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SECONDARY PROCESSES INDUCED BY FINITE BIRTH-AND-DEATH PROCESSES

by

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SUMMARY.

The aim of this thesis is to present a unified study of some of the fundamental secondary processes induced by a finite birth-and-death process. The motivation for the study comes from queueing theory, since a number of elementary queueing models of finite capacity can be viewed as finite birth-and-death processes. However, so as not to limit the scope of the analysis, the study is approached more abstractly using the formal stochastic process, the finite birth-and-death process.

The overflow process is an obvious, and very important, induced secondary process, and the early portion of the thesis is devoted to its study. Orthogonal polynomial theory is used extensively in the analysis of the inter-overflow time distribution, and applications of the results to queueing theory are discussed.

A feature of the thesis is the discovery in the final section of chapter 3 of a new characterisation of the hyperexponential family of distributions.

In chapter 4 we regard arrivals which cause the process to enter some prescribed state as constituting a secondary process. The analysis thus reveals results concerning the time between successive entries to the boundary states and also intermediate states.

The final section of chapter 4 demonstrates how a number of isolated results in the literature concerning first passage times can be derived as corollaries to our analysis.

A similar initial approach can be adopted to analyse the overflow from certain queueing models which are not finite birth-and-death processes. This is demonstrated in chapter 5 where the overflow from the single server queue with finite waiting space and renewal input is discussed.

SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university. To the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text of the thesis.

Alan John Branford.

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CHAPTER 1. INTRODUCTION.

1.1 Introduction.

1.2 The Birth-and-Death Process.

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1.1 INTRODUCTION

It is appropriate to begin a study of secondary processes induced by finite birth-and-death processes by discussing the motivation for such a study. A number of questions concerning queueing models of finite capacity can be interpreted as or related to secondary processes. When considering a finite capacity queueing model there is the inevitable question of overflow: what is its nature and how is it affected by the parameters of the model? However the overflow is a secondary process induced by the queueing process.

Aside from this obvious example, there are several more subtle applications. One can regard the arrivals which cause the model to enter some prescribed state (or set of states) as events of a secondary process. Thus problems related to first passage times, sojourn times, busy periods and so on can be viewed in the light of induced secondary processes.

However the scope of the study would be restricted were it approached simply from a queueing theory viewpoint. Thus, although we draw on queueing theory considerations for motivation (and at times terminology), the study is approached more abstractly using a formal stochastic process, the finite birth-and-death process. The aim of the thesis is therefore to present a unified study of some of the fundamental secondary processes induced by a finite birth-and-death process. The specific queueing theoretic applications will only be mentioned, or at most used as illustrations or examples. The only exception to this occurs in chapter 5 where a related problem is in fact approached from the queueing theory aspect.

Throughout this thesis formal statements of definitions, theorems, proofs, et cetera, are terminated with the symbol ■, so as to distinguish them from the main text.

The remainder of this chapter is devoted to discussions of certain preliminary concepts.

1.2 THE BIRTH-AND-DEATH PROCESS.

The analysis of a specific queueing model can often be achieved by associating it with some class of formal processes within the general theory of stochastic processes. Results proved for the general class of processes can then be directly applied to the model.

Birth-and-death processes form one important class of stochastic processes, and a substantial portion of this thesis is devoted to investigating certain secondary processes associated with finite birth-and-death processes. Examples are given in section 3.2 to illustrate how the general theory which we will develop can be applied to certain specific queueing models. The principal reference for this section is Feller [1968].

1.2.1 Definition. A *birth-and-death process* is a continuous time, discrete state process which obeys the following postulates:

Denote the state space of the system by $\mathcal{N}$ (as $\mathcal{N}$ must be finite or countably infinite we can without loss of generality take $\mathcal{N} = \{0,1,2,...,N\}$ or $\mathcal{N} = \{0,1,2,...\}$ respectively). Suppose at some time t the process is in state $j \in \mathcal{N}$.

(i) The conditional probability that during the period $(t,t+h)$ the transition $j \to (j+1)$ occurs equals

$$\lambda_j h + o(h), \quad \text{as} \quad h \to 0,$$

if $j+1 \in \mathcal{N}$, where $\lambda_j > 0$ depends only on j.

(ii) The conditional probability that during the period $(t,t+h)$ the transition $j \to (j-1)$ occurs equals

$$\mu_j h + o(h), \quad \text{as} \quad h \to 0,$$

if $j-1 \in \mathcal{N}$, where $\mu_j > 0$ depends only on j.

(iii) The probability that during the period $(t,t+h)$ the transition $j \to k$ occurs (where $k \in N$, $k \neq j+1$, $k \neq j-1$) is

$$o(h), \quad \text{as} \quad h \to 0.$$

We call the elements of the set $\{\lambda_j ; j,j+1 \in N\}$ *birth rates* and those of the set $\{\mu_j ; j,j-1 \in N\}$ *death rates*. We will use the term *finite birth-and-death process* to indicate that the state space of the process is finite. ∎

The most important property of a birth-and-death process is that it exhibits a memoryless nature. The future behaviour of the system subsequent to time t depends only on the state of the system at time $t+0$. (We have written $t+0$ to cover the cases when an event occurs at t.) We shall now write this property formally for later reference.

<u>1.2.2 The Memoryless Property</u>. The future behaviour of a birth-and-death process from some time t depends only on the state of the process at time $t+0$. ∎

1.3 ORTHOGONAL POLYNOMIALS.

In later chapters we will make use of orthogonal polynomials and their properties as a tool in the analysis. In this section some elementary theory of orthogonal polynomials will be discussed as a preliminary to this later work. The principal references for this discussion are Szegö [1939], Erdélyi, et. al., [1953] and Chihara [1978].

1.3.1 Definition. (See Szegö [1939], section 2.2)

Let $\psi(x)$ be some given non-decreasing function, with infinitely many points of increase, and suppose the *moments*

$$m_n = \int_{-\infty}^{\infty} x^n \, d\psi(x), \quad n=0,1,2,..., \tag{3.1}$$

exist as real numbers. Then a sequence of polynomials $(p_n(x))_{n=0}^{\infty}$ is an *orthogonal polynomial sequence* for the *distribution function* $\psi(x)$ if and only if

(i) $p_n(x)$ is a polynomial of exact degree n, $n=0,1,2,...,$

(ii) $\int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, d\psi(x) = 0$, for $m \neq n$,

(iii) $\int_{-\infty}^{\infty} [p_n(x)]^2 \, d\psi(x) \neq 0$, $n=0,1,2,...$.

■

The condition that $\psi(x)$ has infinitely many points of increase is equivalent to the condition that the set

$$G(\psi) = \{x; \psi(x+\delta)-\psi(x-\delta) > 0 \text{ for all } \delta > 0\}, \tag{3.2}$$

called the *spectrum* of ψ, be infinite (see Chihara [1978], p. 51).

Given any distribution function $\psi(x)$, then there exists a sequence of polynomials $(p_n(x))_{n=0}^{\infty}$ which is orthogonal with respect to $\psi(x)$ (Szegö [1939], section 2.2(1)), and each $p_n(x)$ is uniquely determined up to an arbitrary non-zero factor (Chihara [1978], p. 9).

That is, if $(\tilde{p}_n(x))_{n=0}^{\infty}$ is also an orthogonal polynomial sequence with respect to $\psi(x)$, then there exist constants $\kappa_n \neq 0$ such that

$$\tilde{p}_n(x) = \kappa_n p_n(x), \quad n=0,1,2,..., \tag{3.3}$$

and conversely given any sequence of constants $(\kappa_n; \kappa_n \neq 0)_{n=0}^{\infty}$ then $(\kappa_n p_n(x))_{n=0}^{\infty}$ is an orthogonal polynomial sequence with respect to $\psi(x)$.

We note that, if the sequence $(p_n(x))_{n=0}^{\infty}$ is orthogonal with respect to $\psi(x)$, then it is also orthogonal with respect to $\kappa\psi(x)$ for any positive constant κ (Chihara [1978], p. 10). However, since $\psi(x)$ is non-decreasing and has infinitely many points of increase, we have that

$$\int_{-\infty}^{\infty} d\psi(x) > 0 ,$$

and so there exists $\kappa > 0$ such that

$$\int_{-\infty}^{\infty} d[\kappa\psi(x)] = 1 .$$

Thus, without loss of generality, we can demand $\psi(x)$ to be such that

$$\int_{-\infty}^{\infty} d\psi(x) = 1 \tag{3.4}$$

in the definition 1.3.1 of orthogonal polynomials. In the remainder of this thesis we shall therefore assume that (3.4) holds when referring to a distribution function.

Suppose $\tilde{\psi}(x)$ is a distribution function for which there exists a constant C such that

$$\tilde{\psi}(x) = \psi(x) + C$$

at all common points of continuity. We say that $\psi(x)$ and $\tilde{\psi}(x)$ are *substantially equal* (Chihara [1978], p. 52), since $\psi(x)$ can be replaced by $\tilde{\psi}(x)$ in any Lebesgue-Stieltjes integral over $(-\infty,\infty)$ with respect to $\psi(x)$, for a continuous integrand, without affecting the value of the integral. For this reason, if the sequence $(p_n(x))_{n=0}^{\infty}$ is orthogonal with respect to $\psi(x)$, then it is also orthogonal with respect to $\tilde{\psi}(x)$.

A sequence $(p_n(x))_{n=0}^{\infty}$ of orthogonal polynomials is called *determinate* if all distribution functions with respect to which it is orthogonal are substantially equal. The question of determinacy is related to the "problem of moments" which is extensively discussed in the literature. We shall not investigate the matter further, save pointing out that there do exist sequences $(p_n(x))_{n=0}^{\infty}$ which are orthogonal with respect to (infinitely many) substantially unequal distribution functions (Chihara [1978], p. 58).

It should be pointed out that Chihara [1978] uses a more general approach to orthogonal polynomials by replacing the Lebesgue-Stieltjes integral in definition 1.3.1 with a general linear functional with finite moments. Our definition of orthogonal polynomials corresponds precisely to a subclass of such linear functionals which Chihara calls "positive-definite" (Chihara [1978], chapter 2).

Orthogonal polynomials have a wealth of properties, particularly involving their zeros. The particular properties and results which we will require for our analysis will be stated and referenced in the text where they are used.

1.4 THE CONCEPT OF TRAFFIC.

When considering queueing models one is concerned with the interaction of an arriving stream of calls(*) and a group of trunks(*) to which it is offered. It is thus desirable to have some measure of the extent to which the arriving calls attempt to "work" the system, and also a measure of the system's ability to handle this "work". Intuitively, one would think of these two related concepts as the "traffic" offered to the system, and the "traffic" carried by the system respectively. This section formalises the concept of traffic.

It is important to realise that any concept of traffic is not inherent to the arriving stream of calls. Moreover, traffic is brought about by the arriving calls' interaction with the equipment installation, and so must be dependent on both.

We will restrict our attention to queueing processes which possess and are in statistical equilibrium. That is, all transient effects can be ignored.

1.4.1 Definition. The *offered traffic* a to the system in equilibrium is the mathematical expectation of the number of arrivals during a time period equal to the average holding time. That is,

$$a = \frac{\lambda}{\mu}, \tag{4.1}$$

where λ is the mean arrival rate to the system, and $\frac{1}{\mu}$ is the mean holding time. ∎

(*) In this section we will use the term "calls" to represent the components of the input process, and the term "trunks" to represent the components of the service installation.

<u>1.4.2 Definition.</u> The *carried traffic* a' by the system in equilibrium is the mathematical expectation of the number of busy trunks. That is,

$$a' = \sum_k kP_k \ , \qquad (4.2)$$

where P_k is the probability of k trunks being occupied at a randomly selected time point. ∎

Both these definitions seem reasonable in terms of the intuitive ideas they endeavour to formalise. However, although one would expect there to be a simple relationship between offered and carried traffic, the definitions of these two quantities, in their present form, do not indicate any such relationship.

The definitions have been presented in the forms in which they are most commonly used. Intuitively, one would expect the carried traffic to be the proportion of offered traffic that is not lost. This statement appears tautological, but, as the definitions stand, requires proof.

<u>1.4.3 Lemma.</u> For a queueing process in equilibrium

$$a' = (1-P_L)a \ , \qquad (4.3)$$

where P_L is the probability that an arriving call is lost. ∎

The proof of this lemma follows readily from Little's Result, which states that, in any equilibrium process,

$$\bar{N}_S = \lambda_S T_S \ , \qquad (4.4)$$

where $\bar{N}_S$ is the mean number of calls in the system, λ_S is the mean rate at which calls enter the system, and T_S is the mean time spent in the system by a call. This result has existed for many years, but was first rigorously proved by Little [1961]. The exact definition of "system" is left flexible, provided all three parameters are for the same "system".

Proof of 1.4.3. Consider the group of trunks, and let this be the "system" of Little's Result. We have immediately $a' = \bar{N}_S$ (by definition) and $\frac{1}{\mu} = T_S$. The mean rate at which calls arrive at the "system" is the mean rate at which *successful* calls arrive, since the "system" is just the trunk group. That is,

$$\lambda_S = \lambda(1-P_L) .$$

Substitution into Little's Result yields

$$a' = \frac{\lambda}{\mu}(1-P_L) ,$$

and so, by definition 1.4.1,

$$a' = a(1-P_L) . \qquad \blacksquare$$

It is worth noting that Little's Result can be similarly used to show that the offered traffic is equal to the expected number of occupied trunks, if there were infinitely many trunks available.

CHAPTER 2. A SURVEY OF RELATED LITERATURE.

CHAPTER 2. A SURVEY OF RELATED LITERATURE.

As previously indicated, the study of secondary processes induced by finite birth-and-death processes has applications to a wide class of finite capacity queueing models, particularly concerning their overflow stream. This will be demonstrated in later chapters, but it is appropriate to begin with a brief overview of some of the principal works which deal with the overflow problem from a queueing theory viewpoint.

One of the pioneering analytical investigations into the overflow from finite capacity queueing models appeared in Kosten [1937]. Kosten [1937] considers two groups of identical trunks with negative exponential holding times. The first or primary group is finite and is offered a Poisson stream of arriving calls. Calls which cannot be accommodated on this primary group engage a trunk on the secondary or overflow group, which is infinite. Kosten [1937] obtains an explicit formula for the joint distribution of occupancy on the primary and secondary groups, from which is obtained the marginal distribution and the first two moments of the occupancy on the secondary group. A version of the analysis of Kosten [1937] in English can be found in Cooper [1972] (pp. 113-119).

Brockmeyer [1954] examines the model of Kosten [1937] in the case when the overflow group is finite, and obtains the joint probability distribution for this case. A summary of the results of Kosten [1937] and Brockmeyer [1954] and some additional formulae are given in Wallström [1966] (pp. 202-209). Chastang [1963] also examines questions related to the Kosten and Brockmeyer models.

A direct derivation of the mean and variance of the occupancy on the overflow group for the Kosten model is given in Riordan's appendix to Wilkinson [1956]. These formulae form the basis for Wilkinson's Equivalent Random Method first proposed in Wilkinson [1956]. Schehrer [1976], also using a joint probability distribution approach, determines the higher order moments of the overflow from the Kosten and Brockmeyer models.

Investigations of the overflow stream and its effect on the overflow group for the Kosten and Brockmeyer models with general renewal input have been made by Takács [1959] and Potter [1979]. A summary of some of the main formulae can be found in Pearce and Potter [1977] . See also Palm [1943] and Syski [1960] (section 3.1 of chapter 5).

Kuczura [1973] approximates the overflow from a system with Poisson input by an "interrupted Poisson process"; this is a process consisting of a Poisson stream which is alternately switched on for a negative exponentially distributed time and then switched off for an independent negative exponentially distributed time.

Numerous studies of the overflow problem from a queueing theory viewpoint, both analytical and approximate, have appeared in the literature. Our discussion here attempts only to mention some of the more relevant studies.

As mentioned earlier, we will be using a sequence of orthogonal polynomials associated with a finite birth-and-death process as a tool in the analysis. Karlin and McGregor, in a series of papers, also use an orthogonal polynomial sequence in their analysis of an infinite birth-and-death process, and related problems. A number of comparisons and comments can be made, and so we now give a survey of these papers.

Karlin and McGregor [1955] serves mainly as an introduction and a promise of results to come. Karlin and McGregor [1957a] develops the basic theory on which the later work rests, and so is of fundamental importance to the series of papers. Using their notation, define $P_{ij}(t)$ to be the probability that the process will be in state j at time $\tau+t$, given that the state at time τ was i. Then, recalling definition 1.2.1,

$$P_{i,i+1}(t) = \lambda_i t + o(t) ,$$

$$P_{ii}(t) = 1 - (\lambda_i + \mu_i)t + o(t) ,$$

$$P_{i,i-1}(t) = \mu_i t + o(t) ,$$

$$\text{as} \quad t \to 0, \quad i=0,1,2,\ldots .$$

(A number of the results in the series of papers can allow $\mu_0 > 0$, and the transition $0 \to -1$ is interpreted as an absorption into some state -1.) Elementary theory of Markov processes results in the following matrix equation for the infinite matrix $P(t) = (P_{ij}(t))$:

$$P'(t) = AP(t) , \quad t \geqslant 0 , \tag{0.1}$$

where A is the (infinite) matrix

$$A = \begin{bmatrix} -(\lambda_0+\mu_0) & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix} . \tag{0.2}$$

(Equation (1.1) of Karlin and McGregor [1957a] .)

Other equations involving $P(t)$ also follow from elementary theory. The main thrust of this paper is to look for an integral representation of $P(t)$ in terms of the eigenvectors of A.

Karlin and McGregor introduce an orthogonal polynomial sequence $(Q_n(x))_{n=0}^{\infty}$ by means of the equation

$$-xQ = AQ \;, \quad Q_0(x) \equiv 1 \;, \tag{0.3}$$

where $Q = [Q_0(x)\ Q_1(x)\ Q_2(x)\ \ldots]^T$. The paper derives the following integral representation for $P_{ij}(t)$:

$$P_{ij}(t) = \pi_j \int_0^{\infty} e^{-xt} Q_i(x) Q_j(x)\, d\psi(x) \;, \tag{0.4}$$

where $\psi(x)$ is a distribution function with respect to which the sequence $(Q_n(x))_{n=0}^{\infty}$ is orthogonal, and

$$\pi_0 = 1, \quad \pi_j = \frac{\lambda_0\lambda_1\ldots\lambda_{j-1}}{\mu_1\mu_2\ldots\mu_j} \quad (j=1,2,3,\ldots) \;.$$

In order to establish (0.4), and also to answer questions related to the existence and uniqueness of $P(t)$, the paper analyses in depth the orthogonal polynomial sequence $(Q_n(x))_{n=0}^{\infty}$ and its associated Stieltjes moment problem. Some of the results are also of independent analytical interest.

Karlin and McGregor [1957b] uses the results of Karlin and McGregor [1957a] "to establish equivalences between properties of the stochastic process and properties of the sequences $\{\lambda_n\},\{\mu_n\}$, and to evaluate, in terms of these sequences, some of the interesting probabilistic quantities associated with the process". Certain ergodic theorems are proved, and the problem of computing $\psi(x)$, given the matrix A, is discussed.

Karlin and McGregor [1958a] applies the theory developed in the previous cases to many server queueing processes with Poisson input and exponential service times (with infinite waiting space for blocked customers). Some of the problems tackled are as follows:

"(1) to obtain a usable formula for the transition probability $P_{ij}(t)$;

(2) to compute the distribution of the length of a busy period;

(3) to compute the distribution of the number of customers served during a busy period;

(4) to compute the distribution of the maximum length of the queue during a busy period; and similar questions."

Karlin and McGregor [1958b] and [1959a] consider linear growth birth-and-death processes and random walks respectively.

Karlin and McGregor [1959b] proves a theorem which finds conditions expressed in terms of the analytic properties of the transition probability function which are equivalent to continuity of the path functions for a wide class of stationary Markov processes whose state space is the set of non-negative integers. As a side remark the paper mentions that, in the special case of a birth-and-death process, the Laplace-Stieltjes transform $F^*_{ij}(s)$ of the distribution function for the length of time until the next entry to state j from an epoch at which the system is in state i, where $i < j$, (the "first passage time from i to j") is given by

$$F^*_{ij}(s) = \frac{Q_i(-s)}{Q_j(-s)} , \qquad i < j . \qquad (0.5)$$

Karlin and McGregor [1959b] finds this result as a corollary to some formulae of Karlin and McGregor [1957a] involving $P_{ij}(t)$. However as a bye-product of some of our preliminary analysis we will find a more direct derivation of formula (0.5). This will be discussed in section 4.4.

In section 3.1 we will observe a simple relationship between the orthogonal polynomial sequence $(Q_n(x))_{n=0}^{\infty}$ and the orthogonal polynomial sequence which we will introduce. For the moment we note that equation (0.3) (assuming $\mu_0=0$) can be written as

$$\lambda_n Q_{n+1}(x) + (x-\lambda_n-\mu_n)Q_n(x) + \mu_n Q_{n-1}(x) = 0, \qquad n=1,2,3,\ldots, \tag{0.6}$$

$$Q_0(x) \equiv 1, \qquad Q_1(x) = \frac{1}{\lambda_0}(-x+\lambda_0) .$$

Karlin and McGregor [1959c] and its companion paper [1959d] consider coincidence properties of birth-and-death processes.

Finally, we discuss some work of Keilson which relates both to some of our analysis and to the work of Karlin and McGregor. Keilson [1979] (section 3.3) obtains a spectral representation for the transition probabilities in a time-reversible ergodic chain. A birth-and-death process is one of the simplest examples of a time-reversible ergodic chain, and so this work is seen to be an extension of the ideas of Karlin and McGregor.

Keilson [1979] (section 3.5B), using his spectral representation, finds an expression for the first passage time density from a state n to its neighbouring state $n+1$ for a birth-and-death process. This expression is a weighted sum of exponentials. As with the expression (0.5) of Karlin and McGregor, our analysis in section 3.1 will provide an alternative derivation of this result.

Keilson [1979] again considers first passage time densities for birth-and-death processes in his chapter 5. The connection between this work and our approach will be indicated in the text of section 3.1 and also in section 4.4.

CHAPTER 3. THE OVERFLOW FROM A FINITE BIRTH-AND-DEATH PROCESS.

3.1 THE OVERFLOW FROM A FINITE BIRTH-AND-DEATH PROCESS.

In this section we consider a stream of secondary events associated with a general finite birth-and-death process. The system is characterised by N states (where N is a positive integer), labelled $0,1,2,\ldots,N-1$. When the system is in some state n $(0 < n < N-1)$ births occur at a rate λ_n, and independently deaths occur at a rate μ_n. The system must stay within the prescribed state space, and so in state 0 only births may occur, with rate λ_0, and in state $N-1$ only deaths may occur, with rate μ_{N-1}. We assume $\lambda_n > 0$ for $0 \leqslant n < N-1$ and $\mu_n > 0$ for $0 < n \leqslant N-1$.

We will impose an additional structure on the birth-and-death process by allowing *overflows* to occur, with rate $\lambda_{N-1} > 0$, when the system is in state $N-1$. These overflows do not cause a change of state, but merely constitute a stream of secondary events, which we will refer to as the *overflow stream* from the finite birth-and-death process. It should be noted that this secondary stream is not simply a Poisson stream, since overflows can only occur when the system is in state $N-1$. We note also that the occurrence of overflows in state $N-1$ in no way affects the behaviour of the underlying birth-and-death process.

For reasons of the obvious physical interpretations, we shall refer to births and overflows collectively as *arrivals*, and so $\lambda_0,\lambda_1,\ldots,\lambda_{N-1}$ are the *arrival rates*. The nature and properties of the overflow stream will now be studied.

An important initial observation is that the overflow stream, although not Poisson, is in fact renewal. This can be readily seen by noting that the future behaviour of the system subsequent to an overflow event is dependent only on the fact that the system is in state N-1 (recall the memoryless property 1.2.2). The overflow stream can thus be characterised by the probability distribution function of the inter-event time, which we will denote by $F(t)$. Our examination of the overflow stream can be achieved by investigating $F(t)$.

Denote by $\{\tau_0^{(n)} < \tau_1^{(n)} < \tau_2^{(n)} < \ldots\}$, where $0 \leqslant n \leqslant N-1$, the random epochs when the system is in state n and an arrival occurs(*). Define $f_n(t)$ $(n=0,1,\ldots,N-1)$ to be the probability distribution function for the time to the next overflow from an instant $\tau_k^{(n)} - 0$. (The memoryless property 1.2.2 gives that $f_n(t)$ $(n=0,1,\ldots,N-1)$ is independent of k.) Then we have trivially that

$$f_{N-1}(t) = u(t) , \tag{1.1}$$

where $u(t)$ is the unit-step or Heaviside function defined by

$$u(t) = \begin{cases} 0 , & \text{if } t < 0 \\ 1 , & \text{if } t \geqslant 0 \end{cases} . \tag{1.2}$$

We write $f_{-1}(t)$ for the probability distribution function for the time until the next overflow from an arbitrary epoch at which the system is in state 0 and no arrival is just about to take place. The memoryless property 1.2.2 gives that $f_{-1}(t)$ is also well-defined.

Suppose that the system is in state n $(0 \leqslant n < N-1)$ and consider an epoch $\tau_k^{(n)}$ (some $k \geqslant 0$). Note that the system will be in state $n+1$ at time $\tau_k^{(n)} + 0$. We are to find an expression for

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(*) Thus the inter-overflow time is the length of the time interval $[\tau_k^{(N-1)}, \tau_{k+1}^{(N-1)})$ (some $k \geqslant 0$).

