SUBMITTED VERSION

Yaping Shan

Incentives for research agents and performance-vested equity-based compensation

Journal of Economic Dynamics and Control, 2019; 102:44-69

© 2019 Elsevier B.V. All rights reserved.

Published at: http://dx.doi.org/10.1016/j.jedc.2019.02.007

PERMISSIONS

https://www.elsevier.com/about/policies/sharing

Preprint

- Authors can share their preprint anywhere at any time.
- If accepted for publication, we encourage authors to link from the preprint to their formal publication via its Digital Object Identifier (DOI). Millions of researchers have access to the formal publications on ScienceDirect, and so links will help your users to find, access, cite, and use the best available version.
- Authors can update their preprints on arXiv or RePEc with their accepted manuscript .

Please note:

- Some society-owned titles and journals that operate double-blind peer review have different preprint policies. Please check the journals Guide for Authors for further information
- Preprints should not be added to or enhanced in any way in order to appear more like, or to substitute for, the final versions of articles.

17 September 2020

http://hdl.handle.net/2440/121699

Incentives for Research Agents and Performance-vested Equity-based Compensation *

Yaping Shan†

Abstract

This paper studies the agency problem between a firm and its research employees in a dynamic optimal contracting setting. We implement the optimal contract by a risky security, which can be created using the equity of the firm, and a sequence of performance-based holding requirements. This result provides a rationale for using performance-vested equity-based compensation in R&D-intensive start-up firms.

Key words: Performance-vesting Provisions, Dynamic Contract, R&D

JEL: D23, D82, D86, J33, L22, O32

*I sincerely thank the editor, B. Ravikumar, and two anonymous reviewers for their constructive comments and suggestions. I would like to thank Srihari Govindan, Ayca Kaya, Kyungmin Kim, Mandar Oak, Raymond Riezman, Yuzhe Zhang, and seminar participants at The University of Iowa, the University of Adelaide, 2017 Asia Meeting of the Econometric Society in Hong Kong, 2017 China Meeting of the Econometric Society in Wuhan, and 2018 China Meeting of the Econometric Society in Shanghai for their valuable advice and comments. Any errors are my own.

†School of Economics, The University of Adelaide, Email: yaping.shan@adelaide.edu.au

1 Introduction

In high-tech start-up firms, equity-based compensations have become an important compensation scheme for research employees. Twitter, one of the most successful start-up firms in the last decade, went public in 2013. During the same year, it spent $380 million in equity-based compensation for its research employees, which accounted for 64% of its total R&D expenses.[1] Equity-based compensation helps to provide incentives to research employees by providing a direct link between the employees' compensation and the firm's performance and is particularly attractive for cash-constrained firms. Regarding provision, equity-based compensations vest over time. Over the last two decades, a general trend observed in equity-based compensation practice is the shifting from traditional simple time-vesting provisions to performance-vesting provisions.[2] Performance-vesting, contrary to time-vesting, helps with incentive provision when the firm's growth depends crucially on the stochastic outcomes of its R&D projects, and hence is particularly useful in R&D-intensive start-up firms.

This paper provides an example of an environment in which, under some conditions, performance-vested equity-based compensation arises as an optimal outcome. We use an optimal contracting approach to analyze the agency problem between a firm and its research employees. A major methodological contribution of this paper is the tractability of the contracting problem which has a closed-form solution. The key question is what kind of compensation scheme could implement the optimal contract, and how it is related to the performance-vested equity-based compensation observed in practice. Our findings indicate that the optimal contract can be implemented by using a risky security with a sequence of holding requirements that will be relaxed once a performance target is achieved. Sharing the main features with performance-vested equity-based compensations, these results provide a motive for using performance-vested equity-based compensation in start-up

[1]This calculation is based on Twitter's Form 10-k for the fiscal year ended December 31, 2013. "Research and development expenses consist primarily of personnel-related costs, including salaries, benefits and stock-based compensation, for engineers and other employees engaged in the research and development of products and services" (Twitter Form 10-k 2013).

[2]In a report by Salary.com, a leading consumer and enterprise resource for compensation data, Whittlesey (2007) stated that the most notable change in equity-based compensation provisions for both executive grants and all-employee programs is "the widespread introduction of performance-based plans with a wide variety of features." Bettis, Bizjak, Coles, and Kalpathy (2018) documented that "the usage of performance-vesting equity awards to top executives in large U.S. companies has grown from 20 to 70 percent from 1998 to 2012."

firms that rely on R&D from the theoretical point of view.

We set up the contracting problem using the model studied in Shan (2017). Briefly, a risk-neutral principal hires a risk-averse agent to perform a multi-stage R&D project. The multi-stage feature captures the observation that the performance of research employees is usually linked to the completion of a sequence of milestones rather than their day-to-day practice. At any point in time, the agent can choose whether to put in effort or shirk. Subject to the agent investing effort, the transition from one stage to the next is a Poisson process with a constant arrival rate. The progress of the innovation process is publicly observable, and the principal cannot monitor the agent's action, which causes a dynamic moral-hazard problem. Shan (2017) characterized the optimal contract under the assumption that the principal has full control over the agent's consumption. In the optimal contract, using a "carrot and stick" strategy, the principal punishes the agent by lowering his compensation over time in case of failure and rewards him by a discrete increase in the payment after each success.

Using this model, the current paper provides an implementation of the optimal contract and discusses how it connects to existing compensation practice. We show that the optimal contract can be implemented by a state-contingent security that appreciates in case of success but depreciates in case of failure. At any point in time, besides the effort choice, the agent also chooses how much to consume and how much to invest in the security for savings subject to a sequence of holding requirements on the risky security. Different from the optimal contract, in which the principal controls the agent's consumption directly, the agent chooses the consumption process in this implementation, which nonetheless generates the same effort and consumption process as the optimal contract. The key finding of the implementation results is how the design of the holding requirement depends on the agent's performance. In the implementation, the principal requires the agent to meet a minimum holding requirement on the state-contingent security till the completion of the project and gradually relaxes the holding requirement as the project progresses. Our model shows that the principal uses the state-contingent security to compensate the agent to encourage him to bear some risks in return for incentives, and the holding requirement in the implementation guarantees the minimum amount of risks that the agent has to take for incentives. When the R&D project progresses, the uncertainty of the project reduces, and hence the holding requirement can be relaxed.

In general, the payoff structure of the state-contingent security that implements the optimal contract depends on the utility function of the agent. There may not exist a financial asset that has the exact payoff of the security. However, the firm can use its equity and other available financial assets to approximate the payoff of the security and use the performance-vesting provision to mimic the performance-based minimum holding requirements. We also consider an example in which, under some conditions, the state-contingent security can be directly linked to the firm's equity. Assuming that the agent has a logarithmic utility function, the contracting model has a closed-form solution. In this case, the state-contingent security has the property that its value increases proportionally after each success, and hence it can be created by a portfolio of the firm's equity and a risk-free asset if the firm's value also grows proportionally after each breakthrough of its R&D project. In this example, the implementation becomes surprisingly simple. The principal only needs to adjust the fraction of equity in the compensation portfolio when the project progresses to the next stage and can leave all other decision problems to the agent. The proportionate growth assumption on the evolution of firm's value is a key assumption to derive these results. In practice, most R&D-intensive start-up firms are backed by venture capital, and whether a firm can receive further rounds of financing depends crucially on the development of its main research project. We show that this proportionate growth assumption is consistent with the growth pattern of firm valuation at each financing round for firms that are backed by venture capital.[3]

Theoretically, the optimality of equity-based implementation requires that the firm's value depends only on the progress of the R&D project and that the firm has an accurate prediction about how its value is affected by the project. In practice, however, the firm's value is also affected by other factors, for example, market aggregate risks, or the performance of other R&D teams when several projects are performed simultaneously. In these cases, equity-based incentive compensation exposes the agent to risks that are not related to his action and becomes less efficient. For these situations, we provide an alternative implementation of the optimal contract using a savings account plus performance-based bonuses after each success. In this implementation, the principal offers the agent a savings account with an initial balance. At any point in time, the agent can withdraw money from the savings account for consumption. The principal rewards the agent with a performance bonus and deposits it into the savings account after each success. Similar to the equity-based implementation, this implementation also generates the same effort and consumption

[3]The proportionate growth property is also called Gibrat's law which states that the proportional rate of growth of a firm is independent of its absolute size.

process as the optimal contract. Comparing these two implementations, the advantage of equity-based implementation is its simplicity for which the principal only needs to adjust the composition of the compensation portfolio according to the progress of the project. It is attractive to cash-constrained start-up firms because it allows them to spend cash in other important areas. However, it may expose the agent to the risks that are not related to his action. If the principal is able to use cash bonuses, the alternative implementation prevents the agent from bearing unnecessary risks, but it requires the principal to monitor the balance of the savings account because the agent is risk-averse and hence the size of bonus depends on the balance of the savings account.

The rest of the paper is organized as follows. Section 2 provides a review of the related literature. Section 3 describes the benchmark model and the optimal contract. In Section 4, we present an implementation of the optimal dynamic contract and discuss how it relates to performance-vested equity-based compensation. Section 5 considers an example in which the agent has a logarithmic utility function. We provide an alternative implementation via performance-based bonuses in Section 6. Section 7 presents the conclusions. Some extensions of the paper are discussed in the Appendix.

2 Literature Review

The CEO compensation literature provides extensive research on equity-based grants. Edmans, Gabaix, Sadzik, and Sannikov (2012) studied the optimal CEO compensation in a dynamic framework and provided an implementation of the optimal contract using a "Dynamic Incentive Account" that comprises cash and the firm's equity. The main difference between their model and our model is the approach to model how the agent's action affects his performance. Since CEO's effort often has an important impact on the operation of the firm, in Edmans, Gabaix, Sadzik, and Sannikov (2012), the earnings of the firm in each period are determined by the CEO's effort and a random noise. In continuous-time settings, this agency problem is usually modeled by the Brownian-motion process in which the agent's unobserved effort controls the drift (for example He (2009)). For research employees' incentive problem, since the effort invested in research today will not necessarily lead to a discovery tomorrow, we assume that the agent's effort affects the probability of success and model the innovation process as a Poisson-type process. In both papers, the equity-based compensation features vesting which ensures that the agent has sufficient equity in the future to induce effort, and the equity compensation fully vests at the time after which the agent's action cannot affect the firm's value anymore (when the agent retires in Edmans, Gabaix, Sadzik, and Sannikov (2012) and when the agent completes the whole project in our model). The difference in assumptions about how the agent's unobservable actions affect the firm's value also leads to different features in implementation regarding vesting. In Edmans, Gabaix, Sadzik, and Sannikov (2012), the vesting of equity-based compensation is time-based because the agent's action affects the drift of the firm's value. In our model, the agent's action controls the arrival of a series of innovations, and hence the vesting is performance-based.

With regard to researchers' compensation, Anderson, Banker, and Ravindran (2000), Ittner, Lambert, and Larcker (2003), and Murphy (2003) have documented that executives and employees in research intensive firms receive more equity-based compensation than their counterparts in traditional industries. Sesil, Kroumova, Blasi, and Kruse (2002) compared the performance of 229 research intensive firms offering broad-based stock options with that of their non-stock option counterparts. They showed that the former have higher shareholder returns. For performance-vesting provisions, Bettis, Bizjak, Coles, and Kalpathy (2010), found that "performance-vesting provisions specify meaningful performance hurdles and provide significant incentives." Also, "performance-vesting firms had significantly better subsequent operating performance than control firms." Our paper contributes to this literature by establishing a specific role for performance-vesting provisions in the optimal contracting problem.

In terms of methodology, this article follows the rich and growing literature on continuous-time dynamic contracting. Sannikov (2008) analyzed a continuous-time principal-agent model, in which the output is a Brownian-motion process with drift determined by the agent's unobserved effort. A similar Brownian motion framework is often used to model agency problems in fields such as CEO compensation and corporate finance (DeMarzo and Sannikov (2006); He (2009); He (2011)). Recently, a few scholars have studied the dynamic moral hazard problem using a Poisson process, where the agent exerts unobservable effort that controls the arrival rate. In Biais, Mariotti, Rochet, and Villeneuve (2010) and Myerson (2015), bad events happen with higher Poisson arrival rate when agents do not put enough effort to prevent such events. In Sun and Tian (2017), the principal needs to provide the incentive for the agent to exert effort to raise the arrival rate of a Poisson process. Most of these studies have assumed that the agent is risk neutral. The risk-neutrality assumption implies that the agent does not receive any payment until the continuation utility reaches a payment threshold (Biais, Mariotti, Rochet, and Villeneuve (2010); Myerson (2015)), or only receives bonuses upon arrivals (Sun and Tian (2017)). Shan (2017) studied a similar contracting problem in which the principal faces multiple risk-averse agents. With a risk-averse agent, besides providing the incentive to work, the optimal contract also needs to account for consumption smoothing. Therefore, the agent's payment is contingent on the entire history and varies over time. In Shan (2017), the optimal contract is written in terms of the agent's continuation utility, which is an abstract term. Also, the agent's consumption is controlled by the principal, which is not realistic. Based on the theoretical model of Shan (2017), the current paper provides an implementation of the optimal contract in which the agent makes both effort and consumption decisions. The implementation uses the standard instruments that are available in practice and provides a justification for using performance-vested equity-based compensation.

The implementation of the optimal contract overcomes the problem pointed out by Rogerson (1985) which is that, if the agent is allowed access to credit, he will adopt a joint deviation of shirking and saving some of his wages, because of a wedge between the agent's Euler equation and the inverse Euler equation implied by the principal's problem. In our implementation, however, the return on savings is state contingent. When the state-contingent rates of return are chosen appropriately, the agent's Euler equation mimics the inverse Euler equation; put differently, the wedge between the Euler equation and the inverse Euler equation disappears. A similar problem arises in the dynamic optimal taxation problem studied by Kocherlakota (2005), in which the agents in the economy are privately informed about their skills. In Kocherlakota (2005), to prevent joint deviations, the return on savings is made to be stochastic by tailoring the tax rates on saving to the agent's announcements of his private information, and hence the government needs to keep track of the entire history of the agent's announcements to set the tax rates. In our model, the problem is much more tractable, especially for the logarithmic utility case in which the principal only needs to know the current stage level of the project because the holding requirement only varies with stage level.

3 The benchmark model

The benchmark model is similar to the single-agent model studied in Shan (2017), in which the principal has full control over the agent's consumption. In this model, time is continuous. At time

0, a principal hires an agent to perform an R&D project. This project has N stages, which must be completed sequentially, i.e., to develop the stage n ($0 < n \leq N$) innovation, the agent must have completed the innovations of stage $n-1$. The transition from one stage to the next is modeled by a Poisson process, which is affected by the agent's choice of effort. For simplicity, the agent is assumed to have only two effort choices: he can either put in effort or shirk. If the agent puts in effort, the arrival rate of completing an innovation is λ. If the agent chooses to shirk, he fails with probability 1, and the Poisson arrival rate is equal to zero.

The agent's action cannot be monitored by the principal. However, the principal can observe exactly when each stage of the R&D project is completed. Let H^t be the history of the agent's performance up to time t. It records the number of stages completed and the time taken by the agent to complete each stage. By assumption, H^t is publicly observable, which is the only information that the principal can use to provide incentives to the agent.

At time 0, the principal offers the agent a contract that specifies a flow of consumption $c_t(H^t)$ based on the principal's observation of the agent's performance. Let T denote the stochastic stopping time when the agent completes the last-stage innovation. After time T, the principal does not need to provide any incentive for the agent to work, and hence the agent receives a constant payment over time, which is equivalent to a lump-sum consumption transfer at time T.

