91, Theorem 5.3] for details).

We define an equivalence relation on this space by saying α and β are equivalent paths if they have the same endpoint and if $\alpha\beta^{-1} \in M$. Let X be the space of equivalence classes of such paths, and define a map $p : X \to Y$ that sends a path to its endpoint. Then p is a covering map. In fact it can be shown that $\pi_1(X) \cong M$ [For91, Theorem 5.9].

Theorem 2.7.11. *Equivalent monodromy subgroups $M_1, M_2 < \pi_1(Y, y_0)$ give rise to equivalent covering maps under Construction 2.7.10.*

Proof. Let $p_1 : X_1 \to Y$ and $p_2 : X_2 \to Y$ be the covering maps associated to M_1 and M_2 respectively according to Construction 2.7.10, and take $\gamma \in \pi_1(Y, y_0)$ such that $M_1 = \gamma^{-1}M_2\gamma$. Let $\alpha \in M_1$ be an equivalence class of curves, then define $h : X_1 \to X_2$ be setting $\alpha \mapsto \gamma^{-1}\alpha\gamma$. This is easily seen to be an equivalence of covering maps. $\square$

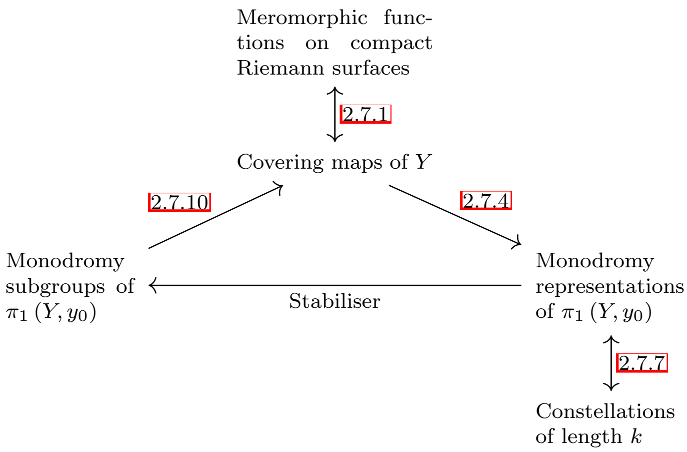

Figure 2.1: Equivalent representations of coverings of $Y = \mathbb{P}^1 \setminus \{c_1, \ldots, c_k\}$.

For more details about any of the above constructions the reader is encouraged to consult [GGD12], [JW16], or [LZ04].

We summarise the above results in Figure 2.1. We have shown that equivalent objects at the base of each arrow give rise to equivalent objects at the tip of the arrow. Since each class of objects is connected to every other class by arrows we have shown that two equivalent objects in a given class give rise to equivalent objects in any other class.

The results of this section can be interpreted in terms of category theory. The classes in Figure 2.1 define the objects of categories. Constructions 2.7.1, 2.7.4 and 2.7.10, and Definition 2.7.7 define the action of functors on the objects of categories. We have shown that these functors are essentially surjective. A more careful treatment of this theory would also define the morphisms in each category and the action of each functor on these morphisms, and show that the functors are full and faithful, thus showing that the categories are in fact equivalent.

2.7.2 Dessins d'enfants

The following lemma shows that every meromorphic function on a compact Riemann surface of genus $g \geq 1$ corresponds to a covering of the sphere with at least three punctures.

Lemma 2.7.12. *Let X be a compact Riemann surface of genus $g > 0$ and $f : X \to \mathbb{P}^1$ be a meromorphic function. Then f has three or more critical values.*

The proof of this lemma can be found in [GGD12, page 170]. We reproduce it here for the convenience of the reader.

Proof. If f has no critical values then it is a proper local homeomorphism and hence a covering map. Since $\mathbb{P}^1$ is simply connected f is a biholomorphism and $X = \mathbb{P}^1$.

If f has a single critical value then, after composing with an automorphism of $\mathbb{P}^1$ if necessary, we can say that value is ∞. Then $f|(X \setminus f^{-1}(\infty))$ is an unramified cover of $\mathbb{C}$. Since $\mathbb{C}$ is simply connected this must be an isomorphism, so $X \setminus f^{-1}(\infty) = \mathbb{C}$ and hence $X = \mathbb{P}^1$, contrary to assumption.

Suppose f has two critical values. Without loss of generality say those values are 0 and ∞. Then $X \setminus f^{-1}\{0, \infty\}$ is a finite sheeted holomorphic covering of $\mathbb{C}^*$ and thus biholomorphic to $\mathbb{C}^*$, so $X \xrightarrow{f} \mathbb{P}^1$ is equivalent to $\mathbb{P}^1 \xrightarrow{z^n} \mathbb{P}^1$. □

Thus, aside from self-coverings, the "simplest" coverings of punctured spheres are those of the triply punctured sphere. Note that since the automorphism group of $\mathbb{P}^1$ acts triply transitively, given a covering of the sphere minus any three points, we can post-compose by an automorphism to get a covering of $Y = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Definition 2.7.13 (Belyi functions, surfaces and pairs)**.** A meromorphic function f on a compact Riemann surface X is called a *Belyi function* if it has at most three critical values (which we usually assume to be $0, 1$ and ∞). A compact Riemann surface X is called a *Belyi surface* if it admits a Belyi function. A *Belyi pair* is a pair (X, f) consisting of a Belyi function and a Belyi surface.

We now show how coverings of $Y = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ can be represented as embedded graphs.

Definition 2.7.14 (Dessins d'enfants)**.** A *dessin d'enfants* (or simply *dessin*) is a pair (X, D) where X is an oriented compact surface and D is a finite connected bipartite graph embedded in X such that $X \setminus D$ is a union of finitely many topological discs. We say two dessins (X, D) and (X', D') are *equivalent* if there is an orientation-preserving homeomorphism $h : X \to X'$ such that the restriction $h|D$ is a graph isomorphism.

By a bipartite graph we mean that the set of vertices can be partitioned into two non-empty disjoint sets V_1 and V_2 such that every edge has one endpoint in each of V_1 and V_2. We will think of the vertices as being coloured either white or black to distinguish the two subsets.

Construction 2.7.15. Given a Belyi function $f : X \to \mathbb{P}^1$ ramified over the set $\{0, 1, \infty\}$ we can construct a bipartite graph D on X by letting $f^{-1}(0)$ be the white vertices of D, $f^{-1}(1)$ be the black vertices and $f^{-1}((0, 1))$ be the edges. The edges thus defined cannot cross over one another since if they did the intersection point would be a critical point (f must be a two-to-one covering in a neighbourhood of the intersection) and so f would have a fourth critical value. Thus D is an embedded finite bipartite graph on X.

The faces of D are biholomorphic to discs since on each connected component of $X \setminus D$, f is a ramified holomorphic covering of $\mathbb{P}^1 \setminus [0, 1]$ with a single critical value.

To show that D is connected let $y_0 = 1/2$ and $Y = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and consider a curve $\gamma \in \pi_1(Y, y_0)$. Let $\tilde{\gamma}$ be a lifting of γ by f. After a moment's reflection the reader will see that $\tilde{\gamma}$ is homotopic to a path contained in D. Since every edge in D contains a point in the preimage of $1/2$ and the monodromy action associated to f is transitive, this shows that D is connected.

Therefore (X, D) defines a dessin. It is not difficult to see that equivalent Belyi pairs give rise to equivalent dessins.

Construction 2.7.16. Given a dessin (X, D) with n edges we can define a constellation $[\sigma, \tau, (\sigma\tau)^{-1}]$ in S_n. Let $E = \{e_1, \ldots, e_n\}$ be the set of edges. Now each edge has exactly one white and one black vertex. Suppose e_i has white endpoint v. Imagine a small positively oriented circle around v (recall that each Riemann surface comes equipped with an orientation). Beginning at e_i, we can follow the circle until it meets another edge e_j. Set $\sigma(i) = j$. Note that e_i and e_j are not necessarily different edges. We define the action of σ on the rest of $\{1, \ldots, n\}$ similarly. The action of τ is defined in the same way using the black vertices of D.

Since D is connected the permutations σ and τ generate a group that acts transitively on E so $[\sigma, \tau, (\sigma\tau)^{-1}]$ is a constellation in S_n.

Thus, in the special case of $Y = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ we can add the category of dessins d'enfants to the previous diagram, resulting in Figure 2.2.

Theorem 2.7.17. *The functors defined in Constructions 2.7.15 and 2.7.16 are essentially surjective.*

For full details of the proof the reader may consult [GGD12, Proposition 4.29], [JW16, Theorem 3.15], or [LZ04]. After properly defining the morphisms in these categories and the action of these functors on these morphisms, one can see that they define an equivalence of categories.

$$\begin{tikzcd}
& \text{Belyi pairs} \arrow[r, "2.7.15"] \arrow[d, leftrightarrow, "2.7.1"'] & \text{Dessins d'enfants} \arrow[dd, "2.7.16"] \\
& \text{Covering maps of } Y \arrow[d, "2.7.4"] & \\
\text{Monodromy subgroups of } \pi_1(Y, y_0) \arrow[ur, "2.7.10"] & \text{Monodromy representations of } \pi_1(Y, y_0) \arrow[l, "\text{Stab}"] \arrow[r, leftrightarrow, "2.7.7"'] & \text{Constellations of length } k
\end{tikzcd}$$

Figure 2.2: Equivalent representations of coverings of $Y = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Dessin (X, D)	Belyi function $f : X \to \mathbb{P}^1$	Constellation $[\sigma, \tau, (\sigma\tau)^{-1}]$ in S_n
White vertices	$f^{-1}(0)$	Cycles of σ
Black vertices	$f^{-1}(1)$	Cycles of τ
Faces	$f^{-1}(\infty)$	Cycles of $(\sigma\tau)^{-1}$
Edges	$f^{-1}((0, 1))$	The set $\{1, \ldots, n\}$

Table 2.1: The relationship between different representations of coverings

Throughout this section we have constructed several equivalent representations a covering of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Under further scrutiny, these constructions reveal how certain features of a dessin manifest themselves in these alternate representations. A short list of these correspondences is presented in Table 2.1.

The following theorem is not strictly necessary for any later results in this thesis, however we would be remiss to make no mention of it having spent so much time developing related theory. It shows that Belyi surfaces are exceptional among compact Riemann surfaces. It even caused Alexander Grothendieck to remark:

> "I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact." [SL97, page 253]

We refer to [GGD12, chapter 3] for the proof.

Theorem 2.7.18 (Belyi's theorem)**.** *A compact Riemann surface can be defined (as a projective curve) over the algebraic numbers if and only if it is a Belyi surface.*

Chapter 3

Algebraic and finite order functions

3.1 Definitions

We begin by extending the concept of finite order growth developed in Section 2.6 to functions on Riemann surfaces.

Definition 3.1.1 (Finite order at a point)**.** We say that a function f defined on a punctured neighbourhood of a point $z_0 \in \mathbb{C}$ is of *finite order at* z_0 if $f(1/(z - z_0))$ is of finite order in the sense of Definition 2.6.1. The *order* of f at z_0 is defined as the order of $f(1/(z - z_0))$.

Definition 3.1.2 (Finite order on a Riemann surface)**.** Let X be a Riemann surface and p be a point in X. A holomorphic function f defined on a punctured neighbourhood of p is said to be of *finite order* at p if there is a holomorphic coordinate neighbourhood (U, z) of p such that $f \circ z^{-1}|z(U \setminus \{p\})$ is of finite order at $z(p)$. The *order* of f is then defined as the order of $f \circ z^{-1}$.

All of the elementary results proved in Section 2.6 carry over to finite order functions on Riemann surfaces.

Lemma 3.1.3. *The above definition of order is independent of the chart.*

Proof. Let p be a point on a Riemann surface X and f be holomorphic in a punctured neighbourhood of p. Let (U, z) and (V, w) be coordinate neighbourhoods of p with $z(p) = 0 = w(p)$ and let $\varphi = z \circ w^{-1}$ be the transition function between the two charts. Suppose $f \circ z^{-1}$ has order μ. Then for $\varepsilon > 0$ and all $a \in \mathbb{C}$ sufficiently close to 0

$$\exp|\varphi(a)|^{-(\mu-\varepsilon)} \leq |f \circ z^{-1} \circ \varphi(a)| = |f \circ w^{-1}(a)| \leq \exp|\varphi(a)|^{-(\mu+\varepsilon)}.$$

Since φ is a biholomorphism there are numbers $m, M > 0$ such that $m|a| \leq |\varphi(a)| \leq M|a|$ for all a. So for all a sufficiently close to 0

$$\exp(M|a|)^{-(\mu-\varepsilon)} \leq |f \circ w^{-1}(a)| \leq \exp(m|a|)^{-(\mu+\varepsilon)}.$$

Since ε was arbitrary we see that $f \circ w^{-1}$ has order μ also. □

With the notion of finite order functions in hand we can define two sheaves that will be of central importance throughout the rest of this thesis.

Definition 3.1.4 (The sheaf of algebraic and finite order functions)**.** Let X be a Riemann surface and $C \subset X$ be a discrete subset. We define the sheaves $\mathscr{A}_C$, the *sheaf of algebraic functions*, and $\mathscr{F}_C$, the *sheaf of finite order functions*, on X as follows. Let $U \subset X$ be an open subset. Define abelian groups

$$\mathscr{A}_C(U) = \{f \in \mathscr{M}(U) : f \text{ has no poles outside of } C\}$$

and $\mathscr{F}_C(U)$ as the set of functions $f \in \mathscr{O}(U \setminus C)$ that are of finite order at all $p \in C$ that are isolated boundary points of $U \setminus C$ (where the group operation is addition). Along with the obvious restriction maps these groups define the sheaves $\mathscr{A}_C$ and $\mathscr{F}_C$ on X.

