Fuzzy Functional Observer Based Controller Design and Fault Detection for Nonlinear Systems

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Contents

List of Figures

List of Symbols

$x(t)$	state vector of a system
$u(t)$	input vector
$y(t)$	output vector
$\hat{x}(t)$	estimated state vector
$\xi_l(t)$	lth premise or schedule variable
$\xi(t)$	premise or schedule vector
M_i^l	fuzzy set for the lth premise variable in the ith rule
$M_i^l(\cdot)$	membership function for the lth premise variable in the ith rule
$\mu_i(\cdot)$	firing strength of the ith rule of a T-S fuzzy system
$w(t)$	state vector of a linear functional observer
$w_i(t)$	state vector of the ith functional observer of a fuzzy functional observer
$u_j(t)$	input vector of the jth subsystem of a T-S fuzzy system
$\hat{u}_j(t)$	estimated input vector of the jth subsystem of a T-S fuzzy system
$e_j(t)$	estimation error of function of states for the jth subsystem of a T-S fuzzy system
τ	constant time-delay
$\tau(t)$	time varying time-delay
τ_M	upper bound of the time varying time-delay
τ_m	lower bound of the time varying time-delay
ρ	upper bound of the derivative of time varying the time-delay
$f(t)$	fault vector

$\hat{f}(t)$	estimated fault vector
$r(t)$	residual to detect a fault
$d(t)$	external disturbance
X_e	transmission line reactance
R_e	transmission line resistance
V_t	terminal voltage of the generator
I_t	line current
$V_{b\infty}$	voltage at the reference bus
ω	angular velocity of the rotor
ω_o	synchronous speed
δ	power angle, the angle between q-axis and the infinite bus voltage
M	matrix variable in Chapters 1 to 5; inertia coefficient of the generator in Chapter 6
T_m	mechanical torque
T_e	electrical torque
T'_{do}	open circuit time constant of the generator
T_E	time constant of the exciter
K_E	gain of the excitation system
I_d	direct axis component of the current
I_q	quadrature axis component of the current
v_d	direct axis component of the terminal voltage
v_q	quadrature axis component of the terminal voltage
X_d	direct axis reactance
X_q	quadrature axis reactance
X'_d	transient reactance
E'_q	induced voltage
E_{fd}	excitation/field voltage
$\Delta\delta$	deviation of the power angle
$\Delta\omega$	deviation of the rotor speed
$\Delta E'_q$	deviation of the induced voltage

ΔE_{fd}	deviation of the excitation/field voltage
$k_1^i, \ldots, k_6^i$	constants for the Heffron-Philip model for the ith rule
P	matrix variable in Chapters 1 to 5; real power in Chapter 6
Q	matrix variable in Chapters 1 to 5; reactive power in Chapter 6
$\bar{P}$	fuzzy set describing the upper limit of P
$\underline{P}$	fuzzy set describing the lower limit of P
$\bar{Q}$	fuzzy set describing the upper limit of Q
$\underline{Q}$	fuzzy set describing the lower limit of Q
$\bar{X}_e$	fuzzy set describing the upper limit of X_e
$\underline{X}_e$	fuzzy set describing the lower limit of X_e

Abstract

This thesis presents fuzzy model based stability analysis and controller design techniques for nonlinear systems using functional observer considering time-delay, external disturbances and model uncertainty. A novel fuzzy functional observer based robust fault detection technique for delayed nonlinear systems is also included in this thesis.

Takagi-Sugeno (T-S) fuzzy model represents a nonlinear system as a fuzzy summation of linear models around the operating points that are expressed by respective fuzzy rules. As this approach uses singleton consequent parts, the defuzzification process of the whole system is straightforward: the overall system dynamics and output are determined by the fuzzy summations of the linear consequent parts. As a result, existing linear tools and techniques can be applied for analysing the stability of the system and designing controller accordingly. Parallel distributed compensation (PDC), a fuzzy blending of the linear compensators designed for the linear subsystems of the fuzzy model, is an effective technique for synthesising a fuzzy controller for a nonlinear system. In case the system states are not readily available, fuzzy observers are employed to obtain PDC controllers.

Functional observer directly estimates a function of states in one step rather than doing it in two steps, i.e., estimating the states and computing the function of the estimated states. It reduces the real-time computational effort of the observer. Unknown input observer can decouple external disturbances from the observer error dynamics. This research investigates the existence and stability conditions for the functional observers for nonlinear systems represented by T-S fuzzy models. The main challenge of designing a functional observer for a T-S fuzzy system is obtaining the observer parameters so that the estimation error approaches zero not only for the individual linear subsystems but also for the whole

observer after fuzzy blending. The fuzzy functional observer is employed to obtain a PDC controller. Apart from reducing the real-time computational burden, this controller design technique reduces the observer size to the dimension of the controller. Considering time-delays and model uncertainties, a new set of stability conditions are presented. Lyapunov-Krasovskii functionals are used to obtain the stability conditions for delayed systems. Free-weighting matrices are used for obtaining delay dependent stability conditions to increase the solution domain of the stability conditions. This technique is applied to design fuzzy power system stabilisers for single machine infinite bus system.

The proposed fault detection technique uses a fuzzy functional observer to obtain a residual. The main advantage is that this technique does not require any calculation of a threshold for the real-time comparison with the residual. The concept of the unknown input observer is used to decouple external disturbances from the error dynamics of residual generation and fault estimation observers.

The stability conditions obtained from the Lyapunov stability analysis approach appear in the form of convex inequality conditions. If the inequalities are not linear, the stability conditions are transformed as linear matrix inequalities so that observer parameters can be constructed by solving these inequalities.

Keywords: Nonlinear systems, T-S fuzzy systems, fuzzy controller, functional observer, unknown input observer, time-delay, fault detection.

HDR Thesis Declaration

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint-award of this degree.

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Signed: .... Date: .... 10 December 2018 ....

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Publications

Journal Articles

Islam, S. I., Lim, C. C. , & Shi, P. (2018). Functional observer based controller for stabilizing Takagi-Sugeno fuzzy systems with time-delays. Journal of the Franklin Institute, 355(8), 3619-3640.

Islam, S. I., Lim, C. C. , & Shi, P. (2018). Functional observer-based fuzzy controller design for continuous nonlinear systems. International Journal of Systems Science, 49(5), 1047-1060.

Islam, S. I., Shi, P., & Lim, C. C. (2018) Robust functional observer for stabilising uncertain fuzzy systems with time-delay. Granular Computing, doi: https://doi.org/10.1007/s41066-018-0138-x.

Islam, S. I., Lim, C. C. , & Shi, P. Robust fault detection of T-S fuzzy systems with time-delay using fuzzy functional observer. Fuzzy Sets and Systems, under review (revision).

Conference Articles

Islam, S. I., Lim, C. C. , & Shi, P.. Existence of Fuzzy Functional Observer of Takagi-Sugeno Fuzzy Model. In 2016 Australian Control Conference (AuCC) (pp. 353-357). IEEE. November 3-4, 2016. Newcastle, Australia.

Thesis Conventions

Typesetting. This thesis is typeset using the MikTex2.9 software. Processed plots were generated using MATLAB R2015b (Mathworks Inc.).

Spelling. Australian English spelling has been adopted throughout, as defined by the Macquarie English Dictionary.

Referencing. The numeric style is used for referencing and citation in this Thesis.

Chapter 1

Introduction

The dynamics of many real systems are nonlinear. Designing controllers for the stability of nonlinear systems is a difficult problem because of the complex dynamical behavior of the systems. This problem becomes more acute when the systems are subject to time-delays, model uncertainties, and external disturbances, which are inherently present in most of the systems. Fuzzy logic based modeling of nonlinear systems is found to be more effective because the system can be described as a group of locally linear models using the plant operators knowledge, and controllers can be designed by employing existing techniques applicable for linear systems.

1.1 Preliminaries and recent research directions

1.1.1 Fuzzy logic controller

The idea of fuzzy logic based controllers originated from the theories of fuzzy sets [1], and now it is quite a mature area of research. The fuzzy logic controller was first introduced in control engineering to control a steam engine [2, 3]. Many modern systems are successfully controlled by fuzzy controllers. A simple fuzzy logic controller comprises of four basic units: fuzzification block, fuzzy knowledge-base block, fuzzy inference engine and defuzzification block as depicted in Figure 1.1. The input, which is crisp in nature, is fuzzified using membership function that uses linguistic variables to describe different ranges of crisp data. A fuzzy inference engine is the knowledge base containing a set of rules that are fired

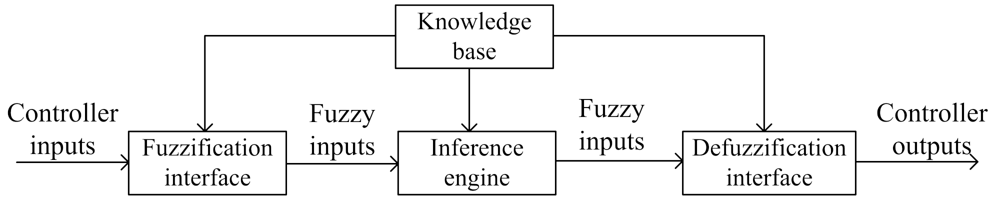

Figure 1.1: Basic structure of fuzzy model

depending on conditions of those rules. After the decision is made, the actual control signal is produced by using the defuzzification process.

Fuzzy logic controllers can be broadly categorised as the traditional fuzzy controller, fuzzy PID controller, neuro-fuzzy controller, fuzzy sliding-mode controller, adaptive fuzzy controller, and Takagi-Sugeno model based fuzzy controller [4, 5]. A fuzzy PID controller uses different sets of proportional, integral and derivative gains depending on the rules used to define different operating points of the system [6–8]. The gains can be calculated using the conventional techniques for obtaining PID controllers for linear systems. A neuro-fuzzy controller combines the precise learning ability of a neural network with the imprecise qualitative knowledge obtained from the plant operators' expert knowledge. In a fuzzy neural network, either the input signals and/or connection weights and/or the outputs are fuzzy subsets or a set of membership values to fuzzy sets [9–12]. A fuzzy sliding mode controller uses a variable structure controller to drive the system trajectories onto the so-called sliding surface in a finite time and maintain on it thereafter [13]. Sometimes a fuzzy PID controller or any model based fuzzy controller is accompanied with a supervisory sliding mode controller to guarantee robust stability in case of modeling uncertainties [4].

When the parameters of the plant dynamic model are unknown, or plant parameters change in time, the controller has to be able to adjust with the modified situation in real-time. The combination of a fixed controller and adaptive fuzzy controller can track the desired trajectory in the vicinity of an uncertain environment and change in manipulator dynamics [14]. The error learning is a vital part of this kind of controller. The fuzzy error learning approach is better than a crisp error learning scheme, although it requires increased control activity. A model free fuzzy adaptive controller along with a feedback error learning strategies is a good combination for handling situation with uncertainties [15]. A fuzzy model reference learning control [14], on the other hand, uses the reference model for the smooth tracking of the fixed reference trajectory. The learning system observes the plant errors in the outputs and adjusts the membership functions of the rules in a direct fuzzy controller by using another fuzzy inverse controller. The ability of learning from the system through interaction with the environment has made fuzzy adaptive controller better fitted for controlling disturbances of systems that depend on the states [16, 17].

In recent years, a significant interest has been shown to the T-S fuzzy model to analyse nonlinear systems and to design controllers for the systems [18–27]. This approach defines system dynamics as linear time invariant models as consequent parts for different rules representing different operating points of the system. As a result, the overall system dynamics is expressed as a convex summation of linear models. Fuzzy controllers obtained using this T-S fuzzy model have been successfully applied in control problems [28–37].

1.1.2 T-S fuzzy model

T-S fuzzy models are regarded as universal function approximators for their ability to approximate any nonlinear functions to any degree of accuracy in any convex compact region [4, 18, 38, 39]. With linear models as the consequent parts, this approach enjoys the advantage of using existing linear tools and techniques for investigating the overall stability of a nonlinear system.

Mathematical expression of a T-S fuzzy model

A T-S fuzzy model is basically described by a number of “IF-THEN” statements. The “IF” statement consists of premise variables that are compared with different possible linguistic variables, and are connected to each other by logical operator “AND” to define a particular operating condition of the system described by a rule. The consequent part, “THEN” statement, consists of a linear state-space representation of the system at the particular operating point for which the rule has been stated. The ith rule of a T-S fuzzy model is expressed as

$$\begin{aligned} &\text{IF} && \xi_1(t) \text{ is } M_i^1 \text{ and } \cdots \text{ and } \xi_l(t) \text{ is } M_i^l \\ &\text{THEN} && \dot{x}(t) = A_i x(t) + B_i u(t) \\ & && y(t) = C_i x(t), \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, $\xi_1(t), \ldots, \xi_l(t)$ are the premise variables, $M_i^1, \ldots, M_i^l$ are the fuzzy sets for the respective premise variables for the ith rule, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ and $C_i \in \mathbb{R}^{p\times n}$ are the matrices representing the linear state space model of the ith rule.

As the consequent parts of a T-S fuzzy model are not fuzzy, the overall representation of the system dynamics is straight forward. Considering r number of rules, and using product T-norm and center-average defuzzifier, the overall system dynamics is expressed as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + B_i u(t)\} \\ y(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{C_i x(t)\}, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} \xi(t) &= \begin{bmatrix} \xi_1(t) & \xi_2(t) & \cdots & \xi_l(t) \end{bmatrix}, \\ h_i(\xi(t)) &= \prod_{k=1}^{l} M_i^k(\xi_k(t)) \text{ and} \\ \mu_i(\xi(t)) &= \frac{h_i(\xi(t))}{\sum_{i=1}^{r} h_i(\xi(t))} \end{aligned}$$

with $M_i^k(\xi_k(t))$ being the membership function that defines the degree of belongingness of premise variable $\xi_k(t)$ to fuzzy set M_l^k.

Construction of a T-S Fuzzy model

System matrices A_i, B_i and C_i may not be readily available for expressing a nonlinear system as a T-S fuzzy model. These matrices can be obtained by following two distinct ways: by identifying parameters using input-output data; or by deriving from a given nonlinear system equations [18]. The first procedure mainly includes structure identification and parameter identification. It is applicable when phys-

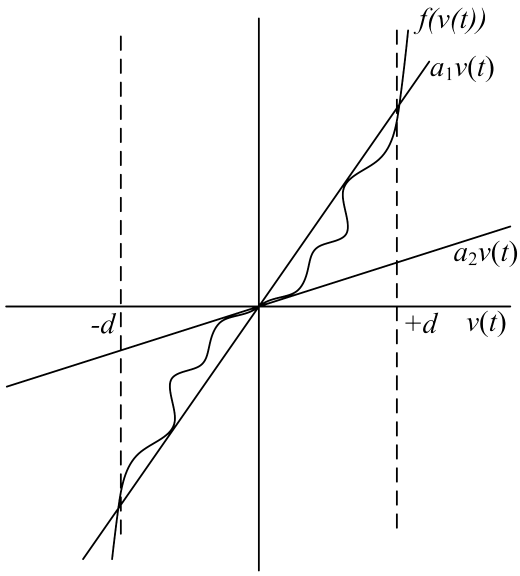

Figure 1.2: Local sector linearity

ical plants are too complex to be defined by mathematical models. The second approach can be used when the plant can be described by a true mathematical model. The premise parameters are determined from the true nonlinear model by applying linearisation technique of the nonlinear model around stable operating points [40, 41].

Obtaining a global fuzzy model for a nonlinear system is a difficult task. Even, in some cases, it may not be possible. More importantly, the global model may be too conservative. A pragmatic way would be defining the system in a local region of the operating points in which the system can be linearised with less conservativeness. This technique is known as "sector nonlinearity". A detailed procedure to obtain a T-S fuzzy model using the sector nonlinearity concept can be found in [42]. For a nonlinear system $\dot{v}(t) = f(v(t))$ with $v(t) \in \mathbb{R}$ and $f(0) = 0$, the local sector $-d < v(t) < d$ can be found such that $\dot{v}(t) = f(v(t)) \in [a_1,\ a_2]v(t)$ as depicted in Figure 1.2. Consequently, the fuzzy model can be obtained to represent the nonlinear system more accurately in the local region described by $-d < v(t) < d$.

Stability of T-S fuzzy systems

The individual stability of local linear models of a T-S fuzzy system does not necessarily imply the overall stability of the system after fuzzy summation. A sufficient condition that guarantees the stability of such a system can be obtained from Lyapunov's direct method [43, 44]. The equilibrium of T-S fuzzy system (1.2) with no control effort, i.e., with $u(t) = 0$, is asymptotically stable if a common positive definite matrix P can be found such that

$$A_i^T P + P A_i < 0 \tag{1.3}$$

for $i = 1, 2, \ldots, r$.

Finding an efficient way for searching this common P matrix attracted researchers for a long time. In [45], the authors investigated a fuzzy model of two spring-mass systems with damping for the existence of a real common P matrix that would satisfy the condition. They proposed a simple algebraic approach to find critical conditions for the existence of a common P-region. However, this approach is too exhaustive, and it becomes very complex to tackle when the number of subsystems of the model is high. Indeed, the stability conditions in (1.3) is a set of matrix inequalities and convex in P. The recent development of convex optimisation introduced mathematical tools that are numerically efficient to solve these linear matrix inequalities (LMIs) [46]. If the stability conditions are not readily in the form of LMIs, different techniques can be applied to recast the inequalities into LMIs [42, 47], which can be numerically solved by using existing tools.

If the system is not stable, a stabilising controller can be designed using parallel distributed compensation (PDC) technique. In this technique, linear feedback controllers are obtained for individual subsystems, and the fuzzy control law is determined by aggregating the feedback controllers by applying the fuzzy summation to ensure the stability of the whole system [48–51]. The following equation describes the PDC controller for the system described in (1.2):

$$u(t) = \sum_{j=1}^{r} \mu_j(\xi(t)) K_j x(t),$$

where $K_j \in \mathbb{R}^{m \times n}$ is the gain matrix for the jth subsystem. Stability conditions can be obtained for the closed loop system in the form of LMIs by using a Lyapunov function, and controller gains can be determined from the solution of the LMIs [47, 52, 53]. In the recent decades, there has been a significant development in terms of reducing the conservativeness of the stability conditions for T-S fuzzy systems by using different types of Lyapunov functions such as quadratic Lyapunov functions, piece-wise Lyapunov functions, fuzzy Lyapunov functions, and Lyapunov functions with higher order derivatives [54–56].

A PDC controller requires the states for obtaining stabilising control vector $u(t)$ for the system. If the states are not directly measurable, observer based PDC controllers are employed to stabilise the system [23, 57, 58]. The observer design procedure for T-S fuzzy systems uses the similar procedure of distributed compensation technique: obtain the linear observers for the subsystems, and aggregate the observers using the fuzzy summation. Most of the work to date uses full order state observer in what measured output and control input are used to estimate the states so that the estimation error approaches zero asymptotically [59].

An observer based PDC controller for the system defined in (1.2) can be expressed as

$$\begin{aligned}
\dot{\hat{x}}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i\hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))\} \\
\hat{y}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{C_i\hat{x}(t)\} \\
u(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{K_i\hat{x}(t)\},
\end{aligned}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimated state and $L_i \in \mathbb{R}^{n\times p}$ is the observer gain. Order reduction of this fuzzy observer has been an interesting area of research. Considering the full row rankness of C_i, a reduced order observer can be obtained by partitioning system matrices A_i, B_i, and C_i [60–62]. The stability conditions for the observers are presented as LMIs, and the observer gains are obtained by solving the inequalities.

1.1.3 Linear matrix inequalities

Considering $F_i = F_i^T \in \mathbb{R}^{n\times n}$ are known, an LMI is [46]

$$F(v) \triangleq F_0 + \sum_{i=1}^{m} v_i F_i > 0, \tag{1.4}$$

where v_i is the ith element of decision vector $v \in \mathbb{R}^m$. The positive definiteness of $F(v)$ is defined as $x^T F(v) x > 0$ for all $x \in \mathbb{R}^n$. This is a special type of LMI where decision variable v_i is scalar. However, an LMI can be of the form

$$F(X) = A^T X + XA < 0,$$

where $A \in \mathbb{R}^{n\times n}$ is known and $X = X^T > 0$ is unknown. This LMI can be converted to the form of (1.4) by using the elementary matrices of the symmetric matrix, $F(X)$. This matrix inequality is called an LMI in X, i.e., the decision variable is matrix X.

The LMI problem is convex in its decision variables. The recent advancement in convex optimisation techniques has made it possible to obtain the global optimum solution of LMIs by using numerical optimisation methods. The LMI problems can be classified into two major kinds: the feasibility problems and optimisation problems. Feasibility problems deal with finding only the existence of a solution satisfying the set of constraints. Optimisation problems, on the other hand, not only look for the feasibility of the problem but also obtain the best solution that optimises the objective function.

In control systems, sometimes, matrix inequalities may appear in non-convex form as

$$Q(v) - S^T(v) R^{-1}(v) S(v) > 0, \tag{1.5}$$

where $R(v) = R^T(v) > 0$, $Q(v) = Q^T(v) > 0$, and $S(v)$ are affine in v. By Schur complement, this matrix inequality problem can be converted as an LMI problem as below:

$$\begin{bmatrix} Q(v) & S^T(v) \\ S(v) & R(v) \end{bmatrix} > 0.$$

Once the constraints are converted into LMIs, the problems can be solved efficiently in polynomial time. By solving the problem we understand looking for the feasibility of the problem, and obtaining a feasible point for which the objective function exceeds the global minimum only by a prefixed precision. There are existing software packages for MATLAB that can be used to solve LMI problems, such as SOSTOOL [63], LMITOOLBOX [64] and CVX [65].

1.1.4 Functional observer

Functions of states are often required to be obtained in many dynamical systems. Although a function of states can be obtained once all of the states are estimated using full state observers, the direct estimation of the function of states is found to be efficient in terms of saving real-time computational effort. A functional observer is a special kind of observer particularly designed for estimating a function of states using the input and output measurements [66]. Considering a linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y &= Cx(t) \end{aligned}$$

with state vector $x(t) \in \mathbb{R}^n$, input vector $u(t) \in \mathbb{R}^m$, output vector $y(t) \in \mathbb{R}^p$, and matrices A, B, and C of appropriate dimensions, a function of states $z(t) \in \mathbb{R}^q$ can be estimated by employing the following functional observer:

$$\begin{aligned} \dot{w}(t) &= Nw(t) + Jy(t) + Hu(t) \\ \hat{z}(t) &= Dw(t) + Ey(t), \end{aligned} \tag{1.6}$$

where $\hat{z}(t) \in \mathbb{R}^q$ is the estimated function of states, $w(t) \in \mathbb{R}^q$ is the state of the observer dynamics, and N, J, H, D and E are real matrices of appropriate dimensions that are determined such that estimation error $e(t) = z(t) - \hat{z}(t)$ approaches zero asymptotically. The operating principle of a linear functional observer can be understood from the block diagram presented in Figure 1.3.

The ability of a linear functional observer of estimating the function of states attracted considerable interest in the recent past [68–72]. The observer construction procedure including the necessary and sufficient conditions for the existence of these observers are well established [66, 73–77]. The main advantage of a func-

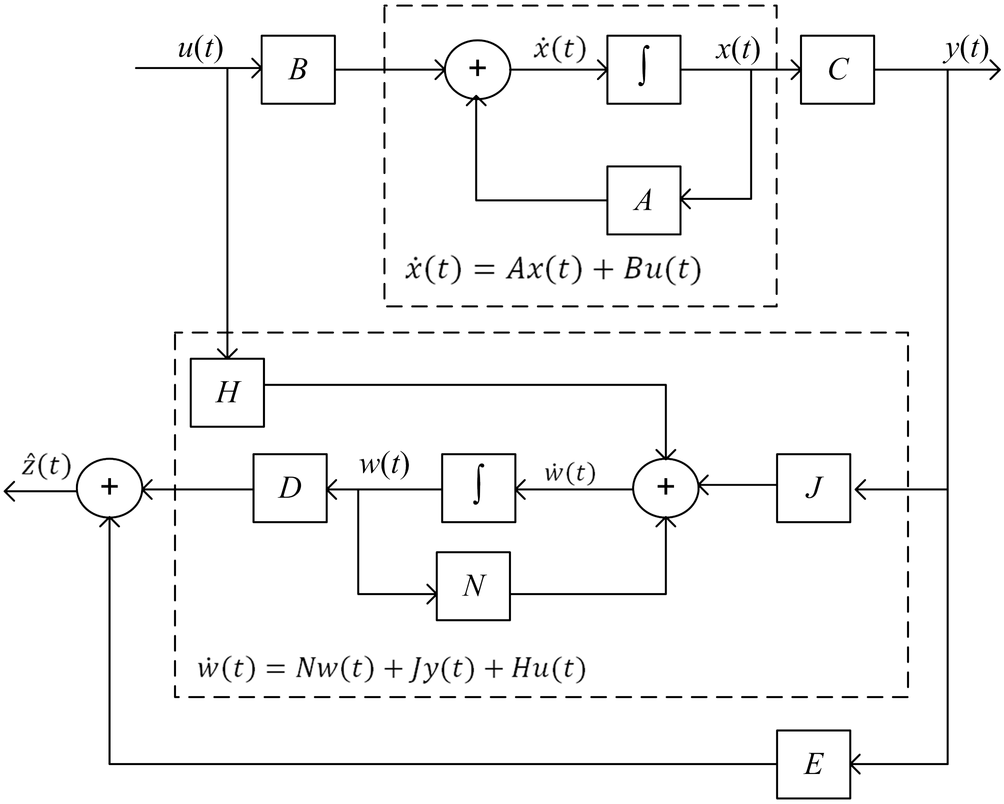

Figure 1.3: Linear functional observer [67]

tional observer is that it estimates only the function of our interest instead of all states of the system. The dimension of observer state $w(t)$ defines the order of the observer. This dimension can be reduced to the dimension of the function of states $z(t)$ if the dimension of the function of states is less than the dimension of the measured output vector of the system, and output matrix C has full row rank [66, 68, 78].

1.2 Motivation and purposes

In many cases, the mathematical model of a nonlinear system is known. However, due to the complexity of the nonlinearity, designing a suitable controller analytically may not always be possible. In such cases, the T-S fuzzy model based control design technique is more pragmatic than model identification based or model free control techniques. A PDC controller can be used for stabilising the system. The competitive advantage of using the T-S fuzzy model based approach compared with any model identification approach is that the stability of the system can be analysed using Lyapunov functions. As the PDC approach of designing a controller uses local feedback controllers of all linear models of the fuzzy model, this controller, in fact, is a fuzzy summation of linear functions of states. Therefore, we can employ linear functional observers to obtain a fuzzy controller for a nonlinear system.

The existence of the fuzzy functional observer for T-S fuzzy systems has recently been studied, and the observer is used as a fuzzy controller for the system in [60, 79]. The stability of the observers is investigated and stability conditions for the observers are presented. However, these works do not consider time-delays, model uncertainty and/or external disturbances. More importantly, the observer construction procedures require some algebraic calculations that may violate the stability conditions. Hence, the stability conditions need to be improved.

Considering the current trend of convex optimisation techniques, LMIs are better options for the formats of the stability conditions because the observer parameters can be constructed from the numerical solutions of the inequalities. The stability conditions can be improved to accommodate time-delays and model uncertainties in the plant models. Besides, the linear functional observer is not yet used for fault detection of nonlinear systems. These research gaps are the key motivations for investigating the application of functional observers for the stability and fault detection of nonlinear systems using the T-S fuzzy model.

The main purposes of this research include improving design techniques of the fuzzy functional observer by involving LMI based stability conditions considering time-delays and model uncertainty, and employing the observer for obtaining PDC controller and fault detection scheme. The formation of stability conditions as LMIs capitalises the recent advancements of convex optimisation techniques.

1.3 Main contributions

Considering a given mathematical model of a nonlinear system, the matrices that describe the T-S fuzzy model are assumed to be known. This thesis investigates the existence and stability conditions for the fuzzy functional observer for a nonlinear system. This fuzzy functional observer is employed for designing controllers and detecting faults of the system considering time-delays, external disturbances and model uncertainty. The main contributions of this research are listed below.

1. A fuzzy functional observer is proposed to estimate the function of states of a nonlinear system described by a T-S fuzzy model. The proposed functional observer is obtained by fuzzy summation of linear functional observers for individual subsystems. The stability conditions are formulated as LMIs so that the observer gains obtained from the solution of these constraints guarantee the asymptotic convergence of the estimation error to zero.

2. The observer is applied to estimate the control vector of a T-S fuzzy system considering time-delays and model uncertainty. As the functional observer estimates the control vector directly, the fuzzy aggregation of the control signal for a PDC controller is not required. More importantly, the observer size is reduced to the size of the dimension of the control signal.

3. The observer is employed for the fault detection and fault estimation of the nonlinear system considering time-delay and external disturbances. The residual generator of the fault detection scheme is designed using the fuzzy functional observer so that the real-time calculation of threshold is not required. External disturbances are decoupled from the estimation error dynamics by applying the unknown input observer concept.

4. Power system stabilisers are designed by applying the functional observer based PDC controller. The effectiveness of the stabiliser is verified using benchmark examples.

1.4 Structure of Thesis

The rest of this thesis is organised as follows. Chapter 2 presents the functional observer design procedure for nonlinear systems using T-S fuzzy models. This chapter also includes functional observer based PDC controller design technique. Chapter 3 investigates the functional observer based PDC controller for the T-S fuzzy system with constant and time varying time-delays. The combined effect of time-delay and model uncertainty is studied and the robust observer construction procedure is presented in Chapter 4. Delay dependent stability conditions are derived for the asymptotic stability of the observer. Chapter 5 deals with the functional observer based robust fault detection technique for T-S fuzzy systems considering time-delay and external disturbances. Chapter 6 describes the application of the proposed functional observer based PDC controller for obtaining a power system stabiliser. Chapter 7 summarises the research findings and concludes the thesis with remarks for future research directions.

The description of the symbols that have common meanings across the chapters is explained in List of Symbols. However, considering the mathematical involvement of this thesis, each chapter introduces the variables and the plant models separately. The notations are explicitly described in the respective chapters to enhance the readability.

Chapter 2

Fuzzy functional observer based controller

This chapter is concerned with the existence and stability conditions of the fuzzy functional observer for a T-S fuzzy model. A fuzzy functional observer can be constructed as a fuzzy summation of linear functional observers for the respective linear subsystems of a T-S fuzzy model. Necessary and sufficient conditions are provided for the existence and stability of the observer. The fuzzy functional observer is employed to obtain a PDC controller for the T-S fuzzy model. The main advantage of designing a fuzzy controller using functional observer is that the fuzzy functional observer estimates the control vector directly, and the order of the observer is equal to the dimension of the control vector. Lyapunov functions are used to obtain stability conditions for the observers. The stability conditions are derived in LMI form; the observer parameters are determined using the solution of the LMIs. The main results of this chapter are published in [80, 81].

2.1 Introduction

A linear functional observer estimates the function of states directly. The existence and stability conditions of the linear functional observer are well established. A T-S fuzzy model is expressed as a fuzzy summation of linear subsystems. Therefore, a set of linear functional observers can be constructed for the respective linear subsystems of a T-S fuzzy model, and the linear functional observers can be fuzzy summed together to obtain a fuzzy functional observer. The main difficulty of designing a fuzzy functional observer is to guarantee the stability of the observer that we get after the fuzzy summation of the linear observers. By the stability of an observer, we mean the asymptotic convergence of the estimation error to zero.

A PDC controller is constructed as a fuzzy summation of the state feedback controllers of the linear subsystems of a T-S fuzzy model [47]. The state feedback controller, by the nature of its construction, is a linear function of states. When the states are not measurable, all the states are estimated using fuzzy observer; and the PDC controller is obtained from the estimated states [22]. As a PDC controller is a fuzzy summation of feedback controllers, which are linear functions of states, this controller can be obtained by applying the fuzzy functional observer if the states are not directly measurable. This technique reduces the observer order and the real-time computational effort of the controller.

The existence of fuzzy functional observer and its application for T-S fuzzy systems is an interesting research topic. In [60], authors investigated the problem of construction of a fuzzy functional observer for nonlinear systems represented by T-S fuzzy model and provided an observer construction procedure by solving interconnected algebraic equations. The application of a functional observer for designing a PDC controller is studied in [79]. Both of the procedures require stability checking after the calculation of the observer parameters. To the best of the authors' knowledge, systematic synthesis procedure of functional observer for T-S fuzzy model to obtain PDC controller has not been studied fully.

This chapter proposes improved stability conditions for the fuzzy functional observer to ensure the asymptotic convergence of the estimation error to zero. The fuzzy functional observer is applied to obtain a PDC controller. It is shown that the separation principle holds. Therefore, the control gain for each linear subsystem is determined by applying the existing LMI based controller design technique for T-S fuzzy model [42]. Using this control gain, a functional observer is designed and stability conditions are attained in the sense that estimation error approaches zero asymptotically. The order of the observer is equal to the control gain matrix of the linear subsystems. To improve the functional observer based controller construction method for T-S fuzzy systems in [60, 79], the stability conditions are transformed into LMIs, so that the observer parameters can be obtained by solving these LMIs.

Notation: In this chapter $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ mean n dimensional real vector and $n\times m$ dimensional real matrix respectively. Superscript $(.)^-$, $(.)^+$ and $(.)^\perp$ denote inverse, Moore-Penrose generalised inverse and orthogonal basis of corresponding matrix, respectively. I_p denotes identity matrix of $p\times p$ dimension.

2.2 Model description and problem formulation

Consider a continuous T-S fuzzy model of a nonlinear system with r number of rules. The ith rule of this model is given by

$$\begin{aligned}
\text{IF} \quad & \xi_1(t) \text{ is } M_i^1 \text{ and } \cdots \text{ and } \xi_l(t) \text{ is } M_i^l \\
\text{THEN} \quad & \dot{x}(t) = A_i x(t) + B_i u(t) \\
& y(t) = Cx(t),
\end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output and $\xi_1(t), \ldots, \xi_l(t)$ are premise variables. Real matrices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ represent the ith local model of the system, and M_i^k represents the fuzzy set for $\xi_k(t)$ in the ith rule with $M_i^k(\cdot)$ being the respective membership function. Considering $\xi(t) \in \mathbb{R}^l$ represents the vector $\left[\xi_1(t)\ldots\xi_l(t)\right]$, the system dynamics is expressed by

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + B_i u(t)\} \\
y(t) &= Cx(t),
\end{aligned} \tag{2.2}$$

where

$$\mu_i(\xi(t)) = \frac{\prod_{k=1}^{l} M_i^k(\xi_k(t))}{\sum_{i=1}^{r}\prod_{k=1}^{l} M_i^k(\xi_k(t))}$$

with

$$\mu_i(\xi(t)) \geq 0, \quad \sum_{i=1}^{r} \mu_i(\xi(t)) = 1.$$

Given a pair of $x(t)$ and $u(t)$, the final output of the T-S fuzzy model is inferred by (2.2). Each subsystem represented by each rule in (2.1) is a linear model.

A fuzzy functional observer can be expressed as

$$\begin{aligned} \dot{w}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{N_i w(t) + J_i y(t) + H_i u(t)\} \\ \hat{z}(t) &= w(t) + Fy(t) \end{aligned} \tag{2.3}$$

where $z(t) \in \mathbb{R}^q, w(t) \in \mathbb{R}^q, F \in \mathbb{R}^{q\times p}, N_i \in \mathbb{R}^{q\times q}, J_i \in \mathbb{R}^{q\times p}$, and $H_i \in \mathbb{R}^{q\times m}$. In this observer, $\hat{z}(t)$ is the estimation of $z(t)$, which is a linear function of $x(t)$ and is defined by

$$z(t) = Lx(t),$$

where $L \in \mathbb{R}^{q\times n}$ is a known matrix. Membership functions of this observer are considered to be similar to those of the plant model described in (2.1). Existence and stability of the observer is ensured in the sense that estimation error approaches zero asymptotically if the observer is properly designed. The estimation error of observer (2.3) is defined as

$$\begin{aligned} e(t) &= Lx(t) - \hat{z}(t) \\ &= Tx(t) - w(t), \end{aligned} \tag{2.4}$$

where $T = L - FC$. After taking derivative in (2.4), the error dynamics of the functional observer is expressed as

$$\begin{aligned} \dot{e}(t) = &\sum_{i=1}^{r} \mu_i(\xi(t))\{N_i e(t) + (TA_i - N_i T - J_i C)x(t) \\ &+ (TB_i - H_i)u(t)\}. \end{aligned} \tag{2.5}$$

The error dynamics in (2.5) reduces to

$$\dot{e}(t) = \sum_{i=1}^{r} \mu_i(\xi(t)) N_i e(t) \tag{2.6}$$

if we have

$$TA_i - N_i T - J_i C = 0 \tag{2.7a}$$

$$TB_i - H_i = 0. \tag{2.7b}$$

As H_i can be obtained from (2.7b), the functional state reconstruction problem is turned into finding F, J_i and N_i so that (2.7a) holds and the error dynamics in (2.6) is asymptotically stable.

2.3 Fuzzy functional observer

2.3.1 Existence of fuzzy functional observer

Without loss of generality it can be assumed that C is a full row rank matrix. Therefore, we can obtain a nonsingular matrix $P = [C^{+} \; C^{\perp}]$, such that $CC^{+} = I_p$ and $CC^{\perp} = 0_{p \times n-p}$. Using invertible matrix P, the conditions in (2.7a) and (2.7b) are written as

$$(L_1 - F)A_{i11} + L_2 A_{i21} - N_i(L_1 - F) = J_i \tag{2.8a}$$

$$N_i L_2 - (L_1 - F)A_{i12} - L_2 A_{i22} = 0 \tag{2.8b}$$

$$TB_i - H_i = 0, \tag{2.8c}$$

where $CP = \begin{bmatrix} I_p & 0 \end{bmatrix}$, $P^{-1}A_i P = \begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix}$ and $LP = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$. Considering the fuzzy functional observer as a fuzzy summation of linear functional observers for each linear subsystem of the T-S fuzzy model, it is required that the existence condition of each linear functional observer holds. The existence condition of the fuzzy functional observer is given in the following lemma.

Lemma 2.3.1. *Necessary and sufficient condition for the existence of the observer*

(2.3) *for each rule can be given by*

$$\operatorname{rank}\begin{bmatrix} L_2A_{i22} \\ A_{i12} \\ L_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_{i12} \\ L_2 \end{bmatrix} \tag{2.9}$$

Proof. According to [66], necessary and sufficient condition for the existence of linear functional observer for each linear model of the T-S model can be given by

$$\operatorname{rank}\begin{bmatrix} LA_i \\ CA_i \\ C \\ L \end{bmatrix} = \operatorname{rank}\begin{bmatrix} CA_i \\ C \\ L \end{bmatrix} \tag{2.10}$$

Multiplying the full ranked matrix P with the right hand side of (2.10) we get [67]

$$\begin{aligned}
\operatorname{rank}\left(\begin{bmatrix} CA_i \\ C \\ L \end{bmatrix} P\right) &= \operatorname{rank}\begin{bmatrix} CA_iP \\ CP \\ LP \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} CPP^{-1}A_iP \\ CP \\ LP \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} \begin{bmatrix} I_p & 0 \end{bmatrix}\begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix} \\ \begin{bmatrix} I_p & 0 \end{bmatrix} \\ \begin{bmatrix} L_1 & L_2 \end{bmatrix} \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} A_{i11} & A_{i12} \\ I_p & 0 \\ L_1 & L_2 \end{bmatrix}
\end{aligned}$$

Therefore,

$$\operatorname{rank}\begin{bmatrix} CA_i \\ C \\ L \end{bmatrix} = p + \operatorname{rank}\begin{bmatrix} A_{i12} \\ L_2 \end{bmatrix} \tag{2.11}$$

Using similar procedure in the left hand side of (2.10) it can be shown that

$$\operatorname{rank}\begin{bmatrix} LA_i \\ CA_i \\ C \\ L \end{bmatrix} = p + \operatorname{rank}\begin{bmatrix} L_2A_{i22} \\ A_{i12} \\ L_2 \end{bmatrix} \tag{2.12}$$

Therefore, (2.9) follows from the equality of (2.11) and (2.12). □

2.3.2 Stability of fuzzy functional observer

The functional observer is expected to estimate function of states $z(t)$ asymptotically. Therefore, it is required that the error system is asymptotically stable, i.e., the estimation error approaches zero asymptotically. Considering error dynamics (2.6), and the identities in (2.8a), (2.8b) and (2.8c) the stability condition for the observer is stated in the following theorem.