$f_n(t)$, which is the probability of the event that the next overflow subsequent to time $\tau_k^{(n)} - 0$ occurs by time $\tau_k^{(n)} + t$. Since the overflow itself corresponds to an arrival, and moreover cannot be the arrival at $\tau_k^{(n)}$ itself (recall that $0 \leqslant n < N-1$), then the desired event can only occur if there is at least one arrival or death subsequent to time $\tau_k^{(n)} + 0$ but by time $\tau_k^{(n)} + t$. We shall consider separately the two cases corresponding to which occurs first.

Suppose that the next event to occur subsequent to time $\tau_k^{(n)} + 0$ is an arrival in the time interval $(\tau_k^{(n)}+y, \tau_k^{(n)}+y+dy)$, where $y+dy \leqslant t$. The probability of this occurrence, noting the independence of arrivals and deaths, is simply

$$\lambda_{n+1} \exp(-\lambda_{n+1} y - \mu_{n+1} y) dy + o(dy) .$$

The time which now remains for the next overflow to occur in is $t-y+0(dy)$, and the system, having been in state $n+1$ since time $\tau_k^{(n)} + 0$, is just about to have an arrival occur. Thus the probability that the first event subsequent to time $\tau_k^{(n)} + 0$ is an arrival and the next overflow occurs by time $\tau_k^{(n)} + t$ is given by

$$\int_0^t \exp(-\lambda_{n+1} y - \mu_{n+1} y) \lambda_{n+1} f_{n+1}(t-y) dy , \qquad (1.3)$$

$$n=0,1,\ldots,N-2.$$

Similarly, suppose that the first event subsequent to time $\tau_k^{(n)} + 0$ is a death occurring in the time interval $(\tau_k^{(n)}+y, \tau_k^{(n)}+y+dy)$, where $y+dy \leqslant t$. The system, having been in state $n+1$ since time $\tau_k^{(n)} + 0$, now is just about to return to state n. But the time from this epoch until the next overflow is stochastically indistinguishable from the time until the next overflow from an epoch at which the system is in state $n-1$ and an arrival is just about to occur (or, in the case of $n=0$, from an epoch at which the system is in state 0 and no arrival is just about to occur); this follows from the memoryless property 1.2.2 since the future behaviour after either the transition $(n+1) \to n$ or the transition $(n-1) \to n$ depends only on the fact that the system is now in state n. Thus the probability that the first event subsequent to time $\tau_k^{(n)} + 0$ is a death and the next overflow occurs by time $\tau_k^{(n)} + t$ is given by

$$\int_0^t \exp(-\lambda_{n+1} y - \mu_{n+1} y) \mu_{n+1} f_{n-1}(t-y)\,dy \,, \qquad n=0,1,\ldots,N-2 \,. \tag{1.4}$$

The probabilities (1.3) and (1.4) correspond to two mutually exclusive but exhaustive events, and so

$$f_n(t) = \int_0^t \exp(-\lambda_{n+1} y - \mu_{n+1} y) [\lambda_{n+1} f_{n+1}(t-y) + \mu_{n+1} f_{n-1}(t-y)]\,dy \,, \quad t \geq 0, \qquad n=0,1,\ldots,N-2. \tag{1.5}$$

Consider now an epoch at which the system is in state 0 and no arrival is just about to occur. In this case the next subsequent event must be an arrival, and so

$$f_{-1}(t) = \int_0^t \exp(-\lambda_0 y) \lambda_0 f_0(t-y)\,dy, \quad t \geq 0. \tag{1.6}$$

We note that Keilson [1979] (p. 58) obtains an expression for the Laplace transform of the first passage time density from state n to state $(n+1)$ in terms of that from state $(n-1)$ to state n. Keilson's approach can thus be seen to be a special case of our method.

Although the nature of the overflow from the finite birth-and-death process can be deduced from a consideration of first passage time densities from states to their higher neighbour, we have adopted the more general approach for two reasons:

(1) it will facilitate our later analysis;

(2) it will provide the key for the connection of the result (0.5) of chapter 2 with other results concerning first passage times. (This will be dealt with in section 4.4.)

Note that the integrals on the right-hand-side of equations (1.5) and (1.6) are (Lebesgue-Stieltjes) convolutions, and so, in terms of the Laplace-Stieltjes transforms

$$f_n^*(x) = \int_0^\infty \exp(-xt)\,df_n(t) \,, \qquad (1.7)$$

$$\text{Re } x \geqslant 0,\ n=-1,0,1,\ldots,N-1,$$

these equations read

$$f_{-1}^*(x) = \frac{\lambda_0}{\lambda_0+x} f_0^*(x) \qquad (\text{Re } x \geqslant 0) \,, \qquad (1.8)$$

$$f_n^*(x) = \frac{1}{\lambda_{n+1}+\mu_{n+1}+x}[\lambda_{n+1} f_{n+1}^*(x)+\mu_{n+1} f_{n-1}^*(x)] \,, \qquad (1.9)$$

$$\text{Re } x \geqslant 0,\ n=0,1,2,\ldots,N-2 \,.$$

A simple rearrangement of equations (1.8) and (1.9) gives the following system of recurrence relations:

$$\lambda_0 f_0^*(x) - (x+\lambda_0) f_{-1}^*(x) = 0 \qquad (\text{Re } x \geqslant 0) \,, \qquad (1.10)$$

$$\lambda_{n+1} f_{n+1}^*(x) - (x+\lambda_{n+1}+\mu_{n+1}) f_n^*(x) + \mu_{n+1} f_{n-1}^*(x) = 0 \,, \qquad (1.11)$$

$$\text{Re } x \geqslant 0,\ n=0,1,2,\ldots,N-2 \,.$$

Favard's Theorem (see Favard [1935] or Chihara [1978], pp.21-2) states that a set of relations of the form

$$p_{n+1}(x) - (A_n x+B_n)p_n(x) + C_n p_{n-1}(x) = 0 ,$$
$$n=1,2,3,\ldots,$$

with $A_n, C_n > 0$ and initial specifications

$$p_0(x) \equiv 1 , \ p_1(x) = A_0 x+B_0 \quad (A_0 > 0) ,$$

defines a sequence $(p_n(x))_{n=0}^{\infty}$ of orthogonal polynomials, $p_n(x)$ being of exact degree n in x. Thus the relations

$$\lambda_n p_{n+1}(x) - (x+\lambda_n+\mu_n)p_n(x) + \mu_n p_{n-1}(x) = 0 ,^{(*)} \qquad n=1,2,\ldots, \qquad (1.12)$$
$$p_0(x) \equiv 1, \qquad p_1(x) = \frac{1}{\lambda_0}(x+\lambda_0),$$

define such a sequence $(p_n(x))_{n=0}^{\infty}$ of orthogonal polynomials.

From equations (1.10) and (1.11) we therefore have that

$$f_n^*(x) = \alpha(x)p_{n+1}(x) , \qquad n=-1,0,1,\ldots,N-1 , \qquad (1.13)$$

for some function $\alpha(x)$ independent of n. The supplementary condition (1.1), or equivalently the condition

$$f_{N-1}^*(x) = 1 ,$$

acts as a boundary condition which fixes

$$\alpha(x) = \frac{1}{p_N(x)} .$$

The relation (1.13) therefore reads

$$f_n^*(x) = \frac{p_{n+1}(x)}{p_N(x)} , \qquad n=-1,0,1,\ldots,N-1 . \qquad (1.14)$$

(*) For $n > N-1$ take λ_n, μ_n to be any positive real numbers for the sake of the definition (1.12) of $(p_n(x))_{n=0}^{\infty}$. The polynomials $p_{N+1}(x)$, $p_{N+2}(x), \ldots$ are irrelevant to the problem.

The purpose for the introduction of the functions $f_n(t)$ $(-1 \leqslant n \leqslant N-1)$ can be seen by expressing the probability distribution function for the time between successive overflows, which we have denoted by $F(t)$, in terms of $f_n(t)$ $(-1 \leqslant n \leqslant N-1)$. Consider an epoch $\tau_k^{(N-1)}$ (some $k \geqslant 0$). The distribution of time from $\tau_k^{(N-1)} - 0$ until the next overflow subsequent to time $\tau_k^{(N-1)} + 0$ (that is, the inter-overflow time) is the same as the distribution of time from $\tau_\ell^{(N-2)} - 0$ (some $\ell \geqslant 0$) until the next subsequent overflow. This follows from the memoryless property 1.2.2, since at both $\tau_k^{(N-1)} + 0$ and $\tau_\ell^{(N-2)} + 0$ the system is in state $N-1$. (Recall that an overflow does not cause a change of state.) Hence

$$F(t) = f_{N-2}(t) \ . \tag{1.15}$$

If we denote by $F^*(x)$ the Laplace-Stieltjes transform of $F(t)$, given by

$$F^*(x) = \int_0^\infty \exp(-xt)\,dF(t) \ , \qquad \operatorname{Re} x \geqslant 0 \ , \tag{1.16}$$

then equations (1.14) and (1.15) imply that

$$F^*(x) = \frac{p_{N-1}(x)}{p_N(x)} \ . \tag{1.17}$$

The orthogonality of the polynomials $p_n(x)$ gives rise to some important results, which are summarised in the following lemma.

<u>3.1.1 Lemma.</u>

(i) $p_n(0) = 1$, $n=0,1,2,\ldots$.

(ii) The polynomial $p_n(x)$, where n is a non-negative integer, possesses n distinct negative real zeros. Denote these zeros by $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$.

(iii) The following decomposition into partial fractions holds:

$$\frac{p_{n-1}(x)}{p_n(x)} = \sum_{i=1}^{n} \frac{\alpha_i^{(n)}}{x-x_i^{(n)}} \,, \qquad n=1,2,3,\ldots,$$

where $\{x_i^{(n)}\,;\ i=1,2,\ldots,n\}$ is the set of zeros of $p_n(x)$ and

$$\alpha_i^{(n)} > 0 \,, \quad i=1,2,\ldots,n \,. \qquad ■$$

Proof of 3.1.1.

(i) The proof uses (1.12) and induction on n. We have $p_0(0) = p_1(0) = 1$, and the inductive hypothesis is the assumption that $p_i(0) = 1$ $(i=0,1,\ldots,n)$. Setting $x = 0$ in (1.12) immediately yields $p_{n+1}(0) = 1$, and so, by the Principle of Mathematical Induction, the result is proved.

(ii) The fact that the zeros of $p_n(x)$ are real and distinct is a standard property of sequences of orthogonal polynomials (see Erdélyi, et al., [1953], p. 158). Another property common to orthogonal polynomials is that between any two consecutive zeros of $p_{n-1}(x)$ there lies exactly one zero of $p_n(x)$, and conversely, (see Szegö [1939], sec. 3.3). We will use this to prove that all the zeros of $p_n(x)$ are negative, proceeding by induction on n.

Note that the zero of $p_1(x)$ is negative, and take as the inductive hypothesis the assumption that the zeros of $p_{n-1}(x)$ are negative.

Denote the zeros of $p_{n-1}(x)$ by $x_1^{(n-1)}, x_2^{(n-1)}, \ldots, x_{n-1}^{(n-1)}$, and the zeros of $p_n(x)$ by $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$, where the zeros are ordered so that

$$x_{n-1}^{(n-1)} < x_{n-2}^{(n-1)} < \ldots < x_2^{(n-1)} < x_1^{(n-1)}$$

and

$$x_n^{(n)} < x_{n-1}^{(n)} < \ldots < x_2^{(n)} < x_1^{(n)} .$$

Then the orthogonality of the polynomials implies that

$$x_n^{(n)} < x_{n-1}^{(n-1)} < x_{n-1}^{(n)} < x_{n-2}^{(n-1)} < \ldots < x_2^{(n-1)} < x_2^{(n)} < x_1^{(n-1)} < x_1^{(n)} ,$$

and so, from the inductive hypothesis we conclude that

$$x_i^{(n)} < 0 , \quad i=2,3,\ldots,n .$$

We now show that if $x_1^{(n)} \geqslant 0$ then a contradiction follows. Suppose $x_1^{(n)} \geqslant 0$. Then

$$p_n(0) = \prod_{i=1}^{n} (-x_i^{(n)}) \leqslant 0 ,$$

which contradicts (i) of this lemma. Thus, $x_1^{(n)} < 0$, and so

$$x_i^{(n)} < 0 , \quad i=1,2,\ldots,n .$$

Hence, by the Principle of Mathematical Induction, the result is proved.

(iii) The decomposition into partial fractions of this type is also a direct consequence of the orthogonality of the polynomials (see Szegö [1939], theorem 3.3.5). It should be noted that Szegö is more restrictive than most authors in his definition of orthogonal polynomials in that he requires the polynomials to be normalised in such a way that the leading coefficient is positive. This is the case for our polynomials $(p_n(x))_{n=0}^{\infty}$, and so Szegö's theorem 3.3.5, which relies on this fact, does apply. ■

Applying lemma 3.1.1 to equation (1.17) yields

$$F^*(x) = \sum_{i=1}^{N} \frac{\alpha_i}{x-x_i} , \tag{1.18}$$

where $\{x_i = x_i^{(N)} ; i=1,2,\ldots,N\}$ is the set of zeros of $p_N(x)$ and $\alpha_i = \alpha_i^{(N)} > 0$ $(i=1,2,\ldots,N)$. Note that if the Laplace-Stieltjes transform of a function $f(t)$ is $\frac{\kappa}{x+\kappa}$, for some constant κ, then

$$f(t) - f(0) = (1 - \exp(-\kappa t)) \ , \quad t \geqslant 0 \ .$$

Thus, as $F(0) = 0$, we can invert (1.18) to yield

$$F(t) = \sum_{i=1}^{N} \left(-\frac{\alpha_i}{x_i}\right)[1 - \exp(x_i t)] \ , \ t \geqslant 0 \ . \tag{1.19}$$

A hyperexponential distribution is defined to be a convex linear combination of exponential distributions. Applying lemma 3.1.1(i) to equation (1.17) reveals that

$$F^*(0) = 1 \ ,$$

and hence, using (1.18),

$$\sum_{i=1}^{N} \left(-\frac{\alpha_i}{x_i}\right) = 1 \ . \tag{1.20}$$

We have also shown that $x_i < 0$ and $\alpha_i > 0$, and so

$$\left(-\frac{\alpha_i}{x_i}\right) > 0 \ , \quad i=1,2,\ldots,N \ . \tag{1.21}$$

The results (1.20) and (1.21) imply that the probability distribution function for the time between successive overflows, which is given by relation (1.19), is hyperexponential.

The theory of partial fractions entails an explicit expression for α_i (see Kreyszig [1972], p. 158):

$$\alpha_i = \frac{p_{N-1}(x_i)}{p_N'(x_i)} \ , \quad i=1,2,\ldots,N, \tag{1.22}$$

where $'$ implies differentiation with respect to x. Thus the weights, β_i $(i=1,2,\ldots,N)$, in the convex linear combination of exponential distributions are given by

$$\beta_i = -\frac{\alpha_i}{x_i} = -\frac{p_{N-1}(x_i)}{x_i\, p_N'(x_i)} , \qquad (1.23)$$

$$i=1,2,...,N.$$

The results of this section can be summarised in the following theorem:

3.1.2 Theorem.

The distribution function of the time between successive overflows from a finite birth-and-death process of N states, with associated overflow process, is a convex linear combination of N exponential distributions. Specifically,

$$F(t) = \sum_{i=1}^{N} \beta_i \,[1 - \exp(x_i t)] , \quad t \geqslant 0 . \qquad (1.24)$$

The parameters x_i $(i=1,2,...,N)$ of the component exponential distributions and the weights β_i $(i=1,2,...,N)$ of the convex linear combination are uniquely determined by the birth-and-death process through the sequence of orthogonal polynomials $(p_n(x))_{n=0}^{\infty}$, given by the recurrence relations (1.12). The parameters are simply the zeros of $p_N(x)$, and the weights can be computed by equation (1.23). ∎

We will now note the relationship between the orthogonal polynomial sequence $(p_n(x))_{n=0}^{\infty}$ and the orthogonal polynomial sequence $(Q_n(x))_{n=0}^{\infty}$ used by Karlin and McGregor in their series of papers. (See chapter 2.) Comparison of (1.12) of this section and relation (0.6) of chapter 2 reveals that

$$p_n(x) = Q_n(-x) , \quad n=0,1,2,... . \qquad (1.25)$$

The fact that our analysis has generated (essentially) the same orthogonal polynomial sequence as that found by Karlin and McGregor when considering the eigenvectors of the matrix A (recall chapter 2, equation (0.2)) gives a further insight into the fundamental relationship which exists between the birth-and-death process and the orthogonal polynomial sequence. We will discuss this connection further in section 4.4.

In chapter 2 it was mentioned that Keilson [1979] (p. 40) writes the first passage time density from a state to its higher neighbour as a weighted sum of exponentials, using his spectral representation for the transition probabilities. Since the state of the process at $T_k^{(N-2)} + 0$ (some $k \geqslant 0$) is $(N-1)$, it is thus evident that $f_{N-2}(t)$ is simply the first passage time distribution from state $(N-1)$ to some state N, were there such a state. Thus the analysis leading to theorem 3.1.2 provides an alternative derivation of the formula of Keilson, without making reference to the transition probabilities.

3.2 APPLICATIONS IN QUEUEING THEORY.

It is well known that many elementary queueing models can be framed as birth-and-death processes. (See, for example, Cooper [1972] or Kleinrock [1975].) Thus the theory developed in the previous section can be applied to a wide family of queueing models of finite capacity to examine the nature of the time between successive overflows. An overflow here refers to an arrival at the service installation when there are no vacant positions, and which is consequently rejected. This particularly applies to models in telephony.

3.2.1 Telephone Trunking Model.

Consider the following basic telephone trunking model: a Poisson stream of calls of intensity λ arrives at a group of T_1 identical trunks. A call which arrives when the group is not at capacity engages one of the vacant trunks for a time, called its holding time, which is distributed according to some (fixed) negative exponential distribution. For convenience, we will use a time scale which has as its unit the mean holding time.

If we define the state of the system to be the number of calls engaged, then the model is a birth-and-death process of $T_1 + 1$ states (*viz.* $0,1,2,\ldots,T_1$) with rates

$$\begin{aligned} \lambda_n &= \lambda \quad , \quad n=0,1,2,\ldots,T_1 \, , \\ \mu_n &= n \quad , \quad n=1,2,3,\ldots,T_1 \, . \end{aligned} \tag{2.1.1}$$

(Note that the call termination rate for each occupied trunk is unity, and so the total call termination rate is n; hence $\mu_n = n$.)

The recurrence relations for the orthogonal polynomials $p_n(x)$ $(n=0,1,2,\ldots)$ become

$$\lambda p_{n+1}(x) - (x+\lambda+n)p_n(x) + np_{n-1}(x) = 0, ^{(*)} \qquad n=1,2,\ldots, \tag{2.1.2}$$

$$p_0(x) \equiv 1 \ , \quad p_1(x) = 1 + \frac{x}{\lambda} \ .$$

The recurrence relations which define the Charlier polynomials $c_n(x;\lambda)$ (see Erdélyi, et. al., [1953], section 10.25) are

$$\lambda c_{n+1}(x;\lambda) + (x-\lambda-n)c_n(x;\lambda) + nc_{n-1}(x;\lambda) = 0 \ , \qquad n=1,2,\ldots, \tag{2.1.3}$$

$$c_0(x;\lambda) \equiv 1 \ , \quad c_1(x;\lambda) = 1 - \frac{x}{\lambda} \ .$$

Comparison of (2.1.2) and (2.1.3) reveals that

$$p_n(x) = c_n(-x;\lambda) \ , \qquad n=0,1,2,\ldots \ . \tag{2.1.4}$$

The Charlier polynomials are a well-established family of orthogonal polynomials associated with Poisson's distribution of rare events. The occurrence of these Charlier polynomials in investigations of this basic telephone trunking model has also been noted by Karlin and McGregor [1958a] (section 3), and Potter [1979] (section 2.3.1). In fact, the result (2.1.4) agrees with the results of section 2.3.1 of Potter [1979], which discovers that the Laplace-Stieltjes transform of the inter-overflow time distribution function is given by

$$F^*(x) = \frac{c_{T_1}(-x;\lambda)}{c_{T_1+1}(-x;\lambda)} \ . \tag{2.1.5}$$

Potter derives the result (2.1.5) as a special case of a formula for

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(*) The natural extension of λ_n, μ_n for $n > T_1$ has been used (refer to the footnote to equation (1.12) of section 3.1).

the Laplace-Stieltjes transform of the inter-overflow time distribution function for the telephone trunking model offered a general renewal input process.

The hyperexponential nature of the overflow from the basic telephone trunking model has previously been noted by Khintchine [1969] in his investigations of the Palm functions $\varphi_r(t)$ associated with the model. (See pp. 87-95 of Khintchine [1969].) Potter [1979] (p. 19) shows that Khintchine's formula for the Palm function $\varphi_1(t)$ is consistent with the results stemming from equation (2.1.5) in the case $T_1 = 1$.

3.2.2 Telephone Trunking Model with Holding Registers.

The telephone trunking model of section 3.2.1 can be readily generalised to allow a finite number of holding registers which store incoming calls when all trunks are engaged. These calls remain on the holding registers until a trunk becomes vacant. Thus calls are only rejected when both the trunk group and the holding register group are at capacity.

If we assume that there are T_2 holding registers then the rates of the birth-and-death process are

$$\lambda_n = \lambda \ , \quad n=0,1,2,\ldots,T_1+T_2 \ ,$$
$$\mu_n = \begin{cases} n \ , & n=0,1,2,\ldots,T_1, \\ T_1 \ , & n=T_1,T_1+1,\ldots,T_1+T_2 \ . \end{cases} \tag{2.2.1}$$

The recurrence relations for the associated orthogonal polynomials $p_n(x)$ $(n=0,1,2,\ldots)$ become

$$\lambda p_{n+1}(x) - (x+\lambda+n)p_n(x) + np_{n-1}(x) = 0 \ , \quad n=1,2,\ldots,T_1 \ ,$$
$$\lambda p_{n+1}(x) - (x+\lambda+T_1)p_n(x) + T_1 p_{n-1}(x) = 0 \ , \quad n=T_1,T_1+1,\ldots,^{(*)} \tag{2.2.2}$$

(*) The natural extension of λ_n, μ_n for $n > T_1+T_2$ has been used (refer to the footnote to equation (1.12) of section 3.1).

$$p_0(x) \equiv 1 \ , \quad p_1(x) = 1 + \frac{x}{\lambda} \ .$$

We note that

$$p_n(x) = c_n(-x;\lambda) \ , \quad n=0,1,2,\ldots,T_1+1 \ , \tag{2.2.3}$$

but for $n > T_1+1$ this relationship does not hold.

3.2.3 Finite Source Models.

Implicit in the previous two models has been the assumption that there is an infinite population from which service requests originate. In many practical applications such as an assumption serves as a reasonable approximation in situations where the population is very large in relation to the size of the service installation. However, in some applications account must be made of the fact that a customer in service cannot generate another service request until it has again left the system.

In these cases we consider each customer to be an identical source of service requests. When not in service, the source generates service requests at a rate γ, say. The rate when the source is in service is, of course, zero. If we suppose there are M such sources, and S $(S \leqslant M)$ servers (with no waiting room) then the rates for the birth-and-death process are

$$\begin{aligned} \lambda_n &= (M-n)\gamma \ , \quad n=0,1,\ldots,S \ , \\ \mu_n &= n \ , \qquad\qquad n=1,2,\ldots,S \ . \end{aligned} \tag{2.3.1}$$

To illustrate the construction of the parameters and weights of the inter-overflow time distribution function let us consider the case of two sources $(M=2)$ and one server $(S=1)$. From (2.3.1) we have

$$\lambda_0 = 2\gamma \ , \quad \lambda_1 = \gamma \ , \quad \mu_1 = 1 \ . \tag{2.3.2}$$

The associated orthogonal polynomials $p_n(x)$ are easily found from the recurrence relations (1.12) of section 3.1 to be

$$p_0(x) = 1 ,$$
$$p_1(x) = \frac{1}{2\gamma}(x+2\gamma) , \tag{2.3.3}$$
$$p_2(x) = \frac{1}{2\gamma^2}[x^2+(3\gamma+1)x+2\gamma^2] .$$

According to theorem 3.1.2, the parameters of the two component exponential distributions are the two zeros of $p_2(x)$, which are

$$x_1 = \frac{1}{2}[-(3\gamma+1)+\sqrt{\gamma^2+6\gamma+1}] \tag{2.3.4}$$

and

$$x_2 = \frac{1}{2}[-(3\gamma+1)-\sqrt{\gamma^2+6\gamma+1}] . \tag{2.3.5}$$

We note that

$$x_1 x_2 = 2\gamma^2 . \tag{2.3.6}$$

Now

$$p_2'(x_1) = \frac{1}{2\gamma^2}[\sqrt{\gamma^2+6\gamma+1}] , \tag{2.3.7}$$

and

$$p_2'(x_2) = -\frac{1}{2\gamma^2}[\sqrt{\gamma^2+6\gamma+1}] . \tag{2.3.8}$$

From equation (1.23) of section (3.1)

$$\beta_1 = -\frac{1}{x_1} \cdot \frac{p_1(x_1)}{p_2'(x_1)}$$

$$= -\frac{1}{x_1} \cdot \frac{\frac{1}{2\gamma}(x_1+2\gamma)}{\frac{1}{2\gamma^2}\sqrt{\gamma^2+6\gamma+1}}$$

using (2.3.7)

$$= -\frac{\gamma}{\sqrt{\gamma^2+6\gamma+1}}\left(\frac{x_1+2\gamma}{x_1}\right)$$

$$= -\frac{1}{\sqrt{\gamma^2+6\gamma+1}}(\gamma+x_2)$$

using (2.3.6)

$$= \frac{1}{2} + \frac{\gamma+1}{2\sqrt{\gamma^2+6\gamma+1}} \quad . \qquad (2.3.9)$$

Similarly,

$$\beta_2 = \frac{1}{2} - \frac{\gamma+1}{2\sqrt{\gamma^2+6\gamma+1}} \quad . \qquad (2.3.10)$$

Thus the distribution function of the time between successive unsuccessful requests for service is

$$F(t) = \left(\frac{1}{2} + \frac{\gamma+1}{2\sqrt{\gamma^2+6\gamma+1}}\right)\left[1 - \exp\left(\frac{-3\gamma-1+\sqrt{\gamma^2+6\gamma+1}}{2} \cdot t\right)\right]$$

$$+ \left(\frac{1}{2} - \frac{\gamma+1}{2\sqrt{\gamma^2+6\gamma+1}}\right)\left[1 - \exp\left(\frac{-3\gamma-1-\sqrt{\gamma^2+6\gamma+1}}{2} \cdot t\right)\right], \qquad (2.3.11)$$

$$t \geqslant 0.$$

3.2.4 Other Models.

The scope for inventing models to which our theory applies is quite endless, and only the most important basic models have been presented here. More involved models can be achieved by combining some of the simpler ones, and also by adjusting the birth and death rates to allow for such phenomena as baulking or discouraged arrivals (arrival rate decreases as the number in the system increases) and accelerated service (service rate per server increases as the number in the system increases).