We can also define sheaves of algebraic and finite order differential forms on a Riemann surface in a similar way.

Definition 3.1.5 (Algebraic and finite order 1-forms)**.** Let X be a Riemann surface, $C \subset X$ be a discrete subset, and $R = X \setminus C$. A holomorphic 1-form $\omega \in \mathscr{O}^{(1)}(R)$ is said to be *algebraic* (respectively *of finite order*) if for every point $c \in C$ there is a holomorphic coordinate chart (U, z) centred on c such that in local coordinates

$$\omega|U = f dz$$

where $f \in \mathscr{A}_C(U)$ (respectively $f \in \mathscr{F}_C(U)$). We can then define the sheaves $\mathscr{A}_C^{(1)}$ of algebraic 1-forms and $\mathscr{F}_C^{(1)}$ of finite order 1-forms by letting $V \subset X$ be open, and letting $\mathscr{A}_C^{(1)}(V)$ be the group of algebraic 1-forms on V and $\mathscr{F}_C^{(1)}(V)$ be the group of finite order 1-forms on V.

3.2 Algebraic functions

Every algebraic function on a compact Riemann surface is, by definition, holomorphic on the corresponding punctured Riemann surface. It is perhaps not surprising then, that many results concerning holomorphic functions on open Riemann surfaces also hold in the restricted case of algebraic functions.

The sheaf of algebraic functions

As a consequence of Dolbeault's theorem it can be seen that for an open Riemann surface R, $H^1(R, \mathscr{O}) = 0$ [For91, Theorem 26.1]. A similar result holds in the restricted setting of algebraic functions.

Lemma 3.2.1. *Let X be a compact Riemann surface of genus g and $C \subset X$ a non-empty finite subset. Then*

$$H^1(X, \mathscr{A}_C) = 0.$$

The proof relies on the theory of compact Riemann surfaces and the following short lemma proved in [For91, Lemma 12.4].

Lemma 3.2.2. *Let $\mathfrak{U} = (U_i)$ and $\mathfrak{V} = (V_j)$ be open covers of a Riemann surface X such that every V_j is contained in at least one U_i and let $\mathscr{S}$ be a sheaf over X. Then the map $r : H^1(\mathfrak{U}, \mathscr{S}) \to H^1(\mathfrak{V}, \mathscr{S})$ induced by restriction is injective, and hence, so is the canonical mapping $H^1(\mathfrak{U}, \mathscr{S}) \to H^1(X, \mathscr{S})$.*

Proof of Lemma 3.2.1. Let $\mathfrak{V} = (U_j)_{j=1}^n$ be a finite open cover of X and let (f_{jk}) be a cocycle in $Z^1(\mathfrak{V}, \mathscr{A}_C)$. Let K_{jk} be the order of the highest order pole of f_{jk} and let $K = \max_{j,k}\{K_{jk}, 2g-1\}$.

Now using χ_C to denote the characteristic function of C, let $D = K\chi_C$ considered as a divisor. We can then view (f_{jk}) as a cocycle in $Z^1(\mathfrak{U}, \mathscr{O}_D)$. But by Theorem 2.1.2 $\deg D > 2g-2$ implies that $H^1(X, \mathscr{O}_D) = 0$ and since the canonical mapping $H^1(\mathfrak{V}, \mathscr{O}_D) \to H^1(X, \mathscr{O}_D)$ is injective we have that $H^1(\mathfrak{V}, \mathscr{O}_D) = 0$. Therefore (f_{jk}) splits in $\mathscr{O}_D$ and hence also in $\mathscr{A}_C$ and so $H^1(X, \mathscr{A}_C) = 0$. □

Prescribing the periods of algebraic 1-forms

In [BS49] Behnke and Stein showed it is possible to prescribe the periods of a holomorphic 1-form on an open Riemann surface. We now show that this is also possible in the algebraic setting.

Theorem 3.2.3. *Given a compact Riemann surface X, a non-empty finite subset $C \subset X$, and a group homomorphism $\rho : \pi_1(X) \to \mathbb{C}$, there is an algebraic 1-form $\omega \in \mathscr{A}_C^{(1)}(X)$ with*

$$\int_\sigma \omega = \rho(\sigma) \text{ for every } \sigma \in \pi_1(X).$$

A proof of this theorem in the holomorphic case can be found in [For91, Theorem 28.6]. The algebraic case follows along the same lines as the more general setting with only minor adaptations.

Proof. Consider the universal covering map $p : \tilde{X} \to X$ of X. The group of covering transformations of p is isomorphic to the fundamental group $\pi_1(X)$. This isomorphism induces an action of $\pi_1(X)$ on $\tilde{X}$.

For every point $x \in X$ there is a neighbourhood U_x such that $p^{-1}(U_x) = \bigsqcup V_{x,j}$ where $V_{x,j} \subset \tilde{X}$ is open and $p|V_{x,j}$ is a homeomorphism. Letting $Y_x = p^{-1}(U_x)$ we can thus construct a homeomorphism $\varphi_x : Y_x \to U_x \times \pi_1(X)$ that is compatible with the action of $\pi_1(X)$ on $\tilde{X}$. That is, we can write $\varphi_x = (p, \eta_x)$ where $\eta_x : Y_x \to \pi_1(X)$ satisfies

$$\eta_x(\sigma y) = \sigma \cdot \eta_x(y)$$

for every $y \in Y_x$ and $\sigma \in \pi_1(X)$ where $\cdot$ denotes the group multiplication in $\pi_1(X)$.

Now we can define locally constant functions $f_x : Y_x \to \mathbb{C}$ by

$$f_x(y) = \rho(\eta_x(y)).$$

The group $\pi_1(X)$ acts on $\mathscr{A}_{p^{-1}(C)}(\tilde{X})$ by $\sigma f = f \circ \sigma^{-1}$. Thus we have

$$\sigma f_x(y) \ = f_x(\sigma^{-1} y) = \rho(\eta_x(\sigma^{-1} y)) = \rho(\sigma^{-1} \cdot \eta_x(y)).$$

So

$$\begin{aligned}
(f_x - \sigma f_x)(y) &= \rho\left(\eta_x(y)\right) - \rho\left(\sigma^{-1} \cdot \eta_x(y)\right) \\
&= \rho\left(\eta_x(y) \cdot \left(\sigma^{-1} \cdot \eta_x(y)\right)^{-1}\right) \\
&= \rho\left(\eta_x(y) \cdot (\eta_x(y))^{-1} \cdot \sigma\right) \\
&= \rho(\sigma)
\end{aligned}$$

Therefore for $x, y \in X$, we have $(f_x - f_y) = \sigma(f_x - f_y)$ on $Y_x \cap Y_y$. Then the function $g_{xy} = f_x - f_y \in \mathscr{A}_{p^{-1}(C)}(Y_x \cap Y_y)$ can be considered as an element of $\mathscr{A}_C(U_x \cap U_y)$. These g_{xy} form a cocycle in $Z^1\big((U_x)_{x \in X}, \mathscr{A}_C\big)$. By Lemma 3.2.1 this cocycle splits, so there are functions $g_x \in \mathscr{A}_C(U_x)$ with $g_x - g_y = g_{xy}$ on $U_x \cap U_y$. Pulling back by p we get functions in $\mathscr{A}_{p^{-1}(C)}(Y_x)$ (also denoted g_x) that are invariant under the action of $\pi_1(X)$. Now let $F_x = f_x - g_x \in \mathscr{A}_{p^{-1}(C)}(Y_x)$. These functions satisfy $F_x - \sigma F_x = \rho(\sigma)$, and on $U_x \cap U_y$

$$F_x - F_y = f_x - f_y - (g_x - g_y) = g_{xy} - (g_x - g_y) = 0.$$

Hence the F_x piece together to form a global function $F \in \mathscr{A}_{p^{-1}(C)}(\tilde{X})$ with $F - \sigma F = \rho(\sigma)$.

Now consider the differential dF. This 1-form is invariant under the action of $\pi_1(X)$ since

$$dF - \sigma dF = dF - d(\sigma F) = d(F - \sigma F) = 0,$$

so dF descends to a 1-form $\omega \in \mathscr{A}_C^{(1)}(X)$ with periods $\int_\sigma \omega = \rho(\sigma)$. □

The corollary below, which follows immediately from the theorem above, generalises Proposition 2.4 of [FO13] which was stated without proof.

Corollary 3.2.4. *Let X be a compact Riemann surface and C be a non-empty finite subset. Every element of $H^1(X \setminus C, \mathbb{C})$ is represented by an element of $\mathscr{A}_C^{(1)}(X)$ as a de Rham cohomology class.*

Algebraic approximation

To approximate a function defined on a compact subset of a compact Riemann surface using global holomorphic functions we must of course allow for some kind of singularity (otherwise the approximants would be constant). The next theorems show that we only need to allow for the mildest kind of singularity that we could hope for, that is, we can approximate by algebraic functions.

Theorem 3.2.5 (Algebraic Runge)**.** *Let X be a compact Riemann surface, $C \subset X$ be a non-empty finite subset, and $R = X \setminus C$. Take a holomorphically convex compact subset K of R and $f \in \mathscr{O}(K)$ (recall Definition 2.2.1). Then f can be uniformly approximated on K by algebraic functions on R, that is, the restriction map $\mathscr{A}_C(X) \to \mathscr{O}(K)$ has dense image with respect to the topology of uniform convergence on K.*

Proof. By Runge's theorem there are functions $f_n \in \mathscr{O}(R)$ converging uniformly to f on K. A priori we cannot say anything about the growth of these functions near the points in C.

Let $E : R \to \mathbb{C}^m$ be an algebraic embedding into some $\mathbb{C}^m$ (as per Remark 2.5.3). By Theorem 2.5.11 the functions $f_n \circ E^{-1}$ extend to holomorphic functions F_n on $\mathbb{C}^m$. There are Taylor polynomials $P_{nk} : \mathbb{C}^m \to \mathbb{C}$ converging uniformly to F_n on compact subsets of $\mathbb{C}^m$. Now define functions $g_n : R \to \mathbb{C}$ by

$$g_n = P_{nn} \circ E.$$

These functions are algebraic. Under E, the set C is mapped to the hyperplane at infinity so $\lim\limits_{x \to c} \|E(x)\| = \infty$ for $c \in C$. Then P_{nn} is either constant, in which case g_n is constant, or has a pole at the hyperplane at infinity, in which case g_n has a pole at every $c \in C$.

Finally since the functions g_n converge uniformly to f on K the theorem is proved. □

Remark 3.2.6. Shortly before submitting this thesis I became aware of the papers [Roy67] and [Sch79] in which the above theorem is proved using solely one-dimensional theory.

Theorem 3.2.7 (Algebraic Mergelyan–Bishop)**.** *Let X be a compact Riemann surface, $C \subset X$ a non-empty finite subset, and $R = X \setminus C$. Take K a holomorphically convex compact subset of R and $f \in \mathscr{O}_{int}(K)$. Then f can be uniformly approximated on K by algebraic functions on R, that is, the restriction map $\mathscr{A}_C(X) \to \mathscr{O}_{int}(K)$ has dense image with respect to the topology of uniform convergence on K.*

Proof. This theorem follows from Theorem 3.2.5 once we note that by the original Mergelyan–Bishop theorem $\mathscr{O}_{int}(K) = \overline{\mathscr{O}}(K)$. Thus we can approximate f by functions in $\mathscr{O}(K)$ and then approximate these functions by functions in $\mathscr{A}_C(X)$. □

3.3 Finite order functions

Prescribing the divisor of finite order functions

On an open Riemann surface we can always find a meromorphic function with a prescribed divisor. We will now see that on an affine curve we can do this in such a way that the function has finite order at the punctures provided the divisor has restrictions on the rate of accumulation of zeros. The following is our own definition.

Definition 3.3.1. Let X be a compact Riemann surface, $C \subset X$ a non-empty finite subset, $c \in C$ be a point, (U, φ) be a coordinate disc centred on c with $U \cap C = \{c\}$ and $\varphi(c) = 0$, and D be a non-negative divisor on $X \setminus C$. Let $(d_j)_{j=1}^{\infty}$ be a listing of the points in $\mathrm{supp}(D) \cap U$ repeated according to their multiplicity. We define the *accumulation order* of D at c as the infimum of numbers ν such that

$$\sum_{j=1}^{\infty} |\varphi(d_j)|^{\nu} < \infty.$$

If the above sum does not converge for any ν we say D has infinite accumulation order. If $\mathrm{supp}(D) \cap U$ is finite we say D has zero accumulation order.

Lemma 3.3.2. *The accumulation order of a divisor at a point is independent of the chart used.*

Proof. The proof is similar to that of Lemma 3.1.3. Suppose X is a Riemann surface, $c \in X$ is a point, and D is a divisor on $X \setminus \{c\}$ with accumulation order μ according to the coordinate neighbourhood (U, φ) of c. Let (V, ψ) be another coordinate neighbourhood of c. Then since $\psi \circ \varphi^{-1}$ is a biholomorphism there are numbers $m, M > 0$ such that

$$m|z| \leq |\psi \circ \varphi^{-1}(z)| \leq M|z|$$

for all $z \in \varphi(U \cap V)$. Now let $(d_j)_{j=1}^{\infty}$ be a listing of the points in $\operatorname{supp}(D) \cap (U \cup V)$ (note that the theorem is obvious if D has finite support). Then for $\nu > 0$

$$m \sum_{j=1}^{\infty} |\varphi(d_j)|^{\nu} \leq \sum_{j=1}^{\infty} |\psi(d_j)|^{\nu} \leq M \sum_{j=1}^{\infty} |\varphi(d_j)|^{\nu}.$$

So D has accumulation order μ at c according to ψ also. □

The above definition is inspired by Hadamard's factorisation theorem. Indeed we can rephrase part of Hadamard's theorem by saying that for any non-negative divisor D on $\mathbb{C}$ with accumulation order μ at ∞ there is a function $f \in \mathscr{F}_{\{\infty\}}(\mathbb{P}^1)$ of order μ with $\operatorname{div}(f) = D$.