Theorem 2.3.1. *The observer defined in* (2.3) *is asymptotically stable if there exists a positive definite symmetric matrix X such that*

$$XN_i + N_i^T X < 0, \tag{2.13}$$

and the conditions in (2.8a), (2.8b) *and* (2.8c) *hold.*

Proof. The error dynamics in (2.6) directly follows from (2.5) if conditions (2.8a), (2.8b) and (2.8c), which are other forms of conditions (2.7a) and (2.7b), are satisfied. Consider a Lyapunov function $V = e^T(t)Xe(t)$ for (2.6) such that X is a positive definite symmetric matrix. Consideration of $X > 0$ implies that

$$V > 0.$$

Taking the derivative of V we obtain

$$\begin{aligned}\dot{V} &= \dot{e}^T(t)Xe(t) + e^T(t)X\dot{e}(t) \\ &= \sum_{i=1}^{r} \mu_i(\xi(t))(N_i e(t))^T Xe(t) + e^T(t)X\sum_{i=1}^{r} \mu_i(\xi(t))(N_i e(t)) \\ &= \sum_{i=1}^{r} \mu_i(\xi(t))e^T(t)(N_i^T X + XN_i)e(t).\end{aligned}$$

It can be concluded that $\dot{V} < 0$ if (2.13) holds and the error dynamics will asymptotically approaches zero. □

From Theorem 2.3.1 it can be seen that the LMIs in (2.13) are not the only constraints that determine the observer parameters; it also requires that the identities in (2.8a), (2.8b) and (2.8c) hold. This problem can be tackled in a different way: identities (2.8a) and (2.8c) can be used to obtain J and H in terms of N such that the calculation of N ensures (2.8b) and (2.13) hold. Therefore, we post-multiply (2.8b) by a full rank matrix $\begin{bmatrix} L_2^+ & L_2^\perp \end{bmatrix}$, and separate this identity into two parts as below:

$$N_i = T_1 A_{i12} L_2^+ + L_2 A_{i22} L_2^+ \tag{2.14a}$$

$$T_1 A_{i12} L_2^\perp = -L_2 A_{i22} L_2^\perp, \tag{2.14b}$$

where $\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} L_1 - F & L_2 \end{bmatrix}$. Considering $\Phi_i = -L_2 A_{i22} L_2^\perp$, $\Omega_i = A_{i12} L_2^\perp$, $N_{i1} = L_2 A_{i22} L_2^+$ and $N_{i2} = A_{i12} L_2^+$, (2.14a) and (2.14b) can be rewritten as

$$N_i = N_{i1} + T_1 N_{i2} \tag{2.15a}$$

$$T_1 \Omega_i = \Phi_i \tag{2.15b}$$

As a consequence, the stability conditions of the observer can be redefined using the relationships expressed in (2.15a) and (2.15b).

Corollary 2.3.1. *If conditions in* (2.8a) *and* (2.8c) *hold, the observer defined in* (2.3) *will be asymptotically stable if there exist a positive definite symmetric matrix* X *and a matrix* Y *of appropriate dimension such that*

$$XN_{i1} + N_{i1}^T X + YN_{i2} + N_{i2}^T Y^T < 0 \tag{2.16a}$$

$$Y\Omega_i - X\Phi_i = 0, \tag{2.16b}$$

where $Y = XT_1$.

Proof. Replacing N_i of (2.13) by its expression in (2.15a) and considering $Y = XT_1$, we obtain (2.16a). Furthermore, (2.16b) is obtained from (2.15b) by applying $T_1 = X^{-1}Y$. Therefore, (2.13) and (2.8b) hold if (2.16a) and (2.16b) hold. This completes the proof. □

Having the stability conditions stated in (2.16a) and (2.16b), it is evident that the functional observer can be obtained by solving those equations. One can obtain the functional observer by following the design steps outlined below.

Fuzzy functional observer construction procedure

Step 1: Obtain $P = \begin{bmatrix} C^+ & C^\perp \end{bmatrix}$ and calculate $A_{i11}, A_{i12}, A_{i21}, A_{i22}, L_1$ and L_2. Using condition (2.9) check existence of the observer;

Step 2: Obtain N_{i1}, N_{i2}, Ω_i and Φ_i from their definitions in (2.15a) and (2.15b);

Step 3: Compute T_1 from the solutions of LMIs (2.16a) for X and Y with constraints (2.16b);

Step 4: Obtain $N_i = N_{i1} + T_1 N_{i2}$, and subsequently F from the definition $T_1 = L_1 - F$; and

Step 5: Find J_i and H_i from (2.8a) and (2.8c), respectively.

Remark 2.3.1. *The LMI conditions of* (2.16a) *with equality constraints in* (2.16b) *can be numerically solved [82]. There are a readily available software packages, which can be used to solve such LMIs numerically.*

Remark 2.3.2. *Equation* (2.15b) *for all r number of rules can be augmented as below.*

$$T_1 \boldsymbol{\Omega} = \boldsymbol{\Phi}, \tag{2.17}$$

where

$$\boldsymbol{\Omega} = \begin{bmatrix} \Omega_1 & \Omega_2 & \dots & \Omega_r \end{bmatrix},$$

$$\mathbf{\Phi} = \begin{bmatrix} \Phi_1 & \Phi_2 & \dots & \Phi_r \end{bmatrix}.$$

There exists a solution of T_1 *in equation* (2.17) *if and only if [83]*

$$\text{rank} \begin{bmatrix} \mathbf{\Phi} \\ \mathbf{\Omega} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{\Omega} \end{bmatrix}$$

General solution of T_1 *is given by*

$$T_1 = \mathbf{\Phi}\mathbf{\Omega}^+ + Z(I - \mathbf{\Omega}\mathbf{\Omega}^+)$$

where Z *is an arbitrary matrix, and* I *is identity matrix of appropriate dimension. This general solution can be applied to obtain stability conditions for the observer as LMIs without any identity constraints.*

Remark 2.3.3. *When* $\text{rank}(L_2) = q$, (2.8b) *can be directly converted into the following form by post multiplying the equation by* L_2^{-1}*:*

$$N_i = T_1 A_{i12} L_2^{-1} + L_2 A_{i22} L_2^{-1} \tag{2.18}$$

The new definition of N_i *in* (2.18) *can be expressed as* (2.15a) *by considering* $N_{i1} = L_2 A_{i22} L_2^{-1}$ *and* $N_{i2} = A_{i12} L_2^{-1}$*. In this case, there will be no equality constraints. Therefore,* T_1 *can be obtained by solving* (2.16a) *only. Same design procedure is applicable with new definition of* N_{i1} *and* N_{i2} *as stated in this remark.*

Remark 2.3.4. *A fuzzy functional observer for a discrete time T-S fuzzy model can also be designed by following the similar line of design procedure. The stability conditions for the discrete time model will be different from the ones for continuous time models. In that case* (2.16a) *will be replaced by the following equation while all other conditions will remain the same.*

$$\begin{bmatrix} X & (XN_{i1} + YN_{i2})^T \\ XN_{i1} + YN_{i2} & X \end{bmatrix} > 0.$$

Example 2.1

Consider a dynamical system with two rules having their corresponding consequents are as below.

$$A_1 = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 29.2529 & -0.3149 & 0 & 44.1811 \\ 0 & 0 & 0 & 1.0000 \\ -1.2637 & 0.0136 & 0 & -16.7096 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -1.9280 \\ 0 \\ 0.7292 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 50.0000 & 0 & 0 \\ 22.7267 & -0.2958 & 0 & 20.7525 \\ 0 & 0 & 0 & 2.5000 \\ -0.9818 & 0.0064 & 0 & -15.6975 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -0.9056 \\ 0 \\ 0.6850 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Membership functions for the plant for two rules are assumed to be

$$\mu_1(x_1(t)) = \left(1 - \frac{1.0}{1.0 + e^{-7.0(x_1(t)-\pi/6)}}\right) \frac{1.0}{1.0 + e^{-7.0(x_1(t)+\pi/6)}}, \text{ and}$$
$$\mu_2(x_1(t)) = 1 - \mu_1(x_1(t)).$$

Consider that the plant is controlled by parallel distributed compensator, and corresponding control gain matrices are as below:

$$K_1 = \begin{bmatrix} 48.8886 & 18.0979 & 1.2032 & 26.8854 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 810.2971 & 324.7614 & 28.0250 & 110.7137 \end{bmatrix}.$$

Our objective is to find a functional observer such that it can estimate the $z(t)$ as a function of states, i.e. $z(t) = Lx(t)$, where L is

$$L = \begin{bmatrix} 1.0000 & 0.0100 & 1.0000 & 0.0300 \end{bmatrix}.$$

By following the steps stated in fuzzy functional observer construction procedure, the observer parameters can be obtained as below. LMITOOLBOX of MATLAB has been used to solve the matrix inequalities.

$$\begin{aligned}
&N_1 = -03.8016, \quad H_1 = 0.0026, \quad J_1 = \begin{bmatrix} 0.2440 & -0.2486 \end{bmatrix}, \\
&N_2 = -14.2300, \quad H_2 = 0.0115, \quad J_2 = \begin{bmatrix} 0.1553 & -0.9306 \end{bmatrix}, \\
&F = \begin{bmatrix} 1.0028 & 1.0654 \end{bmatrix}.
\end{aligned}$$

Considering initial states of the plant $x(0) = \begin{bmatrix} -.65 & 0 & 0 & 0 \end{bmatrix}$, the performance of the observer is simulated in MATLAB environment, and the estimated function of states is compared with the desired function of states in Figure 2.1. It is evident that the proposed functional observer estimates the desired function of states very closely, and estimation error approaches zero asymptotically.

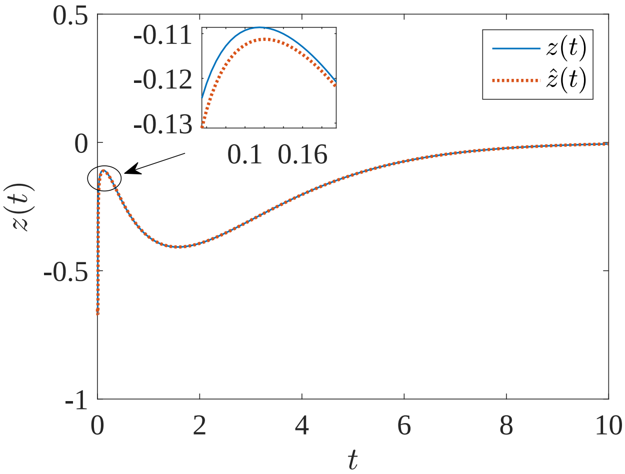

Figure 2.1: Function of states $z(t)$, and estimated function of states $\hat{z}(t)$

2.4 Functional observer based fuzzy controller

Using the same set of premise variables of (2.1), a PDC controller, $u(t)$, can be designed as

$$\begin{aligned}
u(t) &= \sum_{j=1}^{r} \mu_j(\xi(t))\{u_j(t)\} \\
&= \sum_{j=1}^{r} \mu_j(\xi(t))\{K_j x(t)\},
\end{aligned} \tag{2.20}$$

where $u_j(t) = K_j x(t)$, and K_j can be obtained by solving the following LMIs for X and Y_i [42]:

$$XA_i^T + A_i X + Y_i^T B_i^T + B_i Y_i < 0, \quad i = 1, 2, \ldots, r \tag{2.21a}$$

$$\begin{aligned} &XA_i^T + A_i X + XA_j^T + A_j X \\ &\quad + Y_j^T B_i^T + B_i Y_j + Y_i^T B_j^T + B_j Y_i \leq 0, \quad i < j, \end{aligned} \tag{2.21b}$$

where $X = X^T > 0$ and $Y_i = K_i X$. Considering all states are not measurable, a linear functional observer can be employed for each rule to estimate $u_j(t)$ as a function of states, i.e., $K_j x(t)$. Then, the overall PDC control vector can be calculated from the fuzzy summation of estimated control vectors $\hat{u}_j(t)$ for all $j = 1, \ldots, r$. The proposed functional observer based PDC controller is as follows:

$$\begin{aligned} \dot{w}_j(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{N_{ij} w_j(t) + J_{ij} y(t) + H_{ij} \hat{u}(t)\} \\ \hat{u}_j(t) &= w_j(t) + F_j y(t) \\ \hat{u}(t) &= \sum_{j=1}^{r} \mu_j(\xi(t))\{\hat{u}_j(t)\}, \end{aligned} \tag{2.22}$$

where $w_j(t) \in \mathbb{R}^m$, $F_j \in \mathbb{R}^{m\times p}$, $N_{ij} \in \mathbb{R}^{m\times m}$, $J_{ij} \in \mathbb{R}^{m\times p}$, and $H_{ij} \in \mathbb{R}^{m\times m}$. Here $\hat{u}_j(t)$ is the estimation of $u_j(t)$, which is a linear function of states $x(t)$ and is defined in (2.20). The estimation error can be expressed as

$$\begin{aligned} e_j(t) &= u_j(t) - \hat{u}_j(t) \\ &= K_j x(t) - (w_j(t) + F_j y(t)) \\ &= (K_j - F_j C) x(t) - w_j(t) \\ &= T_j x(t) - w_j(t), \end{aligned}$$

where $T_j = K_j - F_j C$. Taking derivative of $e_j(t)$ we obtain

$$\begin{aligned} \dot{e}_j(t) = T_j &\sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + B_i \sum_{l=1}^{r} \mu_l(\xi(t))(w_l(t) + F_l y(t))\} \\ &- \sum_{i=1}^{r} \mu_i(\xi(t))\{N_{ij} w(t) + J_{ij} y(t) + H_{ij} \sum_{l=1}^{r} \mu_l(\xi(t))(w_l(t) + F_l y(t))\} \end{aligned}$$

$$= \sum_{i=1}^{r}\sum_{l=1}^{r}\mu_i(\xi(t))\mu_l(\xi(t))\{(T_jA_i - N_{ij}T_j - J_{ij}C)x(t) + (T_jB_i - H_{ij})F_lCx(t) + (T_jB_i - H_{ij})w(t) + N_{ij}e_j(t)\}.$$

This error dynamics reduces to

$$\begin{aligned}\dot{e}_j(t) &= \sum_{i=1}^{r}\sum_{l=1}^{r}\mu_i(\xi(t))\mu_l(\xi(t))\{N_{ij}e_j(t)\}\\ &= \sum_{i=1}^{r}\mu_i(\xi(t))\{N_{ij}e_j(t)\},\end{aligned} \tag{2.25}$$

provided the following conditions hold:

$$T_jA_i - N_{ij}T_j - J_{ij}C = 0 \tag{2.26a}$$

$$T_jB_i - H_{ij} = 0. \tag{2.26b}$$

Therefore, the functional observer construction problem reduces to finding matrices T_j, N_{ij} and J_{ij} such that N_{ij} is stable, and conditions in (2.26a) and (2.26b) hold. With a proper choice of K_j, it can be shown that the overall system is stable with this functional observer.

The closed loop system dynamics of the T-S fuzzy model can be expressed as

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r}\mu_i(\xi(t))\{A_ix(t) + B_i\hat{u}(t)\}\\ &= \sum_{i=1}^{r}\mu_i(\xi(t))\{A_ix(t) + B_i\sum_{k=1}^{r}(u_k(t) - e_k(t))\}\\ &= \sum_{i=1}^{r}\sum_{l=1}^{r}\mu_i(\xi(t))\mu_l(\xi(t))\{(A_i + B_iK_l)x(t) - \sum_{k=1}^{r}\mu_k(\xi(t))B_ie_k(t)\}.\end{aligned}$$

Therefore, the state and error dynamics can be expressed as

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\mathbf{e}}(\mathbf{t})\end{bmatrix} = \sum_{i=1}^{r}\sum_{l=1}^{r}\mu_i(\xi(t))\mu_l(\xi(t))\begin{bmatrix}A_i + B_iK_l & \mathbf{B_i}\\ 0 & \mathbf{N_i}\end{bmatrix}\begin{bmatrix}x(t)\\ \mathbf{e}(\mathbf{t})\end{bmatrix}, \tag{2.27}$$

where

$$\mathbf{B_i} = \begin{bmatrix} -\mu_1(\xi(t))B_i & -\mu_2(\xi(t))B_i & \dots & -\mu_r(\xi(t))B_i \end{bmatrix},$$

$$\mathbf{N_i} = \begin{bmatrix} N_{i1} & 0 & \dots & 0 \\ 0 & N_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & N_{ir} \end{bmatrix}, \text{ and } \mathbf{e(t)} = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_r(t) \end{bmatrix}.$$

Since, eigenvalues of each linear model of the augmented system (2.27) are the union of eigenvalues of $A_i + B_iK_l$ and $\mathbf{N_i}$, it follows that seperation principle holds for this functional observer based PDC controller of the T-S fuzzy model. Therefore, the controller and the observer can be designed independently.

Remark 2.4.1. *In existing observer based PDC controller design techniques, an observer estimates system state $x(t)$. Estimated state $\hat{x}(t)$ is used to obtain the control input as $\hat{u}(t) = \sum_{j=1}^{r} \mu_j(\xi(t))\{K_j\hat{x}(t)\}$. The observer is constructed by using known system matrices and an unknown observer gain matrix that is obtained so that estimation error, $e_x(t) = x(t) - \hat{x}(t)$, converges to zero. The observer dynamics is described by $\hat{x}(t)$ which is of the same order of system state $x(t)$. The proposed functional observer, on the other hand, estimates control vector $u_j(t)$ directly as a function of states. Estimation error $e_j(t)$ of the functional observer based PDC controller is different from estimation error $e_x(t)$ of existing full order observers. The order of observer state $w_j(t)$ of the functional observer may be different from the order of system state $x(t)$. Unlike existing observer construction procedures, the proposed functional observer employs different observer parameters, N_{ij}, J_{ij}, H_{ij}, and F_j for the observer dynamics. These observer parameters are unknown and are to be constructed so that the estimation error approaches zero.*

The main objective of obtaining the functional observer based PDC controller for a T-S fuzzy system is converted into obtaining observer parameters such that the error dynamics in (2.25) is stable provided that the conditions in (2.26a) and (2.26b) hold. Considering that state feedback gain K_j can be calculated from (2.21) to stabilise a T-S fuzzy model, the following subsections focus on obtaining functional observer parameters, N_{ij}, J_{ij}, H_{ij} and F_j to fulfill this objective. The key idea is to use the equality in (2.26a) to express N_{ij} such a way that the solution of the stability condition can be used to obtain the unknown parameters of the observer. Depending on the transformation of the equality in (2.26a), following subsections provide two different approaches of synthesising the functional observer based PDC controller.

2.4.1 Synthesising functional observer based controller: Approach I

Without loss of generality we assume that output matrix C has full row rank. Observer size can be reduced considering the rank of output matrix. A nonsingular matrix can be constructed as $P = [C^+ \; C^\perp]$, where $CC^+ = I_p$ and $CC^\perp = 0_{p\times n-p}$. This implies $CP = \begin{bmatrix} I_p & 0 \end{bmatrix}$. Consider $K_jP = \begin{bmatrix} K_{j1} & K_{j2} \end{bmatrix}$, $T_jP = \begin{bmatrix} T_{j1} & T_{j2} \end{bmatrix}$ and $P^{-1}A_iP = \begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix}$, where $T_{j1} = K_{j1} - F_j$, $T_{j2} = K_{j2}$. By post-multiplying (2.26a) by P, it can be converted into the following two equations:

$$T_{j1}A_{i11} + T_{j2}A_{i21} - N_{ij}T_{j1} = J_{ij} \tag{2.29a}$$

$$N_{ij}T_{j2} - T_{j1}A_{i12} - T_{j2}A_{i22} = 0. \tag{2.29b}$$

The observer parameters can be found by solving (2.29b) such that N_{ij} ensures stability of the observer to guarantee that the estimation error approaches zero asymptotically. The following theorem states the stability condition for the observers in the form of LMIs.

Theorem 2.4.1. *The functional observer described in* (2.22) *is asymptotically stable if there exist a positive definite symmetric matrix X and matrices Y_j of appropriate dimensions such that*

$$XN_{ij1} + N_{ij1}^TX - Y_jN_{ij2} - N_{ij2}^TY_j^T < 0 \tag{2.30a}$$

$$Y_j\Omega_{ij} - X\Phi_{ij} = 0 \tag{2.30b}$$

$$\textit{for all } i, j,$$

where

$$N_{ij1} = K_{j1}A_{i12}K_{j2}^+ + K_{j2}A_{i22}K_{j2}^+, \qquad N_{ij2} = A_{i12}K_{j2}^+,$$

$$\Phi_{ij} = K_{j1}A_{i12}K_{j2}^{\perp} + K_{j2}A_{i22}K_{j2}^{\perp}, \qquad \Omega_{ij} = A_{i12}K_{j2}^{\perp} \textit{ and}$$
$$Y_j = XF_j.$$

Proof. The identity in (2.29b) can be equivalently written as

$$(N_{ij}T_{j2} - T_{j1}A_{i12} - T_{j2}A_{i22}) \begin{bmatrix} K_{j2}^{+} & K_{j2}^{\perp} \end{bmatrix} = 0 \tag{2.31}$$

since $\begin{bmatrix} K_{j2}^{+} & K_{j2}^{\perp} \end{bmatrix}$ is of full-rank. After some algebraic manipulation (2.31) can be converted into two equations:

$$N_{ij} = T_{j1}A_{i12}K_{j2}^{+} + K_{j2}A_{i22}K_{j2}^{+} \tag{2.32a}$$
$$T_{j1}A_{i12}K_{j2}^{\perp} = -K_{j2}A_{i22}K_{j2}^{\perp}. \tag{2.32b}$$

Considering $\Phi_{ij} = K_{j1}A_{i12}K_{j2}^{\perp}+K_{j2}A_{i22}K_{j2}^{\perp}$, $\Omega_{ij} = A_{i12}K_{j2}^{\perp}$, $N_{ij1} = K_{j1}A_{i12}K_{j2}^{+}+ K_{j2}A_{i22}K_{j2}^{+}$ and $N_{ij2} = A_{i12}K_{j2}^{+}$, (2.32a) and (2.32b) can be reformulated as

$$N_{ij} = N_{ij1} - F_jN_{ij2} \tag{2.33a}$$
$$F_j\Omega_{ij} = \Phi_{ij}, \tag{2.33b}$$

respectively. Equation (2.33a) is a familiar observer equation where F_j can be obtained such that N_{ij} is stable. Recall that stability conditions of the T-S fuzzy model are independent of the stability conditions of the proposed functional observer, which allows us to obtain N_{ij} and K_j independently. Therefore, we consider a Lyapunov function $V = e_j^T(t)Xe_j(t)$ for error dynamics (2.25) of the observer, where X is a symmetric positive definite matrix.

Consideration of $X > 0$ implies that

$$V > 0.$$

The derivative of V is

$$\begin{aligned}\dot{V} &= \dot{e_j}^T(t)Xe_j(t) + e_j^T(t)X\dot{e_j}(t) \\ &= \sum_{i=1}^{r} \mu_i(\xi(t))(N_{ij}e_j(t))^TXe_j(t) + e_j^T(t)X\sum_{i=1}^{r} \mu_i(\xi(t))(N_{ij}e_j(t))\end{aligned}$$

$$= \sum_{i=1}^{r} \mu_i(\xi(t)) e_j^T(t)(N_{ij}^T X + X N_{ij}) e_j(t).$$

Using $N_{ij} = N_{ij1} - F_j N_{ij2}$, we obtain

$$\dot{V} = \sum_{i=1}^{r} \mu_i(\xi(t)) e_j^T(t)(X N_{ij1} + N_{ij1}^T X - Y_j N_{ij2} - N_{ij2}^T Y_j^T) e_j(t), \tag{2.34}$$

where $Y_j = X F_j$. From (2.34) it is evident that $\dot{V} < 0$ if (2.30a) holds, and the error dynamics will be asymptomatically zero. The identity of (2.30b) can be obtained from (2.33b) by multiplying X on its both sides.

It requires to show that the error dynamics of the observer can be expressed as (2.25). If (2.33a) is used to obtain N_{ij} using F_j that satisfies (2.30a) and (2.30b), (2.29b) is guaranteed to hold. Condition (2.29a) can always be ensured to hold if J_{ij} is calculated by using N_{ij} and T_{j1}, which are obtained by using F_j. Therefore, (2.26a) holds. Finally, calculation of H_{ij} using (2.26b) implies that this condition also holds, and the error dynamics can be expressed in the form of (2.25). $\square$

Corollary 2.4.1. *A necessary condition for the existence of the functional observer is*

$$\operatorname{rank} \begin{bmatrix} \begin{bmatrix} K_{j1} & K_{j2} \end{bmatrix} \begin{bmatrix} \mathcal{A}_{12} \\ \mathcal{A}_{22} \end{bmatrix} \mathcal{K}_j \\ \mathcal{A}_{12}\mathcal{K}_j \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mathcal{A}_{12}\mathcal{K}_j \end{bmatrix} \tag{2.35}$$

where $\mathcal{A}_{12} = \begin{bmatrix} A_{112} & A_{212} & \dots & A_{r12} \end{bmatrix}$, $\mathcal{A}_{22} = \begin{bmatrix} A_{122} & A_{222} & \dots & A_{r22} \end{bmatrix}$ *and* $\mathcal{K}_j = \operatorname{diag}(\begin{bmatrix} K_{j2}^{\perp} & K_{j2}^{\perp} & \dots & K_{j2}^{\perp} \end{bmatrix})$.

Proof. The identity (2.30b) holds if (2.33b) holds. By definition,

$$\begin{aligned} \Phi_{ij} &= K_{j1} A_{i12} K_{j2}^{\perp} + K_{j2} A_{i22} K_{j2}^{\perp} \\ &= \begin{bmatrix} K_{j1} & K_{j2} \end{bmatrix} \begin{bmatrix} A_{i12} \\ A_{i22} \end{bmatrix} K_{j2}^{\perp}. \end{aligned}$$

Therefore, (2.33b) can be rewritten as

$$F_j A_{i12} K_{j2}^{\perp} = \begin{bmatrix} K_{j1} & K_{j2} \end{bmatrix} \begin{bmatrix} A_{i12} \\ A_{i22} \end{bmatrix} K_{j2}^{\perp}, \quad i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, r. \tag{2.36}$$

Considering all i for each j, (2.36) can be rewritten as

$$F_j \mathcal{A}_{12} \mathcal{K}_j = \begin{bmatrix} K_{j1} & K_{j2} \end{bmatrix} \begin{bmatrix} \mathcal{A}_{12} \\ \mathcal{A}_{22} \end{bmatrix} \mathcal{K}_j, \tag{2.37}$$

where $\mathcal{A}_{12}$, $\mathcal{A}_{22}$ and $\mathcal{K}_j$ are defined in (2.35). The equality conditions in (2.33b) is equivalent to (2.37). In (2.37), the only unknown is F_j. Therefore, the equality condition in (2.33b) requires the existence of solution of F_j in (2.37). A necessary and sufficient condition for existence of solution of F_j is [83]

$$\operatorname{rank} \begin{bmatrix} \begin{bmatrix} K_{j1} & K_{j2} \end{bmatrix} \begin{bmatrix} \mathcal{A}_{12} \\ \mathcal{A}_{22} \end{bmatrix} \mathcal{K}_j \\ \mathcal{A}_{12} \mathcal{K}_j \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mathcal{A}_{12} \mathcal{K}_j \end{bmatrix}. \tag{2.38}$$

This rank equality ensures the solution of (2.33b), i.e. the equality condition, but it does not necessarily ensure the inequality condition of (2.33a) will have a solution. Therefore, this is a necessary condition for existence of the functional observer. □

Remark 2.4.2. *Inspection of* (2.33a) *reveals that F_j can be obtained such that N_{ij} is Hurwitz by assigning suitable poles with negative real parts, and equality constraints in* (2.33b) *hold. In this approach general solution of* (2.33b) *can be obtained and* (2.33a) *can be reformulated using this general solution. Then, F_j can be obtained such that N_{ij} is Hurwitz for each rule to ensure that each subsystem of the T-S fuzzy model is stable [79]. However, it will require to check whether the whole system is stable.*

Remark 2.4.3. *The equality condition in* (2.30b) *can be modified as minimising problem [84]. The whole set of feasibility problem can be converted into a minimising problem as*

$$\underset{X,Y_1,Y_2,\ldots,Y_r}{\text{minimise}}\ \delta$$

subject to

$$\begin{aligned} XN_{ij1}+N_{ij1}^TX-Y_jN_{ij2}-N_{ij2}^TY_j^T &< 0,\ \textit{for all } i,j, \\ \begin{bmatrix} \delta I & Y_j\Omega_i-X\Phi_i \\ (Y_j\Omega_i-X\Phi_i)^T & \delta I \end{bmatrix} &\geq 0,\ \textit{for all } i,j, \end{aligned} \tag{2.39}$$

where X is a symmetric positive definite matrix. By minimising δ to zero observer parameters can be obtained.

Remark 2.4.4. *When K_{j2} is a square matrix and* $\operatorname{rank}(K_{j2})=m$*, we can obtain*

$$N_{ij}=T_{j1}A_{i12}K_{j2}^{-1}+K_{j2}A_{i22}K_{j2}^{-1} \tag{2.40}$$

by post multiplying (2.29b) *by K_{j2}^{-1} . This equation can be expressed as* (2.33a) *if we consider $N_{ij1}=K_{j1}A_{i12}K_{j2}^{-1}+K_{j2}A_{i22}K_{j2}^{-1}$ and $N_{ij2}=A_{i12}K_{j2}^{-1}$. As a result, the equality constraints will be eliminated. Therefore, we can obtain F_j by solving* (2.30a) *only.*

Remark 2.4.5. *As finding existence conditions and designing a functional observer based fuzzy controller of a T-S model are the primary objectives, this work considers the simplest form of quadratic Lyapunov function with its first-order derivative to construct the stability condition. However, higher order derivative of Lyapunov functions, and other kinds of Lyapunov functions, such as fuzzy Lyapunov function and piece-wise Lyapunov function can be explored to find the controller and consequently the observer parameters to increase the solution domain by reducing conservativeness. Stability conditions in LMI form can be modified further to reduce conservativeness.*

Functional observer based controller design procedure: Approach I

Step 1: Obtain K_j by using (2.21). Determine A_{i11}, A_{i12}, A_{i21}, A_{i22}, K_{j1} and K_{j2} from their definitions;

Step 2: Find F_j by solving LMIs in (2.30). Calculate N_{ij} from (2.33a);

Step 3: Calculate T_{j1}, and T_{j2} and find T_j by using its definition in (2.29);

Step 4: Find J_{ij} from (2.29a); and

Step 5: Obtain H_{ij} from (2.26b).

2.4.2 Synthesising functional observer based controller: Approach II

In this approach (2.26a) is treated in a different way. The following theorem gives stability conditions by which functional observer based controller can be obtained. Same control gain K_j obtained from (2.21) is used for calculating the observer parameters.

Theorem 2.4.2. *The observer defined in* (2.22) *is asymptotically stable if there exist a positive definite symmetric matrix X and matrices Y_{1j} and Y_{2ij} of appropriate dimensions such that*

$$X\Xi_{ij} + \Xi_{ij}^T X - Y_{1j}\Psi_{ij} - \Psi_{ij}^T Y_{1j}^T - Y_{2ij}\Gamma_j - \Gamma_j^T Y_{2ij} < 0 \tag{2.41a}$$

$$Y_{1j}C\bar{A}_{ij} + Y_{2ij}\bar{C}_j - XK_j\bar{A}_{ij} = 0 \tag{2.41b}$$

for all i, j,

where

$$\begin{aligned} Y_{1j} &= XF_j, & Y_{2ij} &= XM_{ij}, \\ M_{ij} &= J_{ij} - N_{ij}F_j, & \Xi_{ij} &= K_jA_iK_j^+, \\ \Psi_{ij} &= CA_iK_j^+, & \Gamma_j &= CK_j^+, \\ \bar{A}_{ij} &= A_i(I - K_j^+K_j), & \bar{C}_j &= C(I - K_j^+K_j). \end{aligned}$$

Proof. Considering K_j^+ the Moore-Penrose generalised inverse of K_j, (2.26a) can be equivalently written as

$$(T_jA_i - N_{ij}T_j - J_{ij}C)\begin{bmatrix} K_j^+ & I - K_j^+K_j \end{bmatrix} = 0 \tag{2.42}$$

since $\begin{bmatrix} K_j^+ & I - K_j^+ K_j \end{bmatrix}$ is a full row rank matrix. Using the definition $T_j = K_j - F_j C$, (2.42) can be expressed as a set of two equations:

$$(K_j - F_jC)A_iK_j^+ - N_{ij}(K_j - F_jC)K_j^+ - J_{ij}CK_j^+ = 0 \tag{2.43a}$$

$$(K_j - F_jC)A_i(I - K_j^+K_j) - N_{ij}(K_j - F_jC)(I - K_j^+K_j) \\ -J_{ij}C(I - K_j^+K_j) = 0. \tag{2.43b}$$

Considering $K_j^+K_j = I$, (2.43a) and (2.43b) can be converted into

$$N_{ij} = \Xi_{ij} - F_j\Psi_{ij} - M_{ij}\Gamma_j \tag{2.44a}$$

$$0 = F_jC\bar{A}_{ij} + M_{ij}\bar{C}_j - K_j\bar{A}_{ij}, \tag{2.44b}$$

where $\Xi_{ij} = K_jA_iK_j^+$, $\Psi_{ij} = CA_iK_j^+$, $\Gamma_j = CK_j^+$ and $M_{ij} = J_{ij} - N_{ij}F_j$. Consider $V = e_j^T(t)Xe_j(t)$ as the Lyapunov function for error dynamics (2.25) such that X is a positive definite symmetric matrix. When $X > 0$, we have $V > 0$.

The derivative of V is

$$\begin{aligned}
\dot{V} &= \dot{e_j}^T(t)Xe_j(t) + e_j^T(t)X\dot{e_j}(t) \\
&= \sum_{i=1}^{r} \mu_i(\xi(t))(N_{ij}e_j(t))^T Xe_j(t) + e_j^T(t)X\sum_{i=1}^{r} \mu_i(\xi(t))(N_{ij}e_j(t)) \\
&= \sum_{i=1}^{r} \mu_i(\xi(t))e_j^T(t)(N_{ij}^T X + XN_{ij})e_j(t).
\end{aligned}$$

Applying $N_{ij} = \Xi_{ij} - F_j\Psi_{ij} - M_{ij}\Gamma_j$, we obtain

$$\begin{aligned}
\dot{V} = \sum_{i=1}^{r} \mu_i(\xi(t))e_j^T(t)(X\Xi_{ij} + \Xi_{ij}^T X - Y_{1j}\Psi_{ij} - \Psi_{ij}^T Y_{1j}^T \\
- Y_{2ij}\Gamma_j - \Gamma_j^T Y_{2ij})e_j(t),
\end{aligned} \tag{2.45}$$

where $Y_{1j} = XF_j$ and $Y_{2ij} = XM_{ij}$. From (2.45) it is evident that $\dot{V} < 0$ if (2.41a) holds, and the error dynamics is asymptomatically stable. Furthermore, (2.41b) can be obtained from (2.44b) by pre-multiplying it by X.

It requires to show that the error dynamics of the observer can be expressed as (2.25). If (2.44a) is used to obtain N_{ij} from the solutions of F_j and M_{ij}, which are obtained from the solutions of the conditions in (2.41a) and (2.41b), (2.26a) eventually holds. H_{ij} can be calculated by using (2.26b) to ensure that this condition holds. Therefore, it is ensured that the error dynamics can be expressed in the form of (2.25). □

Remark 2.4.6. *Using similar line of proof in Corollary 2.4.1 it can be shown that the controller exists if the following condition holds:*

$$\operatorname{rank}\begin{bmatrix} K_j\bar{\mathcal{A}}_j \\ C\bar{\mathcal{A}}_j \\ \bar{\mathcal{C}}_j \end{bmatrix} = \operatorname{rank}\begin{bmatrix} C\bar{\mathcal{A}}_j \\ \bar{\mathcal{C}}_j \end{bmatrix}, \tag{2.46}$$

where $\bar{\mathcal{A}}_j = \begin{bmatrix} \bar{A}_{1j} & \bar{A}_{2j} & \dots & \bar{A}_{rj} \end{bmatrix}$, $\mathcal{M}_j = \begin{bmatrix} M_{1j} & M_{2j} & \dots & M_{rj} \end{bmatrix}$ *and*

$$\bar{\mathcal{C}}_j = \begin{bmatrix} \bar{C}_j & 0 & \dots & 0 \\ 0 & \bar{C}_j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{C}_j \end{bmatrix}.$$

Design steps of the functional observer based controller using Approach II are described below.

Functional observer based controller design procedure: Approach II

Step 1: Obtain K_j from (2.21). Calculate Ξ_{ij}, Ψ_{ij}, Γ_j, $\bar{A}_{ij}$ and $\bar{C}_j$;

Step 2: Find F_j and M_{ij} by solving LMIs with the equality conditions in (2.41);

Step 3: Calculate N_{ij} from (2.44a);

Step 4: Find J_{ij} from the relation $M_{ij} = J_{ij} - N_{ij}F_j$; and

Step 5: Obtain H_{ij} from (2.26b).

Remark 2.4.7. *The main difficulty of obtaining functional observer is to ensure that the error dynamics in* (2.25) *converges while the conditions in* (2.26a) *and* (2.26b) *hold. The identity of* (2.26b) *is used directly to obtain* H_{ij}*. However,* (2.26a) *requires some transformation so that* N_{ij} *can be expressed such that the calculation of* N_{ij} *ensures* (2.26a) *holds. The two approaches, described in Subsections 2.4.1 and 2.4.2, basically differ from each other on the methods of transformations of condition* (2.26a)*. In Approach I, the advantage of full row rank of matrix* C *is utilised. An invertible matrix* $P = \begin{bmatrix} C^+ & C^\perp \end{bmatrix}$ *is constructed, and* (2.26a) *is converted into two equations as given in* (2.29a) *and* (2.29b)*. It requires some numerical calculation for finding pseudo inverse and orthogonal basis of* C*. Afterwards,* (2.29b) *is again transformed into two sets of equations in* (2.32a) *and* (2.32b)*. Approach II, on the other hand, does not use the transformation of* (2.26a) *using* P*. It transforms* (2.26a) *using a full row rank matrix* $\begin{bmatrix} K_j^+ & I - K_j^+ K_j \end{bmatrix}$*, and obtains two sets of equations* (2.43a) *and* (2.43b)*, which are then used to construct the stability conditions. Therefore, each of the approaches has its own numerical characteristics. Approach I is convenient when output matrix* C *is in the canonical form, such as* $\begin{bmatrix} I_p & 0 \end{bmatrix}$*, and the whole procedure is less error prone. More importantly, Approach II uses one set of transformation that requires numerical calculation of pseudo inverse of* K_j*, whereas Approach I uses two sets of transformations that requires calculation of pseudo inverse and orthogonal basis of* C *and* K_{j2}*. Therefore, Approach I is convenient to use when* C *is in the canonical form.*

Remark 2.4.8. *The proposed method of obtaining functional observer based PDC controllers for T-S fuzzy systems in this work is more convenient than the existing methods presented in [60] and [79]. The method proposed in [60] requires to solve number of equations to obtain the observer parameters, and then to check the stability of the whole system. The method in [79], on the other hand, uses transformation of* (2.26a) *and obtains* N_{ij} *such that it is Hurwitz to ensure local stability of each subsystem of the error dynamics, and then uses the stability condition for checking the stability of the whole system. If the whole system is not stable, another set of observer parameters have to be calculated and the stability of the whole system has to be checked again. To overcome this difficulty, the proposed procedure uses LMIs as the stability conditions such that the observer parameters can be obtained solving these LMIs. Therefore, the proposed method* *is more efficient than the existing methods, because N_{ij} are calculated to ensure overall stability of the observer.*

Example 2.2

This example demonstrates the proposed method of synthesising functional observer and to verify the effectiveness of the method. LMITOOLBOX and MATLAB are used to solve the LMIs and to simulate the systems. A two-rule based T-S fuzzy system is considered to illustrate the functional observer based PDC controller construction procedure. System data is as given below:

$$A_1 = \begin{bmatrix} -0.2 & 7 & -1 & 1 \\ -1 & -8 & -1 & 1 \\ -2 & 0 & -1 & 1 \\ -1 & 1.1 & 2 & -10 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 6 & -1.1 & 2 \\ -2 & -6 & -1.1 & 2 \\ -1 & 0 & -1.1 & 2 \\ 1 & 1.1 & 2 & -10.5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.1 \\ -0.2 \\ 0.5 \\ 0.2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.15 \\ -0.30 \\ 0.40 \\ 0.25 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The membership functions of the T-S fuzzy model are as below:

$$\mu_1(\xi(t)) = \left\{1 - \frac{1}{1 + e^{-7(x_1(t) - \pi/6}}\right\} \frac{1}{1 + e^{-7(x_1(t) + \pi/6}},$$
$$\mu_2(\xi(t)) = 1 - \mu_1(\xi(t)).$$

In this system, states $x_1(t)$ and $x_3(t)$ are measurable. Therefore, we need to estimate the states to obtain the conventional observer based PDC controller. However, considering the fact that the PDC controller is the fuzzy summation of the state feedback controllers of the linear subsystems of the T-S fuzzy model, we construct functional observer to estimate the control signal directly. Considering the structure of output matrix C, we can apply Approach I. We find invertible transformation matrix P and PDC control gain matrices K_1 and K_2 as below:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{aligned} K_1 &= \begin{bmatrix} 22.0295 & 33.8170 & 4.8844 & -14.0792 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} 25.8307 & 54.1962 & 9.7697 & -31.1781 \end{bmatrix}. \end{aligned}$$

Using transformation matrix P we get the following matrices:

$$\begin{aligned}
A_{111} &= \begin{bmatrix} -0.2 & -1 \\ -2 & -1 \end{bmatrix}, & A_{112} &= \begin{bmatrix} 7 & 1 \\ 0 & 1 \end{bmatrix}, \\
A_{121} &= \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}, & A_{122} &= \begin{bmatrix} -8 & 1 \\ 1.1 & -10 \end{bmatrix}, \\
A_{211} &= \begin{bmatrix} -1 & -1.1 \\ -1 & -1.1 \end{bmatrix}, & A_{212} &= \begin{bmatrix} 6 & 2 \\ 0 & 2 \end{bmatrix}, \\
A_{221} &= \begin{bmatrix} -2 & -1.1 \\ 1 & 2 \end{bmatrix}, & A_{222} &= \begin{bmatrix} -6 & 2 \\ 1.1 & -10.5 \end{bmatrix}, \\
K_{11} &= \begin{bmatrix} 22.0295 & 4.8844 \end{bmatrix}, & K_{12} &= \begin{bmatrix} 33.8170 & -14.0792 \end{bmatrix}, \\
K_{21} &= \begin{bmatrix} 25.8307 & 9.7697 \end{bmatrix}, & K_{22} &= \begin{bmatrix} 54.1962 & -31.1781 \end{bmatrix}.
\end{aligned}$$

After solving the LMIs with the equality conditions in (2.30), we obtain the following parameters for the functional observer:

$$\begin{aligned}
F_1 &= \begin{bmatrix} 17.9800 & 76.2628 \end{bmatrix}, & F_2 &= \begin{bmatrix} 18.1564 & 145.1705 \end{bmatrix}, \\
N_{11} &= -7.6197, & N_{12} &= -7.6416, \\
N_{21} &= -5.7395, & N_{22} &= -5.7832, \\
J_{11} &= \begin{bmatrix} 153.0648 & -538.5316 \end{bmatrix}, & J_{12} &= \begin{bmatrix} 304.8924 & -1023.505 \end{bmatrix}, \\
J_{21} &= \begin{bmatrix} 8.8576 & -400.9709 \end{bmatrix}, & J_{22} &= \begin{bmatrix} 32.5379 & -764.5234 \end{bmatrix}, \\
H_{11} &= -44.8635, & H_{12} &= -84.0079, \\
H_{21} &= -41.6088, & H_{22} &= -77.0626.
\end{aligned}$$

The time response of the system states of the closed loop system with the

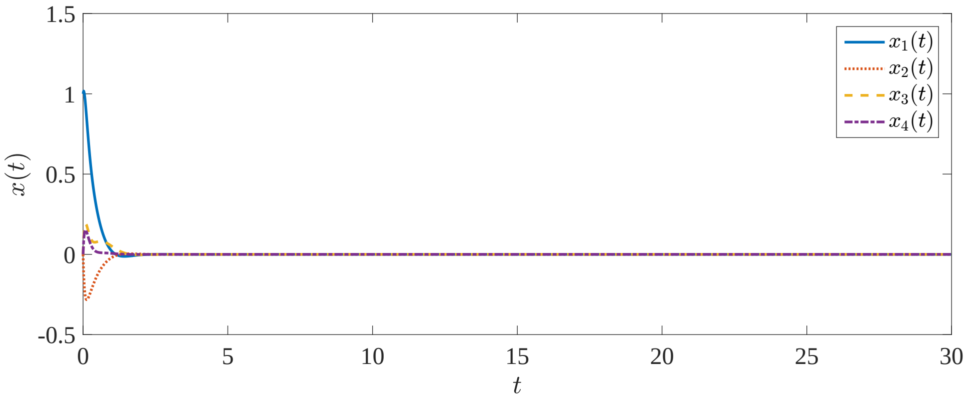

Figure 2.2: States of the system with proposed functional observer based controller

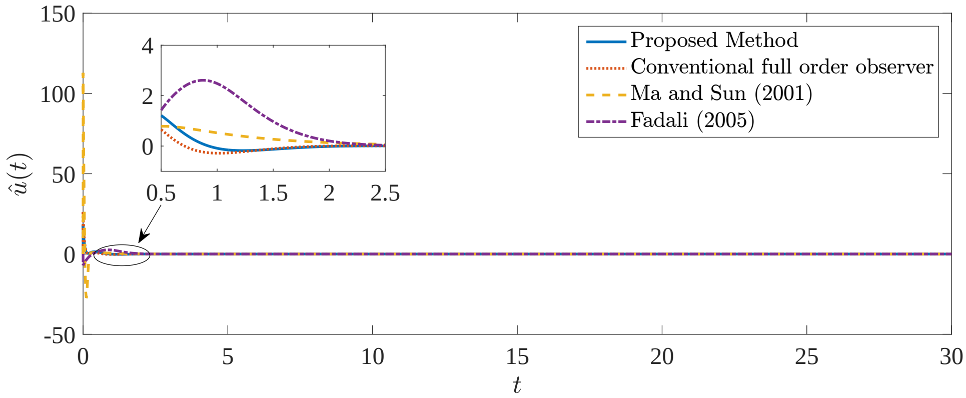

Figure 2.3: Estimated control signal, $\hat{u}(t)$, using different methods

functional observer based controller is displayed in Figure 2.2. The closed loop system is stabilised asymptotically. It should be noted that, the main objective of the functional observer is to estimate the control signal asymptotically, where the control signal stabilises the system. The performance of the proposed procedure for obtaining observer parameters can be compared with the procedures proposed in [60, 79] by comparing the estimated control signal with the conventional full order observer based PDC controller. Figure 2.3 depicts the control signal estimated by different methods with the conventional full state observer based PDC controller. It can be witnessed that the proposed method estimates the control signal effectively to stabilise the system.

