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Fei Han, Varghese Mathai

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Advances in Mathematics, 2019; 358:106860-1- 106860-25

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Final publication at: <http://dx.doi.org/10.1016/j.aim.2019.106860>

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**1 February 2022**

<http://hdl.handle.net/2440/122800>

# PROJECTIVE ELLIPTIC GENERA AND ELLIPTIC PSEUDODIFFERENTIAL GENERA

FEI HAN AND VARGHESE MATHAI

ABSTRACT. In this paper, we construct for the first time the projective elliptic genera for a compact oriented manifold equipped with a projective complex vector bundle. Such projective elliptic genera are rational  $q$ -series that have topological definition and also have analytic interpretation via the fractional index theorem in [25] without requiring spin condition. We prove the modularity properties of these projective elliptic genera. As an application, we construct elliptic pseudodifferential genera for any elliptic pseudodifferential operator. This suggests the existence of putative  $S^1$ -equivariant elliptic pseudodifferential operators on loop space whose equivariant indices are elliptic pseudodifferential genera.

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2010 *Mathematics Subject Classification.* Primary 58J26, 58J40, Secondary 55P35, 11F55.

*Key words and phrases.* Projective elliptic genera, projective elliptic pseudodifferential genera, graded twisted Chern character, modularity, Schur functors.

## INTRODUCTION

In 1980's, Witten studied two-dimensional quantum field theories and the index of Dirac operator in free loop spaces. In [31], Witten argued that the partition function of a type II superstring as a function depending on the modulus of the worldsheet elliptic curve, is an elliptic genus. In [30], Witten derived a series of twisted Dirac operators from the free loop space  $LZ$  on a compact spin manifold  $Z$ . The elliptic genera constructed by Landweber-Stong [20] and Ochanine [27] in a topological way turn out to be the indices of these elliptic operators. Motivated by physics, Witten conjectured that these elliptic operators should be rigid. The Witten conjecture was first proved by Taubes [29] and Bott-Taubes [7]. In [21], using the modular invariance property, Liu presented a simple and unified proof of the Witten conjecture. A useful reference in this area is the book by Hirzebruch, Berger and Jung [16]. We also mention the recent generalisation of these genera by the authors in [15] to noncompact manifolds with noncompact almost connected Lie groups acting properly and cocompactly. Let us be more precise as follows.

Let  $Z$  be a  $4r$  dimensional compact smooth spin manifold and  $V$  be a rank  $2l$  spin vector bundle over  $Z$ . As in [30], let

$$(0.1) \quad \Theta(\widetilde{T_{\mathbb{C}}Z}) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}Z - \mathbb{C}^{4k}),$$

where  $q = e^{2\pi i\tau}$ , be the Witten bundle, which is an element in  $K(Z)[[q]]$ . Construct the bundles

$$(0.2) \quad \Theta(\widetilde{V}) = \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(V_{\mathbb{C}} - \mathbb{C}^{2l}), \quad \Theta_1(\widetilde{V}) = \bigotimes_{u=1}^{\infty} \Lambda_{q^u}(V_{\mathbb{C}} - \mathbb{C}^{2l}),$$

which are also elements in  $K(Z)[[q]]$ , and

$$(0.3) \quad \Theta_2(\widetilde{V}) = \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(V_{\mathbb{C}} - \mathbb{C}^{2l}), \quad \Theta_3(\widetilde{V}) = \bigotimes_{v=1}^{\infty} \Lambda_{q^{v-\frac{1}{2}}}(V_{\mathbb{C}} - \mathbb{C}^{2l}),$$

which are elements in  $K(Z)[[q^{1/2}]]$ . Let  $\hat{A}(Z)$  be the  $\hat{A}$ -class of  $TZ$  and  $\Delta^{\pm}(V)$  the spinor bundles of  $V$ . The bundle twisted elliptic genera are defined to be the **integral**  $q$ -series as follows,

$$(0.4) \quad Ell(Z, V, \tau) := \int_Z \hat{A}(Z) \text{Ch}(\Theta(\widetilde{T_{\mathbb{C}}Z})) \text{Ch}((\Delta^+(V) - \Delta^-(V)) \otimes \Theta(\widetilde{V})) \in \mathbb{Z}[[q]],$$

$$(0.5) \quad Ell_1(Z, V, \tau) := \int_Z \hat{A}(Z) \text{Ch}(\Theta(\widetilde{T_{\mathbb{C}}Z})) \text{Ch}((\Delta^+(V) + \Delta^-(V)) \otimes \Theta_1(\widetilde{V})) \in \mathbb{Z}[[q]],$$

$$(0.6) \quad Ell_2(Z, V, \tau) := \int_Z \hat{A}(Z) \text{Ch}(\Theta(\widetilde{T_C Z})) \text{Ch}(\Theta_2(\widetilde{V})) \in \mathbb{Z}[[q^{1/2}]],$$

$$(0.7) \quad Ell_3(Z, V, \tau) := \int_Z \hat{A}(Z) \text{Ch}(\Theta(\widetilde{T_C Z})) \text{Ch}(\Theta_3(\widetilde{V})) \in \mathbb{Z}[[q^{1/2}]].$$

Without the presence of  $V$ ,

$$(0.8) \quad W(Z) = \int_Z \hat{A}(Z) \text{Ch}(\Theta(\widetilde{T_C Z})) \in \mathbb{Z}[[q]]$$

is the famous Witten genus. By the Atiyah-Singer index theorem, these bundle twisted elliptic genera have analytical interpretation as follows. Let  $\not\partial^+$  be the spin Dirac operator on  $Z$ . Then

$$(0.9) \quad Ell(Z, V, \tau) = \text{Index}(\not\partial^+ \otimes \Theta(\widetilde{T_C Z}) \otimes (\Delta^+(V) - \Delta^-(V)) \otimes \Theta(\widetilde{V})) \in \mathbb{Z}[[q]],$$

$$(0.10) \quad Ell_1(Z, V, \tau) = \text{Index}(\not\partial^+ \otimes \Theta(\widetilde{T_C Z}) \otimes (\Delta^+(V) + \Delta^-(V)) \otimes \Theta_1(\widetilde{V})) \in \mathbb{Z}[[q]],$$

$$(0.11) \quad Ell_2(Z, V, \tau) = \text{Index}(\not\partial^+ \otimes \Theta(\widetilde{T_C Z}) \otimes \Theta_2(\widetilde{V})) \in \mathbb{Z}[[q^{1/2}]],$$

$$(0.12) \quad Ell_3(Z, V, \tau) = \text{Index}(\not\partial^+ \otimes \Theta(\widetilde{T_C Z}) \otimes \Theta_3(\widetilde{V})) \in \mathbb{Z}[[q^{1/2}]],$$

$$(0.13) \quad W(Z) = \text{Index}(\not\partial^+ \otimes \Theta(\widetilde{T_C Z})) \in \mathbb{Z}[[q]].$$

One can show that when  $p_1(Z) = p_1(V)$ , these bundle twisted elliptic genera are modular forms of weight  $2k$  over  $SL(2, \mathbb{Z}), \Gamma_0(2), \Gamma^0(2)$  and  $\Gamma_\theta$  respectively and when  $p_1(Z) = 0$ ,  $W(Z)$  is a modular form of weight  $2k$  over  $SL(2, \mathbb{Z})$  (see Appendix). Witten showed in [30, 32] that formally  $\not\partial^+ \otimes \Theta(\widetilde{T_C Z})$  can be viewed as the Dirac operator on loop space;  $(\Delta^+(V) - \Delta^-(V)) \otimes \Theta(\widetilde{V}), (\Delta^+(V) + \Delta^-(V)) \otimes \Theta_1(\widetilde{V}), \Theta_2(\widetilde{V}), \Theta_3(\widetilde{V})$  can be viewed as vector bundles over loop space; and

$$\begin{aligned} & \not\partial^+ \otimes \Theta(\widetilde{T_C Z}) \otimes (\Delta^+(V) - \Delta^-(V)) \otimes \Theta(\widetilde{V}), \\ & \not\partial^+ \otimes \Theta(\widetilde{T_C Z}) \otimes (\Delta^+(V) + \Delta^-(V)) \otimes \Theta_1(\widetilde{V}), \\ & \not\partial^+ \otimes \Theta(\widetilde{T_C Z}) \otimes \Theta_2(\widetilde{V}), \\ & \not\partial^+ \otimes \Theta(\widetilde{T_C Z}) \otimes \Theta_3(\widetilde{V}) \end{aligned}$$

can be viewed as the Dirac operator on loop space coupled to these bundles. Taubes [29], Bott-Taubes [7] and Liu [21] proved the rigidity of these operators conjectured by Witten. In [22] Liu discovered a profound vanishing theorem for the Witten genus  $W(Z)$ .