We assume that the agent's utility function has a separable form $U(c) - L(a)$, where $U(c)$ is the utility from consumption, and $L(a)$ is the disutility of exerting effort. We assume that $U : [0, +\infty) \to [0, +\infty)$ is an increasing, concave, and C^2 function, and satisfies the Inada condition $\lim_{c \to +\infty} U'(c) = 0$. The agent's choice of effort is binary, indicated by $a \in \{0, 1\}$. $a = 1$ means that the agent chooses to put in effort, and $a = 0$ means that the agent chooses to shirk. Moreover, the disutility of putting in effort equals some $l > 0$, and the disutility of shirking equals zero, i.e., $L(1) = l$ and $L(0) = 0$.

Given the contract, at any time t, the agent makes the effort choice based on the observation of H^t. The effort process is denoted as $a = \{a_t(H^t), 0 \leq t < \infty\}$. The agent's objective is to choose the effort process a to maximize the total expected utility. Thus, the agent's problem is

$$\max_{\{a_t, 0 \leq t < +\infty\}} E\left[\int_0^T re^{-rt}(U(c_t) - L(a_t))dt + e^{-rT}U(c_T) \right],$$

where r is the discount rate.[4] Moreover, the agent has a reservation-utility v_0. If the maximum

[4]We normalize the flow term by multiplying it by the discount rate so that the total discounted utility equals the

expected utility he can get from the contract is less than v_0, then the agent will reject the principal's offer.

We assume that the agent and the principal have the same discount rate. Hence, the principal's expected cost is given by

$$E\left[\int_0^T re^{-rt}c_t dt + e^{-rT}c_T\right].$$

We assume that the completion of R&D is quite valuable to the principal; therefore, he always wants to induce the agent to work.[5] Hence, the principal's objective is to minimize the expected cost by choosing an incentive-compatible payment scheme subject to delivering the agent the requisite initial value of expected utility v_0. Therefore, the principal's problem is

$$\min_{\{c_t, 0\leq t<+\infty\}} E\left[\int_0^T re^{-rt}c_t dt + e^{-rT}c_T\right]$$

s.t.

$$E\left[\int_0^T re^{-rt}(U(c_t)-l)dt + e^{-rT}U(c_T)\right] \geq v_0.$$

Finally, to simplify the analysis, we could recast the problem as one where the principal directly transfers utility to the agent instead of consumption. In the transformed problem, the principal chooses a stream of utility transfers $u_t(H^t)$ $(0 \leq t < +\infty)$ to minimize the expected cost of implementing positive effort. Then, the principal's problem becomes

$$\min_{\{u_t, 0\leq t<+\infty\}} E\left[\int_0^T re^{-rt}S(u_t)dt + e^{-rT}S(u_T)\right]$$

s.t.

$$E\left[\int_0^T re^{-rt}(u_t-l)dt + e^{-rT}u_T\right] \geq v_0,$$

where $S(u) = U^{-1}(u)$, which is the principal's cost of providing the agent with utility u. It can be shown that $S(u)$ is an increasing and strictly convex function and $\lim_{u\to+\infty} S'(u) = +\infty$.

The contracting problem can be analyzed recursively using the agent's continuation utility v, which is the total utility that the principal expects the agent to derive at a given point in time. At any moment in time, given the continuation utility, the contract specifies the agent's utility flow u,

utility flow when the flow is constant over time. Thus, the agent's total discounted utility at time T equals $U(c^T)$.

[5] By assumption, the project has finite number of stages. Moreover, the arrival rate of success when the agent exerts effort is fixed. Hence, if the revenue of completing the project is sufficient high, it is always optimal to induce the agent to work.

the continuation utility $\bar{v}$ if he completes an innovation, and the law of motion of the continuation utility if he fails.

The details of the derivation of the recursive form can be found in Shan (2017). Intuitively, when the agent exerts effort, he raises the arrival rate of success from 0 to λ. After a success, his continuation utility changes from v to $\bar{v}$. Hence his expected benefits of exerting effort is $\lambda(\bar{v}-v)$. His costs of exerting effort is rl. To provide incentive to work, the contract should satisfy the following incentive-compatibility condition:

$$\lambda(\bar{v}-v) \geq rl.$$

Thus, in any incentive compatible contract, the agent's continuation utility increases by at least $\frac{rl}{\lambda}$ after each success. For the evolution the agent's continuation utility in case of failure, since the continuation utility can be explained as the value that the principal owes the agent, when the agent exerts effort, his continuation utility grows at the discount rate r and falls because of the net flow of utility $r(u-l)$ plus the gain of utility $\bar{v}-v$ at rate λ if the agent completes an innovation. Thus, his continuation utility in case of failure evolves according to

$$\frac{dv}{dt} = rv - r(u-l) - \lambda(\bar{v}-v).$$

Let $C_n(v)$ be the principal's minimum cost of delivering continuation utility v when the project is at stage n. Next, we characterize the evolution of the principal's continuation value $C_n(v)$. Since the principal discounts the future at rate r, his expected flow of value at a given point in time is given by

$$rC_n(v).$$

This must equals to sum of the expected instantaneous cash flows $rS(u)$ and the expected rate of change in the continuation value. The later equals to the sum of the variation of the principal's costs brought by the change in the agent's continuation utility and the variation of costs when the project progresses to the next stage at rate λ. This yields

$$rS(u) + C_n'(v)\frac{dv}{dt} + \lambda[C_{n+1}(\bar{v}) - C_n(v)].$$

The principal controls u and $\bar{v}$ to minimize his continuation value. In the recursive form, the principal's problem is to solve the following Hamilton-Jacobi-Bellman (HJB) equation

$$rC_n(v) = \min_{u,\bar{v}} rS(u) + C_n'(v)\frac{dv}{dt} + \lambda[C_{n+1}(\bar{v}) - C_n(v)]$$

s.t.

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u-l) - \lambda(\bar{v}-v), \\ \lambda(\bar{v}-v) &\geq rl. \end{aligned}$$

As the agent is assumed to have limited liability, the continuation utility cannot be less than 0, because the agent can guarantee a utility level of 0 by not putting in any effort. Therefore, a negative continuation utility is not viable.

In the HJB equation, to solve cost function C_n, we need to know the functional form of C_{n+1}. Note that after the agent completes the final stage, he receives a lump-sum transfer, which implies that $C_{N+1}(v) = S(v)$. Then the whole problem can be solved by backward induction starting from the last stage-N problem. Shan (2017) uses a diagrammatic analysis to characterize the solution of the HJB equation. The main properties of the optimal contract are summarized in Proposition 3.1.

Proposition 3.1 *The optimal contract has the following property:*

(i) The principal's expected cost at any point is given by an increasing, convex and differentiable function $C_n(v)$, which satisfies

$$rC_n(v) = rS(u^*(v)) + C_n'(v)[r(v - u^*(v))] + \lambda[C_{n+1}(\bar{v}) - C_n(v)],$$

and the boundary conditions: $C_n'(0) = S'(0)$ and $C_n(0) = \frac{\lambda C_{n+1}(\frac{rl}{\lambda})}{r+\lambda}$. The cost function when the agent completes the last stage innovation is given by $C_{N+1}(v) = S(v)$.

(ii) The instantaneous payment $u^*(v)$ satisfies $S'(u^*(v)) = C_n'(v)$.

(iii) When the agent completes an innovation, he enters the next stage and starts with the continuation utility $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda}$.

(iv) In case of failure, the continuation utility v smoothly decreases over time and stays at 0 when it reaches the lower bound 0.

(v) The minimum-cost functions satisfy $C_n(v) > C_{n+1}(v)$ and $C_n(v) < C_{n+1}(v + \frac{rl}{\lambda})$ for all $v \geq 0$. Its derivative satisfies $\lim_{v\to+\infty} C_n'(v) = +\infty$.

Proposition 3.1 indicates that the optimal contract combines rewards and punishments. The principal rewards the agent by an upward adjustment in the compensation after each success and punishes the agent by cutting his compensation for unsatisfactory performance. Thus, the principal induces the risk-averse agent to bear some risks by introducing some uncertainties into his compensation. Otherwise, the agent lacks an incentive to work. Proposition 3.1 also shows that the costs of delivering the same level of continuation utility is higher at an earlier stage of the project (Figure 1). This is because, at an earlier stage, the uncertainties about the future are higher. Hence, the cost of delivering the same level of continuation utility to a risk-averse agent is higher.

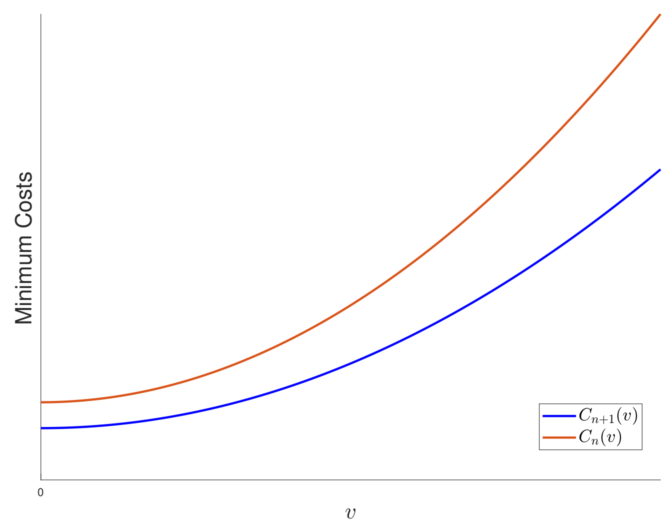

Figure 1: Cost functions.

4 Implementation of the optimal contract

The optimal contract presented in the benchmark model relies entirely on the continuation utility, which is an abstract concept. Moreover, in the benchmark model, we make a strong assumption that the principal controls the agent's consumption directly, i.e., the agent consumes all the payments from the principal at any point in time. In this section, we present an implementation of the optimal contract, which uses monetary terms rather than the abstract continuation utility, and in which the agent also makes consumption decisions besides choosing the effort. Yet, we show that the implementation generates the same consumption path as the original optimal contract. In this implementation, a primary component of the agent's compensation is a state-contingent security that appreciates when the project succeeds and depreciates when it fails. The agent is required to meet a sequence of minimum holding requirements that is relaxed after each success until the whole project is completed. Capturing the main features of performance-vested equity-based compensation, the implementation results show that the principal can use this compensation scheme to mimic the theoretical optimal contract derived with the assumption that the principal has full control over the agent's consumption, thereby providing a rationale for using performance-vested equity-based compensation from a theoretical point of view.

The setup is the same as the benchmark model except that the consumption is decided by the agent rather than controlled by the principal. To implement the optimal contract, the principal designs a state-contingent security, whose return is higher in case of success than in case of failure. Before the project starts, the principal provides the agent with initial wealth y^0, a part of which is paid in terms of the security. The agent can also invest in this security for saving purpose.[6] At any point in time before the whole project is completed, the agent decides whether to exert effort or shirk, how much to consume, and how much to invest in this security subject to a minimum holding requirement $\underline{y}_n$ which depends on the stage level n. The principal's objective is to design the security and the minimum holding requirements properly so that the agent will always exert effort and, more importantly, choose the same consumption path as the one in the optimal contract derived in Section 3 that minimizes the principal's costs.

To describe the design of state-contingent security, we first explain how the value of the security changes over time across different states in a more intuitive discrete-time approximation of the continuous-time setting. In the discrete-time approximation, each period lasts Δt. At the beginning of each period, the outcome of the project and the value of the security that the agent takes from the last period are realized. Then, the agent makes effort, consumption, and investment decisions based on his observation of the outcomes (Figure 2). If the agent exerts effort, the project succeeds with probability approximately $\lambda\Delta t$ and fails with probability $1 - \lambda\Delta t$, and the result will be realized at the beginning of the next period. If the agent shirks, the project fails with probability

[6] We first assume that investing in this security is the only saving technology of the agent. A case with hidden saving is studied in the appendix.

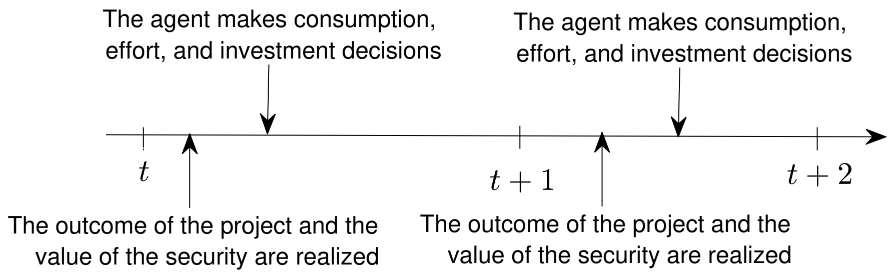

Figure 2: The timeline

1. For the valuation of the security, suppose the project is at stage n in period t and that the agent holds in period t the amount of security that would be worth y_{t+1} in period $t+1$ if the project fails. Denote by $Y_{n+1}(y_{t+1})$ the value of such a security in state of success. The value of this amount of securities in the current period t is determined by the fair-price rule assuming that the agent exerts effort, i.e., the value equals the expected present value, which is given by $P_n(y_{t+1}) = \frac{1}{1+r\Delta t}[(1-\lambda\Delta t)y_{t+1} + \lambda\Delta t Y_{n+1}(y_{t+1})]$ (Figure 3). For easier tracking of the agent's

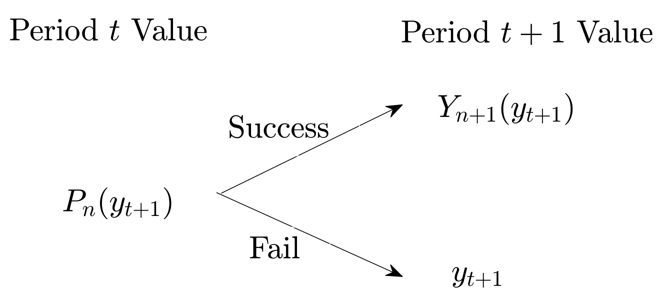

Figure 3: The design of the state-contingent security

wealth level, we write the value of the security in the current period and in the next period in case of success as functions of its value in the next period in case of failure. Given this design of state-contingent security, if the agent allocates $P_n(y_{t+1})$ of his current wealth to the security, then in next period his wealth level equals y_{t+1} in case of failure and $Y_{n+1}(y_{t+1})$ in case of success. Letting y_t denote the agent's wealth in period t, his budget constraint in period t is

$$rc_t\Delta t + P_n(y_{t+1}) = y_t,$$

where the first term on the left-hand side is his consumption in the current period, and the second term is his investment in the security if he wants a guaranteed wealth level of y_{t+1} in case of failure in the next period.[7]

To derive the evolution of the agent's wealth in continuous time, we first substitute the expression of $P_n(y_{t+1})$ into the agent's budget constraint

$$rc_t\Delta t + \frac{1}{1+r\Delta t}[(1-\lambda\Delta t)y_{t+1} + \lambda\Delta t Y_{n+1}(y_{t+1})] = y_t.$$

Multiplying both sides by $1 + r\Delta t$ and rearranging the equation, we can get

$$y_{t+1} - y_t = r\Delta t y_t - (1 + r\Delta t)rc_t\Delta t - \lambda\Delta t[Y_{n+1}(y_{t+1}) - y_{t+1}].$$

Dividing both sides by Δt and letting Δt converge to 0, we can obtain the evolution of the agent's wealth in case of failure

$$\frac{dy}{dt} = ry - rc - \lambda[Y_{n+1}(y) - y].$$

Thus, when the project is at stage n, the agent's wealth in case of failure grows at rate r and decreases because of consumption spending c and the loss on investment in security $\lambda(Y_{n+1}(y) - y)$. If the agent succeeds, his wealth raises to $Y_{n+1}(y)$.