2.5 Conclusion

The fuzzy functional observer is obtained by applying the fuzzy summation to the linear functional observers of linear subsystems of a T-S fuzzy system. This observer can estimate the function of states of nonlinear systems expressed by T-S fuzzy models. Considering the PDC controller as a fuzzy summation of linear functions of states of a T-S fuzzy model, a fuzzy functional observer has been employed to stabilise the system asymptotically. The proposed functional observer based PDC controller reduces the observer size and directly obtains the control vector by treating the control vector as a function rather than doing it in two steps: estimating the states and computing the function of estimated states. Stability conditions are obtained using a quadratic Lyapunov function. LMIs in the stability conditions can be solved by readily available tools and observer parameters can be obtained.

Chapter 3

Functional observer based controller for T-S fuzzy systems with time-delay

This chapter investigates the construction of a fuzzy functional observer for nonlinear systems with time-delays, and the application of the observer to estimate the state functions of the PDC controller for stabilising the system. Two types of time-delays are considered: constant and time-varying delays. The time-varying time-delay is bounded by upper and lower bounds, and its time-derivative is bounded above. Stability conditions are obtained using the Lyapunov-Krasovskii functional approach; and the conditions are transformed into linear matrix inequalities. Functional observer construction procedures are presented considering both constant and time-varying time-delays. The proposed methods are illustrated and verified using an example. The main results of this chapter are published in [85]

3.1 Introduction

As time-delays generally cause instability of dynamical systems, designing a controller for a system with delays has been an attractive research area. Time-delay can be constant or time varying. If the time-delay is time varying the stability conditions can be obtained using the upper bound of the time-derivative of the time delay. Depending on the nature of the time-delay and the choice of Lyapunov-Krasovskii functional to analyze the stability, stability conditions may appear in different forms, which can be classified into two broad categories: delay dependent conditions; and delay independent conditions [86].

The functional observer and unknown-input observer for linear systems with time-delay are studied, and observer construction procedures are proposed in [87–89]. In [90], the authors investigated the existence of delay free low order observers using the concept of designing a functional observer. In [73, 91], the existence of the linear functional observer with time-delay has been investigated, and necessary and sufficient conditions for the observer have been proposed in the form of rank equality. In [72], the authors presented a functional observer construction procedure considering multiple time-varying time-delays of interconnected systems; and provided an LMI based stability condition using Lyapunov-Krasovskii approach. The time-delays were considered to include the internal state delay as well as the delays introduced due to the communication between the subsystems. Although the functional observer construction problem with time-delays for linear systems has been investigated in many works, its application for nonlinear systems with time-delay is still an open field of research.

To the author's knowledge, the controller design for nonlinear time-delay systems using functional observers has not been investigated fully. This chapter presents the existence of the functional observer based PDC controller for T-S fuzzy systems with time-delay. Using Lyapunov-Krasovskii functional, stability conditions of the fuzzy functional observer are obtained so that estimation error converges to zero asymptotically. Considering constant type delay and bounded time varying delay, stability conditions are formulated in terms of LMIs separately.

3.2 Model description and problem formulation

Consider a continuous time T-S fuzzy model of a nonlinear system with time-delay with r number of rules. In this work, we consider time-delay in the state. The ith rule of this model is

$$\begin{aligned}
&\text{IF} && \xi_1(t) \text{ is } M_i^1 \text{ and } \cdots \text{ and } \xi_l(t) \text{ is } M_i^l \\
&\text{THEN} && \dot{x}(t) = A_i x(t) + A_{di} x(t-\tau) + B_i u(t) \\
& && y(t) = Cx(t) \\
& && x(t) = \phi(t), t \in [-\tau, 0], i = 1, \ldots, r,
\end{aligned} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ is the output vector, $\xi_1(t), \ldots, \xi_l(t)$ are premise variables, and $0 < \tau < \infty$ is the time delay. Real matrices $A_i \in \mathbb{R}^{n\times n}$, $A_{di} \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are the system matrices to represent the ith state space model of the system. M_i^k is the fuzzy set for $\xi_k(t)$ in ith rule. Considering $\xi(t) = \left[\xi_1(t), \ldots, \xi_l(t)\right]$, the overall dynamics can be represented by

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + A_{di} x(t-\tau) + B_i u(t)\}, \\
y(t) &= Cx(t),
\end{aligned} \tag{3.2}$$

where

$$\mu_i(\xi(t)) = \frac{\prod_{k=1}^{l} M_i^k(\xi_k(t))}{\sum_{i=1}^{r}\prod_{k=1}^{l} M_i^k(\xi_k(t))}$$

$$\text{with } \mu_i(\xi(t)) \geq 0 \ \text{ and } \sum_{i=1}^{r} \mu_i(\xi(t)) = 1.$$

Each rule in (3.1) represents a linear model of respective subsystem. Using same premise variable vector $\xi(t)$, PDC control vector $u(t)$ can be designed to stabilise the fuzzy system as [42]

$$\begin{aligned}
u(t) &= \sum_{j=1}^{r} \mu_j(\xi(t))\{u_j(t)\} \\
&= \sum_{j=1}^{r} \mu_j(\xi(t))\{K_j x(t)\},
\end{aligned} \tag{3.3}$$

where $u_j(t) = K_j x(t)$. Considering the constant type time-delay, K_j can be obtained by solving the following stability condition of the system [42]:

$$\begin{bmatrix} A_i X + X A_i^T + B_i \bar{Y}_j + \bar{Y}_j^T B_i + A_{di} W_1 A_{di}^T & X \\ \star & -W \end{bmatrix} < 0 \tag{3.4}$$

$$\text{for } i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, r,$$

where $X = X^T > 0$, $W = W^T > 0$ and $\bar{Y}_j = K_j X$. By the definition, $u_j(t)$ is a linear function of states. Our goal is to obtain the control vector $u_j(t)$ for each rule using a linear functional observer when all or some of the states are not accessible. We apply the PDC concept to design the fuzzy functional observer of the T-S fuzzy model. Using the same degree of firing strength $\mu_i(\xi(t))$ of the ith rule and time-delay τ, we propose the following functional observer to obtain the control signal:

$$\begin{aligned} \dot{w}_j(t) &= \sum_{i=1}^{r} \mu_i(\xi(t)) \{ N_{ij} w_j(t) + N_{dij} w_j(t-\tau) \\ &\qquad + J_{ij} y(t) + J_{dij} y(t-\tau) + H_{ij} \hat{u}(t) \} \\ \hat{u}_j(t) &= w_j(t) + F_j y(t) \\ \hat{u}(t) &= \sum_{j=1}^{r} \mu_j(\xi(t)) \{ \hat{u}_j(t) \} \\ w_j(t) &= 0 \quad \text{ for all } t \in [-\tau, 0], \end{aligned} \tag{3.5}$$

where $w_j(t) \in \mathbb{R}^m$, $F_j \in \mathbb{R}^{m \times p}$, $N_{ij} \in \mathbb{R}^{m \times m}$, $N_{dij} \in \mathbb{R}^{m \times m}$, $J_{ij} \in \mathbb{R}^{m \times p}$, $J_{dij} \in \mathbb{R}^{m \times p}$, and $H_{ij} \in \mathbb{R}^{m \times m}$. Considering the fact that $u_j(t)$ is the linear function of states $x(t)$, and $\hat{u}_j(t)$ is the estimated function of states, the estimation error can be expressed as

$$\begin{aligned} e_j(t) &= u_j(t) - \hat{u}_j(t) \\ &= T_j x(t) - w_j(t), \end{aligned}$$

where $T_j = K_j - F_j C$. Consequently, the error dynamics of the functional observer is

$$
\begin{aligned}
\dot{e}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{ & N_{ij}e_j(t) + N_{dij}e_j(t-\tau) \\
& + (T_jA_i - N_{ij}T_j - J_{ij}C)x(t) \\
& + (T_jA_{di} - N_{dij}T_j - J_{dij}C)x(t-\tau) \\
& + (T_jB_i - H_{ij})\hat{u}(t)\}.
\end{aligned} \tag{3.6}
$$

The dynamics in (3.6) can be simplified as

$$
\dot{e}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{N_{ij}e_j(t) + N_{dij}e_j(t-\tau)\} \tag{3.7}
$$

if the following conditions hold:

$$
T_jA_i - N_{ij}T_j - J_{ij}C = 0, \tag{3.8a}
$$

$$
T_jA_{di} - N_{dij}T_j - J_{dij}C = 0, \tag{3.8b}
$$

$$
T_jB_i - H_{ij} = 0. \tag{3.8c}
$$

Therefore, our functional observer based controller construction problem is now converted into the problem of finding observer parameters such that the error dynamics in (3.7) approaches zero while the conditions in (3.8a), (3.8b) and (3.8c) hold.

3.3 Stability conditions

Two types of time-delays are considered in this work: constant type time-delay, and time-varying interval type time-delay with upper and lower bounds. The following subsections deal with obtaining the stability conditions for the observer considering these two types of time-delays individually.

3.3.1 Constant time-delay case

We need to design N_{ij} and N_{dij} such that the error system described in (3.7) asymptotically stable, and the conditions in (3.8a) (3.8b) and (3.8c) hold. To achieve this objective, we obtain general expressions of N_{ij} and N_{dij} by using the conditions in (3.8a) and (3.8b), and then establish stability conditions by employing a suitable Lyapunov-Krasovskii functional to guarantee the asymptotic convergence of the error system to zero. The following theorem states the stability condition for the observer.

Theorem 3.3.1. *The functional observer described by* (3.5) *is asymptotically stable if there exist positive definite symmetric matrices P and R, and matrices Y_{1j}, Y_{2ij} and Y_{2dij} of appropriate dimensions such that*

$$\begin{bmatrix} \begin{matrix} P\Xi_{ij} + \Xi_{ij}^T P - Y_{1j}\Psi_{ij} - \Psi_{ij}^T Y_{1j}^T \\ -Y_{2ij}\Gamma_j - \Gamma_j^T Y_{2ij}^T + R \end{matrix} & P\Xi_{dij} - Y_{1j}\Psi_{dij} - Y_{2dij}\Gamma_j \\ \star & -R \end{bmatrix} < 0 \tag{3.9a}$$

$$Y_{1j}\Omega_{ij} + Y_{2ij}\Phi_j - P\Lambda_{ij} = 0 \tag{3.9b}$$

$$Y_{1j}\Omega_{dij} + Y_{2dij}\Phi_j - P\Lambda_{dij} = 0 \tag{3.9c}$$

$$\textit{for } i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, r,$$

where

$$\begin{aligned} &\Xi_{ij} = K_j A_i K_j^+, && \Xi_{dij} = K_j A_{di} K_j^+, \\ &\Psi_{ij} = C A_i K_j^+, && \Psi_{dij} = C A_{di} K_j^+, \\ &M_{ij} = J_{ij} - N_{ij} F_j, && M_{dij} = J_{dij} - N_{dij} F_j, \\ &\Gamma_j = C K_j^+, && \Phi_j = C K_j^\perp, \\ &\Omega_{ij} = C A_i K_j^\perp, && \Omega_{dij} = C A_{di} K_j^\perp, \\ &\Lambda_{ij} = K_j A_i K_j^\perp \ \textit{and} && \Lambda_{dij} = K_j A_{di} K_j^\perp. \end{aligned}$$

Proof. Without the loss of generality, K_j is assumed to be a full row rank matrix. Hence, post multiplying (3.8a) by the full rank matrix $\begin{bmatrix} K_j^+ & K_j^\perp \end{bmatrix}$, we obtain

$$\begin{aligned} (K_j - F_j C) A_i \begin{bmatrix} K_j^+ & K_j^\perp \end{bmatrix} - N_{ij}(K_j - F_j C) \begin{bmatrix} K_j^+ & K_j^\perp \end{bmatrix} \\ - J_{ij} C \begin{bmatrix} K_j^+ & K_j^\perp \end{bmatrix} = 0. \end{aligned} \tag{3.10}$$

After some algebraic manipulation, (3.10) can be rewritten as

$$N_{ij} = \Xi_{ij} - F_j \Psi_{ij} - M_{ij}\Gamma_j \tag{3.11a}$$

$$0 = F_j \Omega_{ij} + M_{ij}\Phi_j - \Lambda_{ij}, \tag{3.11b}$$

where

$$\Xi_{ij} = K_j A_i K_j^+, \qquad \Psi_{ij} = C A_i K_j^+,$$
$$M_{ij} = J_{ij} - N_{ij} F_j, \qquad \Gamma_j = C K_j^+,$$
$$\Omega_{ij} = C A_i K_j^\perp, \qquad \Phi_j = C K_j^\perp \text{ and}$$
$$\Lambda_{ij} = K_j A_i K_j^\perp.$$

Similarly, (3.8b) can be converted into

$$N_{dij} = \Xi_{dij} - F_j \Psi_{dij} - M_{dij} \Gamma_j \tag{3.12a}$$
$$0 = F_j \Omega_{dij} + M_{dij} \Phi_j - \Lambda_{dij}, \tag{3.12b}$$

where

$$\Xi_{dij} = K_j A_{di} K_j^+, \qquad \Psi_{dij} = C A_{di} K_j^+,$$
$$M_{dij} = J_{dij} - N_{dij} F_j, \quad \Omega_{dij} = C A_{di} K_j^\perp \text{ and}$$
$$\Lambda_{dij} = K_j A_{di} K_j^\perp.$$

We define

$$V(e_j(t)) = e_j^T(t) P e_j(t) + \int_{t-\tau}^{t} e_j^T(s) R e_j(s) ds$$

as the Lyapunov function, where $P = P^T$ and $R = R^T$ are positive definite matrices. Taking derivative of $V(e_j(t))$ along the trajectory of the error dynamics, we obtain

$$\begin{aligned}
\dot{V}(e_j(t)) &= 2e_j^T(t) P \dot{e}_j(t) + e_j^T(t) R e_j(t) - e_j^T(t-\tau) R e_j(t-\tau) \\
&= 2e_j^T(t) P \sum_{i=1}^{r} \mu_i(\xi(t)) \{N_{ij} e_j(t) + N_{dij} e_j(t-\tau)\} \\
&\qquad + e_j^T(t) R e_j(t) - e_j^T(t-\tau) R e_j(t-\tau) \\
&= \sum_{i=1}^{r} \mu_i(\xi(t)) \{e_j^T(t)(P N_{ij} + N_{ij}^T P + R) e_j(t) - e_j^T(t-\tau) R e_j(t-\tau) \\
&\qquad + e_j^T(t) P N_{dij} e_j(t-\tau) + e_j^T(t-\tau) N_{dij}^T P e_j(t)\}.
\end{aligned}$$

From the fact that for any matrix Z, the positive definite matrix R, and any vectors x_1 and x_2, we have [42]

$$x_1^T Z x_2 + x_2^T Z^T x_1 \le x_1^T Z R^{-1} Z^T x_1 + x_2^T R x_2, \tag{3.13}$$

we can write

$$\begin{aligned}\dot{V}(e_j(t)) &\leq \sum_{i=1}^{r} \mu_i(\xi(t))\{e_j^T(t)(PN_{ij} + N_{ij}^T P + R)e_j(t) - e_j^T(t-\tau)Re_j(t-\tau) \\ &\quad + e_j^T(t)PN_{dij}R^{-1}N_{dij}^T Pe_j(t) + e_j^T(t-\tau)Re_j(t-\tau)\} \\ &= \sum_{i=1}^{r} \mu_i(\xi(t))\{e_j^T(t)(PN_{ij} + N_{ij}^T P + R + PN_{dij}R^{-1}N_{dij}^T P)e_j(t).\end{aligned}$$

Therefore, if the inequalities

$$PN_{ij} + N_{ij}^T P + R + PN_{dij}R^{-1}N_{dij}^T P < 0 \tag{3.14}$$

hold, we have $\dot{V}(e_j(t)) < 0$, which implies the asymptotic convergence of the error dynamics to zero. By applying the Schur complement, (3.14) can be re-expressed as

$$\begin{bmatrix} PN_{ij} + N_{ij}^T P + R & PN_{dij} \\ \star & -R \end{bmatrix} < 0. \tag{3.15}$$

Using the expressions of N_{ij} and N_{dij} in (3.11a) and (3.12a), respectively, we can rewrite (3.15) as (3.9a) where $Y_{1j} = PF_j$, $Y_{2ij} = PM_{ij}$ and $Y_{2dij} = PM_{dij}$. Multiplying (3.11b) and (3.12b) by P we have (3.9b) and (3.9c), respectively.

To write the error dynamics as (3.7), N_{ij} and N_{dij} are obtained from identities (3.11a) and (3.12a), respectively. J_{ij} and J_{dij} are calculated from the definitions of M_{ij} and M_{dij}, respectively. Finally, H_{ij} can be obtained from (3.8c), and all the requirements for simplifying the error dynamics are fulfilled. □

The functional observer based PDC controller for T-S fuzzy systems with constant time-delays can be constructed by following the procedure outlined below.

Functional observer based PDC controller synthesis for constant time-delay systems

Step 1: Find K_j using (3.4). Calculate Ξ_{ij}, Ξ_{dij}, Ψ_{ij}, Ψ_{dij}, Γ_j, Ω_{ij}, Ω_{dij}, Φ_j, Λ_{ij} and Λ_{dij} as defined in (3.11) and (3.12);

Step 2: Solve the LMIs with the equality constraints of (3.9). Calculate F_j, M_{ij} and M_{dij} from Y_{1j}, Y_{2ij} and Y_{2dij}, respectively;

Step 3: Calculate N_{ij} and N_{dij} from (3.11a) and (3.12a), respectively.

Obtain J_{ij} and J_{dij} from $J_{ij} = M_{ij} + N_{ij}F_j$ and $J_{dij} = M_{dij} + N_{dij}F_j$, respectively; and

Step 4: Find H_{ij} using (3.8c).

3.3.2 Interval type time varying time-delay case

We consider the time varying time-delay with lower and upper limits, and a bounded rate of change. Time-delay τ in (3.7) is assumed to be a continuous time-varying function $\tau = \tau(t)$ that satisfies

$$\tau_M \geq \tau(t) \geq \tau_m \tag{3.16a}$$

$$\dot{\tau}(t) \leq \rho < 1, \tag{3.16b}$$

where τ_M and τ_m are known positive scalars that define upper and lower bounds of the time-delay, and ρ determines the upper bound of the derivative of the time-delay function. PDC controller gain K_j of the T-S fuzzy model with time varying time-delay can be obtained by solving the LMIs [42, 86]

$$\begin{bmatrix} \begin{array}{c} Q_1A_i^T + A_iQ_1 \\ +\tilde{Y}_j^TB_i^T + B_i\tilde{Y}_j + Q_2 \end{array} & A_{di}Q_1 & \beta Q_1A_i^T + \beta\tilde{Y}_j^TB_i^T \\ \star & -(1-\rho)Q_2 & \beta Q_1A_{di}^T \\ \star & \star & -\beta Q_1 \end{bmatrix} < 0 \tag{3.17}$$

for some positive definite symmetric matrices Q_1 and Q_2, and matrices $\tilde{Y}_j$ of appropriate dimensions, where

$$\tilde{Y}_j = K_jQ_1 \quad \text{and } \beta = \tau_M^2\sigma_1 + (\tau_M - \tau_m)^2\sigma_2$$

with given positive scalars σ_1 and σ_2. Our main focus is to design the functional observer such that it estimates the function of states, $u_j = K_jx(t)$, directly. The following theorem gives the stability condition.

Theorem 3.3.2. *The functional observer described by* (3.5) *with interval type time-delay with assumptions in* (3.16) *is asymptotically stable if for given positive scalars ζ_1 and ζ_2 there exist positive definite symmetric matrices P_1 and P_2, and* *matrices Y_{1j}, Y_{2ij} and Y_{2dij} of appropriate dimensions such that*

$$\begin{bmatrix} \begin{array}{c} \Xi_{ij}^T P_1 + P_1 \Xi_{ij} \\ -\Psi_{ij}^T Y_{1j}^T - Y_{1j}\Psi_{ij} \\ -\Gamma_j^T Y_{2ij}^T - Y_{2ij}\Gamma_j \\ -(\zeta_1+\zeta_2)P_1 \\ +P_2 \end{array} & \begin{array}{c} P_1\Xi_{dij} - Y_{1j}\Psi_{dij} \\ -Y_{2dij}\Gamma_j - \zeta_2 P_1 \end{array} & \begin{array}{c} \zeta_1\tau_M(\Xi_{ij}^T P_1 \\ -\Psi_{ij}^T Y_{1j}^T \\ -\Gamma_j^T Y_{2ij}^T) \end{array} & \begin{array}{c} \zeta_2(\tau_M-\tau_m) \\ (\Xi_{ij}^T P_1 - \Psi_{ij}^T Y_{1j}^T \\ -\Gamma_j^T Y_{2ij}^T) \end{array} \\ \star & \begin{array}{c} -(1-\rho)P_2 \\ -2\zeta_2 P_1 \end{array} & \begin{array}{c} \zeta_1\tau_M(\Xi_{dij}^T P_1 \\ -\Psi_{dij}^T Y_{1j} \\ -\Gamma_j^T Y_{2dij}) \end{array} & \begin{array}{c} \zeta_2(\tau_M-\tau_m) \\ (\Xi_{dij}^T P_1 - \Psi_{dij}^T Y_{1j}^T \\ -\Gamma_j^T Y_{2dij}^T) \end{array} \\ \star & \star & -\zeta_1 P_1 & 0 \\ \star & \star & \star & -\zeta_2 P_1 \end{bmatrix} < 0 \tag{3.18a}$$

$$Y_{1j}\Omega_{ij} + Y_{2ij}\Phi_j - P_1\Lambda_{ij} = 0 \tag{3.18b}$$

$$Y_{1j}\Omega_{dij} + Y_{2dij}\Phi_j - P_1\Lambda_{dij} = 0 \tag{3.18c}$$

$$\textit{for } i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, r,$$

where

$$\begin{aligned} \Xi_{ij} &= K_j A_i K_j^+, & \Xi_{dij} &= K_j A_{di} K_j^+, \\ \Psi_{ij} &= C A_i K_j^+, & \Psi_{dij} &= C A_{di} K_j^+, \\ M_{ij} &= J_{ij} - N_{ij} F_j, & M_{dij} &= J_{dij} - N_{dij} F_j, \\ \Gamma_j &= C K_j^+, & \Phi_j &= C K_j^\perp, \\ \Omega_{ij} &= C A_i K_j^\perp, & \Omega_{dij} &= C A_{di} K_j^\perp, \\ \Lambda_{ij} &= K_j A_i K_j^\perp, & \Lambda_{dij} &= K_j A_{di} K_j^\perp. \end{aligned}$$

Proof. Consider a Lyapunov-Krasovskii functional

$$V(e_j(t)) = V_1(e_j(t)) + V_2(e_j(t)) + V_3(e_j(t)) + V_4(e_j(t)),$$

where

$$\begin{aligned}
V_1(e_j(t)) &= e_j^T(t)P_1e_j(t),\\
V_2(e_j(t)) &= \int_{t-\tau(t)}^{t} e_j^T(s)P_2e_j(s)ds,\\
V_3(e_j(t)) &= \tau_M\int_{-\tau_M}^{0}\int_{t+\theta}^{t}\dot{e}_j^T(s)P_3\dot{e}_j(s)dsd\theta,\\
V_4(e_j(t)) &= (\tau_M-\tau_m)\int_{-\tau_M}^{-\tau_m}\int_{t+\theta}^{t}\dot{e}_j^T(s)P_4\dot{e}_j(s)dsd\theta,
\end{aligned}$$

with symmetric positive definite matrices P_1, P_2, P_3 and P_4. By taking the derivative of $V(t)$ along the trajectories of the error dynamics, we obtain

$$\begin{aligned}
\dot{V}(t) &= 2e_j^T(t)P_1\dot{e}_j(t) + e_j^T(t)P_2e_j(t) - (1-\dot{\tau}(t))e_j^T(t-\tau(t))P_2e_j(t-\tau(t))\\
&\quad + \tau_M^2\dot{e}_j^T(t)P_3\dot{e}_j(t) - \tau_M\int_{t-\tau_M}^{t}\dot{e}_j^T(s)P_3\dot{e}_j(s)ds + (\tau_M-\tau_m)^2\dot{e}_j^T(t)P_4\dot{e}_j(t)\\
&\quad - (\tau_M-\tau_m)\int_{t-\tau_M}^{t-\tau_m}\dot{e}_j^T(s)P_4\dot{e}_j(s)ds\\
&= 2e_j^T(t)P_1\dot{e}_j(t) + e_j^T(t)P_2e_j(t) - (1-\dot{\tau}(t))e_j^T(t-\tau(t))P_2e_j(t-\tau(t))\\
&\quad + \dot{e}_j^T(t)(\tau_M^2P_3 + (\tau_M-\tau_m)^2P_4)\dot{e}_j(t) - \tau_M\int_{t-\tau_M}^{t}\dot{e}_j^T(s)P_3\dot{e}_j(s)ds\\
&\quad - (\tau_M-\tau_m)\int_{t-\tau_M}^{t-\tau(t)}\dot{e}_j^T(s)P_4\dot{e}_j(s)ds - (\tau_M-\tau_m)\int_{t-\tau(t)}^{t-\tau_m}\dot{e}_j^T(s)P_4\dot{e}_j(s)ds.
\end{aligned}$$

Applying Jensen's inequality and the assumption $\dot{\tau}(t) \le \rho < 1$, we have

$$\begin{aligned}
\dot{V}(t) &\le 2e_j^T(t)P_1\dot{e}_j(t) + e_j^T(t)P_2e_j(t) - (1-\rho)e_j^T(t-\tau(t))P_2e_j(t-\tau(t))\\
&\quad + \dot{e}_j^T(t)(\tau_M^2P_3 + (\tau_M-\tau_m)^2P_4)\dot{e}_j(t)\\
&\quad - \left[\int_{t-\tau_M}^{t}\dot{e}_j(s)ds\right]^T P_3\left[\int_{t-\tau_M}^{t}\dot{e}_j(s)ds\right]\\
&\quad - \left[\int_{t-\tau_M}^{t-\tau(t)}\dot{e}_j(s)ds\right]^T P_4\left[\int_{t-\tau_M}^{t-\tau(t)}\dot{e}_j(s)ds\right]\\
&\quad - \left[\int_{t-\tau(t)}^{t-\tau_m}\dot{e}_j(s)ds\right]^T P_4\left[\int_{t-\tau(t)}^{t-\tau_m}\dot{e}_j(s)ds\right].
\end{aligned} \tag{3.21}$$

Taking $\eta_{j1}(t) = e_j(t) - e_j(t-\tau_M)$, $\eta_{j2}(t) = e_j(t-\tau(t)) - e_j(t-\tau_M)$ and $\eta_{j3}(t) = e_j(t-\tau_m) - e_j(t-\tau(t))$, (3.21) can be written as

$$\dot{V}(t) \le \eta_j^T(t)\mathcal{G}\eta_j(t), \tag{3.22}$$

where

$$\mathcal{G} = \begin{bmatrix} \tau_M^2 P_3 + (\tau_M - \tau_m)^2 P_4 & P_1 & 0 & 0 & 0 & 0 \\ P_1 & P_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(1-\rho)P_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -P_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -P_4 \end{bmatrix}$$

and

$$\eta_j(t) = \begin{bmatrix} \dot{e}_j(t)^T & e_j(t)^T & e_j(t-\tau(t))^T & \eta_{j1}(t)^T & \eta_{j2}(t)^T & \eta_{j3}(t)^T \end{bmatrix}^T.$$

Furthermore, considering the error dynamics (3.7), and defining

$$\mathcal{N}_j = \begin{bmatrix} I & -\sum_{i=1}^r \mu_i(\xi(t))N_{ij} & -\sum_{i=1}^r \mu_i(\xi(t))N_{dij} & 0 & 0 & 0 \\ 0 & -I & I & I & -I & 0 \end{bmatrix},$$

we can write

$$\mathcal{N}_j \eta_j(t) = 0.$$

As a consequence, the error dynamics in (3.7) asymptotically approaches zero if for all $\eta_j(t)$,

$$\eta_j^T(t)\mathcal{G}\eta_j(t) < 0$$

subject to $\mathcal{N}_j\eta_j(t) = 0$. Using Finsler's lemma this condition can be equivalently expressed as $\mathcal{N}_j^{\perp T}\mathcal{G}_j\mathcal{N}_j^{\perp} < 0$. Therefore, considering

$$\mathcal{N}_j^{\perp} = \begin{bmatrix} \sum_{i=1}^r \mu_i(\xi(t))N_{ij} & \sum_{i=1}^r \mu_i(\xi(t))N_{dij} \\ I & 0 \\ 0 & I \\ I & 0 \\ 0 & I \\ I & I \end{bmatrix},$$

the inequality, $\mathcal{N}_j^{\perp T}\mathcal{G}_j\mathcal{N}_j^{\perp} < 0$, can be written as

$$\begin{bmatrix} \mathcal{H}_{11j} & \mathcal{H}_{12j} \\ \star & \mathcal{H}_{22j} \end{bmatrix} < 0, \tag{3.23}$$

where

$$\begin{aligned}
\mathcal{H}_{11j} &= P_2 - P_3 - P_4 + P_1 \sum_{k=1}^{r} \mu_k(\xi(t)) N_{kj} + \sum_{i=1}^{r} \mu_i(\xi(t)) N_{ij}^T P_1 \\
&\quad + \tau_M^2 \sum_{i=1}^{r} \mu_i(\xi(t)) N_{ij}^T P_3 \sum_{k=1}^{r} N_{kj} \\
&\quad + (\tau_M - \tau_m)^2 \sum_{i=1}^{r} \mu_i(\xi(t)) N_{ij}^T P_4 \sum_{k=1}^{r} \mu_k(\xi(t)) N_{kj}, \\
\mathcal{H}_{12j} &= -P_4 + P_1 \sum_{k=1}^{r} \mu_k(\xi(t)) N_{dkj} + \tau_M^2 (\sum_{i=1}^{r} \mu_i(\xi(t)) N_{ij})^T P_3 \sum_{k=1}^{r} \mu_k(\xi(t)) N_{dkj} \\
&\quad + (\tau_M - \tau_m)^2 (\sum_{i=1}^{r} \mu_i(\xi(t)) N_{ij})^T P_4 \sum_{k=1}^{r} \mu_k(\xi(t)) N_{dkj} \text{ and} \\
\mathcal{H}_{22j} &= -(1-\rho) P_2 - 2P_4 + \tau_M^2 (\sum_{i=1}^{r} \mu_i(\xi(t)) N_{dij})^T P_3 \sum_{k=1}^{r} \mu_k(\xi(t)) N_{dkj} \\
&\quad + (\tau_M - \tau_m)^2 (\sum_{i=1}^{r} \mu_i(\xi(t)) N_{dij})^T P_4 \sum_{k=1}^{r} \mu_k(\xi(t)) N_{dkj}.
\end{aligned}$$

We can obtain the stability condition using (3.23). Taking quadratic terms apart, and applying the Schur complement and the expressions of N_{ij} and N_{dij} from (3.10) and (3.11), we obtain the stability condition as

$$\begin{bmatrix}
\begin{matrix} \Xi_{ij}^T P_1 + P_1 \Xi_{ij} \\ -\Psi_{ij}^T F_j^T P_1 - P_1 F_j \Psi_{ij} \\ -\Gamma_j^T M_{ij}^T P_1 - P_1 M_{ij} \Gamma_j \\ +P_2 - P_3 - P_4 \end{matrix} & \begin{matrix} P_1 \Xi_{dij} \\ -P_1 F_j \Psi_{dij} \\ -P_1 M_{dij} \Gamma_j \\ -P_4 \end{matrix} & \begin{matrix} \tau_M (\Xi_{ij}^T P_3 \\ -\Psi_{ij}^T F_j^T P_3 \\ -\Gamma_j^T M_{ij}^T P_3) \end{matrix} & \begin{matrix} (\tau_M - \tau_m) \\ (\Xi_{ij}^T P_4 \\ -\Psi_{ij}^T F_j^T P_4 \\ -\Gamma_j^T M_{ij}^T P_4) \end{matrix} \\
\star & \begin{matrix} -(1-\rho) P_2 \\ -2P_4 \end{matrix} & \begin{matrix} \tau_M (\Xi_{dij}^T P_3 \\ -\Psi_{dij}^T F_j^T P_3 \\ -\Gamma_j^T M_{dij}^T P_3) \end{matrix} & \begin{matrix} (\tau_M - \tau_m) \\ (\Xi_{dij}^T P_4 \\ -\Psi_{dij}^T F_j^T P_4 \\ -\Gamma_j^T M_{dij}^T P_4) \end{matrix} \\
\star & \star & -P_3 & 0 \\
\star & \star & \star & -P_4
\end{bmatrix} < 0 \tag{3.24}$$

for $i = 1, 2, \ldots, r,$ and $j = 1, 2, \ldots, r.$

The LMI of (3.24) is a sufficient condition for the error dynamics to approach zero. Our objective is to obtain F_j, M_{ij} and M_{dij} from the solution of this LMI. It can be seen that F_j, M_{ij} and M_{dij} are multiplied with multiple LMI variables P_1, P_3 and P_4, which would lead to multiple values of F_j, M_{ij} and M_{dij}. Therefore, considering $P_3 = \zeta_1 P_1$ and $P_4 = \zeta_2 P_1$ for positive scalars ζ_1 and ζ_2, and defining new matrix variables $Y_{1j} = P_1 F_j$, $Y_{2ij} = P_1 M_{ij}$, and $Y_{2dij} = P_1 M_{dij}$, we obtain (3.18a) where ζ_1 and ζ_2 are given scalars. Furthermore, multiplying (3.11b) and (3.12b) by P_1 we have (3.18b) and (3.18c), respectively.

So far, the asymptotic convergence of estimation error to zero is ensured. To write the error dynamics as (3.7), N_{ij} and N_{dij} are calculated from (3.11a) and (3.12a), respectively. J_{ij} and J_{dij} are calculated from the definitions of M_{ij} and M_{dij}, respectively, and H_{ij} is obtained from (3.8c). □

The fuzzy functional observer based PDC controller for the T-S fuzzy model with time-varying time-delay can be constructed by following the procedure as outlined below.

Functional observer based PDC controller synthesis for time varying time-delay systems

Step 1: Find K_j from (3.17). Calculate Ξ_{ij}, Ξ_{dij}, Ψ_{ij}, Ψ_{dij}, Γ_j, Ω_{ij}, Ω_{ij}, Φ_j, Λ_{ij} and Λ_{dij} from their definitions in (3.11) and (3.12);

Step 2: Specify the range of ζ_1 and ζ_2, and respective increments $\Delta\zeta_1$ and $\Delta\zeta_2$. Set ζ_1 and ζ_2 to the minimum values of the range;

Step 3: Solve the LMIs with the equality constraints of (3.18). If a solution is found go to Step 5 else go to next step;

Step 4: Increase ζ_1 and ζ_2 by their respective increments and go to Step 3;

Step 5: Calculate F_j, M_{ij} and M_{dij} from Y_{1j}, Y_{2ij} and Y_{2dij}, respectively;

Step 6: Calculate N_{ij} and N_{dij} using (3.11a) and (3.12a), respectively. Obtain J_{ij} and J_{dij} from $J_{ij} = M_{ij} + N_{ij}F_j$ and $J_{dij} = M_{dij} + N_{dij}F_j$, respectively; and

Step 7: Find H_{ij} using (3.8c).

Remark 3.3.1. *Observing Theorem 3.3.1 and Theorem 3.3.2 reveals that the equations,*

$$Y_{1j}\Omega_{ij} + Y_{2ij}\Phi_j - P_1\Lambda_{ij} = 0 \tag{3.25a}$$

$$Y_{1j}\Omega_{dij} + Y_{2dij}\Phi_j - P_1\Lambda_{dij} = 0, \tag{3.25b}$$

require to have solutions for Y_{1j}, Y_{2ij} *and* Y_{2dij}. *Considering all values of* i *and multiplying by* P_1^{-1}, *the identity in* (3.25a) *can be expressed as*

$$\begin{bmatrix} F_j & M_{1j} & M_{2j} & \cdots & M_{rj} \end{bmatrix} \begin{bmatrix} \Omega_{1j} & \Omega_{2j} & \cdots & \Omega_{rj} \\ \Phi_j & 0 & \dots & 0 \\ 0 & \Phi_j & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_j \end{bmatrix} = \begin{bmatrix} \Lambda_{1j} & \Lambda_{2j} & \cdots & \Lambda_{rj} \end{bmatrix}. \tag{3.26}$$

Clearly, (3.25a) *has a solution if* (3.26) *has a solution. Therefore, one necessary condition for verifying the existence of the fuzzy functional observer is to check the rank equality [83]*

$$\text{rank} \begin{bmatrix} \Lambda_{1j} & \Lambda_{2j} & \cdots & \Lambda_{rj} \\ \Omega_{1j} & \Omega_{2j} & \cdots & \Omega_{rj} \\ \Phi_j & 0 & \cdots & 0 \\ 0 & \Phi_j & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_j \end{bmatrix} = \text{rank} \begin{bmatrix} \Omega_{1j} & \Omega_{2j} & \cdots & \Omega_{rj} \\ \Phi_j & 0 & \cdots & 0 \\ 0 & \Phi_j & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_j \end{bmatrix}.$$

Using (3.25b) *and by following the similar line of proof, another necessary existence condition is given in* (3.27).