In this paper, we construct *projective elliptic genera* in the case that  $Z$  is a compact oriented manifold *not necessarily spin* and  $E$  is a *projective vector bundle* rather than an

ordinary vector bundle on  $Z$ . More precisely, for such  $Z$  and  $E$ , we construct **rational**  $q$ -series

$$(0.14) \quad PEll(Z, E, \tau) \in \mathbb{Q}[[q]], \quad PEll_1(Z, E, \tau) \in \mathbb{Q}[[q]],$$

$$(0.15) \quad PEll_2(Z, E, \tau) \in \mathbb{Q}[[q^{1/2}]], \quad PEll_3(Z, E, \tau) \in \mathbb{Q}[[q^{1/2}]],$$

which still have both *topological definition* and *analytic interpretation*. We also establish the modularity properties of these genera. The key new idea in the topological side is to introduce the *graded twisted Chern character* for Witten bundles constructed from projective vector bundles (see the definition in (1.15) to (1.18)). For the analytic interpretation, we use the projective spin Dirac operator introduced in [25, 26] and the fractional index theorem proved there. More precisely, let  $\not\partial^+$  be the projective spin Dirac operator associated to a fixed projective spin structure on the oriented compact manifold  $Z$ . Let  $E$  be a projective complex vector bundle. The twisted projective spin Dirac operator  $\not\partial_E^+$  acts on  $S \otimes E$ , where  $S$  is the projective vector bundle of spinors associated to the Azumaya bundle given by the complex Clifford algebra bundle  $\text{Cliff}(T_{\mathbb{C}}M)$ . The index of  $\not\partial_E^+$  has the usual expression in terms of characteristic classes, it is no longer an integer, but only a fraction in general.

In 1983, the physicists Alvarez-Gaumé and Witten [1] discovered the “miraculous cancellation” formula for gravitational anomaly, relating index of the signature operator to indices of twisted Dirac operators in dimension 12. Liu [23] generalised their formula to higher dimension and also allow general bundle twisting rather than the tangent bundle by developing modularities of certain characteristic forms. In this paper, we give a “projective miraculous cancellation” formulae for indices of projective Dirac operators twisted by projective vector bundles (Theorem 2.4) and the 12 dimensional local formula (Theorem 2.5) following Liu’s method.

The Witten genus can be viewed as a morphism from the String bordism ring to the ring of integral modular forms,  $W : \Omega_{4r}^{String} \longrightarrow MF^{\mathbb{Z}}(SL(2, \mathbb{Z}))$ . It has a lift (called  *$\sigma$ -orientation*) [2] in homotopy theory  $\sigma : MString \longrightarrow tmf$ , where  $tmf$  is the deep and powerful theory of topological modular forms, constructed originally by Hopkins and Miller [17], with a new construction due to Lurie [24]. Our projective elliptic genus  $PEll(Z, E, \tau)$  is a rational modular form over  $SL(2, \mathbb{Z})$  when the first rational Pontryagin classes of the projective bundle  $E$  and  $TZ$  are equal. It seems likely that there is a similar lift in homotopy theory for  $PEll(Z, E, \tau)$  and a refinement of our projective genera to a version of elliptic cohomology, but we will not address this here.

Other important approaches for construction of Witten genus and elliptic genera include chiral de Rham complex [13], [14], [6], [10] and the application of factorization homology

[11]. Our projective genera are twisted version of the usual genera in the presence of a  $B$ -field. We plan to look at the construction of our projective genera in these approaches in the presence of a  $B$ -field.

As an interesting application of the projective elliptic genera, we give a construction of the *elliptic pseudodifferential genera* for any elliptic pseudodifferential operator. More precisely, let  $Z$  be a  $4r$ -dimensional compact oriented manifold. Choose and fix a projective  $\text{spin}^c$  structure on  $Z$  [25,26]. Let  $P$  be any elliptic pseudodifferential operator on  $Z$ . We are able to construct elliptic genera type invariants for  $P$ :  $Ell(P, \tau)$  and  $Ell_i(P, \tau), i = 1, 2, 3$ . When  $Z$  is a  $\text{spin}^c$  manifold and  $P$  is the  $\text{spin}^c$  Dirac operator,  $Ell(P, \tau) = 0$  and  $Ell_i(P, \tau)$  degenerate to the Witten genus of  $Z$ , which is a rational  $q$ -series on  $\text{spin}^c$  manifold (see Example 3.4). This is similar to looking at the  $\hat{A}$ -genus on  $\text{spin}^c$  manifolds in [25]. The key step in our construction is to implicitly use a projective vector bundle coming from  $P$  by using the projective  $\text{spin}^c$  structure. We also use the *Schur functors* (c.f. [12]) to understand the Witten bundles of tensor product. Actually we give our construction in a more general setting, namely for projective elliptic pseudodifferential operator which has its own twist.

The paper is organized as follows. In Section 1, we introduce the graded twisted Chern character on the Witten bundles constructed from a projective vector bundle and then construct the projective elliptic genera as well as study their modularities. In Section 2, we first review the index theory for projective elliptic operators in [25, 26] and then give the analytic interpretation of the projective elliptic genera. We also give the “projective miraculous cancellation” formula in this section. As an application, we construct projective elliptic pseudodifferential genera for projective elliptic pseudodifferential operator in Section 3.

**Acknowledgements.** Fei Han was partially supported by the grant AcRF R-146-000-218-112 from National University of Singapore. He is indebted to Prof. Weiping Zhang for helpful suggestions and discussions. Varghese Mathai was supported by funding from the Australian Research Council, through the Australian Laureate Fellowship FL170100020. He gave a talk based on this paper at the conference, Microlocal methods in Analysis and Geometry (In honor of Richard Melrose's 70th birthday) CIRM, Luminy, May 6-10 2019, and would like to thank Isadore Singer and Richard Melrose for past collaboration related to this research.

## 1. PROJECTIVE ELLIPTIC GENERA

In this section, we give the topological construction of projective elliptic genera and study their modular properties. We will give the analytic interpretation of them in the next section by using the index theorem of projective elliptic operators in [25, 26].

**1.1. Projective vector bundles.** Let  $Z$  be a smooth manifold with Riemannian metric and the Levi-Civita connection  $\nabla^Z$ . Let  $Y$  be a principal  $PU(N)$  bundle over  $Z$ ,

$$\begin{array}{ccc} PU(N) & \longrightarrow & Y \\ & & \phi \downarrow \\ & & Z \end{array}$$

The *Dixmier-Douady invariant* of  $Y$ ,

$$DD(Y) = \delta(Y) \in \text{Torsion}(H^3(Z, \mathbb{Z}))$$

is the obstruction to lifting the principal  $PU(N)$ -bundle  $Y$  to a principal  $U(n)$ -bundle (the construction also works for any principal  $G$  bundle  $P$  over  $Z$ , together with a central extension  $\widehat{G}$  of  $G$ ). Let  $M_N(\mathbb{C})$  be the algebra of  $N \times N$  complex matrices. The associated algebra bundle

$$\mathcal{A} = Y \times_{PU(N)} M_N(\mathbb{C})$$

is called the associated *Azumaya bundle*.

A *projective vector bundle* on  $Z$  is *not* a global bundle on  $Z$ , but rather it is a vector bundle  $E \rightarrow Y$ , where  $E$  also satisfies

$$(1.1) \quad \mathcal{L}_g \otimes E_y \cong E_{g \cdot y}, \quad g \in PU(N), y \in Y,$$

where  $\mathcal{L} = U(n) \times_{U(1)} \mathbb{C} \rightarrow PU(N)$  is the primitive line bundle,

$$\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2} \cong \mathcal{L}_{g_1 g_2}, \quad g_i \in PU(N).$$

This gives a projective action of  $PU(N)$  on  $E$ , i.e. an action of  $U(n)$  on  $E$  s.t. the center  $U(1)$  acts as scalars. One can define the *twisted Chern character* [8],  $\text{Ch}_{\delta(Y)}(E) \in H^{\text{even}}(Z, \mathbb{Q})$  for the projective bundle  $E$ .

**1.2. Projective elliptic genera.** Suppose  $Z$  to be closed, oriented and  $4r$ -dimensional. Let  $E$  be an Hermitian projective vector bundle of rank  $l$  over  $Z$ , which is a Hermitian vector bundle over  $Y$  with the action in (1.1). Let  $\nabla^E$  be an Hermitian connection on  $E$  compatible with the action. Let  $B \in \Omega^2(Y)$  be a curving of  $Y$ . Let  $H \in \Omega^3(Z)$  representing the Dixmier-Douady class of  $Y$  such that  $\pi^*H = dB$ . As the Dixmier-Douady class is torsion element,  $H$  is exact on  $Z$ .

The condition (1.1) implies that  $\text{Tr}(BI + R^E)^n$  descends to a degree  $2n$  differential form on  $Z$  for all  $n \geq 0$ .

Define the first rational projective Pontryagin class of  $E$ ,  $p_1(E) \in H^4(Z, \mathbb{Q})$ , such that

$$(1.2) \quad \pi^*(p_1(E)) = \frac{1}{4\pi^2} [\text{Tr}(BI + R^E)^2] \in H^4(Y, \mathbb{Q}).$$

The tensor product  $E^{\otimes k}$  satisfies

$$(1.3) \quad \mathcal{L}_g^{\otimes k} \otimes E_y^{\otimes k} \cong E_{g,y}^{\otimes k}, \quad g \in PU(N), y \in Y.$$

Therefore we see that  $\text{Tr}(kBI + R^{E^{\otimes k}})^n$  descends to a degree  $2n$  differential form on  $Z$  for all  $n \geq 0$ . We can define the twisted Chern character

$$(1.4) \quad \text{Ch}_{kH}(E^{\otimes k}) = \text{Tr} \left( \exp \left[ \frac{i}{2\pi} (kBI + R^{E^{\otimes k}}) \right] \right).$$

As the exterior bundle  $\wedge^k E$  is a subbundle of  $E^{\otimes k}$ , one can also define the twisted Chern character  $\text{Ch}_{kH}(\wedge^k E)$ .