Now, the agent's problem is to choose an effort process and a consumption plan to maximize his discounted expected utility. In the recursive form of the agent's problem, the state variable becomes his wealth level y. Let $V_n(y)$ be the maximum expected utility that the agent can get given wealth level y when the project is at stage n. The HJB equation of the agent's problem is

$$rV_n(y) = \max_{a,c} r[U(c) - al] + V_n'(y)\frac{dy}{dt} + a\lambda[V_{n+1}(Y_{n+1}(y)) - V_n(y)]$$

s.t.

$$\begin{aligned} \frac{dy}{dt} &= ry - rc - \lambda[Y_{n+1}(y) - y], \\ y &\geq \underline{y}_n. \end{aligned}$$

Different from the benchmark model, the agent now chooses both the action and consumption. If the agent decides to work ($a = 1$), he incurs the costs of exerting effort in exchange for a higher

[7]The functional form of $Y_{n+1}(y)$ and $P_n(y)$ depend on the agent's utility function, and therefore they are nonlinear for general risk-averse utility functions. In the next section, we will provide an example for a special case where both $Y_{n+1}(y)$ and $P_n(y)$ take a simple linear form.

return on his securities when the project progresses to the next stage. If the agent shirks ($a = 0$), although he does not suffer any costs of working, he loses the chance to receive the higher return from his securities. In continuous time, the value of the agent's investment in the security converges to his wealth level y at any point in time. Since the agent is required to meet a minimum holding requirement that he invests at least $\underline{y}_n$ of his wealth in the security, it imposes a lower bound of the state variable y at $\underline{y}_n$ when the project is at stage n.

The next proposition shows that if the principal sets the initial wealth, the payoff in case of success, and the minimum holding requirement appropriately, this implementation is able to generate the same consumption path and effort choice as the original optimal contract. The proof is in the appendix.

Proposition 4.1 *Suppose the principal provides the agent with initial wealth* y^0

$$y^0 = C_1(v^0),$$

and at stage n

$$Y_{n+1}(y) = \begin{cases} C_{n+1}\left(C_n^{-1}(y) + \frac{rl}{\lambda}\right) & \text{if } y \geq C_n(0), \\ \frac{C_{n+1}(\frac{rl}{\lambda})}{C_n(0)} y & \text{if } 0 \leq y < C_n(0), \end{cases}$$

$$\underline{y}_n = C_n(0).$$

Then, given income y*, the highest discounted expected utility the agent can achieve is*

$$V_n(y) = C_n^{-1}(y),$$

and he chooses the same consumption process as the one in the optimal contract and always exerts effort until he completes the last-stage innovation. The minimum holding requirement satisfies $\underline{y}_n > \underline{y}_{n+1}$.

For the payoff in case of success $Y_{n+1}(y)$, note that $C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})$ is well defined for $y \geq C_n(0)$. When $y \geq C_n(0)$, we have $Y_{n+1}(y)$ is increasing in y, and $Y_{n+1}(y) = C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda}) > C_n(C_n^{-1}(y)) = y$, which means the payoff of the security in case of success is higher than its payoff in case of failure. For $y < C_n(0)$, intuitively, the payoff $Y_{n+1}(y)$ should satisfy the following conditions. Firstly, the payoff should be higher when the agent holds more security, which means $Y_{n+1}(y)$ is strictly increasing in y. Secondly, the payoff in case of success should be higher than the payoff in case of failure, which requires that $Y_{n+1}(y) > y$. Finally, the payoff should be zero when the agent does not hold any security, and hence $Y_{n+1}(y) = 0$ when $y = 0$. In this "off equilibrium region", we could choose any function that satisfies these conditions. In Proposition 4.1, we choose the simplest linear function that connects the origin and $(C_n(0), C_{n+1}(\frac{rl}{\lambda}))$.

The premise of this implementation lies in the fact that the agent's utility maximization problem is the dual problem of the principal's cost minimization problem in Section 3. Given continuation utility v, $C_n(v)$ is the minimum expected cost to finance the incentive-compatible compensation scheme. From the dual perspective, given the expected wealth $y = C_n(v)$, the maximum expected utility that the agent can achieve should equal v. Further, the consumption allocation should be the same. In this implementation, the agent invests in the risky security for saving purpose, and hence the return on savings is state contingent. When the state-dependent rates of return are chosen appropriately, the agent's Euler equation mimics the inverse Euler equation implied by the principal's problem. In other words, the wedge between the Euler equation and the inverse Euler equation, as stated in Rogerson (1985), disappears.

In this implementation, the state-contingent security plays a key role in incentives. As discussed in Section 3, the principal has to let the agent bear some risks; otherwise, the agent will shirk his work. In the implementation, the risks are embedded in the state-contingent security. The gap between the value in case of success and in case of failure guarantees that the agent will exert effort. The minimum holding requirement arises because, by assumption, the agent has limited liability and hence can guarantee a utility level of 0. It is binding when the highest expected utility the agent can achieve reaches the lower bound 0. At this point, the principal has to make sure that the agent holds enough securities so that the payoff of these securities in case of success is sufficient to deliver the agent with continuation utility $\frac{rl}{\lambda}$, which is the lowest level in case of success for the agent to exert effort. Otherwise, the agent will not have any incentive to work. Hence, the minimum holding requirement ensures the lowest level of risk that can incentivize the agent to exert effort. Proposition 4.1 shows that the minimum holding requirement is relaxed after each innovation. This is because when the project progresses to the next stage, the uncertainty of the project reduces, and the minimum level of risks to be borne by the agent for incentive purposes also becomes less.

The design of security depends on the agent's attitude towards risk, which is determined by his utility function from consumption. In the next section, we explicitly show how to use the equity of the firm to create the security when the agent has logarithmic utility. For general cases, in the financial market, there is no asset that has the exact same payoff structure as the state-contingent security used in this implementation. However, firms can still design equity-based compensation to approximate the security that implements the optimal contract. Since these firms rely heavily on R&D, the performance of the employees in the R&D units greatly influences the firms' performance outcomes, which closely links employees' performance and the return on firms' equities. In particular, each breakthrough in R&D is followed by a notable increase in the firm's equity price. The absence of such developments in a firm over a period generally leads to a drop in its equity price. Thus, among all available assets, the firm's equity has the closest payoff-pattern to the state-contingent security. Another feature of this implementation is the sequence of decreasing minimum holding requirements that the agent has to meet until the completion of the project. In practice, this feature is mimicked by using performance-vesting provisions, under which a part of equity grants is vested when the research employee achieves a predetermined performance target.

5 The optimality of equity-based compensation under logarithmic utility

In this section, we consider an example in which the agent has the logarithmic utility function. In this case, the contracting problem has a closed-form solution, which allows us to create the security that implements the optimal contract in Section 4 using the equity of a firm under some assumptions about how the development of the project affects the firm's value.

5.1 The optimal contract and equity-based implementation

If the agent's utility from consumption is $U(c) = \ln c$, we can show that the principal's minimum cost function takes a simple form—a constant times e^v, where the constant only depends on the parameters of the model and the stage level of the project. The following proposition summarizes the property of the optimal contract.

Proposition 5.1 *When the agent's utility from consumption is $U(c) = \ln c$, the minimum cost of delivering continuation utility v when the project is at stage n is given by $C_n(v) = p_n e^v$, where the constant p_n is determined recursively by*

$$p_{N+1} = 1,$$

$$rp_n \ln p_n + \lambda p_n = \lambda p_{n+1} e^{\frac{rl}{\lambda}},$$

and satisfies $p_n > p_{n+1}$. When the agent completes an innovation, he enters the next stage and starts with continuation utility $\bar{v}$ which satisfies $\bar{v} = v + \frac{rl}{\lambda}$. In case of failure, the continuation utility v evolves according to $\frac{dv}{dt} = -r \ln p_n < 0$.

The optimal contract for the logarithmic utility example is consistent with the results of Proposition 3.1: (1) the continuation utility decreases over time in case of failure and increases by $\frac{rl}{\lambda}$ after each success; (2) $p_n > p_{n+1}$ implies that the cost of delivering the same level of continuation utility is higher when the project is at an earlier stage.

The closed-form solution allows us to derive some comparative statics results regarding how the principal's cost is affected by the agent's cost of exerting effort and the difficulty of the R&D project, which are captured by l and λ respectively. The principal's cost will be higher when the agent incurs a higher cost of exerting effort because the principal needs to compensate the agent more to cover his cost of effort. How the difficulty of the R&D project affects the principal's cost is unclear. When the arrival rate of success λ is very small, which means the R&D project is very challenging, the principal needs to provide a stronger incentive for the agent to exert effort. On the one hand, the agent will receive a higher reward in case of success which increases the principal's cost, but on the other hand, the principal will punish the agent harder when he fails by lowering his continuation utility quicker, which leads to a lower and more rapidly decreasing consumption path in case of failure. In other words, the principal provides a stronger incentive for the agent by making his consumption path more volatile. The following corollary shows that the net effect is to increase the principal's cost.

Corollary 5.2 *The principal's cost of delivering continuation utility v at any stage n is higher when the agent has a higher cost of exerting effort, or a lower chance of success, i.e.,*

$$\frac{\partial p_n}{\partial l} > 0 \text{ and } \frac{\partial p_n}{\partial \lambda} < 0.$$

To implement the optimal contract, we consider a security with the same payoff structure described in Section 4. As shown in Section 4, when the agent invests in this security for saving purpose, at stage n, the agent's wealth in case of failure grows at rate r and decreases because of consumption spending c and the loss on investment in security $\lambda(Y_{n+1}(y) - y)$. Hence, the evolution of the agent's wealth in case of failure satisfies

$$\frac{dy}{dt} = ry - rc - \lambda[Y_{n+1}(y) - y].$$

If the agent succeeds, his wealth raises to $Y_{n+1}(y)$. To replicate the consumption path of the optimal contract, based on the results of Proposition 4.1, the value of the risky security should increase from y to

$$Y_{n+1}(y) = C_{n+1}\left(C_n^{-1}(y) + \frac{rl}{\lambda}\right) = \frac{p_{n+1}}{p_n} e^{\frac{rl}{\lambda}} y = \left(\frac{r \ln p_n}{\lambda} + 1\right) y$$

in case of success, which is a linear function of y.[8] The following proposition confirms that the implementation indeed replicates the optimal contract.

Proposition 5.3 *Suppose the principal provides the agent with initial wealth* y^0

$$y^0 = C_1(v^0) = p_1 e^{v_0},$$

and at stage n

$$Y_{n+1}(y) = \left(\frac{r \ln p_n}{\lambda} + 1\right) y.$$

Then, given income y*, the highest discounted expected utility the agent can achieve is*

$$V_n(y) = \ln y - \ln p_n,$$

and he chooses the same consumption process as the one in the optimal contract and always exerts effort until he completes the last-stage innovation.

From Proposition 5.3, the risky security that implements the optimal contract has a simple structure: (1) the value of the security increases proportionally by $(\frac{r \ln p_n}{\lambda} + 1)$ times when the project progresses from stage n to stage $n+1$; (2) by fair-pricing rule, in case of failure, the value of the security evolves according to

$$\frac{dy}{dt} = ry - \lambda(\bar{y} - y) = ry - \lambda\left[\left(\frac{r \ln p_n}{\lambda} + 1\right) y - y\right] = r(1 - \ln p_n) y.$$

Note that, in case of failure, the value of the risky security also changes proportionally, and its return equals $r(1 - \ln p_n)$. Since the logarithmic utility function is unbounded from below, the minimum holding requirement in Proposition 4.1 no-longer exists. In the logarithmic utility example, to

[8]The last equality is due to $rp_n \ln p_n + \lambda p_n = \lambda p_{n+1} e^{\frac{rl}{\lambda}}$ from Proposition 5.1.

provide incentive, the value of the security increases proportionally by $\frac{r \ln p_n}{\lambda} + 1$ times in case of success and depreciates at rate $r \ln p_n$ in case of failure. Since $p_n > p_{n+1}$, Proposition 5.3 implies the security is more volatile at an earlier stage. The intuition behind this result is again that the principal needs the agent to bear some risks in order to provide an incentive to work. When the project progresses to the next stage, the uncertainty of the project reduces, and hence the risk that the agent needs to take for incentive purpose also reduces.

Proposition 5.3 shows that the value of the risky security that implements the optimal contract increases proportionally after each success. If the firm's value changes in a similar pattern, then we might be able to create the security using the firm's equity. It requires that: 1) the firm's value is only affected by the performance of the project; 2) the firm's value increases proportionally when the project progresses to the next stage. This pattern is consistent with the development of R&D-intensive start-up firms because the development of these firms usually starts with one main research project and the firms' value depends crucially on the performance of the project. Regarding how the performance of the project changes the firm's value, although it is difficult to find data about how a specific project affects the value of the firm, most R&D-intensive start-up firms are backed by venture capital, and the valuation of the firms at each financing round is publicly available. Since whether a start-up firm can receive further rounds of financing depends on the development of its main project, an alternative approach to examine how the development of its main project affects the firm's value is to look at the change of the valuation of the firm at each financing round. From 2007 to 2011, Twitter received seven rounds of financing, and its valuation increased roughly three times at each financing round. A similar proportionate growth pattern is documented in *Venture Pulse Report* published quarterly by KPMG.

If the firm's value increases proportionally by R_n times when its project progresses from stage n to stage $n+1$. We first assume that R_n is certain and discuss how the uncertainty of R_n affects the results in the next subsection. Let k be the value of the firm. Under this assumption, the firm's value in case of failure evolves according to

$$\frac{dk}{dt} = rk - \lambda(R_n k - k) = [r - \lambda(R_n - 1)]k,$$

which implies that the return of the firm's equity in case of failure equals $[r - \lambda(R_n - 1)]$. Consider the following portfolio of the firm's equity and a risk-free asset with interest rate r, in which the fraction of equity in the portfolio α_n satisfies

$$\alpha_n \cdot [r - \lambda(R_n - 1)] + (1 - \alpha_n) \cdot r = r(1 - \ln p_n),$$

or equivalently

$$\alpha_n = \frac{r \ln p_n}{\lambda(R_n - 1)}.$$

By construction, the return of the portfolio in case of failure equals to $r(1 - \ln p_n)$, and the value of the portfolio increases by $\alpha_n \cdot R_n + (1 - \alpha_n) \cdot 1 = \frac{r \ln p_n}{\lambda} + 1$ times when the project progresses from stage n to stage $n + 1$.[9] Thus, the payoff of the portfolio matches exactly with the payoff of the risky security that implements the optimal contract. The construction of the portfolio also requires that $R_n \geq \frac{r \ln p_n}{\lambda} + 1$ so that $\alpha_n \in [0, 1]$ for any n. It means that the growth rate of the firm's value in case of success is higher than the required return of the security in case of success, which is also a sufficient condition on the return of the project so that it is always optimal to implement the positive effort. The following proposition summarizes these results.