Remark 3.3.2. *The LMI condition in Theorem 3.3.2 is a feasibility problem. Solution of the LMIs in* (3.18) *depends on the choice of* ζ_1 *and* ζ_2. *The ranges and the increments of the two constants play an important role in finding the*

$$\operatorname{rank}\begin{bmatrix}\Lambda_{d1j} & \Lambda_{d2j} & \cdots & \Lambda_{drj}\\ \Omega_{d1j} & \Omega_{d2j} & \cdots & \Omega_{drj}\\ \Phi_j & 0 & \cdots & 0\\ 0 & \Phi_j & \cdots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \Phi_j\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\Omega_{d1j} & \Omega_{d2j} & \cdots & \Omega_{drj}\\ \Phi_j & 0 & \cdots & 0\\ 0 & \Phi_j & \cdots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \Phi_j\end{bmatrix}. \tag{3.27}$$

---

solutions. The solution domain can be increased by choosing smaller values of the increments and by increasing the ranges. In addition, it should be noted that the stability conditions are derived by using quadratic Lyapunov-Kravoskii functional. However, more relaxed stability conditions can be obtained by using fuzzy Lyapunov functions [92, 93]. Future work will consider using this technique for obtaining functional observer of T-S fuzzy systems.

Example 3.1

A T-S fuzzy model with two rules is used to illustrate the synthesis procedures and to verify the stability conditions stated in Theorem 3.3.1 and Theorem 3.3.2. The plant data for the two-rule model borrowed from [42] are as below:

$$A_1 = \begin{bmatrix}1 & -0.5\\ 1 & 0\end{bmatrix}, \qquad A_2 = \begin{bmatrix}-1 & -0.5\\ 1 & 0\end{bmatrix},$$

$$A_{d1} = \begin{bmatrix}0 & -0.2\\ 0.2 & 0\end{bmatrix}, \qquad A_{d2} = \begin{bmatrix}0 & -0.1\\ 0.1 & 0\end{bmatrix},$$

$$B_1 = \begin{bmatrix}1\\ 0\end{bmatrix}, \qquad B_2 = \begin{bmatrix}0.6\\ 0\end{bmatrix},$$

$$C = \begin{bmatrix}1 & 1\end{bmatrix}.$$

In this example, the premise variable is state $x_1(t)$, and the fuzzy sets for rule 1 and rule 2 are as depicted in Figure 3.1. We calculate K_1 and K_2 using (3.4) to obtain the PDC controller gains. Then following the functional observer based PDC synthesising steps considering constant time-delay, we find the observer parameters as below:

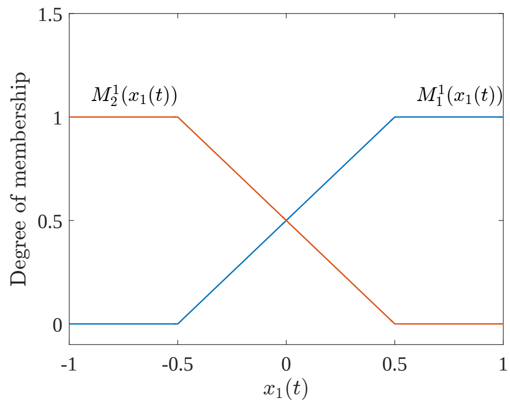

Figure 3.1: Fuzzy sets of membership functions

$$
\begin{aligned}
K_1 &= \begin{bmatrix}-6.3592 & -5.0072\end{bmatrix}, & F_1 &= -6.4750,\\
N_{11} &= -1.2142, & N_{21} &= -1.0428,\\
J_{11} &= 1.7243, & J_{21} &= 1.4728,\\
H_{11} &= 0.1158, & H_{21} &= 0.0695,\\
N_{d11} &= -0.2343, & N_{d21} &= -0.1171,\\
J_{d11} &= 0.3207, & J_{d21} &= 0.1604,\\
K_2 &= \begin{bmatrix}-6.3592 & -5.0072\end{bmatrix}, & F_2 &= -6.4750,\\
N_{12} &= -1.2142, & N_{21} &= -1.0428,\\
J_{12} &= 1.7243, & J_{22} &= 1.4728,\\
H_{12} &= 0.1158, & H_{22} &= 0.0695\\
N_{d12} &= -0.2343, & N_{d22} &= -0.1171,\\
J_{d12} &= 0.3207, & J_{d22} &= 0.1604.
\end{aligned}
$$

The dynamics of the closed loop system with the functional observer based PDC controller is simulated for constant time delay $\tau = 1$ considering different initial conditions $\phi(t) = [1 \quad 1]^T$ and $\phi(t) = [3 \quad 1.5]^T$. Figures 3.2 and 3.3 show the time responses of the states. It is evident that the functional observer based controller stabilises the system asymptotically as expected. The performance of the functional observer based PDC controller is compared with the conventional PDC controller in both the figures. It shows that the functional observer based controller is comparable with the PDC controller in the event that all states are

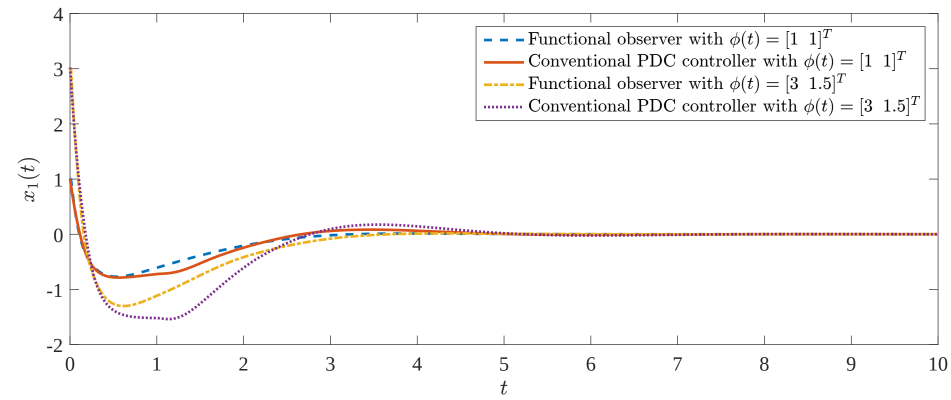

Figure 3.2: Time response of state $x_1(t)$ for different input conditions considering constant time-delay

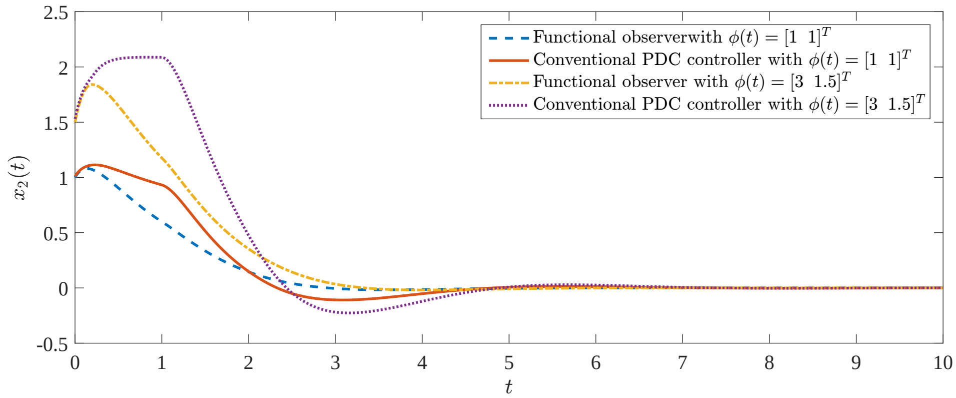

Figure 3.3: Time response of state $x_2(t)$ for different input conditions considering constant time-delay

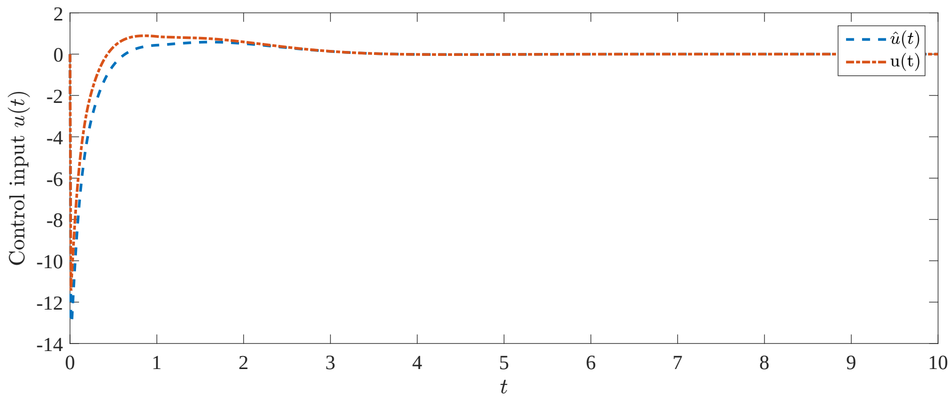

Figure 3.4: Functional observer based control input $\hat{u}(t)$, and PDC control input $u(t)$ for initial condition $\phi(t) = [1 \quad 1]^T$ considering constant time-delay

not available. It should be noted that the convergence time can be enhanced by the suitable choice of LMI conditions considering the pole placement in the preferred region. The main focus of this study is to construct the functional

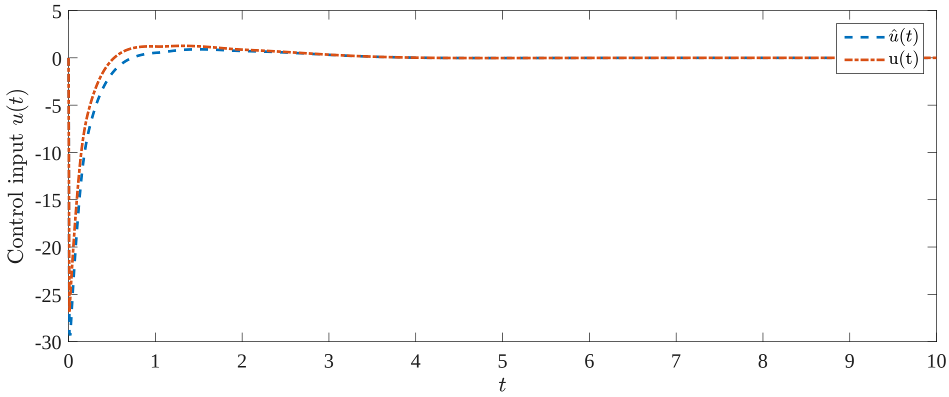

Figure 3.5: Functional observer based control input $\hat{u}(t)$, and conventional PDC control input $u(t)$ for initial condition $\phi(t) = [3 \quad 1.5]^T$ considering constant time-delay

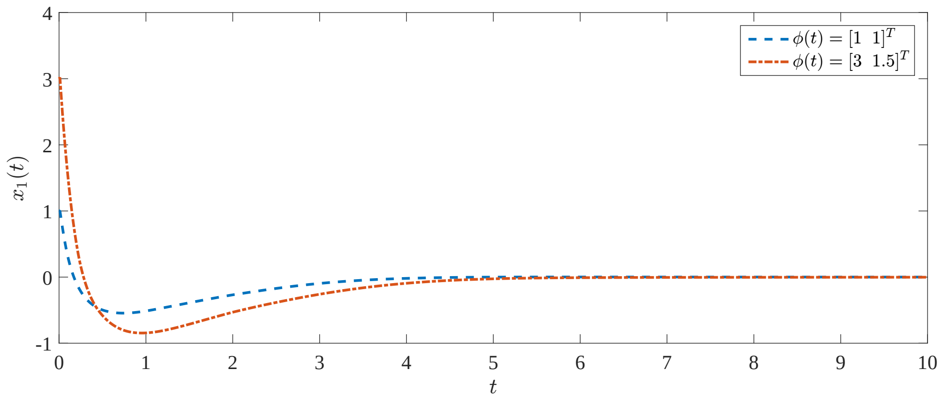

Figure 3.6: Time response of state $x_1(t)$ considering time varying time-delay

observer that estimates the control signal directly without estimating the states. It can be witnessed from the simulation output that the objective has been achieved successfully. Figures 3.4 and 3.5 compare the estimated control input $\hat{u}(t)$ with the objective PDC control input $u(t)$. In both the cases $\hat{u}(t)$ converges to $u(t)$ asymptotically as the estimation error of the functional observer approaches zero asymptotically.

Now we consider the time varying time-delay for the same plant. The time-varying delay is considered to have upper and lower bounds as $\tau_M = 1.5$ and $\tau_m = 0.1$, respectively. The time derivative of the delay is considered to have the upper bound as $\rho = 0.9$. Two parameters for the LMI conditions are considered to be $\zeta_1 = 0.1$ and $\zeta_2 = 0.5$. The observer parameters are obtained as below:

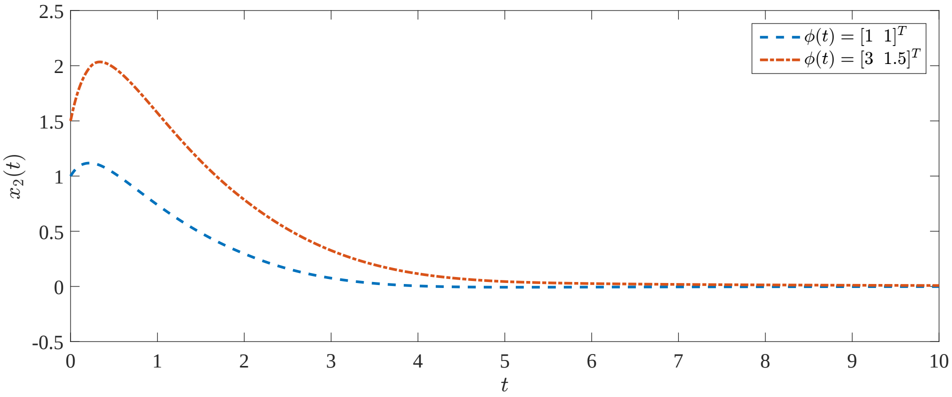

Figure 3.7: Time response of state $x_2(t)$ considering time varying time-delay

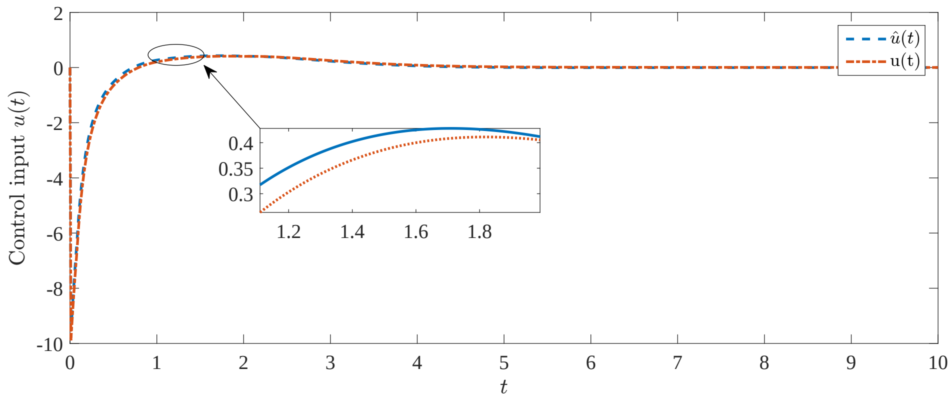

Figure 3.8: Functional observer based control input $\hat{u}(t)$, and PDC control input $u(t)$ for initial condition $\phi(t) = [1 \quad 1]^T$ considering time varying time-delay

$$
\begin{aligned}
K_1 &= \begin{bmatrix} -6.1671 & -3.6549 \end{bmatrix}, & F_1 &= -4.5957, \\
N_{11} &= 0.3651, & N_{21} &= -0.7193, \\
J_{11} &= 0.1288, & J_{21} &= 1.4118, \\
H_{11} &= -1.5714, & H_{21} &= -0.9428, \\
N_{d11} &= 0.0122, & N_{d21} &= 0.0048, \\
J_{d11} &= 0.2429, & J_{d21} &= 0.1206, \\
K_2 &= \begin{bmatrix} -4.3757 & -2.7581 \end{bmatrix}, & F_2 &= -3.4306, \\
N_{12} &= 0.2163, & N_{21} &= -0.7569, \\
J_{12} &= 0.1519, & J_{22} &= 0.0762, \\
H_{12} &= -0.9451, & H_{22} &= -0.5671, \\
N_{d12} &= -0.0115, & N_{d22} &= -0.0049,
\end{aligned}
$$

$$J_{d12} = 0.1515, \qquad J_{d22} = 0.0758.$$

Figure 3.6 and Figure 3.7 depict the time response of the plant. In both cases, the system state stabilises asymptotically. In addition, Figure 3.8 confirms that the estimation error approaches zero asymptotically.

3.4 Conclusion

This work explores the effect of time-delay on a fuzzy functional observer and its application as a PDC controller for T-S fuzzy model. Both the constant type and time varying time-delay are considered. The time varying time-delay is considered to have upper and lower bounds with a bounded rate of change. Stability conditions are derived in the form of LMIs with equality constraints. Design methodologies for the observer are based on solving the LMIs. Stability conditions are verified by simulations of numerical examples.

Chapter 4

Robust fuzzy functional observer for stabilising uncertain T-S fuzzy systems with time-delay

This chapter presents the stabilisation of an uncertain T-S fuzzy model with time-delay using a functional observer based fuzzy controller. The model uncertainty is considered to be norm bounded. The sensitivity of the estimation error to the model uncertainty is reduced by minimising a cost function, which is formulated by using an L_2 gain constraint. The time-delay is considered to be time-varying with upper and lower bounds, and a bounded time derivative. Lyapunov-Krasovskii functionals are used to construct the stability conditions as LMIs. Free-weighting matrices are employed to obtain delay dependent stability conditions. Due to the use of the free-weighting matrices, the stability conditions in this chapter are more relaxed compared with the ones in the previous chapter. Furthermore, the equality conditions are eliminated from the set of stability conditions for the observer. Solutions of the LMIs are used to obtain the observer parameters. The proposed method is verified and illustrated using examples. The main results of this chapter are published in [94]

4.1 Introduction

Mathematical models of systems may not always describe the dynamics of the systems accurately because of the approximation of the true dynamics and the existence of some natural phenomena, e.g., neglected nonlinearities, unmodeled high frequency dynamics, variation of system parameters due to the change of the environment, shifting of the operating points, wear and tear of the plant components, etc. The modeling error affects the stability and performance of control systems and observers. This modeling inaccuracy, in general, is time-varying and can be expressed as norm bounded uncertainty in the mathematical model [95]. The robust stabilisation of a system is acknowledged as guaranteeing the stability of the closed loop system in spite of the parameter uncertainties. The robust controller design problem for T-S fuzzy system is a mature field of research [42, 96, 97].

A functional observer is said to be robust if it estimates the function of states asymptotically even in the presence of the parameter uncertainty. In [98], authors investigated the existence and design of robust functional observer for uncertain fractional order system and presented stability conditions in the sense that the estimation error approaches zero asymptotically even in the presence of the uncertainties.

This chapter presents the problem of obtaining a PDC controller using a robust functional observer considering the time varying time-delay and model uncertainty. The sensitivity of the estimation error to the model uncertainty is minimised by employing the L_2 gain minimisation technique. Having the upper and lower bounds of the time varying time-delay, delay dependent stability condition is formulated as LMIs by using Lyapunov-Krasovskii functional. The functional observer can be constructed by solving the LMIs.

Notation: $\mathbb{R}^{n\times m}$ denotes $n \times m$ dimensional real matrix and $\mathbb{R}^n$ denotes n dimensional real vector. I_p represents $p \times p$ identity matrix. Superscripts $(.)^+$, $(.)^\perp$ and $(.)^-$ mean Moore-Penrose generalised inverse, orthogonal basis of corresponding matrix and inverse, respectively. Symmetric components of respective blocks of a symmetric matrix are denoted by $\star$, and $\text{diag}(X, X, \ldots, X)$ represents a block diagonal matrix.

4.2 Model description and problem formulation

Consider a nonlinear system approximated by a T-S fuzzy model with time-delay and model uncertainty. The ith rule of this T-S fuzzy model is

$$\begin{aligned}
&\text{IF} \quad \xi_1(t) \text{ is } M_i^1 \text{ and } \cdots \text{ and } \xi_l(t) \text{ is } M_i^l \\
&\text{THEN } \dot{x}(t) = (A_i + \Delta A_i(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t - \tau(t)) + B_i u(t) \\
&\qquad\quad y(t) = Cx(t) \\
&\qquad\quad x(t) = \phi(t), t \in [-\tau(t), 0], i = 1, \dots, r,
\end{aligned} \tag{4.1}$$

where $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, $x(t) \in \mathbb{R}^n$ is the state, $\tau(t)$ is the time varying time-delay, and $\xi_1(t), \dots, \xi_l(t)$ are the premise variables. Premise variable $\xi_k(t)$ belongs to fuzzy set M_i^k in the ith rule with the degree of membership defined by membership function $M_i^k(\xi_k(t))$. Matrices $A_i \in \mathbb{R}^{n\times n}$, $A_{di} \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ represent the ith linear sub system. Time varying terms $\Delta A_i(t)$ and $\Delta A_{di}(t)$ represent the model uncertainty. Taking $\xi(t) = [\xi_1(t), \dots, \xi_l(t)]$, the fuzzy summation,

$$\begin{aligned}
\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{&(A_i + \Delta A_i(t))x(t) \\
&+ (A_{di} + \Delta A_{di}(t))x(t - \tau(t)) + B_i u(t)\} \\
y(t) = Cx(t),&
\end{aligned} \tag{4.2}$$

where

$$\mu_i(\xi(t)) = \frac{\prod_{k=1}^{l} M_i^k(\xi_k(t))}{\sum_{i=1}^{r}\prod_{k=1}^{l} M_i^k(\xi_k(t))} \quad \text{with } \mu_i(\xi(t)) \geq 0 \ \text{ and } \sum_{i=1}^{r} \mu_i(\xi(t)) = 1,$$

represents the overall nonlinear plant dynamics.

The upper and lower bounds of the time-delay are denoted by τ_M and τ_m, respectively. The time derivative of time-delay $\tau(t)$ is bounded above; $\dot{\tau}(t) \leq \rho < 1$. The time varying model uncertainties are assumed to be

$$\begin{aligned}
\Delta A_i(t) &= R_i U_i(t) S_i, \quad i = 1, 2, \dots, r \text{ and} \\
\Delta A_{di}(t) &= R_{di} U_{di}(t) S_{di}, \quad i = 1, 2, \dots, r
\end{aligned}$$

such that time-varying uncertain parameters $U_i(t)$ and $U_{di}(t)$ of appropriate dimensions satisfy

$$U_i^T(t)U_i(t) \leq I, \quad i = 1, 2, \ldots, r \text{ and} \tag{4.3a}$$

$$U_{di}^T(t)U_{di}(t) \leq I, \quad i = 1, 2, \ldots, r, \tag{4.3b}$$

where R_i, S_i, R_{di} and S_{di} are known real constant matrices of appropriate dimensions.

Remark 4.2.1. *The plant model uncertainties, which in many cases may not be exactly modeled by mathematical expressions, can be generally treated as uncertainties over-bounded by the condition $U_i^T(t)U_i(t) < I$. While $U_i(t)$ carries the actual information of the uncertain nature of the systems, matrices R_i and S_i link this uncertainty with the nominal system [99].*

A PDC controller for system (4.1) can be expressed as

$$\begin{aligned} u(t) &= \sum_{j=1}^{r} \mu_j(\xi(t))u_j(t) \\ &= \sum_{j=1}^{r} \mu_j(\xi(t))K_j x(t), \end{aligned}$$

where K_j corresponds to the linear feedback gain of the respective subsystem. The procedure for calculating stabilising feedback controller gain K_j is described in the next section. Considering all states are not accessible, our main focus is to employ a linear functional observer to estimate $u_j(t)$ for each linear subsystem, and to obtain $u(t)$ using the fuzzy summation. The proposed functional observer based PDC controller is as described below:

$$\begin{aligned} \dot{w}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{&N_{ij}w_j(t) + N_{dij}w_j(t-\tau) \\ &+ J_{ij}y(t) + J_{dij}y(t-\tau) + H_{ij}\hat{u}(t)\} \end{aligned} \tag{4.4a}$$

$$\hat{u}_j(t) = w_j(t) + F_j y(t) \tag{4.4b}$$

$$\hat{u}(t) = \sum_{j=1}^{r} \mu_j(\xi(t))\{\hat{u}_j(t)\} \tag{4.4c}$$

$$w_j(t) = 0 \text{ for all } t \in [-\tau, 0], \tag{4.4d}$$

where $w_j(t) \in \mathbb{R}^m$, $F_j \in \mathbb{R}^{m\times p}$, $N_{ij} \in \mathbb{R}^{m\times m}$, $N_{dij} \in \mathbb{R}^{m\times m}$, $J_{ij} \in \mathbb{R}^{m\times p}$, $J_{dij} \in \mathbb{R}^{m\times p}$, and $H_{ij} \in \mathbb{R}^{m\times m}$. Here $\hat{u}_j(t)$ is the estimated function of states, where $u_j(t)$ is a linear combination of the states $x(t)$ and is defined by $u_j(t) = K_j x(t)$. The estimation error can be expressed as

$$\begin{aligned} e_j(t) &= u_j(t) - \hat{u}_j(t) \\ &= T_j x(t) - w_j(t), \end{aligned}$$

where $T_j = K_j - F_j C$.

The error dynamics can be expressed as

$$\begin{aligned} \dot{e}_j(t) = \sum_{i=1}^{r} \mu(\xi)\{ & N_{ij} e_j(t) + N_{dij} e_j(t-\tau(t)) \\ & + (T_j A_i - N_{ij} T_j - J_{ij} C + T_j R_i U_i(t) S_i) x(t) \\ & + (T_j A_{di} - N_{dij} T_j - J_{dij} C + T_j R_{di} U_{di}(t) S_{di}) x(t-\tau(t)) \\ & + (T_j B_i - H_{ij}) \hat{u}(t) \}. \end{aligned} \tag{4.5}$$

Error dynamics (4.5) reduces to

$$\begin{aligned} \dot{e}_j(t) = \sum_{i=1}^{r} \mu(\xi)\{ & N_{ij} e_j(t) + N_{dij} e_j(t-\tau(t)) \\ & + T_j R_i U_i(t) S_i x(t) + T_j R_{di} U_{di}(t) S_{di} x(t-\tau(t)) \} \end{aligned} \tag{4.6}$$

if we have

$$T_j A_i - N_{ij} T_j - J_{ij} C = 0 \tag{4.7a}$$

$$T_j A_{di} - N_{dij} T_j - J_{dij} C = 0 \tag{4.7b}$$

$$T_j B_i - H_{ij} = 0. \tag{4.7c}$$

Therefore, the controller design problem for the uncertain T-S fuzzy system with time-delay using the functional observer turns into obtaining matrices N_{ij}, N_{dij}, J_{ij}, J_{dij}, H_{ij} and F_{ij} such that error system (4.6) is asymptotically stable and conditions (4.7) hold.

4.3 Stability condition for the uncertain model

PDC gain matrices K_j can be calculated by solving the LMIs presented in the following lemma.

Lemma 4.3.1. *The fuzzy time-delay system described by* (4.2) *is stable if, for some given constants $\bar{\sigma}_1$, $\bar{\sigma}_2$, τ_m, τ_M and ρ, there exist positive definite symmetric matrices $\bar{P}_1$ and $\bar{P}_2$, and matrices $\bar{Y}_j$ of appropriate dimension such that LMIs in* (4.8) *hold*

$$\begin{bmatrix}
\begin{matrix}\bar{P}_2 + A_i\bar{P}_1 \\ +\bar{P}_1 A_i^T + B_i\bar{Y}_j \\ +\bar{Y}_j^T B_i^T\end{matrix} & A_{di}\bar{P}_1 & \frac{1}{2}\kappa(\bar{P}_1 A_i^T + \bar{Y}_j^T B_i^T) & R_i & R_{di} & 0 & 0 & \bar{P}_1 S_i^T & 0 \\
\star & -(1-\bar{\rho})\bar{P}_2 & \frac{1}{2}\kappa\bar{P}_1 A_{di}^T & 0 & 0 & 0 & 0 & 0 & \bar{P}_1 S_{di}^T \\
\star & \star & -\tau_m\bar{\sigma}_2\bar{P}_1 & 0 & 0 & \kappa R_i & \kappa R_{di} & 0 & 0 \\
\star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & -I & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -I & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & -\frac{1}{2}I & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & -\frac{1}{2}I
\end{bmatrix} < 0 \tag{4.8}$$

for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, r$.

for $\kappa = \tau_M(\bar{\sigma}_1 + \bar{\sigma}_2)$. PDC controller gain matrices K_j can be obtained from the relation $\bar{Y}_j = K_j\bar{P}_1$.

Proof. Proof is given in the appendix. □

Having obtained gain matrices K_j, our goal is to construct robust functional observer (4.4) that estimates the fuzzy summation of function of states $K_j x(t)$ directly. The error dynamics of the observer is sensitive to the model uncertainty. Therefore, our goal includes two aspects: first, to ensure that the estimation error approaches zero asymptotically if there is no model uncertainty; second, to minimise the sensitivity of estimation error to uncertainty. The sensitivity minimisation problem is formulated in the form of minimising a cost function subject to L_2 gain bound constraint. We say that the functional observer is robust if there exists a positive scalar γ such that

$$\frac{\|u_j(t) - \hat{u}_j(t)\|_2}{\|u_j(t)\|_2} = \frac{\|e_j(t)\|_2}{\|u_j(t)\|_2} < \gamma,$$

where $\|\cdot\|_2$ is an L_2 norm as expressed below:

$$\|u_j(t)\|_2^2 = \int_0^\infty u_j^T(t) u_j(t) dt.$$

The following theorem describes the stability condition of the functional observer.

Theorem 4.3.1. *The functional observer described by* (4.4) *with time varying time-delay with upper bound τ_M and lower bound τ_m is robustly asymptotically stable if, for given scalars σ_1 and σ_2, there exist positive definite symmetric matrices P_1, P_2^1, P_2^2, P_2^3 and P_2^4, and matrices W_{11}, W_{12}, W_{21}, W_{22}, W_{31}, W_{33} and Y_j of appropriate dimensions such that the optimisation problem in* (4.9) *has a solution considering*

$$\begin{aligned}
\Xi_{ij}^{1,1} &= I_p + P_1 N_{ij}^1 + Y_j N_{ij}^2 + (N_{ij}^1)^T P_1 + (N_{ij}^2)^T Y_j + P_2^1 + P_2^2 + P_2^3 + W_{11} + W_{11}^T, \\
\Xi_{ij}^{1,2} &= P_1 N_{dij}^1 + Y_j N_{dij}^2 - W_{11} + W_{12}^T + W_{21} - W_{31}, \\
\Xi_{ij}^{1,5} &= \frac{1}{2}\tau_M(\sigma_1 + \sigma_2)((N_{ij}^1)^T P_1 + (N_{ij}^2)^T Y_j), \\
\Xi_{ij}^{1,11} &= (P_1 T_j^1 + Y_j T_j^2) R_i, \quad \Xi_{ij}^{1,12} = (P_1 T_j^1 + Y_j T_j^2) R_{di}, \\
\Xi_{ij}^{2,2} &= -(1-\rho) P_2^1 - W_{12} - W_{12}^T + W_{22} + W_{22}^T - W_{32} - W_{32}^T \\
\Xi_{ij}^{2,5} &= \frac{1}{2}\tau_M(\sigma_1 + \sigma_2)((N_{dij}^1)^T P_1 + (N_{dij}^2)^T Y_j),
\end{aligned}$$

$$
\begin{aligned}
&\text{minimise } \gamma > 0 \\
&\text{subject to} \\
&\left[\begin{array}{cccccccccccccc}
\Xi_{ij}^{1,1} & \Xi_{ij}^{1,2} & -W_{21} & W_{31} & \Xi_{ij}^{1,5} & 0 & 0 & \tau_M W_{11} & \tilde{\tau} W_{21} & \tilde{\tau} W_{31} & \Xi_{ij}^{1,11} & \Xi_{ij}^{1,12} & 0 & 0 \\
\star & \Xi_{ij}^{2,2} & -W_{22} & W_{32} & \Xi_{ij}^{2,5} & 0 & 0 & \tau_M W_{11} & \tilde{\tau} W_{22} & \tilde{\tau} W_{32} & 0 & 0 & 0 & 0 \\
\star & \star & -P_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & -P_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \Xi_{ij}^{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Xi_{ij}^{5,13} & \Xi_{ij}^{5,14} \\
\star & \star & \star & \star & \star & \Xi_{ij}^{6,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \Xi_{ij}^{7,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \Xi_{ij}^{8,8} & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \Xi_{ij}^{9,9} & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \Xi_{ij}^{10,10} & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -I & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -I & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -I
\end{array}\right] < 0.
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\Xi_{ij}^{5,5} &= -\tau_m \sigma_2 P_1, \quad \Xi_{ij}^{5,13} = \tau_M(\sigma_1 + \sigma_2)(P_1 T_j^1 + Y_j T_j^2) R_i, \\
\Xi_{ij}^{5,14} &= \tau_M(\sigma_1 + \sigma_2)(P_1 T_j^1 + Y_j T_j^2) R_{di}, \\
\Xi_{ij}^{6,6} &= -\gamma^2 K_j^T K_j + \frac{3}{2} S_i^T S_i, \quad \Xi_{ij}^{7,7} = -(1-\rho) P_2^4 + \frac{3}{2} S_{di}^T S_{di}, \\
\Xi_{ij}^{8,8} &= -\tau_M \sigma_1 P_1, \quad \Xi_{ij}^{9,9} = -\tilde{\tau}(\sigma_1 + \sigma_2) P_1, \quad \Xi_{ij}^{10,10} = -\tilde{\tau} \sigma_2 P_1, \\
T_j^1 &= K_j - F_j^1 C, \quad T_j^2 = -F_j^2 C, \quad Y_j = P_1 Z_j, \quad \tilde{\tau} = \tau_M - \tau_m,
\end{aligned}
$$

and N_{ij}^1, N_{ij}^2, N_{dij}^1, N_{dij}^2, F_j^1 *and* F_j^2 *are defined in* (4.12a), (4.12b) *and* (4.12e).

Proof. Using the definition, $T_j = K_j - F_j C$, and considering all values of i, (4.7a) and (4.7b) can be expressed as

$$F_j \begin{bmatrix} C\mathcal{A} & C\mathcal{A}_d \end{bmatrix} + \begin{bmatrix} \mathcal{N}_j & \mathcal{N}_{dj} \end{bmatrix} \begin{bmatrix} \mathcal{K}_j & 0 \\ 0 & \mathcal{K}_j \end{bmatrix} + \begin{bmatrix} \mathcal{M}_j & \mathcal{M}_{dj} \end{bmatrix} \begin{bmatrix} \mathcal{C} & 0 \\ 0 & \mathcal{C} \end{bmatrix} = \begin{bmatrix} K_j\mathcal{A} & K_j\mathcal{A}_d \end{bmatrix}, \tag{4.10}$$

where

$$\mathcal{N}_j = \begin{bmatrix} N_{1j} & N_{2j} & \dots & N_{rj} \end{bmatrix}, \qquad \mathcal{N}_{dj} = \begin{bmatrix} N_{d1j} & N_{d2j} & \dots & N_{drj} \end{bmatrix},$$
$$M_{ij} = J_{ij} - N_{ij}F_j, \qquad M_{dij} = J_{dij} - N_{dij}F_j,$$
$$\mathcal{M}_j = \begin{bmatrix} M_{1j} & M_{2j} & \dots & M_{rj} \end{bmatrix}, \qquad \mathcal{M}_{dj} = \begin{bmatrix} M_{d1j} & M_{d2j} & \dots & M_{drj} \end{bmatrix},$$
$$\mathcal{A}_j = \begin{bmatrix} A_1 & A_2 & \dots & A_r \end{bmatrix}, \qquad \mathcal{A}_{dj} = \begin{bmatrix} A_{d1} & A_{d2} & \dots & A_{dr} \end{bmatrix},$$
$$\mathcal{K}_j = \begin{bmatrix} K_j & 0 & \dots & 0 \\ 0 & K_j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K_j \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} C & 0 & \dots & 0 \\ 0 & C & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C \end{bmatrix}.$$

The general solution of the unknown matrices of (4.10) can be given as

$$\begin{bmatrix} F_j & \mathcal{N}_j & \mathcal{M}_j & \mathcal{N}_{dj} & \mathcal{M}_{dj} \end{bmatrix} = \Phi_j\Psi_j^+ - Z_j(I - \Psi_j\Psi_j^+), \tag{4.11}$$

where $\Psi_j = \begin{bmatrix} C\mathcal{A} & C\mathcal{A}_d \\ \mathcal{K}_j & 0 \\ \mathcal{C} & 0 \\ 0 & \mathcal{K}_j \\ 0 & \mathcal{C} \end{bmatrix}$ and $\Phi_j = \begin{bmatrix} K_j\mathcal{A} & K_j\mathcal{A}_d \end{bmatrix}$ are known, and Z_j is an arbitrary matrix of appropriate dimension. N_{ij}, N_{dij}, M_{ij}, M_{dij} and F_j can be expressed as

$$N_{ij} = N_{ij}^1 + Z_jN_{ij}^2 \tag{4.12a}$$
$$N_{dij} = N_{dij}^1 + Z_jN_{dij}^2 \tag{4.12b}$$
$$M_{ij} = M_{ij}^1 + Z_jM_{ij}^2 \tag{4.12c}$$
$$M_{dij} = M_{dij}^1 + Z_jM_{dij}^2 \tag{4.12d}$$
$$F_j = F_j^1 + Z_jF_j^2, \tag{4.12e}$$

where N_{ij}^1, N_{ij}^2, N_{dij}^1, N_{dij}^2, M_{dij}^1, M_{dij}^2, F_j^1 and F_j^2 are extracted from (4.11) by partitioning Φ_j and Ψ_j properly. Consider a Lyapunov-Krasovskii functional

$$\begin{aligned} V(t) =& e_j^T(t)P_1e_j(t) + \int_{t-\tau(t)}^t e_j^T(s)P_2^1e_j(s)ds + \int_{t-\tau_M}^t e_j^T(s)P_2^2e_j(s)ds \\ &+ \int_{t-\tau_m}^t e_j^T(s)P_2^3e_j(s)ds + \int_{t-\tau(t)}^t x^T(s)P_2^4x(s)ds \\ &+ \int_{-\tau_M}^0 \int_{t+\theta}^t \dot{e}_j^T(s)P_3\dot{e}_j(s)dsd\theta + \int_{-\tau_M}^{-\tau_m} \int_{t+\theta}^t \dot{e}_j^T(s)P_4\dot{e}_j(s)dsd\theta, \end{aligned} \tag{4.13}$$

where P_1, P_2^1, P_2^2, P_2^3, P_2^4, P_3 and P_4 are positive definite symmetric matrices, and τ_m and τ_M are lower and upper bounds of delay $\tau(t)$, respectively. Taking the derivative of (4.13) along the error dynamics, we have

$$\begin{aligned} \dot{V}(t) =& 2e_j^T(t)P_1\dot{e}_j(t) + e_j^T(t)P_2^1e_j(t) - (1-\dot{\tau}(t))e_j^T(t-\tau(t))P_2^1e_j(t-\tau(t)) \\ &+ e_j^T(t)P_2^2e_j(t) - e_j^T(t-\tau_M)P_2^2e_j(t-\tau_M) + e_j^T(t)P_2^3e_j(t) \\ &- e_j^T(t-\tau_m)P_2^3e_j(t-\tau_m) + x^T(t)P_2^4x(t) \\ &- (1-\dot{\tau}(t))x^T(t-\tau(t))P_2^4x(t-\tau(t)) \\ &+ \dot{e}_j^T(t)(\tau_MP_3 + (\tau_M-\tau_m)P_4)\dot{e}_j(t) \\ &- \int_{t-\tau_M}^t \dot{e}_j^T(s)P_3\dot{e}_j(s)ds - \int_{t-\tau_M}^{t-\tau_m} \dot{e}_j^T(s)P_4\dot{e}_j(s)ds \\ \leq& 2e_j^T(t)P_1\dot{e}_j(t) + e_j^T(t)(P_2^1+P_2^2+P_2^3)e_j(t) \\ &- (1-\rho)e_j^T(t-\tau(t))P_2^1e_j(t-\tau(t)) - e_j^T(t-\tau_M)P_2^2e_j(t-\tau_M) \\ &- e_j^T(t-\tau_m)P_2^3e_j(t-\tau_m) + x^T(t)P_2^4x(t) - (1-\rho)x^T(t-\tau(t))P_2^4x(t-\tau(t)) \\ &+ \dot{e}_j^T(t)(\tau_MP_3 + (\tau_M-\tau_m)P_4)\dot{e}_j(t) - \int_{t-\tau(t)}^t \dot{e}_j^T(s)P_3\dot{e}_j(s)ds \\ &- \int_{t-\tau_M}^{t-\tau(t)} \dot{e}_j^T(s)(P_3+P_4)\dot{e}_j(s)ds - \int_{t-\tau(t)}^{t-\tau_m} \dot{e}_j^T(s)P_4\dot{e}_j(s)ds. \end{aligned}$$