Given the rates for a finite birth-and-death process, it is a simple procedure to find the numerical values of the parameters and weights of the hyperexponential inter-overflow time distribution function. A computer program to achieve these computations is given in appendix 1.

3.3 SOME NOTES ON THE ASSOCIATED ORTHOGONAL POLYNOMIALS.

For the sake of completeness we now make some notes and observations regarding the orthogonality of the polynomials $p_n(x)$ $(n=0,1,2,...)$ associated with the finite birth-and-death process. We recall that in chapter 2 it was mentioned that Karlin and McGregor, in their series of papers, make use of the orthogonal polynomial sequence $(Q_n(x))_{n=0}^{\infty}$, which we observed in section 3.1 (equation (1.25)) is simply the sequence $(p_n(-x))_{n=0}^{\infty}$. The properties of the polynomials $Q_n(x)$ $(n=0,1,2,...)$, and hence those of $p_n(x)$ $(n=0,1,2,...)$, from an orthogonal polynomial theory viewpoint, are discussed at length in Karlin and McGregor [1957a] and [1957b], and so, in this section, only some additional observations of specific interest will be made.

It has been noted in section 3.2.1 that for the telephone trunking model the polynomials $p_n(x)$ are simply related to the Charlier polynomials. In the telephone trunking model with holding registers of section 3.2.2 the recurrence relations (2.2.2) for the $p_n(x)$ are also quite simple, particularly in the case when the number of trunks, T_1, is unity. It is the latter special case which we will investigate.

The recurrence relations (2.2.2) for the case $T_1 = 1$ are

$$\lambda p_{n+1}(x) - (x+\lambda+1)p_n(x) + p_{n-1}(x) = 0, \quad n=1,2,3,..., \tag{3.1}$$

$$p_0(x) \equiv 1, \quad p_1(x) = 1 + \frac{x}{\lambda}.$$

Define the generating function

$$G(x,z) = \sum_{n=0}^{\infty} p_n(x)z^n, \quad |z| < 1. \tag{3.2}$$

By multiplying (3.1) by z^{n+1} and summing over n from 1 to ∞ we obtain

$$\lambda[G(x,z)-1-(1+\frac{x}{\lambda})z] - (x+\lambda+1)z[G(x,z)-1] + z^2G(x,z) = 0 ,$$

which yields, on rearrangement,

$$G(x,z) = \frac{\lambda-z}{\lambda-(x+\lambda+1)z+z^2} . \qquad (3.3)$$

The generating function of the Chebyshev polynomials of the second kind, $U_n(x)$, is

$$\sum_{n=0}^{\infty} U_n(x)\zeta^n = \frac{1}{1-2x\zeta+\zeta^2} , \qquad (3.4)$$

(see Abramowitz and Stegun [1964], p. 783). Comparison of (3.3) and (3.4) reveals that

$$p_n(x) = \lambda^{-\frac{n}{2}}[U_n(\frac{x+\lambda+1}{2\sqrt{\lambda}}) - \frac{1}{\sqrt{\lambda}} U_{n-1}(\frac{x+\lambda+1}{2\sqrt{\lambda}})] , \quad n=1,2,\ldots . \qquad (3.5)$$

Proof of Relation (3.5).

Consider the right-hand-side of (3.5). Multiplying by z^n and summing over n from 1 to ∞ we obtain

$$\sum_{n=1}^{\infty} \lambda^{-\frac{n}{2}} U_n(\frac{x+\lambda+1}{2\sqrt{\lambda}}) z^n - \sum_{n=1}^{\infty} \lambda^{-\frac{n+1}{2}} U_{n-1}(\frac{x+\lambda+1}{2\sqrt{\lambda}}) z^n$$

$$= \sum_{n=0}^{\infty} U_n(\frac{x+\lambda+1}{2\sqrt{\lambda}}) \cdot (\frac{z}{\sqrt{\lambda}})^n - U_0(\frac{x+\lambda+1}{2\sqrt{\lambda}}) - \frac{z}{\lambda} \sum_{n=0}^{\infty} U_n(\frac{x+\lambda+1}{2\sqrt{\lambda}}) (\frac{z}{\sqrt{\lambda}})^n$$

$$= (1 - \frac{z}{\lambda}) \cdot \sum_{n=0}^{\infty} U_n(\frac{x+\lambda+1}{2\sqrt{\lambda}}) (\frac{z}{\sqrt{\lambda}})^n - 1$$

since $U_0(x) = 1$

$$= \frac{(1 - \frac{z}{\lambda})}{1 - 2(\frac{x+\lambda+1}{2\sqrt{\lambda}})(\frac{z}{\sqrt{\lambda}}) + \frac{z^2}{\lambda}} - 1$$

using (3.4)

$$= \frac{\lambda-z}{\lambda-(x+\lambda+1)z+z^2} - 1$$

$$= G(x,z) - 1$$ using (3.3)

$$= \sum_{n=0}^{\infty} p_n(x)z^n - 1 \qquad \text{using (3.2)}$$

$$= \sum_{n=1}^{\infty} p_n(x)z^n \qquad \text{since } p_0(x) = 1 .$$

This last expression is the left-hand-side of (3.5) after multiplication by z^n and summing from $n=1$ to ∞. Thus expression (3.5) is true.

∎

We shall now investigate further the relationship between $p_n(x)$ and the Chebyshev polynomials. We note that the two orthogonal polynomial sequences $(p_n(x))_{n=0}^{\infty}$ and $(\kappa_n p_n(\sigma_1 x+\sigma_2))_{n=0}^{\infty}$ (where $\kappa_n \neq 0$ and $\sigma_1 \neq 0$) are essentially the same. If we picture the polynomials $p_n(x)$ graphically then κ_n provides a vertical dilation while σ_1 and σ_2 provide a horizontal dilation and shift respectively. Thus, knowing any result concerning $(p_n(x))_{n=0}^{\infty}$, the corresponding result for $(\kappa_n p_n(\sigma_1 x+\sigma_2))_{n=0}^{\infty}$ can be found immediately. However, the recurrence relations of these two systems may be markedly different on first inspection.

Bearing this in mind, and recalling the result (3.5), let us define the orthogonal polynomial sequence $(r_n(x))_{n=0}^{\infty}$ by

$$r_n(x) = \lambda^{\frac{n}{2}} p_n(2\lambda^{\frac{1}{2}}x-\lambda-1) . \qquad (3.6)$$

Then from (3.1) we obtain the recurrence relations

$$r_{n+1}(x) = 2xr_n(x) - r_{n-1}(x) , \quad n=1,2,\ldots, \qquad (3.7)$$

$$r_0(x) \equiv 1, \quad r_1(x) = 2x - \frac{1}{\sqrt{\lambda}} .$$

<u>Proof of (3.7).</u>

$$r_{n+1}(x) = \lambda^{\frac{n+1}{2}} p_{n+1}(2\lambda^{\frac{1}{2}}x-\lambda-1) \qquad \text{using (3.6)}$$

$$= \lambda^{\frac{n-1}{2}} [(2\lambda^{\frac{1}{2}}x-\lambda-1+\lambda+1)p_n(2\lambda^{\frac{1}{2}}x-\lambda-1) - p_{n-1}(2\lambda^{\frac{1}{2}}x-\lambda-1)] \text{ using (3.1)}$$

$$= 2x\lambda^{\frac{n}{2}} \, p_n(2\lambda^{\frac{1}{2}}x-\lambda-1) - \lambda^{\frac{n-1}{2}} \, p_{n-1}(2\lambda^{\frac{1}{2}}x-\lambda-1)$$

$$= 2xr_n(x) - r_{n-1}(x) \qquad \text{using (3.6)} . \qquad \blacksquare$$

Note that the recurrence relation (3.7) is simply the Chebyshev recurrence relation (Abramowitz and Stegun [1964], p. 782, 22.7.4 and 22.7.5). The two "kinds" of Chebyshev polynomials, $T_n(x)$ and $U_n(x)$, both satisfy the same recurrence relation, differing only in the initial specifications.

Chihara [1978] (p. 204) quotes a result of Geronimus [1930] which states that a sequence of orthogonal polynomials $(P_n(x))_{n=0}^{\infty}$ satisfying the Chebyshev recurrence relation, but with general initial specifications

$$P_0(x) \equiv 1, \quad P_1(x) = ax-b , \quad (a \neq 0)$$

can be represented by

$$P_n(x) = aT_n(x) + (a-1)U_{n-2}(x) - bU_{n-1}(x) , \qquad n=2,3,\ldots .$$

Thus the sequence $(r_n(x))_{n=0}^{\infty}$ can be written as

$$r_n(x) = 2T_n(x) + U_{n-2}(x) - \frac{1}{\sqrt{\lambda}} U_{n-1}(x) , \qquad n=2,3,\ldots . \tag{3.8}$$

It should be pointed out that this does not conflict with the result (3.5), as can be seen from the following argument.

Proof that (3.8) is consistent with (3.5).

Equation 22.5.6 of Abramowitz and Stegun [1964] reads

$$T_n(x) = U_n(x) - xU_{n-1}(x) .$$

Thus

$$2T_n(x) + U_{n-2}(x) - \frac{1}{\sqrt{\lambda}} U_{n-1}(x)$$

$$= 2[U_n(x) - xU_{n-1}(x)] + U_{n-2}(x) - \frac{1}{\sqrt{\lambda}} U_{n-1}(x)$$

$$= 2U_n(x) - [2xU_{n-1}(x) - U_{n-2}(x)] - \frac{1}{\sqrt{\lambda}} U_{n-1}(x)$$

$$= 2U_n(x) - U_n(x) - \frac{1}{\sqrt{\lambda}} U_{n-1}(x)$$

using the Chebyshev recurrence relation

$$= U_n(x) - \frac{1}{\sqrt{\lambda}} U_{n-1}(x) .$$

If we use (3.6) to write (3.5) in terms of $r_n(x)$ we obtain

$$r_n(x) = U_n(x) - \frac{1}{\sqrt{\lambda}} U_{n-1}(x) ,$$

and so the two results (3.5) and (3.8) are consistent. ∎

Chihara [1957] investigated the distribution function, $\psi(x)$, for orthogonal polynomial sequences $(P_n(x))_{n=0}^{\infty}$ satisfying the Chebyshev recurrence relation but with initial specifications

$$P_0(x) \equiv 1, \quad P_1(x) = 2x-b , \quad (b \neq 0).$$

Using his result for the case $b = \frac{1}{\sqrt{\lambda}}$ we find that the distribution function $\psi_r(x)$ corresponding to the orthogonal polynomial sequence $(r_n(x))_{n=0}^{\infty}$ is given by

$$\psi_r(x) = -\frac{2\lambda}{\pi} \int_{-1}^{x} \frac{\sqrt{1-t^2}}{2\lambda^{\frac{1}{2}}t-\lambda-1} dt \quad \text{for} \quad -1 \leqslant x \leqslant 1 , \tag{3.9}$$

while $\psi_r(x)$ is constant on $(-\infty,-1]$ and also on $[1,\infty)$, except when $\lambda < 1$ in which case $\psi_r(x)$ has a single jump of magnitude $1-\lambda$ at the point $x = \frac{\lambda+1}{2\sqrt{\lambda}}$.

We must comment on the determinacy of $(r_n(x))_{n=0}^{\infty}$ (recall section 1.3). We note that there exist finite a,b such that [a,b] contains the spectrum of $\psi(x)$. Chihara [1978], p. 29, defines a set he calls the *true interval of orthogonality*, and his theorem 3.2, p. 58, gives that this interval is a subset of [a,b], and is thus bounded. Chapter 2, section 5 of Chihara [1978] shows that if the true interval of orthogonality is bounded then the system is determinate. Thus $\psi_r(x)$ as given is unique, save for distribution functions which are substantially equal to $\psi_r(x)$.

We have defined $\psi_r(x)$ to be the distribution function corresponding to the orthogonal polynomial sequence $(r_n(x))_{n=0}^{\infty}$, which means that

$$\int_{-\infty}^{\infty} r_m(x)r_n(x)d\psi_r(x) = 0 \quad \text{for} \quad m \neq n, \tag{3.10}$$

and

$$\int_{-\infty}^{\infty} [r_n(x)]^2 d\psi_r(x) \neq 0, \quad n=0,1,2,\ldots \; . \tag{3.11}$$

By using (3.6) to express the statements (3.10) and (3.11) in terms of $p_n(x)$ $(n=0,1,2,\ldots)$, we obtain

$$\int_{-\infty}^{\infty} p_m(2\lambda^{\frac{1}{2}}x-\lambda-1)p_n(2\lambda^{\frac{1}{2}}x-\lambda-1)d\psi_r(x) = 0 \tag{3.12}$$

$$\text{for} \quad m \neq n,$$

and

$$\int_{-\infty}^{\infty} [p_n(2\lambda^{\frac{1}{2}}x-\lambda-1)]^2 d\psi_r(x) \neq 0 , \tag{3.13}$$

$$n=0,1,2,\ldots \; .$$

A simple linear change of variable in (3.12) and (3.13) gives

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)d\psi_r\left(\frac{x+\lambda+1}{2\sqrt{\lambda}}\right) = 0 \tag{3.14}$$

$$\text{for} \quad m \neq n,$$

and

$$\int_{-\infty}^{\infty} [p_n(x)]^2 d\psi_r\left(\frac{x+\lambda+1}{2\sqrt{\lambda}}\right) \neq 0 , \qquad (3.15)$$

$$n=0,1,2,\ldots .$$

Thus, denoting the distribution function corresponding to the orthogonal polynomial sequence $(p_n(x))_{n=0}^{\infty}$ by $\psi_p(x)$, the results (3.14) and (3.15) imply by definition that

$$\psi_p(x) = \psi_r\left(\frac{x+\lambda+1}{2\sqrt{\lambda}}\right) .^{(*)} \qquad (3.16)$$

Thus from equation (3.9) we have

$$\psi_p(x) = -\frac{2\lambda}{\pi}\int_{-1}^{\frac{x+\lambda+1}{2\sqrt{\lambda}}} \frac{\sqrt{1-t^2}}{2\lambda^{\frac{1}{2}}t-\lambda-1}\, dt \qquad (3.17)$$

$$\text{for} \quad -2\lambda^{\frac{1}{2}}-\lambda-1 \leqslant x \leqslant 2\lambda^{\frac{1}{2}}-\lambda-1,$$

which is the same as

$$\psi_p(x) = \frac{1}{2\pi}\int_{-x}^{2\sqrt{\lambda}+\lambda+1} \frac{\sqrt{4t-(t-\lambda+1)^2}}{t}\, dt \qquad (3.18)$$

$$\text{for } -2\lambda^{\frac{1}{2}}-\lambda-1 \leqslant x \leqslant 2\lambda^{\frac{1}{2}}-\lambda-1 .$$

Proof of equation (3.18).

If we make the linear change of variable $t \to \tilde{t} = -2\lambda^{\frac{1}{2}}t+\lambda+1$ in (3.17) we obtain

$$\psi_p(x) = -\frac{2\lambda}{\pi}\int_{2\sqrt{\lambda}+\lambda+1}^{-x} \frac{[1-\frac{1}{4\lambda}(\lambda+1-\tilde{t})^2]^{\frac{1}{2}}}{(-\tilde{t})}\,\frac{d\tilde{t}}{(-2\sqrt{\lambda})}$$

$$= \frac{1}{2\pi}\int_{-x}^{2\sqrt{\lambda}+\lambda+1} \frac{1}{\tilde{t}}[4\lambda-(\lambda+1-\tilde{t})^2]^{\frac{1}{2}}\, d\tilde{t}$$

$$= \frac{1}{2\pi}\int_{-x}^{2\sqrt{\lambda}+\lambda+1} \frac{1}{\tilde{t}}[4\tilde{t}-(\tilde{t}-\lambda+1)^2]^{\frac{1}{2}}\, d\tilde{t}$$

(*) Note that the determinacy of $(r_n(x))_{n=0}^{\infty}$ implies the determinacy of $(p_n(x))_{n=0}^{\infty}$.

and thus (3.18) holds. ∎

Also from (3.9) we have that $\psi_p(x)$ is constant on $(-\infty, -2\lambda^{\frac{1}{2}}-\lambda-1]$ and also on $[2\lambda^{\frac{1}{2}}-\lambda-1, \infty)$, except when $\lambda < 1$ in which case $\psi_p(x)$ has a single jump of magnitude $1-\lambda$ at the point $x = 0$.

Thus, to summarise, the orthogonal polynomial sequence $(p_n(x))_{n=0}^{\infty}$ associated with the telephone trunking model with holding registers but only one trunk is simply related to the Chebyshev polynomials, specifically

$$p_n(x) = \lambda^{-\frac{n}{2}} \left[U_n\left(\frac{x+\lambda+1}{2\sqrt{\lambda}}\right) - \frac{1}{\sqrt{\lambda}} U_{n-1}\left(\frac{x+\lambda+1}{2\sqrt{\lambda}}\right)\right], \qquad n=1,2,\ldots . \tag{3.19}$$

The distribution function $\psi_p(x)$ with respect to which the polynomial sequence $(p_n(x))_{n=0}^{\infty}$ is orthogonal is given by

$$\psi_p(x) = \frac{1}{2\pi} \int_{-x}^{2\sqrt{\lambda}+\lambda+1} \frac{\sqrt{4t-(t-\lambda+1)^2}}{t}\, dt \qquad \text{for } -2\lambda^{\frac{1}{2}}-\lambda-1 \leqslant x \leqslant 2\lambda^{\frac{1}{2}}-\lambda-1, \tag{3.20}$$

while $\psi_p(x)$ is constant on $(-\infty, -2\lambda^{\frac{1}{2}}-\lambda-1]$ and also on $[2\lambda^{\frac{1}{2}}-\lambda-1, \infty)$, except when $\lambda < 1$ in which case $\psi_p(x)$ has a single jump of magnitude $1-\lambda$ at the point $x = 0$.

We note, since the unit of time has been taken as the mean holding time, that λ is the offered traffic to the system. (Recall section 1.4.) If, in queueing systems with infinite waiting room, the offered traffic per server equals or exceeds unity, then the system is unstable, with queues growing beyond bound. Since in our model there is only one server, the offered traffic per server is simply λ, and, although there cannot be any instability since the system is finite, we nonetheless see from our formula for $\psi_p(x)$ that $\lambda=1$ is still a "critical" value.

3.4 THE HYPEREXPONENTIAL DISTRIBUTION AS AN OVERFLOW.

Theorem 3.1.2 states that the overflow from a finite birth-and-death process of N states is a hyperexponential distribution with N component exponential distributions. We now seek to establish its converse: that any hyperexponential distribution with N component exponential distributions can be interpreted as the overflow from a finite birth-and-death process of N states.

Consider the sequence $(p_n(x))_{n=0}^{\infty}$ of orthogonal polynomials introduced in section 3.1. Associated with this sequence is the sequence $(q_n(x))_{n=0}^{\infty}$ of monic orthogonal polynomials defined by

$$\begin{aligned} q_n(x) &= \lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}p_n(x) , \quad n=1,2,\ldots, \\ q_0(x) &= p_0(x) \equiv 1 . \end{aligned} \tag{4.1}$$

This sequence is simply a renormalisation of the original sequence, and using (4.1) in (1.12) we obtain the recurrence relations

$$\begin{aligned} q_{n+1}(x) &= (x+\lambda_n+\mu_n)q_n(x) - \lambda_{n-1}\mu_n q_{n-1}(x) , \\ &\qquad\qquad n=1,2,\ldots, \\ q_0(x) &\equiv 1 , \qquad q_1(x) = x+\lambda_0 . \end{aligned} \tag{4.2}$$

(We note that a simple induction argument using (4.2) gives that $q_n(x)$ is in fact monic.) Equation (1.17) of section 3.1 can thus be written as

$$F^*(x) = \lambda_{N-1}\frac{q_{N-1}(x)}{q_N(x)} . \tag{4.3}$$

Suppose we are given some arbitrary hyperexponential distribution defined by its probability distribution function, say

$$F(t) = \sum_{i=1}^{N} \beta_i \, [1 - \exp(x_i t)] \ , \quad t \geqslant 0, \tag{4.4}$$

where $x_i < 0$, x_i all distinct, $\beta_i > 0$, and

$$\sum_{i=1}^{N} \beta_i = 1 \ . \tag{4.5}$$

If we denote by $F^*(x)$ the Laplace-Stieltjes transform of $F(t)$, then (4.4) implies that

$$F^*(x) = \sum_{i=1}^{N} \frac{\alpha_i}{x-x_i} \ , \quad \text{Re } x \geqslant 0, \tag{4.6}$$

where

$$\alpha_i = -x_i \beta_i > 0 \ , \quad i=1,2,\ldots,N \ . \tag{4.7}$$

Expression (4.6) can be written in the form

$$F^*(x) = \lambda_{N-1} \frac{q_{N-1}(x)}{q_N(x)} \ (*) \ , \tag{4.8}$$

where

$$\lambda_{N-1} = \sum_{i=1}^{N} \alpha_i > 0 \ , \tag{4.9}$$

$$q_N(x) = \prod_{i=1}^{N} (x-x_i) \ , \tag{4.10}$$

and

$$q_{N-1}(x) = \frac{1}{\lambda_{N-1}} \sum_{i=1}^{N} [\alpha_i \prod_{\substack{j=1 \\ j \neq i}}^{N} (x-x_j)] \ . \tag{4.11}$$

We note that $q_{N-1}(x)$ and $q_N(x)$ are monic polynomials in x of exact degree $N-1$ and N respectively, and that $\lambda_{N-1} > 0$.

(*) *c.f.* equation (4.3).

In order to establish the converse of theorem 3.1.2 we must find positive real numbers $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$, $\mu_1, \mu_2, \ldots, \mu_{N-1}$ such that the sequence of orthogonal polynomials $(q_n(x))_{n=0}^{\infty}$, defined by (4.2) has $q_{N-1}(x)$ and $q_N(x)$ as its (N-1)st and Nth members respectively.

The zeros of $q_N(x)$ are $x_1, x_2, \ldots, x_N$, and suppose they are ordered so that

$$x_N < x_{N-1} < \ldots < x_2 < x_1 .$$

The theory of partial fractions (see Kreyszig [1972], p. 158) gives that

$$\alpha_i = \lambda_{N-1} \frac{q_{N-1}(x_i)}{q_N'(x_i)} , \quad i=1,2,\ldots,N, \tag{4.12}$$

where ' implies differentiation with respect to x. As $\lambda_{N-1} > 0$, $\alpha_i > 0$ $(i=1,2,\ldots,N)$, then equation (4.12) implies that

$$\frac{q_{N-1}(x_i)}{q_N'(x_i)} > 0 , \quad i=1,2,\ldots,N. \tag{4.13}$$

Now $q_N(x)$ is a polynomial of exact degree N with N distinct zeros, and so

$$q_N'(x_i)\, q_N'(x_{i+1}) < 0 , \quad i=1,2,\ldots,N-1. \tag{4.14}$$

(That is, the gradient of the polynomial changes in sign from one zero to the next.) Combining the inequalities (4.13) and (4.14) we see that

$$q_{N-1}(x_i)\,q_{N-1}(x_{i+1}) < 0 , \qquad (4.15)$$

$$i=1,2,\ldots,N-1,$$

and so, by the Intermediate Value Theorem, there is at least one zero of $q_{N-1}(x)$ in the interval (x_i , x_{i+1}), $i=1,2,\ldots,N-1$. But $q_{N-1}(x)$ has at most $N-1$ real zeros, and there are $N-1$ such intervals. Hence all the zeros of $q_{N-1}(x)$ are real and distinct, with exactly one in each of the intervals (x_i , x_{i+1}), $i=1,2,\ldots,N-1$.