Proposition 5.4 *Suppose the value of the firm increases by R_n times when the project progresses from stage n to stage $n + 1$. If $R_n \geq \frac{r \ln p_n}{\lambda} + 1$ for all n, then the risky security that implements the optimal contract can be created by a portfolio of the firm's equity and a risk-free asset with interest rate r. The fraction of equity, α_n, satisfies*

$$\alpha_n = \frac{r \ln p_n}{\lambda(R_n - 1)}.$$

Proposition 5.4 shows that the fraction of equity in the portfolio depends only on the stage level of the project and the parameters of the model. To implement the optimal contract, the principal can offer the agent a wealth level of $p_1 e^{v_0}$ before the project starts, let him have access to the portfolio for investment for future consumption, and adjust the fraction of equity in the portfolio according to the stage level of the project. The agent makes all the remaining decisions, including consumption, investment, and effort choices. Proposition 5.3 shows that the agent will choose the same consumption path as the one in the optimal contract and always exerts effort. This result again shows that the composition of the equity-based incentive compensation should depend on the agent's performance. From the determination of the fraction of equity in Proposition 5.4, if the

[9] Since the value of the risk-free asset dons not depend on the outcome of the project, the fraction of the risk-free asset $1 - \alpha_n$ is multiplied by 1.

firm's value increases by the same proportion after each success, then the fraction of equity in the compensation portfolio decreases when the project progresses to a higher stage and a fraction of equity vests after each success.

A direct implication of Corollary 5.2 is how the share of equity in the optimal portfolio changes with the cost of exerting effort, the difficulty of innovation, as well as the impact of innovation on the firm's value R_n.

Corollary 5.5 *The share of equity in the optimal portfolio is higher when the agent has a higher cost of exerting effort, a lower chance of success, or a lower growth rate of firm value in case of success, i.e.,*

$$\frac{\partial \alpha_n}{\partial l} > 0, \frac{\partial \alpha_n}{\partial \lambda} < 0, \text{ and } \frac{\partial \alpha_n}{\partial R_n} < 0.$$

The intuition of these comparative statics results is straightforward. When the agent has a higher cost of exerting effort or a lower chance of success, the principal needs to provide stronger incentive for the agent to work, and hence the optimal compensation portfolio should include more of the firm's equity. Holding everything else constant, when the firm's value is very sensitive to the development of the R&D project, a small share of equity is enough to ensure incentive. Some testable implications of these results are that research employees receive more equity-based compensation when each breakthrough of the project takes longer time on average, or when the variation of equity price is smaller when any news of the development of the project is revealed.

5.2 The limitation of equity-based compensation

In this subsection, we briefly discuss the limitations of the equity implementation results. In the previous subsection, we assumed that the firm's value grow by a certain proportion when the project progresses from one stage to the next. In reality, however, the firm's value faces two important types of uncertainty. Firstly, the firm faces the aggregate uncertainty of the market which will also affect its equity price. Secondly, the valuation of the firm's R&D project may also be uncertain, i.e., the growing proportion of the firm's value in case of success, R_n, is uncertain. For these situations, using equity-based incentive compensation lets the agent face the uncertainties that he can not control, and as a result, the principal needs to compensate the agent more for bearing the unnecessary risks. It implies that the second-best consumption allocation of the optimal contract cannot be achieved by using equity-based incentive compensation when these two types of uncertainty exist. When the firm is cash constrained and chooses to use equity-based incentive compensation for its research employee, to increase efficiency, the firm should try to reduce the unnecessary risks faced by its employees as much as possible. For aggregate uncertainty, one approach is to add a short position of a market portfolio (for example a short position of index future) in the compensation portfolio to hedge the aggregate risk so that the research employees are effectively paid according to the firm's performance relative to a benchmark. The approach of using the relative-performance evaluation scheme to remove the market aggregate risk inherent in equity-based incentive compensation has been extensively discussed in executive compensation literature since the theoretical prediction in Holmstrom (1982), which suggests that "the market component of a firm's returns should be removed from the compensation package since executives cannot affect the overall market by their actions and it is costly for executives to bear the related risks." For the uncertainty of valuation of the firm's research project, however, it is difficult to find a financial tool to hedge the risk. Next, we use a two-period model to explain how it affects the efficiency of equity-based compensation.

Now, suppose a research project lasts for two periods. In period 0, the agent decides whether to exert effort or shirk. Conditional on exerting effort, the agent succeeds with probability μ in period 1. If he chooses to shirk, he fails with probability 1. The agent consumes in both period, and the utility function from consumption equals $U(c) = \ln c$. The agent's initial requisite utility equals v_0. Let l be the disutility of exerting effort and β be the discount rate. As to how the performance of the project affects the firm's value, in general, the firm's value equals to its real value plus a random noise. Specifically, the firm's period-1 value equals $k + \sigma$ in case of failure, and its value equals to $\tilde{R}k + \sigma$ in case of success, where the random variable $\tilde{R}$ captures the uncertainty of the valuation of the project and random variable σ captures the aggregate uncertainty. Since we want to focus on the effect of the uncertainty of the valuation of the project, we consider an extreme case in which the firm can hedge the aggregate uncertainty perfectly, and the only uncertainty comes from the firm's value when the project succeeds. In this case, for a certain amount of the firm's equity, its period-1 value in case of success equals $\tilde{R}$ times its value in case of failure. We also assume that the return of the project is sufficient high, which satisfies $E(\ln \tilde{R}) > \frac{l}{\beta\mu}$, so that it is optimal to induce positive effort. We have the following result.

Proposition 5.6 *If $\tilde{R}$ is certain and equals $\bar{R}$, then the second-best outcome (the optimal contract) can be implemented by a portfolio of the firm's equity and a risk-free asset. If $\tilde{R}$ is a random variable with mean $\bar{R}$, then the firm still can use a portfolio of the firm's equity and a risk-free asset to* *induce incentive, but incurs higher costs of delivering the same level of initial requisite utility v_0 to the agent than the second-best outcome. The efficiency loss is lower when the distribution of $\tilde{R}$ is more concentrated around its mean $\bar{R}$.*

The implication of Proposition 5.6 is that if the firm can make an more accurate prediction of the valuation of its project, then it can achieve very close to the optimal contract by using equity-based compensation.

6 An implementation via performance-based bonuses

Equity-based compensation is attractive to cash-constrained start-up firms because it can incentivize their research employees without spending their precious cash. However, it also has some limitations as discussed in the previous section. In firms with enough cash flow, performance-based bonus is another commonly used compensation scheme to provide incentive for research employees. In this section, we provide an alternative implementation of the optimal contract via performance-based bonuses for situations in which equity-based compensation becomes less efficient.

We keep the assumption that the agent has logarithmic utility function. Before the project starts, the principal offers a savings account to the agent with an initial balance of y^0. When the project is at stage n, the principal sets the return on this account at r_n in case of failure. In case of success, the principal rewards the agent with a performance bonus and deposits it into the savings account to increase its balance from y to $Y_{n+1}(y)$. Note that the size of the bonus depends on the balance of the savings account and the progress of the project. At any point in time, the agent can withdraw money from the savings account for consumption subject to the constraint that the balance of the savings account is nonnegative. Then, the balance of the savings account in case of failure evolves according to

$$\frac{dy}{dt} = r_n y - rc.$$

In case of success, the balance of the account increases from y to $Y_{n+1}(y)$. Similar to Proposition 5.3, we can show that if the principal chooses y^0, r_n, and $Y_{n+1}(y)$ appropriately, the agent will always exert effort and choose the exact same consumption allocation as the optimal contract.

Proposition 6.1 *Suppose the principal provides the agent with initial balance*

$$y^0 = p_1 e^{v_0},$$

and sets

$$r_n = r(1 - \ln p_n),$$
$$Y_{n+1}(y) = \left(\frac{r \ln p_n}{\lambda} + 1\right)y.$$

Then, the agent chooses the same consumption process as the one in the optimal contract and always exerts effort until he completes the last-stage innovation.

The balance of the savings account y in this implementation plays a similar role to the agent's wealth level y in the equity-based implementation in Subsection 5.1. Both of them capture the agent's expected income from the contract and serve as the state variable in the agent's utility maximization problem. Therefore, if the evolutions of y are the same, then both implementations generate the same consumption allocation as in the optimal contract. The main difference between these two implementations is the approach to control the evolution of y. In equity-based implementation, the principal adjusts the fraction of equity in the compensation portfolio according to the stage level, and then the equity-based compensation scheme automatically determines the evolution of the agent's wealth y. In performance-bonus implementation, the principal manually controls the evolution of the balance of the savings account y through the bonus for success. Comparing these two implementations, the advantage of using performance-based bonuses is that the agent does not bear any unnecessary risks brought by equity-based incentive compensation. However, the principal needs to monitor the balance of the account since the agent is risk-averse and hence the size of the bonus for success depends on the balance. The advantage of equity-based incentive compensation lies in its simplicity for which the principal only needs to adjust the faction of equity in the compensation portfolio depending on the development of the project and can leave all other decision problems to the agent.

7 Conclusion

To examine the optimality of the equity-based compensation scheme that is widely used by R&D-intensive start-up firms for their research employees, we study a dynamic contracting problem in which a principal hires an agent to perform a multi-stage R&D project. The R&D process is modeled by a Poisson process. In the optimal contract, the principal provides incentive to the agents in two ways: (1) the agent's compensation increases to a higher level when he completes an innovation (reward); (2) if the agent fails to complete the innovation, his compensation decreases continuously over time (punishment). We show that the optimal contract could be implemented using a risky security that appreciates when the project succeeds and depreciates when it fails. Until the agent completes the whole project, he is required to meet a sequence of holding requirements which are relaxed each time when the project progresses to the next stage. In this implementation, instead of the principal directly controlling the agent's consumption as in the optimal contract, the agent chooses both consumption level and effort level. We show that this implementation yields the same consumption allocation as the one in the optimal contract. We also provide an example in which the contracting problem has a closed-form solution and explicitly describe how to use the equity of the firm to implement the optimal contract. This implementation suggests that the structure of equity-based compensation should relate to the research employees' performance, and it provides a rationale for using the performance-vested equity-based compensation in R&D-intensive start-up firms from the theoretical point of view.

Appendix A: Proofs

Proof of Proposition 3.1

The proof of Proposition 3.1 in Shan (2017) proves points (i) to (iv). $C_n(v) > C_{n+1}(v)$ for all v is by Corollary 3.2 in Shan (2017). Proposition 3.1 in Shan (2017) shows that the derivative of the cost function $C_n(v)$ satisfies $S'(v) < C_n'(v) < C_{n+1}'(v + \frac{rl}{\lambda})$. Since the utility function $U(c)$ satisfies the Inada condition $\lim_{c\to+\infty} U'(c) = 0$, we have $\lim_{v\to+\infty} S'(v) = +\infty$, which implies that $\lim_{v\to+\infty} C_n'(v) = +\infty$. Since $C_n'(v) < C_{n+1}(v + \frac{rl}{\lambda})$ for all v and $C_n(0) = \frac{\lambda C_{n+1}(\frac{rl}{\lambda})}{r+\lambda} < C_{n+1}(\frac{rl}{\lambda})$, it follows that $C_n(v) < C_{n+1}(v + \frac{rl}{\lambda})$ for all v.

How to replicate the inverse Euler equitation in a two-period model

Before proving Proposition 4.1, we first illustrate the principle of the implementation in a two-period model in which the agent chooses action and consumption in the first period and the outcome of the project is realized in the second period. If the agent works in the first period, the project succeeds with probability μ. If he shirks, it fails with probability 1. In the first period, the agent's chooses the consumption c and security holding $\underline{y}$ given initial wealth level y subject to the following budget constraint

$$c + P(\underline{y}) = y.$$

The price of security are given by the discounted expected value of the security

$$P(\underline{y}) = \beta[(\mu Y(\underline{y}) + (1-\mu)\underline{y})],$$

where $\underline{y}$ is the payoff of the security in case of failure, $Y(\underline{y})$ is the payoff in case of success, and β is the discount factor. Thus, the agent's problem is

$$\max_{c,a} U(c) - al + \beta[a\mu U(Y(\underline{y})) + (1-\mu a)U(\underline{y})]$$

s.t.

$$c + \beta[(\mu Y(\underline{y}) + (1-\mu)\underline{y})] = y.$$

If the principal set $Y(\underline{y}) = U^{-1}(U(\underline{y}) + \frac{l}{\beta\mu})$, consider the agent's choice of effort, we have

$$-l + \beta[\mu U(Y(\underline{y})) + (1-\mu)U(\underline{y})] = \beta U(\underline{y}).$$

It implies that the agent is indifferent between working and shirking no matter what consumption level he chooses. For the consumption choice, if the agent decides to save one unit consumption and invest it in the security in the first period, then in the second period

$$\Delta\underline{y} = \frac{1}{\beta[\mu Y'(\underline{y}) + (1-\mu)]}, \text{ and } \Delta Y(\underline{y}) = \frac{Y'(\underline{y})}{\beta[\mu Y'(\underline{y}) + (1-\mu)]},$$

When the agent chooses to shirk $a = 0$, the optimal consumption choice satisfies the following Euler equation

$$U'(c) = \frac{U'(\underline{y})}{\mu Y'(\underline{y}) + (1-\mu)}.$$

When the agent chooses to work $a = 1$, the Euler equation becomes

$$U'(c) = \frac{\mu Y'(\underline{y})U'(Y(\underline{y}))}{\mu Y'(\underline{y}) + (1-\mu)} + \frac{(1-\mu)U'(\underline{y})}{\mu Y'(\underline{y}) + (1-\mu)}.$$

Note that $Y(\underline{y}) = U^{-1}(U(\underline{y}) + \frac{l}{\beta\mu})$. Hence, $Y'(\underline{y}) = \frac{U'(\underline{y})}{U'(Y(\underline{y}))}$, which implies that

$$\frac{\mu Y'(\underline{y})U'(Y(\underline{y}))}{\mu Y'(\underline{y}) + (1-\mu)} + \frac{(1-\mu)U'(\underline{y})}{\mu Y'(\underline{y}) + (1-\mu)} = \frac{U'(\underline{y})}{\mu Y'(\underline{y}) + (1-\mu)}.$$

Thus, for both actions the agent chooses the same consumption level because they satisfy the same Euler equation

$$U'(c) = \frac{U'(\underline{y})}{\mu Y'(\underline{y}) + (1-\mu)}.$$

Thus, if the principal set $Y(\underline{y}) = U^{-1}(U(\underline{y}) + \frac{l}{\beta\mu})$, the agent is willing to exert effort, and the joint deviation strategy of shirking and saving can be ruled out. Finally, taking the reciprocal of both sides of the Euler equation, we have

$$\frac{1}{U'(c)} = \frac{1}{U'(\underline{y})}[\mu Y'(\underline{y}) + (1-\mu)].$$

Note that

$$\frac{1}{U'(\underline{y})}[\mu Y'(\underline{y}) + (1-\mu)] = \frac{1}{U'(\underline{y})}[\mu \frac{U'(\underline{y})}{U'(Y(\underline{y}))} + (1-\mu)] = \frac{\mu}{U'(Y(\underline{y}))} + \frac{1-\mu}{U'(\underline{y})}.$$

Then, the Euler equation of the agent's consumption choice problem becomes

$$\frac{1}{U'(c)} = \frac{\mu}{U'(Y(\underline{y}))} + \frac{1-\mu}{U'(\underline{y})},$$

which is exactly the inverse Euler equation implied by the principal's problem. This result confirms that the implementation rules out the joint-deviation strategy.