By using the Leibniz-Newton formula, we can obtain the following identities:

$$\begin{aligned} &2(e_j^T(t)W_{11} + e_j^T(t-\tau(t))W_{12}) \\ &\left(e_j(t) - e_j(t-\tau(t) - \int_{t-\tau(t)}^t \dot{e}_j(s)ds\right) = 0 \end{aligned} \tag{4.15a}$$

$$2(e_j^T(t)W_{21} + e_j^T(t-\tau(t))W_{22})$$
$$\left(e_j(t-\tau(t)) - e_j(t-\tau_M) - \int_{t-\tau_M}^{t-\tau(t)} \dot{e}_j(s)ds\right) = 0 \tag{4.15b}$$
$$2(e_j^T(t)W_{31} + e_j^T(t-\tau(t))W_{32})$$
$$\left(e_j(t-\tau_m) - e_j(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_m} \dot{e}_j(s)ds\right) = 0, \tag{4.15c}$$

where W_{kl} with $k = 1, \dots, 3$ and $l = 1, 2$ are matrices of appropriate dimensions with real entries. By using the identities in (4.15) and defining augmented vector

$$\zeta_j^T(t) = \begin{bmatrix} e_j^T(t) & e_j^T(t-\tau(t)) & e_j^T(t-\tau_M) & e_j^T(t-\tau_m) & \dot{e}_j^T(t) & x^T(t) & x^T(t-\tau(t)) \end{bmatrix},$$

we obtain

$$\begin{aligned}
&-\int_{t-\tau(t)}^{t} \dot{e}_j^T(s)P_3\dot{e}_j(s)ds \\
&\quad = 2\zeta_j^T(t)W_1 e_j(t) - 2\zeta_j^T(t)W_1 e_j(t-\tau(t)) + \tau(t)\zeta_j^T(t)W_1 P_3^{-1} W_1^T \zeta_j(t) \\
&\qquad - \int_{t-\tau(t)}^{t} (\zeta_j^T(t)W_1 + \dot{e}_j^T(s)P_3)P_3^{-1}(W_1^T\zeta_j(t) + P_3\dot{e}_j(s))ds, \qquad (4.16a)\\
&-\int_{t-\tau_M}^{t-\tau(t)} \dot{e}_j^T(s)(P_3+P_4)\dot{e}_j(s)ds \\
&\quad = 2\zeta_j^T(t)W_2 e_j(t-\tau(t)) - 2\zeta_j^T(t)W_2 e_j(t-\tau_M) \\
&\qquad + (\tau_M - \tau(t))\zeta_j^T(t)W_2(P_3+P_4)^{-1}W_2^T\zeta_j(t) \\
&\qquad - \int_{t-\tau_M}^{t-\tau(t)} (\zeta_j^T(t)W_2 + \dot{e}_j^T(s)(P_3+P_4))(P_3+P_4)^{-1} \\
&\qquad\quad (W_2^T\zeta_j(t) + (P_3+P_4)\dot{e}_j(s))ds \text{ and} \qquad (4.16b)\\
&-\int_{t-\tau(t)}^{t-\tau_m} \dot{e}_j^T(s)P_3\dot{e}_j(s)ds \\
&\quad = 2\zeta_j^T(t)W_3 e_j(t-\tau_m) - 2\zeta_j^T(t)W_3 e_j(t-\tau(t)) \\
&\qquad + (\tau(t)-\tau_m)\zeta_j^T(t)W_3P_4^{-1}W_3^T\zeta_j(t) - \int_{t-\tau(t)}^{t-\tau_m} (\zeta_j^T(t)W_3 \\
&\qquad + \dot{e}_j^T(s)P_4)P_4^{-1}(W_3^T\zeta_j(t) + P_4\dot{e}_j(s))ds, \qquad (4.16c)
\end{aligned}$$

where

$$W_1 = \begin{bmatrix} W_{11} \\ W_{12} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} W_{21} \\ W_{22} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and } W_3 = \begin{bmatrix} W_{31} \\ W_{32} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using assumptions $U_i^T(t)U_i(t) < I$ and $U_{di}^T(t)U_{di}(t) < I$, the following inequalities can be obtained:

$$\begin{aligned} 2e_j^T(t)P_1T_jR_iU_i(t)S_ix(t) \leq & e_j^T(t)(P_1T_jR_i)(P_1T_jR_i)^Te_j(t) \\ & + x^T(t)S_i^TS_ix(t) \qquad (4.17a) \\ 2e_j^T(t)P_1T_jR_{di}U_{di}(t)S_{di}x(t) \leq & e_j^T(t)(P_1T_jR_{di})(P_1T_jR_{di})^Te_j(t) \\ & + x^T(t)S_{di}^TS_{di}x(t). \qquad (4.17b) \end{aligned}$$

Applying inequalities (4.17a) (4.17b), it can be shown that

$$\begin{aligned} 2e_j^T(t)P_1\dot{e}_j(t) \leq \sum_{i=1}^{r} \mu_i(\xi(t)) & \left\{ e_j^T(t)(P_1N_{ij} + N_{ij}^TP_1)e_j(t) \right. \\ & + 2e_j^T(t)P_1N_{dij}e_j(t-\tau(t)) + e_j^T(t)P_1T_jR_i(P_1T_jR_i)^Te_j(t) \\ & + e_j^T(t)P_1T_jR_{di}(P_1T_jR_{di})^Te_j(t) + x^T(t)S_i^TS_ix(t) \\ & \left. + x^T(t-\tau(t))S_{di}^TS_{di}x(t-\tau(t)) \right\} \end{aligned} \qquad (4.18)$$

and

$$\begin{aligned} & \dot{e}_j^T(t)(\tau_MP_3 + (\tau_M - \tau_m)P_4)\dot{e}_j(t) \\ & = \sum_{i=1}^{r} \mu_i(\xi(t)) \left\{ \dot{e}_j^T(t)\Lambda(\bar{N}_{ij}\bar{e}_j(t) + \begin{bmatrix} T_jR_iU_i(t)S_i & T_jR_{di}U_{di}(t)S_{di} \end{bmatrix} \bar{x}(t)) \right. \\ & \qquad \left. -\tau_m\dot{e}_j^T(t)P_4\dot{e}_j(t) \right\} \\ & \leq \sum_{i=1}^{r} \mu_i(\xi(t)) \left\{ \dot{e}_j^T(t)\Lambda\bar{N}_{ij}\bar{e}_j(t) \right. \\ & \qquad + \frac{1}{2}\dot{e}_j^T(t)\Lambda \begin{bmatrix} T_jR_i & T_jR_{di} \end{bmatrix} \begin{bmatrix} T_jR_i & T_jR_{di} \end{bmatrix}^T \Lambda\dot{e}_j(t) \end{aligned}$$

$$+\frac{1}{2}\bar{x}^T(t)\begin{bmatrix} S_i^T S_i & 0 \\ 0 & S_{di}^T S_{di} \end{bmatrix}\bar{x}(t) - \tau_m \dot{e}_j^T(t) P_4 \dot{e}_j(t) \Bigg\}, \tag{4.19}$$

where $\bar{e}_j^T(t) = [e_j^T(t) \quad e_j^T(e-\tau(t))]$, $\bar{x}^T(t) = [x^T(t) \quad x^T(e-\tau(t))]$, $\bar{N}_{ij} = [N_{ij} \quad N_{dij}]$ and $\Lambda = \tau_M(P_3 + P_4)$. Using the identities in (4.16a), (4.16b) and (4.16c), and the inequalities in (4.18) and (4.19), we can write

$$\dot{V}(t) \leq \sum_{i=1}^{r} \zeta_j^T(t)\left(\mathcal{G}_{ij} + \tau_M W_1 P_3^{-1} W_1^T + (\tau_M - \tau_m) W_2 (P_3 + P_4)^{-1} W_2^T \right.$$
$$\left. + (\tau_M - \tau_m) W_3 P_4^{-1} W_3^T + \Gamma_{ij}\Gamma_{ij}^T\right)\zeta_j(t),$$

where

$$\mathcal{G}_{ij} = \begin{bmatrix} \mathcal{G}^{11} & \mathcal{G}^{12} & -W_{21} & W_{31} & \mathcal{G}_{ij}^{15} & 0 & 0 \\ \star & \mathcal{G}^{23} & -W_{22} & W_{32} & \mathcal{G}_{ij}^{25} & 0 & 0 \\ \star & \star & -P_2^2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -P_2^3 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\tau_m P_4 & 0 & 0 \\ \star & \star & \star & \star & \star & \mathcal{G}_{ij}^{66} & 0 \\ \star & \star & \star & \star & \star & \star & \mathcal{G}_{ij}^{77} \end{bmatrix},$$

with

$$\begin{aligned}
\mathcal{G}_{ij}^{1} &= P_1 N_{ij} + N_{ij}^T P_1 + P_2^1 + P_2^2 + P_2^3 + W_{11} + W_{11}^T, \\
\mathcal{G}_{ij}^{12} &= P_1 N_{dij} - W_{11} + W_{12}^T + W_{21} - W_{31}, \\
\mathcal{G}_{ij}^{15} &= \frac{1}{2} N_{ij}^T \Lambda, \\
\mathcal{G}_{ij}^{22} &= -(1-\rho)P_2^1 - W_{12} - W_{12}^T + W_{22} + W_{22}^T - W_{32} - W_{32}^T, \\
\mathcal{G}_{ij}^{25} &= \frac{1}{2} N_{dij}^T \Lambda, \\
\mathcal{G}_{ij}^{66} &= \frac{3}{2} S_i^T S_i + P_2^4, \quad \mathcal{G}_{ij}^{77} = -(1-\rho)P_2^4 + \frac{3}{2} S_{di}^T S_{di},
\end{aligned}$$

and

$$\Gamma_{ij} = \begin{bmatrix} P_1T_jR_i & P_1T_jR_{di} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\Lambda T_jR_i & \frac{1}{2}\Lambda T_jR_{di} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To minimise the effect of parameter uncertainties on the error dynamics, we assume a positive scalar γ and consider

$$\frac{dV(t)}{dt} + e_j^T(t)e_j(t) - \gamma^2 x^T(t)K_j^TK_jx(t) < 0. \tag{4.20}$$

By integration we can write

$$V(\infty) - V(0) < \int_0^\infty (-e_j^T(s)e_j(s) + \gamma^2 x^T(s)K_j^TK_jx(s))ds. \tag{4.21}$$

Under zero initial condition, (4.21) eventually implies

$$\int_0^\infty (e_j^T(s)e_j(s) - \gamma^2 x^T(s)K_j^TK_jx(s))ds < 0 \iff \frac{\|e_j(t)\|_2}{\|u_j(t)\|_2} < \gamma. \tag{4.22}$$

Therefore, a sufficient condition for the error dynamics to approach zero asymptotically with minimised effect of parameter uncertainty on the convergence of error can be given as minimising γ subject to (4.20). Considering $P_3 = \sigma_1 P_1$ and $P_4 = \sigma_2 P_1$ and applying the Schur complement, it can be shown that the inequalities in (4.20) hold if the inequalities in (4.9) hold, where N_{ij} and N_{dij} are obtained from (4.12a) and (4.12b), respectively. This completes the proof. □

The robust functional observer based controller construction procedure for stabilising a T-S fuzzy system with model uncertainty and time-delay is outlined below.

Synthesising steps for the robust functional observer

Step 1: Calculate K_j from the solution of (4.8). Obtain N_{ij}^1, N_{ij}^2, N_{dij}^1, N_{dij}^2, M_{ij}^1, M_{ij}^2, M_{dij}^1, M_{dij}^2, F_j^1 and F_j^2 from (4.12);

Step 2: Specify the ranges of σ_1 and σ_2, and increments $\Delta\sigma_1$ and $\Delta\sigma_2$. Take the minimum value of the ranges of σ_1 and σ_2;

Step 3: Solve the minimising problem in (4.9). If no solution is obtained, increase σ_1 and σ_2 by their respective increments and repeat Step 3, else follow the next step;

Step 4: Calculate Z_j using the values of Y_j. Then, calculate N_{ij}, N_{dij}, F_j, M_{ij}, and M_{dij} as defined in (4.12);

Step 5: Calculate J_{ij} and J_{dij} using the relations $J_{ij} = M_{ij} + N_{ij}F_j$ and $J_{dij} = M_{dij} + N_{dij}F_j$, respectively; and

Step 6: Obtain H_{ij} using (4.7c).

Remark 4.3.1. *It is evident that* (4.10) *requires to have a solution for some* Z_j *such that* N_{ij}*,* N_{dij}*,* M_{ij}*,* M_{dij} *and* F_j *can be expressed by* (4.12)*. Therefore, one necessary condition for the existence of the functional observer is given as the rank equality as below [83]:*

$$rank \begin{bmatrix} K_j\mathcal{A} & K_j\mathcal{A}_d \\ C\mathcal{A} & C\mathcal{A}_d \\ \mathcal{K}_j & 0 \\ \mathcal{C} & 0 \\ 0 & \mathcal{K}_j \\ 0 & \mathcal{C} \end{bmatrix} = rank \begin{bmatrix} C\mathcal{A} & C\mathcal{A}_d \\ \mathcal{K}_j & 0 \\ \mathcal{C} & 0 \\ 0 & \mathcal{K}_j \\ 0 & \mathcal{C} \end{bmatrix}.$$

Remark 4.3.2. *The solution of the optimisation problem in* (4.9) *depends on choosing two scalars* σ_1 *and* σ_2*. This solution depends on the ranges and the increments of these two scalars. We can increase the solution domain by choosing smaller increments and larger ranges.*

Remark 4.3.3. *This chapter considers an uncertain T-S fuzzy model of a delayed nonlinear system for obtaining the functional observer based PDC controller. If there is no model uncertainty in the system, this problem reduces to the problem investigated in the previous chapters. Therefore, this chapter investigates a more generalised problem compared with the problems dealt in Chapters 2 and 3. Moreover, the equality constraints of the stability conditions for the observer presented*

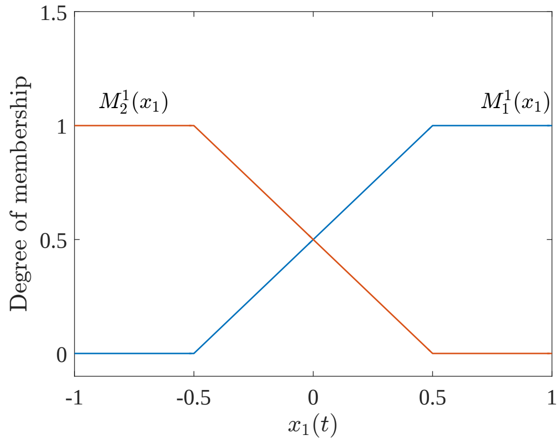

Figure 4.1: Membership functions of fuzzy sets M_2^1 and M_1^1

in Chapter 3 are eliminated in the stability conditions for the observer in this chapter.

Example 4.1

In this example we apply the proposed method to a two rule T-S fuzzy model for illustrating the main results. The matrices of the linear systems representing the two rules are as below:

$$A_1 = \begin{bmatrix} 1 & -0.5 \\ 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 0 & -0.2 \\ 0.2 & 0 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} 0 & -0.1 \\ 0.1 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad R_1 = R_2 = R_{d1} = R_{d1} = \begin{bmatrix} -0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$S_1 = S_2 = \begin{bmatrix} -0.05 & 0.02 \\ 0 & 0.04 \end{bmatrix}, \qquad S_{d1} = S_{d2} = \begin{bmatrix} -0.05 & -0.05 \\ 0.08 & -0.05 \end{bmatrix}.$$

State variable $x_1(t)$ is considered to be the premise variable. Membership functions for the $x_1(t)$ are displayed in Figure 4.1.

We consider $\tau_M = 0.85$, $\tau_m = 0.05$, and $\rho = 0.95$. By following the steps

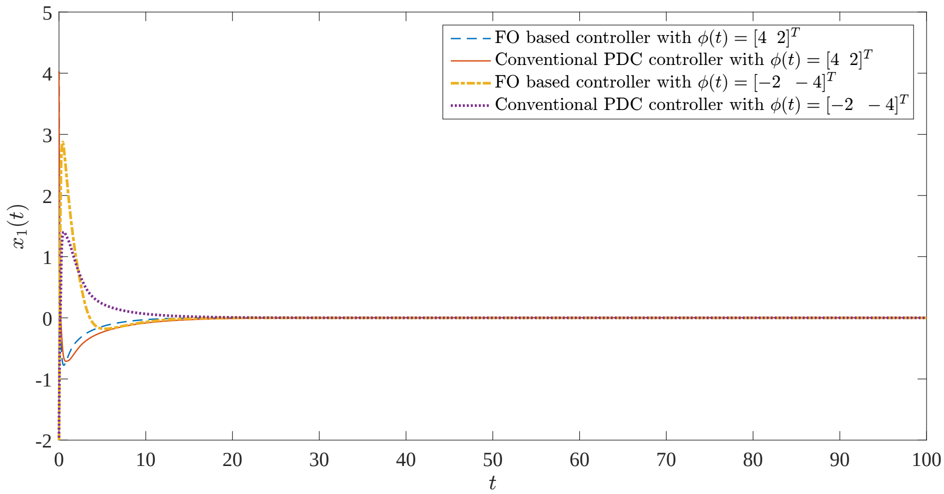

Figure 4.2: Time response of state $x_1(t)$ for the system in Example 4.1

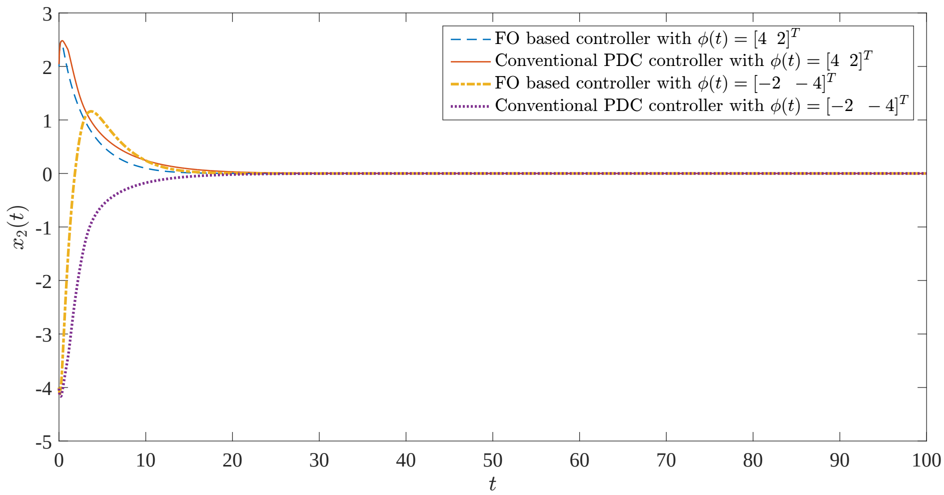

Figure 4.3: Time response of state $x_2(t)$ for the system in Example 4.1

given in Section 4.3, we obtain the observer parameters. SOSTOOLS toolbox in MATLAB has been used to obtain the results of the optimisation problem of (4.9). The observer parameters are as follows:

$$K_1 = \begin{bmatrix} -9.7763 & -2.4474 \end{bmatrix}, \quad N_{11} = -0.3084, \quad N_{21} = -0.7320,$$
$$J_{11} = 2.3363, \quad J_{21} = 5.5453, \quad F_1 = -7.5760,$$
$$N_{d11} = -0.0799, \quad N_{d21} = -0.0400, \quad J_{d11} = 0.6054,$$
$$H_{11} = -2.2003, \quad H_{21} = -1.3202, \quad J_{d21} = 0.3027$$
$$K_2 = \begin{bmatrix} -9.7763 & -2.4474 \end{bmatrix}, \quad N_{12} = -0.3084, \quad N_{21} = -0.7320,$$

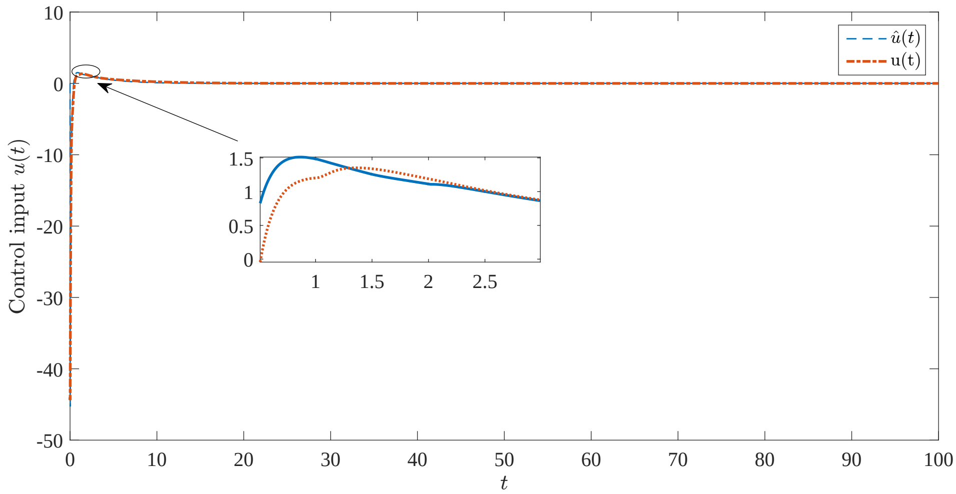

Figure 4.4: Control signals generated for initial condition $\phi = [4 \quad 2]^T$ in Example 4.1

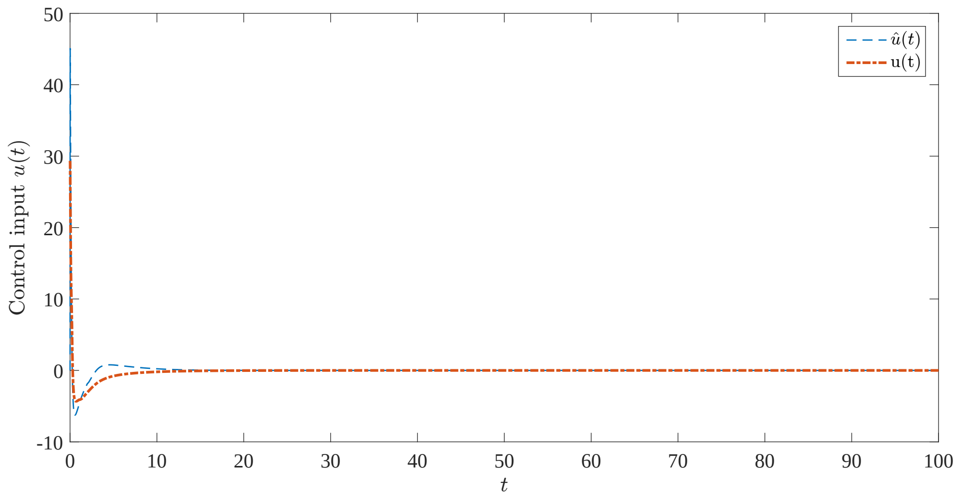

Figure 4.5: Control signals generated for initial condition $\phi = [-2 \quad -4]^T$ in Example 4.1

$$
\begin{aligned}
J_{12} &= 2.3363, & J_{22} &= 5.5453, & F_2 &= -7.5760, \\
N_{d12} &= -0.0799, & N_{d22} &= -0.0400, & J_{d12} &= 0.6054 \\
H_{12} &= -2.2003, & H_{22} &= -1.3202 & J_{d22} &= 0.3027.
\end{aligned}
$$

Considering two input conditions $\phi(t) = [4 \quad 2]^T$ and $\phi(t) = [-2 \quad -4]^T$ system performances are simulated in MATLAB environment. Figures 4.2 and 4.3 display the state responses of the system with the proposed functional observer based PDC controller and the conventional PDC controller [52]. It can be observed that the proposed functional observer based PDC controller stabilises the system asymptotically. Figures 4.4 and 4.5 compare the control signals for these two methods under two initial conditions. In these figures $u(t)$ is the desired control input generated using the conventional PDC controller considering all states are measurable while $\hat{u}(t)$ is the estimated control input obtained by the functional observer. It can be seen that $\hat{u}(t)$ converges with desired $u(t)$. The convergence is depicted with enlarged graphs in Figure 4.4. In comparison with the conventional PDC controller, the proposed method can stabilise the fuzzy system satisfactorily. Nevertheless, its performance can be enhanced by choosing suitable stabilisation conditions. Future work may consider this point.

Example 4.2

In this example we apply the proposed method to the benchmark problem of truck trailer system represented by the delayed uncertain T-S fuzzy model [97] for verifying its applicability. The system is expressed as a two rule T-S fuzzy model with the following matrices.

$$A_1 = \begin{bmatrix} -a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ -a\frac{v^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{lt_0} & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ -ad\frac{v^2\bar{t}^2}{2Lt_0} & \frac{dv\bar{t}}{lt_0} & 0 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} -(1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{lt_0} & 0 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} -(1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{dv^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{lt_0} & 0 \end{bmatrix},$$

$$R_1 = R_2 = R_{d1} = R_{d2} = \begin{bmatrix} 0.255 \\ 0.255 \\ 0.255 \end{bmatrix}, \qquad B_1 = B_2 = \begin{bmatrix} \frac{v\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix},$$

$$S_1 = S_2 = S_{d1} = S_{d2} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix},$$

where $a = 0.7$, $v = -1.0$, $\bar{t} = 2.0$, $t_0 = 0.5$, $L = 5.5$, $l = 2.8$, $d = \frac{10t_0}{\pi}$. By solving the LMIs in Lemma 1 for $\tau_M = 0.5$, $\tau_m = 0.1$, $\rho = 0.4$, $\bar{\sigma}_1 = 0.01$, and $\bar{\sigma}_2 = 0.02$, we find PDC gain matrices $K_1 = K_2 = \begin{bmatrix} 7.0648 & -30.1913 & 0.7873 \end{bmatrix}$. This PDC controller uses state vector $x(t)$ to obtain control law $u(t) = \sum_{j=1}^{r} \mu_i(\theta(t))K_j x(t)$, where

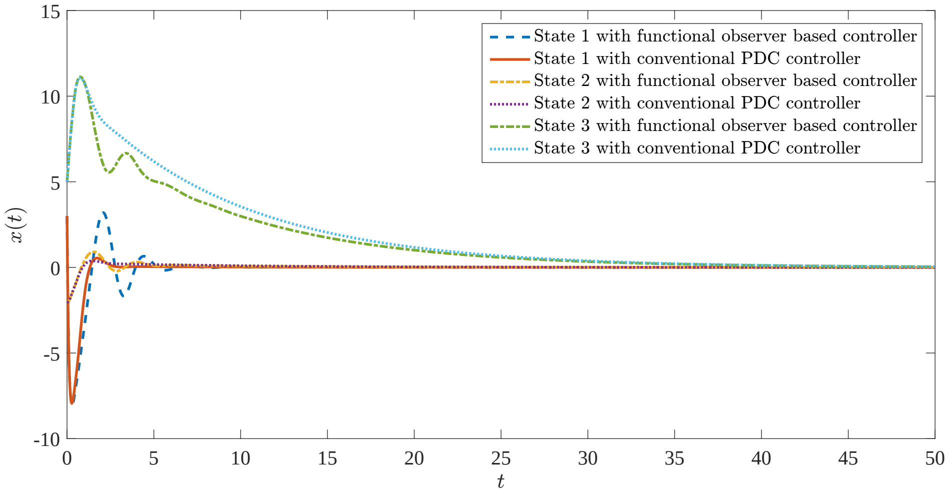

Figure 4.6: State responses of the close loop truck trailer system

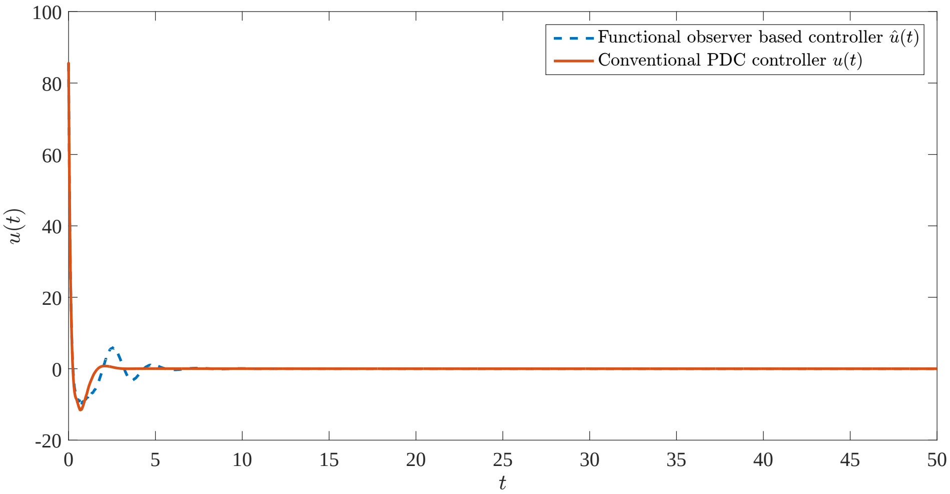

Figure 4.7: Control law for the closed loop truck trailer system

$$\mu_1(\theta(t)) = \left(1 - \frac{1}{1+\exp(3(-\theta(t)-0.5\pi))}\right)\frac{1}{1+\exp(3(-\theta(t)+0.5\pi))},$$
$$\mu_2(\theta(t)) = 1 - \mu_1(\theta(t)).$$

Our objective is to design a functional observer based PDC controller so that the control input is estimated directly without estimating the states. Considering $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and following the steps described in Section 4.3, we find the functional observer parameters as below.

$$N_{11} = -0.8062, \qquad N_{12} = -0.8062,$$
$$N_{21} = 0.4950, \qquad N_{22} = 0.4950,$$
$$N_{d11} = -2.2310, \qquad N_{d12} = -2.2310,$$
$$N_{d21} = -2.2310, \qquad N_{d22} = -2.2310,$$
$$F_1 = \begin{bmatrix} -78.9713 & 31.3140 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} -78.9713 & 31.3140 \end{bmatrix},$$
$$J_{11} = \begin{bmatrix} 161.4356 & -24.6120 \end{bmatrix}, \qquad J_{12} = \begin{bmatrix} 161.4356 & -24.6120 \end{bmatrix},$$
$$J_{21} = \begin{bmatrix} 170.1920 & 15.1114 \end{bmatrix}, \qquad J_{21} = \begin{bmatrix} 170.1920 & 15.1114 \end{bmatrix},$$
$$J_{d11} = \begin{bmatrix} 108.8304 & -68.1065 \end{bmatrix}, \qquad J_{d12} = \begin{bmatrix} 108.8304 & -68.1065 \end{bmatrix},$$
$$J_{d21} = \begin{bmatrix} 108.8304 & -68.1065 \end{bmatrix}, \qquad J_{d22} = \begin{bmatrix} 108.8304 & -68.1065 \end{bmatrix},$$
$$H_{11} = H_{12} = H_{21} = H_{22} = -10.0926.$$

The performance of the proposed controller for the truck trailer system is simulated for an initial condition $\phi(t) = \begin{bmatrix} 3 & -2 & 5 \end{bmatrix}^T$. The time-varying delay is considered to be $\tau(t) = 0.4 + 0.1\sin(t)$. Both the conventional fuzzy control law and proposed functional observer based fuzzy control law are applied to the closed loop system. The state responses for both cases are displayed in Figure 4.6. The system is stable under the effect of model uncertainty and time-delay. The functional observer based controller is compared with the conventional PDC controller in Figure 4.7. Estimated control input $\hat{u}(t)$ obtained by the functional observer converges with the desired control input $u(t)$ as expected.

4.4 Conclusion

A systematic synthesis procedure for obtaining a robust fuzzy functional observer for an uncertain fuzzy system with time varying time-delay is presented. This functional observer is employed to estimate control signal that stabilises the system asymptotically. The stability of the observer is guaranteed in the sense that the estimation error approaches zero asymptotically. The sensitivity of the estimation error to the uncertainty of the model is minimised using a performance index. Lyapunov-Kravoskii functionals are used to ensure asymptotic stability of the observer and the system; the stability conditions are formulated as LMIs.

Solutions of these LMIs are used to construct the observer. The proposed design methodology is illustrated using two examples; the simulation verifies the effectiveness of the proposed method. Future work will consider improving the stability conditions to guarantee the finite-time convergence of the observer.

Chapter 5

Functional observer based fault detection of nonlinear system

This chapter presents a new fuzzy functional observer based fault detection technique for T-S fuzzy systems with time-delay and exogenous disturbance. The estimation error of the fuzzy functional observer is used to obtain a residual to detect the fault. The proposed fault detection scheme does not require any threshold to compare with the residual. The exogenous disturbance is decoupled from the error dynamics using the concept of unknown input observers to ensure the robustness of the residual generator. The time-delay is considered to be time-varying and bounded. Sufficient conditions are presented to ensure the asymptotic stability of the observers by applying LMI approach. A robust fault estimator design technique using the functional observer is also included in this chapter. The proposed design techniques are illustrated and tested using examples to demonstrate the effectiveness. The revision of a manuscript containing the main results of this chapter is under review for publication [100].

5.1 Introduction

Systems are often subject to faults caused by the sudden breakdown or malfunction of actuators, sensors or other components of the system. A wide range of work has appeared in the recent decades for diagnosing the faults of both linear and nonlinear systems considering different aspects including time-delays, uncertainties, exogenous disturbances, and measurement noise [30, 101–107]. The fault diagnosis process consists of three major steps: detection, isolation and estimation [104]. The model-based fault detection technique has attracted much attention as it requires no redundant hardware [108]. The key idea of this technique is to construct a residual that indicates the presence of a fault. The observer based residual generation technique is regarded as the most useful technique when the system model is known. An observer is employed to estimate the output, and the residual is generated using the estimation error, the difference between the estimated output and measured output. The residual is compared with a real-time threshold, and the fault is identified if the residual exceeds the threshold. It is worth noting that the calculation of the threshold, in some cases, is a computational burden of a fault detection scheme. In addition, the order of the observer is of concern when the system order is high.

The linear functional observer, which estimates the function of states without estimating the whole set of states, is inherently a lower order observer compared with a full-order observer. Functional observers for diagnosing faults, however, received less attention. In [109], the authors presented a functional observer based residual generator where the observer order is reduced to one. However, it does not consider time-delay in the system. In [110], the authors extended the approach of functional observer based fault detection considering time-delay for a linear system. However, it does not consider the effect of disturbance on the fault detection procedure. In [111], the authors presented a functional observer based fault detection technique for a linear system considering model uncertainty. Although the proposed method considers model uncertainty, it does not consider time-delays. More importantly, most of the work to date on functional observer based fault detection considers linear systems. To the best of the author's knowledge, fuzzy functional observer based fault detection for nonlinear systems with time-delays in the presence of external disturbance has not received much attention.

When the system is subject to external disturbances, an L_2 norm based performance indicator is calculated to measure the impact of the disturbances on the residual. The value of the performance indicator is minimised to an acceptable amount to reduce the sensitivity of the residual to the exogenous disturbance [103, 112]. Nonetheless, the disturbances can be decoupled from the residual by considering the disturbances as unknown inputs [113]. In [101], the authors showed that the unknown input observer can be used to design robust fault detection filter where the residual possesses the directional property so that the faults can be isolated as well. Time-delay is another unavoidable phenomenon that makes the fault diagnosis scheme vulnerable. A holistic approach of fault diagnosis technique should not only focus on the residual generation, but also consider that the system may be subject to the time-delay and external disturbances. The residual should be sensitive by an acceptable amount or, if possible, should be insensitive to the external noise.

Considering the recent interest of using T-S fuzzy modeling approach for fault detection of nonlinear systems [103, 106, 107, 112, 114–117], this chapter presents a novel residual generator and fault estimator for a nonlinear system using the fuzzy functional observer. The system is subject to time-delay and external disturbances. The time delay is bounded by known upper and lower bounds. Lyapunov-Krasovskii functional is used to construct stability conditions for the observers in the form of LMIs. The exogenous disturbance is decoupled from the residual using the concept of unknown input observers. The main advantage of using the proposed fuzzy functional observer based fault detection method is that, unlike existing fault detection procedures of nonlinear systems, the proposed method does not require the calculation of any thresholds. Therefore, considering the real-time calculation burden, the proposed method is less computationally involved. Furthermore, it is an inherently reduced order observer.

Notation: $\mathbb{R}^n$ denotes n dimensional real vectors, R_+ denotes positive real numbers, and $\mathbb{R}^{n\times m}$ represents $n \times m$ dimensional real matrices. I_p denotes a $p \times p$ identity matrix. Symbolic superscripts $(\cdot)^-$, $(\cdot)^+$, $(\cdot)^\perp$, and $(\cdot)^T$ mean inverse, Moore-Penrose generalised inverse, orthogonal basis, and transpose of corresponding matrix, respectively. Numeric superscripts and subscripts $(\cdot)_{ij}^{k}$ are used to distinguish among different matrices. In a symmetric matrix, $\star$ represents the symmetric elements of respective blocks.

5.2 Model description and preliminaries

Consider a T-S fuzzy system with the following r number of rules:

$$\begin{aligned}
\text{IF} \quad & \xi_1(t) \text{ is } M_i^1 \text{ and } \cdots \text{ and } \xi_l(t) \text{ is } M_i^l \\
\text{THEN} \quad & \dot{x}(t) = A_i x(t) + A_{di} x(t-\tau(t)) + B_i u(t) + E_i f(t) \\
& y(t) = Cx(t) \\
& x(t) = \phi(t), t \in [-\tau(t), 0), i = 1, \ldots, r,
\end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors respectively, $\xi_1(t), \ldots, \xi_l(t)$ are the premise variables, and $\tau(t)$ is the time delay. System matrices $A_i \in \mathbb{R}^{n\times n}$, $A_{di} \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ describe the ith linear subsystem of the T-S fuzzy system for the ith rule. Fault vector $f(t) \in \mathbb{R}^{m_f}$ affects the state dynamics of a subsystem through a known fault signature matrix $E_i \in \mathbb{R}^{n\times m_f}$. The fuzzy set for premise variable $\xi_k(t)$ in the ith rule is denoted by M_i^k. Considering $\xi(t) = \left[\xi_1(t) \ldots \xi_l(t)\right]$, the overall system dynamics can be represented by

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + A_{di} x(t-\tau(t)) + B_i u(t) + E_i f(t)\} \\
y(t) &= Cx(t),
\end{aligned} \tag{5.2}$$

where $\mu_i(\xi(t)) = \frac{\prod_{k=1}^{l} M_i^k(\xi_k(t))}{\sum_{i=1}^{r}\prod_{k=1}^{l} M_i^k(\xi_k(t))}$ with $\mu_i(\xi(t)) \geq 0$ and $\sum_{i=1}^{r} \mu_i(\xi(t)) = 1$.

The upper and lower bounds of the time-delay are τ_M and τ_m, respectively, i.e., $\tau_M \geq \tau(t) \geq \tau_m$. Without loss of generality, it is assumed that $\text{rank}(C) = p$, $\text{rank}(E_i) = m_f$, and the faults are linearly independent. The premise variables and time-delay parameters are known at all times.

Suppose $w(t) \in \mathbb{R}^{n_o}$ is the fuzzy summation of $w_j(t) \in \mathbb{R}^{n_o}$, which are linear functions of states $x(t)$ for respective rules as given below:

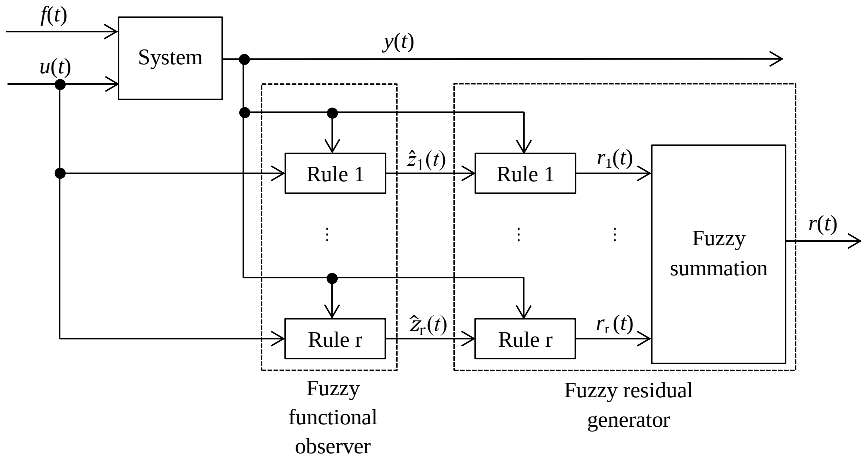

Figure 5.1: Fuzzy functional observer based residual generation scheme of a T-S fuzzy system

$$
\begin{aligned}
w(t) &= \sum_{j=1}^{r} \mu_j(\xi(t)) w_j(t) \\
&= \sum_{j=1}^{r} \mu_j(\xi(t)) L_j x(t),
\end{aligned}
$$

where $L_j \in \mathbb{R}^{n_o \times n}$ is chosen such that $\operatorname{rank}(L_j) = n_o$. The procedure of obtaining L_j is discussed in the following. The following fuzzy functional observer is proposed for detecting the fault of system (5.2):

$$
\begin{aligned}
\dot{\hat{w}}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t)) \{ & N_{ij} \hat{w}_j(t) + N_{dij} \hat{w}(t - \tau(t)) \\
& + J_{ij} y(t) + J_{dij} y(t - \tau(t)) + H_{ij} u(t) \} \\
\hat{w}_j(t) = \hat{w}_{j0}, t \in & \left[-\tau(t), 0\right),
\end{aligned}
\tag{5.3}
$$

where $N_{ij} \in \mathbb{R}^{n_o \times n_o}$, $N_{dij} \in \mathbb{R}^{n_o \times n_o}$, $J_{ij} \in \mathbb{R}^{n_o \times p}$, $J_{dij} \in \mathbb{R}^{n_o \times p}$, and $H_{ij} \in \mathbb{R}^{n_o \times m}$. In the no-fault condition, i.e., $f(t) = 0$, $\hat{w}(t)$ is an asymptotic estimation of $w(t)$. The residual can be obtained as

$$
r(t) = \sum_{j=1}^{r} \mu_j(\xi(t)) (R_j \hat{w}_j(t) + F_j y(t)), \tag{5.4}
$$

where $R_j \in \mathbb{R}^{1 \times n_o}$ and $F_j \in \mathbb{R}^{1 \times p}$ are determined such that

$$\lim_{t\to\infty} r(t) = \begin{cases} 0 \text{ if } f(t) = 0 \\ c \text{ if } f(t) \neq 0, \end{cases} \tag{5.5}$$

with $c \neq 0$. Therefore, $r(t) = 0$ implies a fault-free condition of the system, whereas $r(t) \neq 0$ implies faulty condition of the system. The functional observer based residual generation scheme for a T-S fuzzy system is depicted in Figure 5.1.