Recall that for an indeterminate  $t$  (c.f. [3]),

$$(1.5) \quad \Lambda_t(E) = \mathbb{C}|_M + tE + t^2 \wedge^2(E) + \dots, \quad S_t(E) = \mathbb{C}|_M + tE + t^2 S^2(E) + \dots,$$

are the total exterior and symmetric powers of  $E$  respectively.

Let [30]

$$(1.6) \quad \Theta(T_{\mathbb{C}}Z) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}Z).$$

be the Witten bundle, which is an element in  $K(Z)[[q]]$ .

Let  $\bar{E}$  be the complex conjugate of  $E$ , which carries the induced Hermitian metric and connection.

Set (c.f. [21, 22])

$$(1.7) \quad \Theta(E) = \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(E) \otimes \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\bar{E}), \quad \Theta_1(E) = \bigotimes_{u=1}^{\infty} \Lambda_{q^u}(E) \otimes \bigotimes_{u=1}^{\infty} \Lambda_{q^u}(\bar{E}),$$

which are elements in  $K(Y)[[q]]$ ;

$$(1.8) \quad \Theta_2(E) = \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(E) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\bar{E}), \quad \Theta_3(E) = \bigotimes_{v=1}^{\infty} \Lambda_{q^{v-\frac{1}{2}}}(E) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{q^{v-\frac{1}{2}}}(\bar{E}),$$

which are elements in  $K(Y)[[q^{1/2}]]$ .



In the  $q$ -expansion of  $(\wedge^{\text{even}} E - \wedge^{\text{odd}} E) \otimes \Theta(E)$ , the coefficient of  $q^n$  is integral linear combination of terms of the form

$$\wedge^{i_1}(E) \otimes \wedge^{i_2}(E) \otimes \cdots \otimes \wedge^{i_k}(E) \otimes \wedge^{j_1}(\bar{E}) \otimes \wedge^{j_2}(\bar{E}) \otimes \cdots \otimes \wedge^{j_l}(\bar{E}).$$

Pick out the the terms such that  $(i_1 + i_2 + \cdots + i_k) - (j_1 + j_2 + \cdots + j_l) = m$  and denote their sum by  $W_{m,n}(E)$ . Then we have the expansion

$$(1.9) \quad (\wedge^{\text{even}} E - \wedge^{\text{odd}} E) \otimes \Theta(E) = \sum_{n=0}^{\infty} \left( \sum_{m=-\infty}^{\infty} W_{m,n}(E) \right) q^n.$$

$W_{m,n}(E)$  is a vector bundle over  $Z$  carrying induced Hermitian metric and connection for each  $m$ . It is clear that for each fixed  $n$ , there are only finite many  $m$  such that  $W_{m,n}(E)$  is nonzero.  $W_{m,n}(E)$  satisfies

$$(1.10) \quad \mathcal{L}_g^{\otimes m} \otimes W_{m,n}(E)_y \cong W_{m,n}(E)_{g,y}, \quad g \in PU(N), y \in Y.$$

Therefore one can define the twisted Chern character  $\text{Ch}_{mH}(W_{m,n}(E))$ .

Similarly in the  $q$ -expansion of  $(\wedge^{\text{even}} E + \wedge^{\text{odd}} E) \otimes \Theta_1(E)$ , the coefficient of  $q^n$  is integral linear combination of terms of of the form

$$\wedge^{i_1}(E) \otimes \wedge^{i_2}(E) \otimes \cdots \otimes \wedge^{i_k}(E) \otimes \wedge^{j_1}(\bar{E}) \otimes \wedge^{j_2}(\bar{E}) \otimes \cdots \otimes \wedge^{j_l}(\bar{E}).$$

Pick out the the terms such that  $(i_1 + i_2 + \cdots + i_k) - (j_1 + j_2 + \cdots + j_l) = m$  and denote their sum by  $A_{m,n}(E)$ . Then we have the expansion

$$(1.11) \quad (\wedge^{\text{even}} E + \wedge^{\text{odd}} E) \otimes \Theta_1(E) = \sum_{n=0}^{\infty} \left( \sum_{m=-\infty}^{\infty} A_{m,n}(E) \right) q^n.$$

$A_{m,n}(E)$  is a vector bundle over  $Z$  carrying induced Hermitian metric and connection for each  $m$ . For each fixed  $n$ , there are only finite many  $m$  such that  $A_{m,n}(E)$  is nonzero.  $A_{m,n}(E)$  satisfies

$$(1.12) \quad \mathcal{L}_g^{\otimes m} \otimes A_{m,n}(E)_y \cong A_{m,n}(E)_{g,y}, \quad g \in PU(N), y \in Y.$$

We can therefore define the twisted Chern character  $\text{Ch}_{mH}(A_{m,n}(E))$ .

One can decompose  $\Theta_2(E)$  and  $\Theta_3(E)$  in a similar way as

$$(1.13) \quad \Theta_2(E) = \sum_{n=0}^{\infty} \left( \sum_{m=-\infty}^{\infty} B_{m,n}(E) \right) q^{n/2},$$

$$(1.14) \quad \Theta_3(E) = \sum_{n=0}^{\infty} \left( \sum_{m=-\infty}^{\infty} C_{m,n}(E) \right) q^{n/2},$$

and define the twisted Chern characters  $\text{Ch}_{mH}(B_{m,n}(E))$  and  $\text{Ch}_{mH}(C_{m,n}(E))$  as well.

Define the *graded twisted Chern character*

(1.15)

$$\text{GCh}_H \left( (\wedge^{\text{even}} E - \wedge^{\text{odd}} E) \otimes \Theta(E) \right) = \sum_{n=0}^{\infty} \left( \sum_{m=-\infty}^{\infty} \text{Ch}_{mH}(W_{m,n}(E)) \right) q^n \in \Omega^*(M)[[q]],$$

(1.16)

$$\text{GCh}_H \left( (\wedge^{\text{even}} E + \wedge^{\text{odd}} E) \otimes \Theta_1(E) \right) = \sum_{n=0}^{\infty} \left( \sum_{m=-\infty}^{\infty} \text{Ch}_{mH}(A_{m,n}(E)) \right) q^n \in \Omega^*(M)[[q]],$$

(1.17)

$$\text{GCh}_H(\Theta_2(E)) = \sum_{n=0}^{\infty} \left( \sum_{m=-\infty}^{\infty} \text{Ch}_{mH}(B_{m,n}(E)) \right) q^{n/2} \in \Omega^*(M)[[q^{1/2}]],$$

(1.18)

$$\text{GCh}_H(\Theta_3(E)) = \sum_{n=0}^{\infty} \left( \sum_{m=-\infty}^{\infty} \text{Ch}_{mH}(C_{m,n}(E)) \right) q^{n/2} \in \Omega^*(M)[[q^{1/2}]].$$

Let (compare with (0.1))

(1.19)

$$\Theta(T_{\mathbb{C}}Z) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}Z).$$

Let  $\det \bar{E}$  be the determinant line bundle of  $\bar{E}$ , which carries the induced Hermitian metric and connection. As the determinant is the highest exterior power, one can define the twisted Chern character  $\text{Ch}_{-lH}(\det \bar{E})$ .

Define the *projective elliptic genera* by

(1.20)

$$\begin{aligned} & \text{PEll}(Z, E, \tau) \\ & := \left( \prod_{j=1}^{\infty} (1 - q^j) \right)^{4r-2l} \cdot \int_Z \hat{A}(Z) \text{Ch}(\Theta(T_{\mathbb{C}}Z)) \sqrt{\text{Ch}_{-lH}(\det \bar{E})} \text{GCh}_H \left( (\wedge^{\text{even}} E - \wedge^{\text{odd}} E) \otimes \Theta(E) \right) \\ & \in \mathbb{Q}[[q]], \end{aligned}$$

(1.21)

$$\begin{aligned}
& PELL_1(Z, E, \tau) \\
& := \frac{\left(\prod_{j=1}^{\infty} (1 - q^j)\right)^{4r}}{\left(\prod_{j=1}^{\infty} (1 + q^j)\right)^{2l}} \cdot \int_Z \hat{A}(Z) \text{Ch}(\Theta(T_{\mathbb{C}}Z)) \sqrt{\text{Ch}_{-IH}(\det \bar{E})} \text{GCh}_H \left( (\wedge^{\text{even}} E + \wedge^{\text{odd}} E) \otimes \Theta_1(E) \right) \\
& \in \mathbb{Q}[[q]],
\end{aligned}$$

(1.22)

$$PELL_2(Z, E, \tau) := \frac{\left(\prod_{j=1}^{\infty} (1 - q^j)\right)^{4r}}{\left(\prod_{j=1}^{\infty} (1 - q^{j-1/2})\right)^{2l}} \cdot \int_Z \hat{A}(Z) \text{Ch}(\Theta(T_{\mathbb{C}}Z)) \text{GCh}_H(\Theta_2(E)) \in \mathbb{Q}[[q^{1/2}]],$$

(1.23)

$$PELL_3(Z, E, \tau) := \frac{\left(\prod_{j=1}^{\infty} (1 - q^j)\right)^{4r}}{\left(\prod_{j=1}^{\infty} (1 + q^{j-1/2})\right)^{2l}} \cdot \int_Z \hat{A}(Z) \text{Ch}(\Theta(T_{\mathbb{C}}Z)) \text{GCh}_H(\Theta_3(E)) \in \mathbb{Q}[[q^{1/2}]].$$

We have the following results.