Proposition 3.3.3. *Let X be a compact Riemann surface of genus $g \geq 1$, $C = \{c_1, \ldots, c_k\} \subset X$ a finite non-empty subset, and $R = X \setminus C$. Take a non-negative divisor D on R with accumulation order $\mu_j < \infty$ at c_j for each $j = 1, \ldots, k$. Then there is a function $f \in \mathscr{F}_C(X)$ such that $\operatorname{div}(f) = D$ and so that the order of f at c_j is at least μ_j and at most $\mu_j + 2g - 1$.*

This proposition generalises [FO13, Proposition 2.1]. It can also be seen as an extension of Hadamard's theorem to affine curves.

Proof. By Weierstrass' theorem there is $f_0 \in \mathscr{O}(R)$ with $\operatorname{div}(f_0) = D$. For $j = 1, \ldots, k$ let V_j be a coordinate disc centred on c_j so that the V_j are pairwise disjoint. Let $W_j \Subset V_j$ be a smaller disc containing c_j and let $V_0 = X \setminus \bigcup_{j=1}^{k} \overline{W}_j$. Then

$$\mathfrak{V} = \{V_0, V_1, \ldots, V_k\}$$

is an open cover of X. By Theorem 2.6.9 there is a function $f_j \in \mathscr{F}_C(V_j)$ with $\operatorname{div}(f_j) = D|V_j$ and order μ_j at c_j for $j = 1, \ldots, k$. Let m_j denote the winding number of $\dfrac{f_0}{f_j}$ at c_j and let $h_j \in \mathscr{O}(V_j)$ be a holomorphic function with a zero of order m_j at c_j and no zeros. Then $\dfrac{f_0}{f_j h_j}$ has a holomorphic logarithm on $V_j \setminus \{c_j\}$. Now for $j = 1, \ldots, k$ define

$$\zeta_{0j} = \log \frac{f_0}{f_j h_j} \in \mathscr{O}(V_j \setminus \overline{W}_j).$$

Since $V_i \cap V_j = \emptyset$ for $1 \leq i < j$ this defines a cocycle $\zeta = (\zeta_{ij}) \in Z^1(\mathfrak{V}, \mathscr{O})$.

Now let E be any non-zero divisor on X supported on C with $\deg E = 2g - 1$. Then since the canonical map $H^1(\mathfrak{V}, \mathscr{O}_E) \to H^1(X, \mathscr{O}_E)$ is injective by Lemma 3.2.2 and $H^1(X, \mathscr{O}_E) = 0$ by Theorem 2.1.2, there are functions $u_j \in \mathscr{O}_E(V_j)$ for $j = 0, \ldots, k$ with

$$u_0 - u_j = \log \frac{f_0}{f_j h_j}$$

(that is, the 0-cochain $(u_0, \ldots, u_k)$ splits ζ). Then on $V_j \setminus \overline{W}_j$ we have

$$f_j h_j \exp(-u_j) = f_0 \exp(-u_0).$$

Define $f \in \mathscr{F}_C(X)$ by $f|V_j = f_j h_j \exp(u_j)$ for $j = 1, \ldots, k$ and $f|V_0 = f_0 \exp(-u_0)$. Then f has the desired properties. □

Proposition 3.3.4. *Let X be a compact Riemann surface of genus $g \geq 1$, $C = \{c_1, \ldots, c_k\} \subset X$ a finite non-empty subset, and $R = X \setminus C$. Take a divisor D on R with accumulation order $\mu_j < \infty$ at c_j for $j = 1, \ldots, k$. There is a 1-form $\omega \in \mathscr{F}_C(X)$ with $\operatorname{div}(\omega) = D$ and so that the order of ω at c_j is at least μ_j and at most $\mu_j + 2g - 1$.*

Proof. Since $H^1(X, \mathscr{A}_C) = 0$ there is an algebraic 1-form $\omega' \in \mathscr{A}_C^{(1)}(X)$ with no zeros in R. Let $f \in \mathscr{F}_C(X)$ be a finite order function as constructed in Proposition 3.3.3. Then $\omega = f\omega'$ has the desired properties. □

Chapter 4

The theorem of Forstnerič and Ohsawa

4.1 The main theorem

The following theorem was proved in [FO13] as Theorem 1.1.

Theorem 4.1.1 (Forstnerič and Ohsawa)**.** *If X is a compact Riemann surface and $x_0 \in X$, then the punctured Riemann surface $R = X \setminus \{x_0\}$ admits a nowhere critical holomorphic function of finite order.*

The proof presented in [FO13] relies on Theorem 3.2.3 and Proposition 3.2.7 which were stated without proof. Moreover some steps of the proof are delicate and only discussed briefly. In this section we will provide a full proof of a generalisation of the theorem, elaborating on those details where we found the original exposition sparse.

The following lemma is necessary for the proof of the main theorem but was left unstated in [FO13].

Lemma 4.1.2. *Let*

$$D = \{x \in \mathbb{R}^n : \|x\| \leq R\}$$

for $R > 0$. Suppose $f : D \to \mathbb{R}^n$ is a continuous map such that $\|f(x) - x\| < R$ for all $x \in D$. Then there is $x_0 \in D$ for which $f(x_0) = 0$.

Proof. Firstly we claim that $f|\partial D$ is homotopic to the inclusion map $\iota : \partial D \hookrightarrow \mathbb{R}^n \setminus \{0\}$ as a map $\partial D \to \mathbb{R}^n \setminus \{0\}$. To see this define

$$H(t, x) = f(x) + t(x - f(x))$$

for $0 \leq t \leq 1$ and $x \in \partial D$. Clearly, H is continuous so defines a homotopy. Also by the reverse triangle inequality

$$\begin{aligned} \|H(t,x)\| &= \|x + H(t,x) - x\| \\ &\geq \Big| \|x\| - \|H(t,x) - x\| \Big| \\ &= \Big| \|x\| - (1-t)\|f(x) - x\| \Big| \\ &> R - (1-t)R \geq 0 \end{aligned}$$

for all $0 \leq t \leq 1$ and all $x \in \partial D$; in particular, $H(t,x) \neq 0$ for all $x \in \partial D$.

Now suppose $0 \notin f(D)$. Then f factors through D by ι, that is, the following diagram commutes.

$$\begin{tikzcd} \partial D \arrow[r, hook, "\iota"] \arrow[dr, "f|\partial D"'] & D \arrow[d, dashed, "f"] \\ & \mathbb{R}^n \setminus \{0\} \end{tikzcd}$$

Then f is null-homotopic since D is contractible. This contradicts f being homotopic to ι. So there must be some $x_0 \in D$ with $f(x_0) = 0$. □

Now we let X be a compact Riemann surface of genus g, $C = \{c_1, \ldots, c_k\} \subset X$ be a non-empty finite subset and $R = X \backslash C$. We aim to prove the following generalisation of Theorem 4.1.1.

Theorem 4.1.3. *Given a non-negative divisor D on R with finite accumulation order at every point in C, there is an exact holomorphic 1-form ω on R with* $\operatorname{div}(\omega) = D$ *and so that ω is of finite order at every point of C.*

Proof. For a curve $\gamma : [0,1] \to X$ we will use the symbol γ to denote both the map and its image in X. If γ is a loop we will denote its homology class in $H_1(X, \mathbb{Z})$ by $[\gamma]$.

Let $N = 2g + k - 1$. We claim there exist simple piecewise differentiable loops $\gamma_1, \ldots, \gamma_N$ in R that generate $H_1(R, \mathbb{Z})$ and such that, letting $K = \bigcup_{i=1}^{N} \gamma_i$, $R \setminus K$ has no relatively compact connected components (that is, K is holomorphically convex in R), $D(x) = 0$ for all $x \in K$, and so that $\bigcap_{i=1}^{N} \gamma_i = \{p\}$ for some $p \in R$.

Suppose X has genus $g \geq 1$ and view X as a $4g$-gon P in $\mathbb{C}$ or $\mathbb{D}$ with sides identified in the usual way. Choose P so that the points of C and $\operatorname{supp}(D)$ are contained in the interior of P.

The boundary of P is made up of $2g$ curves $\gamma_1, \dots, \gamma_{2g}$ that generate $H_1(X, \mathbb{Z})$ (note that each side of P is identified with exactly one other). There are k points in C. If $k = 1$ the loops $\gamma_1, \dots, \gamma_{2g}$ will also generate $H_1(R, \mathbb{Z})$. Assuming $k > 1$ we need to find additional loops to complete the generating set.

We can find a neighbourhood V_i of each $c_i \in C$ so that the V_i are pairwise disjoint and are contained in the interior of P. These neighbourhoods will contain all but finitely many points in $\mathrm{supp}(D)$.

Now we can find curves $\gamma_{2g+1}, \dots, \gamma_{2g+k-1}$ in P with endpoints in the vertices of P that avoid the neighbourhoods V_i and the finitely many points in $\mathrm{supp}(D) \setminus \bigcup_i V_i$. Let

$$K = \bigcup_{i=1}^{2g+k-1} \gamma_i.$$

We can choose $\gamma_{2g+1}, \dots, \gamma_{2g+k-1}$ so that each connected component of $P \setminus K$ contains exactly one point of C. Then the curves $\gamma_1, \dots, \gamma_{2g+k-1}$ satisfy the desired properties. An example of a simple case is depicted in Figure 4.1.

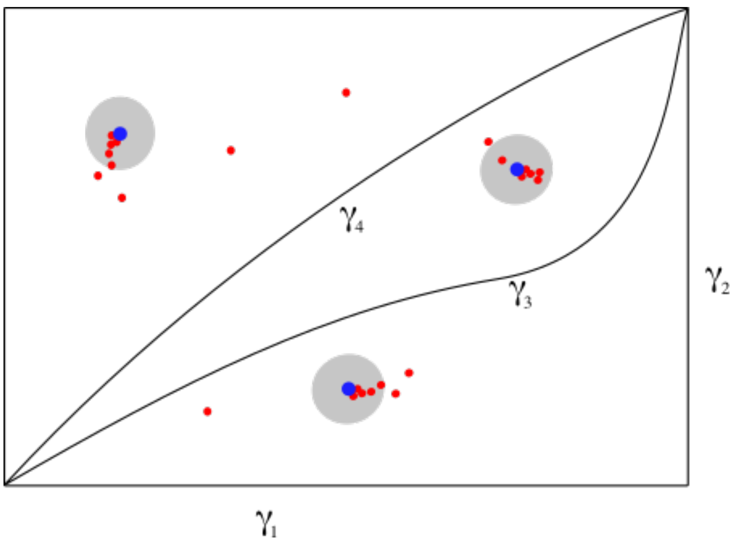

Figure 4.1: Suppose X is a torus and C consists of three distinct points. The fundamental domain for X is a parallelogram in $\mathbb{C}$. In the above figure the blue dots represent the three points of C, and the red dots represent the support of the divisor D. The areas shaded in grey represent the neighbourhoods V_i. The lines in black represent the loops that generate the homology group $H_1(R, \mathbb{Z})$.

If $X = \mathbb{P}^1$ and $k \geq 2$ the loops can be constructed by taking small circles around all but one of the points in C and deforming them so that the other conditions are met.

Now let us take $\omega_0 \in \mathscr{F}_C^{(1)}(X)$ with divisor D as constructed in Proposition 3.3.4. For each curve γ_i there is a neighbourhood $U_i \subset R$ containing γ_i and a biholomorphic map φ_i from an annulus $A_r = \{w \in \mathbb{C} : 1 - r < |w| < 1 + r\}$ onto U_i for sufficiently small $r > 0$ such that the positively oriented unit circle is mapped by φ_i onto γ_i, with $\varphi_i(1) = p$, and so that $D|U_i = 0$ for all i.

For $i = 1, \ldots, N$ take $H_i : A_r \to \mathbb{C}$ so that

$$\varphi_i^* \omega_0 = H_i(w)dw,$$

and let

$$n_i = \frac{1}{2\pi i} \int_{|w|=1} d \log H_i. \tag{4.1}$$

By Corollary 3.2.4 there is $\xi \in \mathscr{A}_C^{(1)}(X)$ such that for $i = 1, \ldots, N$

$$\frac{1}{2\pi i} \int_{\gamma_i} \xi = n_i.$$

Define $u : R \to \mathbb{C}$ by

$$u(x) = \exp\left(-\int_p^x \xi\right).$$

Note that u is well defined since

$$\exp\left(-\int_{\gamma_i} \xi\right) = 1.$$

Also u is nowhere vanishing and of finite order on R.

Let $\omega = u\omega_0$. Then ω is a finite order 1-form with $\operatorname{div}(\omega) = D$ whose winding numbers as defined by Equation (4.1) are all zero. Thus for every $i = 1, \ldots, N$ there is a holomorphic function $h_i : A_r \to \mathbb{C}$ with $h_i(1) = 0$ and a complex number c_i such that

$$\varphi_i^* \omega = \exp(h_i + c_i)dw.$$

Note that the functions $h_i \circ \varphi_i^{-1} : \gamma_i \to \mathbb{C}$ agree at the unique intersection point p of the curves γ_i and hence define a continuous function H on $K = \bigcup_i \gamma_i$.