Defining the estimation error $e_j(t) = L_j x(t) - \hat{w}_j(t)$, the error dynamics can be expressed as

$$\begin{aligned} \dot{e}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{&N_{ij}e_j(t) + N_{dij}e_j(t-\tau(t)) \\ &+ (L_jA_{ij} - N_{ij}L_j - J_{ij}C)x(t) \\ &+ (L_jA_{dij} - N_{dij}L_j - J_{dij}C)x(t-\tau(t)) \\ &+ (L_jB_i - H_{ij})u(t) + L_jE_if(t)\}. \end{aligned} \tag{5.6}$$

The dynamics in (5.6) reduces to

$$\dot{e}_j(t) = \sum_{i=1}^{r} \mu(\xi)\{N_{ij}e_j(t) + N_{dij}e_j(t-\tau(t)) + L_jE_if(t)\}, \tag{5.7}$$

if the following conditions hold:

$$L_jA_i - N_{ij}L_j - J_{ij}C = 0 \tag{5.8a}$$

$$L_jA_{dij} - N_{dij}L_j - J_{dij}C = 0 \tag{5.8b}$$

$$L_jB_i - H_i = 0. \tag{5.8c}$$

Observer (5.3) estimates $L_jx(t)$ asymptotically in the absence of fault $f(t)$ if the error system

$$\dot{e}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{N_{ij}e_j(t) + N_{dij}e_j(t-\tau(t))\} \tag{5.9}$$

is asymptotically stable and the conditions in (5.8a), (5.8b) and (5.8c) hold. Furthermore, by using the definition of the estimation error, the residual generator in (5.4) reduces to

$$r(t) = -\sum_{j=1}^{r} \mu_j(\xi(t)) R_j e_j(t), \tag{5.10}$$

if

$$R_j L_j + F_j C = 0. \tag{5.11}$$

By rewriting condition (5.11) as

$$\begin{bmatrix} R_j & F_j \end{bmatrix} \begin{bmatrix} L_j \\ C \end{bmatrix} = 0, \tag{5.12}$$

we can chose L_j such that (5.11) holds.

Remark 5.2.1. *As C is a full row rank matrix, L_j can be chosen to be any linear combinations of the rows of C such that (5.12) holds. Then we can obtain the left null space of the matrix $\begin{bmatrix} L_j \\ C \end{bmatrix}$, and take any row of the null space matrix to obtain R_j and F_j.*

The residual generation problem is transformed into a problem of obtaining the parameters of the fuzzy functional observer such that error system (5.9) is asymptotically stable and conditions in (5.8) and (5.11) hold.

The following lemma is used to develop the stability conditions for the fault detection observers in the next section.

Lemma 5.2.1. *[118] For any real matrices X_i, Y_i, and $S > 0$ with appropriate dimensions, we have*

$$2\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j X_i^T S Y_j \leq \sum_{i=1}^{r} h_i (X_i^T S X_i + Y_i^T S Y_i) \tag{5.13}$$

where $h_i \geq 0$ and $\sum_{i=1}^{r} h_i = 1$.

5.3 Fault detection and estimation

5.3.1 Fault detection

We present the following theorem that describes the stability condition for the observer as LMIs. The observer parameters can be obtained using the solution of these inequalities.

Theorem 5.3.1. *For given scalars τ_M, τ_m, σ_1 and σ_2, fault detection observer* (5.3) *is asymptotically stable if there exist positive definite symmetric matrices P_1, P_2^1, P_2^2 and P_2^3, and matrices Y_{ij}, W_{11}, W_{12}, W_{21}, W_{22}, W_{31} and W_{32} with appropriate dimensions such that*

$$\begin{bmatrix} \Xi_{ij}^{1,1} & \Xi_{ij}^{1,2} & -W_{21} & W_{31} & \tau_M\sigma_1 W_{11} & \tilde{\tau}(\sigma_1+\sigma_2)W_{21} & \tilde{\tau}\sigma_2 W_{31} & \Xi_{ij}^{1,8} \\ \star & \Xi_{ij}^{2,2} & -W_{22} & W_{32} & \tau_M\sigma_1 W_{12} & \tilde{\tau}(\sigma_1+\sigma_2)W_{22} & \tilde{\tau}\sigma_2 W_{32} & \Xi_{ij}^{2,8} \\ \star & \star & -P_2^2 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -P_2^3 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\tau_M\sigma_1 P_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{\tau}(\sigma_1+\sigma_2)P_1 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\tilde{\tau}\sigma_2 P_1 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\kappa P_1 \end{bmatrix} < 0 \tag{5.14}$$

for all $i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, r$, where

$$\begin{aligned} \Xi_{ij}^{1,1} &= P_1 N_{ij}^1 + (N_{ij}^1)^T P_1 + Y_{ij} N_{ij}^2 + (N_{ij}^2)^T Y_{ij}^T + P_2^1 + P_2^2 + P_2^3 + W_{11} + W_{11}^T, \\ \Xi_{ij}^{1,2} &= P_1 N_{dij}^1 + Y_{ij} N_{dij}^2 + W_{12}^T - W_{11} + W_{21} - W_{31}, \\ \Xi_{ij}^{2,2} &= -(1-\rho)P_2^1 + W_{12} + W_{12}^T + W_{22} + W_{22}^T - W_{32} - W_{32}^T, \\ \tilde{\tau} &= \tau_M - \tau_m, \quad \kappa = \sigma_1\tau_M + \sigma_2\tilde{\tau}, \\ \Xi_{ij}^{1,8} &= \kappa((N_{ij}^1)^T P_1 + (N_{ij}^2)^T Y_{ij}^T), \quad \Xi_{i,j}^{2,8} = \kappa((N_{dij}^1)^T P_1 + (N_{dij}^2)^T Y_{ij}^T) \end{aligned}$$

with N_{ij}^1, N_{ij}^2, N_{dij}^1 and N_{dij}^2 being defined in (5.20a) *and* (5.20b).

Proof. Without loss of generality, we assume that L_j has full row rank. Therefore, we can obtain a full rank matrix $Q_j = \begin{bmatrix} L_j^+ & L_j^\perp \end{bmatrix}$. Post multiplication of (5.8a) by Q_j gives

$$N_{ij} L_j \begin{bmatrix} L_j^+ & L_j^\perp \end{bmatrix} + J_{ij} C \begin{bmatrix} L_j^+ & L_j^\perp \end{bmatrix} - L_j A_i \begin{bmatrix} L_j^+ & L_j^\perp \end{bmatrix} = 0. \tag{5.15}$$

After some algebraic manipulation, (5.15) can be rewritten as

$$N_{ij} = L_j A_i L_j^+ - J_{ij} C L_j^+, \tag{5.16a}$$

$$J_{ij} C L_j^\perp = L_j A_i L_j^\perp. \tag{5.16b}$$

Similarly, (5.8b) can be converted into

$$N_{dij} = L_j A_i L_j^+ - J_{dij} C L_j^+, \tag{5.17a}$$

$$J_{dij} C L_j^\perp = L_j A_{dij} L^\perp. \tag{5.17b}$$

By augmenting (5.16b) and (5.17b) we obtain

$$\begin{bmatrix} J_{ij} & J_{dij} \end{bmatrix} \Phi_j = \Psi_{ij}, \tag{5.18}$$

where $\Phi_j = \begin{bmatrix} CL_j^\perp & 0 \\ 0 & CL_j^\perp \end{bmatrix}$ and $\Psi_{ij} = \begin{bmatrix} L_j A_{ij} L_j^\perp & L_j A_{dij} L_j^\perp \end{bmatrix}$. By using the general solution of (5.18), we can write

$$J_{ij} = \left[\Psi_{ij}\Phi_j^+ + Z_{ij}(I_{2p} - \Phi_j\Phi_j^+)\right] \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \tag{5.19a}$$

$$J_{dij} = \left[\Psi_{ij}\Phi_j^+ + Z_{ij}(I_{2p} - \Phi_j\Phi_j^+)\right] \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \tag{5.19b}$$

where Z_{ij} is an arbitrary matrix. Using (5.16a), (5.17a), (5.19a), and (5.19b), observer parameters N_{ij} and N_{dij} can be expressed as

$$N_{ij} = N_{ij}^1 + Z_{ij} N_{ij}^2 \tag{5.20a}$$

$$N_{dij} = N_{dij}^1 + Z_{ij} N_{dij}^2, \tag{5.20b}$$

where $N_{ij}^1 = L_j A_i L_j^+ - \Psi_{ij}\Phi_j^+ \begin{bmatrix} I_p \\ 0 \end{bmatrix} CL_j^+, \qquad N_{ij}^2 = (\Phi_j\Phi_j^+ - I_{2p}) \begin{bmatrix} I_p \\ 0 \end{bmatrix} CL_j^+,$

$$N_{dij}^1 = L_j A_i L_j^+ - \Psi_{ij}\Phi_j^+ \begin{bmatrix} 0 \\ I_p \end{bmatrix} CL_j^+, \quad N_{dij}^2 = (\Phi_j\Phi_j^+ - I_{2p}) \begin{bmatrix} 0 \\ I_p \end{bmatrix} CL_j^+.$$

Consider a Lyapunov-Krasovskii functional,

$$\begin{aligned}V_j(t) &= e_j^T(t)P_1e_j(t) + \int_{t-\tau(t)}^{t} e_j^T(s)P_2^1e_j(s)ds + \int_{t-\tau_M}^{t} e_j^T(s)P_2^2e_j(s)ds \\ &+ \int_{t-\tau_m}^{t} e_j^T(s)P_2^3e_j(s)ds + \int_{-\tau_M}^{0}\int_{t+\theta}^{t} \dot{e}_j^T(s)P_3\dot{e}_j(s)dsd\theta \\ &+ \int_{-\tau_M}^{-\tau_m}\int_{t+\theta}^{t} \dot{e}_j^T(s)P_4\dot{e}_j(s)dsd\theta,\end{aligned} \tag{5.21}$$

where matrices P_1, P_2^1, P_2^2, P_2^3, P_3 and P_4 are symmetric and positive definite. By taking derivative of $V_j(t)$, we obtain

$$\begin{aligned}\dot{V}_j(t) &= 2e_j^T(t)P_1\dot{e}_j(t) + e_j^T(t)P_2^1e_j(t) - (1-\dot{\tau}(t))e_j^T(t-\tau(t))P_2^1e_j(t-\tau(t)) \\ &+ e_j^T(t)P_2^2e_j(t) - e_j^T(t-\tau_M)P_2^2e_j(t-\tau_M) + e_j^T(t)P_2^3e_j(t) \\ &- e_j^T(t-\tau_m)P_2^3e_j(t-\tau_m) + \dot{e}_j^T(t)(\tau_MP_3 + (\tau_M-\tau_m)P_4)\dot{e}_j(t) \\ &- \int_{t-\tau_M}^{t} \dot{e}_j^T(s)P_3\dot{e}_j(s)ds - \int_{t-\tau_M}^{t-\tau_m} \dot{e}_j^T(s)P_4\dot{e}_j(s)ds \\ &\leq 2e_j^T(t)P_1\dot{e}_j(t) + e_j^T(t)(P_2^1+P_2^2+P_2^3)e_j(t) \\ &- (1-\rho)e_j^T(t-\tau(t))P_2^1e_j(t-\tau(t)) - e_j^T(t-\tau_M)P_2^2e_j(t-\tau_M) \\ &- e_j^T(t-\tau_m)P_2^3e_j(t-\tau_m) + \dot{e}_j^T(t)(\tau_MP_3 + (\tau_M-\tau_m)P_4)\dot{e}_j(t) \\ &- \int_{t-\tau(t)}^{t} \dot{e}_j^T(s)P_3\dot{e}_j(s)ds - \int_{t-\tau_M}^{t-\tau(t)} \dot{e}_j^T(s)(P_3+P_4)\dot{e}_j(s)ds \\ &- \int_{t-\tau(t)}^{t-\tau_m} \dot{e}_j^T(s)P_4\dot{e}_j(s)ds.\end{aligned}$$

By using the Leibniz-Newton formula, the following identities can be written:

$$\begin{aligned}&2(e_j^T(t)W_{11} + e_j^T(t-\tau(t))W_{12}) \\ &\qquad \left(e_j(t) - e_j(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{e}_j(s)ds\right) = 0,\end{aligned} \tag{5.23a}$$

$$\begin{aligned}&2(e_j^T(t)W_{21} + e_j^T(t-\tau(t))W_{22}) \\ &\qquad \left(e_j(t-\tau(t)) - e_j(t-\tau_M) - \int_{t-\tau_M}^{t-\tau(t)} \dot{e}_j(s)ds\right) = 0,\end{aligned} \tag{5.23b}$$

$$\begin{aligned}&2(e_j^T(t)W_{31} + e_j^T(t-\tau(t))W_{32}) \\ &\qquad \left(e_j(t-\tau_m) - e_j(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_m} \dot{e}_j(s)ds\right) = 0,\end{aligned} \tag{5.23c}$$

where W_{kl} for $k = 1, \ldots, 3$ and $l = 1, 2$ are any matrices with appropriate dimensions. By using the identities of (5.23a), (5.23b), and (5.23c), and defining an augmented vector

$$\zeta_j^T(t) = \begin{bmatrix} e_j^T(t) & e_j^T(t-\tau(t)) & e_j^T(t-\tau_M) & e_j^T(t-\tau_m) \end{bmatrix},$$

we obtain

$$\begin{aligned}
&-\int_{t-\tau(t)}^{t} \dot{e}_j^T(s)P_3\dot{e}_j(s)ds \\
&= 2\zeta_j^T(t)W_1e_j(t) - 2\zeta_j^T(t)W_1e_j(t-\tau(t)) + \tau(t)\zeta_j^T(t)W_1P_3^{-1}W_1^T\zeta_j(t) \\
&\quad - \int_{t-\tau(t)}^{t} (\zeta_j^T(t)W_1 + \dot{e}_j^T(s)P_3)P_3^{-1}(W_1^T\zeta_j(t) + P_3\dot{e}_j(s))ds, \qquad (5.24a) \\
&-\int_{t-\tau_M}^{t-\tau(t)} \dot{e}_j^T(s)(P_3+P_4)\dot{e}_j(s)ds \\
&= 2\zeta_j^T(t)W_2e_j(t-\tau(t)) - 2\zeta_j^T(t)W_2e_j(t-\tau_M) \\
&\quad + (\tau_M - \tau(t))\zeta_j^T(t)W_2(P_3+P_4)^{-1}W_2^T\zeta_j(t) \\
&\quad - \int_{t-\tau_M}^{t-\tau(t)} (\zeta_j^T(t)W_2 + \dot{e}_j^T(s)(P_3+P_4))(P_3+P_4)^{-1} \\
&\qquad\qquad (W_2^T\zeta_j(t) + (P_3+P_4)\dot{e}_j(s))ds, \qquad (5.24b) \\
&-\int_{t-\tau(t)}^{t-\tau_m} \dot{e}_j^T(s)P_4\dot{e}_j(s)ds \\
&= 2\zeta_j^T(t)W_3e_j(t-\tau_m) - 2\zeta_j^T(t)W_3e_j(t-\tau(t)) \\
&\quad + (\tau(t) - \tau_m)\zeta_j^T(t)W_3P_4^{-1}W_3^T\zeta_j(t) \\
&\quad - \int_{t-\tau(t)}^{t-\tau_m} (\zeta_j^T(t)W_3 + \dot{e}_j^T(s)P_4)P_4^{-1}(W_3^T\zeta_j(t) + P_4\dot{e}_j(s))ds, \qquad (5.24c)
\end{aligned}$$

where

$$W_1 = \begin{bmatrix} W_{11} \\ W_{12} \\ 0 \\ 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} W_{21} \\ W_{22} \\ 0 \\ 0 \end{bmatrix} \quad \text{and } W_3 = \begin{bmatrix} W_{31} \\ W_{32} \\ 0 \\ 0 \end{bmatrix}.$$

By using (5.9) and (5.13) we have

$$\dot{e}_j^T(t)(\tau_M P_3 + (\tau_M - \tau_m)P_4)\dot{e}_j(t)$$

$$
\begin{aligned}
&= \sum_{i=1}^{r}\sum_{k=1}^{r} \mu_i(\xi(t))\mu_k(\xi(t))(N_{ij}e_j(t) + N_{dij}e_j(t-\tau(t))^T\Lambda \\
&\qquad\qquad (N_{kj}e_j(t) + N_{dkj}e_j(t-\tau(t)) \\
&\le \sum_{i=1}^{r} \mu_i(\xi(t))(N_{ij}e_j(t) + N_{dij}e_j(t-\tau(t))^T\Lambda \\
&\qquad\qquad (N_{ij}e_j(t) + N_{dij}e_j(t-\tau(t)),
\end{aligned}
\tag{5.25}
$$

where $\Lambda = \tau_M P_3 + (\tau_M - \tau_m)P_4$. By using (5.24a), (5.24b), (5.24c) and (5.25), we can write

$$
\begin{aligned}
\dot{V}_j(t) \le \sum_{i=1}^{r} \zeta^T(t)\left\{\mathcal{G}_{ij} + \tau_M W_1 P_3^{-1} W_1^T + (\tau_M - \tau_m)W_2(P_3+P_4)^{-1}W_2^T\right. \\
\left. +(\tau_M - \tau_m)W_3P_4^{-1}W_3^T + \mathcal{N}_{ij}\Lambda^{-1}\mathcal{N}_{ij}^T\right\}\zeta_j(t),
\end{aligned}
\tag{5.26}
$$

where

$$
\mathcal{G}_{ij} = \begin{bmatrix} \mathcal{G}_{ij}^{1,1} & \mathcal{G}_{ij}^{1,2} & -W_{21} & W_{31} \\ \star & \mathcal{G}_{ij}^{2,2} & -W_{22} & W_{32} \\ \star & \star & -P_2^2 & 0 \\ \star & \star & \star & -P_2^3 \end{bmatrix} \text{ and } \mathcal{N}_{ij} = \begin{bmatrix} N_{ij}^T\Lambda \\ N_{dij}^T\Lambda \\ 0 \\ 0 \end{bmatrix}
$$

with

$$
\begin{aligned}
\mathcal{G}_{ij}^{1,1} &= P_1N_{ij} + N_{ij}^TP_1 + P_2^1 + P_2^2 + P_2^3 + W_{11} + W_{11}^T, \\
\mathcal{G}_{ij}^{1,2} &= PN_{dij} + W_{12}^T - W_{11} + W_{21} - W_{31} \text{ and} \\
\mathcal{G}_{ij}^{2,2} &= -(1-\rho)P_2^1 + W_{12} + W_{12}^T + W_{22} + W_{22}^T - W_{32} - W_{32}^T.
\end{aligned}
$$

Therefore, the asymptotic stability condition, $\dot{V}(t) < 0$, holds if we have

$$
\begin{aligned}
\mathcal{G}_{ij} + \tau_M W_1P_3^{-1}W_1^T + (\tau_M - \tau_m)W_2(P_3+P_4)^{-1}W_2^T \\
+ (\tau_M - \tau_m)W_3P_4^{-1}W_3^T + \mathcal{N}_{ij}\Lambda^{-1}\mathcal{N}_{ij}^T < 0
\end{aligned}
\tag{5.27}
$$

for all $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, r$. Applying the Schur complement, inequality (5.27) can be rewritten as

$$\begin{bmatrix} \Xi_{ij}^{1,1} & \Xi_{ij}^{1,2} & -W_{21} & W_{31} & \tau_M W_{11} & \tilde{\tau} W_{21} & \tilde{\tau} W_{31} & N_{ij}^T \Lambda \\ \star & \Xi_{ij}^{2,2} & -W_{22} & W_{32} & \tau_M W_{12} & \tilde{\tau} W_{22} & \tilde{\tau} W_{32} & N_{dij}^T \Lambda \\ \star & \star & -P_2^2 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -P_2^3 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\tau_M P_3 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{\tau}(P_3 + P_4) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\tilde{\tau} P_4 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\Lambda \end{bmatrix} < 0, \quad (5.28)$$

where

$$\begin{aligned} \Xi_{ij}^{1,1} &= P_1 N_{ij} + N_{ij}^T P_1 + P_2^1 + P_2^2 + P_2^3 + W_{11} + W_{11}^T, \\ \Xi_{ij}^{1,2} &= P_1 N_{dij} + W_{12}^T - W_{11} + W_{21} - W_{31} \text{ and} \\ \Xi_{ij}^{2,2} &= -(1-\rho) P_2^1 + W_{12} + W_{12}^T + W_{22} + W_{22}^T - W_{32} - W_{32}^T. \end{aligned}$$

Using the expressions of N_{ij} and N_{dij} in (5.20a) and (5.20b) respectively, and defining $P_3 = \sigma_1 P_1$, $P_4 = \sigma_2 P_4$ and $Y_{ij} = P_1 Z_{ij}$ for some positive scalars σ_1 and σ_2 the LMIs in (5.28) can be written as (5.14). Furthermore, if J_{ij} and J_{dij} are obtained from (5.19a) and (5.19b), respectively, using the solution of Y_{ij} of the LMIs in (5.14), and H_{ij} is obtained from (5.8c), then all of the conditions of (5.8) will be satisfied. □

Remark 5.3.1. *Observer matrices N_{ij} and N_{dij} can be expressed in the form of* (5.20a) *and* (5.20b)*, respectively, if J_{ij} and J_{dij} have solutions. The existence conditions of these solutions can be given as a rank equality condition as below:*

$$rank \begin{bmatrix} CL_j^{\perp} & 0 \\ 0 & CL_j^{\perp} \\ L_j A_i L_j^{\perp} & L_j A_{di} L_j^{\perp} \end{bmatrix} = rank \begin{bmatrix} CL_j^{\perp} & 0 \\ 0 & CL_j^{\perp} \end{bmatrix}. \quad (5.29)$$

The procedure for constructing a functional observer based residual generator of a time-delay T-S fuzzy system is outlined below.

Fuzzy functional observer based residual generator design algorithm

Step 1: Find L_j following the procedure described in Remark 5.2.1;

Step 2: Specify the ranges and increments of σ_1 and σ_2. Set σ_1 and σ_2 to the lower limits of the ranges;

Step 3: Obtain N_{ij}^1, N_{ij}^2, N_{dij}^1 and N_{dij}^2 using (5.20a) and (5.20b);

Step 4: Solve the LMIs of (5.14). If there is a solution, jump to Step 6, else proceed to the next step;

Step 5: Add the increments with σ_1 and σ_2 and jump to Step 3;

Step 6: Obtain Z_{ij} using the solutions of the LMIs and calculate N_{ij} and N_{dij} using (5.20a) and (5.20b) respectively;

Step 7: Calculate J_{ij} and J_{dij} using (5.19a) and (5.19b), respectively; and

Step 8: Obtain H_{ij} using (5.8c).

Remark 5.3.2. *The proposed method for constructing the observer applies the technique of parameterising observer matrices N_{ij}, N_{dij}, J_{ij} and J_{dij} using the general solution of respective equations in terms of unknown matrix Z_{ij} and known matrices N_{ij}^1, N_{ij}^2, N_{dij}^1, N_{dij}^2, Φ_j and Ψ_{ij} calculated from known system matrices A_i, A_{di} and C. We can obtain Z_{ij} from the solution of the LMI based stability condition for the observer. Therefore, the construction of the observer matrices ensures the asymptotic stability of the observer. Although the parameterisation involves more algebraic calculation compared with existing observer based residual generation techniques, the proposed residual generation method eliminates the requirement of real-time threshold calculation as well as reduces the observer size significantly.*

Remark 5.3.3. *If the system has a single output, i.e., C has one row, we cannot obtain L_j by following the procedure stated in Remark 5.2.1. We can obtain L_j by applying the concept of functional observability. Each subsystem of* (5.2) *is functional observable for L_0 if and only if [76]*

$$\operatorname{rank}\begin{bmatrix} C \\ CA_j \\ \vdots \\ CA_j^{n-1} \\ L_0 \\ L_0 A_j \\ \vdots \\ L_0 A_j^{n-1} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} C \\ CA_j \\ \vdots \\ CA_j^{n-1} \end{bmatrix}.$$

Considering $L_0 = C$, we can obtain $L_j = \begin{bmatrix} L_0 \\ \tilde{L}_j \end{bmatrix}$, where $\tilde{L}_j$ is constructed such that it belongs to the row space of

$$\begin{bmatrix} C \\ CA_j \\ \vdots \\ CA_j^{n-1} \end{bmatrix}.$$

Remark 5.3.4. *If L_j is nonsingular, we do not need to calculate pseudo inverse L_j^+ and orthogonal matrix $L_J^\perp$. As a consequence, we have*

$$N_{ij}^1 = L_j A_i L_j^{-1}, \qquad N_{ij}^2 = -\begin{bmatrix} I_p \\ 0 \end{bmatrix} CL_j^{-1},$$

$$N_{dij}^1 = L_j A_{di} L_j^{-1}, \qquad N_{dij}^2 = -\begin{bmatrix} I_p \\ 0 \end{bmatrix} CL_j^{-1},$$

$$Z_{ij} = \begin{bmatrix} J_{ij} & 0 \\ 0 & J_{dij} \end{bmatrix}.$$

Once we find Z_{ij} from the solution of the stability condition for the observer, J_{ij} and J_{dij} can be calculated by partitioning Z_{ij}.

5.3.2 Robust fault detection

It is common that the fault detection process is subject to external unknown disturbances. In this subsection we present a functional observer based residual generator that is robust against external disturbances. Consider a T-S fuzzy model:

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + A_{di} x(t-\tau(t)) + B_i u(t) + E_i f(t) + D_i d(t)\} \\
y(t) &= Cx(t),
\end{aligned}
\tag{5.30}
$$

where $d(t) \in \mathbb{R}^{m_d}$ is unknown external disturbance, $D_i \in \mathbb{R}^{n \times m_d}$, and the other matrices are as defined in (5.1). We use the following fuzzy functional observer for constructing a robust residual generator.

$$
\begin{aligned}
\dot{z}_j(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\{N_{ij} z_j(t) + N_{dij} z_j(t-\tau(t)) \\
&\qquad + J_{ij} y(t) + J_{dij} y(t-\tau(t)) + H_{ij} u(t)\} \\
z_j(t) &= z_{j_0}, t \in [-\tau(t), 0) \\
\hat{w}_j(t) &= z_j(t) + M_j y(t),
\end{aligned}
\tag{5.31}
$$

where $N_{ij} \in \mathbb{R}^{n_o \times n_o}$, $N_{dij} \in \mathbb{R}^{n_o \times n_o}$, $J_{ij} \in \mathbb{R}^{n_o \times p}$, $J_{dij} \in \mathbb{R}^{n_o \times p}$, $H_{ij} \in \mathbb{R}^{n_o \times m}$, and $M_j \in \mathbb{R}^{n_o \times p}$. The estimation error dynamics can be written as

$$
\begin{aligned}
\dot{e}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{&N_{ij} e(t) + N_{dij} e(t-\tau(t)) \\
&+ (T_j A_{ij} - N_{ij} T_j - J_{ij} C) x(t) \\
&+ (T_j A_{dij} - N_{dij} T_j - J_{dij} C) x(t-\tau(t)) \\
&+ (T_j B_i - H_{ij}) u(t) \\
&+ T_j E_i f(t) + T_j D_i d(t)\},
\end{aligned}
\tag{5.32}
$$

where $T_j = L_j - M_j C$. The dynamics in (5.32) reduces to

$$
\dot{e}_j(t) = \sum_{i=1}^{r} \mu(\xi)\{N_{ij} e_j(t) + N_{dij} e_j(t-\tau) + T_j E_i f(t)\} \tag{5.33}
$$

if the following conditions hold:

$$
T_j A_i - N_{ij} T_j - J_{ij} C = 0 \tag{5.34a}
$$

$$
T_j A_{dij} - N_{dij} T_j - J_{dij} C = 0 \tag{5.34b}
$$

$$T_jB_i - H_i = 0 \tag{5.34c}$$

$$L_jD_i - M_jCD_i = 0. \tag{5.34d}$$

Remark 5.3.5. *If $T_j = 0$, the estimation error $e(t)$ approaches zero when the observer is stable even if $f(t) \neq 0$; as a result,* (5.10) *will not produce any residual. Therefore, one condition that the proposed observer can be used to generate residual is $T_j \neq 0$. Furthermore,* (5.34d) *holds if*

$$\mathcal{D} = \begin{bmatrix} D_1 & D_2 & \dots & D_r \end{bmatrix}$$

has left null space. Therefore, considering the order of the observer (5.31)*, T_j can be obtained as any combination of rows of the left null space of $\mathcal{D}$. Considering the full row rank of C, one solution of M_j can be given as*

$$M_j = (L_j - T_j)C^T(CC^T)^{-1}. \tag{5.35}$$

We use the residual generator in (5.4) to construct the residual for detecting the fault. The stability condition for observer (5.31) is provided in the following theorem.

Theorem 5.3.2. *For given scalars τ_M, τ_m, σ_1 and σ_2, fault detection observer* (5.31) *is asymptotically stable if there exist positive definite symmetric matrices P_1, P_2^1, P_2^2 and P_2^3, and matrices $\tilde{Y}_{ij}$, W_{11}, W_{12}, W_{21}, W_{22}, W_{31} and W_{32} with appropriate dimensions such that*

$$\begin{bmatrix} \tilde{\Xi}_{ij}^{1,1} & \tilde{\Xi}_{ij}^{1,2} & -W_{21} & W_{31} & \tau_M\sigma_1W_{11} & \tilde{\tau}(\sigma_1+\sigma_2)W_{21} & \tilde{\tau}\sigma_2W_{31} & \tilde{\Xi}_{ij}^{1,8} \\ \star & \tilde{\Xi}_{ij}^{2,2} & -W_{22} & W_{32} & \tau_M\sigma_1W_{12} & \tilde{\tau}(\sigma_1+\sigma_2)W_{22} & \tilde{\tau}\sigma_2W_{32} & \tilde{\Xi}_{ij}^{2,8} \\ \star & \star & -P_2^2 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -P_2^3 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\tau_M\sigma_1P_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{\tau}(\sigma_1+\sigma_2)P_1 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\tilde{\tau}\sigma_2P_1 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\kappa P_1 \end{bmatrix} < 0 \tag{5.36}$$

for all $i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, r,$ *where*

$$
\begin{aligned}
\tilde{\Xi}_{ij}^{1,1} &= P_1\tilde{N}_{ij}^1 + (\tilde{N}_{ij}^1)^T P_1 + \tilde{Y}_{ij}\tilde{N}_{ij}^2 + (\tilde{N}_{ij}^2)^T\tilde{Y}_{ij}^T + P_2^1 + P_2^2 + P_2^3 + W_{11} + W_{11}^T,\\
\tilde{\Xi}_{ij}^{1,2} &= P_1\tilde{N}_{dij}^1 + \tilde{Y}_{ij}\tilde{N}_{dij}^2 + W_{12}^T - W_{11} + W_{21} - W_{31},\\
\tilde{\Xi}_{ij}^{2,2} &= -(1-\rho)P_2^1 + W_{12} + W_{12}^T + W_{22} + W_{22}^T - W_{32} - W_{32}^T,\\
\tilde{\tau} &= \tau_M - \tau_m, \quad \kappa = \sigma_1\tau_M + \sigma_2\tilde{\tau},\\
\tilde{\Xi}_{ij}^{1,8} &= \kappa((\tilde{N}_{ij}^1)^T P_1 + (\tilde{N}_{ij}^2)^T\tilde{Y}_{ij}^T), \quad \tilde{\Xi}_{i}^{2,8} = \kappa((\tilde{N}_{dij}^1)^T P_1 + (\tilde{N}_{dij}^2)^T\tilde{Y}_{ij}^T),
\end{aligned}
$$

and $\tilde{N}_{ij}^1$, $\tilde{N}_{ij}^2$, $\tilde{N}_{dij}^1$ *and* $\tilde{N}_{dij}^2$ *are as defined in* (5.40a) *and* (5.40b)*.*

Proof. Post multiplication of (5.34a) and (5.34b) by Q_j gives

$$N_{ij} = T_jA_iL_j^+ - S_{ij}CL_j^+, \tag{5.37a}$$

$$S_{ij}CL_j^\perp = T_jA_iL_j^\perp, \tag{5.37b}$$

$$N_{dij} = T_jA_iL_j^+ - S_{dij}CL_j^+, \tag{5.37c}$$

$$S_{dij}CL_j^\perp = T_jA_{dij}L_j^\perp, \tag{5.37d}$$

where $S_{ij} = J_{ij} - N_{ij}M_{ij}$ and $S_{dij} = J_{dij} - N_{dij}M_{dij}$. By augmenting (5.37b) and (5.37d) we obtain

$$\begin{bmatrix} S_{ij} & S_{dij} \end{bmatrix} \Phi_j = \tilde{\Psi}_{ij}, \tag{5.38}$$

where $\tilde{\Psi}_{ij} = \begin{bmatrix} T_jA_{ij}L_j^\perp & T_jA_{dij}L_j^\perp \end{bmatrix}$ and Φ_j is defined in (5.18). By using the general solution of (5.38), we can write

$$S_{ij} = \begin{bmatrix} \tilde{\Psi}_{ij}\Phi_j^+ + \tilde{Z}_{ij}(I_{2p} - \Phi_j\Phi_j^+) \end{bmatrix} \begin{bmatrix} I_p \\ 0 \end{bmatrix} \tag{5.39a}$$

$$S_{dij} = \begin{bmatrix} \tilde{\Psi}_{ij}\Phi_j^+ + \tilde{Z}_{ij}(I_{2p} - \Phi_j\Phi_j^+) \end{bmatrix} \begin{bmatrix} 0 \\ I_p \end{bmatrix} \tag{5.39b}$$

where $\tilde{Z}_{ij}$ is an arbitrary matrix. Using (5.37a), (5.37c), (5.39a) and (5.39a), N_{ij} and N_{dij} can be expressed as below

$$N_{ij} = \tilde{N}_{ij}^1 + \tilde{Z}_{ij}\tilde{N}_{ij}^2 \tag{5.40a}$$

$$N_{dij} = \tilde{N}^1_{dij} + \tilde{Z}_{ij}\tilde{N}^2_{dij}, \tag{5.40b}$$

where $\tilde{N}^1_{ij}$, $\tilde{N}^2_{ij}$, $\tilde{N}^1_{dij}$ and $\tilde{N}^2_{dij}$ can be expressed by following the partition technique used in (5.20). Following the similar line of proof in Theorem 5.3.1 we can obtain the stability condition in (5.36) □

The procedure for constructing a functional observer based robust residual generator for a T-S fuzzy system is outlined below.

Fuzzy functional observer based robust residual generator design algorithm

Step 1: Find L_j following the procedure in Remarks 5.2.1 and 5.3.3. Calculate T_j and M_j as described in Remark 5.3.5;

Step 2: Specify the ranges and the increments of σ_1 and σ_2. Set σ_1 and σ_2 to the lower limits of the ranges;

Step 3: Obtain $\tilde{N}^1_{ij}$, $\tilde{N}^2_{ij}$, $\tilde{N}^1_{dij}$ and $\tilde{N}^2_{dij}$ using their definition in (5.40a) and (5.40b);

Step 4: Solve the LMIs of (5.36) and calculate $\tilde{Z}_{ij}$. If there is a solution jump to Step 6, else proceed to the next step;

Step 5: Add the increments with σ_1 and σ_2 and jump to Step 3;

Step 6: Obtain $\tilde{N}_{ij}$ and $\tilde{N}_{dij}$ using (5.40a) and (5.40b) respectively;

Step 7: Obtain S_{ij} and S_{dij} using (5.39a) and (5.39b) respectively. Then calculate $J_{ij} = S_{ij} + N_{ij}M_j$, $J_{dij} = S_{dij} + N_{dij}M_j$; and

Step 8: Find H_{ij} using (5.34c).

5.3.3 Fault isolation

This subsection presents the fault isolation technique of a T-S fuzzy system by applying a functional observer. If the qth fault is to be detected, the other faults can be considered as disturbances for the fault residual generator dedicated for the qth fault. Therefore, the model described by (5.30) can be re-written as

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + A_{di} x(t-\tau(t)) + B_q u(t) + e_i^q f_q(t) + \bar{D}_i \bar{d}(t)\}$$
$$y(t) = Cx(t), \tag{5.41}$$

where e_i^q is the qth column of E_i, $f_q(t)$ is the qth fault of the fault vector $f(t)$, $\bar{E}_i^q \in \mathbb{R}^{n\times(m_f-1)}$ is the matrix composed of the columns of E_i except the qth column, $\bar{f}_q \in \mathbb{R}^{m_f-1}$ is the fault vector except the qth fault,

$$\bar{D}_i = \begin{bmatrix} D_i & 0 \\ 0 & \bar{E}_i^q \end{bmatrix}, \quad \text{and } \bar{d}(t) = \begin{bmatrix} d(t) \\ \bar{f}_q \end{bmatrix}.$$

Using the similar technique for residual generation considering external disturbance described in the previous subsection, we can design the functional observer to isolate the faults. In this case, we need to construct m_f number of observers to detect the faults independently. The procedure for constructing the proposed fault isolator is outlined below.

Fuzzy functional observer based robust fault isolator design algorithm

Step 1: Calculate $\bar{D}_i$ using its definition in (5.41); and

Step 2: Follow the steps described in Steps for obtaining fuzzy functional observer based robust residual generator in Subsection 5.3.2 by replacing D_i with $\bar{D}_i$.

5.3.4 Fault estimation

We consider the T-S fuzzy system with time delay subject to external disturbance as described in (5.30). The fault estimator estimates faults asymptotically. The time-derivative of the time delay is assumed to have an upper bound, $\rho < 1$. The following fuzzy functional observer is proposed to estimate the qth fault:

$$\dot{z}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{N_{ij} z_j(t) + N_{dij} z_j(t-\tau(t))$$
$$+ J_{ij} y(t) + J_{dij} y(t-\tau(t)) + H_{ij} u(t) + K_{ij} \hat{f}_j^q(t)\}$$
$$z_j(t) = z_{j0}, t \in [-\tau(t), 0)$$

$$\begin{aligned}\hat{w}_j(t) &= z_j + M_j y(t) \\ \dot{\hat{f}}_j^q(t) &= -\Gamma \sum_{i=1}^{r} \mu_i(\xi(t)) G_{ij} e_j(t) \\ \hat{f}^q(t) &= \sum_{j=1}^{r} \mu_j(\xi(t)) \hat{f}_j^q(t),\end{aligned} \tag{5.42}$$

where $N_{ij} \in \mathbb{R}^{n_o \times n_o}$, $N_{dij} \in \mathbb{R}^{n_o \times n_o}$, $J_{ij} \in \mathbb{R}^{n_o \times p}$, $J_{dij} \in \mathbb{R}^{n_o \times p}$, $H_{ij} \in \mathbb{R}^{n_o \times m}$, and $\Gamma \in \mathbb{R}_+$ is the learning rate.

By defining the estimation error of the functional observer as $e_j(t) = \hat{w}_j(t) - L_j x(t)$, the overall error dynamics can be expressed as

$$\dot{e}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t)) \{N_{ij} e_j(t) + N_{dij} e_j(t - \tau(t)) + K_{ij} \hat{f}_j^q(t) - T_j E_i f^q(t)\}, \tag{5.43}$$

if the following conditions hold:

$$T_j A_i - N_{ij} T_j - J_{ij} C = 0, \tag{5.44a}$$

$$T_j A_{dij} - N_{dij} T_j - J_{dij} C = 0, \tag{5.44b}$$

$$T_j B_i - H_i = 0, \tag{5.44c}$$

$$L_j \bar{D}_i - M_j C \bar{D}_i = 0, \tag{5.44d}$$

where $T_j = L_j - M_j C$. Defining the estimation error of the qth fault as $e_{f_j}^q(t) = \hat{f}_j^q(t) - f^q(t)$, and considering $\dot{f}^q(t) = 0$, error dynamics (5.43) can be further reduced to

$$\dot{e}_j(t) = \sum_{i=1}^{r} \mu_i(\xi(t)) \{N_{ij} e_j(t) + N_{dij} e_j(t - \tau) + K_{ij} e_{f_j}^q(t)\} \tag{5.45}$$

if we have

$$K_{ij} - T_j E_i = 0. \tag{5.46}$$

The stability condition for the proposed fault estimator is presented in the following theorem.