**Theorem 1.1.** (i) If  $p_1(TZ) = \mathfrak{p}_1(E)$ , then  $PELL(Z, E, \tau)$  is a modular form of weight  $2r$  over  $SL(2, \mathbb{Z})$ .

(ii) If  $p_1(TZ) = \mathfrak{p}_1(E)$ , then  $PELL_1(Z, E, \tau)$  is a modular form of weight  $2r$  over  $\Gamma_0(2)$ ,  $PELL_2(Z, E, \tau)$  is a modular form of weight  $2r$  over  $\Gamma^0(2)$  and  $PELL_3(Z, E, \tau)$  is a modular form of weight  $2r$  over  $\Gamma_{\theta}(2)$ ; moreover, we have

$$(1.24) \quad PELL_1(Z, E, -1/\tau) = \tau^{2r} PELL_2(Z, E, \tau), \quad PELL_2(Z, E, \tau + 1) = PELL_3(Z, E, \tau).$$

*Proof.* Let  $g^{TZ}$  be a Riemann metric on  $TZ$ ,  $\nabla^{TZ}$  be the Levi-Civita connection and  $R^{TZ} = (\nabla^{TZ})^2$  be the curvature of  $\nabla^{TZ}$ . The  $\hat{A}$ -form can be expressed as

$$\hat{A}(Z, \nabla^{TZ}) = \det^{1/2} \left( \frac{\frac{\sqrt{-1}}{4\pi} R^{TZ}}{\sinh \left( \frac{\sqrt{-1}}{4\pi} R^{TZ} \right)} \right)$$

By the Chern-Weil theory (c.f. [33]), the Chern-Weil expression of twisted Chern characters ([8]) and the definitions of Jacobi theta functions and the Jacobi identity (see Appendix), we have

$$(1.25) \quad PELL(Z, E, \tau) = \int_Z \det^{\frac{1}{2}} \left( \frac{R^{TZ}}{4\pi^2} \frac{\theta'(0, \tau)}{\theta \left( \frac{R^{TZ}}{4\pi^2}, \tau \right)} \right) \det \left( \frac{4\pi^2}{BI + R^E} \frac{\theta \left( \frac{BI + R^E}{4\pi^2}, \tau \right)}{\theta'(0, \tau)} \right),$$

$$(1.26) \quad P\text{Ell}_1(Z, E, \tau) = \int_Z \det^{\frac{1}{2}} \left( \frac{R^{TZ}}{4\pi^2} \frac{\theta'(0, \tau)}{\theta\left(\frac{R^{TZ}}{4\pi^2}, \tau\right)} \right) \det \left( \frac{\theta_1\left(\frac{BI+R^E}{4\pi^2}, \tau\right)}{\theta_1(0, \tau)} \right),$$

$$(1.27) \quad P\text{Ell}_2(Z, E, \tau) = \int_Z \det^{\frac{1}{2}} \left( \frac{R^{TZ}}{4\pi^2} \frac{\theta'(0, \tau)}{\theta\left(\frac{R^{TZ}}{4\pi^2}, \tau\right)} \right) \det \left( \frac{\theta_2\left(\frac{BI+R^E}{4\pi^2}, \tau\right)}{\theta_2(0, \tau)} \right),$$

$$(1.28) \quad P\text{Ell}_3(Z, E, \tau) = \int_Z \det^{\frac{1}{2}} \left( \frac{R^{TZ}}{4\pi^2} \frac{\theta'(0, \tau)}{\theta\left(\frac{R^{TZ}}{4\pi^2}, \tau\right)} \right) \det \left( \frac{\theta_3\left(\frac{BI+R^E}{4\pi^2}, \tau\right)}{\theta_3(0, \tau)} \right).$$

It is known that the generators of  $\Gamma_0(2)$  are  $T, ST^2ST$ , the generators of  $\Gamma^0(2)$  are  $STS, T^2STS$  and the generators of  $\Gamma_\theta$  are  $S, T^2$ . Applying the Chern root algorithm on the level of forms (over certain ring extension  $\mathbb{C}[\wedge_x^2 T^*M] \subset R'$  for each  $x \in M$ , cf. [19] for details) and the transformation laws of the theta functions, one can see that that when  $\text{Tr}(R^{TZ})^2$  and  $\text{Tr}(BI + R^E)^2$  are cohomologous, the anomalies arising from modular transformation  $S : \tau \rightarrow -1/\tau$  vanish. □

## 2. ANALYTIC DEFINITION OF PROJECTIVE ELLIPTIC GENERA

In this section, we first briefly review the fractional index theorem of Mathai-Melrose-Singer [25, 26] and then use it to give an analytic construction of the projective elliptic genera.

**2.1. Projective elliptic operators.** For a compact manifold,  $Z$ , and vector bundles  $E$  and  $F$  over  $Z$ , the *Schwartz kernel theorem* gives a 1-1 correspondence,

$$\text{continuous linear operators, } T : C^\infty(Z, E) \longrightarrow C^{-\infty}(Z, F)$$

$$\begin{array}{c} \Uparrow \\ \Downarrow \end{array} \qquad \begin{array}{c} \Uparrow \\ \Downarrow \end{array}$$

$$\text{distributional sections, } k_T \in C^{-\infty}(Z^2, \text{Hom}(E, F) \otimes \Omega_R)$$

where  $\text{Hom}(E, F)_{(z, z')} = F_z \boxtimes E_{z'}^*$  is the ‘big’ homomorphism bundle over  $Z^2$  and  $\Omega_R$  is the density bundle from the right factor.

When restricted to pseudodifferential operators,  $\Psi^m(Z, E, F)$ , get an isomorphism with the space of *conormal distributions* with respect to the diagonal,  $I^m(Z^2, \Delta; \text{Hom}(E, F))$ . i.e.

$$\Psi^m(Z, E, F) \iff I^m(Z^2, \Delta; \text{Hom}(E, F))$$

When further restricted to differential operators  $\text{Diff}^m(Z, E, F)$  (which by definition have the property of being local operators) this becomes an isomorphism with the space of conormal distributions,  $I_\Delta^m(Z^2, \Delta; \text{Hom}(E, F))$ , with respect to the diagonal, *supported within the diagonal*,  $\Delta$ . i.e.

$$\text{Diff}^m(Z, E, F) \iff I_\Delta^m(Z^2, \Delta; \text{Hom}(E, F))$$

The previous facts motivates our definition of projective differential and pseudodifferential operators when  $E$  and  $F$  are only projective vector bundles associated to a fixed finite-dimensional Azumaya bundle  $\mathcal{A}$ .

Since a projective vector bundle  $E$  is not global on  $Z$ , one *cannot* make sense of sections of  $E$ , let alone operators acting between sections! However, it still makes sense to talk about Schwartz kernels even in this case, as we explain.

Notice that  $\text{Hom}(E, F) = F \boxtimes E^*$  is a projective bundle on  $Z^2$  associated to the Azumaya bundle,  $\mathcal{A}_L \boxtimes \mathcal{A}'_R$ .

The restriction  $\Delta^* \text{Hom}(E, F) = \text{hom}(E, F)$  to the diagonal is an ordinary vector bundle, it is therefore reasonable to expect that  $\text{Hom}(E, F)$  also restricts to an ordinary vector bundle in a tubular nbd  $N_\varepsilon$  of the diagonal.

In [25], it is shown that there is a canonical such choice,  $\text{Hom}^{\mathcal{A}}(E, F)$ , such that the composition properties hold.

This allows us to define the space of projective pseudo-differential operators  $\Psi_\varepsilon^\bullet(Z; E, F)$  with Schwartz kernels supported in an  $\varepsilon$ -neighborhood  $N_\varepsilon$  of the diagonal  $\Delta$  in  $Z^2$ , with the space of conormal distributions,  $I_\varepsilon^\bullet(N_\varepsilon, \Delta; \text{Hom}^{\mathcal{A}}(E, F))$ .

$$\Psi_\varepsilon^\bullet(Z; E, F) := I_\varepsilon^\bullet(N_\varepsilon, \Delta; \text{Hom}^{\mathcal{A}}(E, F)).$$

Despite *not* being a space of operators, this has precisely the same local structure as in the standard case and has similar composition properties provided supports are restricted to appropriate neighbourhoods of the diagonal.

The space of *projective smoothing operators*,  $\Psi_\varepsilon^{-\infty}(Z; E, F)$  is defined as the smooth sections,  $C_c^\infty(N_\varepsilon; \text{Hom}^{\mathcal{A}}(E, F) \otimes \pi_R^* \Omega)$ .

The space of all projective differential operators,  $\text{Diff}^\bullet(Z; E, F)$  is defined as those conormal distributions that are supported within the diagonal  $\Delta$  in  $Z^2$ ,

$$\text{Diff}^\bullet(Z; E, F) := I_\Delta^\bullet(N_\varepsilon, \Delta; \text{Hom}^{\mathcal{A}}(E, F)).$$

In fact,  $\text{Diff}^\bullet(Z; E, F)$  is even a *ring* when  $E = F$ .