By Proposition 3.2.7 we can find an algebraic function $h \in \mathscr{A}_C(X)$ that approximates H on K arbitrarily well. The periods of the nowhere vanishing finite order 1-form $e^{-h}\omega$

$$\int_{\gamma_j} e^{-h}\omega = e^{c_j} \int_{|w|=1} e^{h_j - h\circ\varphi_j} dw$$

can thus be made arbitrarily small by choosing h sufficiently close to H. We will use a perturbation argument to show that such a form can be altered further so that it is exact. The perturbation factor we will use is parameterised by N complex numbers $\tau_1, \ldots, \tau_N$ and has the form

$$\exp\left(\sum_{i=1}^{N} \tau_i f_i\right),$$

where the f_i are algebraic functions. Our goal is thus to find complex numbers $\zeta_1, \ldots, \zeta_N$, and algebraic functions $h, f_1, \ldots, f_N \in \mathscr{A}_C(X)$ satisfying

$$\int_{\gamma_j} \exp\left(\sum_{i=1}^{N} \zeta_i f_i - h\right) \omega = 0$$

for all $j = 1, \ldots, N$. The details in [FO13] about the construction of these functions and numbers are quite brief. The following discussion is intended to make this construction more explicit.

We begin by fixing the functions $f_i \in \mathscr{A}_C(X)$ for $i = 1, \ldots, N$ so that for $j = 1, \ldots, N$

$$\int_{|w|=1} f_i \circ \varphi_j(w) dw = \begin{cases} e^{-c_j} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

To do this take a continuous function $b : S^1 \to \mathbb{C}$ such that $b(1) = 0$ and

$$\int_{|w|=1} b(w) dw = 1.$$

Then for $i, j = 1, \ldots, N$ define continuous functions $b_i : K \to \mathbb{C}$ on γ_j by

$$b_i = \begin{cases} e^{-c_j} b \circ \varphi_j & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 3.2.7 for any $\varepsilon > 0$ we can find $\hat{f}_i \in \mathscr{A}_C(X)$ such that $\sup_K |\hat{f}_i - b_i| < \varepsilon$. Take ε small enough that the matrix B whose (i, j)-entry is given by

$$B_{ij} = \int_{|w|=1} \hat{f}_i \circ \varphi_j dw$$

is invertible. Then for every i there is a linear combination f_i of $\hat{f}_1, \ldots, \hat{f}_N$ with complex coefficients such that

$$\int_{|w|=1} f_i \circ \varphi_j dw = \begin{cases} e^{-c_j} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

These f_i will now remain fixed.

Now letting $h \in \mathscr{A}_C(X)$ be an arbitrary algebraic function and $\tau_1, \dots, \tau_N$ be arbitrary complex numbers, define functions $E_1 : K \to \mathbb{C}$ by

$$E_1 = \exp\left(\sum_{i=1}^{N} \tau_i f_i\right) - 1 - \sum_{i=1}^{N} \tau_i f_i$$

and $E_2 : K \to \mathbb{C}$ given on γ_i by

$$E_2 = \exp(h_i \circ \varphi_i^{-1} - h) - 1.$$

Note that E_1 and E_2 can be thought of as error terms in Taylor approximations to

$$\exp\left(\sum_{i=1}^{N} \tau_i f_i\right) \text{ and } \exp(h_i \circ \varphi_i^{-1} - h)$$

respectively.

Consider the map $\Phi : \mathbb{C}^N \to \mathbb{C}^N$ defined by

$$\tau = (\tau_1, \dots, \tau_N) \mapsto (\Phi_1(\tau), \dots, \Phi_N(\tau)),$$

where Φ_j is given by

$$\Phi_j(\tau) = \int_{\gamma_j} \exp\left(\sum_{i=1}^{N} \tau_i f_i - h\right) \omega.$$

We can rewrite Φ_j using E_1 and E_2:

$$\begin{aligned}
\Phi_j(\tau) &= e^{c_j} \int_{|w|=1} \left(1 + \sum_{i=1}^{N} \tau_i f_i + E_1\right) (1 + E_2) \circ \varphi_j dw \\
&= e^{c_j} \int_{|w|=1} \left(\sum_{i=1}^{N} \tau_i f_i + E_2 \left(1 + \sum_{i=1}^{N} \tau_i f_i\right) + E_1 + E_1 E_2\right) \circ \varphi_j dw \\
&= e^{c_j} \sum_{i=1}^{N} \tau_i \int_{|w|=1} f_i \circ \varphi_j^{-1} dw \\
&+ e^{c_j} \int_{|w|=1} \left(E_2 \left(1 + \sum_{i=1}^{N} \tau_i f_i\right) + E_1 + E_1 E_2\right) \circ \varphi_j dw \\
&= \tau_j + e^{c_j} \int_{|w|=1} \left(E_2 \left(1 + \sum_{i=1}^{N} \tau_i f_i\right) + E_1 + E_1 E_2\right) \circ \varphi_j dw.
\end{aligned}$$

Thus if we are able to take E_1 and E_2 to be "small" (in a sense that will soon be made precise), we find that $\Phi(\tau)$ is close to τ. Then by applying Lemma 4.1.2 we will prove the existence of $\zeta \in \mathbb{C}^N$ with $\Phi(\zeta) = 0$. It remains to determine an algebraic function h to suit this purpose.

Choose numbers $0 < \varepsilon_1 < 1$ and $C_1 > 1$ such that

$$\sup_K |E_1| \leq C_1 \|\tau\|^2$$

whenever $\|\tau\| \leq \varepsilon_1$. Let $c = \max_i |c_i|$. Then for $\tau \in \mathbb{C}^N$ with $\|\tau\| \leq \varepsilon_1$,

$$\begin{aligned}
|\tau_j - \Phi_j(\tau)| &= \left| e^{c_j} \int_{|w|=1} \left(E_2 \left(1 + \sum_{i=1}^{N} \tau_i f_i \right) + E_1 + E_1 E_2 \right) \circ \varphi_j dw \right| \\
&\leq e^c 2\pi \sup_K \left(|E_2| \left(1 + \sum_{i=1}^{N} |\tau_i f_i| \right) + |E_1| + |E_1 E_2| \right) \\
&\leq e^c 2\pi \left(\|E_2\|_K \left(1 + \sum_{i=1}^{N} \sup_K |f_i| \right) + C_1 \varepsilon_1^2 + C_1 \varepsilon_1 \|E_2\|_K \right)
\end{aligned}$$

where $\|E_2\|_K = \sup_K |E_2|$. Now decrease ε_1 if necessary so that $\varepsilon_1 < \dfrac{1}{C_1 6\pi e^c}$. Then the above inequality becomes

$$|\tau_j - \Phi_j(\tau)| < e^c 2\pi \|E_2\|_K \left(1 + \sum_{i=1}^{N} \sup_K |f_i| \right) + \frac{\varepsilon_1}{3} + \frac{\|E_2\|_K}{3}.$$

Now choose a number ε_2 such that

$$0 < \varepsilon_2 < \frac{\varepsilon_1}{e^c 6\pi \left(1 + \sum\limits_{i=1}^{N} \sup\limits_K |f_i| \right)},$$

and so that

$$|e^z - 1| \leq |z| \text{ for } |z| < \varepsilon_2.$$

Then take $h \in \mathscr{A}_C(R)$ such that $\sup_K |h - H| < \varepsilon_2$. Then $\|E_2\|_K < \varepsilon_2$ and we find $|\tau_j - \Phi_j(\tau)| < \varepsilon_1$ for all $\|\tau\| \leq \varepsilon_1$. Applying Proposition 4.1.2 we find that there must exist $\zeta \in \mathbb{C}^N$ with $\Phi(\zeta) = 0$, thus proving Theorem 4.1.3. Note that Theorem 4.1.1 follows immediately by letting $D = 0$. $\square$

4.2 Extensions of the main theorem

The following corollary is a generalisation of Theorem 4.1 from [FO13] which was stated without proof.

Corollary 4.2.1. *Let X be a compact Riemann surface, $C = \{c_1, \ldots, c_k\} \subset X$ be a non-empty finite subset, $R = X \setminus C$, and D be a non-negative divisor on R with finite accumulation order at the points of C. Every element of the cohomology group $H^1(R, \mathbb{C})$ is represented by a holomorphic 1-form ω on R that is of finite order at the points of C and having* $\operatorname{div}(\omega) = D$.

For the proof of this corollary we use the method pioneered by Gunning and Narasimhan in [GN67]. This method was used to prove a similar result in [KS71].

Proof. Let $N = 2g+k-1$ then take complex numbers $\alpha_1, \ldots, \alpha_N$ and an exact finite order 1-form $\omega_0 \in \mathscr{F}_C^{(1)}(X)$ having $\operatorname{div}(\omega_0) = D$ as in Theorem 4.1.3.

Take simple piecewise differentiable loops $\gamma_1, \ldots, \gamma_N$ in R so that the γ_i form a basis for the group $H_1(R, \mathbb{Z})$, so that $K = \gamma_1 \cup \cdots \cup \gamma_N$ is holomorphically convex, and so that $D|K = 0$. We aim to alter ω_0 so that it integrates to α_i along the curve γ_i without introducing zeros or increasing the growth at the points of C beyond finite order.

Suppose $\int_{\gamma_i} \omega_0 \neq \alpha_i$ for $i = 1, \ldots, M$ and $\int_{\gamma_i} \omega_0 = \alpha_i$ for $i = M+1, \ldots, N$ where $1 \leq M \leq N$ (if $M = 0$ we are done). By Lemma 2.3.2 there are continuous functions $u_1, \ldots, u_M : K \to \mathbb{C}$ such that $\operatorname{supp}(u_i) \cap \gamma_j = \emptyset$ for $i \neq j$ and so that

$$\int_{\gamma_i} e^{u_i} \omega_0 = \alpha_i$$

and

$$\int_{\gamma_i} u_i e^{u_i} \omega_0 \neq 0$$

for $i = 1, \ldots, M$.

Now for $s = (s_1, \ldots, s_M) \in \mathbb{C}^M$ define holomorphic functions

$$\varphi_i(s) = \int_{\gamma_i} \omega_0 \exp(s_1 u_1 + \cdots + s_M u_M).$$

Then for $a = (1, \dots, 1) \in \mathbb{C}^M$,

$$\begin{aligned}\varphi_i(a) &= \int_{\gamma_i} \omega_0 e^{u_i} = \alpha_i,\\ \frac{\partial \varphi_i}{\partial s_i}(a) &= \int_{\gamma_i} u_i \omega_0 e^{u_i} \neq 0,\\ \frac{\partial \varphi_i}{\partial s_j}(a) &= \int_{\gamma_i} u_j \omega_0 e^{u_i} = 0 \text{ for } i \neq j.\end{aligned}$$

Let $\varphi = (\varphi_1, \dots, \varphi_M) : \mathbb{C}^M \to \mathbb{C}^M$. Then $\varphi(a) = (\alpha_1, \dots, \alpha_M)$ and the Jacobian matrix $\left[\frac{\partial \varphi}{\partial s}(a)\right]$ is non-singular.

For each $i = 1, \dots, M$ using Proposition 3.2.7, take a sequence of algebraic functions $f_i^{(\nu)} \in \mathscr{A}_C(X)$ indexed by ν, so that $f_i^{(\nu)} \to u_i$ uniformly on K. Then for $i = 1, \dots, M$ define holomorphic functions $\varphi_i^{(\nu)} : \mathbb{C}^M \to \mathbb{C}$ by

$$\varphi_i^{(\nu)}(s) = \int_{\gamma_i} \omega_0 \exp\left(s_1 f_1^{(\nu)} + \dots + s_M f_M^{(\nu)}\right)$$

and a holomorphic map $\varphi^{(\nu)} = (\varphi_1^{(\nu)}, \dots, \varphi_N^{(\nu)}) : \mathbb{C}^M \to \mathbb{C}^M$. Then for $i = 1, \dots, M$, $\varphi_i^{(\nu)}$ converges to φ_i uniformly on compact subsets of $\mathbb{C}^M$. So for $\varepsilon > 0$ and sufficiently large ν there is a point $p = (p_1, \dots, p_M) \in \mathbb{C}^M$ with $\|p - a\| < \varepsilon$ such that $\varphi^{(\nu)}(p) = \varphi(a)$. For such ν and p define an algebraic function $f \in \mathscr{A}_C(X)$ by

$$f = p_1 f_1^{(\nu)} + \dots + p_M f_M^{(\nu)}.$$

Let $\omega = \omega_0 e^f$. Then $\int_{\gamma_i} \omega = \alpha_i$ and ω is of finite order at each point of C. Since $\alpha_1, \dots, \alpha_N$ were arbitrary we can choose them so that ω represents any class in $H^1(R, \mathbb{C})$. Since $\operatorname{div}(\omega) = D$, ω satisfies the conditions of the corollary. $\square$

We should point out some of the differences between the above corollary and Theorem 4.1 in [FO13]. In the above we refer to elements of $H^1(R, \mathbb{C})$ whereas in Theorem 4.1 of [FO13] the authors refer to elements of $H^1(X, \mathbb{C})$. In the case of a single puncture the difference is irrelevant since the two cohomology groups are isomorphic by the restriction map. For subsets $V \subset U \subset X$ let $r_V^U : H^1(U, \mathbb{C}) \to H^1(V, \mathbb{C})$ denote the map induced by restriction. Suppose for a moment that $C = \{c\}$. Take a class $[\omega] \in \ker r_R^X$ and let U be a simply connected neighbourhood of c. Then there is $f \in \mathscr{E}(R)$ with $df = \omega|R$ and $\tilde{f} \in \mathscr{E}(U)$ with $d\tilde{f} = \omega|U$ (since U is simply connected). Now f and $\tilde{f}$ only differ by a constant on $U \setminus \{c\}$ so we can piece them together to show that ω is exact on all of X. Thus r_R^X is injective when C consists of a single point. In this case we also have that R has the homotopy type of a wedge sum of $2g$ circles where g is the genus of X. So $H^1(R, \mathbb{C})$ and $H^1(X, \mathbb{C})$ are both $2g$-dimensional and r_R^X is an isomorphism.