If we denote the zeros of $q_{N-1}(x)$ by $y_1,y_2,\ldots,y_{N-1}$, and suppose they are ordered so that

$$y_{N-1} < y_{N-2} < \ldots < y_2 < y_1 ,$$

then the above result gives that

$$x_N < y_{N-1} < x_{N-1} < y_{N-2} < \ldots < y_2 < x_2 < y_1 < x_1 . \qquad (4.16)$$

The string of inequalities (4.16) enables us to invoke a theorem of Wendroff [1961], which guarantees the existence of a sequence of monic orthogonal polynomials in x with $q_{N-1}(x)$ and $q_N(x)$ as the $(N-1)$st and Nth members respectively. These polynomials satisfy a recurrence relation of the form

$$q_{n+1}(x) = (x-c_n)q_n(x) - d_n q_{n-1}(x) , \qquad (4.17)$$

$$n=0,1,2,\ldots,$$

where c_n is real, $d_n > 0$ $(n=0,1,2,\ldots)$ and $q_{-1}(x) \equiv 0$.

We now use a descending inductive argument (for $n=N-1,N-2,\ldots,1$) to show that (4.17) can be written in the form of (4.2), with $\lambda_0,\lambda_1,\ldots,\lambda_{N-1}$, $\mu_1,\mu_2,\ldots,\mu_{N-1}$ all positive, and also that

$$q_{i+1}(0) = \lambda_i q_i(0), \quad i=0,1,2,\ldots,N-1 . \qquad (4.18)$$

As a basis for the induction (the case $n=N-1$), we note that λ_{N-1}

exists and is positive. Substituting (4.7) into (4.5) we see that

$$\sum_{i=1}^{N} \left(-\frac{\alpha_i}{x_i}\right) = 1 \ ,$$

and so, combining (4.6) and (4.8),

$$\lambda_{N-1} \frac{q_{N-1}(0)}{q_N(0)} = 1 \ .$$

That is,

$$q_N(0) = \lambda_{N-1} q_{N-1}(0) \ .$$

Take as the inductive hypothesis, for $n \in \{N-1, N-2, \ldots, 1\}$, the assumption that $\lambda_n, \lambda_{n+1}, \ldots, \lambda_{N-1}$ and $\mu_{n+1}, \mu_{n+2}, \ldots, \mu_{N-1}$ all exist and are positive, and also that

$$q_{n+1}(0) = \lambda_n q_n(0) \ . \tag{4.19}$$

Comparison of (4.2) and (4.17) reveals that

$$\mu_n = -c_n - \lambda_n \tag{4.20}$$

exists. In order to show $\mu_n > 0$ it is thus necessary and sufficient to show that

$$-c_n > \lambda_n \ ,$$

which is also equivalent to the condition

$$-\lambda_n c_n > \lambda_n^2 \ , \tag{4.21}$$

since $\lambda_n > 0$ by the inductive hypothesis.

Denote the zeros of $q_{n+1}(x)$ by $\xi_1, \xi_2, \ldots, \xi_{n+1}$. Then the orthogonality properties (see Szegö [1939], theorem 3.3.5) give that

$$\lambda_n \frac{q_n(x)}{q_{n+1}(x)} = \sum_{i=1}^{n+1} \frac{\eta_i}{x - \xi_i} \ , \tag{4.22}$$

where $\eta_i > 0$ $(i=1,2,\ldots,n+1)$. Let

$$\theta_i = -\frac{\eta_i}{\xi_i} , \qquad (4.23)$$

$$i=1,2,3,\ldots,n+1,$$

and so, by setting $x=0$ in (4.22) and using the relation (4.19), we see that

$$\sum_{i=1}^{n+1} \theta_i = 1 . \qquad (4.24)$$

Now from (4.22), since $q_n(x)$ is a monic polynomial, we have

$$\lambda_n = \sum_{i=1}^{n+1} \eta_i , \qquad (4.25)$$

and so, using (4.23),

$$\lambda_n^2 = (-\sum_{i=1}^{n+1} \theta_i \xi_i)^2 ,$$

and hence also

$$\lambda_n^2 = (\sum_{i=1}^{n+1} \theta_i \xi_i)^2 . \qquad (4.26)$$

Write

$$q_{n+1}(x) = x^{n+1} + a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 , \qquad (4.27)$$

and

$$q_n(x) = x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \ldots + b_1 x + b_0 . \qquad (4.28)$$

Now from relation (4.22) we see that

$$q_n(x) = \prod_{i=1}^{n+1} (x-\xi_i) , \qquad (4.29)$$

and

$$q_{n-1}(x) = \frac{1}{\lambda_n} \sum_{i=1}^{n+1} \eta_i \prod_{\substack{j=1 \\ j\neq i}}^{n+1} (x-\xi_j) . \qquad (4.30)$$

Comparison of (4.27) and (4.29) reveals that

$$a_n = -\sum_{i=1}^{n+1} \xi_i , \qquad (4.31)$$

and likewise comparison of (4.28) and (4.30) reveals that

$$b_{n-1} = \frac{1}{\lambda_n} \sum_{i=1}^{n+1} \eta_i \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (-\xi_j) \ . \tag{4.32}$$

Now from (4.31) we have that

$$\sum_{\substack{j=1 \\ j \neq i}}^{n+1} (-\xi_j) = a_n + \xi_i \ , \qquad i=1,2,\ldots,n+1,$$

which, on substitution into (4.32), gives

$$b_{n-1} = \frac{1}{\lambda_n} \sum_{i=1}^{n+1} \eta_i (a_n + \xi_i) \ . \tag{4.33}$$

Rearrangement of (4.33) yields

$$\lambda_n b_{n-1} - a_n \sum_{i=1}^{n+1} \eta_i = \sum_{i=1}^{n+1} \eta_i \xi_i \ ,$$

which, by virtue of (4.25) and (4.23), can be written as

$$\lambda_n (b_{n-1} - a_n) = - \sum_{i=1}^{n+1} \theta_i \xi_i^2 \ . \tag{4.34}$$

But by equating the coefficients of x^n in (4.17) we see that

$$c_n = b_{n-1} - a_n \ ,$$

and so, substituting into (4.34), we have

$$- \lambda_n c_n = \sum_{i=1}^{n+1} \theta_i \xi_i^2 \ . \tag{4.35}$$

Now a theorem of Mitrinović [1970] (theorem 1, p. 76) gives, by virtue of (4.24), that

$$\sum_{i=1}^{n+1} \theta_i \xi_i^2 > \left(\sum_{i=1}^{n+1} \theta_i \xi_i \right)^2 \ ,$$

and so, by (4.26) and (4.35),

$$-\lambda_n c_n > \lambda_n^2 \quad . \tag{4.36}$$

It has already been demonstrated that (4.36) is equivalent to the statement that

$$\mu_n > 0 \ .$$

Thus

$$\lambda_{n-1} = \frac{d_n}{\mu_n} \tag{4.37}$$

exists and is positive, since $d_n > 0$.

Setting $x = 0$ in the recurrence relation (4.2) gives

$$q_{n+1}(0) = (\lambda_n + \mu_n) q_n(0) - \lambda_{n-1} \mu_n q_{n-1}(0) \ ,$$

and so, using (4.19),

$$\lambda_n q_n(0) = (\lambda_n + \mu_n) q_n(0) - \lambda_{n-1} \mu_n q_{n-1}(0) \ . \tag{4.38}$$

Rearrangement of (4.38) yields

$$q_n(0) = \lambda_{n-1} q_{n-1}(0) \ , \tag{4.39}$$

which completes the inductive argument.

All that remains to be shown is the initial specifications

$$q_0(x) \equiv 1, \quad q_1(x) = x + \lambda_0 \ . \tag{4.40}$$

The fact that the polynomials are monic gives immediately that $q_0(x) \equiv 1$. Now equation (4.39) in the case $n=1$ states

$$q_1(0) = \lambda_0 q_0(0) = \lambda_0 \ ,$$

and hence, as $q_1(x)$ is a monic first degree polynomial,

$$q_1(x) = x + \lambda_0 \ ,$$

as required.

Thus we have established the existence of positive real numbers $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}, \mu_1, \mu_2, \ldots, \mu_{N-1}$ such that

$$q_{n+1}(x) = (x+\lambda_n+\mu_n)q_n(x) - \lambda_{n-1}\mu_n q_{n-1}(x) , \qquad n=1,2,\ldots, \tag{4.41}$$

with $q_0(x) \equiv 1$, $q_1(x) = x+\lambda_0$.

We note that, from the given hyperexponential distribution, $q_{N-1}(x)$, $q_N(x)$ and λ_{N-1} are uniquely determined. By equating coefficients of powers of x in (4.41) for the case $n=N-1$, and noting that $q_{N-2}(x)$ is monic, we see that λ_{N-2}, μ_{N-1} and $q_{N-2}(x)$ are all uniquely determined, and so inductively all the rates and polynomials are likewise uniquely determined.

We can now summarise the results of the investigation of the converse of theorem 3.1.2 as follows.

3.4.1 Theorem .

A hyperexponential distribution with N component exponential distributions can be interpreted as the overflow from a finite birth-and-death process of N states, with associated overflow process. The rates of the process are uniquely determined, and can be computed recursively from (4.41), using equations (4.6) and (4.8) to provide a starting point. ∎

Theorem 3.1.2 and its converse, theorem 3.4.1, establish a one-to-one correspondence between finite birth-and-death processes of N states, with associated overflow process, and hyperexponential distributions with N component exponential distributions. This correspondence is achieved via the overflow from the finite birth-and-death process.

Given the parameters and weights of a hyperexponential distribution, it is a simple procedure to find the numerical values of the arrival and death rates, and a computer program to achieve this is given in appendix 2.

CHAPTER 4. OTHER SECONDARY PROCESSES ASSOCIATED WITH FINITE BIRTH-AND-DEATH PROCESSES.

4.1 THE TIME BETWEEN SUCCESSIVE ENTRIES TO THE FULL STATE.

We have examined, in the previous chapter, the nature of the overflow stream from a finite birth-and-death process. We may however also be interested in questions relating to the length of time between entries to the boundary state, and also, having left the boundary state, the length of time the system spends in other states before it again enters the boundary state.

Consider a birth-and-death process with $N+1$ states (where N is a positive integer), labelled $0,1,2,\ldots,N$. (Note that in the previous chapter we considered a birth-and-death process with N states, $0,1,2,\ldots,N-1$; we now consider a process with $N+1$ states, $0,1,2,\ldots,N$, merely to facilitate comparisons between results of this section and results from the previous chapter.) When the system is in state n $(0 < n < N)$ births occur at a rate λ_n, and independently deaths occur at a rate μ_n. When the system is in state 0 only births may occur, with rate λ_0, and when in state N only deaths may occur, with rate μ_N. We again assume $\lambda_n > 0$ for $0 \leqslant n < N$ and $\mu_n > 0$ for $0 < n \leqslant N$. For our present discussion it will not be necessary to impose an overflow structure on the finite birth-and-death process.

Denote by $\{\tau_0^{(n)} < \tau_1^{(n)} < \tau_2^{(n)} < \ldots\}$, where $0 \leqslant n < N$, the random epochs at which a birth occurs when the system is in state n, and also denote by $\{\sigma_0^{(n)} < \sigma_1^{(n)} < \sigma_2^{(n)} < \ldots\}$, where $0 < n \leqslant N$, the random epochs when the system is in state n and a death occurs.

For obvious intuitive reasons we will refer to the boundary state N as the *full state*, and on entering the full state the system becomes *blocked*. Define a *blocking period* to be the time from when the system is just about to become blocked until it next is just about to leave the full state. A *blocking cycle* will be the period of time from when the system is just about to become blocked until, after leaving the full state, the system is again just about to become blocked. That is, a blocking cycle is the time interval $[\tau_k^{(N-1)}, \tau_{k+1}^{(N-1)})$, for some $k \geq 0$. Note that at an instant when one blocking cycle ends the next blocking cycle begins. We will refer to the time from when the system is just about to leave the full state until it next is just about to become blocked as a *vacancy period*. That is, a vacancy period is the time interval $[\sigma_k^{(N)}, \tau_{k+1}^{(N-1)})^{(*)}$, for some $k \geq 0$. Thus a blocking cycle consists of two parts: a blocking cycle commences with a blocking period and is then followed by a vacancy period. These concepts are illustrated in figure 4.1.1.

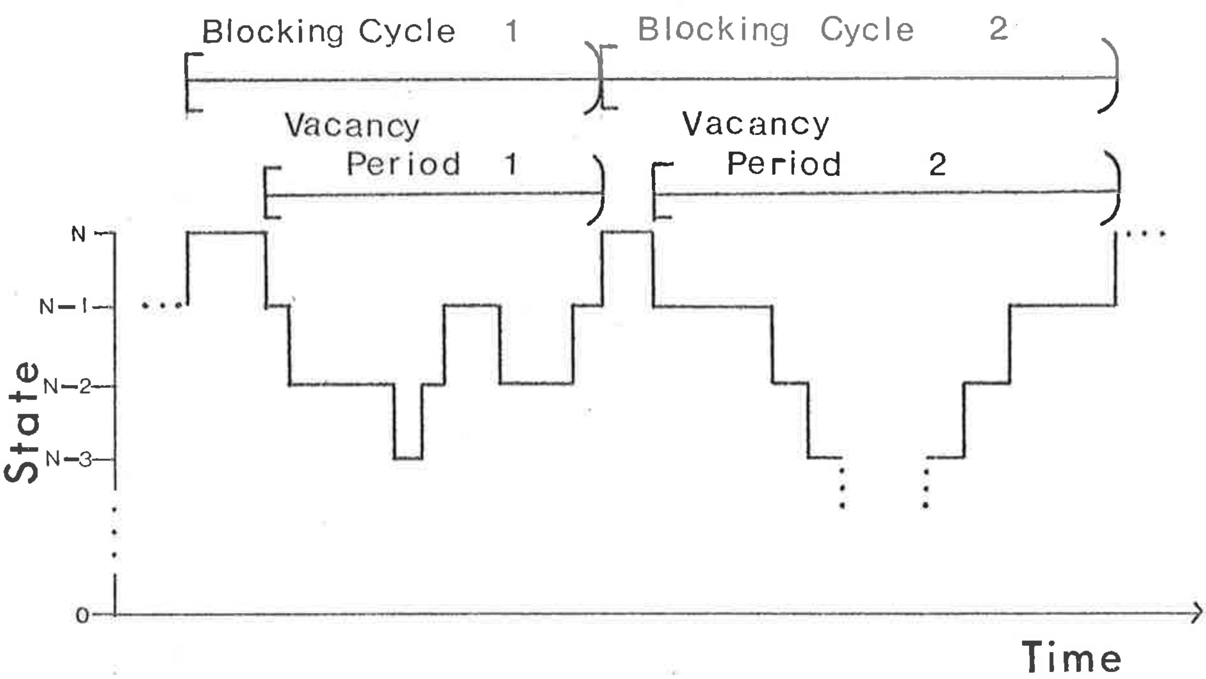

Fig. 4.1.1
An illustration of blocking cycles and vacancy periods.

(*) If the process originally started in state N this interval should read $[\sigma_k^{(N)}, \tau_k^{(N-1)})$.

The length of any blocking cycle has the same distribution as that of any other blocking cycle (recall the memoryless property 1.2.2). The analogous observations for blocking periods and vacancy periods likewise applies. Accordingly we will denote the probability distribution function for the length of a blocking cycle by $B(t)$ and that for the length of a vacancy period by $V(t)$. Of course, the distribution function for the length of a blocking period is simply $1 - e^{-\mu N t}$ $(t \geqslant 0)$.

Denote by $b_n(t)$, $0 \leqslant n \leqslant N-1$, the probability distribution function for the time to the next entry into the full state from $\tau_k^{(n)} - 0$; that is, from an epoch at which the transition $n \to (n+1)$ is just about to occur. (Compare with the functions $f_n(t)$ of section 3.1.) The system becomes blocked immediately upon the transition $(N-1) \to N$, and so trivially

$$b_{N-1}(t) = u(t) , \tag{1.1}$$

where $u(t)$ is the unit-step or Heaviside function given by the expression (1.2) of section 3.1. We write $b_{-1}(t)$ for the probability distribution function for the time until the next entry into state N from an arbitrary epoch at which the system is in state 0 and no birth is just about to occur. These functions $b_n(t)$ $(n=-1,0,1,\ldots,N-1)$ are well-defined due to the memoryless property 1.2.2. We now use the same procedure as in section 3.1 to find expressions for these functions $b_n(t)$ $(n=-1,0,1,\ldots,N-1)$.

Suppose that the system is in state n, $0 \leqslant n < N-1$, and consider an epoch $\tau_k^{(n)}$ (some $k \geqslant 0$). Note that the system will be in state $n+1$ at time $\tau_k^{(n)} + 0$. Thus $b_n(t)$ is the probability of the event that the system next enters state N by time

$\tau_k^{(n)} + t$. Since the entry into state N must itself be a birth, then the desired event can only occur if there is at least one birth or death subsequent to time $\tau_k^{(n)} + 0$ but by time $\tau_k^{(n)} + t$.

If we refer to the derivation in section 3.1 of the equations (1.3) and (1.4) which lead to the expressions (1.5) and (1.6) for $f_n(t)$, we see that exactly the same argument applies to $b_n(t)$ with "overflow" replaced by "entry into state N". Since $f_n(t)$ and $b_n(t)$ $(n=-1,0,1,\ldots,N-1)$ satisfy the same recurrence relation and have the same boundary condition, we thus conclude that

$$b_n(t) = f_n(t), \quad n=-1,0,1,\ldots,N-1. \tag{1.2}$$

This result, although it may be surprising at first glance, is actually to be expected. Both these distribution functions relate to time intervals which commence as a transition $n \to (n+1)$ is just about to occur and end at the instant of the next subsequent arrival in state $N-1$. Whether this arrival constitutes an overflow or causes a transition to some state N is of no consequence to the time interval being considered. In particular we note that $b_n(t)$, $n=-1,0,1,\ldots,N-1$, is independent of the value of μ_N.

Thus, from equation (1.14) of section 3.1, the Laplace-Stieltjes transform of $b_n(t)$, defined by

$$b_n^*(x) = \int_0^\infty e^{-xt}\, db_n(t), \tag{1.3}$$

$$\text{Re } x \geqslant 0, \; n=-1,0,1,\ldots,N-1,$$

is given by

$$b_n^*(x) = \frac{p_{n+1}(x)}{p_N(x)}, \quad n=-1,0,1,\ldots,N-1, \tag{1.4}$$

where $(p_n(x))_{n=0}^{\infty}$ is the sequence of orthogonal polynomials defined by (recall equation (1.12) of section 3.1 and its footnote)

$$\lambda_n p_{n+1}(x) - (x+\lambda_n+\mu_n)p_n(x) + \mu_n p_{n-1}(x) = 0, \qquad (1.5)$$

$$n=1,2,\ldots,$$

$$p_0(x) \equiv 1, \quad p_1(x) = \frac{x+\lambda_0}{\lambda_0} .$$

Consider an epoch $\sigma_k^{(N)}$ (some $k \geqslant 0$) at which a vacancy period commences. Now the distribution of time from $\sigma_k^{(N)} - 0$ until the next entry into the full state (that is, the length of the vacancy period) is the same as the distribution of time until the next entry into state N from an epoch $\tau_\ell^{(N-2)} - 0$ (some $\ell \geqslant 0$). This follows immediately from the memoryless property 1.2.2, since at both $\sigma_k^{(N)} + 0$ and $\tau_\ell^{(N-2)} + 0$ the system is in state N-1. That is,

$$V(t) = b_{N-2}(t) , \qquad (1.6)$$

which implies, when combined with result (1.2) of this section and expression (1.15) of section 3.1, that

$$V(t) = F(t) , \qquad (1.7)$$

where $F(t)$ is the inter-overflow time distribution function for the corresponding birth-and-death process of N states with associated overflow process. Thus equation (1.24) of section 3.1 gives also an expression for $V(t)$.

We can summarise the results concerning the vacancy period as follows:

4.1.1 Theorem.

The distribution function of the length of a vacancy period for a finite birth-and-death process of $N+1$ states is a convex linear combination of N exponential distributions. Specifically,

$$V(t) = \sum_{i=1}^{N} \beta_i [1 - \exp(x_i t)] , \quad t \geq 0, \tag{1.8}$$

where $\{x_i ; i=1,\ldots,N\}$ is the set of zeros of $p_N(x)$ and

$$\beta_i = - \frac{p_{N-1}(x_i)}{x_i p_N'(x_i)} , \quad i=1,2,\ldots,N. \tag{1.9}$$

We note that $V(t)$ is independent of the value of μ_N, and also is exactly the same as the distribution function for the time between successive overflows from the corresponding birth-and-death process of N states formed by deleting state N and considering births in state $N-1$ as overflows. ∎

Now the length of a blocking cycle is equal to the sum of the length of the blocking period and the length of the following vacancy period. Thus the distribution function of the length of a blocking cycle, $B(t)$, is the (Lebesgue-Stieltjes) convolution of the distribution function of the time spent in state N with that of the length of a vacancy period, $V(t)$. That is,

$$B(t) = \int_0^t V(t-y)\, d[1-e^{-\mu_N y}]$$

$$= \int_0^t V(t-y) \mu_N e^{-\mu_N y} dy , \quad t \geq 0. \tag{1.10}$$

If we denote by $B^*(x)$ and $V^*(x)$ the Laplace-Stieltjes transforms of $B(t)$ and $V(t)$ respectively, given by

$$B^*(x) = \int_0^\infty e^{-xt} dB(t), \quad \text{Re } x \geq 0, \tag{1.11}$$

and

$$V^*(x) = \int_0^\infty e^{-xt} dV(t), \quad \text{Re } x \geq 0, \tag{1.12}$$

then equation (1.10) implies that

$$B^*(x) = \frac{\mu_N}{x+\mu_N} V^*(x) \quad . \tag{1.13}$$

From (1.7) we have

$$V^*(x) = F^*(x) \quad , \tag{1.14}$$

and equation (1.17) of section 3.1 gives that

$$F^*(x) = \frac{p_{N-1}(x)}{p_N(x)} \quad . \tag{1.15}$$

Combining (1.14) and (1.15) and substituting into (1.13) reveals that

$$B^*(x) = \frac{\mu_N}{x+\mu_N} \frac{p_{N-1}(x)}{p_N(x)} \quad . \tag{1.16}$$

Using the notation and results of lemma 3.1.1, we can write equation (1.16) as

$$B^*(x) = \frac{\mu_N}{(x+\mu_N)} \sum_{i=1}^{N} \frac{\alpha_i}{(x-x_i)} \quad . \tag{1.17}$$

Recalling equation (1.23) of section (3.1), which defines β_i $(i=1,2,...,N)$ as

$$\beta_i = -\frac{\alpha_i}{x_i} , \quad i=1,2,...,N,$$

we can express equation (1.17) as

$$B^*(x) = \sum_{i=1}^{N} \frac{-x_i \beta_i \mu_N}{(x+\mu_N)(x-x_i)} \quad . \tag{1.18}$$

In order to invert the Laplace-Stieltjes transform we must decompose the expression (1.18) for $B^*(x)$ into partial fractions. We note that if $\mu_N \neq -x_i$ for some i $(1 \leqslant i \leqslant N)$ then

$$\frac{-x_i \beta_i \mu_N}{(x+\mu_N)(x-x_i)} = \left(\frac{x_i \beta_i}{\mu_N+x_i}\right) \frac{\mu_N}{(x+\mu_N)} + \left(\frac{\beta_i \mu_N}{\mu_N+x_i}\right) \frac{(-x_i)}{(x-x_i)} . \qquad (1.19)$$

Thus from (1.18) we have

$$B^*(x) = \left(\sum_{i=1}^{N} \frac{x_i \beta_i}{\mu_N+x_i}\right) \frac{\mu_N}{(x+\mu_N)} + \sum_{i=1}^{N} \left[\frac{\beta_i \mu_N}{(\mu_N+x_i)} \cdot \frac{(-x_i)}{(x-x_i)}\right] , \qquad (1.20)$$

if $\mu_N \neq -x_i$ for all $i=1,2,\ldots,N$,

and

$$B^*(x) = \left(\sum_{\substack{i=1 \\ i\neq j}}^{N} \frac{x_i \beta_i}{\mu_N+x_i}\right) \frac{\mu_N}{(x+\mu_N)} + \sum_{\substack{i=1 \\ i\neq j}}^{N} \left[\frac{\beta_i \mu_N}{(\mu_N+x_i)} \cdot \frac{(-x_i)}{(x-x_i)}\right] + \frac{\beta_j \mu_N^2}{(x+\mu_N)^2} , \qquad (1.21)$$

if $\mu_N = -x_j$ for some j $(1 \leqslant j \leqslant N)$.

(Recall that the zeros of $p_N(x)$ are distinct, and so μ_N can equal $-x_j$ for at most one j $(1 \leqslant j \leqslant N)$.)