Proof of Proposition 4.1

Since C_n is a strictly increasing and differentiable function by Proposition 3.1, C_n^{-1} exists and is also differentiable. We first show that $V_n(y)$, which is the maximum expected utility that the agent can achieve given the expected wealth y, equals $C_n^{-1}(y)$ for any n $(0 < n \leq N+1)$. This is obviously true when the agent completes the last stage and receives a lump-sum transfer, because $V_{N+1}(y) = U(y) = S^{-1}(y) = C_{N+1}^{-1}(y)$. Next, for any stage n, taking $V_{n+1}(y) = C_{n+1}^{-1}(y)$ as a known function, we verify that $V_n(y) = C_n^{-1}(y)$ is one solution to the value function of the following HJB equation for the agent's problem under the conditions of Proposition 4.1,

$$rV_n(y) = \max_{a,c} r[U(c) - al] + V_n'(y)\frac{dy}{dt} + a\lambda[C_{n+1}^{-1}(Y_{n+1}(y)) - V_n(y)]$$

s.t.

$$\begin{aligned} \frac{dy}{dt} &= ry - rc - \lambda[Y_{n+1}(y) - y], \\ y &\geq \underline{y}_n. \end{aligned}$$

Next, we show that $V_n(y) = C_n^{-1}(y)$ is the true value function for the agent's utility maximization problem in stage n, and hence the consumption path implied by the value function is a true solution to the agent's problem. If this is true, then we can show that $V_n(y) = C_n^{-1}(y)$ is the true value function for the agent's problem for any n $(0 < n \leq N+1)$ by backward induction.

Step 1: *Verify that $V_n(y) = C_n^{-1}(y)$ is one solution to the HJB equation.*

To verify that $V_n(y) = C_n^{-1}(y)$ satisfies the HJB equation, we plug $V_n(y) = C_n^{-1}(y)$ and its derivative $V_n'(y) = \frac{1}{C_n'(C_n^{-1}(y))}$ into both sides of the HJB equation and show that the equation holds. For the right-hand side, we first consider the agent's decision for action a. Letting $V_n(y) = C_n^{-1}(y)$, we have

$$\begin{aligned}
\lambda[C_{n+1}^{-1}(Y_{n+1}(y)) - V_n(y)] - rl &= \lambda[C_{n+1}^{-1}(C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})) - C_n^{-1}(y)] - rl \\
&= \lambda[C_n^{-1}(y) + \frac{rl}{\lambda} - C_n^{-1}(y)] - rl \\
&= 0.
\end{aligned}$$

Then, for either choice of action a, the right-hand side of the HJB equation becomes

$$RHS = \max_c rU(c) + V_n'(y)\{ry - rc - \lambda[Y_{n+1}(y) - y]\}.$$

Taking $V_n'(y) = \frac{1}{C_n'(C_n^{-1}(y))}$ and $Y_{n+1}(y) = C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})$ into the expression above, we have

$$\begin{aligned}
RHS &= \max_c rU(c) + \frac{ry - rc - \lambda[C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda}) - y]}{C_n'(C_n^{-1}(y))}, \\
&= rU(c^*(y)) + \frac{(r+\lambda)y - rc^*(y) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}{C_n'(C_n^{-1}(y))},
\end{aligned}$$

where $c^*(y)$ is the optimal choice of consumption which is determined by the first-order condition $U'(c^*(y)) = \frac{1}{C_n'(C_n^{-1}(y))}$.

The next step is to find the expression for $\frac{1}{C_n'(C_n^{-1}(y))}$. Since, from principal's problem, $C_n(v)$ satisfies the following differential equation

$$(r+\lambda)C_n(v) = rS(u^*(v)) + C_n'(v)[r(v - u^*(v))] + \lambda C_{n+1}(v + \frac{rl}{\lambda}),$$

where $u^*(v)$ is the optimal choice of utility flow and satisfies $S'(u^*(v)) = C_n'(v)$. Then we have

$$\frac{1}{C_n'(v)} = \frac{r(v - u^*(v))}{(r+\lambda)C_n(v) - rS(u^*(v)) - \lambda C_{n+1}(v + \frac{rl}{\lambda})}.$$

Letting the continuation utility v equal $C_n^{-1}(y)$ in the equation above, we can get

$$\begin{aligned}
\frac{1}{C_n'(C_n^{-1}(y))} &= \frac{r[C_n^{-1}(y) - u^*(C_n^{-1}(y))]}{(r+\lambda)C_n(C_n^{-1}(y)) - rS(u^*(C_n^{-1}(y)) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})} \\
&= \frac{r[C_n^{-1}(y) - u^*(C_n^{-1}(y))]}{(r+\lambda)y - rS(u^*(C_n^{-1}(y))) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})},
\end{aligned}$$

and $u^*(C_n^{-1}(y))$ satisfies $S'(u^*(C_n^{-1}(y))) = C_n'(C_n^{-1}(y))$. Note that $S(u) = U^{-1}(u)$, it implies that $S'(u) = \frac{1}{U'(S(u))}$. Then we have $C_n'(C_n^{-1}(y)) = S'(u^*(C_n^{-1}(y))) = \frac{1}{U'(S(u^*(C_n^{-1}(y))))}$. Since $c^*(y)$ satisfies $U'(c^*(y)) = \frac{1}{C_n'(C_n^{-1}(y))}$, it follows that $U'(S(u^*(C_n^{-1}(y)))) = U'(c^*(y))$. Because the utility function U is strictly concave, we then have $S(u^*(C_n^{-1}(y))) = c^*(y)$, and hence $u^*(C_n^{-1}(y)) = U(c^*(y))$. This result shows that $c^*(y) = S(u^*(C_n^{-1}(y)))$ because they satisfy the same first-order condition. Therefore,

$$\frac{1}{C_n'(C_n^{-1}(y))} = \frac{r[C_n^{-1}(y) - U(c^*(y))]}{(r+\lambda)y - rc^*(y) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}.$$

Taking this expression for $\frac{1}{C_n'(C_n^{-1}(y))}$ into the right-hand side of the HJB equation, we have

$$\begin{aligned} RHS &= rU(c^*(y)) + \frac{(r+\lambda)y - rc^*(y) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}{C_n'(C_n^{-1}(y))} \\ &= rU(c^*(y)) + r[C_n^{-1}(y) - U(c^*(y))] \\ &= rC_n^{-1}(y) \end{aligned}$$

For the left-hand side, we have

$$LHS = rV_n(y) = rC_n^{-1}(y).$$

Thus, $V_n(y) = C_n^{-1}(y)$ is one solution to the HJB equation of the agent's problem.

Step 2: *Check that the path of wealth y (or security holding) implied by the HJB equation given value function $V_n(y) = C_n^{-1}(y)$ does not violate the minimum holding requirement $y \geq \underline{y}_n$.*

Since the payoff of the security in case of success is strictly increasing in y, if the agent invests less than $\underline{y}_n$ in the security, by the design of the security, the payoff in case of success is less than $C_{n+1}(\frac{rl}{\lambda})$, and hence the expected utility that the agent can derive from this amount of wealth when he enters stage $n+1$ is less than $C_{n+1}^{-1}(C_{n+1}(\frac{rl}{\lambda})) = \frac{rl}{\lambda}$. To induce incentive, the agent's continuation utility needs to increase by at least $\frac{rl}{\lambda}$. Since in stage n the agent can always guarantee 0 utility by doing noting, if he knows that the highest utility he can receive in case of success is less than $\frac{rl}{\lambda}$, he will not have any incentive to work. Intuitively, the minimum holding requirement ensures that the agent has sufficient equity in the future to induce effort. The minimum holding requirement imposes a condition at the lower bound of y that $\frac{dy}{dt} \geq 0$ when y reaches the lower bound $\underline{y}_n$ because y cannot decrease any further when it hits the lower bound. Our next task is to check this condition is satisfied given $V_n(y) = C_n^{-1}(y)$. Since $\underline{y}_n = C_n(0)$, we have

$$Y_{n+1}(\underline{y}_n) = C_{n+1}(C_n^{-1}(\underline{y}_n) + \frac{rl}{\lambda}) = C_{n+1}(\frac{rl}{\lambda}).$$

When y reaches the lower bound $\underline{y}_n$, the agent's choice of consumption satisfies

$$c^*(\underline{y}_n) = S(u^*(C_n^{-1}(\underline{y}_n))) = S(u^*(0)) = S(0),$$

where $u^*(0) = 0$ is because from Proposition 3.1 we have $S'(u^*(0)) = C_n'(0)$ and the boundary condition $C_n'(0) = S'(0) = 0$ when the continuation utility reaches the lower bound 0 in the principal's problem. Then, at $y = \underline{y}_n$,

$$\frac{dy}{dt} = r\underline{y}_n - rc^*(\underline{y}_n) - \lambda[Y_{n+1}(\underline{y}_n) - \underline{y}_n] = rC_n(0) - \lambda[C_{n+1}(\frac{rl}{\lambda}) - C_n(0)] = 0,$$

where the last equality is because $C_n(0) = \frac{\lambda C_{n+1}(\frac{rl}{\lambda})}{r+\lambda}$ from Proposition 3.1. Therefore, the boundary condition for y is satisfied.

Step 3: *Verify that $V_n(y) = C_n^{-1}(y)$ is the true value function of the agent's maximization problem and the consumption path implied by the HJB equation given value function $V_n(y) = C_n^{-1}(y)$ is the same as the consumption path of the optimal contract.*

Let the time when the stage n problem starts be 0 and let y_0 be the wealth at the beginning of stage n. When the project remains in stage n, the state variable y_t evolves according to

$$\frac{dy}{dt} = ry - rc - \lambda[Y_{n+1}(y) - y].$$

Since $C_n(v) < C_{n+1}(v + \frac{rl}{\lambda})$, we have $Y_{n+1}(y) = C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda}) > C_n(C_n^{-1}(y)) = y$, which implies that

$$\frac{dy}{dt} = ry - rc - \lambda[Y_{n+1}(y) - y] \leq ry.$$

Hence, for any feasible path of the state variable $\{y_t\}_{t\geq 0}$, we have $y_t \leq e^{rt}y_0$. Since $C_n(v)$ is a strictly increasing function and satisfies $\lim_{v\to+\infty} C_n'(v) = +\infty$, we have $V_n(y) = C_n^{-1}(y)$ is a strictly increasing function and satisfies and $\lim_{y\to+\infty} V_n'(y) = 0$. Since $V_n(\underline{y}_n) = 0$, we have $V_n(y) \geq 0$ for all $y \geq \underline{y}_n$. It follows that for all feasible paths of the state variable $\{y_t\}_{t\geq 0}$,

$$0 \leq \lim_{t\to+\infty} e^{-rt}V_n(y_t) \leq \lim_{t\to+\infty} \frac{V_n(e^{rt}y_0)}{e^{rt}} = \lim_{t\to+\infty} \frac{V_n'(e^{rt}y_0)y_0re^{rt}}{re^{rt}} = 0,$$

where the first equality is by L'Hôpital's rule. It shows that $V_n(y) = C_n^{-1}(y)$ satisfies the transversality condition $\lim_{t\to+\infty} e^{-rt}V_n(y_t) = 0$ for any feasible path of the state variable $\{y_t\}_{t\geq 0}$.[10]

[10] This condition is the continuous-time version of the condition in Theorem 4.3 in Stokey and Lucas (1989). In the supplement of this paper, we provide a proof that if a function is one solution to the HJB equation and satisfies the transversality condition, then the function is the true value function.

Therefore, $V_n(y) = C_n^{-1}(y)$ is the true value function of the agent's maximization problem, and the consumption path implied by the HJB equation given value function $V_n(y) = C_n^{-1}(y)$ is the true solution to the agent's problem.

Our next task is to show that the consumption path chosen by the agent is the same as the consumption path of the optimal contract. We can interpret $V_n(y)$ as the agent's "continuation utility" given wealth y. In case of success, the agent's "continuation utility" changes from $V_n(y)$ to $C_{n+1}^{-1}(Y_{n+1}(y))$, which satisfies

$$C_{n+1}^{-1}(Y_{n+1}(y)) = C_{n+1}^{-1}\left(C_{n+1}\left(C_n^{-1}(y) + \frac{rl}{\lambda}\right)\right) = C_n^{-1}(y) + \frac{rl}{\lambda} = V_n(y) + \frac{rl}{\lambda}.$$

Thus, if $Y_{n+1}(y) = C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})$ then the agent is always indifferent between working and shirking no matter what his consumption choice is. Moreover, the agent's optimal choices of consumption for the two actions are the same because they satisfy the same first-order condition $U'(c) = V_n'(y)$. Thus, the agent is always willing to exert effort and cannot achieve higher utility through the joint-deviation strategy by shirking and saving. In case of failure, his "continuation utility" changes smoothly and evolves according to

$$\frac{dV_n(y)}{dt} = V_n'(y)\frac{dy}{dt} = rV_n(y) - rU(c^*(y)),$$

where $c^*(y)$ is the optimal choice of consumption given y and satisfies $U'(c^*(y)) = V_n'(y)$, and the last equality is derived from the agent's HJB equation. From the principal's problem in Section 3, since the incentive-compatibility condition is always binding so that $\bar{v} = v + \frac{rl}{\lambda}$, in case of failure, the continuation utility evolves according to

$$\frac{dv}{dt} = rv - ru^*(v),$$

where $u^*(v)$ is the optimal choice of utility flow and satisfies $S'(u^*(v)) = C_n'(v)$. The previous proof has shown that if $v = V_n(y) = C_n^{-1}(y)$ then $u^*(v) = u^*(C_n^{-1}(y)) = U(c^*(y))$, which means that the optimal utility flow given continuation utility level $C_n^{-1}(y)$ in the principal's problem equals the agent's utility from the optimal consumption choice given wealth level y in the agent's problem. Therefore, given the same continuation utility, the agent chooses the same level of consumption as the optimal contract, which further induces the same dynamics of the continuation utility. Thus, the optimal consumption path for the agent's problem is the same as the consumption path of the optimal contract.

Finally, since we have verified that $V_{N+1}(y) = C^{-1}_{N+1}(y)$, we can show that $V_n(y) = C^{-1}_n(y)$ is the true value function for the agent's problem for any n $(0 < n \leq N+1)$ by backward induction.

Therefore, given the same continuation utility, the implementation and the optimal contract choose the same consumption level, which further induces the same dynamics of the continuation utility. $Y_{n+1}(y) = C_{n+1}(C^{-1}_n(y) + \frac{rl}{\lambda})$ guarantees that the agent is always indifferent between working and shirking. The initial condition that $y^0 = C_1(v^0)$ guarantees that the agent starts with initial continuation-utility v^0. Thus, the implementation and the optimal contract generate the same consumption path under all possible realization of the agent's performance and the agent is always willing to exert effort.

For the value of the minimum holding requirements, since $\underline{y}_n = C_n(0)$ and $C_n(0) > C_{n+1}(0)$ by Proposition 3.1, it follows that the minimum holding requirement satisfies $\underline{y}_n > \underline{y}_{n+1}$.

Proof of Proposition 5.1

For logarithmic utility function $U(c) = \ln(c)$, the cost of delivering u is $S(u) = e^u$, and the cost of delivering the one-time transfer when the project is completed is $C_{N+1}(v) = e^v$. Suppose the stage $n+1$ cost function is $C_{n+1}(v) = p_{n+1}e^v$, where p_{n+1} is a constant. We first use a guess-and-verify method to show that the solution to the stage n HJB equation $C_n(v)$ also takes the form of $p_n e^v$—a constant times e^v.