Using the same argument inductively we see that r_R^X is injective for any finite set C. When C contains more than one point, however, the dimension of $H^1(R, \mathbb{C})$ increases with the number of points in C.

Chapter 5

Algebraic immersions

5.1 Elementary results

The preceding chapter showed that we can always find a holomorphic immersion of an affine curve that has restricted growth. We might wonder if this is the best upper bound on the growth of the immersion. For example is it always possible to find an algebraic immersion of an affine curve into the complex plane? We will see shortly that the answer, in general, is no, however we are able to say something about which affine curves admit algebraic immersions.

All algebraic immersions of affine curves into the complex plane can be obtained in the following way. Let X be a compact Riemann surface and $f : X \to \mathbb{P}^1$ a non-constant meromorphic function. Let $C \subset X$ be the finite set of critical points and poles of f and let $R = X \setminus C$. Then $f|R \to \mathbb{C}$ is an algebraic immersion of the affine curve R.

The converse follows from Construction 2.7.1. Let R be an affine curve and $f : R \to \mathbb{C}$ be an algebraic immersion. There is a unique compact Riemann surface X such that $X = R \sqcup \{x_1, \ldots, x_N\}$ where $x_1, \ldots, x_N$ are finitely many points that "fill in the punctures". Then f extends across the points $x_1, \ldots, x_N$ to a meromorphic function on X whose critical points and poles lie in the finite set $\{x_1, \ldots, x_N\}$.

For the above construction to be useful we need to know which finite subsets of a compact Riemann surfaces are the critical sets of meromorphic functions. A general solution to this problem is difficult but we now present some results that deal with special cases.

The following result was proved in [FO13].

Proposition 5.1.1. *If X is a compact Riemann surface of genus $g \geq 1$ and $x_0 \in X$, then every algebraic function $f : X \setminus \{x_0\} \to \mathbb{C}$ has a critical point.*

A stronger result can be seen to follow from Lemma 2.7.12. We provide another proof below for the sake of variety.

Proposition 5.1.2. *Let X be a compact Riemann surface with genus $g \geq 1$. Every meromorphic function $f : X \to \mathbb{P}^1$ has at least three distinct branch points.*

Proof. We consider f as a branched holomorphic covering of $\mathbb{P}^1$. Suppose f has n sheets. Then by the Riemann–Hurwitz formula f has total branching order

$$b = 2(g + n - 1).$$

Each branch point $x_1, \dots, x_N$ has branch order

$$b(f, x_j) \leq n - 1.$$

We use $\lceil x \rceil$ to denote the integer k with $k - 1 < x \leq k$. So f has at least $\left\lceil \frac{b}{n-1} \right\rceil = 2 + \left\lceil \frac{2g}{n-1} \right\rceil \geq 3$ branch points (note that $n \neq 1$ since $g \geq 1$ and a one-sheeted holomorphic covering map is a biholomorphism). $\square$

Thus we have the following necessary condition:

Corollary 5.1.3. *Let R be an affine curve that admits a holomorphic immersion into $\mathbb{P}^1$. Then there is a compact Riemann surface X with a finite subset $C \subset X$ consisting of at least three distinct points so that $R \cong X \setminus C$.*

Remark 5.1.4. Let X be a compact Riemann surface and $C \subset X$ consist of one or two points. By Theorem 4.1.3 the affine curve $R = X \setminus C$ admits a holomorphic immersion into $\mathbb{C}$ that is of finite order at the points of C. It follows from Proposition 5.1.2 and the chain rule that this immersion cannot be the exponential of an algebraic function.

The following theorem due to [Mei60] will allow us to obtain an upper bound on the number of points we need to remove in order to find an immersion into $\mathbb{P}^1$. We use $\lfloor x \rfloor$ to denote the integer k with $k \leq x < k + 1$.

Theorem 5.1.5. *Every compact Riemann surface of genus $g \geq 1$ admits an n-sheeted branched holomorphic covering map of $\mathbb{P}^1$ where*

$$2 \leq n \leq \left\lfloor \frac{g+3}{2} \right\rfloor.$$

We refer to [Mei60] for the proof.

Corollary 5.1.6. *Every compact Riemann surface X of genus $g \geq 1$ admits a branched holomorphic covering map of $\mathbb{P}^1$ with $3g$ or fewer branch points if g is even and $3g+1$ or fewer branch points if g is odd.*

Proof. By Theorem 5.1.5 there is an n-sheeted branched holomorphic covering map with $n \leq \left\lfloor \frac{g+3}{2} \right\rfloor$. Then by the Riemann–Hurwitz formula

$$b \leq 2\left(g + \left\lfloor \frac{g+3}{2} \right\rfloor - 1\right).$$

Then each branch point has branch order at least 1 so the result follows. $\square$

Combining the upper and lower bounds we get the following result.

Theorem 5.1.7. *Every compact Riemann surface of genus $g \geq 1$ admits a branched holomorphic covering map of $\mathbb{P}^1$ with between 3 and $3g$ branch points if g is even and between 3 and $3g+1$ branch points if g is odd.*

So for a compact Riemann surface X of genus $g \geq 1$ to accommodate a holomorphic immersion into $\mathbb{P}^1$ we must remove between 3 and $3g$ points if g is even or between 3 and $3g+1$ points if g is odd. Finally to get an algebraic immersion into $\mathbb{C}$ we remove the poles, of which there need not be more than $\left\lfloor \frac{g+3}{2} \right\rfloor$.

5.2 Algebraic immersions of thrice punctured surfaces

The results of the previous section raise the question: how commonly do compact Riemann surfaces admit meromorphic functions with exactly three branch points? The answer is: very rarely. Using the theory of dessins d'enfants we will now show that for a given genus there are only finitely many Riemann surfaces that admit such functions.

Recall that a meromorphic function f on a compact Riemann surface X is called a Belyi function if it has no more than three critical values. A compact Riemann surface is called a Belyi surface if it admits a Belyi function. A pair (X, f) consisting of a Belyi surface X and a Belyi function $f : X \to \mathbb{P}^1$ is called a Belyi pair. Since the automorphism group of $\mathbb{P}^1$ acts triply transitively we can assume that the critical values of a Belyi function are 0, 1, and ∞.

Definition 5.2.1. We say a Belyi function f on a Belyi surface X is *simple* if X has genus $g \geq 1$, f has exactly three critical points, and its critical values are $0, 1$ and ∞. We will also call a Belyi surface *simple* if it admits a simple Belyi function, and a pair (X, f) *simple* if X and f are simple. (This is our own definition.)

Suppose (X, f) is a simple Belyi pair of genus g. Recall that by Construction 2.7.15, (X, f) can equivalently be represented as a dessin (X, D). We saw in Table 2.1 that the white vertices, black vertices and faces of D are in one-to-one correspondence with the points in the preimage $f^{-1}(0)$, $f^{-1}(1)$ and $f^{-1}(\infty)$ respectively. Since (X, f) is simple the associated bipartite graph D has exactly one white vertex, one black vertex, and one face. By the formula for the Euler characteristic, such a graph has $n = 2g + 1$ edges. Note that the formula for the Euler characteristic applies to dessins since they are CW complexes.

By Construction 2.7.16, the pair (X, f) can also be represented as a constellation $[\sigma, \tau, (\sigma\tau)^{-1}]$ unique up to joint conjugation. Since the graph D has n edges, the permutations σ and τ are elements of S_n. Again, we saw in Table 2.1, that the cycles of σ, τ, and $(\sigma\tau)^{-1}$ are in one-to-one correspondence with the white vertices, the black vertices, and the faces of D respectively. There is only one of each of these so σ, τ, and $(\sigma\tau)^{-1}$ must have order n.

Clearly there are only finitely many constellations in S_n satisfying this condition so we have the following theorem.

Theorem 5.2.2. *There are only finitely many non-equivalent simple Belyi pairs of a given genus.*

The problem of enumeration of simple Belyi pairs thus boils down to a problem in combinatorics, namely, how many non-equivalent constellations $[\sigma, \tau, (\sigma\tau)^{-1}]$ in S_n are composed entirely of n-cycles?

Using a brute force approach it is only feasible to explicitly compute this number for low genera. The results of these computations are presented in Table 5.1. Refer to Appendix B for a discussion on the approach we take.

For higher genera we will now show how to find an upper bound on this number. There are two straightforward avenues that we can explore.

Since constellations are only unique up to joint conjugation we can fix $(\sigma\tau)^{-1} = (1\ 2\ \cdots\ n)$. We then ask: in how many ways can we decompose $\sigma\tau$ as a product of two n-cycles? This question was answered in [Boc80, Corollary 4.8]. For $n = 2g + 1$ there are precisely $\dfrac{(2g)!}{g+1}$ ways to decompose an n-cycle as a product of two n-cycles.

Many of these decompositions will result in equivalent constellations. For example consider the constellations

$$a = [(1\ 4\ 5\ 2\ 3), (1\ 4\ 2\ 3\ 5), (1\ 2\ 3\ 4\ 5)]$$

and

$$b = [(1\ 3\ 4\ 2\ 5), (1\ 2\ 5\ 3\ 4), (1\ 2\ 3\ 4\ 5)].$$

These constellations are equivalent since $(1\ 2\ 3\ 4\ 5)a(1\ 5\ 4\ 3\ 2) = b$, but are counted as distinct decompositions. Thus there are at most $\frac{(2g)!}{g+1}$ non-equivalent simple Belyi pairs.

We can find another upper bound by counting the equivalence classes of constellations whose first two entries are n-cycles (but the third not necessarily). Again, since constellations are only unique up to joint conjugation, we can always fix the first entry to be $\sigma = (1\ 2\ \cdots\ n)$. Let τ denote the second entry of the constellation. Our only condition on τ is that it be an n-cycle so we have $(n-1)!$ possible candidates. Many of these will result in equivalent constellations however. For example if we denote the centraliser of σ in S_n by $C_{S_n}(\sigma)$ and take $\rho \in C_{S_n}(\sigma)$, then the constellations $[\sigma, \tau, (\sigma\tau)^{-1}]$ and $[\sigma, \rho^{-1}\tau\rho, (\sigma\rho^{-1}\tau\rho)^{-1}]$ will be equivalent.

It is not difficult to see that $C_{S_n}(\sigma) = \{\mathrm{id}, \sigma, \ldots, \sigma^{n-1}\}$: the number of elements in the conjugacy class of σ is equal to the index of the centraliser of σ in S_n. Since there are $(n-1)!$ elements in the conjugacy class and $n!$ in S_n there must be n elements in $C_{S_n}(\sigma)$.

Thus in order to enumerate the non-equivalent constellations in S_n whose first two entries are n-cycles we only need to count the number of n-cycles in S_n up to conjugation by an element of $C_{S_n}(\sigma)$, that is, up to conjugation by some power of σ.

This number was calculated in [GW60]. Let $\phi(n)$ be the number of positive integers less than n that are relatively prime to n (ϕ is known as Euler's totient function). Then, when n is odd, there are

$$\frac{1}{n^2} \sum_{d|n} \phi^2\left(\frac{n}{d}\right) d! \left(\frac{n}{d}\right)^d \tag{5.1}$$

different possibilities. Equation (5.1) differs by a factor of 2 from the formula given in [GW60]. This is because we need to distinguish between a permutation and its inverse ([GW60] treats these as equivalent). This number serves as another upper bound on the number of non-equivalent simple Belyi pairs. So we have the following theorem.

Genus g	# non-equivalent simple Belyi pairs	$\dfrac{(2g)!}{g+1}$	$\dfrac{1}{n^2}\sum_{d\mid n}\phi^2\left(\frac{n}{d}\right)d!\left(\frac{n}{d}\right)^d$
1	1	1	2
2	4	8	8
3	30	120	108
4	900	8,064	4,492
5	54,990	604,800	329,900
6	5,263,764	68,428,800	36,846,288
7	?	10,897,286,400	5,811,886,656

Table 5.1: Upper bounds on the number of non-equivalent Belyi pairs and the exact number for small genus. We have set $n = 2g+1$ in the last column.

Theorem 5.2.3. *Let g be an integer and $n = 2g+1$. There are at most*

$$\min\left\{\frac{(2g)!}{g+1}, \frac{1}{n^2}\sum_{d\mid n}\phi^2\left(\frac{n}{d}\right)d!\left(\frac{n}{d}\right)^d\right\}$$

non-equivalent simple Belyi pairs of genus g.

5.3 The action of automorphisms of $\mathbb{P}^1$

Having calculated the number of simple Belyi pairs for low genera we might wonder how many of these pairs have the same underlying surface. In this section we aim to find an upper bound on the number of simple Belyi surfaces of a given genus.

The six automorphisms of the sphere that permute the set $\{0, 1, \infty\}$ form a group that is isomorphic to the dihedral group of order 6. We denote this group by Φ. These automorphisms have a canonical action on the set of equivalence classes of simple Belyi pairs. Given a simple Belyi pair (X, f) and an automorphism $\varphi \in \Phi$ the action is given by $(X, f) \mapsto (X, \varphi \circ f)$. Clearly the underlying Belyi surface is left invariant under this action. Thus if we had some way to compute the orbits of Φ and we found that the orbits were not trivial, we would be able to attain a smaller upper bound on the number of Belyi surfaces of a particular genus.

Without an explicit formula for a Belyi function f it is perhaps not immediately obvious how to determine the orbit of f under Φ. We now demonstrate how to calculate the orbit using the constellation representation of a Belyi pair. Let $[\sigma, \tau, (\sigma\tau)^{-1}]$ be a constellation in S_n and consider the automorphism $\varphi \in \Phi$ given by $\varphi(z) = 1 - z$. Clearly φ fixes ∞ and swaps 0 and

1. Since the cycles of σ correspond to the preimages of 0 and the cycles of τ to the preimages of 1, we might naively expect that the action of φ should be given by

$$[\sigma, \tau, (\sigma\tau)^{-1}] \mapsto [\tau, \sigma, (\sigma\tau)^{-1}].$$

This cannot be the case however since the result is not necessarily even a constellation.