Now if the Laplace-Stieltjes transform of some function $f(t)$ is $\frac{\kappa}{x+\kappa}$, for some constant κ, then

$$f(t) - f(0) = (1 - e^{-\kappa t}) , \quad t \geqslant 0.$$

Also if the Laplace-Stieltjes transform of some function $g(t)$ is $\frac{\kappa}{(x+\kappa)^2}$, then

$$g(t) - g(0) = -te^{-\kappa t} + \frac{1}{\kappa}(1 - e^{-\kappa t}) , \quad t \geqslant 0.$$

However,

$$B(0) = 0 ,$$

and so, inverting (1.20) and (1.21), we have

$$B(t) = \left(\sum_{i=1}^{N} \frac{x_i \beta_i}{\mu_N + x_i}\right) \cdot (1 - e^{-\mu_N t}) + \sum_{i=1}^{N} \left[\frac{\beta_i \mu_N}{(\mu_N + x_i)} \cdot (1 - e^{x_i t})\right], \quad t \geq 0, \tag{1.22}$$

$$\text{if} \quad \mu_N \neq -x_i \quad \text{for all} \quad i=1,2,\ldots,N,$$

and

$$B(t) = \left(\sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{x_i \beta_i}{\mu_N + x_i}\right) \cdot (1 - e^{-\mu_N t}) + \sum_{\substack{i=1 \\ i \neq j}}^{N} \left[\frac{\beta_i \mu_N}{(\mu_N + x_i)} \cdot (1 - e^{x_i t})\right] + \beta_j (1 - e^{-\mu_N t}) - \beta_j \mu_N t e^{-\mu_N t}, \quad t \geq 0, \tag{1.23}$$

$$\text{if} \quad \mu_N = -x_j \quad \text{for some} \quad j \quad (1 \leq j \leq N).$$

It is self-evident that the latter expression for $B(t)$ (for the case $\mu_N = -x_j$ for some j $(1 \leq j \leq N)$) is not hyperexponential. Suppose that $\mu_N \neq -x_i$ for all $i=1,2,\ldots,N$, so that expression (1.22) for $B(t)$ applies. Define γ_i $(i=1,2,\ldots,N+1)$ by

$$\gamma_i = \frac{\beta_i \mu_N}{\mu_N + x_i}, \quad i=1,2,\ldots,N,$$

$$\gamma_{N+1} = \sum_{i=1}^{N} \frac{x_i \beta_i}{\mu_N + x_i} . \tag{1.24}$$

Then, from (1.22),

$$B(t) = \gamma_{N+1}(1 - e^{-\mu_N t}) + \sum_{i=1}^{N} \gamma_i (1 - e^{x_i t}),$$

$$t \geq 0, \; \mu_N \neq -x_i \quad \text{for all} \quad i=1,2,\ldots,N. \tag{1.25}$$

Thus, in this case, $B(t)$ is a weighted sum of exponential distributions, and these weights of course sum to unity since $B(t)$ is a distribution function. For $B(t)$ to be hyperexponential we require in addition that the weights be positive. Now $\gamma_i > 0$ $(i=1,2,\ldots,N)$ if and only if $(\mu_N+x_i) > 0$ $(i=1,2,\ldots,N)$, since $\beta_i > 0$ $(i=1,2,\ldots,N)$ and $\mu_N > 0$. But if $(\mu_N+x_i) > 0$ for all $i=1,2,\ldots,N$, then $\gamma_{N+1} < 0$, since $x_i < 0$ $(i=1,2,\ldots,N)$. Thus we will have always at least one negative weight, and so $B(t)$ is not hyperexponential.

We can summarise the results concerning the blocking cycle as follows:

4.1.2 Theorem.

The distribution function of the length of a blocking cycle for a finite birth-and-death process of $N+1$ states is given by

$$B(t) = \left(\sum_{i=1}^{N} \frac{x_i \beta_i}{\mu_N+x_i}\right) \cdot (1 - e^{-\mu_N t})$$

$$+ \sum_{i=1}^{N} \left[\frac{\beta_i \mu_N}{(\mu_N+x_i)} \cdot (1 - e^{x_i t})\right], \quad t \geq 0, \tag{1.26}$$

$$\text{if} \quad \mu_N \neq -x_i \quad \text{for all} \quad i=1,2,\ldots,N,$$

and

$$B(t) = \left(\sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{x_i \beta_i}{\mu_N+x_i}\right) \cdot (1 - e^{-\mu_N t})$$

$$+ \sum_{\substack{i=1 \\ i \neq j}}^{N} \left[\frac{\beta_i \mu_N}{(\mu_N+x_i)} \cdot (1 - e^{x_i t})\right] \tag{1.27}$$

$$+ \beta_j (1 - e^{-\mu_N t}) - \beta_j \mu_N t e^{-\mu_N t}, \quad t \geq 0,$$

$$\text{if} \quad \mu_N = -x_j \quad \text{for some} \quad j \quad (1 \leqslant j \leqslant N),$$

where $\{x_i\ ;\ i=1,2,\ldots,N\}$ is the set of zeros of $p_N(x)$ and

$$\beta_i = -\frac{p_{N-1}(x_i)}{x_i\, p_N'(x_i)}\ ,\ i=1,2,\ldots,N. \qquad (1.28)$$

We note that $B(t)$ is never hyperexponential. ∎

4.2 THE TIME BETWEEN SUCCESSIVE ENTRIES TO THE EMPTY STATE.

In this section we will examine the same birth-and-death process of $N+1$ states, but we will now concentrate on phenomena associated with the 0 boundary state. Again we will denote by $\{\tau_0^{(n)} < \tau_1^{(n)} < \tau_2^{(n)} < \ldots\}$, where $0 \leqslant n < N$, the random epochs at which a birth occurs when the system is in state n, and also denote by $\{\sigma_0^{(n)} < \sigma_1^{(n)} < \sigma_2^{(n)} < \ldots\}$, where $0 < n \leqslant N$, the random epochs when the system is in state n and a death occurs.

We shall now make some definitions related to the boundary state 0 which are analogous to the definitions of section 4.1 concerning the boundary state N. We will refer to the boundary state 0 as the *empty state*, and on entering the empty state the system becomes *idle*. Define an *idle period* to be the time from when the system is just about to become idle until it next is just about to leave the empty state. An *idle cycle* will be the period of time from when the system is just about to become idle until, after leaving the empty state, the system is again just about to become idle. That is, an idle cycle is the time interval $[\sigma_k^{(1)}, \sigma_{k+1}^{(1)})$, for some $k \geqslant 0$. Note that at an instant when one idle cycle ends the next idle cycle begins. We will refer to the time from when the system is just about to leave the empty state until it next is just about to become idle as an *engaged period*. That is, an engaged period is the time interval $[\tau_k^{(0)}, \sigma_{k+1}^{(1)})$ (*) for some $k \geqslant 0$. Thus an idle cycle consists of two parts: an idle cycle commences with an idle period and is then followed by an engaged period. These concepts are illustrated in figure 4.2.1.

(*) If the process originally started in state 0 this interval should read $[\tau_k^{(0)}, \sigma_k^{(1)})$.

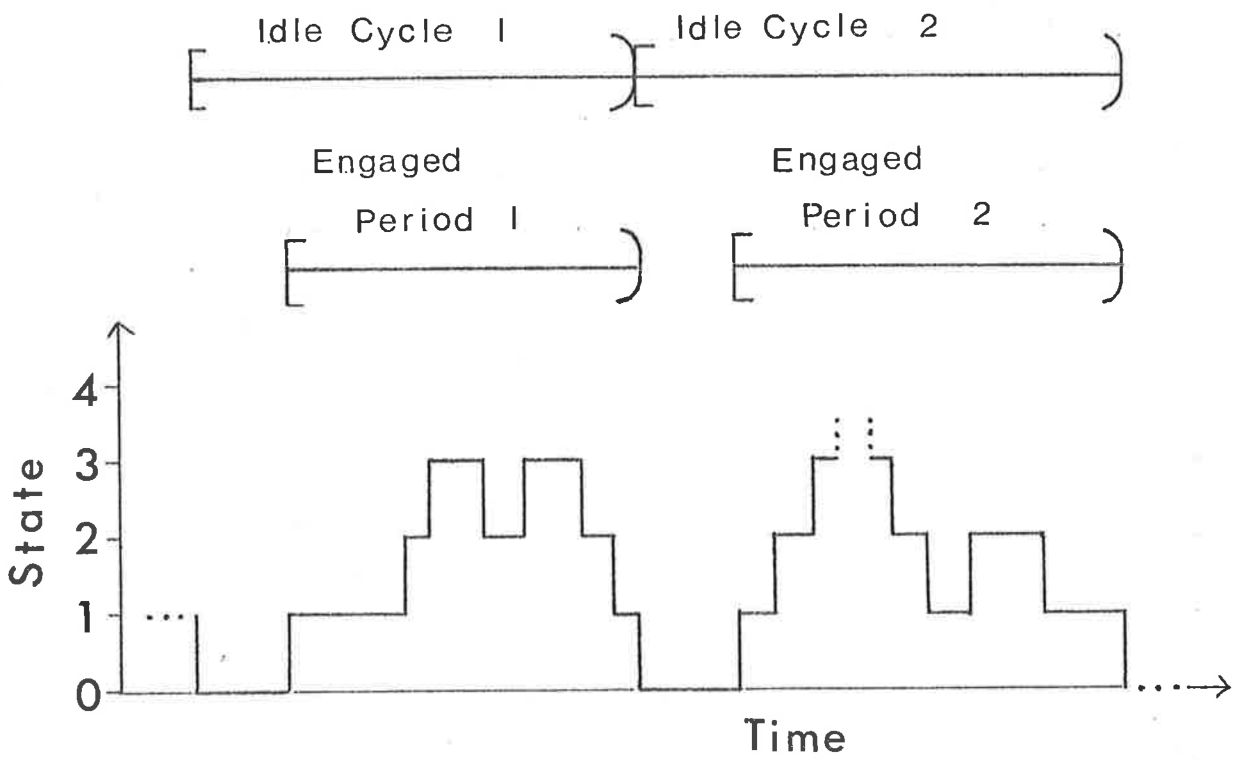

Fig. 4.2.1

An illustration of idle cycles and engaged periods.

The length of any idle cycle has the same distribution as that of any other idle cycle (recall the memoryless property 1.2.2). The analogous observations for idle periods and engaged periods likewise applies. Accordingly we will denote the probability distribution function for the length of an idle cycle by $I(t)$ and that for the length of an engaged period by $E(t)$. Of course, the distribution function for the length of an idle period is simply $1 - e^{-\lambda_0 t}$ $(t \geq 0)$.

Consider a birth-and-death process of $N+1$ states, labelled $\hat{0},\hat{1},\hat{2},\ldots,\hat{N}$, with birth rates $\lambda_{\hat{n}}$ $(n=0,1,\ldots,N-1)$ given by

$$\lambda_{\hat{n}} = \mu_{N-n} \quad , \quad n=0,1,2,\ldots,N-1, \qquad (2.1)$$

and death rates $\hat{\mu}_{\hat{n}}$ (n=1,2,...,N) given by

$$\hat{\mu}_{\hat{n}} = \lambda_{N-n} , \quad n=1,2,3,...,N. \tag{2.2}$$

Define the orthogonal polynomial sequence $(W_n(x))_{n=0}^{\infty}$ by

$$\hat{\lambda}_{\hat{n}} W_{n+1}(x) - (x+\hat{\lambda}_{\hat{n}}+\hat{\mu}_{\hat{n}})W_n(x) + \hat{\mu}_{\hat{n}} W_{n-1}(x) = 0, \quad n=1,2,3,...,^{(*)} \tag{2.3}$$

$$W_0(x) \equiv 1, \quad W_1(x) = \frac{1}{\hat{\lambda}_{\hat{0}}}(x+\hat{\lambda}_{\hat{0}}) .$$

Let $\{y_i ; i=1,2,...,N\}$ be the set of zeros of $W_N(x)$ and define ρ_i (i=1,2,...,N) by

$$\rho_i = - \frac{W_{N-1}(y_i)}{y_i \, W'_N(y_i)} . \tag{2.4}$$

Let $\hat{V}(t)$ be the probability distribution function for the length of a vacancy period, and let $\hat{B}(t)$ be that for the length of a blocking cycle, for this birth-and-death process.

Applying theorems 4.1.1 and 4.1.2 to this new process we have immediately that

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(*) For $n > N-1$ take $\hat{\lambda}_{\hat{n}}$ and $\hat{\mu}_{\hat{n}}$ to be any positive real numbers for the sake of the definition (2.3) of $(W_n(x))_{n=0}^{\infty}$. The polynomials $W_{N+1}(x)$, $W_{N+2}(x)$,..., are irrelevant to the analysis.

$$\hat{V}(t) = \sum_{i=1}^{N} \rho_i \left[1 - \exp(y_i t)\right] , \quad t \geqslant 0, \tag{2.5}$$

$$\hat{B}(t) = \left(\sum_{i=1}^{N} \frac{y_i \rho_i}{\hat{\mu}_{\hat{N}}+y_i}\right) \cdot (1 - \exp(-\hat{\mu}_{\hat{N}} t)) + \sum_{i=1}^{N} \left[\frac{\rho_i \hat{\mu}_{\hat{N}}}{(\hat{\mu}_{\hat{N}}+y_i)} \cdot (1 - \exp(y_i t))\right], \quad t \geqslant 0, \tag{2.6}$$

if $\hat{\mu}_{\hat{N}} \neq -y_i$ for all $i=1,2,\ldots,N$,

$$\hat{B}(t) = \left(\sum_{\substack{i=1 \\ i\neq j}}^{N} \frac{y_i \rho_i}{\hat{\mu}_{\hat{N}}+y_i}\right) \cdot (1 - \exp(-\hat{\mu}_{\hat{N}} t)) + \sum_{\substack{i=1 \\ i\neq j}}^{N} \left[\frac{\rho_i \hat{\mu}_{\hat{N}}}{(\hat{\mu}_{\hat{N}}+y_i)} \cdot (1 - \exp(y_i t))\right] \tag{2.7}$$

$$+ \rho_j \left[1-\exp(-\hat{\mu}_{\hat{N}} t)\right] - \rho_j \hat{\mu}_{\hat{N}} t \exp(-\hat{\mu}_{\hat{N}} t), \quad t \geqslant 0,$$

if $\hat{\mu}_{\hat{N}} = -y_j$ for some j $(1 \leqslant j \leqslant N)$.

Now if we associate state $\hat{n}$ of the new process with state $N-n$ of the original process $(n=0,1,2,\ldots,N)$, then, by virtue of relations (2.1) and (2.2), we have that the two processes are stochastically identical. However the blocking cycles and vacancy periods of the new process are precisely the idle cycles and engaged periods respectively of the original process. Thus

$$E(t) = \hat{V}(t) , \tag{2.8}$$

and

$$I(t) = \hat{B}(t) . \tag{2.9}$$

Using (2.1) and (2.2) we can express the recurrence relations (2.3), which define the orthogonal polynomial sequence $(W_n(x))_{n=0}^{\infty}$, in terms of quantities associated with the original process:

$$\mu_{N-n} W_{n+1}(x) - (x+\mu_{N-n}+\lambda_{N-n})W_n(x) + \lambda_{N-n} W_{n-1}(x) = 0, \quad n=1,2,3,\ldots,^{(*)} \tag{2.10}$$

$$W_0(x) \equiv 1, \quad W_1(x) = \frac{1}{\mu_N}(x+\mu_N) .$$

Combining (2.8) with (2.5) and (2.9) with (2.6) and (2.7) (and again applying (2.1) and (2.2)) we have that

$$E(t) = \sum_{i=1}^{N} \rho_i [1 - \exp(y_i t)] , \quad t \geqslant 0, \tag{2.11}$$

$$I(t) = \left(\sum_{i=1}^{N} \frac{y_i \rho_i}{\lambda_0+y_i} \right) \cdot (1 - \exp(-\lambda_0 t)) + \sum_{i=1}^{N} \left[\frac{\rho_i \lambda_0}{(\lambda_0+y_i)} \cdot (1 - \exp(y_i t)) \right], \quad t \geqslant 0, \tag{2.12}$$

if $\lambda_0 \neq -y_i$ for all $i=1,2,\ldots,N$,

$$I(t) = \left(\sum_{\substack{i=1 \\ i\neq j}}^{N} \frac{y_i \rho_i}{\lambda_0+y_i} \right) \cdot (1 - \exp(-\lambda_0 t)) + \sum_{\substack{i=1 \\ i\neq j}}^{N} \left[\frac{\rho_i \lambda_0}{(\lambda_0+y_i)} \cdot (1 - \exp(y_i t)) \right] \tag{2.13}$$

$$+ \rho_j [1 - \exp(-\lambda_0 t)] - \rho_j \lambda_0 t \exp(-\lambda_0 t), \quad t \geqslant 0,$$

if $\lambda_0 = -y_j$ for some j $(1 \leqslant j \leqslant N)$.

Theorem 4.1.2 gives also that $\hat{B}(t)$ and hence $I(t)$ is never hyperexponential.

(*) For $n > N-1$ take λ_{N-n}, μ_{N-n} to be any positive real numbers for the sake of the definition (2.10) of $(W_n(x))_{n=0}^{\infty}$. (Refer to the footnote to (2.3).

We can summarise the results of this section as follows:

<u>4.2.1 Theorem.</u>

The distribution function of the length of an engaged period for a finite birth-and-death process of $N+1$ states is a convex linear combination of N exponential distributions.

Specifically,

$$E(t) = \sum_{i=1}^{N} \rho_i \,[1 - \exp(y_i t)] \;, \quad t \geq 0, \tag{2.14}$$

where $\{y_i \,;\, i=1,2,\ldots,N\}$ is the set of zeros of $W_N(x)$ and

$$\rho_i = -\frac{W_{N-1}(y_i)}{y_i\, W_N'(y_i)} \,, \quad i=1,2,\ldots,N. \tag{2.15}$$

We note that $E(t)$ is independent of the value of λ_0. ∎

<u>4.2.2 Theorem.</u>

The distribution function for the length of an idle cycle for a finite birth-and-death process of $N+1$ states is given by

$$\begin{aligned} I(t) &= \left(\sum_{i=1}^{N} \frac{y_i\,\rho_i}{\lambda_0+y_i}\right) \cdot (1 - e^{-\lambda_0 t}) \\ &\quad + \sum_{i=1}^{N} \left[\frac{\rho_i\,\lambda_0}{(\lambda_0+y_i)} \cdot (1 - e^{y_i t})\right], \quad t \geq 0, \end{aligned} \tag{2.16}$$

$$\text{if} \quad \lambda_0 \neq -y_i \quad \text{for all} \quad i=1,2,\ldots,N,$$

and

$$\begin{aligned} I(t) &= \left(\sum_{\substack{i=1 \\ i\neq j}}^{N} \frac{y_i\,\rho_i}{\lambda_0+y_i}\right) \cdot (1 - e^{-\lambda_0 t}) \\ &\quad + \sum_{\substack{i=1 \\ i\neq j}}^{N} \left[\frac{\rho_i\,\lambda_0}{(\lambda_0+y_i)} \cdot (1 - e^{y_i t})\right] \qquad (2.17) \\ &\quad + \rho_j\,(1 - e^{-\lambda_0 t}) - \rho_j\,\lambda_0 t e^{-\lambda_0 t}, \quad t \geq 0, \end{aligned}$$

$$\text{if} \quad \lambda_0 = -y_j \quad \text{for some} \quad j \quad (1 \leq j \leq N),$$

where $\{y_i ; i=1,2,...,N\}$ is the set of zeros of $W_N(x)$ and

$$\rho_i = -\frac{W_{N-1}(y_i)}{y_i W'_N(y_i)} , \quad i=1,2,...,N . \tag{2.18}$$

We note that $I(t)$ is never hyperexponential. ∎

We point out that Keilson [1971] also finds that the length of an engaged period (or, as he terms it, "the sojourn time on $\{1,2,...,N\}$") is hyperexponential. However our method, which is essentially constructive, not only provides a simple procedure for the numerical determination of the distribution, but also relates to the methods and results of the previous sections thereby giving greater insight into the structure of the process.

4.3 THE TIME BETWEEN SUCCESSIVE ENTRIES TO AN INTERMEDIATE STATE.

As illustrated in section 3.2, a number of limited capacity queueing models can be framed as finite birth-and-death processes. The theory developed in the previous sections, which we recall concerned overflows, entries to the full state, and entries to the empty state, is of obvious relevance to the models being studied. However in some queueing models there may be some other states which are of particular interest. For example, let us recall the telephone trunking model with holding registers of section 3.2.2. The theory which we have already developed concerns overflows, the full state $T_1 + T_2$, and the empty state 0. Another state, of interest in its own right, is the state T_1, which is the state in which all trunks are occupied but all holding registers are vacant. For this reason we now turn our attention to the study of phenomena associated with some (arbitrarily selected) intermediate state.

As in sections 4.1 and 4.2 we will consider a finite birth-and-death process of $N+1$ states, labelled $0,1,2,\ldots,N$, with positive birth rates λ_n $(n=0,1,\ldots,N-1)$ and positive death rates μ_n $(n=1,2,\ldots,N)$. Again we will denote by $\{\tau_0^{(n)} < \tau_1^{(n)} < \tau_2^{(n)} < \ldots\}$, for $0 \leqslant n < N$, and $\{\sigma_0^{(n)} < \sigma_1^{(n)} < \sigma_2^{(n)} < \ldots\}$, for $0 < n \leqslant N$, the random epochs at which the system is in state n and a birth and death occur respectively.

Let us arbitrarily select, but then fix, an intermediate state, r say $(0 < r < N)$. We define an *r-cycle* to be an interval of time which commences as the system is just about to enter state r until it next is just about to enter state r. It is clear from our definition that any instant of time belongs to exactly one r-cycle, and the memoryless property 1.2.2 gives that the distribution of the length of any r-cycle is the same as that for any other r-cycle. However, as the system may enter or leave state r with a birth or a death, any particular r-cycle can have one of the following four forms: (for some $k \geqslant 0$, $\ell \geqslant 0$)

(i) $[\tau_k^{(r-1)}, \tau_{k+1}^{(r-1)})$, if $\sigma_\ell^{(r+1)} < \tau_k^{(r-1)} < \tau_{k+1}^{(r-1)} < \sigma_{\ell+1}^{(r+1)}$;

(ii) $[\tau_k^{(r-1)}, \sigma_\ell^{(r+1)})$, if $\sigma_{\ell-1}^{(r+1)} < \tau_k^{(r-1)} < \sigma_\ell^{(r+1)} < \tau_{k+1}^{(r-1)}$;

(iii) $[\sigma_k^{(r+1)}, \sigma_{k+1}^{(r+1)})$, if $\tau_\ell^{(r-1)} < \sigma_k^{(r+1)} < \sigma_{k+1}^{(r+1)} < \tau_{\ell+1}^{(r-1)}$;

(iv) $[\sigma_k^{(r+1)}, \tau_\ell^{(r-1)})$, if $\tau_{\ell-1}^{(r-1)} < \sigma_k^{(r+1)} < \tau_\ell^{(r-1)} < \sigma_{k+1}^{(r+1)}$.

The concept of r-cycle is illustrated in figure 4.3.1, which gives an example for each of the four possible types.

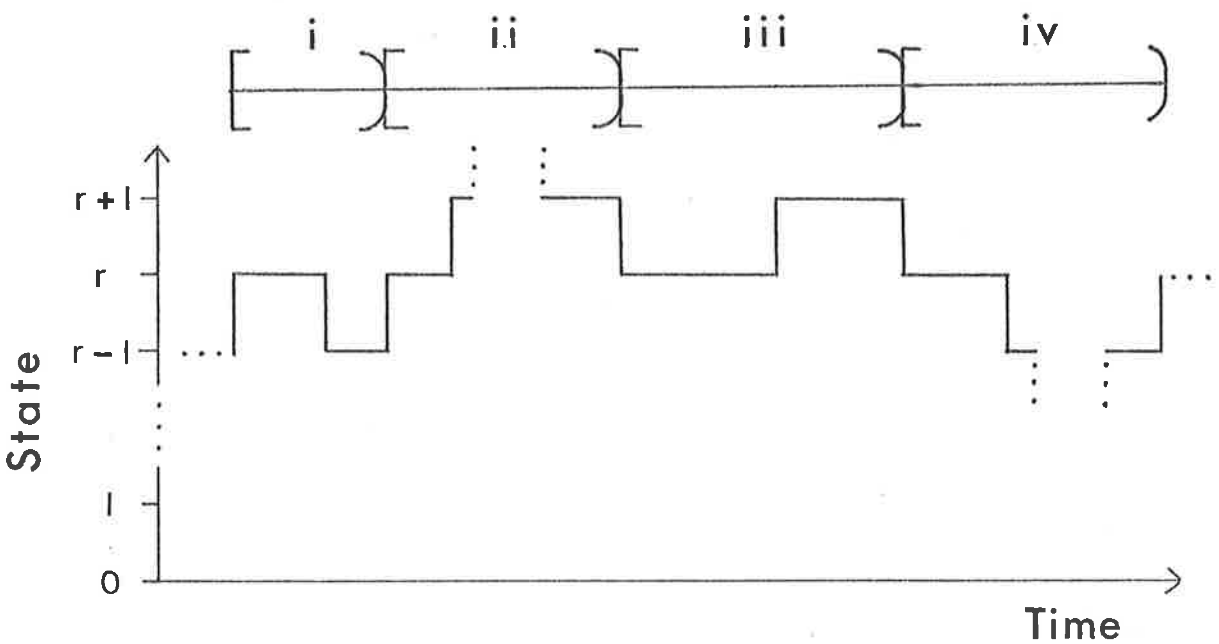

Fig. 4.3.1

An illustration of the concept of r-cycles.