Theorem 5.3.3. *Fuzzy functional observer* (5.42) *estimates the fault asymptotically if the conditions in* (5.44c), (5.44d) *and* (5.46) *hold, and for some given* *positive scalar $\rho < 1$ there exist positive definite matrix P_1 and matrices Y_{ij} such that*

$$\begin{bmatrix} P_1\tilde{N}^1_{ij} + (\tilde{N}^1_{ij})^T P_1 + Y_{ij}\tilde{N}^2_{ij} + (\tilde{N}^2_{ij})^T Y^T_{ij} & P_1\tilde{N}^1_{dij} + Y_{ij}\tilde{N}^2_{dij} \\ \star & -(1-\rho)P_2 \end{bmatrix} < 0, \tag{5.47}$$

for all $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, r$, where $\tilde{N}^1_{ij}$, $\tilde{N}^2_{ij}$, $\tilde{N}^1_{dij}$, and $\tilde{N}^2_{dij}$ are given in (5.40a) *and* (5.40b).

Proof. We construct $\bar{D}_i$ and $\bar{d}(t)$ as described in Subsection 5.3.3, and obtain N_{ij} and N_{dij} by following the similar line of proof in Theorem 5.3.2. We consider the following Lyapunov-Krasovskii functional to establish the stability condition such that the estimation error approaches zero asymptotically:

$$V_j(t) = e^T_j(t)P_1 e_j(t) + \int_{t-\tau(t)}^{t} e^T_j(s)P_2 e_j(s)ds + e_{f_j}(t)^T \Gamma^{-1} e_{f_j}(t), \tag{5.48}$$

where matrices P_1 and P_2 are positive definite and symmetric. Taking the derivative of $V(t)$, we have

$$\begin{aligned} \dot{V}_j(t) = {} & 2e^T_j(t)P_1\dot{e}_j(t) + e^T_j(t)P_2 e_j(t) \\ & - (1-\dot{\tau}(t))e^T_j(t-\tau(t))P_2 e_j(t-\tau(t)) + 2e_{f_j}(t)^T\Gamma^{-1}\dot{e}_{f_j}(t) \\ \le {} & 2e^T_j(t)P_1 \sum_{i=1}^{r} \mu_i(\xi(t))\{N_{ij}e_j(t) + N_{dij}e_j(t-\tau(t)) + K_{ij}e_{f_j}(t)\} \\ & + e^T_j(t)P_2 e_j(t) - (1-\rho)e^T_j(t-\tau(t))P_2 e_j(t-\tau(t)) \\ & + 2e_{f_j}(t)^T\Gamma^{-1}\dot{e}_{f_j}(t). \end{aligned} \tag{5.49}$$

Using (5.45) and considering $\dot{f}(t) = 0$, we can have

$$\begin{aligned} 2e_{f_j}(t)^T\Gamma^{-1}\dot{e}_{f_j}(t)) &= 2e_{f_j}(t)^T\Gamma^{-1}(\dot{\hat{f}}_j(t) - \dot{f}(t)) \\ &= 2e_{f_j}(t)^T\Gamma^{-1}(-\Gamma\sum_{i=1}^{r}\mu_i(\xi(t))G_{ij}e_j(t)) \\ &= -2\sum_{i=1}^{r}\mu_i(\xi(t))e^T_{f_j}(t)G_{ij}e_j(t). \end{aligned} \tag{5.50}$$

Using (5.40a) and (5.40b), inequality (5.49) can be expressed as

$$\begin{aligned}\dot{V}_j(t) &\leq 2e_j^T(t)P_1\sum_{i=1}^{r}\mu_i(\xi(t))\{(\tilde{N}_{ij}^1+\tilde{Z}_{ij}\tilde{N}_{ij}^2)e_j(t)+(\tilde{N}_{dij}^1+\tilde{Z}_{ij}\tilde{N}_{dij}^2)e_j(t-\tau(t))\}\\ &\quad+e_j^T(t)P_2e_j(t)-(1-\rho)e_j^T(t-\tau(t))P_2e_j(t-\tau(t))\\ &=\eta_j^T(t)\begin{bmatrix}P_1\tilde{N}_{ij}^1+(\tilde{N}_{ij}^1)^TP_1+Y_{ij}\tilde{N}_{ij}^2+(\tilde{N}_{ij}^2)^TY_{ij}^T & P_1\tilde{N}_{dij}^1+Y_{ij}\tilde{N}_{dij}^2\\ \star & -(1-\rho)P_2\end{bmatrix}\eta_j(t),\end{aligned} \tag{5.51}$$

where $Y_{ij} = P_1\tilde{Z}_{ij}$ and $\eta_j^T(t) = [e_j^T(t)\ \ e_j^T(t-\tau(t))]$ if we have $G_{ij} = K_{ij}^TP_1$. Therefore, the LMIs in (5.47) ensure $\dot{V}_j(t) < 0$, which eventually ensures asymptotic convergence of the estimation error to zero and guarantees the asymptotic estimation of the qth fault $f^q(t)$. □

The procedure for constructing the proposed functional observer based fault estimator for nonlinear system is outlined below.

Fuzzy functional observer based robust fault estimator design algorithm

Step 1: Calculate $\bar{D}_i$ using its definition in (5.41). Take a positive real number for ρ;

Step 2: Solve the LMIs in (5.47); and

Step 3: Follow the Steps 6, 7, and 8 described in "Steps for obtaining fuzzy functional observer based robust residual generator" in Subsection 5.3.2.

Example 5.1

Consider a T-S fuzzy system with two rules to illustrate the fault detection, isolation and estimation technique described in this chapter. The system matrices of the T-S fuzzy model are as follows:

$$A_1 = \begin{bmatrix}-0.2 & 7 & -1 & 1\\ -1 & -8 & -1 & 1\\ -2 & 0 & -1 & 1\\ -1 & 1.1 & 2 & -10\end{bmatrix}, \qquad A_{d1} = \begin{bmatrix}1 & 0 & -1 & 0\\ -8 & -4 & -1 & 0\\ 0 & 0 & -1 & 0\\ -2 & 1 & 3 & -3\end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 6 & -1.1 & 2 \\ -2 & -6 & -1.1 & 2 \\ -1 & 0 & -1.1 & 2 \\ 1 & 1.1 & 2 & -10.5 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} 1.2 & 0 & -1 & 0 \\ -3 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ -2.2 & 1.4 & 2 & -3.5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.1 \\ -0.2 \\ 0.5 \\ 0.2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.15 \\ -0.30 \\ 0.40 \\ 0.25 \end{bmatrix},$$

$$E_1 = E_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad D_1 = D_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The subsystems of the fuzzy model are interconnected by weighting functions $\mu_1(x_1(t)) = \frac{1-\tanh((x_1(t)-1)/10)}{2}$ and $\mu_2(x_1(t)) = 1 - \mu_1(x_1(t))$. The external disturbance is assumed to be zero-mean Gaussian white noise with standard deviation 5. The time-delay is considered to be $\tau(t) = 0.2 + 0.1 \sin(t)$. Suppose the fault vector has two faults. Fault 1 is a step fault starting at the 10-second mark until the 50-second mark. The second fault, Fault 2, starts at the 30-second mark and continues until the 70-second mark. There is an overlapping between these two faults during the period from 30 seconds to 50 seconds. A unit step input is applied to the system at $t = 2$ seconds. The system outputs subject to the faults and the disturbance are shown in Figure 5.2.

Our objective is to obtain residuals such that each fault can be identified individually even though there is an overlapping period between the faults. The residuals should be insensitive to the disturbances. Using the technique described in Remark 5.2.1, we consider the functions for the fuzzy functional observer to be $L_1 = L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$. By following the procedure described in Subsection 5.3.3 we obtain the observer parameters. The LMIs are solved using LMITOOLBOX of MATLAB. Considering the bounds of time delays $\tau_M = 0.4$ second and $\tau_m = 0.01$

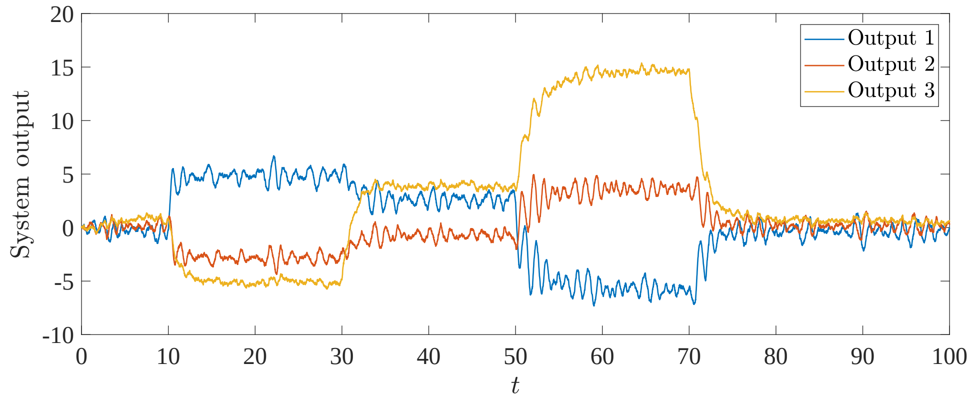

Figure 5.2: Outputs of the plant

second, and scalars $\sigma_1 = 0.1$ and $\sigma_2 = 0.3$ the observer parameters are obtained as given below.

The observer for generating the residual to detect Fault 1:

$$
\begin{aligned}
&N_{11} = N_{12} = N_{21} = N_{22} = -1.3910, && N_{d11} = N_{d12} = N_{d21} = N_{d22} = -1.2188,\\
&J_{11} = J_{12} = \begin{bmatrix} 2.1910 & 13.6090 & 0 \end{bmatrix}, && J_{21} = J_{22} = \begin{bmatrix} 2.3910 & 10.6090 & 0 \end{bmatrix},\\
&J_{d11} = J_{d12} = \begin{bmatrix} 10.2188 & 2.7812 & 0 \end{bmatrix}, && J_{d21} = J_{d22} = \begin{bmatrix} 5.4188 & -5.2188 & 0 \end{bmatrix},\\
&H_{11} = H_{12} = 0.30, && H_{21} = H_{22} = 0.45,\\
&M_1 = M_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, && R_1 = R_2 = 2,\\
&F_1 = F_2 = \begin{bmatrix} -2 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

The observer for generating the residual to detect Fault 2:

$$
\begin{aligned}
&N_{11} = N_{12} = N_{21} = N_{22} = -1.3910, && N_{d11} = N_{d12} = N_{d21} = N_{d22} = -1.2188,\\
&J_{11} = J_{12} = \begin{bmatrix} -1.0000 & 6.6091 & 1.3909 \end{bmatrix}, && J_{21} = J_{22} = \begin{bmatrix} 1.0000 & 4.6091 & 1.3909 \end{bmatrix},\\
&J_{d11} = J_{d12} = \begin{bmatrix} 8.0000 & 2.7813 & 1.2187 \end{bmatrix}, && J_{d21} = J_{d22} = \begin{bmatrix} 3.0000 & -5.2187 & 1.2187 \end{bmatrix},\\
&H_{11} = H_{12} = H_{21} = H_{22} = 0.70, && M_1 = M_2 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix},\\
&R_1 = R_2 = 2, && F_1 = F_2 = \begin{bmatrix} -2 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

The fault detection residuals generated by the proposed functional observer is depicted in Figure 5.3. The fault detectors can generate residuals to isolate the faults from each other. The residuals are not affected by the disturbance.

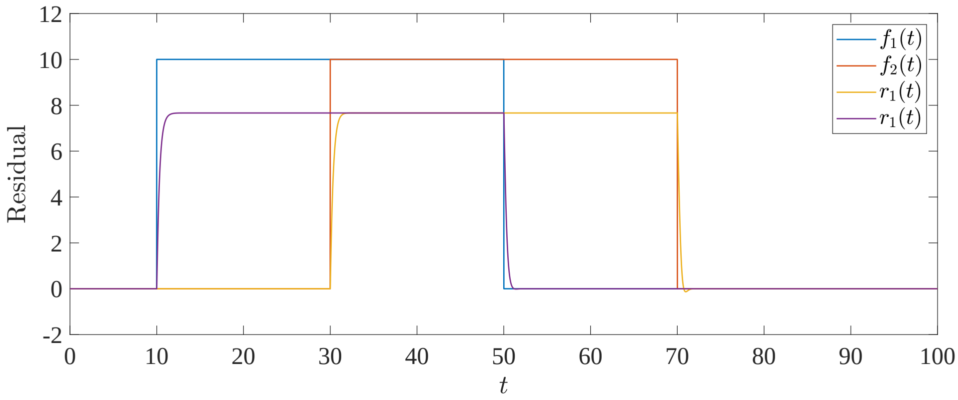

Figure 5.3: Residuals to detect and isolate the faults

The main feature of the functional observer based fault detection is that this method does not require any calculation of thresholds. The graphs of the residuals demonstrate this feature.

Now, the faults are estimated using the functional observer. The upper bound of the derivative of the time-delay ρ is taken as 0.5, and we consider $\Gamma = 1$. The observer is constructed by following the procedure stated in Subsection 5.3.4. The fault estimation observer parameters are as below.

The observer parameters for estimating Fault 1:

$$N_{11} = N_{12} = N_{21} = N_{22} = -4.8596, \qquad N_{d11} = N_{d12} = N_{d21} = N_{d22} = 0,$$
$$J_{11} = J_{21} = \begin{bmatrix} -5.6596 & -10.1404 & 0 \end{bmatrix}, \quad J_{21} = J_{22} = \begin{bmatrix} -5.8596 & -7.1404 & 0 \end{bmatrix},$$
$$J_{d11} = J_{d21} = \begin{bmatrix} -9.0000 & -4.0000 & 0 \end{bmatrix}, \quad J_{d21} = J_{d22} = \begin{bmatrix} -4.2000 & 4.0000 & 0 \end{bmatrix},$$
$$H_{11} = H_{12} = -0.3000, \qquad H_{21} = H_{22} = -0.4500,$$
$$K_{11} = K_{12} = K_{21} = K_{22} = 1.0000, \qquad M_1 = M_2 = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}.$$

The observer parameters for estimating Fault 2:

$$N_{11} = N_{12} = N_{21} = N_{22} = -4.8596, \qquad N_{d11} = N_{d12} = N_{d21} = N_{d22} = 0,$$
$$J_{11} = J_{21} = \begin{bmatrix} -1.0000 & 3.1404 & 4.8596 \end{bmatrix}, \quad J_{21} = J_{22} = \begin{bmatrix} 1.0000 & 1.1404 & 4.8596 \end{bmatrix},$$
$$J_{d11} = J_{d21} = \begin{bmatrix} 8.0000 & 4.0000 & 0 \end{bmatrix}, \qquad J_{d21} = J_{d22} = \begin{bmatrix} 3.0000 & -4.0000 & 0 \end{bmatrix},$$
$$H_{11} = H_{12} = 0.7, \qquad H_{21} = H_{22} = 0.7,$$
$$K_{11} = K_{12} = K_{21} = K_{22} = -1.0000, \qquad M_1 = M_2 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}.$$

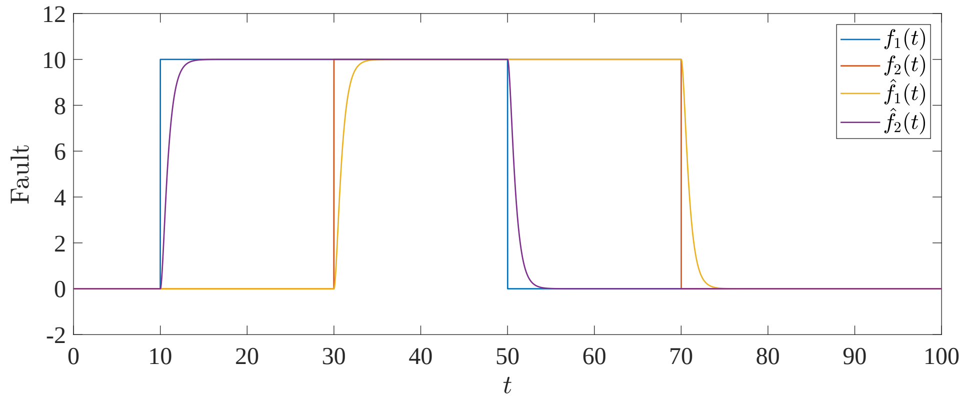

Figure 5.4: Estimated faults using the fuzzy functional observer

The simulation output of the fuzzy functional observer based fault estimator is displayed in Figure 5.4. It can be seen that the proposed method estimates the faults asymptotically without being affected by the external disturbances and time-delay.

Example 5.2: Fault detection of a truck-trailer system

In this example we apply the proposed fault estimation method to a benchmark problem of the delayed T-S fuzzy model of a truck-trailer system with exogenous disturbance [119]. The matrices that describe the linear subsystems are

$$A_1 = \begin{bmatrix} -a\frac{v\bar{t}}{L_l t_0} & 0 & 0 \\ a\frac{v\bar{t}}{L_l t_0} & 0 & 0 \\ -a\frac{v^2\bar{t}^2}{2L_l t_0} & \frac{v\bar{t}}{t_0} & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -a\frac{v\bar{t}}{L_l t_0} & 0 & 0 \\ a\frac{v\bar{t}}{L_l t_0} & 0 & 0 \\ -ad\frac{v^2\bar{t}^2}{2L_l t_0} & \frac{dv\bar{t}}{t_0} & 0 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} -(1-a)\frac{v\bar{t}}{L_l t_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{L_l t_0} & 0 & 0 \\ (1-a)\frac{v^2\bar{t}^2}{2L_l t_0} & 0 & 0 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} -(1-a)\frac{v\bar{t}}{L_l t_0} & 0 & 0 \\ (1-a)\frac{v\bar{t}}{L_l t_0} & 0 & 0 \\ (1-a)\frac{dv^2\bar{t}^2}{2L_l t_0} & 0 & 0 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} \frac{v\bar{t}}{l t_0} \\ 0 \\ 0 \end{bmatrix}, \qquad D_1 = D_2 = \begin{bmatrix} 0 \\ \frac{v\bar{t}}{l t_0} \\ 0 \end{bmatrix},$$

where $x_1(t)$ is the angle difference between the truck and the trailer, $x_2(t)$ is the angle of the trailer, $x_3(t)$ is the vertical position of the rear end of the trailer, $u(t)$ is the steering angle, a is the retarded coefficient, v is the constant backing up speed, l is the length of the truck, and L_l is the length of the trailer. The premise

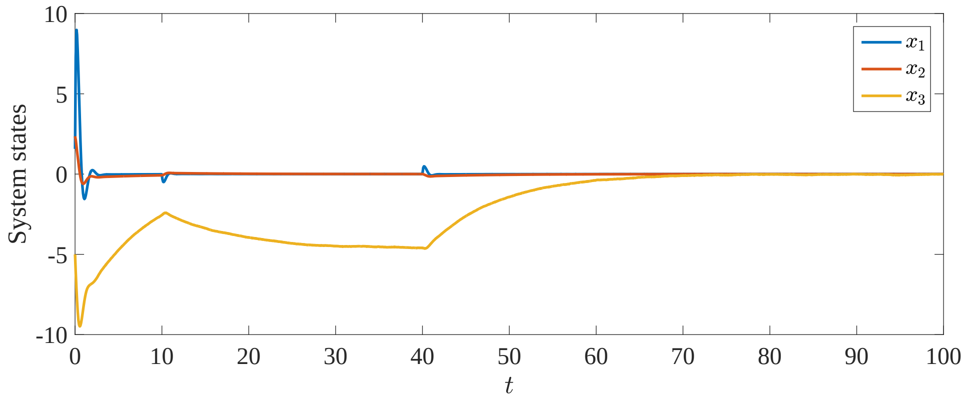

Figure 5.5: Time response of the truck-trailer system

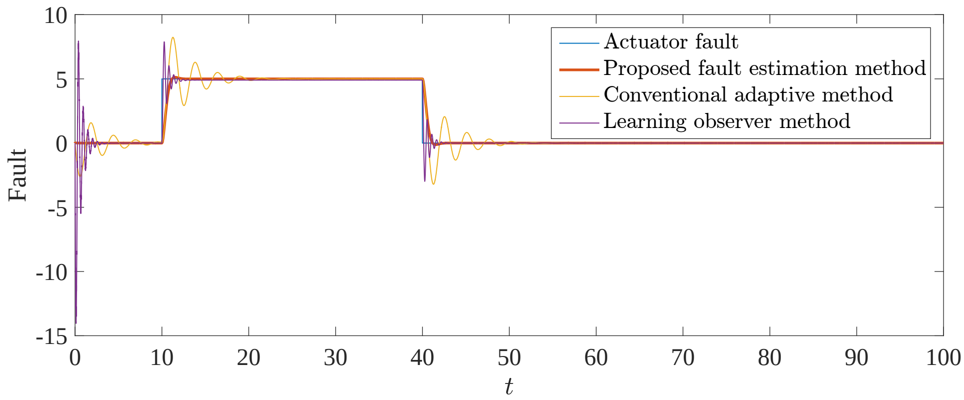

Figure 5.6: Comparison of estimated faults with the actual fault for the truck-trailer system

variable $\xi(t) = x_2(t) + a\frac{v\bar{t}}{2L_l}x_1(t) + (1-a)\frac{v\bar{t}}{2L_l}x_1(t-\tau(t))$ and the membership functions are

$$\mu_1(\xi(t)) = \left(1 - \frac{1}{1+\exp(3(-\xi(t)-0.5\pi))}\right)\frac{1}{1+\exp(3(-\xi(t)+0.5\pi))},$$
$$\mu_2(\xi(t)) = 1 - \mu_1(\xi(t)).$$

Considering $a = 0.7$, $v = -1.0$, $\bar{t} = 2.0$, $t_0 = 0.5$, $L_l = 5.5$, $l = 2.8$, $d = \frac{10t_0}{\pi}$ and following the observer construction steps described in Subsection 5.3.4. for $\tau_M = 0.5$, $\tau_m = 0.1$, $\rho = 0.4$, and $L_1 = L_2 = [1 \quad 1 \quad 1]$, we find the following observer parameters:

$$N_{11} = -4.8596, \qquad N_{12} = -4.8596,$$

$$N_{21} = -4.8596, \qquad N_{22} = -4.8596,$$
$$N_{d11} = 0, \qquad N_{d12} = 0,$$
$$N_{d21} = 0, \qquad N_{d22} = 0,$$
$$J_{11} = \begin{bmatrix} 5.3687 & 0 & 0 \end{bmatrix}, \qquad J_{12} = \begin{bmatrix} 5.3687 & 0 & 0 \end{bmatrix},$$
$$J_{21} = \begin{bmatrix} 5.3687 & 0 & 0 \end{bmatrix}, \qquad J_{21} = \begin{bmatrix} 5.3687 & 0 & 0 \end{bmatrix},$$
$$J_{d11} = \begin{bmatrix} 0.2182 & 0 & 0 \end{bmatrix}, \qquad J_{d12} = \begin{bmatrix} 0.2182 & 0 & 0 \end{bmatrix},$$
$$J_{d21} = \begin{bmatrix} 0.2182 & 0 & 0 \end{bmatrix}, \qquad J_{d22} = \begin{bmatrix} 0.2182 & 0 & 0 \end{bmatrix},$$
$$H_{11} = H_{12} = H_{21} = H_{22} = -1.4286, \qquad K_{11} = K_{12} = K_{21} = K_{22} = -1.4286,$$
$$M_1 = M_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \qquad G_{11} = G_{12} = G_{21} = G_{22} = -7.1429.$$

The performance of the observer is simulated for estimating a step fault applied to the system. Figure 5.5 displays the time response of the system states. Figure 5.6 compares the performance of the proposed method with conventional adaptive fault estimation method [120] and learning observer based fault estimated method [119]. It can be seen that the estimated fault converges to the actual fault asymptotically. It is evident that the proposed method estimates the fault satisfactorily.

Example 5.3: Fault detection of bearing fault of DC Motor

This example employs the proposed fault detection scheme for detecting the bearing fault of a permanent magnet DC motor. The dynamics of a permanent magnet DC motor is expressed by

$$\begin{aligned}
\dot{i}_a(t) &= -\frac{R_a}{L} i_a(t) - \frac{K_e}{L}\omega(t) + \frac{1}{L}u(t) \\
\dot{\omega}(t) &= \frac{K_T}{J_1} i_a(t) - \frac{f_r\omega(t) + f_p\omega^2(t)}{J_1} - \frac{T_0(t) + T_1}{J_1},
\end{aligned}$$

where $i_a(t)$ is the armature current, $\omega(t)$ is the rotor speed, $u(t)$ is the armature voltage, R_a is the armature resistance, L is the inductance of the armature, J_1 is the normalised inertial moment, f_r is the friction coefficient due to the bearing lubrication condition, f_p is the friction coefficient due to aerodynamics, T_0 is the friction torque, and T_1 is the load torque. A common source of fault for this kind of machine is the breakage of the bearing system. The fault at the bearing causes a sudden high friction torque T_0. The torque produced due to the bearing fault is significantly high compared with the torque during a normal operating condition. Therefore, it can be assumed that the total torque produced by the motor is equal to the fault torque $f(t)$.

The dynamics of the permanent magnet DC motor is nonlinear. Hence, the dynamics of this motor can be expressed as a two-rule T-S fuzzy model. The matrices representing the linear models of respective subsystems are

$$A_1 = \begin{bmatrix} -\frac{R_a}{L} & -\frac{K_e}{L} \\ \frac{K_T}{J_1} & \frac{f_r + \bar{\omega} f_p}{J_1} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -\frac{R_a}{L} & -\frac{K_e}{L} \\ \frac{K_T}{J_1} & \frac{f_r + \underline{\omega} f_p}{J_1} \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \qquad E_1 = E_2 = \begin{bmatrix} 0 \\ -\frac{1}{J_1} \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

where system state $x(t) = \begin{bmatrix} i_a(t) & \omega(t) \end{bmatrix}^T$, premise variable $\xi(t) = \omega(t)$, and weighting functions $\mu_1(t) = \frac{\bar{\omega} - \omega(t)}{\bar{\omega} - \underline{\omega}}$ and $\mu_2(t) = 1 - \mu_1(t)$ with $\bar{\omega}$ and $\underline{\omega}$ being the upper and lower limits of the rotor speed respectively. It should be noted that output matrix C has only one row. Therefore, we cannot obtain a one-dimensional observer for the residual generation purpose. We follow the procedure outlined in Remark 5.3.3 to obtain L_j for $j = 1, 2$. Considering $R_a = 0.6\ \Omega$, $L = 0.012$ H, $K_e = 0.001$ V/rpm, $K_T = 0.3$ N.m/A, $J_1 = 0.20$ N.m.s, $f_r = 0.35$ N.m/rpm, $f_p = 0.0007$ N.m/rpm^2, $\bar{\omega} = 100$ rpm, and $\underline{\omega} = 120$ rpm, we obtain the following observer parameters by applying the observer construction procedure described in Subsection 5.3.1.

$$L_1 = \begin{bmatrix} 0 & 1.0000 \\ 1.5000 & -2.1700 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 0 & 1.0000 \\ 1.5000 & -2.1000 \end{bmatrix}$$

$$N_{11} = \begin{bmatrix} -16.0128 & 1.0000 \\ -42.6755 & -52.1700 \end{bmatrix}, \qquad N_{12} = \begin{bmatrix} -16.0828 & 1.0000 \\ -39.0285 & -52.1000 \end{bmatrix},$$

$$N_{21} = \begin{bmatrix} -16.0128 & 1.0000 \\ -42.6755 & -52.1700 \end{bmatrix}, \qquad N_{22} = \begin{bmatrix} -16.0828 & 1.0000 \\ -39.0285 & -52.1000 \end{bmatrix},$$

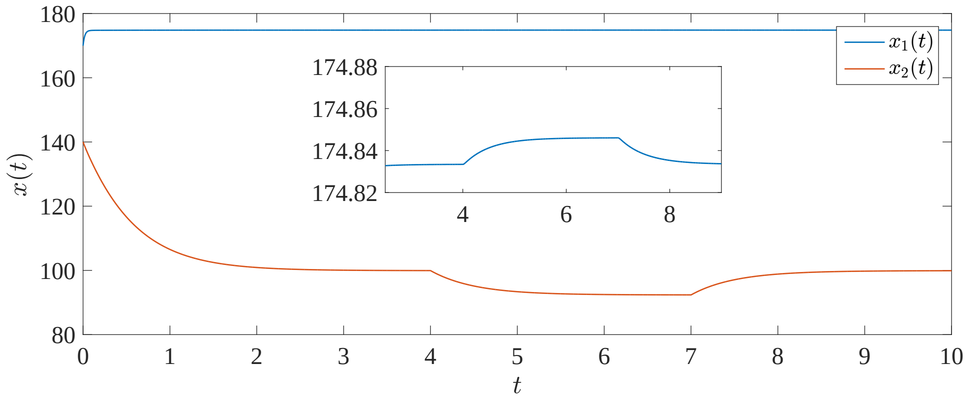

Figure 5.7: States of the permanent magnet DC motor under fault and normal running condition

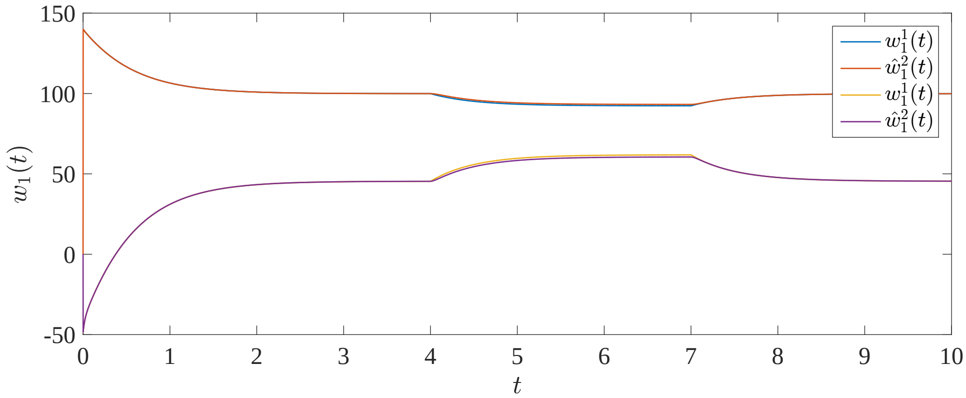

Figure 5.8: Function of states for Rule 1

$$J_{11} = \begin{bmatrix} 16.0128 \\ -65.9495 \end{bmatrix}, \qquad J_{11} = \begin{bmatrix} 16.0128 \\ -65.9495 \end{bmatrix},$$

$$J_{21} = \begin{bmatrix} 16.0828 \\ -66.1014 \end{bmatrix}, \qquad J_{21} = \begin{bmatrix} 16.0828 \\ -66.1014 \end{bmatrix},$$

$$R_1 = R_2 = \begin{bmatrix} 1.4142 & 0 \end{bmatrix}, \qquad H_{11} = H_{12} = H_{21} = H_{22} = \begin{bmatrix} 0 \\ 125.0000 \end{bmatrix},$$

$$F_1 = F_2 = -1.4142, \text{ and}$$

N_d and J_d are zero matrices with appropriate dimensions.

The performance of the system is simulated with the torque due to the fault equal to 3 N.m, where the torque due to the friction and load under normal operating condition of the motor is assumed to be 0.01 N.m. The state response of

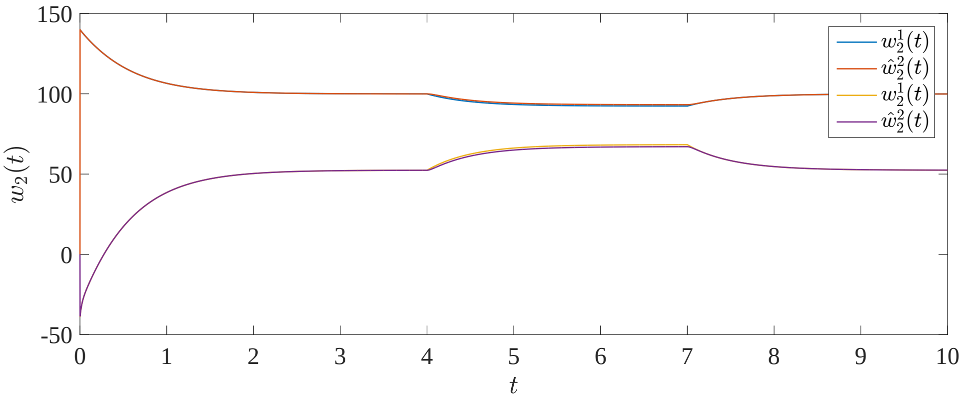

Figure 5.9: Function of states for Rule 2

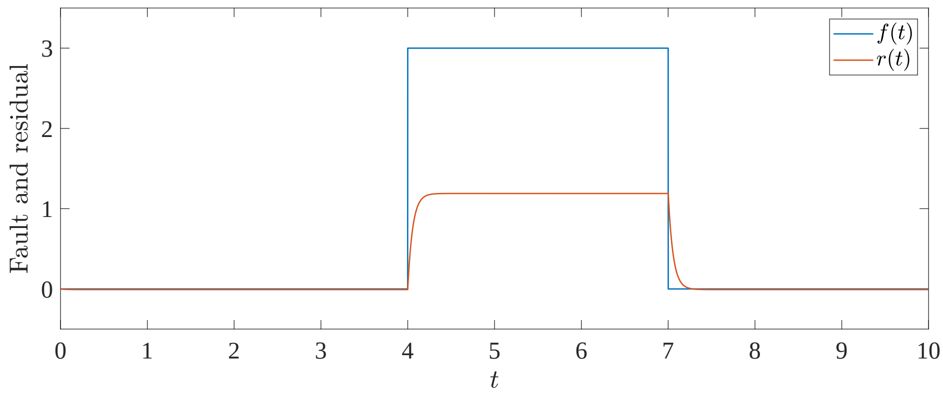

Figure 5.10: Fault and residual for detecting bearing fault

the system is displayed in Figure 5.7. It can be seen that the rotor speed reduces significantly and the armature current increases during the faulty condition. The change in the armature current during the fault is depicted by enlarging the vertical axis in this figure. Figures 5.8 and 5.9 show the functions of the states and the estimated functions of states. Figure 5.10 shows residual $r(t)$ and actual fault $f(t)$. It is evident that the fault can be detected from the residual generated by the fuzzy functional observer based residual generator.

5.4 Conclusion

Fuzzy functional observers can be used effectively for fault detection, isolation and estimation of nonlinear systems subject to time-delays and external disturbances. The functional observers are lower order compared with the observers used in existing observer based fault diagnosis techniques of T-S fuzzy systems. This fault detection method does not require the calculation of any thresholds. Taking the disturbances as unknown inputs enables us to decouple the exogenous disturbances from the residual generator. Using suitable Lyapunov-Krasovskii functionals, the stability conditions are formulated such that the conditions are delay dependent to ensure more generality compared with delay independent conditions. The examples verify the effectiveness of the proposed technique.

Chapter 6

Functional observer based power system stabiliser

Power systems are complex networks with synchronous generators as the major contributors as power sources. It is of prime concern that the generators operate with asymptotic stability in case of any sudden changes in the network. A power system stabiliser is used to damp out the oscillation of a synchronous generator due to the sudden changes. If the power systems network is too large and complex, the whole network is simplified as a generator connected to an infinite bus, which is recognised as a single machine infinite bus system in the literature. The mathematical model of a single machine infinite bus system is nonlinear; hence it requires linearisation for obtaining the power system stabiliser. T-S fuzzy model can be used to express the system as a fuzzy combination of linear models so that the system is defined for a range of operating points instead of a single operating point. This chapter presents fuzzy functional observer based power system stabiliser for a single machine infinite bus system. The contents of this chapter are presented as practical examples for the theoretical developments of functional observer based PDC controllers published in [81, 85].

6.1 Introduction

In general, the synchronous generators are connected to the loads through a large and complex network. The capacity of the network is much greater than that of a synchronous generator alone; it is often assumed that the network is not affected at all by any change in the synchronous generator. While designing a power system stabiliser for a generator, it is generally considered that the generator is connected to an infinite bus through a transmission line. There are existing methods to simplify the whole network as a single machine infinite bus system. A block diagram of such a system is displayed in Figure 6.1. The mathematical

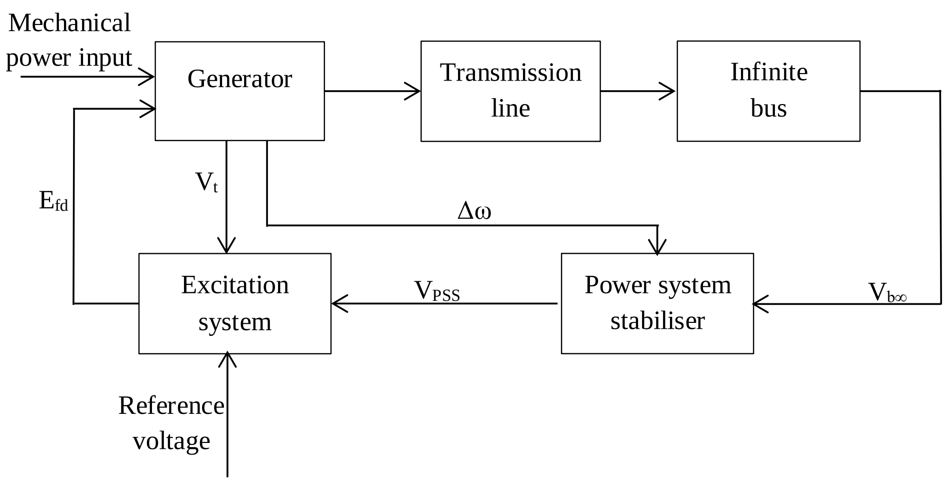

Figure 6.1: Single synchronous generator connected to infinite bus

model of a single machine infinite bus system is nonlinear. Therefore, the model is linearised at a stable operating point, and a power system stabiliser is designed using this linearised model.

Heffron-Philip [121] model is the most widely used linearising technique for this emulated system to design such stabiliser. This model uses six constant parameters that are obtained using the stable operating condition of the network. As a result, the power system stabiliser may not be robust against the changes in the operating conditions of the network. A T-S fuzzy model can be used to obtain the constant parameters of the Heffron-Philip model so that the model will remain valid for a range of operating conditions rather than a specific operating point [122–124]. Consequently, a functional observer based PDC controller can be obtained for the T-S fuzzy model of a single machine infinite bus system to design the power system stabiliser. The following sections present the design technique of the functional observer based power system stabiliser.

6.2 Single machine infinite bus system

A single machine infinite bus system simplifies the network complexity and is viewed as a synchronous generator connected to an infinite bus through a transmission line. A one line circuit diagram is presented in Figure 6.2 considering $V_{b\infty}$ is the infinite bus voltage, X_e and R_e represent the transmission line impedance, V_t is the terminal voltage of the generator, I_t is the line current, $P + jQ$ is the complex power generated by the generator.

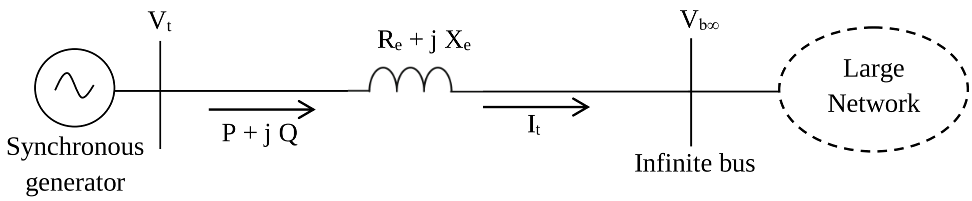

Figure 6.2: One line circuit diagram of a single machine infinite bus system

6.2.1 System model

The dynamics of a three phase synchronous machine is generally expressed with respect to two synchronously rotating quadrature axes, known as d-q axes where d-axis is the direct axis and q-axis represent the quadrature axis. The derivation of the d-q axis transformation of the generator model can be found in any standard power system textbook. This model gives the relation among real and reactive components of power, terminal voltage, terminal current, line impedance, and infinite bus voltage. Considering the third order model of a synchronous generator, the system is expressed by (6.1) where $V_{b\infty}$, X_e, V_t and I_t are the terms as defined in Figure 6.2, and X_d is the direct axis reactance, X_q is the quadrature axis reactance, X'_d is the transient reactance, δ is the angle between q-axis and $V_{b\infty}$, M is the inertia coefficient, T'_{d0} is the open circuit time constant, ω_0 is the synchronous speed, K_E is the gain of the excitation system and T_E is the time constant of the

$$
\begin{aligned}
\dot{\omega} &= \frac{1}{M}(T_m - T_e) \\
\dot{\delta} &= \omega_o \omega \\
\dot{E}'_q &= \frac{1}{T'_{do}}\left(E_{fd} - \frac{X_d + X_e}{X'_d + X_e}E'_q + \frac{X_d + X'_d}{X'_d + X_e}V_{b\infty}\cos\delta\right) \\
\dot{E}_{fd} &= \frac{1}{T_E}(k_E V_{ref} - k_E V_t - E_{fd}),
\end{aligned}
\tag{6.1}
$$

exciter.