Recall that there is a projective bundle of spinors  $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$  on any even dimensional oriented manifold  $Z$ .

There are natural spin connections on the Clifford algebra bundle  $Cl(Z)$  and  $\mathcal{S}^\pm$  induced from the Levi-Civita connection on  $T^*Z$ .

Recall also that  $\text{hom}(\mathcal{S}, \mathcal{S}) \cong Cl(Z)$ , has an extension to  $\tilde{Cl}(Z)$  in a tubular neighbourhood of the diagonal  $\Delta$ , with an induced connection  $\nabla$ .

The *projective spin Dirac operator* is defined as the distributional section

$$\not{D} = cl \cdot \nabla_L(\kappa_{Id}), \quad \kappa_{Id} = \delta(z - z') Id_{\mathcal{S}}$$

Here  $\nabla_L$  is the connection  $\nabla$  restricted to the left variables with  $cl$  the contraction given by the Clifford action of  $T^*Z$  on the left.

As in the usual case, the projective spin Dirac operator  $\not{D}$  is *elliptic* and odd with respect to  $\mathbb{Z}_2$  grading of  $\mathcal{S}$ .

The *principal symbol map* is well defined for conormal distributions, leading to the globally defined symbol map,

$$\sigma : \Psi_\varepsilon^m(Z; E, F) \longrightarrow C^\infty(T^*Z, \pi^* \text{hom}(E, F)),$$

homogeneous of degree  $m$ ; here  $\text{hom}(E, F)$ , is a globally defined *ordinary vector bundle* with fibre  $\text{hom}(E, F)_z = F_z \otimes E_z^*$ . Thus *ellipticity* is well defined, as the *invertibility of this symbol*.

Equivalently,  $A \in \Psi_{\varepsilon/2}^m(Z; E, F)$  is elliptic if there exists a parametrix  $B \in \Psi_{\varepsilon/2}^{-m}(Z; F, E)$  and smoothing operators  $Q_R \in \Psi_\varepsilon^{-\infty}(Z; E, E)$ ,  $Q_L \in \Psi_\varepsilon^{-\infty}(Z; F, F)$  such that

$$BA = I_E - Q_R, \quad AB = I_F - Q_L$$

The *trace functional* is defined on projective smoothing operators  $\text{Tr} : \Psi_\varepsilon^{-\infty}(Z; E) \rightarrow \mathbb{C}$  as

$$\text{Tr}(Q) = \int_Z \text{tr} Q(z, z).$$

It vanishes on commutators, i.e.  $\text{Tr}(QR - RQ) = 0$ , if

$$Q \in \Psi_{\varepsilon/2}^{-\infty}(Z; F, E), R \in \Psi_{\varepsilon/2}^{-\infty}(Z; E, F),$$

which follows from Fubini's theorem.

The *fractional analytic index* of the projective elliptic operator  $A \in \Psi_\varepsilon^\bullet(Z; E, F)$  is defined in the essentially analytic way as,

$$\text{Index}_a(A) = \text{Tr}([A, B]) \in \mathbb{R},$$

where  $B$  is a parametrix for  $A$ , and the RHS is the notation for  $\text{Tr}_F(AB - I_F) - \text{Tr}_E(BA - I_E)$ .

For  $A \in \Psi_{\varepsilon/4}^m(Z; E, F)$ , the Guillemin-Wodzicki *residue trace* is,

$$\text{Tr}_R(A) = \lim_{z \rightarrow 0} z \text{Tr}(AD(z)),$$

where  $D(z) \in \Psi_{\varepsilon/4}^z(Z; E)$  is an entire family of  $\Psi$ DOs of complex order  $z$  which is elliptic and such that  $D(0) = I$ . The residue trace is independent of the choice of such a family.

- (1) The residue trace  $\text{Tr}_R$  vanishes on all  $\Psi$ DOs of sufficiently negative order.
- (2) The residue trace  $\text{Tr}_R$  is also a trace functional, that is,

$$\text{Tr}_R([A, B]) = 0,$$

$$\text{for } A \in \Psi_{\varepsilon/4}^m(Z; E, F), B \in \Psi_{\varepsilon/4}^{m'}(Z; F, E).$$

The *regularized trace*, is defined to be the residue,

$$\text{Tr}_D(A) = \lim_{z \rightarrow 0} \frac{1}{z} (z \text{Tr}(AD(z)) - \text{Tr}_R(A)).$$

For general  $A$ ,  $\text{Tr}_D$  does depend on the regularizing family  $D(z)$ . But for smoothing operators it coincides with the standard operator trace,

$$\text{Tr}_D(S) = \text{Tr}(S), \quad \forall S \in \Psi_\varepsilon^{-\infty}(Z, E).$$

Therefore *the fractional analytic index* is also given by,

$$\text{Index}_a(A) = \text{Tr}_D([A, B]),$$

for a projective elliptic operator  $A$ , and  $B$  a parametrix for  $A$ .

The regularized trace  $\text{Tr}_D$  is *not* a trace functional, but however it satisfies the *trace defect formula*,

$$\text{Tr}_D([A, B]) = \text{Tr}_R(B\delta_D A),$$

where  $\delta_D$  is a *derivation* acting on the full symbol algebra. It also satisfies the condition of being closed,

$$\text{Tr}_R(\delta_D a) = 0 \quad \forall a.$$

Using the derivation  $\delta_D$  and the trace defect formula, we prove:

(1) the *homotopy invariance* of the index,

$$\frac{d}{dt} \text{Index}_a(A_t) = 0,$$

where  $t \mapsto A_t$  is a smooth 1-parameter family of projective elliptic  $\Psi$ DOs;

(2) the *multiplicativity property* of the index,

$$\text{Index}_a(A_2 A_1) = \text{Index}_a(A_1) + \text{Index}_a(A_2),$$

where  $A_i$  for  $i = 1, 2$  are projective elliptic  $\Psi$ DOs.

An analogue of the *McKean-Singer formula* holds,

$$\text{Index}_a(\not{\partial}_E^+) = \lim_{t \downarrow 0} \text{Tr}_s(H_\chi(t)),$$

where  $H_\chi(t) = \chi(H_t)$  is a globally defined, *truncated heat kernel*, both in space (in a neighbourhood of the diagonal) and in time. The local index theorem can then be applied, thanks to the McKean-Singer formula, to obtain the *index theorem* for projective spin Dirac operators.

**Theorem 2.1** ([25]). *The projective spin Dirac operator on an even-dimensional compact oriented manifold  $Z$ , has fractional analytic index,*

$$\text{Index}_a(\not{\partial}^+) = \int_Z \widehat{A}(Z) \in \mathbb{Q}.$$

Recall that  $Z = \mathbb{C}P^{2n}$  is an oriented but *non-spin* (actually  $\text{spin}^c$ ) manifold such that  $\int_Z \widehat{A}(Z) \notin \mathbb{Z}$ , e.g.

$$Z = \mathbb{C}P^2 \quad \implies \quad \text{Index}_a(\not{\partial}^+) = -1/8.$$

$$Z = \mathbb{C}P^4 \quad \implies \quad \text{Index}_a(\not{\partial}^+) = 3/128.$$

**2.2. Transversally elliptic operators and projective elliptic operators.** Recall that projective half spinor bundles  $\mathcal{S}^\pm$  on  $Z$  can be realized as  $\text{Spin}(n)$ -equivariant honest vector bundles,  $(\widehat{\mathcal{S}}^+, \widehat{\mathcal{S}}^-)$ , over the total space of the oriented frame bundle  $\mathcal{P}$  and in which the center,  $\mathbb{Z}_2$ , acts as  $\pm 1$ , as follows:

the conormal bundle  $N$  to the fibres of  $\mathcal{P}$  has vanishing  $w_2$ -obstruction, and  $\widehat{\mathcal{S}}^\pm$  are just the  $1/2$  spin bundles of  $N$ .

One can define the  $\text{Spin}(n)$ -equivariant transversally elliptic Dirac operator  $\widehat{\not{\partial}}^\pm$  using the Levi-Civita connection on  $Z$  together with the Clifford contraction, where *transverse ellipticity* means that the principal symbol is invertible when restricted to directions that are conormal to the fibres.



The nullspaces of  $\hat{\partial}^\pm$  are infinite dimensional unitary representations of  $Spin(n)$ . The transverse ellipticity implies that the characters of these representations are *distributions* on the group  $Spin(n)$ . In particular, the *multiplicity* of each irreducible unitary representation in these nullspaces is *finite*, and grows at most polynomially.