Taking $C_{n+1}(v) = p_{n+1}e^v$ and the guess $C_n(v) = p_n e^v$ into the HJB equation, the left-hand becomes $rp_n e^v$. If we can show that the right-hand side also takes the form of a constant times e^v, then we can pin down the constant p_n from the HJB equation, and the guess is verified. The right-hand side of the HJB equation is given by

$$RHS = \min_{u,\bar{v}} re^u + p_n e^v \frac{dv}{dt} + \lambda(p_{n+1}e^{\bar{v}} - p_n e^v)$$

s.t.

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u-l) - \lambda(\bar{v} - v), \\ \lambda(\bar{v} - v) &\geq rl. \end{aligned}$$

Utility-flow u satisfies the first-order condition $S'(u) = C'_n(v)$. Therefore,

$$e^u = p_n e^v,$$

which implies that $u = v + \ln p_n$.

The incentive compatibility constraint must be binding, otherwise the principal can lower costs by offering a lower $\bar{v}$. Hence, $\bar{v} = v + \frac{rl}{\lambda}$, which implies that $\frac{dv}{dt} = rv - ru = -r\ln p_n$. Taking the solution for u and $\bar{v}$ into the right-hand side of the HJB equation, it becomes

$$\begin{aligned} RHS &= rp_n e^v + p_n e^v(-r\ln p_n) + \lambda(p_{n+1}e^{v+\frac{rl}{\lambda}} - p_n e^v) \\ &= (rp_n - rp_n \ln p_n + \lambda p_{n+1}e^{\frac{rl}{\lambda}} - \lambda p_n)e^v, \end{aligned}$$

which also takes the form of a constant times e^v. Finally, letting the left-hand side of the HJB equation equal the right-hand side, we have

$$rp_n e^v = (rp_n - rp_n \ln p_n + \lambda p_{n+1}e^{\frac{rl}{\lambda}} - \lambda p_n)e^v,$$

which implies that

$$rp_n \ln p_n + \lambda p_n = \lambda p_{n+1}e^{\frac{rl}{\lambda}}.$$

Therefore, if $C_{n+1}(v) = p_{n+1}e^v$, then $C_n(v)$ also takes the form of $p_n e^v$, where the constant p_n is determined by the above equation given p_{n+1}. When the agent completes the project, the principal's cost of delivering the one-time transfer is $C_{N+1}(v) = e^v$, and hence $p_{N+1} = 1$. Then, by backward induction, the cost function at any stage n equals $C_n(v) = p_n e^v$, where p_n is determined recursively starting from $p_{N+1} = 1$.

Next, we show that the constants satisfy $p_n > p_{n+1}$ by backward induction. Since $p_{N+1} = 1$, p_N satisfies

$$rp_N \ln p_N + \lambda p_N = \lambda e^{\frac{rl}{\lambda}}.$$

If $p_N = 1$, then we have

$$rp_N \ln p_N + \lambda p_N = r\ln 1 + \lambda = \lambda < \lambda e^{\frac{rl}{\lambda}}.$$

Since $rp_N \ln p_N + \lambda p_N$ is an increasing function of p_N, it implies that $p_N > 1 = p_{N+1}$.

For any $0 < n \leq N$, we have

$$rp_n \ln p_n + \lambda p_n = \lambda p_{n+1}e^{\frac{rl}{\lambda}},$$

$$rp_{n-1} \ln p_{n-1} + \lambda p_{n-1} = \lambda p_n e^{\frac{rl}{\lambda}}.$$

Thus, $p_n > p_{n+1}$ implies that $p_{n-1} > p_n$. We have shown that $p_N > p_{N+1}$. Applying backward induction, we can show that $p_n > p_{n+1}$ for all $0 < n \leq N$.

Proof of Corollary 5.2

Since p_n is determined recursively by

$$rp_n \ln p_n + \lambda p_n = \lambda p_{n+1} e^{\frac{rl}{\lambda}}.$$

By Implicit Function Theorem, we have

$$\frac{\partial p_n}{\partial l} = \frac{rp_{n+1}e^{\frac{rl}{\lambda}} + \lambda e^{\frac{rl}{\lambda}} \frac{\partial p_{n+1}}{\partial l}}{r \ln p_n + r + \lambda}$$

Since $p_n \geq 1$ and $p_{n+1} \geq 1$, it implies that if $\frac{\partial p_{n+1}}{\partial l} \geq 0$, then $\frac{\partial p_n}{\partial l} > 0$. Note that $p_{N+1} = 1$ and hence $\frac{\partial p_{N+1}}{\partial l} = 0$. Then, $\frac{\partial p_n}{\partial l} > 0$ for all n by backward induction.

Similarly, by Implicit Function Theorem, we have

$$\frac{\partial p_n}{\partial \lambda} = -\frac{p_n + (\frac{rl}{\lambda} - 1)e^{\frac{rl}{\lambda}} p_{n+1} - \lambda e^{\frac{rl}{\lambda}} \frac{\partial p_{n+1}}{\partial \lambda}}{r \ln p_n + r + \lambda} < -\frac{p_n - p_{n+1} - \lambda e^{\frac{rl}{\lambda}} \frac{\partial p_{n+1}}{\partial \lambda}}{r \ln p_n + r + \lambda},$$

where the last step is because $(\frac{rl}{\lambda} - 1)e^{\frac{rl}{\lambda}} > -1$.[11] Since $p_n > p_{n+1}$ by Proposition 5.1, it implies that if $\frac{\partial p_{n+1}}{\partial \lambda} \leq 0$, then $\frac{\partial p_n}{\partial \lambda} < 0$. Once again, we can show that $\frac{\partial p_n}{\partial \lambda} < 0$ for all n by backward induction starting from the fact that $\frac{\partial p_{N+1}}{\partial \lambda} = 0$.

Proof of Proposition 5.3

The proof is similar to the proof of Proposition 4.1. When the agent completes the last stage, his utility from the lump-sum payment equals $V_{N+1}(y) = \ln y - \ln p_{N+1}$, where $p_{N+1} = 1$. Next, given $V_{n+1}(y) = \ln y - \ln p_{n+1}$ and $Y_{n+1}(y) = (\frac{r \ln p_n}{\lambda} + 1)y$, we verify that $V_n(y) = \ln y - \ln p_n$ is the solution to the value function of the following HJB equation for the agent's problem,

$$rV_n(y) = \max_{a,c} r[\ln c - al] + V_n'(y)\frac{dy}{dt} + a\lambda\{[\ln(Y_{n+1}(y)) - \ln p_{n+1}] - V_n(y)\}$$

s.t.

$$\frac{dy}{dt} = (r - r \ln p_n)y - rc.$$

If this is true, then we can show that $V_n(y) = \ln y - \ln p_n$ for any n $(0 < n \leq N + 1)$ by backward induction.

[11] Let $f(x) = (x - 1)e^x + 1$. We have $f(0) = 0$ and $f'(x) = xe^x > 0$ for all $x > 0$. Hence, $f(x) = (x - 1)e^x + 1 > 0$ for all $x > 0$. This implies that $(\frac{rl}{\lambda} - 1)e^{\frac{rl}{\lambda}} > -1$.

To verify $V_n(y) = \ln y - \ln p_n$ is the solution of the HJB equation, we plug $V_n(y) = \ln y - \ln p_n$ and its derivative $V_n'(y) = \frac{1}{y}$ into both sides and show that they are equal. For the right-hand side, we first consider the agent's decision for action a. Letting $V_n(y) = \ln y - \ln p_n$, we have

$$\begin{aligned}
\lambda\{[\ln(Y_{n+1}(y)) - \ln p_{n+1}] - V_n(y)\} - rl &= \lambda\{[\ln(\frac{r \ln p_n}{\lambda} + 1)y - \ln p_{n+1}] - (\ln y - \ln p_n)\} - rl \\
&= \lambda\{[\ln(\frac{r \ln p_n}{\lambda} + 1) + \ln y - \ln p_{n+1}] - (\ln y - \ln p_n)\} - rl \\
&= \lambda \ln(\frac{rp_n \ln p_n + \lambda p_n}{\lambda p_{n+1}}) - rl \\
&= \lambda \ln e^{\frac{rl}{\lambda}} - rl \\
&= 0,
\end{aligned}$$

where the forth equality is because p_n satisfies

$$rp_n \ln p_n + \lambda p_n = \lambda p_{n+1} e^{\frac{rl}{\lambda}}.$$

Then, for either choice of action a, the right-hand side of the HJB equation becomes

$$RHS = \max_c r \ln c + V_n'(y)[(r - r \ln p_n)y - rc].$$

Taking $V_n'(y) = \frac{1}{y}$ into the above expression, we have

$$RHS = r \ln y + \frac{1}{y}[(r - r \ln p_n)y - rc].$$

The optimal choice of consumption satisfies the first-order condition $\frac{r}{c^*(y)} = \frac{r}{y}$, and hence $c^*(y) = y$. Then,

$$RHS = r \ln y + \frac{1}{y}[(r - r \ln p_n)y - ry] = r(\ln y - \ln p_n).$$

For the left-hand side, we have

$$LHS = rV_n(y) = r(\ln y - \ln p_n).$$

Thus, $V_n(y) = \ln y - \ln p_n$ is one solution to the HJB equation of the agent's problem.

For the dynamics of state variable y, we have

$$\frac{dy}{dt} = [(r - r \ln p_n)y - rc^*(y)] = (-r \ln p_n)y.$$

Thus, suppose the stage n problem starts at time 0 with wealthy level y_0. If the project remains in stage n till time t, the wealth implied by the first-order condition of the HJB equation is $y_t^* =$

$e^{-r \ln p_n t} y_0$. Given value function $V_n(y) = \ln y - \ln p_n$, on the path of wealth $\{y_t^*\}_{t \geq 0}$, we have

$$\lim_{t \to +\infty} e^{-rt} V_n'(y_t^*) y_t^* = \lim_{t \to +\infty} e^{-rt} \frac{1}{e^{-r \ln p_n t} y_0} e^{-r \ln p_n t} y_0 = 0.$$

This transversality condition implies that $\{y_t^*\}_{t \geq 0}$ is the true solution of the agent's maximization problem, and hence $V_n(y) = \ln y - \ln p_n$ is the true value function of the agent's maximization problem.[12]

Next, we show that the implementation generates the same consumption path as the optimal contract. As in the proof of Proposition 4.1, we can interpret $V_n(y)$ as the agent's "continuation utility" given wealth y. We have shown that the agent is always indifferent between working and shirking because his "continuation utility" increases by $\frac{rl}{\lambda}$ after each success. Hence, the agent is always willing to exert effort. Given wealth level y, the optimal choice of consumption satisfies $c^*(y) = y$. Given "continuation utility" $V_n(y)$, the utility flow from consumption equals $r \ln c^*(y) = r \ln y = V_n(y) + \ln p_n$. In case of failure, his "continuation utility" changes smoothly and evolves according to

$$\frac{dV_n(y)}{dt} = V_n'(y) \frac{dy}{dt} = rV_n(y) - rU(c^*(y)) = r(\ln y - \ln p_n) - r \ln y = -r \ln p_n,$$

where $c^*(y)$ is the optimal choice of consumption given y and satisfies $c^*(y) = y$. For the principal's problem, Proposition 5.1 shows that given continuation utility v, the optimal choice of utility flow equals $v + \ln p_n$ and, in case of failure, the continuation utility evolves according to

$$\frac{dv}{dt} = -r \ln p_n,$$

Therefore, given the same continuation utility, the agent chooses the same level of consumption as the optimal contract, which further induces the same dynamics of the continuation utility. The initial condition that $y^0 = C_1(v^0)$ guarantees that the agent starts with initial continuation-utility v^0. Thus, the implementation and the optimal contract generate the same consumption path under all possible realization of the agent's performance, and the agent is always willing to exert effort.

Proof of Proposition 5.6

To implement the optimal contract, the proof of a two-period implementation problem before the proof of Proposition 4.1 shows that the optimal contract can be implemented by a risky security that

[12] We provide a proof that the transversality condition plus the first-order conditions are sufficient in the supplement of this paper.

satisfies the following property: 1) if the payoff of a certain amount of the security in case of failure equals $\underline{y}$, then its payoff in case of success equals $Y(\underline{y}) = U^{-1}(U(\underline{y}) + \frac{l}{\beta\mu}) = e^{\frac{l}{\beta\mu}}\underline{y}$; 2) the period-0 price of this amount of security is determined by fair-pricing rule and equals $\beta(\mu Y(\underline{y}) + (1-\mu)\underline{y}) = \beta[\mu e^{\frac{l}{\beta\mu}} + (1-\mu)]\underline{y}$. We first consider the case in which $\tilde{R}$ is certain and equals $\bar{R}$. For a certain amount of the firm's equity, if its value equals y in case of failure, then its value equals $\bar{R}y$ in case of success. In this case, the risky security can be created by a portfolio of the firm's equity and a risk-free asset where the fraction of equity α satisfies $\alpha\bar{R} + (1-\alpha) = e^{\frac{l}{\beta\mu}}$. Therefore, if there is no uncertainty about the value of the project, the optimal contract (the second-best allocation) can be implemented by the equity of the firm. Given this portfolio, the agent is always indifferent between working and shirking. If his initial wealth is y_0, the highest expected utility he can achieve is the solution to the following optimization problem

$$\bar{V}(y_0) = \max_c \ln c + \beta \ln(\underline{y})$$

s.t.

$$c + \beta\{[\mu[\alpha\bar{R} + (1-\alpha)] + (1-\mu)\}\underline{y} = y_0.$$

Next, we study the case in which the value of the project is uncertain so that $\tilde{R}$ is random and calculate the third-best outcome when the principal can only use the firm's equity and a risk-free asset to compensate the agent. Consider a portfolio of the firm's equity and a risk-free asset where the fraction of equity equals α'. Then, if the value of the portfolio in case of failure equals $\underline{y}$ its value in case of success equals $[\alpha'\tilde{R} + (1-\alpha')]\underline{y}$. The period-0 cost of this portfolio equals

$$\beta E\{\mu[\alpha'\tilde{R} + (1-\alpha')]\underline{y} + (1-\mu)\underline{y}\} = \beta\{\mu[\alpha'\bar{R} + (1-\alpha')] + (1-\mu)\}\underline{y}.$$

If the agent exerts effort in period 0, his expected utility in period 1 equals $E\{\mu \ln[\alpha'\tilde{R} + (1-\alpha')]\underline{y} + (1-\mu)\ln \underline{y}\}$. If he chooses to shirk, his utility in period 1 equals $\ln \underline{y}$. Thus, to provide incentive for working, the portfolio needs to satisfies the following IC constraint

$$\beta\big\{E\{\mu \ln[\alpha'\tilde{R} + (1-\alpha')]\underline{y} + (1-\mu)\ln \underline{y}\} - \ln \underline{y}\big\} \geq l,$$

which implies that

$$E\{\ln[\alpha'\tilde{R} + (1-\alpha')]\} \geq \frac{l}{\beta\mu}.$$

To minimize the cost, the IC constraint must be binding, and hence the principal should choose a α' that satisfies $E\{\ln[\alpha'\tilde{R} + (1-\alpha')]\} = \frac{l}{\beta\mu}$. Since logarithmic function is concave, we have

$$E\{\ln[\alpha'\tilde{R} + (1-\alpha')]\} < \ln\{E[\alpha'\tilde{R} + (1-\alpha')]\} = \ln[\alpha'\bar{R} + (1-\alpha')].$$

Note that $\ln[\alpha\bar{R} + (1-\alpha)] = \frac{l}{\beta\mu}$. Then, we have $\ln[\alpha'\bar{R} + (1-\alpha')] > \ln[\alpha\bar{R} + (1-\alpha)]$, which implies that $\alpha' > \alpha$. The difference between α and α' depends on the distribution of $\tilde{R}$, which becomes smaller when the distribution of $\tilde{R}$ is more concentrated around its mean $\bar{R}$. Similar to the certainty case, given this portfolio, the agent is always indifferent between working and shirking. The highest expected utility that he can achieve with initial wealth y_0 is the solution to the following optimization problem

$$\tilde{V}(y_0) = \max_c \ln c + \beta \ln(\underline{y})$$

s.t.

$$c + \beta\{[\mu[\alpha'\bar{R} + (1-\alpha')] + (1-\mu)\}\underline{y} = y_0.$$

Comparing the above problem with the agent's maximization problem when $\tilde{R}$ is certain, the only difference is the "price" of the portfolio, which is higher when $\tilde{R}$ is random because $\alpha' > \alpha$. Hence, we have $\tilde{V}(y_0) < \bar{V}(y_0)$.