Fix a base point $y_0 \in Y$. Let (X, f) be a Belyi pair and denote its associated monodromy representation by $h_{(X,f)} : \pi_1(Y, y_0) \to S_n$. We would like to define an action of Φ on the set of monodromy representations so that $\psi(h_{(X,f)}) = h_{(X,\psi\circ f)}$ for $\psi \in \Phi$. The following proposition follows from Construction 2.7.4 of the monodromy representation associated to a Belyi pair.

Proposition 5.3.1. *Suppose $y_0 \in Y$ is a fixed point of $\psi \in \Phi$ and let (X, f) be a Belyi pair. Then $h_{(X,\psi\circ f)} = h_{(X,f)} \circ \psi_*$ where $\psi_* : \pi_1(Y, y_0) \to \pi_1(Y, y_0)$ is the map induced by ψ.*

Proof. When we construct the monodromy representation of $(X, \psi \circ f)$ we lift loops in Y to paths in X by the map $\psi \circ f$. This is the same as lifting first by ψ and then by f. $\square$

There is no common fixed point of the elements of Φ so one might expect that the above proposition can only be used to find the orbit of subgroups of Φ rather than the whole group. The next proposition shows that this is not the case.

Proposition 5.3.2. *Let (X, f) be a Belyi pair and take two points $a, b \in Y$. Now (X, f) determines monodromy representations $h : \pi_1(Y, a) \to S_n$ and $g : \pi_1(Y, b) \to S_n$. Suppose we choose generating sets $\{\gamma_0, \gamma_1, \gamma_\infty\}$ for $\pi_1(Y, a)$ and $\{\delta_0, \delta_1, \delta_\infty\}$ for $\pi_1(Y, b)$ so that γ_j and δ_j are freely homotopic in Y to small positively oriented circles centred on j for $j = 0, 1, \infty$, and so that $\gamma_0\gamma_1\gamma_\infty = \mathrm{id}$ and $\delta_0\delta_1\delta_\infty = \mathrm{id}$. Then the constellations*

$$c_a = [h(\gamma_0), h(\gamma_1), h(\gamma_\infty)] \text{ and } c_b = [g(\delta_0), g(\delta_1), g(\delta_\infty)]$$

are equivalent.

Proof. This proposition is another consequence of Construction 2.7.4. Let E denote the preimage $f^{-1}((0, 1))$, then E consists of n connected components. Let j equal 0 or 1. Since the curves γ_j and δ_j are freely homotopic in Y their liftings by f determine the same permutation of connected components of E. Thus, up to relabelling of the components of E, the permutations $h(\gamma_j)$ and $g(\delta_j)$ are the same. The same must be true for $h(\gamma_\infty)$ and $g(\delta_\infty)$ because of the relation imposed on the generating sets. $\square$

Thus we see that so long as we choose appropriate generators of fundamental groups we can easily calculate the action of Φ on the set of constellations in a consistent way using Proposition 5.3.1. Take $\psi \in \Phi$, y_0 a fixed point of ψ and let $\{\gamma_0, \gamma_1, \gamma_\infty\}$ be a generating set for $\pi_1(Y, y_0)$ satisfying the conditions of Proposition 5.3.2. Table 5.2 shows how each element of Φ acts on the generators.

$\psi(z)$	$(\psi_*(\gamma_0), \psi_*(\gamma_1), \psi_*(\gamma_\infty))$
$1-z$	$(\gamma_1, \gamma_0, (\gamma_1\gamma_0)^{-1})$
$1/z$	$(\gamma_\infty, (\gamma_0\gamma_\infty)^{-1}, \gamma_0)$
$\dfrac{z}{z-1}$	$((\gamma_\infty\gamma_1)^{-1}, \gamma_\infty, \gamma_1)$
$\dfrac{1}{1-z}$	$(\gamma_1, \gamma_\infty, \gamma_0)$
$\dfrac{z-1}{z}$	$(\gamma_\infty, \gamma_0, \gamma_1)$

Table 5.2: The image of the elements of the generating set under the elements of Φ.

With the aid of a computer we can compute the number of orbits for low genera thus obtaining an upper bound on the number of simple Belyi surfaces (see Appendix B). The results of these computations are presented in Table 5.3.

Genus g	# non-equivalent simple Belyi pairs	# orbits of Φ
1	1	1
2	4	2
3	30	10
4	900	174
5	54,990	9,362
6	5,263,764	879,275

Table 5.3: The number of orbits of Φ for low genus. This number gives an upper bound on the number of simple Belyi surfaces in each genus.

In the case of genus 1 we can give an explicit description the only simple Belyi pair.

Example 5.3.3. Let $X = \mathbb{C}/\Gamma$ be the complex torus with $\Gamma = \mathbb{Z} + \xi\mathbb{Z} \subset \mathbb{C}$ where $\xi = e^{\frac{i\pi}{3}}$, and let $f : X \to \mathbb{P}^1$ be the meromorphic function defined by

$$f = \frac{1 + \wp'/\sqrt{-g_3}}{2}$$

where $\wp$ is the Weierstrass $\wp$-function and

$$g_3 = 140 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^6}.$$

Then (X, f) is the only simple Belyi pair of genus one up to equivalence. We refer to [GGD12, Example 4.28] for a proof.

In the case of genus 2, Φ has two orbits, meaning there are at most two simple Belyi surfaces. One of these orbits consists of a single simple Belyi pair which we will denote (X, f), the other contains three Belyi pairs which we will denote $(Z, g_1), (Z, g_2), (Z, g_3)$.

Since (X, f) is fixed by Φ, for each element $\varphi \in \Phi$ there is an automorphism $h_\varphi : X \to X$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{h_\varphi} & X \\ \downarrow f & & \downarrow f \\ \mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{P}^1 \end{array}$$

The automorphisms h_φ are distinct for distinct φ, so X must have at least six distinct automorphisms.

By explicit calculations we find that each of the pairs (Z, g_i) for $i = 1, 2, 3$ is fixed by a single non-trivial element of Φ which we will denote φ_i for $i = 1, 2, 3$ respectively. Thus there are automorphisms $h_i : Z \to Z$ such that

$$\begin{array}{ccc} Z & \xrightarrow{h_i} & Z \\ \downarrow g_i & & \downarrow g_i \\ \mathbb{P}^1 & \xrightarrow{\varphi_i} & \mathbb{P}^1 \end{array}$$

commutes. However we cannot say whether the automorphisms h_1, h_2, h_3 are distinct. So Z is a genus two surface with at least two automorphisms.

As of yet we do not know of any way to explicitly describe the surfaces X and Z and meromorphic functions g_1, g_2, g_3 and f other than as constellations. It may be the case that X and Z are biholomorphic. This is an interesting avenue for future research.

Appendix A

The Mergelyan–Bishop theorem

A.1 A brief history of the theorem

Many of the constructions throughout this thesis have relied heavily on holomorphic approximation, in particular the Mergelyan–Bishop theorem (stated again below).

Theorem A.1.1 (Mergelyan–Bishop)**.** *Let K be a holomorphically convex compact subset of an open Riemann surface R. Then $\overline{\mathscr{O}}(K) = \mathscr{O}_{int}(K)$. Hence, by Runge's theorem, a function $f \in \mathscr{O}_{int}(K)$ can be uniformly approximated by a sequence (f_k) in $\mathscr{O}(R)$.*

The proof of the above theorem for the special case where $R = \mathbb{C}$ is due to [Mer53], see [Rud87] for an accessible reference. The result was extended to the more general case of an open Riemann surface by [Bis58] using highly measure theoretic techniques. The following theorem, which follows from [Bis58, Lemma 6], was extracted by [Kod65].

Theorem A.1.2 (Bishop's localisation theorem)**.** *Let K be a compact subset of an open Riemann surface R and $f : K \to \mathbb{C}$ be a continuous function. If every point $x \in K$ has a neighbourhood $V_x \Subset R$ such that $f|(K \cap \overline{V_x}) \in \overline{\mathscr{O}}(K \cap \overline{V_x})$ then $f \in \overline{\mathscr{O}}(K)$.*

The localisation theorem has since been used by [Kod65] to provide an alternative measure theoretic proof of Theorem A.1.1, by [Gar68] to simplify the proof of Mergelyan's original theorem for the plane, and again by [Sak72] to prove Theorem A.1.1 using a bounded solution to the $\overline{\partial}$-problem. (See [FFW18] for a more in-depth account of the history of holomorphic approximation as a whole).

The method used by [Sak72] was replicated in a textbook by [JP00]; however, an error was discovered when certain sets were found to be dependent on certain parameters. A correction was presented in an unpublished report by [Gar06]; however, we found this difficult to understand. The purpose of this appendix is to provide a full account of the proof and to clarify exactly where the error was and how it may be rectified.

A.2 Local approximation

Theorem A.1.2 is of course only useful if it is possible to satisfy its hypotheses. We now show that this is the case for an open Riemann surface and a holomorphically convex subset.

Proposition A.2.1. *Let K be a holomorphically convex compact subset of a Riemann surface R. Let $\mathfrak{V} = \{(V_j, \psi_j)\}$ be a finite cover of K by coordinate discs so that ψ_j extends as a biholomorphism to a larger disc U_j with $V_j \Subset U_j$. Take $f \in \mathscr{O}_{int}(K)$. Then $f|(\overline{V}_j \cap K) \in \overline{\mathscr{O}}(\overline{V}_j \cap K)$.*

Proof. If $V_j \subset K$ then we can simply restrict f to a slightly larger open set and we are done.

Suppose $V_j \setminus K \neq \emptyset$. By assumption, $R \setminus K$ and $R \setminus \overline{V}_j$ have no relatively compact connected components, so the same can be said of their union $(R \setminus K) \cup (R \setminus \overline{V}_j) = R \setminus (\overline{V}_j \cap K)$. So $\overline{V}_j \cap K$ is holomorphically convex.

Moving to the complex plane, $\psi_j(\overline{V}_j \cap K)$ is a holomorphically convex compact set in $\mathbb{C}$ and $f \circ \psi_j^{-1}$ is a continuous function that is holomorphic on the interior of $\psi_j(\overline{V}_j \cap K)$. Applying Mergelyan's theorem for the plane we find holomorphic functions $\tilde{f}_n : \psi_j(U_j) \to \mathbb{C}$ uniformly converging to $f \circ \psi_j^{-1}$ on $\overline{V}_j \cap K$.

Now let $f_n = \tilde{f}_n \circ \psi_j | U_j \to \mathbb{C}$. These functions are holomorphic on U_j and converge uniformly to f on $\overline{V}_j \cap K$. □

A.3 A bounded solution to the $\overline{\partial}$-problem

In the complex plane

The following results were well known at the time of Sakai's complex-analytic proof, so were stated without proof. We provide a proof here for completeness.

Lemma A.3.1. *For $z \in \mathbb{C}$ and fixed $R > 0$*

$$\sup_{z \in \mathbb{C}} \left| \int_{D(0,R)} \frac{1}{\zeta - z} d\zeta \wedge d\overline{\zeta} \right| \leq 4\pi R.$$

Proof. Let $E = D(0, R)$ and

$$I_z = \int_E \frac{1}{\zeta - z} d\zeta \wedge d\overline{\zeta}$$

then letting $x = \mathrm{Re}(\zeta)$ and $y = \mathrm{Im}(\zeta)$

$$\begin{aligned} |I_z| &= \left| \int_E \frac{1}{\zeta - z} 2i dx \wedge dy \right| \\ &\leq 2 \int_E \left| \frac{1}{\zeta - z} \right| dx \wedge dy. \end{aligned}$$

Now letting $\zeta' = \zeta - z$ we get

$$|I_z| \leq 2 \int_{E-z} \frac{1}{|\zeta'|} dx \wedge dy$$

where $E - z = \{w \in \mathbb{C} : |w + z| < R\}$. For $\zeta' \notin E$ we have $\dfrac{1}{|\zeta'|} \leq \dfrac{1}{R}$ so

$$\begin{aligned} \int_{E-z} \frac{1}{|\zeta'|} dx \wedge dy &= \int_{(E-z)\cap E} \frac{1}{|\zeta'|} dx \wedge dy + \int_{(E-z)\setminus E} \frac{1}{|\zeta'|} dx \wedge dy \\ &\leq \int_{(E-z)\cap E} \frac{1}{|\zeta'|} dx \wedge dy + \frac{1}{R} \int_{(E-z)\setminus E} dx \wedge dy. \end{aligned}$$

Since E and $E - z$ have the same area

$$\int_{(E-z)\setminus E} dx \wedge dy = \int_{E\setminus(E-z)} dx \wedge dy$$

so

$$\begin{aligned} \int_{E-z} \frac{1}{|\zeta'|} dx \wedge dy &\leq \int_{(E-z)\cap E} \frac{1}{|\zeta'|} dx \wedge dy + \frac{1}{R} \int_{E\setminus(E-z)} dx \wedge dy \\ &\leq \int_{(E-z)\cap E} \frac{1}{|\zeta'|} dx \wedge dy + \int_{E\setminus(E-z)} \frac{1}{|\zeta'|} dx \wedge dy \\ &= \int_E \frac{1}{|\zeta'|} dx \wedge dy. \end{aligned}$$

So

$$\sup_{z\in\mathbb{C}} |I_z| \leq 2 \sup_{z\in\mathbb{C}} \int_{E-z} \frac{1}{|\zeta'|} dx \wedge dy = 2 \int_E \frac{1}{|\zeta|} dx \wedge dy = 4\pi R. \qquad \square$$

Corollary A.3.2. *Let $g d\bar{z} \in \mathscr{E}^{0,1}(\mathbb{C})$ be compactly supported and let $K =$ $\mathrm{supp}(g)$. Then there is a function $f \in \mathscr{E}(\mathbb{C})$ such that $\overline{\partial} f = g d\bar{z}$ and such that $\|f\|_{\mathbb{C}} \leq \mathrm{diam}(K)\|g\|_K$.*

Proof. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

Then $\overline{\partial} f = g d\bar{z}$ (see [For91, Lemma 13.1] for a proof). In fact, since $f(z) \to 0$ as $z \to \infty$, f is the unique solution to $\overline{\partial} f = g d\bar{z}$ with $f(\infty) = 0$. In particular $\|f\|_{\mathbb{C}}$ is finite. Take $\varepsilon > 0$. Then by translating we can assume $K \subset D(0, R)$ where $R = \frac{1}{2} \operatorname{diam}(K) + \varepsilon$. We then have

$$\begin{aligned}
\|f\|_{\mathbb{C}} &= \frac{1}{2\pi} \sup_{z \in \mathbb{C}} \left| \int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta} \right| \\
&\leq \frac{1}{2\pi} \|g\|_{\mathbb{C}} \sup_{z \in \mathbb{C}} \left| \int_{D(0,R)} \frac{1}{\zeta - z} d\zeta \wedge d\overline{\zeta} \right| \\
&\leq 2R \|g\|_K,
\end{aligned}$$

where we have used Lemma A.3.1. This is true for all ε so we can conclude

$$\|f\|_{\mathbb{C}} \leq \operatorname{diam}(K) \|g\|_K. \qquad \square$$

Norms on $\mathscr{E}^{0,1}(G)$

Sakai extended the above result to the setting of an open Riemann surface R using techniques of [BS49]. The approach we take is closer to [JP00] which we feel is more explicit.