6.2.2 Small signal model

The angular positions of different voltage and current parameters with respect to d-q axes are portrayed using a phasor diagram in Figure 6.3. In this figure, I_d is the direct axis component of the terminal current, I_q is the quadrature axis component of the terminal current, v_d is the direct axis component of the terminal voltage and v_q is the quadrature axis component of the terminal voltage. This phasor diagram considers infinite bus voltage $V_{b\infty}$ as the known reference voltage; real power P, reactive power Q and transmission line reactance X_e are known for a stable operating point; and transmission line resistance R_e is negligible. We can obtain the terminal current and the angle between the terminal current and the infinite bus voltage as

$$|I_t| \angle\phi = \frac{P - jQ}{|V_{b\infty}| \angle 0}.$$

Then we can obtain terminal voltage V_t and load angle δ from the following equations:

$$
\begin{aligned}
|V_t| \angle\theta &= V_{b\infty}\angle 0 + jX_e |I_t| \angle\phi \\
E_q\angle\delta &= |V_t| \angle\theta + jX_q |I_t| \angle\phi.
\end{aligned}
$$

Once relative angular position δ of q-axis and the angles of the terminal voltage and current with respect to the infinite bus voltage are calculated, we can obtain the direct and quadrature axes components of the terminal voltage and current from Figure 6.3. These direct and quadrature axis components are used for finding the small signal model of the system.

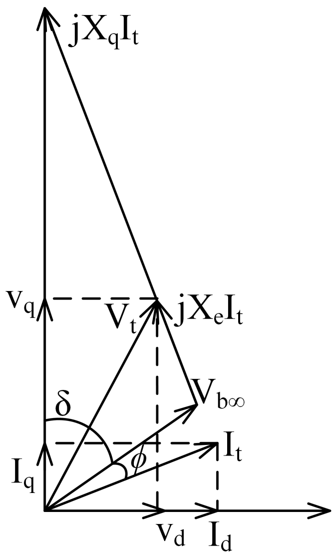

Figure 6.3: Phasor diagram of a single machine infinite bus system

The system often experiences small perturbation resulting in sustained oscillation in the steady state condition. This phenomenon is studied by obtaining a small signal model of the system. This small signal model linearises nonlinear system dynamics (6.1) at a known steady state operating point defined by P, Q and X_e for known $V_{b\infty}$. The state space model of the linearised small signal model of this system at a specific operating point can be expressed as

$$\begin{bmatrix} \dot{\Delta\delta} \\ \dot{\Delta\omega} \\ \dot{\Delta E'_q} \\ \dot{\Delta E_{fd}} \end{bmatrix} = \begin{bmatrix} 0 & \omega_o & 0 & 0 \\ -\frac{k_1}{M} & -\frac{D}{M} & -\frac{k_2}{M} & 0 \\ -\frac{k_4}{T'_{do}} & 0 & -\frac{1}{T'_{do}k_3} & \frac{1}{T'_{do}} \\ -\frac{K_E k_5}{T_E} & 0 & -\frac{K_E k_6}{T_E} & \frac{1}{T_E} \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\omega \\ \Delta E'_q \\ \Delta E_{fd} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_E}{T_E} \end{bmatrix} \Delta V_{ref}, \qquad (6.2)$$

where $\Delta\delta$, $\Delta\omega$, $\Delta E'_q$ and ΔE_{fd} are the deviations of power angle, rotor speed, induced voltage and excitation voltage, respectively, and $k_1, \ldots, k_6$ are constants. These constants can be obtained from the following set of equations for given real power P, reactive power Q and transmission line impedance X_e [121]:

$$k_1 = \frac{X_q - X'_d}{X_e + X'_d} I_q V_{b\infty} \sin\delta + \frac{1}{X_e + X_q} E_q V_{b\infty} \cos\delta,$$
$$k_2 = \frac{V_{b\infty}}{X_e + X'_d},$$
$$k_3 = \frac{X_e + X'_d}{X_e + X_d},$$

$$k_4 = \frac{X_d - X'_d}{X_e + X'_d} V_{b\infty} \sin\delta,$$
$$k_5 = \frac{X_q}{X_e + X_q} \frac{v_d}{V_t} V_{b\infty} \cos\delta - \frac{X'_d}{X_e + X'_d} \frac{v_q}{V_t} V_{b\infty} \sin\delta,$$
$$k_6 = \frac{X_e}{X_e + X'_d} \frac{v_d}{V_t}.$$

Note that v_d, v_q, I_d, I_q, and δ can be determined from the angular positions of the terminal voltage and terminal current with respect to the reference bus voltage, i.e., infinite bus voltage as described at the beginning of this subsection.

6.2.3 Fuzzy small signal model

Constants $k_1, \ldots, k_6$ in (6.2) are calculated using the steady state values of P, Q and X_e. This is common that the operating condition changes due to any changes in the network, such as change of mechanical power input to the generator, any faults in the network or tie line, change of loads, etc. Therefore, it is pragmatic to represent the small signal model not only at a single operating point, but, if possible, at a range of operating points. To accommodate a range of operating points, the following fuzzy rules can be applied considering the upper and lower bounds of parameters P, Q and X_e:

Rule 1: IF P is $\bar{P}$ and Q is $\bar{Q}$ and X_e is $\bar{X}_e$

THEN $\dot{x}(t) = A_1 x(t) + Bu(t)$,

Rule 2: IF P is $\bar{P}$ and Q is $\bar{Q}$ and X_e is $\underline{X}_e$

THEN $\dot{x}(t) = A_2 x(t) + Bu(t)$,

Rule 3: IF P is $\bar{P}$ and Q is $\underline{Q}$ and X_e is $\bar{X}_e$

THEN $\dot{x}(t) = A_3 x(t) + Bu(t)$,

Rule 4: IF P is $\bar{P}$ and Q is $\underline{Q}$ and X_e is $\underline{X}_e$

THEN $\dot{x}(t) = A_4 x(t) + Bu(t)$,

Rule 5: IF P is $\underline{P}$ and Q is $\bar{Q}$ and X_e is $\bar{X}_e$

THEN $\dot{x}(t) = A_5 x(t) + Bu(t)$,

$$\text{Rule 6: IF } P \text{ is } \underline{P} \text{ and } Q \text{ is } \bar{Q} \text{ and } X_e \text{ is } \underline{X}_e$$
$$\text{THEN } \dot{x}(t) = A_6 x(t) + Bu(t),$$
$$\text{Rule 7: IF } P \text{ is } \underline{P} \text{ and } Q \text{ is } \underline{Q} \text{ and } X_e \text{ is } \bar{X}_e$$
$$\text{THEN } \dot{x}(t) = A_7 x(t) + Bu(t),$$
$$\text{Rule 8: IF } P \text{ is } \underline{P} \text{ and } Q \text{ is } \underline{Q} \text{ and } X_e \text{ is } \underline{X}_e$$
$$\text{THEN } \dot{x}(t) = A_8 x(t) + Bu(t),$$

where

$$x(t) = \begin{bmatrix} \Delta\delta \\ \Delta\omega \\ \Delta E'_q \\ \Delta E_{fd} \end{bmatrix},\ A_i = \begin{bmatrix} 0 & \omega_o & 0 & 0 \\ -\frac{k_1^i}{M} & -\frac{D}{M} & -\frac{k_2^i}{M} & 0 \\ -\frac{k_4^i}{T'_{do}} & 0 & -\frac{1}{T'_{do}k_3^i} & \frac{1}{T'_{do}} \\ -\frac{K_E k_5^i}{T_E} & 0 & -\frac{K_E k_6^i}{T_E} & \frac{1}{T_E} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_E}{T_E} \end{bmatrix},$$

for $i = 1, 2, \ldots, 8$ and $u(t)$ is control effort of the power system stabiliser u_{PSS}. In the above rule set, over bar $(\bar{\cdot})$ and under bar $(\underline{\cdot})$ denote the fuzzy sets corresponding to the high and low ranges of the premise variables, respectively. The membership functions for the fuzzy sets are displayed in Figures 6.4, 6.5 and 6.6. Constants $k_1^i, \ldots, k_6^i$ can be obtained using the upper and lower bounds of P, Q and X_e for the respective rules.

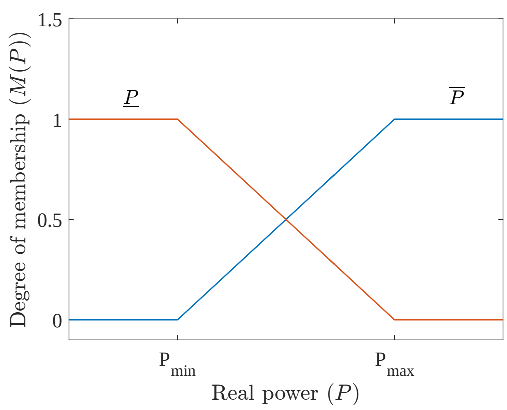

Figure 6.4: Membership or real power P

Considering $\xi(t) = \begin{bmatrix} P & Q & X_e \end{bmatrix}$ the vector of premise variables, and $M_i^k(\xi_k(t))$ the membership functions of the fuzzy set for the kth premise variable of the ith rule, the overall fuzzy small signal model of the single machine infinite bus system

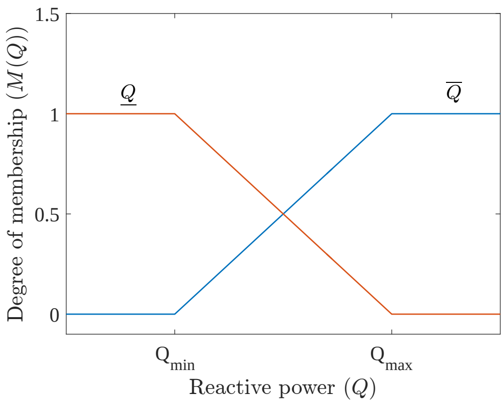

Figure 6.5: Membership of reactive power Q

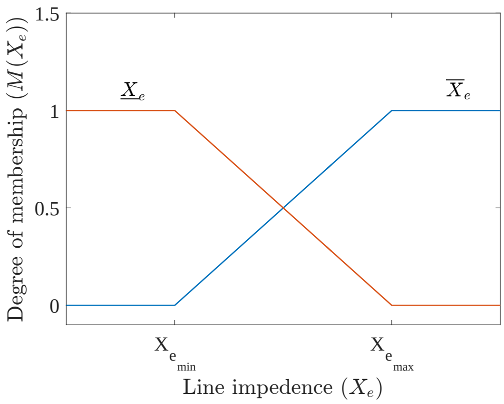

Figure 6.6: Membership of line impedance X_e

can be expressed as

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + Bu(t)\}, \tag{6.3}$$

where $\mu_i(\xi(t)) = \frac{\prod_{k=1}^{l} M_i^k(\xi_k(t))}{\sum_{i=1}^{r}\prod_{k=1}^{l} M_i^k(\xi_k(t))}$.

6.3 Functional observer based fuzzy power system stabiliser

We use the fuzzy functional observer to estimate $u_{PSS}(t)$ such that the system becomes asymptotically stable under sudden perturbation.

6.3.1 Power system stabiliser without time-delay

The fuzzy model derived in Section 6.2 does not consider the time-delay in the feedback loop. We can obtain the stabiliser as a function of the states of the system. Therefore, we can directly apply the fuzzy functional observer based PDC controller described in Chapter 2 for designing a power system stabiliser. The following example illustrates the design procedure and performance of the functional observer based power system stabiliser.

Example 6.1

This example uses the following data in per unit (pu) system for calculating constants $k_1, \ldots, k_6$ and the respective linear time invariant models of each rule of T-S fuzzy model.

$$
\begin{array}{llll}
P \in \begin{bmatrix} 0.7 & 1 \end{bmatrix}, & Q \in \begin{bmatrix} -0.2 & .3 \end{bmatrix}, & X_e \in \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}, & \\
X_d = 1.6, & X'_d = 0.32, & X_q = 1.55, & M = 7, \\
K_E = 100, & T_E = 0.01, & T'_{do} = 6, & \omega_0 = 314.16.
\end{array}
$$

Considering the ranges of real power, reactive power, and line reactance, we can obtain eight sets of constants for different rules and we can obtain the T-S fuzzy model of the system. Table 6.1 displays the values of $k_1^i, \ldots, k_6^i$ for different rules.

Table 6.1: $k_1^i, \ldots, k_6^i$ for different fuzzy rules without time-delay

Rule	P	Q	X_e	k_1^i	k_2^i	k_3^i	k_4^i	k_5^i	k_6^i
1	low	low	low	1.3818	1.2923	0.2889	1.6541	0.0314	0.3168
2	low	low	high	1.1429	0.9731	0.3600	1.2456	0.0133	0.4814
3	low	high	low	1.4464	1.7930	0.2889	2.2950	-0.0057	0.1904
4	low	high	high	1.0148	1.3288	0.3600	1.7009	-0.0508	0.3152
5	high	low	low	1.6188	1.5227	0.2889	1.9490	-0.0069	0.2882
6	high	low	high	1.2937	1.1310	0.3600	1.4476	-0.0400	0.4595
7	high	high	low	1.5303	1.8559	0.2889	2.3756	-0.0651	0.1758
8	high	high	high	1.0509	1.3585	0.3600	1.7388	-0.1252	0.3301

Power system stabiliser is meant to stabilise the electromechanical oscillation in the system caused by any abrupt changes in the power system. The stabiliser takes the change of the rotor speed as input and generates the control signal that

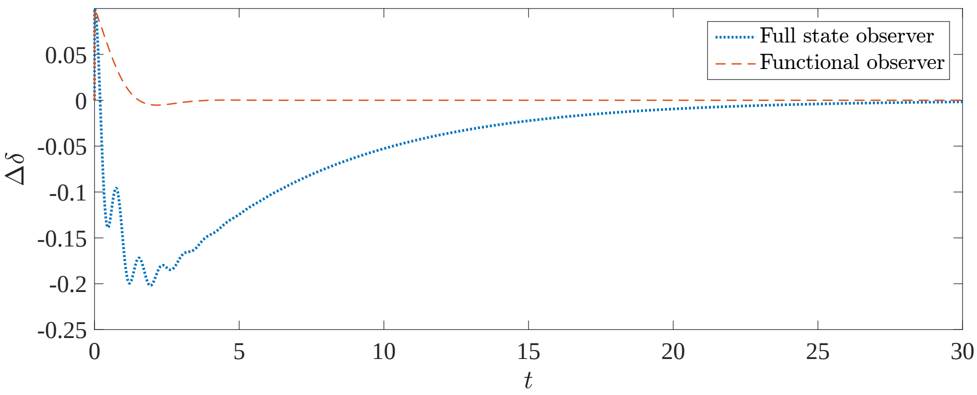

Figure 6.7: Deviation of power angle

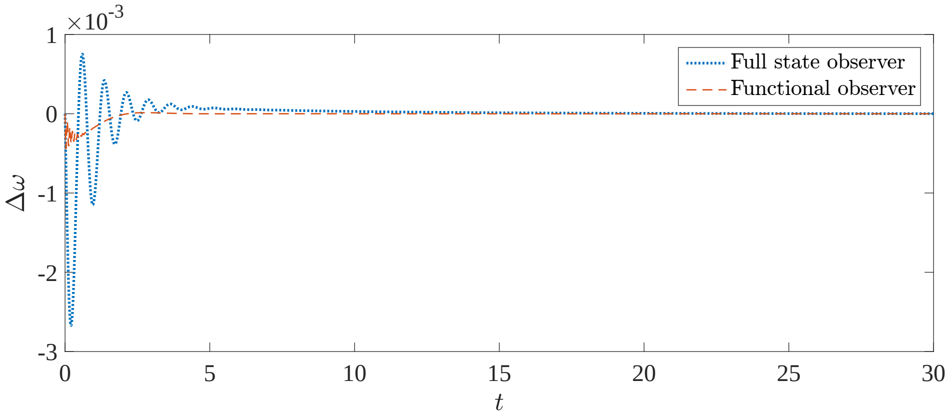

Figure 6.8: Deviation of rotor speed

is fed back to the exciter to damp out the electromechanical oscillation. In this example, we apply the functional observer to generate the control signal as a function of states, which are the deviations of the four parameters: namely power angle, rotor speed, induced voltage, and field voltage. By following the procedure stated in Section 2.4 of Chapter 2, the functional observe parameters are obtained. The effect of a small perturbation in the system for different initial conditions are simulated in MATLAB environment, and the results considering initial condition $\begin{bmatrix} 0.1 & 0 & 0 & 0 \end{bmatrix}^T$ are presented in the following figures. The electromechanical oscillation of the system in terms of the deviations of power angle, rotor speed, induced voltage, and field excitation voltage are depicted in Figures 6.7 to 6.10. Each figure contains two graphs to compare the stabilising performance of the functional observer based controller with the full state observer based controller.

It can be seen that the oscillation damps out asymptotically. Figure 6.11 compares the control signals generated by these two controllers. From the ob-

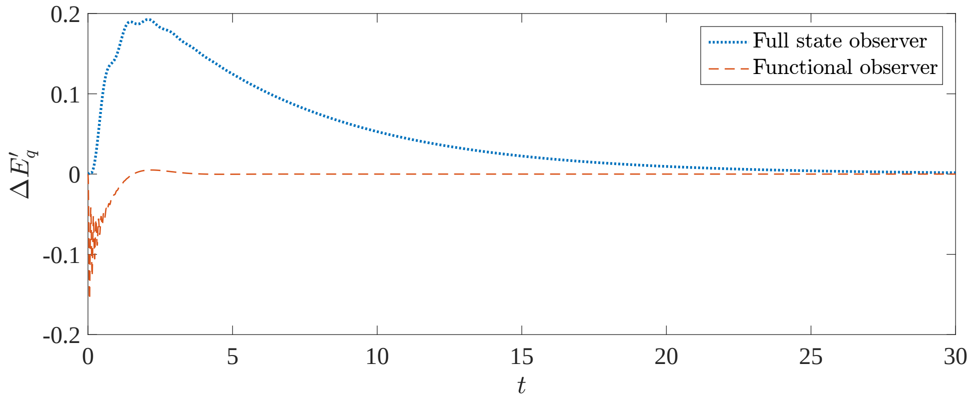

Figure 6.9: Deviation of induced voltage

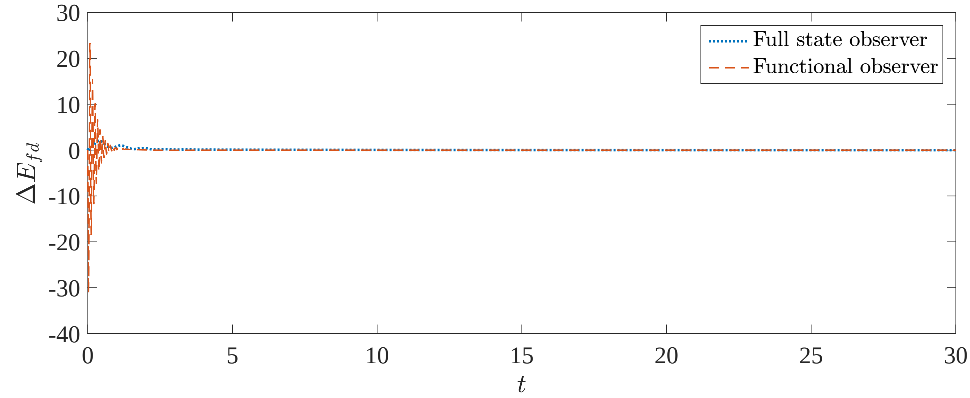

Figure 6.10: Deviation of field voltage

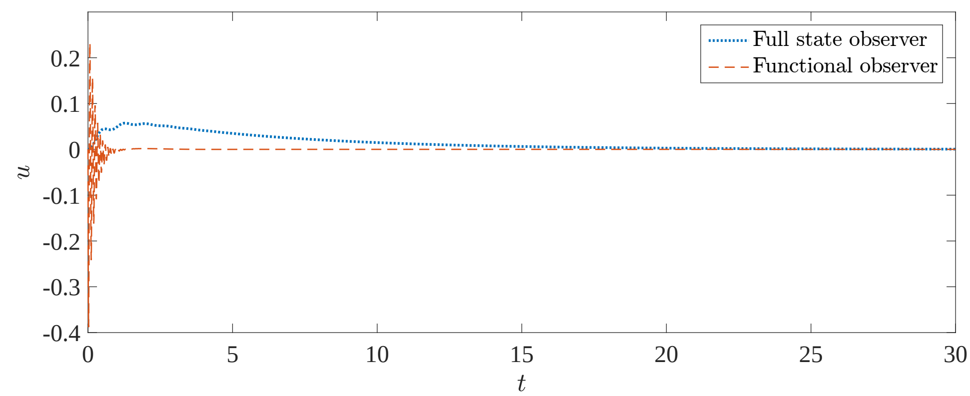

Figure 6.11: Control signal generated by the fuzzy power system stabiliser

servation of all the figures, it is evident that the functional observer based PDC controller stabilises the system asymptotically and it performs better than the full state observer based controller in terms of overshoot and setting time. Note that the convergence of the deviation of system states to zero can be made faster by choosing different control gains K_j using different algorithms, and the functional observer for that set of control gains can be designed accordingly.

6.3.2 Power system stabiliser with time-delay

Considering delays in the excitation system, the overall T-S fuzzy model of the system can be expressed as

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\{A_i x(t) + A_{di} x(t-\tau) + Bu(t)\}, \tag{6.4}$$

where

$$x(t) = \begin{bmatrix} \Delta\delta \\ \Delta\omega \\ \Delta E'_q \\ \Delta E_{fd} \end{bmatrix}, \ A_i = \begin{bmatrix} 0 & \omega_o & 0 & 0 \\ -\frac{k_1^i}{M} & -\frac{D}{M} & -\frac{k_2^i}{M} & 0 \\ -\frac{k_4^i}{T'_{do}} & 0 & -\frac{1}{T'_{do}k_3^i} & \frac{1}{T'_{do}} \\ 0 & 0 & 0 & \frac{1}{T_E} \end{bmatrix},$$

$$A_{di} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{K_E k_5^i}{T_E} & 0 & -\frac{K_E k_6^i}{T_E} & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_E}{T_E} \end{bmatrix},$$

and $u(t)$ is control signal of the power system stabiliser, and $\mu_i(\xi(t))$ is calculated from respective fuzzy rules as described in the previous subsection. The power system stabiliser can be directly obtained by following the procedure for synthesising PDC controller for a T-S fuzzy system with time-delay stated in Subsection 3.3.2 of Chapter 3.

Example 6.2

The following data is used to illustrate the stabiliser construction procedure considering time varying time-delay:

$$P \in \begin{bmatrix} 0.7 & 1.0 \end{bmatrix}, \quad X_e \in \begin{bmatrix} 0.2 & 0.4 \end{bmatrix} \quad Q \in \begin{bmatrix} -0.2 & 0.3 \end{bmatrix},$$

$$X_d = 1.81, \quad X'_d = 0.3, \quad X_q = 1.76, \quad M = 7,$$

$$T'_{do} = 8, \quad T_E = 0.01, \quad K_E = 100, \quad \omega_0 = 314.6,$$

$$V_{b\infty} = 1.0\angle 0^\circ, \quad D = 1.0.$$

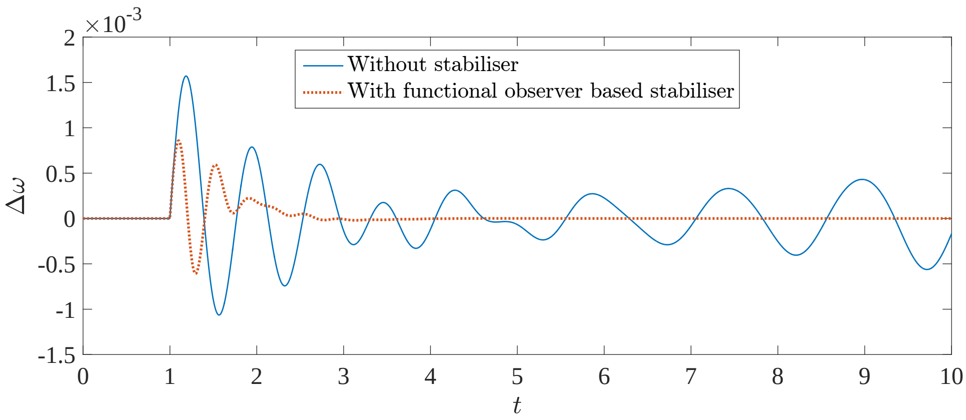

Figure 6.12: Deviation of rotor speed considering time-delay in excitation system

The values are in per unit basis. Eight sets of constants for different rules are obtained for the T-S fuzzy model. Table 6.2 displays the constants, $k_1^i, \ldots, k_6^i$, for different rules.

Table 6.2: $k_1^i, \ldots, k_6^i$ for different fuzzy rules considering time-delay

Rule	P	Q	X_e	k_1^i	k_2^i	k_3^i	k_4^i	k_5^i	k_6^i
1	high	high	high	1.5555	1.8285	0.2488	2.7610	0.0156	0.2131
2	high	high	low	1.1092	1.3373	0.3167	2.0194	-0.0170	0.3480
3	high	low	high	1.4470	1.3075	0.2488	1.9744	0.0452	0.3343
4	high	low	low	1.2043	0.9658	0.3167	1.4583	0.0319	0.5022
5	low	high	high	1.6692	1.9102	0.2488	2.8844	-0.0452	0.1939
6	low	high	low	1.1661	1.3816	0.3167	2.0862	-0.0937	0.3536
7	low	low	high	1.7096	1.5540	0.2488	2.3465	0.0101	0.3051
8	low	low	low	1.3733	1.1358	0.3167	1.7150	-0.0165	0.4792

Considering the upper and lower bounds of the time-delay as $\tau_M = 0.5$ seconds and $\tau_m = 0.005$ seconds, respectively, the upper bound of the derivative of time-delay as $\rho = 0.2$, and the constants $\zeta_1 = 0.5$ and $\zeta_2 = 0.6$, the observer parameters are obtained by following the procedure outlined in Subsection 3.3.2 of Chapter 3. SOSTOOLS [63] is used to solve the LMIs. The closed loop system performance is simulated considering the initial power injection in the infinite bus as $0.85 + j0.25$, and a 10% step increase of the mechanical power input to the synchronous generator at time $t = 1.0$ second. The initial load angle $\delta_0 = 0.8274$ rad. The time varying time-delay, in seconds, is taken as $\tau(t) = 0.1\sin(t) + 0.4$.

The simulation outputs, displayed in Figures 6.12 and 6.13, reveal that the system is stable with the proposed fuzzy functional observer based power system

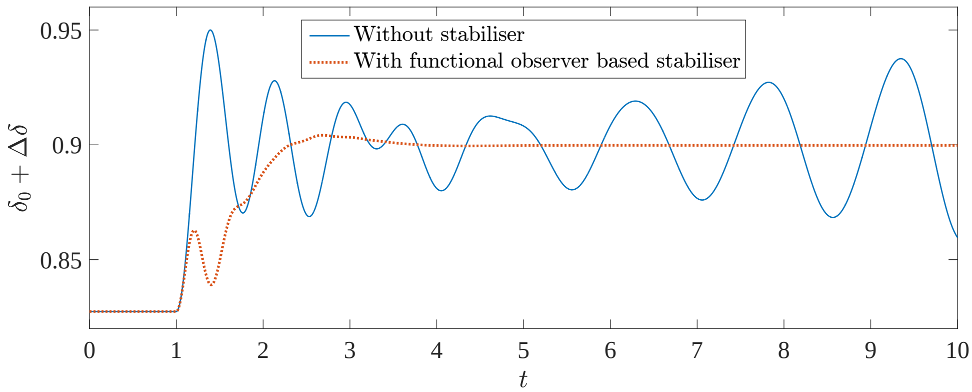

Figure 6.13: Deviation of load angle considering time-delay in excitation system

stabiliser when the system is perturbed. The load angle settles to a new value, and the deviation of the rotor speed approaches zero asymptotically. The system without any stabiliser, on the other hand, is not stable under the perturbation. With this observation, it can be said that the fuzzy functional observer based power system stabiliser can be used for damping out the electromechanical oscillation of a single machine infinite bus system.

6.4 Conclusion

The T-S fuzzy model based approach increases the operation domain of the power system stabiliser because the constants of the Heffron-Philip model are determined using a range of the operating conditions. The fuzzy functional observer is found to be effective and useful to obtain the stabilising signal for the small signal model. Future work may consider external disturbance in the fuzzy model and finite time convergence of the deviation of the system states to zero.

Chapter 7

Conclusions and future work

7.1 Conclusions

A fuzzy functional observer can be designed as a fuzzy summation of linear functional observers for linear subsystems of a T-S fuzzy model of a nonlinear system. Chapter 2 presents the construction procedure of a fuzzy functional observer by solving LMI based conditions that guarantee the stability of the observer. It has been demonstrated that the functional observer can be employed to estimate the control signal for stabilising a nonlinear system directly. The order of the observer reduces to the dimension of the control vector. As the separation principle holds, the observer and controller gains can be obtained separately.

Time-delay, model uncertainty, and external disturbances are naturally present in the plant operating conditions. Chapter 3 and 4 investigate the effects of time-delay and model uncertainty on the functional observer and present the existence and stability conditions for the observer to minimise the effect of time-delay and model uncertainty on the estimation error dynamics. The functional observer is applied for obtaining PDC controllers considering time-delay and model uncertainty in the plant model. The stability conditions for time-delay systems are formulated using Lyapunov-Krasovskii functionals. Free-weighting matrices are introduced to obtain delay dependent stability conditions. An L_2 gain based performance indicator is used to minimise the effect of model uncertainty on the error dynamics. The concept of unknown input observer is applied in Chapter 5 for decoupling external disturbances from the error dynamics of the fuzzy functional observer.

The fuzzy functional observer is employed for fault detection scheme considering the effect of time-delay and external disturbance. The residual is generated using the proposed observer so that the fault detection procedure does not require to compare with any threshold. A fault observer is designed using the functional observer to estimate the fault vector asymptotically. The fault detection scheme is robust against the external disturbance as the disturbance is decoupled from the error dynamics.

The proposed techniques are verified using examples; the performances of the techniques are simulated and compared with existing results. The proposed fuzzy functional observer based PDC controller is applied to design a power system stabiliser for a single machine infinite bus system. The simulation outputs show the satisfactory damping of electromechanical oscillation of the system due to a sudden disturbance in the power network.

7.2 Future research directions

Future work may consider the following improvements to the results presented in this thesis.

- The stability conditions for the observers can be improved by applying fuzzy Lyapunov function, higher order derivatives of Lyapunov functions, or triple integration terms for Lyapunov-Krasovski functionals. The application of these kinds of Lyapunov functions will increase the solution domains of the LMI based stability conditions.

- The time-delays that are considered in this thesis are deterministic. However, the plant may experience stochastic time-delays. The plant itself can be stochastic rather than becoming a deterministic one. These issues are interesting for studying the stability of the system using functional observer.

- This thesis considers continuous time models for designing the observers. Discrete time models, in some cases, may be more pragmatic for defining a system and designing the controller considering the computational efforts for analog to digital, and digital to analog converters. More importantly, some systems are discrete time by nature. Therefore, having the present developments of the theories for continuous time models, future work may consider the extensions of the proposed techniques for applying in discrete time and sampled-data fuzzy models.

- In recent years, the interval type-2 T-S fuzzy model has attracted many researchers for its ability to define the modeling uncertainties as intervals of the degree of membership. The proposed techniques may be extended for nonlinear systems defined by interval type-2 T-S fuzzy models.

Appendix A

Proof of Lemma 4.3.1

With the PDC controller $u = \sum_{j=1}^{r} \mu_j(\xi(t))K_j x(t)$, the system described in (4.2) can be expressed as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(\xi(t))\mu_j(\xi(t))\{(A_i+\Delta A_i+B_iK_j)x(t)+(A_{di}+\Delta A_{di})x(t-\tau(t))\}. \quad (A.1)$$

Consider a Lyapunov-Krasovskii functional

$$\begin{aligned} V(t) =& x^T(t)P_1x(t) + \int_{t-\tau(t)}^{t} x^T(s)P_2x(s)ds + \int_{-\tau_M}^{0}\int_{t+\theta}^{t} x^T(s)P_3x(s)dsd\theta \\ &+ \int_{-\tau_M}^{-\tau_m}\int_{t+\theta}^{t} x^T(s)P_4x(s)dsd\theta, \end{aligned} \quad (A.2)$$

where P_1, P_2, P_3 and P_4 are positive definite symetric matrices. Taking derivative along the state dynamics and considering the assumption $\dot{\tau}(t) \leq \rho$ we can obtain

$$\begin{aligned} \dot{V}(t) =& 2x^T(t)P_1\dot{x}(t) + x^T(t)P_2x(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))P_2x(t-\tau(t)) \\ &+ \dot{x}^T(t)(\tau_M P_3 + (\tau_M - \tau_m)P_4)\dot{x}(t) - \int_{t-\tau_M}^{t} \dot{x}^T(s)P_3\dot{x}(s)ds \\ &- \int_{t-\tau_M}^{t-\tau_m} \dot{x}^T(s)P_4\dot{x}(s)ds \\ \leq& 2x^T(t)P_1\dot{x}(t) + x^T(t)P_2x(t) - (1-\rho)x^T(t-\tau(t))P_2x(t-\tau(t)) \\ &+ \dot{x}^T(t)(\tau_M P_3 + (\tau_M - \tau_m)P_4)\dot{x}(t). \end{aligned} \quad (A.3)$$

By using (A.1), and the assumption of (4.3a) and (4.3b), it can be shown that

$$
\begin{aligned}
&\dot{x}^T(t)(\tau_M P_3 + (\tau_M - \tau_m)P_4)\dot{x}(t) \\
&\quad = \textstyle\sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t))\mu_j(\xi(t)) \left\{ \dot{x}^T(t)\Lambda \begin{bmatrix} A_i + B_iK_j & A_{di} \end{bmatrix} \bar{x}(t) \right. \\
&\qquad \left. + \dot{x}^T(t) \begin{bmatrix} \Lambda R_i & \Lambda R_{di} \end{bmatrix} \begin{bmatrix} U_i(t)S_i & 0 \\ 0 & U_{di}(t)S_{di} \end{bmatrix} \bar{x}(t) - \tau_m \dot{x}^T(t)P_4\dot{x}(t) \right\} \\
&\quad \le \textstyle\sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t))\mu_j(\xi(t)) \left\{ \dot{x}^T(t)\Lambda \begin{bmatrix} A_i + B_iK_j & A_{di} \end{bmatrix} \bar{x}(t) \right. \\
&\qquad + \dot{x}^T(t) \begin{bmatrix} \Lambda R_i & \Lambda R_{di} \end{bmatrix} \begin{bmatrix} \Lambda R_i & \Lambda R_{di} \end{bmatrix}^T \dot{x}(t) \\
&\qquad \left. + \bar{x}^T(t) \begin{bmatrix} S_i^T S_i & 0 \\ 0 & S_{di}^T S_{di} \end{bmatrix} \bar{x}(t) - \tau_m \dot{x}^T(t)P_4\dot{x}(t) \right\},
\end{aligned}
\tag{A.4}
$$

where $\bar{x}^T(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) \end{bmatrix}$ and $\Lambda = \tau_M(P_3 + P_4)$. Applying some algebraic manipulation, it can also be shown that

$$
\begin{aligned}
2x^T(t)P_1\dot{x}(t) \le \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t))\mu_j(\xi(t))\bar{x}^T(t) \\
\left\{ \begin{bmatrix} P_1(A_i + B_iK_j) + (A_i + B_iK_j)^T P_1 & P_1A_{di} \\ \star & 0 \end{bmatrix} \right. \\
\left. + \begin{bmatrix} P_1R_i & P_1R_{di} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (P_1R_i)^T & 0 \\ (P_1R_{di})^T & 0 \end{bmatrix} + \begin{bmatrix} S_i^T S_i & 0 \\ 0 & S_{di}^T S_{di} \end{bmatrix} \right\} \bar{x}(t).
\end{aligned}
\tag{A.5}
$$

Therefore, using (A.3), (A.4) and (A.5) we get

$$
\begin{aligned}
\dot{V}(t) \le \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t))\mu_j(\xi(t))\eta^T(t) \\
\left(\begin{bmatrix} \begin{matrix} P_2 + P_1A_i + A_i^T P_1 \\ +P_1B_iK_j + K_j^T B_i^T P_1 \end{matrix} & P_1A_{di} & \frac{1}{2}(A_i + B_iK_j)^T\Lambda \\ \star & -(1-\rho)P_2 & \frac{1}{2}A_{di}^T\Lambda \\ \star & \star & -\tau_m P_4 \end{bmatrix} \right. \\
+ \begin{bmatrix} P_1R_i & P_1R_{di} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda R_i & \Lambda R_{di} \end{bmatrix} \begin{bmatrix} (P_1R_i)^T & 0 & 0 \\ (P_1R_{di})^T & 0 & 0 \\ 0 & 0 & R_i^T\Lambda \\ 0 & 0 & R_{di}^T\Lambda \end{bmatrix}
\end{aligned}
$$

$$+2\begin{bmatrix} S_i^T S_i & 0 & 0 \\ 0 & S_{di}^T S_{di} & 0 \\ 0 & 0 & 0 \end{bmatrix}\Bigg)\eta(t),$$

where $\eta^T(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & \dot{x}^T(t) \end{bmatrix}$. As a result, the asymptotic stability condition for the closed loop fuzzy system can be given as

$$\begin{bmatrix} \begin{array}{c} P_2 + P_1 A_i + A_i^T P_1 \\ +P_1 B_i K_j + K_j^T B_i^T P_1 \end{array} & P_1 A_{di} & \frac{1}{2}(A_i + B_i K_j)^T \Lambda \\ \star & -(1-\rho)P_2 & \frac{1}{2}A_{di}^T \Lambda \\ \star & \star & -\tau_m P_4 \end{bmatrix}$$

$$+\begin{bmatrix} P_1 R_i & P_1 R_{di} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda R_i & \Lambda R_{di} \end{bmatrix}\begin{bmatrix} (P_1 R_i)^T & 0 & 0 \\ (P_1 R_{di})^T & 0 & 0 \\ 0 & 0 & R_i^T \Lambda \\ 0 & 0 & R_{di}^T \Lambda \end{bmatrix}$$

$$+\,2\begin{bmatrix} S_i^T S_i & 0 & 0 \\ 0 & S_{di}^T S_{di} & 0 \\ 0 & 0 & 0 \end{bmatrix} < 0.$$

By Schur complement, we get (A.6). Considering $P_3 = \bar{\sigma}_1 P_1$ and $P_4 = \bar{\sigma}_2 P_1$, and pre-multiplying and post-multiplying (A.6) by nonsingular block-diagonal matrix

$$\mathrm{diag}(P_1^{-1} \quad P_1^{-1} \quad P_1^{-1} \quad I \quad I \quad I \quad I \quad I \quad I),$$

we obtain (A.7), where $\bar{P}_1 = P_1^{-1}$ $\bar{P}_2 = \bar{P}_1 P_2 \bar{P}_1$, $\bar{Y}_j = K_j \bar{P}_1$, $\kappa = \tau_M(\bar{\sigma}_1 + \bar{\sigma}_2)$ for some known scalars $\bar{\sigma}_1$ and $\bar{\sigma}_2$. This completes the proof.

Equations (A.6) and (A.7) are displayed in the next page.

$$
\begin{bmatrix}
\begin{matrix} P_2 + P_1 A_i + A_i^T P_1 \\ + P_1 B_i K_j + K_j^T B_i^T P_1 \end{matrix} & P_1 A_{di} & \frac{1}{2}(A_i^T + K_j^T B_i^T)\Lambda & P_1 R_i & P_1 R_{di} & 0 & 0 & S_i^T & 0 \\
\star & -(1-\rho)\bar{P}_2 & \frac{1}{2} A_{di}^T \Lambda & 0 & 0 & 0 & 0 & 0 & S_{di}^T \\
\star & \star & -\tau_m P_4 & 0 & 0 & \Lambda R_i & \Lambda R_{di} & 0 & 0 \\
\star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & -I & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -I & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & -\frac{1}{2}I & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & -\frac{1}{2}I
\end{bmatrix} < 0. \tag{A.6}
$$

$$
\begin{bmatrix}
\begin{matrix} \bar{P}_2 + A_i \bar{P}_1 + \bar{P}_1 A_i^T \\ + B_i \bar{Y}_j + \bar{Y}_j^T B_i^T \end{matrix} & A_{di} \bar{P}_1 & \frac{1}{2}\kappa(\bar{P}_1 A_i^T + \bar{Y}_j^T B_i^T) & R_i & R_{di} & 0 & 0 & \bar{P}_1 S_i^T & 0 \\
\star & -(1-\rho)\bar{P}_2 & \frac{1}{2}\kappa \bar{P}_1 A_{di}^T & 0 & 0 & 0 & 0 & 0 & \bar{P}_1 S_{di}^T \\
\star & \star & -\tau_m \bar{\sigma}_2 \bar{P}_1 & 0 & 0 & \kappa R_i & \kappa R_{di} & 0 & 0 \\
\star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & -I & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -I & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & -\frac{1}{2}I & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & -\frac{1}{2}I
\end{bmatrix} < 0. \tag{A.7}
$$

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