To summarize, given the same initial wealth y_0, the agent can achieve higher expected utility when the firm's value in case of success is certain than when the firm's value is random. In other words, when the firm's value is random, the principal needs to compensate the agent more to deliver the same level of promised utility. The efficiency loss is caused by letting the agent bear risks that are not affected by his action. The proof shows that the efficiency loss depends on the difference between α' and α, which further depends on the distribution of $\tilde{R}$. When the distribution of $\tilde{R}$ is more concentrated around its mean, the efficiency loss of equity-based compensation compared to the optimal contract is less.

Appendix B: Extensions

A multi-agent model

So far, we have assumed that the principal faces a single agent, while in practice a research project is usually performed by multiple agents in research teams. In this subsection, using the "team-performance case" studied in Shan (2017), we extend the benchmark model to a multi-agent model. In this multi-agent model, the research project is performed by a research team that consists $I > 2$ research agents, and the principal can only observe the progress of the project. The objective of the principal is to design an incentive-compatible contract for each agent so that every agent is willing to exert effort, i.e., exerting effort is a Nash equilibrium strategy played by all the agents at any point in time. For simplicity, we assume that all these agents in the research team have the same utility function $U(c) - L(a)$, which satisfies the same assumptions in Section 3. Let λ be the arrival rate of success if all agents exert effort and λ_{-i} be the arrival rate if all agent except agent i exert effort. Consider the contracting problem for agent i. Conditional on all other agents exert effort, agent i increases the arrival rate of success of the team from λ_{-i} to λ if he chooses to exert effort. Hence, his benefit for exerting effort is $(\lambda - \lambda_{-i})(\bar{v} - v)$, and his costs of exerting effort is rl. Then, the Nash-incentive-compatibility condition is given by

$$(\lambda - \lambda_{-i})(\bar{v} - v) \geq rl.$$

Let $C_{i,n}(v)$ be the principal's minimum cost of delivering continuation utility v to agent i when the project is at stage n. The cost function satisfies the following HJB equation

$$rC_{i,n}(v) = \min_{u,\bar{v}} rS(u) + C'_{i,n}(v)\frac{dv}{dt} + \lambda[C_{i,n+1}(\bar{v}) - C_{i,n}(v)]$$

s.t.

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u - l) - \lambda(\bar{v} - v), \\ (\lambda - \lambda_{-i})(\bar{v} - v) &\geq rl. \end{aligned}$$

The properties of the optimal contract are summarized in the following proposition.

Proposition B.1 *The principal's expected cost at any point is given by an increasing and convex function $C_{i,n}(v)$ that satisfies*

$$rC_{i,n}(v) = rS(u^*(v)) + C'_{i,n}(v)[rv - ru^*(v) - \lambda_{-i}(\bar{v} - v)] + \lambda[C_{i,n+1}(\bar{v}) - C_{i,n}(v)],$$

and the boundary condition

$$C_{i,n}\left(\frac{\lambda_{-i}l}{\lambda - \lambda_{-i}}\right) = \frac{\lambda C_{i,n+1}\left(\frac{(r+\lambda_{-i})l}{\lambda-\lambda_{-i}}\right)}{r + \lambda}.$$

The cost function when the team completes the last stage innovation is given by $C_{i,N+1}(v) = S(v)$. The instantaneous payment $u^*(v)$ satisfies $S'(u^*(v)) = C'_{i,n}(v)$. When the team completes an innovation, agent i's continuation utility increases to $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda-\lambda_{-i}}$. In case of failure, the continuation utility v decreases over time and stays at $\frac{\lambda_{-i}l}{\lambda-\lambda_{-i}}$ when it reaches the lower bound $\frac{\lambda_{-i}l}{\lambda-\lambda_{-i}}$. The instantaneous payment u has the same dynamics as the continuation utility v.

The only main difference between this case and the single-agent case is the positive lower bound on the implementable continuation utility $\frac{\lambda_{-i} l}{\lambda - \lambda_{-i}}$. To provide an incentive, the principal should reward agent i by raising his continuation utility by $\frac{rl}{\lambda-\lambda_{-i}}$ after success. Thus, even if agent i shirks, he still can receive the reward by free-riding on his coworkers' work and guarantee a positive expected utility of $\frac{\lambda_{-i} l}{\lambda - \lambda_{-i}}$.[13]

Since including multiple agents only affects the lower bound on the implementable continuation utility and the minimum reward in the incentive-compatibility condition, for the implementation results, the only two modifications are the payoff in case of success and the minimum holding requirement, which are given below:

$$Y_{i,n+1}(y) = \begin{cases} C_{i,n+1}\big(C_{i,n}^{-1}(y) + \frac{rl}{\lambda-\lambda_{-i}}\big) & \text{if } y \geq C_{i,n}\big(\frac{\lambda_{-i}l}{\lambda-\lambda_{-i}}\big), \\ \frac{C_{i,n+1}\big(\frac{\lambda_{-i}l}{\lambda-\lambda_{-i}} + \frac{rl}{\lambda-\lambda_{-i}}\big)}{C_{i,n}\big(\frac{\lambda_{-i}l}{\lambda-\lambda_{-i}}\big)} y & \text{if } 0 \leq y < C_{i,n}\big(\frac{\lambda_{-i}l}{\lambda-\lambda_{-i}}\big), \end{cases}$$

$$\underline{y}_{i,n} = C_{i,n}\left(\frac{\lambda_{-i}l}{\lambda - \lambda_{-i}}\right).$$

All other results go through.

For general utility functions, the principal needs an individually-designed security for each agent, which seems unrealistic. Like the single-agent problem, if the agents' utility function from consumption is logarithmic, we can obtain a closed-form solution and provide a practical implementation using the equity of the firm. The agents' compensation packages differ only in the holding requirement (the required fraction of equity) depending on how an agent's action affects the performance of the team. The results of the optimal contract and the implementation for the logarithmic case are summarized in the following proposition.

Proposition B.2 *In the multi-agent model, if the agents' utility from consumption is $U(c) = \ln c$:*

- *The minimum cost of delivering continuation utility v to agent i when the project is at stage n is given by $C_{i,n}(v) = q_{i,n}e^v$, where the constant $q_{i,n}$ is determined recursively by*

$$q_{i,N+1} = 1 \text{ and } rq_{i,n}\ln q_{i,n} + \left(\frac{\lambda_{-i}rl}{\lambda - \lambda_{-i}} + \lambda\right)q_{i,n} = \lambda q_{i,n+1}e^{\frac{rl}{\lambda}},$$

and satisfies $q_{i,n} > q_{i,n+1}$. When the project progresses to the next stage, agent i's continuation utility increases from v to $\bar{v} = v + \frac{rl}{\lambda-\lambda_{-i}}$. In case of failure, the continuation utility v evolves according to $\frac{dv}{dt} = -r\ln q_{i,n} - \frac{\lambda_{-i}rl}{\lambda-\lambda_{-i}} < 0$.

[13] The derivation of the lower bound on the implementable continuation utility can be found in Shan (2017).

- *Suppose the value of the firm increases by R_n times when the project progresses from stage n to stage $n+1$. If $R_n \geq \frac{r \ln q_{i,n}}{\lambda} + \frac{\lambda_{-i} rl}{\lambda(\lambda-\lambda_{-i})} + 1$ for all n, then the risky security that implements the optimal contract can be created by a portfolio of the firm's equity and a risk-free asset with interest rate r. The fraction of equity $\beta_{i,n}$ depends on the stage level and satisfies*

$$\beta_{i,n} = \frac{r \ln q_{i,n} + \frac{\lambda_{-i} rl}{\lambda-\lambda_{-i}}}{(R_n - 1)\lambda}.$$

PROOF OF PROPOSITION B.2: Similar to the proof of Proposition 5.1, we first use a guess-and-verify method to show that the solution to the stage n HJB equation $C_{i,n}(v)$ takes the form of $q_{i,n}e^v$ given $C_{i,n+1}(v) = q_{i,n+1}e^v$.

Taking $C_n(v) = q_n e^v$ and $C_{i,n+1}(v) = q_{i,n+1}e^v$ into the HJB equation, we have

$$RHS = \min_{u,\bar{v}} re^u + q_{i,n}e^v \frac{dv}{dt} + \lambda(q_{i,n+1}e^{\bar{v}} - q_{i,n}e^v)$$

s.t.

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u-l) - \lambda(\bar{v} - v), \\ (\lambda - \lambda_{-i})(\bar{v} - v) &\geq rl. \end{aligned}$$

Utility-flow u satisfies the first-order condition $S'(u) = C'_{i,n}(v)$. Therefore,

$$e^u = q_{i,n}e^v,$$

which implies $u = v + \ln q_{i,n}$. The binding incentive compatibility constraint implies that $\bar{v} = v + \frac{rl}{\lambda-\lambda_{-i}}$, which implies that $\frac{dv}{dt} = rv - ru - \frac{\lambda_{-i} rl}{\lambda-\lambda_{-i}} = -r \ln q_{i,n} - \frac{\lambda_{-i} rl}{\lambda-\lambda_{-i}}$. Taking the solution for u and $\bar{v}$ into the right-hand side of the HJB equation, it becomes

$$\begin{aligned} RHS &= rq_{i,n}e^v + q_{i,n}e^v(-r \ln q_{i,n} - \frac{\lambda_{-i} rl}{\lambda - \lambda_{-i}}) + \lambda(q_{i,n+1}e^{v+\frac{rl}{\lambda-\lambda_{-i}}} - q_{i,n}e^v) \\ &= (rq_{i,n} - rq_{i,n} \ln q_{i,n} - \frac{\lambda_{-i} rl}{\lambda - \lambda_{-i}} q_{i,n} + \lambda q_{i,n+1} e^{\frac{rl}{\lambda-\lambda_{-i}}} - \lambda q_{i,n})e^v. \end{aligned}$$

which also takes the form of a constant times e^v. Finally, letting the left-hand side of the HJB equation equal the right-hand side, we have

$$rq_{i,n}e^v = (rq_{i,n} - rq_{i,n} \ln q_{i,n} - \frac{\lambda_{-i} rl}{\lambda - \lambda_{-i}} q_{i,n} + \lambda q_{i,n+1} e^{\frac{rl}{\lambda-\lambda_{-i}}} - \lambda q_{i,n})e^v,$$

which implies that

$$rq_{i,n} \ln q_{i,n} + \left(\frac{\lambda_{-i} rl}{\lambda - \lambda_{-i}} + \lambda\right) q_{i,n} = \lambda q_{i,n+1} e^{\frac{rl}{\lambda-\lambda_{-i}}}.$$

Hence, we have verified that if $C_{i,n+1}(v) = q_{i,n+1}e^v$, then $C_{n+1}(v)$ also takes the form of a constant $q_{i,n}$ times e^v, where $q_{i,n}$ is determined by the equation above. Finally, since $q_{i,N+1} = 1$, by backward induction, the cost function at any stage n equals $C_{i,n}(v) = q_{i,n}e^v$, where $q_{i,n}$ is determined recursively by

$$rq_{i,n} \ln q_{i,n} + \left(\frac{\lambda_{-i}rl}{\lambda - \lambda_{-i}} + \lambda\right)q_{i,n} = \lambda q_{i,n+1}e^{\frac{rl}{\lambda-\lambda_{-i}}}.$$

Next, we show that the constants satisfy $q_{i,n} > q_{i,n+1}$ by backward induction. Since $q_{i,N+1} = 1$, $q_{i,N}$ satisfies

$$rq_{i,N} \ln q_{i,N} + \left(\frac{\lambda_{-i}rl}{\lambda - \lambda_{-i}} + \lambda\right)q_{i,N} = \lambda e^{\frac{rl}{\lambda-\lambda_{-i}}}.$$

If $q_{i,N} = 1$, then

$$rq_{i,N} \ln q_{i,N} + \left(\frac{\lambda_{-i}rl}{\lambda - \lambda_{-i}} + \lambda\right)q_{i,N} = r\ln 1 + \frac{\lambda_{-i}rl}{\lambda - \lambda_{-i}} + \lambda < \frac{\lambda rl}{\lambda - \lambda_{-i}} + \lambda = \lambda\left(1 + \frac{rl}{\lambda - \lambda_{-i}}\right) < \lambda e^{\frac{rl}{\lambda-\lambda_{-i}}}.$$

Since the left-hand side is an increasing function of $q_{i,N}$, it implies that $q_{i,N} > 1 = q_{i,N+1}$.

For any n, we have

$$rq_{i,n} \ln q_{i,n} + \left(\frac{\lambda_{-i}rl}{\lambda - \lambda_{-i}} + \lambda\right)q_{i,n} = \lambda q_{i,n+1}e^{\frac{rl}{\lambda-\lambda_{-i}}}.$$

$$rq_{i,n-1} \ln q_{i,n-1} + \left(\frac{\lambda_{-i}rl}{\lambda - \lambda_{-i}} + \lambda\right)q_{i,n-1} = \lambda q_{i,n}e^{\frac{rl}{\lambda-\lambda_{-i}}}.$$

Then, $q_{i,n} > q_{i,n+1}$ implies that $q_{i,n-1} > q_{i,n}$. We have shown that $q_{i,N} > q_{i,N+1}$. By backward induction, we can proof that $q_{i,n} > q_{i,n+1}$ for all n.

When the agents have the logarithmic utility function, we have

$$C_{i,n}(v) = q_{i,n}e^v \text{ and } V_{i,n}(y) = C_{i,n}^{-1}(y) = \ln y - \ln q_{i,n}.$$

In case of success, the value of the risky security that can implement the agent i's contract increases from y to

$$Y_{i,n+1}(y) = C_{i,n+1}\left(C_{i,n}^{-1}(y) + \frac{rl}{\lambda - \lambda_{-i}}\right) = \frac{q_{i,n+1}}{q_{i,n}}e^{\frac{rl}{\lambda-\lambda_{-i}}}y = \left(\frac{r\ln q_{i,n}}{\lambda} + \frac{\lambda_{-i}rl}{\lambda(\lambda - \lambda_{-i})} + 1\right)y,$$

which is a linear function of y. Hence, the value of risky security rises by $\frac{r\ln q_{i,n}}{\lambda} + \frac{\lambda_{-i}rl}{\lambda(\lambda-\lambda_{-i})} + 1$ times when the project progresses from stage n to stage $n+1$. We could replicate the payoff of the security using a portfolio of the firm's equity and a risk-free asset with interest rate r, in which the fraction of equity in the portfolio $\beta_{i,n}$ satisfies

$$\beta_{i,n} \cdot R_n + (1 - \beta_{i,n}) \cdot 1 = \frac{r\ln q_{i,n}}{\lambda} + \frac{\lambda_{-i}rl}{\lambda(\lambda - \lambda_{-i})} + 1,$$

or equivalently

$$\beta_{i,n} = \frac{r \ln q_{i,n} + \frac{\lambda_{-i} r l}{\lambda - \lambda_{-i}}}{(R_n - 1)\lambda}.$$

Finally, $R_n \geq \frac{r \ln q_{i,n}}{\lambda} + \frac{\lambda_{-i} r l}{\lambda(\lambda - \lambda_{-i})} + 1$ guarantees that the security can always be created by the firm's equity.