Let $G \subset R$ be open and relatively compact and suppose there is a finite cover $\mathfrak{V} = \{(V_j, z_j)\}$ of G by coordinate charts. Then for any $S \subset G$ we define an operator $\|\cdot\|_{S,\mathfrak{V}}$ on $\mathscr{E}^{0,1}(G)$. For $\omega \in \mathscr{E}^{0,1}(G)$ take $f_j : V_j \to \mathbb{C}$ with $\omega | V_j = f_j d\bar{z}_j$. Then we let

$$\|\omega\|_{S,\mathfrak{V}} = \sum_j \|f_j\|_{V_j \cap S}$$

where $\|f_j\|_{V_j \cap S}$ is the supremum norm. Note that $\|\cdot\|_{S,\mathfrak{V}}$ is not a true norm for all S since it could be infinite. It does however satisfy all other properties of norms, most importantly the triangle inequality.

On an open Riemann surface

We now fix a cover with special properties that will allow us to prove the existence of a bounded solution to the $\overline{\partial}$ problem.

Lemma A.3.3. *Let R be an open Riemann surface and K a compact subset. There is an open cover $(U_j)_{j=1}^N$ of K and a finer cover $(V_j)_{j=1}^N$, and functions $\psi_1, \dots, \psi_N \in \mathscr{O}(R)$ such that for $j = 1, \dots, N$,*

- $\psi_j|V_j \to \frac{1}{2}\mathbb{D}$ *is a biholomorphism,*

- $\psi_j|U_j \to \mathbb{D}$ *is a biholomorphism,*

- $\psi_j \left(\bigcup_{k=1}^N U_k \setminus U_j \right) \cap \mathbb{D} = \emptyset.$

Proof. By the theorem of Weierstrass, for every $z \in R$ there is a holomorphic function $\psi'_z : R \to \mathbb{C}$ with a simple zero at z and no other zeros. By the inverse function theorem there is a relatively compact open neighbourhood U'_z of z for which $\psi'_z|U'_z$ is a biholomorphism onto its image. Since $\psi'_z(U'_z)$ is open in $\mathbb{C}$ and contains the origin we can take U'_z and ψ'_z such that $\psi'_z(U'_z) = \mathbb{D}$. Then $G' = \bigcup_{z \in K} U'_z$ is relatively compact in R.

Since ψ'_z has no zeros except for the one at z and since G' is relatively compact, $\psi'_z(G' \setminus U'_z)$ is bounded away from zero, that is, there is some $\varepsilon_z \in (0, 1]$ such that

$$|\psi'_z(x)| > \varepsilon_z \text{ for all } x \in G' \setminus U'_z. \tag{A.1}$$

Now define $\psi_z : R \to \mathbb{C}$ by

$$\psi_z = \frac{1}{\varepsilon_z} \psi'_z.$$

Then by (A.1), $|\psi_z(x)| > 1$ for all $x \in G' \setminus U'_z$ and $\mathbb{D} \subset \psi_z(U'_z)$. Now let $U_z = \psi_z^{-1}(\mathbb{D})$ and $V_z = \psi_z^{-1}(\frac{1}{2}\mathbb{D}) \Subset U_z$. Then since K is compact there are $z_1, \dots, z_N \in K$ such that $K \subset V_{z_1} \cup \dots \cup V_{z_N}$. Letting $U_j = U_{z_j}$, $V_j = V_{z_j}$, and $\psi_j = \psi_{z_j}$, the collections $(U_j)_{j=1}^N$, $(V_j)_{j=1}^N$, and $(\psi_j)_{j=1}^N$ satisfy the conditions of the lemma. □

For the next proposition, fix the functions $\psi_j : R \to \mathbb{C}$ and covers $\mathfrak{U} = \{(U_j, \psi_j)\}$ and $\mathfrak{V} = \{(V_j, \psi_j)\}$ constructed above, and let

$$G = \bigcup_{j=1}^N U_j, \text{ and } G_0 = \bigcup_{j=1}^N V_j.$$

Proposition A.3.4. *There is a constant C depending only on G and $\mathfrak{V}$ such that for every $\omega \in \mathscr{E}^{0,1}(G)$ there is a function $u \in \mathscr{E}(G_0)$ such that*

$$\overline{\partial} u = \omega | G_0 \text{ and } \|u\|_{G_0} \le C \|\omega\|_{G,\mathfrak{U}}.$$

Proof. Let $(\varphi_j)_{j=1}^N$ be a collection of smooth functions with $\varphi_j : R \to [0,1]$, $\operatorname{supp}(\varphi_j) \subset U_j$ and such that $\sum_{j=1}^N \varphi_j = 1$ on G_0. For $j = 1, \ldots, N$ define $\omega_j \in \mathscr{E}^{0,1}(R)$ by

$$\omega_j = \begin{cases} \varphi_j \omega & \text{on } U_j, \\ 0 & \text{otherwise.} \end{cases}$$

So $\operatorname{supp}(\omega_j) \subset U_j$ and $\omega | G_0 = \sum_{j=1}^N \omega_j | G_0$. Then in local coordinates ω_j can be written

$$\omega_j = \psi_j^*(w_j d\bar{z})$$

where $w_j : \mathbb{D} \to \mathbb{C}$ can be smoothly extended by zero to $w_j : \mathbb{C} \to \mathbb{C}$ with $\operatorname{supp}(w_j) \subset \mathbb{D}$.

Now by Corollary A.3.2 there are functions $g_j \in \mathscr{E}^{0,1}(\mathbb{C})$ such that $\|g_j\|_{\mathbb{C}} \leq 2\|w_j\|_{\mathbb{C}}$. Note that for $z \notin \mathbb{D}$ we have $w_j(z) = 0$ and hence $\dfrac{\partial g_j}{\partial \bar{z}} = 0$. So g_j is holomorphic on $\mathbb{C} \setminus \mathbb{D}$.

Consider the function $\eta_j = g_j \circ \psi_j | G \to \mathbb{C}$. Clearly η_j is smooth, in fact, since $\psi_j(G \setminus U_j) \cap \mathbb{D} = \emptyset$ and g_j is holomorphic on $\mathbb{C} \setminus \mathbb{D}$, η_j is holomorphic on $G \setminus U_j$. Then on G

$$\bar{\partial}\eta_j = \bar{\partial}(g_j \circ \psi_j) = \psi_j^*(\bar{\partial} g_j) = \psi_j^*(w_j d\bar{z}) = \omega_j$$

and

$$\|\eta_j\|_G \leq \|g_j\|_{\mathbb{C}}. \tag{A.2}$$

Now let $u = \eta_1 + \cdots + \eta_N$. Then u is smooth and $\bar{\partial} u | G_0 = \omega | G_0$. By the triangle inequality

$$\|u\|_{G_0} \leq \|\eta_1\|_{G_0} + \cdots + \|\eta_N\|_{G_0} \leq \|\eta_1\|_G + \cdots + \|\eta_N\|_G.$$

By (A.2) we then have

$$\|\eta_1\|_G + \cdots + \|\eta_N\|_G \leq \|g_1\|_{\mathbb{C}} + \cdots + \|g_N\|_{\mathbb{C}},$$

and by the previous calculations

$$\|g_1\|_{\mathbb{C}} + \cdots + \|g_N\|_{\mathbb{C}} \leq 2(\|w_1\|_{\mathbb{C}} + \cdots + \|w_N\|_{\mathbb{C}}).$$

Now by the definition of ω_j

$$\|w_j\|_{\mathbb{C}} = \|\omega_j\|_{G,\mathfrak{U}} \leq \|\omega\|_{G,\mathfrak{U}}$$

so
$$\|w_1\|_{\mathbb{C}} + \cdots + \|w_N\|_{\mathbb{C}} \leq N\|\omega\|_{G,\mathfrak{U}}.$$
Then letting $C = 2N$ we have the result
$$\|u\|_{G_0} \leq C\|\omega\|_{G,\mathfrak{U}}. \qquad \square$$

We now depart slightly from proof presented in [Sak72] and prove the following corollary.

Corollary A.3.5. *With R, K, G and G_0 as before given $\varepsilon > 0$ and $\omega \in \mathscr{E}^{0,1}(G)$, there is a neighbourhood $A \subset G_0$ of K and a function $u \in \mathscr{E}(G_0)$ such that*
$$\overline{\partial}u|A = \omega|A \text{ and } \|u\|_K \leq C(\|\omega\|_{K,\mathfrak{U}} + \varepsilon),$$
where C is as in Proposition A.3.4.

Proof. By continuity there is a neighbourhood $H \subset G_0$ of K for which $\|\omega\|_{H,\mathfrak{U}} \leq \|\omega\|_{K,\mathfrak{U}} + \varepsilon$. Let $\alpha \in \mathscr{E}(G)$ be such that:

- $\alpha(z) = 1$ for z in a neighbourhood $A \subset H$ of K.
- $\alpha(z) = 0$ for $z \notin H$.
- $\|\alpha\|_G = 1$.

Then applying Proposition A.3.4 to the form $\alpha\omega$ we find $u \in \mathscr{E}(G_0)$ such that $\overline{\partial}u|G_0 = \alpha\omega|G_0$ and $\|u\|_{G_0} \leq C\|\alpha\omega\|_{G,\mathfrak{U}}$. So $\overline{\partial}u|A = \omega|A$ and since $\|\alpha\omega\|_{G,\mathfrak{U}} < \infty$
$$\|u\|_K \leq \|u\|_{G_0} \leq C\|\alpha\omega\|_{G,\mathfrak{U}} = C\|\alpha\omega\|_{H,\mathfrak{U}}$$
$$\leq C\|\omega\|_{H,\mathfrak{U}} \leq C(\|\omega\|_{K,\mathfrak{U}} + \varepsilon). \qquad \square$$

A.4 The localisation theorem

Finally we are in a position to prove Theorem A.1.2.

Proof of Theorem A.1.2. Let R, K, G and G_0 be as before, and let f be a continuous function on K such that $f|(K \cap \overline{V}_j) \in \overline{\mathscr{O}}(K \cap \overline{V}_j)$ for all j. Take $\varepsilon > 0$, then by Proposition A.2.1 there are functions $f_j \in \mathscr{O}(U_j)$ such that
$$\|f_j - f\|_{\overline{V}_j \cap K} < \varepsilon.$$

For every j define $\varphi_j : R \to [0,1]$ so that $\operatorname{supp}(\varphi_j) \subset U_j$ and so that for every $z \in G_0$, $\sum_j \varphi_j(z) = 1$. Then define functions $h_{jk} : U_k \to \mathbb{C}$ by

$$h_{jk}(z) = \begin{cases} \varphi_j(z)(f_j(z) - f_k(z)) & \text{if } z \in U_j \cap U_k, \\ 0 & \text{if } z \in U_k \setminus U_j. \end{cases}$$

Let $h_k : U_k \to \mathbb{C}$ be defined by $h_k = \sum\limits_j h_{jk}$. We now find bounds for $\|h_k\|_{\overline{V}_k \cap K}$ and $\|\overline{\partial} h_k\|_{\overline{V}_k \cap K}$. For $z \in \overline{V}_k \cap K$ we have

$$\begin{aligned} |h_k(z)| &\le \sum_{j=1}^N \varphi_j(z)|f_j(z) - f_k(z)| \\ &\le \sum_{j=1}^N \varphi_j(z)\left(|f_j(z) - f(z)| + |f_k(z) - f(z)|\right) \\ &< 2\varepsilon \sum_{j=1}^N \varphi_j(z) = 2\varepsilon. \end{aligned}$$

So $\|h_k\|_{\overline{V}_k \cap K} \le 2\varepsilon$ for all k.