The  $Spin(n)$ -equivariant index of  $\hat{\partial}^+$  is defined to be the following distribution on  $Spin(n)$ ,

$$\text{Index}_{Spin(n)}(\hat{\partial}^+) = \text{Char}(\text{Nullspace}(\hat{\partial}^+)) - \text{Char}(\text{Nullspace}(\hat{\partial}^-))$$

An alternate, *analytic* description of the  $Spin(n)$ -equivariant index of  $\hat{\partial}^+$  is: for a function of compact support  $\chi \in \mathcal{C}_c^\infty(G)$ , the action of the group induces a graded operator

$$T_\chi : C^\infty(\mathcal{P}; \hat{\mathcal{S}}) \longrightarrow C^\infty(\mathcal{P}; \hat{\mathcal{S}}), \quad T_\chi u(x) = \int_G \chi(g) g^* u dg,$$

which is smoothing along the fibres.  $\hat{\partial}^+$  has a microlocal parametrix  $Q$ , in the directions that are conormal to the fibres (ie along  $N$ ). Then for any  $\chi \in \mathcal{C}_c^\infty(G)$ ,

$$T_\chi \circ (\hat{\partial}^+ \circ Q - I_-) \in \Psi^{-\infty}(\mathcal{P}; \hat{\mathcal{S}}^-); \quad T_\chi \circ (Q \circ \hat{\partial}^+ - I_+) \in \Psi^{-\infty}(\mathcal{P}; \hat{\mathcal{S}}^+)$$

are smoothing operators. The  $Spin(n)$ -equivariant index of  $\hat{\partial}^+$ , evaluated at  $\chi \in \mathcal{C}_c^\infty(G)$ , is also given by:

$$\text{Index}_{Spin(n)}(\hat{\partial}^+)(\chi) = \text{Tr}(T_\chi \circ (\hat{\partial}^+ \circ Q - I_-)) - \text{Tr}(T_\chi \circ (Q \circ \hat{\partial}^+ - I_+))$$

**Theorem 2.2** ([26]). *Let  $\pi : \mathcal{P}^2 \rightarrow Z^2$  denote the projection. The pushforward map,  $\pi_*$ , maps the Schwartz kernel of the  $Spin(n)$ -transversally elliptic Dirac operator to the projective Dirac operator: That is,*

$$\pi_*(\hat{\partial}^\pm) = \partial^\pm.$$

We will next relate these two pictures. An easy argument shows that the support of the equivariant index distribution is contained within the center  $\mathbb{Z}_2$  of  $Spin(n)$ .

**Theorem 2.3** ([26]). *Let  $\phi \in \mathcal{C}^\infty(Spin(n))$  be such that :*

- (1)  $\phi \equiv 1$  in a neighborhood of  $e$ , the identity of  $Spin(n)$ ;
- (2)  $-e \notin \text{supp}(\phi)$ . Then

$$\text{Index}_{Spin(n)}(\hat{\partial}^+ \otimes \hat{E})(\phi) = \text{Index}_a(\partial^+ \otimes E)$$

where  $E$  is a projective vector bundle associated to  $\mathcal{P}$  on  $Z$  and  $\hat{E}$  is the lift of  $E$  to  $\mathcal{P}$ .

Informally, the fractional analytic index, of the projective Dirac operator  $\hat{\mathcal{D}}^+$ , is the coefficient of the delta function (distribution) at the identity in  $Spin(n)$  of the  $Spin(n)$ -equivariant index for the associated transversally elliptic Dirac operator  $\hat{\mathcal{D}}^+$  on  $\mathcal{P}$ .

**2.3. Analytic definition of projective elliptic genera.** In view of the Definition 1.20 to 1.23, Theorem 2.1 and Theorem 2.3, we obtain the following analytic expressions

$$\begin{aligned} PEll_2(Z, E, \tau) &= \text{index}_a(\hat{\mathcal{D}}^+ \otimes (\Theta(T_{\mathbb{C}}Z) \otimes \Theta_2(E))) \\ &= \text{index}_{Spin(n)}(\hat{\mathcal{D}}^+ \otimes (\Theta(T_{\mathbb{C}}Z) \otimes \widehat{\Theta_2(E)}))(\phi) \in \mathbb{Q}[[q^{1/2}]]; \\ PEll_3(Z, E, \tau) &= \text{index}_a(\hat{\mathcal{D}}^+ \otimes (\Theta(T_{\mathbb{C}}Z) \otimes \Theta_3(E))) \\ &= \text{index}_{Spin(n)}(\hat{\mathcal{D}}^+ \otimes (\Theta(T_{\mathbb{C}}Z) \otimes \widehat{\Theta_3(E)}))(\phi) \in \mathbb{Q}[[q^{1/2}]], \end{aligned}$$

where  $\phi$  is as in Theorem 2.3.

When  $c_1(E)$  is even and  $l$  is even, the square root line bundle  $\sqrt{\det \bar{E}}$  exists and satisfies

$$(2.1) \quad \mathcal{L}_g^{\otimes l/2} \otimes \sqrt{\det \bar{E}}_y \cong \sqrt{\det \bar{E}}_{g,y}, \quad g \in PU(N), y \in Y.$$

Then we further have

$$\begin{aligned} (2.2) \quad PEll(Z, E, \tau) &= \text{index}_a(\hat{\mathcal{D}}^+ \otimes (\Theta(T_{\mathbb{C}}Z) \otimes \sqrt{\det \bar{E}} \otimes (\wedge^{\text{even}} E - \wedge^{\text{odd}} E) \otimes \Theta(E))) \\ &= \text{index}_{Spin(n)}(\hat{\mathcal{D}}^+ \otimes (\Theta(T_{\mathbb{C}}Z) \otimes \sqrt{\det \bar{E}} \otimes \widehat{(\wedge^{\text{even}} E - \wedge^{\text{odd}} E)} \otimes \Theta(E)))(\phi) \in \mathbb{Q}[[q]]; \end{aligned}$$

$$\begin{aligned} (2.3) \quad PEll_1(Z, E, \tau) &= \text{index}_a(\hat{\mathcal{D}}^+ \otimes (\Theta(T_{\mathbb{C}}Z) \otimes \sqrt{\det \bar{E}} \otimes (\wedge^{\text{even}} E + \wedge^{\text{odd}} E) \otimes \Theta_1(E))) \\ &= \text{index}_{Spin(n)}(\hat{\mathcal{D}}^+ \otimes (\Theta(T_{\mathbb{C}}Z) \otimes \sqrt{\det \bar{E}} \otimes \widehat{(\wedge^{\text{even}} E + \wedge^{\text{odd}} E)} \otimes \Theta_1(E)))(\phi) \in \mathbb{Q}[[q]], \end{aligned}$$

where  $\phi$  is as in Theorem 2.3.

**2.4. Projective miraculous cancellation formula.** We have the following ‘‘projective miraculous cancellation formula’’ for projective Dirac operators, generalizing the celebrated Alvarez-Gaumé-Witten ‘‘miraculous cancellation formula’’ [1] in dimension 12 for ordinary Dirac operators.

Let

$$\begin{aligned}\Theta(T_{\mathbb{C}}Z) \otimes \Theta_2(E) &= \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}Z) \otimes \bigotimes_{\nu=1}^{\infty} \Lambda_{-q^{\nu-\frac{1}{2}}}(E) \otimes \bigotimes_{\nu=1}^{\infty} \Lambda_{-q^{\nu-\frac{1}{2}}}(\bar{E}) \\ &= \sum_{i=0}^{\infty} B_i(T_{\mathbb{C}}Z, E) q^{i/2},\end{aligned}$$

where each  $B_i(T_{\mathbb{C}}Z, E)$  is a virtual projective bundle over  $Z$ .

The following *projective miraculous cancellation* can be similarly proved as Theorem 1 in [23] by using the modularities in Theorem 1.1 and the looking at the basis of rings of modular forms over  $\Gamma_0(2)$  and  $\Gamma^0(2)$ .

**Theorem 2.4.** *If  $p_1(TZ) = p_1(E)$ ,  $c_1(E)$  is even and  $l$  is even, then the following equality holds,*

(2.4)

$$\text{index}_a(\not\partial^+ \otimes \sqrt{\det \bar{E}} \otimes (\wedge^{\text{even}} E + \wedge^{\text{odd}} E)) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} 2^{l+k-6\lfloor \frac{k}{2} \rfloor} \text{index}_a(\not\partial^+ \otimes h_j(T_{\mathbb{C}}Z, E)),$$

where the virtual projective bundles  $h_j(T_{\mathbb{C}}Z, E)$ ,  $0 \leq j \leq \lfloor \frac{k}{2} \rfloor$  are canonical integral linear combination of  $B_i(T_{\mathbb{C}}M, E)$ ,  $0 \leq i \leq j$ .

When the dimension of  $Z$  is 12, the local formula of the projective miraculous cancellation formula for the top (degree 12) forms reads

**Theorem 2.5.**

$$\begin{aligned}(2.5) \quad & \{\hat{A}(TZ, \nabla^{TZ}) \text{Ch}_{(2^{l-1}-\frac{1}{2})H}(\sqrt{\det \bar{E}} \otimes (\wedge^{\text{even}} E + \wedge^{\text{odd}} E))\}^{(12)} \\ &= 2^{l-3} \left( \{\hat{A}(TZ, \nabla^{TZ}) \text{Ch}_H(E, \nabla^E)\}^{(12)} + (8-2l) \{\hat{A}(TZ, \nabla^{TZ})\}^{(12)} \right),\end{aligned}$$

provided  $\text{Tr}(R^{TZ})^2 = \text{Tr}(BI + R^E)^2$ .

### 3. ELLIPTIC PSEUDODIFFERENTIAL GENERA

In this section, we give an application of the projective elliptic genera by constructing *projective elliptic pseudodifferential genera* for any projective elliptic pseudodifferential operator. For references on the basics of pseudodifferential operators, see [18, 28]. Our construction of elliptic pseudodifferential genera suggests the existence of putative  $S^1$ -equivariant elliptic pseudodifferential operators on loop space that localises to the elliptic pseudodifferential genera, by a formal application of the Atiyah-Segal-Singer localisation

theorem, [4, 5]. We also compute the elliptic pseudodifferential genera for some concrete elliptic pseudodifferential operators in this section.