Denote by $G_r(t)$ the probability distribution function for the length of an r-cycle. We now seek to find an expression for $G_r(t)$.

Denote by $g_n(t)$, $0 \leq n \leq N-1$, the probability distribution function for the time to the next entry to state r from $\tau_k^{(n)} - 0$; that is, from an epoch at which the transition $n \to (n+1)$ is just about to occur. Trivially,

$$g_{r-1}(t) = u(t) \ , \qquad (3.1)$$

where $u(t)$ is the unit-step or Heaviside function given by the expression (1.2) of section 3.1. We write $g_{-1}(t)$ for the probability distribution function for the time until the next entry into state r from an arbitrary epoch at which the system is in state 0 and no birth is just about to occur. The memoryless property 1.2.2 implies that the functions $g_n(t)$ $(n=-1,0,1,\ldots,N-1)$ are well-defined; for the sake of notational brevity we have not explicitly indicated the dependence of $g_n(t)$ on r.

We can again apply the technique used in the derivation of equations (1.5) and (1.6) of section 3.1 to find recurrence expressions for $g_n(t)$ $(-1 \leq n < r-1$, and $r < n < N-1)$ simply by replacing "overflow" with "entry into state r". For the case $n=r$, if the next event subsequent to the initial transition $r \to (r+1)$ is a death, then this event is the next entry into state r. However, as we have that $g_{r-1}(t) = u(t)$, then the expression for $r < n < N-1$ also holds for $n=r$. Noting that the next event after a transition $(N-1) \to N$ must be a death, we also have immediately an expression for $g_{N-1}(t)$. Thus,

$$g_{-1}(t) = \int_0^t g_0(t-y)e^{-\lambda_0 y}\lambda_0 dy \ , \quad t \geq 0, \qquad (3.2)$$

$$g_n(t) = \int_0^t e^{-(\lambda_{n+1}+\mu_{n+1})y}[\lambda_{n+1}g_{n+1}(t-y) + \mu_{n+1}g_{n-1}(t-y)]dy\ , \qquad 0 \leqslant n \leqslant r-2\ ,\quad t \geqslant 0, \tag{3.3}$$

$$g_{r-1}(t) = u(t)\ , \tag{3.4}$$

$$g_n(t) = \int_0^t e^{-(\lambda_{n+1}+\mu_{n+1})y}[\lambda_{n+1}g_{n+1}(t-y) + \mu_{n+1}g_{n-1}(t-y)]dy\ , \qquad r \leqslant n \leqslant N-2\ ,\quad t \geqslant 0, \tag{3.5}$$

$$g_{N-1}(t) = \int_0^t g_{N-2}(t-y)e^{-\mu_N y}\mu_N dy\ ,\quad t \geqslant 0. \tag{3.6}$$

In terms of the Laplace-Stieltjes transform of $g_n(t)$ $(-1 \leqslant n \leqslant N-1)$, defined by

$$g_n^*(x) = \int_0^\infty e^{-xt}\,dg_n(t)\ , \qquad \text{Re } x \geqslant 0,\quad n=-1,0,1,\ldots,N-1, \tag{3.7}$$

these recurrence equations read

$$g_{-1}^*(x) = \frac{\lambda_0}{x+\lambda_0}\,g_0^*(x)\ , \tag{3.8}$$

$$g_n^*(x) = \frac{\lambda_{n+1}}{x+\lambda_{n+1}+\mu_{n+1}}\,g_{n+1}^*(x) + \frac{\mu_{n+1}}{x+\lambda_{n+1}+\mu_{n+1}}\,g_{n-1}^*(x)\ , \qquad 0 \leqslant n \leqslant r-2\ , \tag{3.9}$$

$$g_{r-1}^*(x) = 1\ , \tag{3.10}$$

$$g_n^*(x) = \frac{\lambda_{n+1}}{x+\lambda_{n+1}+\mu_{n+1}}\,g_{n+1}^*(x) + \frac{\mu_{n+1}}{x+\lambda_{n+1}+\mu_{n+1}}\,g_{n-1}^*(x)\ , \qquad r \leqslant n \leqslant N-2\ , \tag{3.11}$$

$$g_{N-1}^*(x) = \frac{\mu_N}{x+\mu_N}\,g_{N-2}^*(x)\ . \tag{3.12}$$

Hence, on rearrangement, we have

$$\lambda_0 g_0^*(x) - (x+\lambda_0) g_{-1}^*(x) = 0 \ , \tag{3.13}$$

$$\lambda_{n+1} g_{n+1}^*(x) - (x+\lambda_{n+1}+\mu_{n+1}) g_n^*(x) + \mu_{n+1} g_{n-1}^*(x) = 0 \ , \tag{3.14}$$

$$0 \leqslant n \leqslant r-2,$$

$$g_{r-1}^*(x) = 1 \ , \tag{3.15}$$

$$\lambda_{n+1} g_{n+1}^*(x) - (x+\lambda_{n+1}+\mu_{n+1}) g_n^*(x) + \mu_{n+1} g_{n-1}^*(x) = 0 \ , \tag{3.16}$$

$$r \leqslant n \leqslant N-2,$$

$$-(x+\mu_N) g_{N-1}^*(x) + \mu_N g_{N-2}^*(x) = 0 \ . \tag{3.17}$$

We recall that the orthogonal polynomial sequence $(p_n(x))_{n=0}^{\infty}$ was defined by

$$\lambda_n p_{n+1}(x) - (x+\lambda_n+\mu_n) p_n(x) + \mu_n p_{n-1}(x) = 0 \ , \tag{3.18}$$

$$n=1,2,3,\ldots \ ,$$

$$p_0(x) \equiv 1 \ , \quad p_1(x) = \frac{1}{\lambda_0}(x+\lambda_0)$$

(refer to equation (1.12) of section 3.1). Comparison of (3.18) with (3.13) and (3.14) reveals immediately that

$$g_n^*(x) = \alpha(x) p_{n+1}(x) \ , \quad n=-1,0,1,\ldots,r-1, \tag{3.19}$$

where $\alpha(x)$ is independent of n. The condition (3.15) fixes $\alpha(x)$ to be

$$\alpha(x) = \frac{1}{p_r(x)} \ ,$$

and so from (3.19)

$$g_n^*(x) = \frac{p_{n+1}(x)}{p_r(x)} \ , \quad n=-1,0,1,\ldots,r-1 \ . \tag{3.20}$$

The orthogonal polynomial sequence $(W_n(x))_{n=0}^{\infty}$ was defined by

$$\mu_{N-n} W_{n+1}(x) - (x+\mu_{N-n}+\lambda_{N-n})W_n(x) + \lambda_{N-n} W_{n-1}(x) = 0 , \qquad n=1,2,3,\ldots , \tag{3.21}$$

$$W_0(x) \equiv 1 , \quad W_1(x) = \frac{1}{\mu_N}(x+\mu_N)$$

(refer to equation (2.10) of section 4.2). If we make the substitution N-n-1 for n in (3.21), we see that

$$\mu_{n+1} W_{N-n}(x) - (x+\mu_{n+1}+\lambda_{n+1}) W_{N-n-1}(x) + \lambda_{n+1} W_{N-n-2}(x) = 0 , \qquad n=0,1,2,\ldots,N-2, \tag{3.22}$$

$$W_{N-(N-1)-1}(x) \equiv 1 , W_{N-(N-2)-1}(x) = \frac{1}{\mu_N}(x+\mu_N) .$$

Comparison of (3.22) with (3.16) and (3.17) reveals that

$$g_n^*(x) = \beta(x) W_{N-n-1}(x) , \qquad n=r-1,r,r+1,\ldots,N-1, \tag{3.23}$$

where $\beta(x)$ is independent of n. The condition (3.15) fixes $\beta(x)$ as

$$\beta(x) = \frac{1}{W_{N-r}(x)} ,$$

and so from (3.23)

$$g_n^*(x) = \frac{W_{N-n-1}(x)}{W_{N-r}(x)} , \qquad n=r-1,r,r+1,\ldots,N-1. \tag{3.24}$$

Combining (3.20) and (3.24) we thus have

$$g_n^*(x) = \begin{cases} \dfrac{p_{n+1}(x)}{p_r(x)} , & -1 \leqslant n \leqslant r-1 . \\[2ex] \dfrac{W_{N-n-1}(x)}{W_{N-r}(x)} , & r-1 \leqslant n \leqslant N-1 . \end{cases} \tag{3.25}$$

We now seek an expression for $G_r(t)$ in terms of $g_n(t)$ $(-1 \leqslant n \leqslant N-1)$. Suppose that at time τ an entry into state r occurs. We require the probability that the next entry into state r subsequent to time $\tau+0$ occurs by time $\tau+t$. Suppose that the next event subsequent to time $\tau+0$ occurs in the time interval $(\tau+y, \tau+y+dy)$, where $y+dy \leqslant t$. The time which now remains for the next entry into state r to occur in is $t-y+0(dy)$, and either the transition $r \to (r+1)$ (if the event is a birth) or the transition $r \to (r-1)$ (if the event is a death) is just about to occur. However we note that the future behaviour of the system subsequent to a transition $r \to (r-1)$ is stochastically indistinguishable from the future behaviour subsequent to a transition $(r-2) \to (r-1)$ (or, in the case $r=1$, subsequent to an arbitrary epoch when the system is in state 0 and no birth occurs). (Recall the memoryless property 1.2.2.) Thus we have that

$$G_r(t) = \int_0^t e^{-(\lambda_r+\mu_r)y}[\lambda_r g_r(t-y) + \mu_r g_{r-2}(t-y)]dy , \quad t \geqslant 0. \tag{3.26}$$

In terms of the Laplace-Stieltjes transform of $G_r(t)$, defined by

$$G_r^*(x) = \int_0^\infty e^{-xt} dG_r(t) , \quad \text{Re } x \geqslant 0 , \tag{3.27}$$

expression (3.26) can be written as

$$G_r^*(x) = \frac{1}{(x+\lambda_r+\mu_r)} [\lambda_r g_r^*(x) + \mu_r g_{r-2}^*(x)] . \tag{3.28}$$

Using equation (3.25), we therefore have that

$$G_r^*(x) = \frac{1}{(x+\lambda_r+\mu_r)}\left[\lambda_r \frac{W_{N-r-1}(x)}{W_{N-r}(x)} + \mu_r \frac{p_{r-1}(x)}{p_r(x)}\right]. \quad (3.29)$$

Recalling lemma 3.1.1(iii), the following partial fraction decompositions are immediate:

$$\frac{p_{r-1}(x)}{p_r(x)} = \sum_{i=1}^{r} \frac{\alpha_i^{(r)}}{(x-x_i^{(r)})}, \quad (3.30)$$

where $\{x_i^{(r)};\ i=1,2,\ldots,r\}$ is the set of zeros of $p_r(x)$ and

$$\alpha_i^{(r)} = \frac{p_{r-1}(x_i^{(r)})}{p_r'(x_i^{(r)})} > 0, \quad i=1,2,\ldots,r; \quad (3.31)$$

and

$$\frac{W_{N-r-1}(x)}{W_{N-r}(x)} = \sum_{i=1}^{N-r} \frac{\omega_i^{(N-r)}}{(x-y_i^{(N-r)})}, \quad (3.32)$$

where $\{y_i^{(N-r)};\ i=1,2,\ldots,N-r\}$ is the set of zeros of $W_{N-r}(x)$ and

$$\omega_i^{(N-r)} = \frac{W_{N-r-1}(y_i^{(N-r)})}{W'_{N-r}(y_i^{(N-r)})} > 0, \quad i=1,2,\ldots,N-r. \quad (3.33)$$

If we make the definitions

$$\beta_i^{(r)} = -\frac{\alpha_i^{(r)}}{x_i^{(r)}}, \quad i=1,2,\ldots,r, \quad (3.34)$$

and

$$\rho_i^{(r)} = -\frac{\omega_i^{(r)}}{y_i^{(r)}}, \quad i=1,2,\ldots,r, \quad (3.35)$$

then substitution of (3.30) and (3.32) into (3.29) yields

$$G_r^*(x) = \sum_{i=1}^{N-r} \frac{-y_i^{(N-r)} \rho_i^{(N-r)} \lambda_r}{(x+\lambda_r+\mu_r)(x-y_i^{(N-r)})} + \sum_{i=1}^{r} \frac{-x_i^{(r)} \beta_i^{(r)} \mu_r}{(x+\lambda_r+\mu_r)(x-x_i^{(r)})} . \tag{3.36}$$

Let $j \in \{1,2,\ldots,r\}$. Then, if $\lambda_r+\mu_r \neq -x_j^{(r)}$,

$$\frac{-x_j^{(r)} \beta_j^{(r)} \mu_r}{(x+\lambda_r+\mu_r)(x-x_j^{(r)})} = \frac{x_j^{(r)} \beta_j^{(r)} \mu_r}{(\lambda_r+\mu_r)(\lambda_r+\mu_r+x_j^{(r)})} \cdot \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)} + \frac{\beta_j^{(r)} \mu_r}{(\lambda_r+\mu_r+x_j^{(r)})} \cdot \frac{(-x_j^{(r)})}{(x-x_j^{(r)})} , \tag{3.37}$$

while if $\lambda_r+\mu_r = -x_j^{(r)}$ then

$$\frac{-x_j^{(r)} \beta_j^{(r)} \mu_r}{(x+\lambda_r+\mu_r)(x-x_j^{(r)})} = \beta_j^{(r)} \mu_r \cdot \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)^2} . \tag{3.38}$$

Similarly, a partial fraction decomposition can be found for the summands in the first summation on the right-hand-side of (3.36).

Define

$$\gamma_i = \frac{\beta_i^{(r)} \mu_r}{(\lambda_r+\mu_r+x_i^{(r)})} , \quad i=1,2,\ldots,r , \tag{3.39}$$

$$\hat{\gamma}_i = \frac{x_i^{(r)} \beta_i^{(r)} \mu_r}{(\lambda_r+\mu_r)(\lambda_r+\mu_r+x_i^{(r)})} , \quad i=1,2,\ldots,r , \tag{3.40}$$

$$\delta_i = \frac{\rho_i^{(N-r)} \lambda_r}{(\lambda_r+\mu_r+y_i^{(N-r)})} , \quad i=1,2,\ldots,N-r , \tag{3.41}$$

$$\hat{\delta}_i = \frac{y_i^{(N-r)} \rho_i^{(N-r)} \lambda_r}{(\lambda_r+\mu_r)(\lambda_r+\mu_r+y_i^{(N-r)})} , \quad i=1,2,\ldots,N-r . \tag{3.42}$$

Then, using the partial fraction decomposition as illustrated in (3.37) and (3.38), we can write (3.36) as

$$G_r^*(x) = \sum_{i=1}^{r} \gamma_i \frac{(-x_i^{(r)})}{(x-x_i^{(r)})} + (\sum_{i=1}^{r} \hat{\gamma}_i) \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)}$$

$$+ \sum_{i=1}^{N-r} \delta_i \frac{(-y_i^{(N-r)})}{(x-y_i^{(N-r)})} + (\sum_{i=1}^{N-r} \hat{\delta}_i) \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)}, \tag{3.43}$$

if $\lambda_r+\mu_r \neq -x_i^{(r)}$ for all $i=1,2,\ldots,r$, and $\lambda_r+\mu_r \neq -y_i^{(N-r)}$ for all $i=1,2,\ldots,N-r$;

$$G_r^*(x) = \sum_{\substack{i=1\\ i\neq j}}^{r} \gamma_i \frac{(-x_i^{(r)})}{(x-x_i^{(r)})} + (\sum_{\substack{i=1\\ i\neq j}}^{r} \hat{\gamma}_i) \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)}$$

$$+ \sum_{i=1}^{N-r} \delta_i \frac{(-y_i^{(N-r)})}{(x-y_i^{(N-r)})} + (\sum_{i=1}^{N-r} \hat{\delta}_i) \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)}$$

$$+ \beta_j^{(r)} \mu_r \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)^2}, \tag{3.44}$$

if $\lambda_r+\mu_r = -x_j^{(r)}$ for some j $(1 \leqslant j \leqslant r)$, but $\lambda_r+\mu_r \neq -y_i^{(N-r)}$ for all $i=1,2,\ldots,N-r$;

$$G_r^*(x) = \sum_{i=1}^{r} \gamma_i \frac{(-x_i^{(r)})}{(x-x_i^{(r)})} + (\sum_{i=1}^{r} \hat{\gamma}_i) \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)}$$

$$+ \sum_{\substack{i=1\\ i\neq k}}^{N-r} \delta_i \frac{(-y_i^{(N-r)})}{(x-y_i^{(N-r)})} + (\sum_{\substack{i=1\\ i\neq k}}^{N-r} \hat{\delta}_i) \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)}$$

$$+ \rho_k^{(N-r)} \lambda_r \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)^2}, \tag{3.45}$$

if $\lambda_r+\mu_r = -y_k^{(N-r)}$ for some k $(1 \leqslant k \leqslant N-r)$, but $\lambda_r+\mu_r \neq -x_i^{(r)}$ for all $i=1,2,\ldots,r$;

$$G_r^*(x) = \sum_{\substack{i=1\\ i\neq j}}^{r} \gamma_i \frac{(-x_i^{(r)})}{(x-x_i^{(r)})} + \Big(\sum_{\substack{i=1\\ i\neq j}}^{r} \hat{\gamma}_i\Big) \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)}$$

$$+ \sum_{\substack{i=1\\ i\neq k}}^{N-r} \delta_i \frac{(-y_i^{(N-r)})}{(x-y_i^{(N-r)})} + \Big(\sum_{\substack{i=1\\ i\neq k}}^{N-r} \hat{\delta}_i\Big) \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)}$$

$$+ \left(\beta_j^{(r)}\mu_r + \rho_k^{(N-r)}\lambda_r\right) \frac{(\lambda_r+\mu_r)}{(x+\lambda_r+\mu_r)^2} , \qquad (3.46)$$

if $\lambda_r+\mu_r = -x_j^{(r)} = -y_k^{(N-r)}$, where $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant N-r$.

By noting that $G_r(0) = 0$ we can immediately invert $G_r^*(x)$ as given in equations (3.43), (3.44), (3.45) and (3.46) to yield $G_r(t)$, in the same fashion as we inverted $B^*(x)$ in section 4.1. We note, for similar reasons as for $B(t)$ in section 4.1, that $G_r(t)$ is never hyperexponential.

The functions $G_r(t)$ have been defined for $0 < r < N$. The time between successive entries to state r for $r=0$ and $r=N$ has probability distribution function $I(t)$ (*) and $B(t)$ (**) respectively. Thus we define

$$G_0(t) = I(t) , \qquad (3.47)$$

and

$$G_N(t) = B(t) . \qquad (3.48)$$

(*) See section 4.2.

(**) See section 4.1.

If we make the following natural interpretations

$$\mu_0 = 0 = \lambda_N , \tag{3.49}$$

and

$$W_{-1}(x) \equiv 0 \equiv p_{-1}(x) , \tag{3.50}$$

then, recalling equations (2.9) of section 4.2 and (1.16) of section 4.1, we see that equation (3.29), and hence all subsequent analysis, is true also in the cases $r=0$ and $r=N$.

4.4 FIRST PASSAGE TIME DISTRIBUTIONS.

In the preliminary analysis of the previous sections we have made use of first passage time distributions, although we have not explicitly stated them as such. In this section we bring together these results and relate them to existing formulae.

For two distinct states i and j $(0 \leqslant i,j \leqslant N, \ i \neq j)$ define the first passage time from i to j as the time from an arbitrary epoch at which the process is in state i until the next subsequent entry into state j. (We note that the first passage time is independent of the length of time the process has been in state i before the initial epoch, by the memoryless property 1.2.2.) Denote by $F_{ij}(t)$ the probability distribution function of the first passage time from i to j, and define its Laplace-Stieltjes transform by

$$F^*_{ij}(x) = \int_0^\infty e^{-xt}\, dF_{ij}(t) , \qquad (4.1)$$

$$i \neq j, \quad \text{Re } x \geqslant 0.$$

Recall that in section 4.3 we made use of the functions $g_n(t)$ $(n=-1,0,1,\ldots,N-1)$, which were defined as the probability distribution function of the time from $\tau_k^{(n)}-0$ (some $k \geqslant 0$) until the next subsequent entry into state r, where r is some fixed state. But the state of the process at $\tau_k^{(n)}+0$ is $(n+1)$, and so

$$g_n(t) = F_{n+1,r}(t) , \qquad (4.2)$$

$$n+1 \neq r, \ -1 \leqslant n < N, \ 0 \leqslant r \leqslant N.$$

In terms of the Laplace-Stieltjes transforms we can write (4.2) as

$$g_n^*(x) = F_{n+1,r}^*(x) , \qquad (4.3)$$

$$n+1 \neq r, \ -1 \leqslant n < N, \ 0 \leqslant r \leqslant N.$$

Invoking equation (3.25) of section 4.3 for $g_n^*(x)$ $(-1 \leqslant n \leqslant N-1)$ we have

$$F_{n+1,r}^*(x) = \begin{cases} \dfrac{p_{n+1}(x)}{p_r(x)} , & -1 \leqslant n < r-1, \\[2ex] \dfrac{W_{N-n-1}(x)}{W_{N-r}(x)} , & r-1 < n \leqslant N-1, \end{cases}$$

$$r=0,1,2,\ldots,N,$$

or, more conveniently,

$$F_{ij}^*(x) = \begin{cases} \dfrac{p_i(x)}{p_j(x)} , & 0 \leqslant i < j , \\[2ex] \dfrac{W_{N-i}(x)}{W_{N-j}(x)} , & j < i \leqslant N , \end{cases} \qquad (4.4)$$

$$j=0,1,2,\ldots,N.$$

We thus see that the functions $g_n(t)$ $(n=-1,0,1,\ldots,N-1)$ are simply first passage time distribution functions. The functions $b_n(t)$ $(n=-1,0,1,\ldots,N-1)$ used in section 4.1 are the special cases when the first passage time is to the boundary state N. This is also the case for the functions $f_n(t)$ $(n=-1,0,1,\ldots,N-1)$ used in section 3.1, since whether the arrival in state $N-1$ causes an overflow or a birth to some state N is irrelevant to the time interval being considered.

It is important to note that for the first passage time from state i to some *higher* state j we do not need to know anything about the states above j. Thus we may truncate the state space (as far as $\{0,1,2,\ldots,j\}$) or extend the state space (even to an infinite space) without affecting $F^*_{ij}(x)$ for $i < j$.

In chapter 2 it was mentioned that Karlin and McGregor [1959b] gives a formula for $F^*_{ij}(x)$ $(i < j)$ for an infinite birth-and-death process (equation (0.5) of chapter 2). Bearing in mind the remarks of the previous paragraph, and recalling the relationship between $(p_n(x))_{n=0}^{\infty}$ and the sequence $(Q_n(x))_{n=0}^{\infty}$ of Karlin and McGregor (equation (1.25) of section 3.1), we see that our formula (4.4) in the case $i < j$ agrees with the formula of Karlin and McGregor. Thus our previous analysis provides an alternative and more direct derivation of an existing formula for $F^*_{ij}(x)$ in the case $i < j$.

However, by approaching the problem directly from the point of view of a finite birth-and-death process, rather than simply truncating an infinite one, we find our analysis also gives a formula for $F^*_{ij}(x)$ when $i > j$.

Returning again to $F^*_{ij}(x)$ for $i < j$, the formula (4.4) also provides an alternative derivation of some relevant results of Keilson [1979]. Using our notation, theorem 5.1A of Keilson [1979] reads

$$F^*_{0n}(x) = \frac{\theta_{n1}\theta_{n2}\cdots\theta_{nn}}{(\theta_{n1}+x)(\theta_{n2}+x)\cdots(\theta_{nn}+x)} \tag{4.5}$$

where θ_{nj} $(j=1,2,\ldots,n)$ are distinct and positive. This result is however an immediate corollary to our formula (4.4):

$$F^*_{0n}(x) = \frac{p_0(x)}{p_n(x)} = \frac{1}{p_n(x)} , \tag{4.6}$$

(recall $p_0(x) \equiv 1$). As the zeros of $p_n(x)$ are real, distinct and negative (lemma 3.1.1) we can write

$$p_n(x) = k_n(\theta_{n1}+x)(\theta_{n2}+x)\ldots(\theta_{nn}+x), \tag{4.7}$$

where θ_{nj} $(j=1,2,\ldots,n)$ are distinct and positive. However $p_n(0) = 1$ (lemma 3.1.1) and so

$$k_n = [\theta_{n1}\theta_{n2}\ldots\theta_{nn}]^{-1}, \tag{4.8}$$

thus giving the desired form (4.5). Moreover our analysis provides a means of determining the θ_{nj} $(j=1,2,\ldots,n)$:

$$\theta_{nj} = -x_j^{(n)}, \quad j=1,2,\ldots,n, \tag{4.9}$$

where $\{x_j^{(n)};\ j=1,2,\ldots,n\}$ is the set of zeros of $p_n(x)$. We point out that yet another, but closely related, derivation of (4.5) can be found in Keilson [1971], in which the connection with the orthogonal polynomials of Karlin and McGregor is noted.