Now consider the (0,1)-form $\overline{\partial} h_k$. Since $f_j - f_k$ is holomorphic

$$\overline{\partial} h_{jk}(z) = \begin{cases} \overline{\partial}\varphi_j(f_j - f_k)(z) & \text{if } z \in U_j \cap U_k, \\ 0 & \text{otherwise.} \end{cases}$$

Since φ_j is smooth and compactly supported, $\|\overline{\partial}\varphi_j\|_R$ is finite, so let

$$C_1 = N \max_{j=1,\dots,N} \|\overline{\partial}\varphi_j\|_R.$$

Note that C_1 depends only on the cover $\mathfrak{U}$ and the functions φ_j. Then

$$\begin{aligned} \|\overline{\partial} h_k\|_{\overline{V}_k \cap K} &\le \sum_{j=1}^N \|\overline{\partial}\varphi_k\|_R \|f_k - f_j\|_{\overline{V}_j \cap \overline{V}_k \cap K} \\ &\le 2C_1\epsilon. \end{aligned}$$

Now on $U_j \cap U_k$,

$$\begin{aligned} h_j - h_k &= \sum_{l \ne j} \varphi_l(f_l - f_j) - \sum_{m \ne k} \varphi_m(f_m - f_k) \\ &= \varphi_k f_k - \varphi_j f_j + \sum_{m \ne k} \varphi_m f_k - \sum_{l \ne j} \varphi_l f_j \\ &= f_k - f_j, \end{aligned}$$

where, for ease of notation, we implicitly ignore terms that are not defined on $U_j \cap U_k$.

Therefore since f_j, f_k are holomorphic we get

$$\overline{\partial} h_j | U_j \cap U_k = \overline{\partial} h_k | U_j \cap U_k,$$

so we can define a (0,1)-form $\omega \in \mathscr{E}^{0,1}(G)$ by $\omega | U_j = -\overline{\partial} h_j$. By Corollary A.3.5, there is a neighbourhood $A \subset G_0$ of K and a function $u \in \mathscr{E}(G_0)$ such that

$$\overline{\partial} u | A = \omega | A$$

and

$$\|u\|_K \leq C(\|\omega\|_{K,\mathfrak{V}} + \varepsilon)$$

where C does not depend on ω. But

$$\|\omega\|_{K,\mathfrak{V}} \leq \max_{j=1,\ldots,N} \|\overline{\partial} h_j\| \leq 2C_1\varepsilon$$

so

$$\|u\|_K \leq C\varepsilon(2C_1 + 1).$$

Now consider the function $u + h_k + f_k$ defined on V_k. It is holomorphic since

$$\overline{\partial} u + \overline{\partial} h_k + \overline{\partial} f_k = -\overline{\partial} h_k + \overline{\partial} h_k = 0.$$

Also on $V_j \cap V_k$,

$$h_k + f_k = h_j + f_j,$$

so we can piece together a function $F \in \mathscr{O}(G_0)$ defined by $F|V_k = u + h_k + f_k$ satisfying

$$\begin{aligned}\|f - F\|_{\overline{V}_k \cap K} &\leq \|f - f_k\|_{\overline{V}_k \cap K} + \|u\|_{\overline{V}_k \cap K} + \|h_k\|_{\overline{V}_k \cap K} \\ &\leq \varepsilon + C\varepsilon(2C_1 + 1) + 2\varepsilon \\ &= \varepsilon(2CC_1 + C + 3).\end{aligned}$$

Therefore $\|f - F\|_K \leq \varepsilon(2CC_1 + C + 3)$ and the constant multiplying ε is independent of f. So we have shown that we can uniformly approximate f by functions holomorphic on G_0. Applying Runge's theorem we can approximate F by functions holomorphic on all of R. Thus we can do the same for f. $\square$

The original complex-analytic proof of [Sak72] uses the kernel methods of [BS49]. The method used in the above proof is intended to give an explicit local description of the solutions to the $\overline{\partial}$-equation. The approach taken by [JP00] is similar to the one presented above, however in [JP00], the neighbourhoods G and G_0 are dependent on the value of ε. Also, Corollary A.3.5

is not present; instead the authors use the estimate from Proposition A.3.4. This is where the error occurs. Since G and G_0 depend on ε, so too does the constant C used in the estimate on the solution. This is problematic since C is used to bound $\|f - F\|_K$.

The proof presented in [Gar06] is slightly different to the one presented above. The neighbourhoods G and G_0 still depend on the value of ε used in the proof of the Mergelyan–Bishop theorem, but the author uses a result that is similar to Corollary A.3.5 and obtains an estimate on the solution that is independent of the cover used. We found this corollary difficult to understand, so instead fixed the neighbourhoods independently of ε; then, even though the constant C depends on G and G_0 it is independent of the chosen value of ε. Thus the bound on $\|f - F\|_K$ depends linearly on ε.

Appendix B

Enumerating simple Belyi pairs with small genus

We used the following code to compute the values in Tables 5.1 and 5.3. The code was run using Python version 2.7.15. The goal is to enumerate the non-equivalent simple Belyi pairs of genus g, or equivalently, the non-equivalent constellations in S_n consisting solely of n-cycles where $n = 2g + 1$. We do this using the function `find_belyi_pairs(n)` which we describe below.

We use a brute force algorithm. We begin by assuming that the first entry of each constellation is given by $\sigma = (1\ 2\ \cdots\ n)$. We then make a list of all of the n-cycles in S_n up to conjugation by some power of σ. These n-cycles are all the possible candidates for the second entry of the constellations. For each candidate in the list, we then compute the product with the first entry. If the product is an n-cycle the corresponding constellation represents a simple Belyi pair so we add it to a list. Otherwise we ignore it.

Once we have a list of constellations corresponding to simple Belyi pairs we can compute the orbit of each constellation under the action of Φ. For each simple Belyi pair of a given genus we compute the orbit using the functions `group_action_2` and `group_action_3` (the elements corresponding to these operations generate Φ). Once again we need to take care that we do not count equivalent constellations as being distinct. The `conjugate` function takes any constellation corresponding to a simple Belyi pair and returns a unique representative from the set of equivalent constellations.

```
from itertools import permutations

def rotations_of(tau):
    # Takes in an n-cycle of the set 0, ..., n-1 and returns all
       permutations of the form (sigma^{-k} tau sigma^k), where
```

```
        sigma is the permutation (0, 1, ..., n-1). (We call
        conjugating by a power of sigma 'rotating')
    # tuple -> set of tuples
    # Eg. (0, 1, 2, 3, 4) -> {(0, 1, 2, 3, 4)}
    #     (0, 3, 1, 2, 4) -> {(0, 3, 1, 2, 4), (0, 3, 4, 1, 2),(0,
        1, 3, 4, 2),(0, 2, 3, 1, 4), (0, 1, 4, 2, 3)}

    # Alter tau to make sure 0 is at the start.
    i = tau.index(0)
    pile = {tau[i:]+tau[:i]}
    n = len(tau)

    for i in range(1, n):
        # Rotated cycles can be found by subtracting a constant,
            for example subtract 1 (modulo 5) from each entry of (0,
            3, 1, 2, 4) and we get (4, 2, 0, 1, 3). We can rearrange
            so that it starts with 0 and get (0, 1, 3, 4, 2).
        cycled = [(x - tau[i]) % n for x in tau]
        pile.add(tuple(cycled[i:] + cycled[:i]))
    return pile

def find_taus(n):
    # Finds the second entry of all non-equivalent constellations
        in S_n that have n-cycles as the first two entries and have
        the first entry being (0, 1, ..., n-1).
    # int -> set of tuples
    # Eg. 4 -> {(0, 1, 2, 3), (0, 2, 1, 3), (0, 3, 2, 1)}
    taus = set()
    for p in permutations(range(1,n)):
        candidate = (0,)+p
        taus.add(min(rotations_of(candidate)))
    return taus

def product_with_sigma_is_cyclic(tau):
    # Computes the product sigma*tau where sigma = (0, 1, ...,
        n-1), if the product is an n-cycle, returns the product,
        otherwise returns False.
    # tuple -> tuple or False
    # Eg. (0, 1, 2, 3) -> False (since product is (0,2)(1,3))
    #     (0, 1, 2, 3, 4) -> (0, 2, 4, 1, 3)
    # Note that a non-empty tuple will be evaluated as True in an
        if statement.
    n = len(tau)
```

```
    sigma = range(0,n)
    product = [0]
    is_cyclic = True
    for i in range(1,n):
        a = product[i-1]
        b = sigma[(tau[(tau.index(a)+1)%n]+1)%n]
        product.append(b)
        if b==0:
            return False
    return tuple(product)

def invert(cycle):
    # Takes in a cycle and returns its inverse.
    # tuple -> tuple
    # Eg. (0, 1, ..., n-1) -> (0, n-1, ..., 1)
    #     (0, 3, 1, 2, 4) -> (0, 4, 2, 1, 3)
    return (0,) + tuple(reversed(cycle[1:]))

def find_belyi_pairs(n):
    # Returns a list of all non-equivalent constellations in S_n
        corresponding to simple Belyi pairs. That is all
        constellations consisting only of n-cycles.
    # int -> list of constellations (a constellation is a tuple of
        three tuples)
    sigma = tuple(range(0,n))
    belyi_pairs = []
    possible_taus = find_taus(n)
    for tau in possible_taus:
        prod = product_with_sigma_is_cyclic(tau)
        if prod:
            constellation = (sigma, tau, invert(prod))
            belyi_pairs.append(constellation)
    return belyi_pairs

def conjugate(constellation):
    # Takes in a constellation (that is, a tuple of three tuples)
        consisting of n-cycles then conjugates so that the first
        entry is (0, 1, ..., n-1). This is only unique up to
        conjugation by a power of (0, 1, ..., n-1) so we then find
        all rotations of the resulting constellation and return the
        minimum.
    # constellation -> constellation
    # Eg. ((0,3,1,2,4), (0,2,3,1,4), (0,1,2,3,4)) -> ((0,1,2,3,4),
```

```
        (0,1,3,4,2), (0,3,4,1,2))

    sigma = constellation[0]
    n = len(sigma)

    # conj tells us what to conjugate things by to get the first
        entry equal to (0, ..., n-1).
    conj = [None]*n
    for i in range(0, n):
        a = sigma[i]
        conj[a] = i

    conj_constellation = []
    for perm in constellation:
        conj_constellation.append(tuple([conj[x] for x in perm]))

    equiv_second_element = min(rotations_of(conj_constellation[1]))

    return (conj_constellation[0], equiv_second_element,
        invert(product_with_sigma_is_cyclic(equiv_second_element)))

def group_action_2(constellation):
    # Takes in a constellation and acts on it via the map induced
        by 1/z. This has the effect of swapping the first and third
        entry and adjusting the second so that the result is a
        constellation. We then need to conjugate so that the first
        entry is (0, 1, ..., n-1).
    # Eg. ((0, 1, 2, 3, 4), (0, 2, 4, 1, 3), (0, 2, 4, 1, 3)) ->
        ((0, 1, 2, 3, 4), (0, 1, 2, 3, 4), (0, 3, 1, 4, 2))
    # We call this function group_action_2 since it has order 2.
    # Note that this function assumes that the first element is (0,
        1, ..., n-1) and the third element is an n-cycle.

    # The new second entry will be the inverse of the product of
        the first and third entries.
    prod = product_with_sigma_is_cyclic(constellation[2])
    new_second_element = invert(prod)
    new_constellation = conjugate((constellation[2],
        new_second_element, constellation[0]))
    return new_constellation

def group_action_3(constellation):
    # Takes in a constellation and acts on it via the map induced
```

```
        by 1/(1-z). This has the effect of permuting the all three
        entries, so the first is sent to the second, the second to
        the third and the third to the first. We then need to
        conjugate so that the first entry is (0, 1, ..., n-1).
    # constellation -> constellation
    # Eg. ((0, 1, 2, 3, 4), (0, 2, 4, 1, 3), (0, 2, 4, 1, 3)) ->
        ((0, 1, 2, 3, 4), (0, 1, 2, 3, 4), (0, 3, 1, 4, 2))
    #    ((0, 1, 2, 3, 4), (0, 3, 1, 2, 4), (0, 2, 3, 1, 4)) ->
        ((0, 1, 2, 3, 4), (0, 1, 3, 4, 2), (0, 3, 4, 1, 2))
    # We call this function group_action_3 since it has order 3.
    new_constellation = conjugate((constellation[1],
        constellation[2], constellation[0]))

    return new_constellation

def orbit_of(constellation):
    # Computes the orbit of the automorphisms of the sphere
        containing the given constellation. Returns the minimum
        element in the orbit.
    pile = {constellation}
    a = conjugate(group_action_3(constellation))
    b = conjugate(group_action_3(a))
    pile.add(a)
    pile.add(b)
    pile.add(conjugate(group_action_2(constellation)))
    pile.add(conjugate(group_action_2(a)))
    pile.add(conjugate(group_action_2(b)))

    return min(pile)

def find_orbits(bps):
    # Takes in a list of constellations and returns a set
        containing a representative from each orbit of the
        automorphism group
    # list of constellations -> set of constellations
    # Eg. [((0, 1, 2, 3, 4), (0, 2, 4, 1, 3), (0, 2, 4, 1, 3)),
    #       ((0, 1, 2, 3, 4), (0, 1, 3, 4, 2), (0, 3, 4, 1, 2)),
    #       ((0, 1, 2, 3, 4), (0, 1, 2, 3, 4), (0, 3, 1, 4, 2)),
    #       ((0, 1, 2, 3, 4), (0, 3, 1, 4, 2), (0, 1, 2, 3, 4))]
    #    ->
    #    {((0, 1, 2, 3, 4), (0, 1, 3, 4, 2), (0, 3, 4, 1, 2)),
    #       ((0, 1, 2, 3, 4), (0, 1, 2, 3, 4), (0, 3, 1, 4, 2))}
    orbits = set([])
```

```
    for con in bps:
        orbits.add(orbit_of(con))
    return orbits
```

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