Let  $Z$  be a  $4r$ -dimensional compact oriented smooth manifold. Let  $W_3(Z) \in H^3(M, \mathbb{Z})$  be the third integral Stiefel-Whitney class. Fix a projective  $\text{spin}^c$  structure on  $Z$  and let  $S_c^\pm(TZ)$  be the complex projective  $\text{spin}^c$  bundle of  $TZ$  with twist  $W_3(Z)$ . Denote by  $S_c(TZ)$  the bundle  $S_c^+(TZ) \oplus S_c^-(TZ)$ . Let  $TZ^\perp$  be the stable complement of the tangent bundle  $TZ$ . Let  $S_c^\pm(TZ^\perp)$  be the projective  $\text{spin}^c$  bundle of  $TZ^\perp$  with twist  $-W_3(Z)$ . Denote by  $S_c(TZ^\perp)$  the bundle  $S_c^+(TZ^\perp) \oplus S_c^-(TZ^\perp)$ . Let  $\nabla^{S_c^\pm(TZ^\perp)}$  be projective Hermitian connections on  $S_c^\pm(TZ^\perp)$ . Denote by  $\nabla^{S_c(TZ^\perp)}$  the  $\mathbb{Z}_2$ -graded projective Hermitian connection on  $S_c(TZ^\perp)$ .

Let  $P : C^\infty(F_0) \rightarrow C^\infty(F_1)$  be a projective pseudodifferential elliptic operator with  $F_0$  and  $F_1$  being projective Hermitian vector bundles over  $Z$  with twist  $H$ . Let  $\nabla_0$  and  $\nabla_1$  be projective Hermitian connections on  $F_0, F_1$  respectively. Denote by  $\nabla^{\mathcal{F}}$  the  $\mathbb{Z}_2$ -graded projective Hermitian connection on the bundle  $\mathcal{F} = F_0 \oplus F_1$ . There exists  $m \in \mathbb{Z}^+$  and projective complex vector bundle  $E$  on  $Z$  with twist  $H - W_3(Z)$  such that  $TZ \oplus TZ^\perp \cong Z \times \mathbb{R}^m$  and  $\mathcal{F} \otimes S_c(TZ^\perp) = E^{\oplus 2^m}$ . Suppose the rank of  $E$  is  $l$ . Define *the first rational Pontryagin class of  $P$*  by

$$\mathfrak{p}_1(P) := \mathfrak{p}_1(E) \in H^4(Z, \mathbb{Q}).$$

It is clear that the it is well defined.

Let  $\mathbb{S}_\lambda$  be the *Schur functor* (c.f. Sec. 6.1 in [12]). They are indexed by Young diagram  $\lambda$  and are functors from the category of vector spaces to itself. It is not hard to see that the Schur functor is a continuous functor (c.f. [3]) and therefore if  $(E, \nabla^E)$  is a vector bundle with connection, then applying the Schur functor gives us a vector bundle with connection  $(\mathbb{S}_\lambda(E), \mathbb{S}_\lambda(\nabla^E))$ . If  $U, V$  be two vector spaces, the exterior power of a tensor product has the following nice expression via the Schur functors :

$$(3.1) \quad \Lambda^n(U \otimes V) = \bigoplus \mathbb{S}_\lambda(U) \otimes \mathbb{S}_{\lambda'}(V),$$

where  $\mathbb{S}_\lambda$  is the Schur functor with  $\lambda$  running over all the Young diagram with  $n$  cells, at most  $\dim(U)$  rows,  $\dim(V)$  columns, and  $\lambda'$  being the transposed Young diagram. Hence on the projective bundle  $\Lambda^n(\mathcal{F} \otimes S_c(TZ^\perp))$ , there is a projective Hermitian connection

$$(3.2) \quad \bigoplus \left( \mathbb{S}_\lambda(\nabla^{\mathcal{F}}) \otimes 1 + 1 \otimes \mathbb{S}_{\lambda'}(\nabla^{S_c(TZ^\perp)}) \right),$$

Denote this connection by  $\Lambda^n(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)})$ .

In the following, when we write  $\psi(\nabla^{\mathcal{F}}, S_c(TZ^\perp))$ , where  $\psi$  is certain operations on vector bundles constructed from exterior power, it always means the connections constructed in this way. For instance,

$$\Theta(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) = \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) \otimes \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\overline{\nabla^{\mathcal{F}}}, \overline{\nabla^{S_c(TZ^\perp)}})$$

is a projective Hermitian connection on the  $q$ -series with virtual projective bundle coefficients,

$$\Theta(\mathcal{F} \otimes S_c(TZ^\perp)) = \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\mathcal{F} \otimes S_c(TZ^\perp)) \otimes \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\overline{\mathcal{F} \otimes S_c(TZ^\perp)}).$$

Set

(3.3)

$$\begin{aligned} & \mathcal{H}(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) \\ &= \text{Ch}_{-2^m l(H-W_3(Z))}^{\frac{1}{2^m+1}} \det(\overline{\nabla^{\mathcal{F}}}, \overline{\nabla^{S_c(TZ^\perp)}}) \\ & \cdot \text{GCh}_{H-W_3(Z)}^{\frac{1}{2^m}} \left( (\wedge^{\text{even}}(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) - \wedge^{\text{odd}}(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)})) \otimes \Theta(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) \right), \end{aligned}$$

(3.4)

$$\begin{aligned} & \mathcal{H}_1(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) \\ &= \text{Ch}_{-2^m l(H-W_3(Z))}^{\frac{1}{2^m+1}} \det(\overline{\nabla^{\mathcal{F}}}, \overline{\nabla^{S_c(TZ^\perp)}}) \\ & \cdot \text{GCh}_{H-W_3(Z)}^{\frac{1}{2^m}} \left( (\wedge^{\text{even}}(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) + \wedge^{\text{odd}}(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) \otimes \Theta_1(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) \right) \end{aligned}$$

and for  $i = 2, 3$

$$(3.5) \quad \mathcal{H}_i(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) = \text{GCh}_{H-W_3(Z)}^{\frac{1}{2^m}} \left( \Theta_i(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}) \right).$$

**Definition 3.1.** For the projective elliptic pseudodifferential operator  $P$ , define the projective elliptic pseudodifferential genera  $Ell(P, \tau)$  and  $Ell_i(P, \tau), i = 1, 2, 3$  by the following,

$$(3.6) \quad Ell(P, \tau) = \left( \prod_{j=1}^{\infty} (1 - q^j) \right)^{4r-2l} \cdot \int_Z \hat{A}(Z) \text{Ch}(\Theta(T_{\mathbb{C}}Z)) \mathcal{H}(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}),$$

$$(3.7) \quad Ell_1(P, \tau) = \frac{\left( \prod_{j=1}^{\infty} (1 - q^j) \right)^{4r}}{\left( \prod_{j=1}^{\infty} (1 + q^j) \right)^{2l}} \cdot \int_Z \hat{A}(Z) \text{Ch}(\Theta(T_{\mathbb{C}}Z)) \mathcal{H}_1(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^\perp)}),$$

$$(3.8) \quad Ell_2(P, \tau) = \frac{\left(\prod_{j=1}^{\infty} (1 - q^j)\right)^{4r}}{\left(\prod_{j=1}^{\infty} (1 - q^{j-1/2})\right)^{2l}} \cdot \int_Z \hat{A}(Z) \text{Ch}(\Theta(T_{\mathbb{C}}Z)) \mathcal{H}_2(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^{\perp})}),$$

and

$$(3.9) \quad Ell_3(P, \tau) = \frac{\left(\prod_{j=1}^{\infty} (1 - q^j)\right)^{4r}}{\left(\prod_{j=1}^{\infty} (1 + q^{j-1/2})\right)^{2l}} \cdot \int_Z \hat{A}(Z) \text{Ch}(\Theta(T_{\mathbb{C}}Z)) \mathcal{H}_3(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^{\perp})}).$$

**Remark 3.2.** We will see from the next theorem that

(i) the genera for  $P$  are well defined;

(ii)  $Ell(P, \tau) \in \mathbb{Q}[[q]]$ ,  $Ell_1(P, \tau) \in \mathbb{Q}[[q]]$  and  $Ell_2(P, \tau) \in \mathbb{Q}[[q^{1/2}]]$ ,  $Ell_3(P, \tau) \in \mathbb{Q}[[q^{1/2}]]$ .