Q.E.D.

As the single-agent case, the risky security can be created by a portfolio of the equity of a firm and a risk-free asset, where the fraction of equity, $\beta_{i,n}$, only depends on the stage level and exogenous parameters of the model. To implement the optimal contract, the principal requires that agent i invests $\beta_{i,n}$ fraction of his wealth in firm's equity when the project is at stage n.

Hidden saving

In the main body of the paper, we assume that the agent cannot engage in hidden saving. In the benchmark model, the contract determines a consumption path contingent on the agent's performance. At any point in time, the agent consumes all the payments from the principal and cannot save or borrow. In Section 4 and Section 5, although the agent chooses how much to consume, he can only invest in the state-contingent security for saving purpose. An important feature of the optimal contract in Section 3 is that the principal punishes the agent by cutting his consumption in case of unsatisfactory performance. A well-known result, first documented by Rogerson (1985), shows that the optimal contract is impracticable if the agent can save secretly due to a precautionary saving incentive.[14] Aware of the risk of lower compensation in case of failure, a risk-averse agent would save some of his income for consumption smoothing purpose. In some cases, the agent may adopt a double-deviation strategy by shirking to avoid the costs of working and saving secretly to smooth consumption, which makes the problem even more complicated for the principal.

To see how hidden saving affects the optimal contract derived in the main body of the paper, we first examine the case when the agent can save secretly at the same rate of return r as the principal. For illustration, consider the following one-period deviation in the discrete-time approximation. Suppose from time t to $t + \Delta t$, instead of working and consuming all the payments received from

[14]This problem only arises when the agent can save secretly. If the principal can monitor the agent's saving, then the principal can offer a contract contingent on the agent's saving.

the principal, the agent shirks and saves some of the payments at time t and consumes the saving at $t + \Delta t$. Because of shirking, the agent will fail. The marginal effect of shifting consumption in this way is

$$-ru'(c_t)\Delta t + \frac{1}{1 + r\Delta t}[ru'(\underline{c}_{t+\Delta t})\Delta t](1 + r\Delta t) = r\Delta t[u'(\underline{c}_{t+\Delta t}) - u'(c_t)] > 0.$$

The inequality is due to the result that the principal cuts the agent's compensation in case of failure so that $\underline{c}_{t+\Delta t} < c_t$ and the assumption that the agent is risk averse. This result suggests that if the agent shirks then he could receive higher utility through hidden saving. Under the consumption allocation of the optimal contract, the agent is indifferent to working or shirking because the incentive-compatibility condition is always binding. It further implies that if the agent shirks and shifts some consumption from the current period to the next period, his deviation payoff is higher than the payoff on the equilibrium path. Therefore, if the agent can save secretly at the same rate as the principal, the principal cannot punish the agent by cutting his compensation for unsatisfactory performance. Otherwise, the agent will adopt a double-deviation strategy, and the optimal contract becomes invalid. This result is similar to the observation in He (2012).

However, if the agent incurs a cost on account of hiding his saving, then the low return on hidden saving will mitigate the agent's precautionary saving incentive. If the return is considerable low, it may restore the optimality of the contract derived in the previous sections. Note that the agent's saving incentive depends on his marginal utility of consumption. To simplify the notation, we use m_t, where $m_t = U'(c_t)$, to denote the agent's marginal utility of consumption at any time t given the contract. Suppose the agent can save secretly at rate r'. The following proposition provides sufficient condition under which the agent has no incentive to save.

Proposition B.3 *Given contract $\{c_t(H^t), 0 < t < +\infty\}$, if in case of failure the agent's marginal utility of consumption satisfies*

$$\frac{d \ln m_t}{dt} \leq -(r' - r),$$

then the agent has no incentive to conduct hidden saving. At any point in time, he consumes all the payments from the principal and exerts efforts until the project is completed.

Proof of Proposition B.3: We show that under the condition in Proposition B.3 the agent will not engage in hidden saving by checking the agent's precautionary saving incentive at any time t. Since the contract punishes the agent by cutting his consumption in case of unsatisfactory performance, the lowest consumption path from time t to t' that the agent may receive is the one when he fails to complete any innovation during this period time. Since the agent's utility function is concave, he has the strongest incentive to save when he receives this "worst" consumption path. Thus, if we can show that the agent has no incentive to save even on this "worst" consumption path, then it implies that the agent has no incentive to save on any other consumption paths. The marginal cost of saving at time t equals m_t. Since the rate of return on hidden saving is r', the marginal benefit of saving at time t and consuming it at $t'(t' > t)$ is $e^{-r(t'-t)}e^{r'(t'-t)}m_{t'} = e^{(r'-r)(t'-t)}m_{t'}$. If in case of failure the agent's marginal utility of consumption satisfies

$$\frac{d\ln m_t}{dt} \le -(r'-r),$$

then on this "worst" consumption path

$$\ln m_{t'} - \ln m_t \le -(r'-r)(t'-t).$$

It implies that

$$\ln m_t \ge \ln m_{t'} + (r'-r)(t'-t).$$

Taking exponential to both sides, it becomes

$$m_t \ge e^{(r'-r)(t'-t)}m_{t'}.$$

Thus, the marginal cost of saving exceeds the marginal benefit, which implies that the agent has no incentive to save on the "worst" consumption path. It further implies that the agent has no incentive to saving on any other consumption paths. Therefore, if $\frac{d\ln m_t}{dt} \le -(r'-r)$ in case of failure, the hidden saving problem can be ignored. If the agent will not deviate from the consumption path offered by the principal, the incentive compatibility condition then guarantees that the agent will always exert effort. *Q.E.D.*

Proposition B.3 indicates that if the return on hidden saving is very low so that $r' \le r - \frac{d\ln m_t}{dt}$ for all $\{c_t(H^t), 0 < t < +\infty\}$, then the optimal contract in Section 3 is still optimal as the agent will not deviate from the consumption path suggested by the principal and always put effort at work. In a general setting, this sufficient condition is difficult to ascertain because it has to be held at any time t on all possible consumption paths. However, note that

$$\frac{d\ln m_t}{dt} = \frac{d\ln U'(c_t)}{dt} = \frac{U''(c_t)}{U'(c_t)}\frac{dc_t}{dt}.$$

If the agent utility function has CARA form, then $\frac{U''(c_t)}{U'(c_t)}$ is a constant number. It can be shown that $\frac{dc_t}{dt}$ is bounded.[15] Therefore, for CARA utility function, there exists an upper bound of r' such that the sufficient condition in Proposition B.3 is satisfied.

For logarithmic utility, we are able to derive a closed-form upper bound of r' that satisfies the sufficient condition in Proposition B.3.

Proposition B.4 *If the agent has logarithmic utility, the agent has no incentive to conduct hidden saving if save the rate on hidden saving is not higher than $r(1 - \ln p_1)$ for the single-agent case ($r' \leq r(1 - \ln q_{i,1}) - \frac{\lambda_{-i} rl}{\lambda - \lambda_{-i}}$ for all i for the multi-agent case), and hence he will not deviate from the consumption path offered by the principal.*

PROOF OF PROPOSITION B.4: For logarithmic utility function, the marginal utility from consumption satisfies

$$m = U'(c) = \frac{1}{c} = \frac{1}{e^u} = e^{-u}.$$

Hence, $\frac{d \ln m}{dt} = -\frac{du}{dt}$, and the no saving condition in Proposition B.3 becomes that $\frac{du}{dt} \geq r' - r$.

For the single-agent case, when the project is at stage n, we have $u = v + \ln p_n$, which implies that

$$\frac{du}{dt} = \frac{dv}{dt} = -r \ln p_n.$$

Thus, the no-saving condition becomes $-r \ln p_n \geq r' - r$, which implies $r' \leq r(1 - \ln p_n)$. Since $p_1 > p_n$ for any $1 < n \leq N$, $r' \leq r(1 - \ln p_1)$ guarantees that $r' \leq r(1 - \ln p_n)$ for all n. Hence, the agent has no incentive to conduct hidden saving if the rate on hidden saving is not higher than $r(1 - \ln p_1)$, and hence he will not deviate from the consumption path offered by the principal.

For the multi-agent case, we have

$$\frac{du}{dt} = \frac{dv}{dt} = -r \ln q_{i,n} - \frac{\lambda_{-i} rl}{\lambda - \lambda_{-i}}.$$

Hence, the no-saving condition becomes that $r' \leq r(1 - \ln q_{i,n}) - \frac{\lambda_{-i} rl}{\lambda - \lambda_{-i}}$. Similarly, $r' \leq r(1 - \ln q_{i,1}) - \frac{\lambda_{-i} rl}{\lambda - \lambda_{-i}}$ guarantees that $r' \leq r(1 - \ln q_{i,n}) - \frac{\lambda_{-i} rl}{\lambda - \lambda_{-i}}$ for all n. Hence, agent i has no incentive

[15] Note that $c_t = S(u_t)$ and u_t is determined by $S'(u_t) = C'(v_t)$. Hence, c_t is a continuous function of the continuation utility v_t. Proposition 3.1 shows that $\frac{dv_t}{dt} = r(v_t - u_t)$. Therefore, $\frac{dv_t}{dt}$ is also a continuous function of v_t. The highest level of continuation utility that the agent can achieve is $v_0 + \frac{Nrl}{\lambda}$ when the agent completes all N innovations instantly. Therefore, $\frac{dv_t}{dt}$ is bounded. This implies that $\frac{dc_t}{dt}$ is bounded because c_t is a continuous function of v_t.

to conduct hidden saving if the rate on hidden saving is not higher than $r(1 - \ln q_{i,1}) - \frac{\lambda_{-i} r l}{\lambda - \lambda_{-i}}$. *Q.E.D.*

This result is easier to observe from the aspect of implementation. For the single-agent case, the value of the security raises $\frac{r \ln p_n}{\lambda} + 1$ times when the project progresses from stage n to stage $n+1$, and its return in case of failure equals $r(1 - \ln p_n)$. Then, it is obvious that if the return on hidden saving is not higher than the lowest return on the security in case of failure, $r(1 - \ln p_1)$, then the agent will not have any incentive to engage in hidden saving and deviate from the optimal consumption path. A similar analysis applies to the multi-agent case. The interpretation of this result is that when the firm adopts equity-based compensation and the return on equity-based compensation is higher than the return on hidden saving, then the employees prefer to hold the equities for saving instead of saving secretly. Thus, the firms can almost mimic the optimal contract even if they cannot monitor their employees' hidden saving levels.

References

Anderson, M. C., R. D. Banker, and S. Ravindran (2000): "Executive Compensation in the Information Technology Industry," *Management Science*, 46(4), 530 – 547.

Bergemann, D., and U. Hege (2005): "The Finance of Innovation: Learning and Stopping," *Rand Journal of Economics*, 36(4), 719–752.

Bettis, J. C., J. Bizjak, J. L. Coles, and S. Kalpathy (2010): "Stock and Option Grants with Performance-based Vesting Provisions," *Review of Financial Studies*, 23(10), 3849–3888.

——— (2018): "Performance-vesting provisions in executive compensation," *Journal of Accounting and Economics*, 66(1), 194 – 221.

Biais, B., T. Mariotti, J.-C. Rochet, and S. Villeneuve (2010): "Large Risks, Limited Liability, and Dynamic Moral Hazard," *Econometrica*, 105(1), 73 – 118.

DeMarzo, P. M., and Y. Sannikov (2006): "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," *The Journal of Finance*, 61(6), 2681–2724.

Edmans, A., X. Gabaix, T. Sadzik, and Y. Sannikov (2012): "Dynamic CEO compensation," *The Journal of Finance*, 67(5), 1603–1647.

Green, E. J. (1987): "Lending and the Smoothing of Uninsurable Income," *Contractual Arrangements for Intertemporal Trade*, pp. 3–25.

Grochulski, B., and Y. Zhang (2011): "Optimal Risk Sharing and Borrowing Constraints in a Continuous-time Model with Limited Commitment," *Journal of Economic Theory*, 146, 2356 – 2388.

Halac, M., N. Kartik, and Q. Liu (2016): "Optimal contracts for experimentation," *The Review of Economic Studies*, 83(3), 1040–1091.

He, Z. (2009): "Optimal Executive Compensation when Firm Size Follows Geometric Brownian Motion," *The Review of Financial Studies*, 22(2), 859–892.

——— (2011): "A model of dynamic compensation and capital structure," *Journal of Financial Economics*, 100(2), 351–366.

——— (2012): "Dynamic compensation contracts with private savings," *Review of Financial Studies*, 25(5), 1494–1549.

Holmstrom, B. (1982): "Moral Hazard in Teams," *The Bell Journal of Economics*, 13(2), 324–340.

Hopenhayn, H. A., and J. P. Nicolini (1997): "Optimal Unemployment Insurance," *The Journal of Political Economy*, 105(2), 412–438.

Hörner, J., and L. Samuelson (2013): "Incentives for Experimenting Agents," *The RAND Journal of Economics*, 44(4), 632–663.

Ittner, C. D., R. A. Lambert, and D. F. Larcker (2003): "The structure and performance consequences of equity grants to employees of new economy firms," *Journal of Accounting and Economics*, 34, 89 – 127.

Kocherlakota, N. (2005): "Zero Expected Wealth Taxes: A Mirrlees Approach to Dynamic Optimal Taxation," *Econometrica*, 73(5), 1587–1621.

Manso, G. (2011): "Motivating Innovation," *Journal of Finance*, 66(5), 1823–1860.

Murphy, K. J. (2003): "Stock-based pay in new economy firms," *Journal of Accounting and Economics*, 34, 129 – 147.

Myerson, R. B. (2015): "Moral hazard in high office and the dynamics of aristocracy," *Econometrica*, 83(6), 2083–2126.

Reinganum, J. F. (1984): "Practical Implications of Game Theoretic Models of R&D," *The American Economic Review*, 74(2), 61–66.

Rogerson, W. P. (1985): "Repeated Moral Hazard," *Econometrica*, 53(1), 69 – 76.

Sannikov, Y. (2008): "A Continuous-Time Version of the Principal-Agent Problem," *The Review of Economic Studies*, 75(3), 957–984.

Sesil, J. C., M. K. Kroumova, J. R. Blasi, and D. L. . Kruse (2002): "Broad-based Employee Stock Options in US 'New Economy' Firms," *British Journal of Industrial Relations*, 42(2), 273 – 294.

Shan, Y. (2017): "Optimal contracts for research agents," *The RAND Journal of Economics*, 48(1), 94–124.

Spear, S. E., and S. Srivastava (1987): "On Repeated Moral Hazard with Discounting," *The Review of Economic Studies*, 54(4), 599–617.

Stokey, N. L. (2008): *The Economics of Inaction: Stochastic Control models with fixed costs.* Princeton University Press.

Stokey, N. L., and R. E. Lucas (1989): *Recursive Mathods in Economic Dynamics.* Harvard University Press.

Sun, P., and F. Tian (2017): "Optimal contract to induce continued effort," *Management Science.*

Whittlesey, F. (2007): *The New Era of Equity Compensation Design: Performance Plans.* Salary.com.