Keilson [1979] (section 5.2) discusses the mean first passage time from n to $(n+1)$, denoted by $\bar{T}_n^+$. By using a technique in which $\bar{T}_n^+$ is expressed in terms of $\bar{T}_{n-1}^+$, Keilson [1979] (p. 61) obtains

$$\bar{T}_n^+ = \frac{1}{\lambda_n\pi_n}\sum_{j=0}^{n}\pi_j, \tag{4.10}$$

where

$$\pi_0 = 1,\ \pi_n = \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n} \quad (n > 1). \tag{4.11}$$

This result can also be derived from formula (4.4), since

$$\bar{T}_n^+ = -\frac{d}{dx}F^*_{n,n+1}(x)\Big|_{x=0}. \tag{4.12}$$

Applying (4.4) to (4.12), recalling that $p_n(0) = 1$ (lemma 3.1.1), yields

$$\bar{T}_n^+ = p'_{n+1}(0) - p'_n(0) \quad . \tag{4.13}$$

From the recurrence relation for $(p_n(x))_{n=0}^{\infty}$ (equation (1.12) of section 3.1) we have that

$$p'_{n+1}(0) = \lambda_n^{-1}[1 + (\lambda_n+\mu_n)p'_n(0)-\mu_n p'_{n-1}(0)] \ .$$

Thus,

$$p'_{n+1}(0)-p'_n(0) = \lambda_n^{-1}[1 + \mu_n(p'_n(0)-p'_{n-1}(0))] \ ,$$

which, from (4.13), gives

$$\bar{T}_n^+ = \lambda_n^{-1}[1 + \mu_n\bar{T}_{n-1}^+]. \tag{4.14}$$

Equation (4.14) is precisely the relation used by Keilson [1979] to derive (4.10).

We have seen that the approach adopted in section 3.1 which in passing finds what essentially amount to first passage time distributions leads quite naturally to the orthogonal polynomials of Karlin and McGregor. This in turn provides alternative derivations for a number of existing formulae concerning first passage times, as shown in this section.

CHAPTER 5. THE OVERFLOW STREAM FROM CERTAIN QUEUEING MODELS WITH RENEWAL INPUT.

5.1 INTRODUCTION.

In the preceding chapters, certain secondary processes associated with finite birth-and-death processes have been investigated, particularly the overflow process. In section 3.2 it was illustrated that these results can be applied to certain finite capacity queueing models. However, many elementary queueing models cannot be treated as birth-and-death processes, and so the results of the previous chapters cannot be applied.

One of the most important elementary queueing models is a generalisation of the telephone trunking model with holding registers introduced in section 3.2.2. We recall that the arrival stream to the model in section 3.2.2 was Poisson; that is, the distribution of time between successive arrivals was negative exponential. The obvious generalisation is to allow some other fixed distribution as the distribution of time between successive arrivals. We would therefore have a telephone trunking model with holding registers with general renewal input.

In order to describe this queueing model succinctly we will use a four part descriptor:

$$-/-/-/- .$$

The first two symbols represent the inter-arrival time distribution and holding time distribution respectively. The number of trunks is given in the third part, and the total system capacity (the number of trunks plus the number of holding registers) is given in the fourth part. The symbols used for distributions are

M - negative exponential,

GI - general renewal,

D - deterministic.

Thus the model in section 3.2.2 is the $M/M/T_1/(T_1+T_2)$ queue, and the generalisation which we are now considering is the $GI/M/T_1/(T_1+T_2)$ queue.

The basic property of birth-and-death processes is the memoryless property 1.2.2. This property was used extensively in our analysis, particularly in the derivation of the recurrence equations for the functions $f_n(t)$ $(n=-1,0,1,...,N-1)$ of section 3.1. However a similar approach may be able to be used for queueing models which exhibit a memoryless property not on the whole time continuum but on a discrete subset thereof. For example, the future behaviour of the $GI/M/T_1/(T_1+T_2)$ queue subsequent to an arrival instance depends only on the number then present in the system. This is an immediate consequence of the renewal nature of the arrival stream, remembering that the holding time distribution is negative exponential. Thus this queue exhibits a memoryless property at the set of arrival instances.

In this chapter, the $GI/M/1/(T_2+1)$ queue will be used as an illustration of how a similar analysis as to the one used for birth-and-death processes can be made for certain queueing models which exhibit a memoryless behaviour at only a discrete subset of the time continuum.

The $GI/M/T_1/T_1$ queue (that is, the basic telephone trunking model of section 3.2.1, except that the arrival process is a general renewal stream) has been investigated by this approach. (See Takács [1959], Syski [1960] (section 3.1 of chapter 5) and Potter [1979].) It has already been shown (section 3.2.1) that these results, in the special case of Poisson arrivals, agree with the results obtained by applying our analysis to the $M/M/T_1/T_1$ queue.

5.2 CONCERNING THE OVERFLOW STREAM FROM THE GI/M/1/(T_2+1) QUEUE.

Consider the GI/M/1/(T_2+1) queue introduced in section 5.1, and denote the probability distribution function of the inter-arrival time by $A(t)$. As in section 3.2.2 we will use a time scale which takes as its unit the mean holding time. Denote by $\{\tau_0^{(n)} < \tau_1^{(n)} < \tau_2^{(n)} < \ldots\}$, where $0 \leqslant n \leqslant T_2+1$, the random epochs at which the system contains n calls and an arrival occurs. Thus the set of overflow instances is $\{\tau_k^{(T_2+1)}\,;\, k \geqslant 0\}$, and the time between successive overflows is the length of the time interval $[\tau_k^{(T_2+1)}, \tau_{k+1}^{(T_2+1)})$.

As mentioned in section 5.1, the future behaviour of the system subsequent to an arrival instance depends only on the state the system is then in. Thus, in particular, the behaviour subsequent to an overflow instance is stochastically identical to that subsequent to any other overflow instance. Hence the distribution of the length of the time interval $[\tau_k^{(T_2+1)}, \tau_{k+1}^{(T_2+1)})$ (that is, the distribution of the inter-overflow time) is independent of k. Accordingly, denote the corresponding probability distribution function by $F(t)$. We note in passing that, as an overflow is an arrival which does not result in an increase to the number in the system, $F(t)$ is also the distribution function for the length of the time interval $[\tau_\ell^{(T_2)}, \tau_m^{(T_2+1)})$,(*) since at both $\tau_\ell^{(T_2)}+0$ and $\tau_k^{(T_2+1)}+0$ the system contains T_2+1 calls.

Define $f_n(t)$ $(0 \leqslant n \leqslant T_2+1)$ as the probability distribution function of the time until the next overflow from $\tau_k^{(n)}-0$. From the remark in the previous paragraph,

$$F(t) = f_{T_2}(t) \,. \tag{2.1}$$

(*) Where $\tau_{m-1}^{(T_2+1)} < \tau_\ell^{(T_2)} < \tau_m^{(T_2+1)}$.

We also have immediately that

$$f_{T_2+1}(t) = u(t) \ , \qquad (2.2)$$

where $u(t)$ is the unit-step or Heaviside function defined by equation (1.2) of section 3.1.

Thus far in our analysis we have not been impaired by the fact that the memoryless behaviour only occurs at arrival instances. However the method previously used to derive recurrence relations for these functions considered the next *event* subsequent to $\tau_k^{(n)}$, be it an arrival or a death. In our present situation, we are forced to consider the next *arrival* subsequent to $\tau_k^{(n)}$.

Consider an epoch $\tau_k^{(n)}$ for some n $(0 \leqslant n \leqslant T_2)$. Then $f_n(t)$ is the probability that the next overflow after $\tau_k^{(n)}-0$ occurs at or before time $\tau_k^{(n)}+t$. Since $0 \leqslant n \leqslant T_2$, the arrival which occurs at $\tau_k^{(n)}$ cannot be an overflow, and so there would have to be at least one more arrival by time $\tau_k^{(n)}+t$. Suppose the next such arrival occurs in the time interval $(\tau_k^{(n)}+y, \tau_k^{(n)}+y+dy)$, where $y+dy \leqslant t$. The time which now remains for the next overflow to occur in is $t-y+O(dy)$. However, since the arrival at $\tau_k^{(n)}$ promotes the system to state $(n+1)$, there could have been any number up to $n+1$ departures between the arrival at $\tau_k^{(n)}$ and the next subsequent arrival.

Since we have a single trunk with negative exponential holding times, the stream of departures from the trunk over any period of time is Poisson, provided that the circuit is busy continuously throughout the entire period, including the endpoints. This Poisson stream has unit mean since the mean holding time has been taken as one unit. Thus, the probability that j $(0 \leqslant j < n+1)$

depart from the system during a period of length $y+dy$ commencing at $\tau_k^{(n)}$ is

$$\frac{y^j}{j!} e^{-y} + o(dy) . \qquad (2.3)$$

The probability that all $n+1$ calls depart during the same interval is then of course simply

$$1 - \sum_{j=0}^{n} \frac{y^j}{j!} e^{-y} + o(dy) . \qquad (2.4)$$

Should j $(0 \leqslant j \leqslant n+1)$ calls depart during this period, then the state of the system at the next arrival subsequent to $\tau_k^{(n)}+0$ would be $n+1-j$.

Thus we have

$$\begin{aligned} f_n(t) = & \sum_{j=0}^{n} \int_0^t \frac{y^j}{j!} e^{-y} f_{n+1-j}(t-y)\, dA(y) \\ & + \int_0^t [1 - \sum_{j=0}^{n} \frac{y^j}{j!} e^{-y}] f_0(t-y)\, dA(y) , \quad t \geqslant 0, \end{aligned} \qquad (2.5)$$

$$n=0,1,2,\ldots,T_2.$$

In terms of the Laplace-Stieltjes transform of $f_n(t)$ $(0 \leqslant n \leqslant T_2+1)$, defined by

$$f_n^*(x) = \int_0^\infty e^{-xt} df_n(t) , \qquad (2.6)$$

$$\text{Re } x \geqslant 0, \ n=0,1,2,\ldots,T_2+1,$$

the relations expressed by (2.5) can be written as

$$\begin{aligned} f_n^*(x) = & \sum_{j=0}^{n} f_{n+1-j}^*(x) \int_0^\infty e^{-xy} \frac{y^j}{j!} e^{-y} dA(y) \\ & + f_0^*(x) \int_0^\infty e^{-xy} [1 - \sum_{j=0}^{n} \frac{y^j}{j!} e^{-y}] dA(y) , \end{aligned} \qquad (2.7)$$

$$n=0,1,2,\ldots,T_2.$$

Equation (2.2) can be written as

$$f^*_{T_2+1}(x) = 1 , \qquad (2.8)$$

and so acts as a boundary condition.

For any given x, the recurrence equations (2.7), with supplementary condition (2.8), form a system of T_2+1 linear and clearly independent equations in T_2+1 unknowns. Thus these equations can, in principle, be uniquely solved to yield $f^*_n(x)$ $(n=0,1,2,\ldots,T_2+1)$. This would then immediately give an expression for the Laplace-Stieltjes transform of $F(t)$, defined by

$$F^*(x) = \int_0^\infty e^{-xt}\, dF(t), \qquad (2.9)$$

$$\text{Re } x \geqslant 0,$$

since equation (2.1) implies that

$$F^*(x) = f^*_{T_2}(x) . \qquad (2.10)$$

No attempt has been made to solve equations (2.7) for general T_2 and $A(t)$, but in the next two sections the special cases GI/M/1/2 and M/M/1/(T_2+1) will be discussed.

5.3 THE OVERFLOW STREAM FROM THE GI/M/1/2 QUEUE.

Consider now the case of the GI/M/1/2 queue. The equations (2.7) and (2.8) become

$$f_0^*(x) = f_1^*(x) \int_0^\infty e^{-(x+1)y} dA(y) + f_0^*(x) \int_0^\infty e^{-xy} dA(y) - f_0^*(x) \int_0^\infty e^{-(x+1)y} dA(y) , \quad (3.1)$$

$$f_1^*(x) = f_2^*(x) \int_0^\infty e^{-(x+1)y} dA(y) + f_1^*(x) \int_0^\infty ye^{-(x+1)y} dA(y) + f_0^*(x) \int_0^\infty e^{-xy} dA(y) - f_0^*(x) \int_0^\infty (1+y)e^{-(x+1)y} dA(y) , \quad (3.2)$$

$$f_2^*(x) = 1 . \quad (3.3)$$

Define $\mathcal{L}(x;h)$, where h is some function of y, as

$$\mathcal{L}(x;h) = \int_0^\infty h(y)e^{-(x+1)y} dA(y) . \quad (3.4)$$

Note that $\mathcal{L}(x;h)$ is linear with respect to h. Then using (3.4) and the Laplace-Stieltjes transform of $A(t)$, defined by

$$A^*(x) = \int_0^\infty e^{-xt} dA(t) , \qquad \text{Re } x \geqslant 0, \quad (3.5)$$

we can write (3.1), (3.2) and (3.3) as

$$f_0^*(x) = \mathcal{L}(x;1)f_1^*(x) + A^*(x)f_0^*(x) - \mathcal{L}(x;1)f_0^*(x) \ , \tag{3.6}$$

$$f_1^*(x) = \mathcal{L}(x;1)f_2^*(x) + \mathcal{L}(x;y)f_1^*(x) + A^*(x)f_0^*(x) - \mathcal{L}(x;1+y)f_0^*(x) \ , \tag{3.7}$$

$$f_2^*(x) = 1 \ . \tag{3.8}$$

Substitution of (3.8) into (3.7) yields

$$f_1^*(x) = \mathcal{L}(x;1) + \mathcal{L}(x;y)f_1^*(x) + A^*(x)f_0^*(x) - \mathcal{L}(x;1+y)f_0^*(x) \ . \tag{3.9}$$

Expression (3.6) can be rearranged to read

$$f_0^*(x) = \frac{\mathcal{L}(x;1)}{1-A^*(x)+\mathcal{L}(x;1)} f_1^*(x) \ , \tag{3.10}$$

which, when substituted into (3.9) gives

$$f_1^*(x) = \mathcal{L}(x;1) + \mathcal{L}(x;y)f_1^*(x) + \frac{\mathcal{L}(x;1)[A^*(x) - \mathcal{L}(x;1+y)]}{1 - A^*(x) + \mathcal{L}(x;1)} f_1^*(x) \ . \tag{3.11}$$

Rearrangement of (3.11) yields

$$f_1^*(x) = \frac{\mathcal{L}(x;1)[1-A^*(x)+\mathcal{L}(x;1)]}{\mathcal{L}(x;1)[1-A^*(x)+\mathcal{L}(x;1)]+[1-A^*(x)][1-\mathcal{L}(x;y)]} \ . \tag{3.12}$$

However equation (2.10) implies that

$$F^*(x) = f_1^*(x) \ , \tag{3.13}$$

and so (3.12) provides an expression for $F^*(x)$.

We will now give an example of the application of this result in the special case of deterministic arrivals, the D/M/1/2 queue. Suppose that the inter-arrival time is some constant $\tau > 0$. Since our time scale has as its unit the mean holding time, the mean arrival rate in equilibrium, and hence the offered traffic, is $\frac{1}{\tau}$. (Recall definition 1.4.1.)

The probability distribution function for the time between successive arrivals is given by

$$A(t) = u(t-\tau) \ , \tag{3.14}$$

where $u(t)$ is the unit-step or Heaviside function defined by equation (1.2) of section 3.1. Hence

$$A^*(x) = e^{-x\tau} \ , \tag{3.15}$$

$$\mathcal{L}(x;1) = e^{-(x+1)\tau} \ , \tag{3.16}$$

$$\mathcal{L}(x;y) = \tau e^{-(x+1)\tau} \ . \tag{3.17}$$

Substitution of these quantities into equation (3.12), bearing in mind result (3.13), yields

$$F^*(x) = \frac{e^{-(x+1)\tau}[1-e^{-x\tau} + e^{-(x+1)\tau}]}{e^{-(x+1)\tau}[1-e^{-x\tau}+e^{-(x+1)\tau}] + [1-e^{-x\tau}][1-\tau e^{-(x+1)\tau}]} \ . \tag{3.18}$$

Equation (3.18) does not afford any significant simplification.

The mean and variance of the inter-overflow time can be found directly from $F^*(x)$, as

$$\text{mean} = m_1 = -\frac{d}{dx} F^*(x) \Big|_{x=0} \ , \tag{3.19}$$

$$\text{variance} = \sigma^2 = \frac{d^2}{dx^2} F^*(x) \Big|_{x=0} - m_1^2 \ . \tag{3.20}$$

We note that

$$\frac{d}{dx}[\log F^*(x)] = \frac{1}{F^*(x)} \frac{d}{dx} F^*(x) ,$$

and so from (3.19)

$$m_1 = - F^*(0) \frac{d}{dx} [\log F^*(x)]\Big|_{x=0} . \qquad (3.21)$$

Equation (3.21) provides an easier derivation of m_1 than (3.19), and yields

$$m_1 = \tau e^{\tau} [e^{\tau} - \tau] . \qquad (3.22)$$

The derivation of the variance is likewise straightforward but tedious and so we will simply state the result:

$$\sigma^2 = \tau^2 e^{\tau} [e^{3\tau} - 2e^{2\tau}(\tau-1) + e^{\tau}(\tau+1)(\tau-3) + \tau] . \qquad (3.23)$$

5.4 THE OVERFLOW STREAM FROM THE $M/M/1/(T_2+1)$ QUEUE.

Suppose now that the arrival stream is Poisson, with rate $\lambda > 0$; that is, the inter-arrival time distribution function is

$$A(t) = 1 - e^{-\lambda t}, \quad t \geqslant 0. \tag{4.1}$$

As we are taking the unit of time to be the mean holding time, the parameter λ is the offered traffic. (Recall definition 1.4.1.)

The equations (2.7) and (2.8) of section 5.2 become

$$f_n^*(x) = \lambda \sum_{j=0}^{n} f_{n+1-j}^*(x) \int_0^\infty \frac{y^j}{j!} e^{-(x+\lambda+1)y} dy$$

$$+ \lambda f_0^*(x) \int_0^\infty [1 - \sum_{j=0}^{n} \frac{y^j}{j!} e^{-y}] e^{-(x+\lambda)y} dy, \tag{4.2}$$

$$n=0,1,2,\ldots,T_2,$$

$$f_{T_2+1}^*(x) = 1. \tag{4.3}$$

The integral $\int_0^\infty \frac{y^j}{j!} e^{-(x+\lambda+1)y} dy$ is simply the Laplace transform of the quantity $\frac{y^j}{j!} e^{-(\lambda+1)y}$, and so (4.2) can be written as

$$f_n^*(x) = \lambda \sum_{j=0}^{n} f_{n+1-j}^*(x) \frac{1}{(x+\lambda+1)^{j+1}}$$

$$+ \lambda f_0^*(x) \left\{ \frac{1}{x+\lambda} - \sum_{j=0}^{n} \frac{1}{(x+\lambda+1)^{j+1}} \right\}, \tag{4.4}$$

$$n=0,1,2,\ldots,T_2.$$

Note that, using the formula for the sum of a geometric series, we have

$$\frac{1}{x+\lambda} - \sum_{j=0}^{n} \frac{1}{(x+\lambda+1)^{j+1}} = \frac{1}{(x+\lambda)(x+\lambda+1)^{n+1}}, \tag{4.5}$$

and so

$$f_n^*(x) = \sum_{j=0}^{n} \frac{\lambda}{(x+\lambda+1)^{j+1}} f_{n+1-j}^*(x) + \frac{\lambda}{(x+\lambda)(x+\lambda+1)^{n+1}} f_0^*(x) , \qquad (4.6)$$

$$n=0,1,2,\ldots,T_2 .$$

As indicated in the discussion in section 5.1, the $M/M/1/(T_2+1)$ is the special case of the telephone trunking model with holding registers of section 3.2.2 for $T_1=1$. Thus the results of section 3.1 apply to this queue; that is

$$f_n^*(x) = \frac{p_{n+1}(x)}{p_{T_2+2}(x)} , \quad n=0,1,2,\ldots,T_2+1 , \qquad (4.7)$$

where the orthogonal polynomial sequence $(p_n(x))_{n=0}^{\infty}$ is defined by

$$\lambda p_{n+1}(x) - (x+\lambda+1)p_n(x) + p_{n-1}(x) = 0 ,$$

$$n=1,2,\ldots, \qquad (4.8)$$

$$p_0(x) \equiv 1, \quad p_1(x) = 1 + \frac{x}{\lambda} .$$

We now verify that $f_n^*(x)$ $(n=0,1,2,\ldots,T_2)$ as defined by (4.7) is indeed the solution to (4.6).

<u>5.4.1 Lemma.</u> If $\{f_n^*(x);\ n=0,1,\ldots,T_2+1\}$ is defined by (4.7), then

$$\lambda f_{n+1}^*(x) + \frac{p_n(x)}{p_{n+1}(x)} f_n^*(x) = (x+\lambda+1) \frac{p_{n+1}(x)}{p_{n+2}(x)} f_{n+1}^*(x) , \qquad (4.9)$$

$$n=0,1,2,\ldots,T_2.$$

■

<u>Proof of 5.4.1.</u> We note that we have immediately from (4.7)

$$f_n^*(x) = \frac{p_{n+1}(x)}{p_{n+2}(x)} f_{n+1}^*(x) , \qquad (4.10)$$

$$n=0,1,2,\ldots,T_2 .$$

Thus,

$$\lambda f^*_{n+1}(x) + \frac{p_n(x)}{p_{n+1}(x)} f^*_n(x)$$

$$= \lambda f^*_{n+1}(x) + \frac{p_n(x)}{p_{n+1}(x)} \frac{p_{n+1}(x)}{p_{n+2}(x)} f^*_{n+1}(x)$$

$$= \frac{1}{p_{n+2}(x)} f^*_{n+1}(x)\,[\lambda p_{n+2}(x) + p_n(x)]$$

$$= (x+\lambda+1) \frac{p_{n+1}(x)}{p_{n+2}(x)} f^*_{n+1}(x) \quad \text{using (4.8)},$$

$$n=0,1,2,\ldots,T_2,$$

as required. ∎

<u>5.4.2 Lemma</u>. If $\{f^*_n(x);\ n=0,1,2,\ldots,T_2+1\}$ is defined by (4.7) then

$$f^*_n(x) = \sum_{j=0}^{n} \frac{\lambda}{(x+\lambda+1)^{j+1}} f^*_{n+1-j}(x) + \frac{\lambda}{(x+\lambda)(x+\lambda+1)^{n+1}} f^*_0(x),$$

$$n=0,1,2,\ldots,T_2 .$$ ∎

<u>Proof of 5.4.2</u>. We will use a descending induction argument on k to show that

$$\frac{1}{(x+\lambda+1)^k} \frac{p_{n+1-k}(x)}{p_{n+2-k}(x)} f^*_{n+1-k}(x) \tag{4.11}$$

$$= \sum_{j=k}^{n} \frac{\lambda}{(x+\lambda+1)^{j+1}} f^*_{n+1-j}(x) + \frac{\lambda}{(x+\lambda)(x+\lambda+1)^{n+1}} f^*_0(x),$$

$$n=0,1,2,\ldots,T_2,$$

$$k=0,1,2,\ldots,n.$$

The basis of the inductive argument is the case $k=n$: the right-hand-side of (4.11) reads, for $k=n$,

$$\frac{\lambda}{(x+\lambda+1)^{n+1}} f_1^*(x) + \frac{\lambda}{(x+\lambda)(x+\lambda+1)^{n+1}} f_0^*(x)$$

$$= \frac{1}{(x+\lambda+1)^{n+1}} [\lambda f_1^*(x) + \frac{\lambda}{(x+\lambda)} f_0^*(x)]$$

$$= \frac{1}{(x+\lambda+1)^{n+1}} [\lambda f_1^*(x) + \frac{p_0(x)}{p_1(x)} f_0^*(x)] \quad \text{from (4.8)}$$

$$= \frac{1}{(x+\lambda+1)^{n}} \frac{p_1(x)}{p_2(x)} f_1^*(x) \quad \text{by lemma 5.4.1,}$$

which is the left-hand-side of (4.11) for $k=n$.

Take as the inductive hypothesis the assumption that (4.11) holds for k replaced by $k+1$ $(0 \leqslant k < n)$; we now prove (4.11) holds for k. The right-hand-side of (4.11) can be written as

$$\frac{\lambda}{(x+\lambda+1)^{k+1}} f_{n+1-k}^*(x) + \sum_{j=k+1}^{n} \frac{\lambda}{(x+\lambda+1)^{j+1}} f_{n+1-j}^*(x)$$

$$+ \frac{\lambda}{(x+\lambda)(x+\lambda+1)^{n+1}} f_0^*(x)$$

$$= \frac{\lambda}{(x+\lambda+1)^{k+1}} f_{n+1-k}^*(x) + \frac{1}{(x+\lambda+1)^{k+1}} \frac{p_{n-k}(x)}{p_{n+1-k}(x)} f_{n-k}^*(x)$$

using the inductive hypothesis

$$= \frac{1}{(x+\lambda+1)^{k}} \frac{p_{n+1-k}(x)}{p_{n+2-k}(x)} f_{n+1-k}^*(x)$$

using lemma 5.4.1,

which is the left-hand-side of (4.11). Hence by the Principle of Mathematical Induction (4.11) is true.

By setting k=0 in (4.11), we have

$$\frac{p_{n+1}(x)}{p_{n+2}(x)}\, f^*_{n+1}(x)$$

$$= \sum_{j=0}^{n} \frac{\lambda}{(x+\lambda+1)^{j+1}}\, f^*_{n+1-j}(x) + \frac{\lambda}{(x+\lambda)(x+\lambda+1)^{n+1}}\, f^*_0(x),$$

$$n=0,1,2,\ldots,T_2.$$

But equation (4.10) states that

$$f^*_n(x) = \frac{p_{n+1}(x)}{p_{n+2}(x)}\, f^*_{n+1}(x) ,$$

$$n=0,1,2,\ldots,T_2,$$

and so the lemma is proved. ∎

Lemma 5.4.2 verifies that the analysis of this chapter is consistent with our previous work.

CHAPTER 6. CONCLUDING REMARKS.

CHAPTER 6. CONCLUDING REMARKS.

Our analysis of induced secondary processes has led us quite naturally to the orthogonal polynomial sequence associated with the birth-and-death process. The fact that the sequence played such a key role in the analysis supports the claims made in the literature that the sequence and the process have a very close relationship.

We have seen that our unified approach to the topic of induced secondary processes has provided alternative derivations of some existing results, and indeed has related them to a common theme.

The results of chapter 3 demonstrate a new characterisation of the hyperexponential family. This very strong result complements Khintchine's observation of the hyperexponential nature of the overflow (recall section 3.2.1).

In section 4.2 a natural duality associated with the finite birth-and-death process was set up and exploited to give some of the results.

Finally we saw how a similar initial approach can be adopted to analyse the overflow from certain related queueing models.

We have thus gained an insight into the structure of a finite birth-and-death process by examining some of its fundamental induced secondary processes...

Deus iam omnia scit.

APPENDIX I. LISTING OF COMPUTER PROGRAM FOR DETERMINATION OF THE PARAMETERS AND WEIGHTS FOR THE INTER-OVERFLOW TIME DISTRIBUTION FUNCTION.

The following program has been written in FORTRAN for the Control Data Corporation (C.D.C.) FORTRAN extended version 4 compiler, and implemented on the C.D.C. Cyber 173 machine using a NOS/BE operating system. The reference used when writing this program was Wiley [1976].

```
      PROGRAM BD2HYP(RATES,OUTPUT,PARAMS,TAPE1=RATES,
     *TAPE3=PARAMS)
*
* PURPOSE:-
*      GIVEN FINITE BIRTH-AND-DEATH PROCESS OF N STATES,
*      CALCULATES PARAMETERS AND WEIGHTS OF HYPEREXPONENTIAL
*      INTER-OVERFLOW TIME DISTRIBUTION FUNCTION.
*
* EXTERNAL REFERENCES:-
*      REFERENCES I.M.S.L. ROUTINE ZPOLR - ZEROS OF A
*      POLYNOMIAL WITH REAL COEFFICIENTS (LAGUERRE).
*
* INPUT:-
*      NUMBER OF STATES (N), ARRIVAL RATES OF STATES
*      0,1,2,...,N-1, DEATH RATES FOR STATES 1,2,3,...,N-1
*      IN LIST-DIRECTED FORMAT (*-FORMAT) ON FILE RATES.
*
* OUTPUT:-
*   (1) INPUT DATA IS ECHOPRINTED TO OUTPUT.   TABLE OF
*       COEFFICIENTS OF ASSOCIATED ORTHOGONAL POLYS,
*       PARAMETERS AND WEIGHTS ALSO WRITTEN TO OUTPUT.
*   (2) NUMBER OF STATES (N). PARAMETERS AND WEIGHTS WRITTEN
*       TO FILE PARAMS IN LIST-DIRECTED FORMAT (*-FORMAT).
*
      INTEGER N,NM1,NP1,I,IP1,IP2,J,IER,ISTAR
      REAL L(10),MU(10),A(11),B(11),C(11),AA(11),LI,MUI,
     *X(10),ALF(10),BET(10),XI,Z1,Z2
      COMPLEX Z(10)
*
```

```
* DESCRIPTION OF PRINCIPAL VARIABLES:-
*     N - NUMBER OF STATES.  (INTEGER)
*     L(I) - ARRIVAL RATE FOR STATE I-1.  (REAL)
*     MU(I) - DEATH RATE FOR STATE I-1.  (REAL)
*     A(J),B(J),C(J) - COEFFICIENT OF X**(J-1) IN (I+1)ST,
*      ITH, (I-1)ST  DEGREE ORTHOGONAL POLYNOMIAL RESP.,
*      WHERE 0<I<N.  (REAL)
*     X(I) - X IS ARRAY OF ZEROS OF NTH DEGREE ORTHOGONAL
*      POLYNOMIAL.  (REAL)
*     ALF(I) - NUMERATORS IN PARTIAL FRACTION DECOMPOSITION
*      OF RATIO OF (N-1)ST AND NTH DEG. ORTHOG. POLY.  (REAL)
*     BET(I) - WEIGHTS OF THE HYPEREXPONENTIAL.  (REAL)
*
*
*     READ, ECHOPRINT AND CHECK INPUT DATA.
*
      REWIND 1
      REWIND 3
      READ(1,*) N
      IF((N.GT.0).AND.(N.LT.11))GOTO 30
      PRINT*,≠***<<<ERROR>>>***   ILLEGAL NUMBER OF STATES≠,
     *≠ IN BIRTH-AND-DEATH PROCESS≠
      PRINT*,≠                          NUMBER = ≠,N,≠  -MUST BE≠,
     *≠ STRICTLY GREATER THAN 0 AND LESS THAN 11≠
      PRINT*,≠ ≠
      PRINT*,≠$$$ PROGRAM ABORTED $$$≠
      PRINT*,≠ ≠



      STOP ≠ILLEGAL INPUT≠
   30 PRINT 40,N
   40 FORMAT(1H1,≠INPUT BIRTH-AND-DEATH PROCESS:-≠/1H0,
     *≠NUMBER OF STATES IN BIRTH-AND-DEATH PROCESS = ≠,I3)
      NM1=N-1
      NP1=N+1
      ISTAR=≠******≠
      READ(1,*) (L(I),I=1,N)
      IF(N.GT.1)GOTO 48
      PRINT 50, (ISTAR,I=1,9),L(1)
      IF(L(1).LE.0.)GOTO 56
      GOTO 57
   48 READ(1,*) (MU(I),I=2,N)
      PRINT 50,(ISTAR,I=1,9),L(1),(I-1,L(I),MU(I),I=2,N)
   50 FORMAT(1H0,2X,≠I≠,9X,≠ARRIVAL RATE≠,9X,≠DEATH RATE≠/
     *1X,9A5/1X,≠  0≠,5X,1PE15.4/9(1X,I3,5X,1PE15.4,5X,
     *1PE15.4/))
      IF(L(1).LE.0.)GOTO 56
      DO 55 I=2,N
      IF(L(I).LE.0.)GOTO 56
      IF(MU(I).LE.0.)GOTO 56
   55 CONTINUE
      GOTO 57
```

```
56 PRINT*,≠***<<<ERROR>>>***    ILLEGAL INPUT≠
   PRINT*,≠                     RATES MUST BE POSITIVE≠
   PRINT*,≠ ≠
   PRINT*,≠$$$ PROGRAM ABORTED $$$≠
   PRINT*,≠ ≠
   STOP ≠ILLEGAL INPUT≠
57 PRINT 60,(I-1,I=1,11),(ISTAR,I=1,26)
60 FORMAT(1H0/1H0,≠TABLE OF COEFFICIENTS OF NON-MONIC ≠,
  *≠ORTHOGONAL POLYNOMIALS:-≠/1H0,63X,≠POWER OF X≠/
  *1X,≠DEGREE≠,1X,≠*≠,11I11/1X,26A5)
*
*     COMPUTE ASSOCIATED ORTHOGONAL POLYNOMIALS FROM
*     RECURRENCE RELATION.
*
      C(1)=1.0
      B(1)=1.0
      B(2)=1.0/L(1)
      PRINT 70,C(1)
 70   FORMAT(1X,≠ 0≠,5X,≠*≠,1X,1PE10.3)
      PRINT 80,B(1),B(2)
 80   FORMAT(1X,≠ 1≠,5X,≠*≠,2(1X,1PE10.3))
      IF(N.GT.1)GOTO 90
      A(1)=1.0
      A(2)=1.0/L(1)
      X(1)=-L(1)
      GOTO 650
 90   DO 400 I=1,NM1
      IP1=I+1
      LI=L(IP1)
      MUI=MU(IP1)
      A(1)=1.0
      A(I+2)=B(I+1)/LI
      A(I+1)=(B(I)+(LI+MUI)*B(I+1))/LI



      IF(I.EQ.1)GOTO 200
      DO 100 J=2,I
100   A(J)=(B(J-1)+(LI+MUI)*B(J)-MUI*C(J))/LI
200   DO 300 J=1,IP1
      C(J)=B(J)
300   B(J)=A(J)
      B(I+2)=A(I+2)
      IP2=I+2
      PRINT 350,I+1,(A(J),J=1,IP2)
350   FORMAT(1X,I2,5X,≠*≠,11(1X,1PE10.3))
400   CONTINUE
*
*     COMPUTE ZEROS OF NTH DEGREE ORTHOGONAL POLYNOMIAL.
*
      DO 500 I=1,NP1
500   AA(I)=A(N+2-I)
      CALL ZPOLR(AA,N,Z,IER)
```

```
      DO 600 I=1,N
  600 X(I)=Z(I)
  650 PRINT 700,N,(X(I),I=1,N)
  700 FORMAT(1H0/1H0/1H0,≠OUTPUT HYPEREXPONENTIAL≠,
     *≠ DISTRIBUTION:-≠/
     *1H0,≠ZEROS OF ≠,I2,≠TH DEGREE ORTHOGONAL ≠,
     *≠POLYNOMIAL ARE:-≠/1X,≠  (THESE ARE THE PARAMETERS ≠,
     *≠IN THE CONVEX COMBINATION OF EXPONENTIAL ≠,
     *≠DISTRIBUTIONS)≠/1H0,10(1X,1PE10.3))
*
*     COMPUTE ARRAYS ALF AND BET.
*
      IF(N.GT.1)GOTO 750
      ALF(1)=L(1)
      BET(1)=1.0
      GOTO 950
  750 DO 900 I=1,N
      XI=X(I)
*     COMPUTE Z1 = (N-1)ST DEGREE ORTHOG. POLY. EVALUATED
*     AT X(I).
*     COMPUTE Z2 = DERIVATIVE OF NTH DEGREE ORTHOG. POLY.
*     EVALUATED AT X(I).
      Z1=C(N)
      Z2=N*A(N+1)
      DO 800 J=1,NM1
      Z1=Z1*XI+C(N-J)
  800 Z2=Z2*XI+(N-J)*A(N+1-J)
      ALF(I)=Z1/Z2
      BET(I)=-ALF(I)/X(I)
  900 CONTINUE
  950 PRINT 1000,NM1,N,(ALF(I),I=1,N)
 1000 FORMAT(1H0/1H0,≠NUMERATORS IN PARTIAL FRACTION ≠,
     *≠DECOMPOSITION OF RATIO OF ≠,I2,≠TH DEGREE POLYN≠,
     *≠OMIAL TO ≠,I2,≠TH DEGREE POLYNOMIAL ARE:-≠/
     *1H0,10(1X,1PE10.3))
      PRINT 1100,(BET(I),I=1,N)
 1100 FORMAT(1H0/1H0,≠RATIOS OF NUMERATOR TO MINUS THE≠,
     *≠ ZERO ARE:-≠/1X,≠  (THESE ARE THE WEIGHTS IN THE ≠,
     *≠CONVEX COMBINATION OF EXPON≠,


     *≠ENTIAL DISTRIBUTIONS)≠/1H0,10(1X,1PE10.3))
*
*     WRITE N, PARAMETERS AND WEIGHTS TO FILE PARAMS.
*
      WRITE(3,*) N
      WRITE(3,*) (X(I),I=1,N)
      WRITE(3,*) (BET(I),I=1,N)
      ENDFILE 3
      REWIND 1
      REWIND 3
*
      STOP ≠HYPEREXPONENTIAL FOUND≠
      END
```

APPENDIX II. LISTING OF COMPUTER PROGRAM FOR DETERMINATION OF THE FINITE BIRTH-AND-DEATH PROCESS FOR WHICH THE GIVEN HYPEREXPONENTIAL IS THE INTER-OVERFLOW TIME DISTRIBUTION FUNCTION.

The following program has been written in FORTRAN for the Control Data Corporation (C.D.C.) FORTRAN extended version 4 compiler, and implemented on the C.D.C. Cyber 173 machine using a NOS/BE operating system. The reference used when writing this program was Wiley [1976].

```
      PROGRAM HYP2BD(PARAMS,OUTPUT,RATES,TAPE1=PARAMS,
     *TAPE3=RATES)
*
* PURPOSE:-
*     GIVEN HYPEREXPONENTIAL, COMPUTES RATES OF FINITE
*     BIRTH-AND-DEATH PROCESS FOR WHICH THE HYPEREXPONENTIAL
*     IS THE INTER-OVERFLOW TIME DISTRIBUTION FUNCTION.
*
* INPUT:-
*     NUMBER OF COMPONENT EXPONENTIAL DISTRIBUTIONS (N),
*     PARAMETERS (X(I),I=1,N) AND WEIGHTS (BET(I),I=1,N) -
*     WHERE X(I) CORRESPONDS TO BET(I) - IN LIST-DIRECTED
*     FORMAT (*-FORMAT) ON FILE PARAMS.
*
* OUTPUT:-
*  (1) INPUT DATA IS ECHOPRINTED TO OUTPUT. TABLE OF
*      COEFFICIENTS OF ASSOCIATED  MONIC ORTHOGONAL
*      POLYNOMIALS, ARRIVAL RATES AND DEATH RATES ALSO
*      WRITTEN TO OUTPUT.
*  (2) NUMBER OF COMPONENT EXPONENTIAL DISTRIBUTIONS (N),
*      ARRIVAL RATES AND DEATH RATES WRITTEN TO FILE RATES
*      IN LIST-DIRECTED FORMAT (*-FORMAT).
*
      INTEGER N,NM1,NP1,K,KK,KKM2,I,ISTAR
      REAL X(10),BET(10),A(11),B(11),C(11),L(10),MU(10),XK,Z
*
* DESCRIPTION OF PRINCIPAL VARIABLES:-
*     N - NUMBER OF COMPONENT EXPONENTIAL DISTRIBUTIONS.
*      (INTEGER)
*     X(I) - X IS ARRAY OF PARAMETERS.  (REAL)
*     BET(I) - BET IS ARRAY OF WEIGHTS.  (REAL)
*     A(I),B(I),C(I) - COEFFICIENT OF X**(I-1) IN (K+1)ST,
*      KTH, (K-1)ST DEGREE MONIC ORTHOGONAL POLY. RESP.,
*      WHERE 0<K<N.  (REAL)
*     L(I) - ARRIVAL RATE FOR STATE I-1.  (REAL)
*     MU(I) - DEATH RATE FOR STATE I-1.  (REAL)
*
```

```
*
*     READ, ECHOPRINT AND CHECK INPUT DATA.
*
      REWIND 1
      REWIND 3
      READ(1,*)N
      IF((N.GT.0).AND.(N.LT.11))GOTO 20
      PRINT*,≠***<<<ERROR>>>***    ILLEGAL NUMBER OF ≠,
     *≠COMPONENT EXPONENTIAL DISTRIBUTIONS≠
      PRINT*,≠                     NUMBER = ≠,N,≠ -MUST BE ≠,
     *≠STRICTLY GREATER THAN 0 AND LESS THAN 11≠
      PRINT*,≠ ≠
      PRINT*,≠$$$ PROGRAM ABORTED $$$≠
      PRINT*,≠ ≠
      STOP ≠ILLEGAL INPUT≠
   20 NM1=N-1
      NP1=N+1
      READ(1,*)(X(I),I=1,N)
      READ(1,*)(BET(I),I=1,N)


*     ROUTINE CHECK PRINTS AND VERIFIES VALIDITY OF
*     PARAMETERS AND WEIGHTS.
      CALL CHECK(N,X,BET)
*
*     DETERMINE NTH AND (N-1)ST DEGREE MONIC ORTHOG. POLYS
*     AND OVERFLOW RATE.
*
      CALL SETUP(N,X,BET,A,B,L(N))
      ISTAR=≠*****≠
      PRINT 40,(I-1,I=1,11),(ISTAR,I=1,26)
   40 FORMAT(1H0/1H0/1H0,≠TABLE OF COEFFICIENTS OF MONIC ≠,
     *≠ORTHOGONAL POLYNOMIALS:-≠/1H0,63X,≠POWER OF X≠/
     *1X,≠DEGREE≠,1X,≠*≠,11I11/1X,26A5)
      PRINT 250,N,(A(I),I=1,NP1)
      PRINT 250,NM1,(B(I),I=1,N)
*
*     CAN IMMEDIATELY DEAL WITH CASE N=1:
*
      IF(N.GT.1)GOTO 50
      PRINT 150,(ISTAR,I=1,13),L(1)
      WRITE(3,*) N
      WRITE(3,*) L(1)
      GOTO 160
*
*     DETERMINE MONIC ORTHOGONAL POLYS AND RATES RECURSIVELY.
*     AT EACH ITERATION ARRIVAL RATE FOR STATE K AND KTH AND
*     (K+1)ST DEGREE POLY. ARE KNOWN; DETERMINE DEATH RATE
*     FOR STATE K, ARRIVAL RATE FOR STATE K-1 AND (K-1)ST
*     DEGREE POLYNOMIAL.
*
   50 K=NM1
      KK=N
      IF(K-2)120,110,90
```

```
*
*      K>=3:
   90 XK=A(KK)-B(KK-1)
      MU(KK)=XK-L(KK)
      L(KK-1)=(B(KK-2)-A(KK-1)+B(KK-1)*XK)/MU(KK)
      KKM2=KK-2
      DO 100 I=2,KKM2
  100 C(I)=(B(I-1)-A(I)+B(I)*XK)/(MU(KK)*L(KK-1))
      C(1)=(-A(1)+B(1)*XK)/(MU(KK)*L(KK-1))
      C(K)=1.
      PRINT 250,K-1,(C(I),I=1,K)
      GOTO 200
*
*      K=2:
  110 XK=A(3)-B(2)
      MU(3)=XK-L(3)
      L(2)=(B(1)-A(2)+B(2)*XK)/MU(3)
      C(2)=1.
      C(1)=(-A(1)+B(1)*XK)/(MU(3)*L(2))
      PRINT 250,K-1,(C(I),I=1,K)
      GOTO 200
*
*      K=1:

  120 XK=A(2)-B(1)
      MU(2)=XK-L(2)
      L(1)=(-A(1)+B(1)*XK)/MU(2)
      C(1)=1.
      PRINT 250,K-1,(C(I),I=1,K)
      GOTO 200
*
*      K=0: PRINT RESULTS
  130 PRINT 150,(ISTAR,I=1,13),L(1),(K,L(K+1),MU(K+1),K=1,NM1)
  150 FORMAT(1H0/1H0/1H0,≠OUTPUT BIRTH-AND-DEATH PROCESS:-≠/
     *1H0,≠  K≠,19X,≠ARRIVAL RATE≠,19X,≠DEATH RATE≠/1X,
     *13A5/1X,≠  0≠,10X,F20.6/1X,9(I3,2(10X,F20.6)/1X))
      WRITE(3,*) N
      WRITE(3,*) (L(I),I=1,N)
      WRITE(3,*) (MU(I),I=2,N)
  160 ENDFILE 3
      REWIND 1
      REWIND 3
      STOP ≠BIRTH-AND-DEATH PROCESS FOUND≠
*
```

```
*      UPDATE FOR NEXT ITERATION
  200 DO 210 I=1,K
      A(I)=B(I)
  210 B(I)=C(I)
      A(KK)=B(KK)
      KK=K
      K=K-1
      IF(K.GE.3)GOTO 90
      IF(K-1)130,120,110
*
  250 FORMAT(1X,I2,5X,≠*≠,11(1X,1PE10.3))
      END




*
*$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
*
      SUBROUTINE CHECK(N,X,BET)
*
*     ROUTINE PRINTS PARAMETERS AND WEIGHTS AND CHECKS THEIR
*     VALIDITY.
*
      INTEGER N,NM1,I,J,ERR1,ERR2,ERR3,ERR4
      REAL X(10),BET(10),Z
      DATA TOL/1.E-4/
*
*     PRINT PARAMETERS AND WEIGHTS.
*
      ERR1=0
      ERR2=0
      ERR3=0
      ERR4=0
      PRINT 10,N,(X(I),I=1,N)
   10 FORMAT(1H1,≠INPUT HYPEREXPONENTIAL DISTRIBUTION:-≠/1H0,
     *≠NUMBER OF COMPONENT EXPONENTIAL DISTRIBUTIONS = ≠,I5/
     *1H0,≠PARAMETERS ARE:-      ≠,10(1X,1PE10.3))
      PRINT 20,(BET(I),I=1,N)
   20 FORMAT(1H0,≠WEIGHTS ARE:-         ≠,10(1X,1PE10.3))
*
*     CHECK THAT WEIGHTS ARE POSITIVE.
*
      DO 30 I=1,N
      IF(BET(I).LE.0.)GOTO 40
   30 CONTINUE
      GOTO 50
   40 ERR1=1
*
```

```
*     CHECK THAT PARAMETERS ARE POSITIVE.
*
   50 DO 60 I=1,N
      IF(X(I).GE.0.)GOTO 70
   60 CONTINUE
      GOTO 80
   70 ERR2=1
*
*     CHECK THAT PARAMETERS ARE DISTINCT.
*
   80 IF(N.EQ.1)GOTO 110
      NM1=N-1
      DO 90 I=1,NM1
      IP1=I+1
      DO 90 J=IP1,N
      IF(X(I).EQ.X(J))GOTO 100
   90 CONTINUE
      GOTO 110
  100 ERR3=1
*
*     CHECK THAT WEIGHTS SUM TO UNITY.
*
  110 Z=0.



      DO 120 I=1,N
  120 Z=Z+BET(I)
      IF(ABS(Z-1.0).GT.TOL)ERR4=1
*
*     PRINT ERROR MESSAGES, IF ANY.
*
      I=ERR1+ERR2+ERR3+ERR4
      IF(I.EQ.0)RETURN
      PRINT*,≠***<<<ERROR>>>***   ILLEGAL INPUT≠
      IF(ERR1.EQ.1)PRINT*,≠                    WEIGHTS MUST≠,
     *≠ BE POSITIVE≠
      IF(ERR2.EQ.1)PRINT*,≠                    PARAMETERS ≠,
     *≠MUST BE NEGATIVE≠
      IF(ERR3.EQ.1)PRINT*,≠                    PARAMETERS ≠,
     *≠MUST BE DISTINCT≠
      IF(ERR4.EQ.1)PRINT*,≠                    SUM OF ≠,
     *≠WEIGHTS MUST BE 1.0≠
      PRINT*,≠ ≠
      PRINT*,≠$$$ PROGRAM ABORTED $$$≠
      PRINT*,≠ ≠
      STOP ≠ILLEGAL INPUT≠
      END
```

```
*
*$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
*
      SUBROUTINE SETUP(N,X,BET,A,B,L)
*
*     ROUTINE DETERMINES OVERFLOW RATE (L) AND NTH AND
*     (N-1)ST DEGREE MONIC ORTHOG. POLYS.
*     RECALL THAT LAPLACE-STIELTJES TRANSFORM OF HYPEREXP. =
*     L * ((N-1)ST DEGREE POLY.) / (NTH DEGREE POLY.) .
*
      INTEGER N,NM1,I,J
      REAL X(10),BET(10),A(11),B(11),C(11),Y(10),L,Z
*
      IF(N.EQ.1)GOTO 50
      NM1=N-1
*
*     DETERMINE NTH DEGREE MONIC ORTHOG. POLY. =
*      (X-X(1))*(X-X(2))*...*(X-X(N))
      CALL COEFF(N,X,A)
*
*     DETERMINE OVERFLOW RATE (L), AND INITIALISE ARRAY B.
*
      L=0.
      DO 10 I=1,N
      B(I)=0.
   10 L=L-BET(I)*X(I)
*
*     DETERMINE (N-1)ST DEGREE MONIC ORTHOG. POLY.
*
      DO 40 I=1,N
      DO 20 J=1,N
   20 Y(J)=X(J)
      Z=Y(I)
      Y(I)=Y(N)
      Y(N)=Z
      CALL COEFF(NM1,Y,C)
*     C CORRESPONDS TO:
*      (X-X(1))*...*(X-X(I-1))*(X-X(I+1))*...*(X-X(N)) .
      Z=-BET(I)*X(I)/L
      DO 30 J=1,N
   30 B(J)=B(J)+Z*C(J)
   40 CONTINUE
      RETURN
*
*     DEALS WITH CASE N=1:
*
   50 A(1)=-X(1)
      A(2)=1.
      B(1)=1.
      L=-X(1)
      RETURN
      END
```

```
*
*$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
*
      SUBROUTINE COEFF(K,X,A)
*
*     DETERMINES POLYNOMIAL (X-X(1))*(X-X(2))*...*(X-X(K)) .
*
      INTEGER K,I,J
      REAL X(10),A(11),Z
*
      A(1)=-X(1)
      A(2)=1.
      IF(K.EQ.1)RETURN
*
*     AFTER EACH ITERATION A CORRESPONDS TO:
*      (X-X(1))*(X-X(2))*...*(X-X(I)) .
      DO 20 I=2,K
      Z=X(I)
      A(I+1)=A(I)
      J=I
   10 A(J)=A(J-1)-Z*A(J)
      J=J-1
      IF(J.GT.1)GOTO 10
      A(1)=-Z*A(1)
   20 CONTINUE
      RETURN
      END
```

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