**Theorem 3.3.** (i)  $Ell(P, \tau) = PELL(Z, E, \tau)$ ,  $Ell_i(P, \tau) = PELL_i(Z, E, \tau)$ ,  $i = 1, 2, 3$ ;

(ii) If  $p_1(TZ) = p_1(P)$ , then  $Ell(P, \tau)$  is a modular form of weight  $2r$  over  $SL(2, \mathbb{Z})$ ,  $PELL_1(Z, P, \tau)$  is a modular form of weight  $2r$  over  $\Gamma_0(2)$ ,  $PELL_2(Z, P, \tau)$  is a modular form of weight  $2r$  over  $\Gamma^0(2)$  and  $PELL_3(Z, P, \tau)$  is a modular form of weight  $2r$  over  $\Gamma_{\theta}(2)$ ; moreover, we have

$$Ell_1(P, -1/\tau) = \tau^{2r} Ell_2(P, \tau), \quad Ell_2(P, \tau + 1) = Ell_3(P, \tau).$$

*Proof.* (i) By the multiplicativity of the operations  $\det(V)$ ,  $\wedge^{even} E - \wedge^{odd} E$  and  $\Theta(E)$ , it is not hard to see that the cohomology class

(3.10)

$$\begin{aligned} & \left[ \sqrt{\text{Ch}_{-2^m l(H-W_3(Z))}(\det \bar{E})} \text{GCh}_{H-W_3(Z)} \left( (\wedge^{even} E - \wedge^{odd} E) \otimes \Theta(E) \right) \right]^{2^m} \\ &= \sqrt{\text{Ch}_{-2^m l(H-W_3(Z))}(\det \overline{E^{\oplus 2^m}})} \text{GCh}_{H-W_3(Z)} \left( (\wedge^{even}(E^{\oplus 2^m}) - \wedge^{odd}(E^{\oplus 2^m})) \otimes \Theta(E^{\oplus 2^m}) \right) \\ &= \sqrt{\text{Ch}_{-2^m l(H-W_3(Z))}(\det(\mathcal{F} \otimes S_c(TZ^{\perp})))} \\ & \quad \cdot \text{GCh}_{H-W_3(Z)} \left( (\wedge^{even}(\mathcal{F} \otimes S_c(TZ^{\perp})) - \wedge^{odd}(\mathcal{F} \otimes S_c(TZ^{\perp}))) \otimes \Theta(\mathcal{F} \otimes S_c(TZ^{\perp})) \right) \end{aligned}$$

is represented by the differential form  $(\mathcal{H}(\nabla^{\mathcal{F}}, \nabla^{S_c(TZ^{\perp})}))^{2^m}$ . Then  $Ell(P, \tau) = PELL(Z, E, \tau)$  follows from (3.6) and (1.20). One can similarly use the multiplicativity to prove that  $Ell_i(P, \tau) = PELL_i(Z, E, \tau)$ ,  $i = 1, 2, 3$ .

Combining (i) and Theorem 1.1, (ii) is obtained.  $\square$

In the following, we give two examples of explicit computation of the elliptic pseudo-differential genera.

**Example 3.4.** Let  $Z$  be a  $\text{spin}^c$  manifold and  $P = \partial_c^+$ , the  $\text{spin}^c$  Dirac operator. Then the  $E$  corresponding to  $P$  is just the trivial complex line bundle  $\mathbb{C}$ . Hence by (i) in Theorem 3.3, we have

$$(3.11) \quad \text{Ell}(\partial_c^+, \tau) = \text{PEll}(Z, \mathbb{C}, \tau) = 0 \in \mathbb{Q}[[q]],$$

$$(3.12) \quad \text{Ell}_1(\partial_c^+, \tau) = \text{PEll}_1(Z, \mathbb{C}, \tau) = 2W(Z) \in \mathbb{Q}[[q]],$$

$$(3.13) \quad \text{Ell}_2(\partial_c^+, \tau) = \text{PEll}_2(Z, \mathbb{C}, \tau) = W(Z) \in \mathbb{Q}[[q^{1/2}]],$$

and

$$(3.14) \quad \text{Ell}_3(\partial_c^+, \tau) = \text{PEll}_3(Z, \mathbb{C}, \tau) = W(Z) \in \mathbb{Q}[[q^{1/2}]],$$

where  $W(Z)$  is the Witten genus of  $Z$ . This is similar to looking at the  $\hat{A}$ -genus on  $\text{spin}^c$  manifolds in [25].

**Example 3.5.** On  $\mathbb{C}P^2$ , take the standard  $\text{spin}^c$  structure from the complex structure. let  $\mathcal{F} = \mathcal{O}(1) \otimes (\wedge^{\text{ev}} T \oplus \wedge^{\text{odd}} T)$ , where  $\mathcal{O}(1)$  is the canonical complex line bundle and  $T$  stands for the complex tangent bundle. Let  $P : \mathcal{O}(1) \otimes \wedge^{\text{ev}} T \rightarrow \mathcal{O}(1) \otimes \wedge^{\text{odd}} T$  be an elliptic pseudodifferential operator. Here the  $E$  corresponding to  $P$  happens to be the honest line bundle  $\mathcal{O}(1)$  rather than a projective one. Let  $x \in H^2(\mathbb{C}P^2, \mathbb{Z})$  be the generator and  $z = \frac{x}{2\pi\sqrt{-1}}$ . By (i) in Theorem 3.3, we have

$$(3.15) \quad \text{Ell}(P, \tau) = \text{PEll}(\mathbb{C}P^2, \mathcal{O}(1), \tau) = \int_{\mathbb{C}P^2} \left( \frac{z\theta'(0, \tau)}{\theta(z, \tau)} \right)^3 \frac{\theta(z, \tau)}{\theta'(0, \tau)} = 0$$

due to  $\left( \frac{z\theta'(0, \tau)}{\theta(z, \tau)} \right)^3 \frac{\theta(z, \tau)}{\theta'(0, \tau)}$  is an odd function of  $z$ ;

$$(3.16) \quad \begin{aligned} \text{Ell}_1(P, \tau) &= \text{PEll}_1(\mathbb{C}P^2, \mathcal{O}(1), \tau) \\ &= \int_{\mathbb{C}P^2} \left( \frac{z\theta'(0, \tau)}{\theta(z, \tau)} \right)^3 \frac{\theta_1(z, \tau)}{\theta_1(0, \tau)} \\ &= -\frac{1}{8\pi^2} \left( \frac{\theta_1''(0, \tau)}{\theta_1(0, \tau)} - \frac{\theta_1'''(0, \tau)}{\theta_1'(0, \tau)} \right) \\ &= 2q + \cdots; \end{aligned}$$

$$\begin{aligned}
(3.17) \quad Ell_2(P, \tau) &= P Ell_2(\mathbb{C}P^2, \mathcal{O}(1), \tau) \\
&= \int_{\mathbb{C}P^2} \left( \frac{z\theta'(0, \tau)}{\theta(z, \tau)} \right)^3 \frac{\theta_2(z, \tau)}{\theta_2(0, \tau)} \\
&= -\frac{1}{8\pi^2} \left( \frac{\theta_2''(0, \tau)}{\theta_2(0, \tau)} - \frac{\theta_2'''(0, \tau)}{\theta_2'(0, \tau)} \right) \\
&= -\frac{1}{8} - q^{1/2} + \dots \in \mathbb{Q}[[q^{1/2}]]
\end{aligned}$$

and

$$\begin{aligned}
(3.18) \quad Ell_3(P, \tau) &= P Ell_3(\mathbb{C}P^2, \mathcal{O}(1), \tau) \\
&= \int_{\mathbb{C}P^2} \left( \frac{z\theta'(0, \tau)}{\theta(z, \tau)} \right)^3 \frac{\theta_3(z, \tau)}{\theta_3(0, \tau)} \\
&= -\frac{1}{8\pi^2} \left( \frac{\theta_3''(0, \tau)}{\theta_3(0, \tau)} - \frac{\theta_3'''(0, \tau)}{\theta_3'(0, \tau)} \right) \\
&= -\frac{1}{8} + q^{1/2} + \dots \in \mathbb{Q}[[q^{1/2}]].
\end{aligned}$$

They are all rational  $q$ -series rather than integral  $q$ -series.

#### APPENDIX. SOME NUMBER THEORY PREPARATION

A general reference for this appendix is [9].

Let

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

as usual be the modular group. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be the two generators of  $SL_2(\mathbb{Z})$ . Their actions on  $\mathbb{H}$  are given by

$$S: \tau \rightarrow -\frac{1}{\tau}, \quad T: \tau \rightarrow \tau + 1.$$

Let

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \pmod{2} \right\},$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| b \equiv 0 \pmod{2} \right\}$$



$$\Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2} \right\}$$

be the three modular subgroups of  $SL_2(\mathbb{Z})$ . It is known that the generators of  $\Gamma_0(2)$  are  $T, ST^2ST$ , the generators of  $\Gamma^0(2)$  are  $STS, T^2STS$  and the generators of  $\Gamma_\theta$  are  $S, T^2$ . (cf. [9]).

The four Jacobi theta-functions (c.f. [9]) defined by infinite multiplications are

$$(3.19) \quad \theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^j)(1 - e^{-2\pi\sqrt{-1}v} q^j)],$$

$$(3.20) \quad \theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^j)(1 + e^{-2\pi\sqrt{-1}v} q^j)],$$

$$(3.21) \quad \theta_2(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 - e^{-2\pi\sqrt{-1}v} q^{j-1/2})],$$

$$(3.22) \quad \theta_3(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 + e^{-2\pi\sqrt{-1}v} q^{j-1/2})],$$

where  $q = e^{2\pi\sqrt{-1}\tau}$ ,  $\tau \in \mathbb{H}$ .

They are all holomorphic functions for  $(v, \tau) \in \mathbb{C} \times \mathbb{H}$ , where  $\mathbb{C}$  is the complex plane and  $\mathbb{H}$  is the upper half plane.

Let  $\theta'(0, \tau) = \frac{\partial}{\partial v} \theta(v, \tau)|_{v=0}$ . The *Jacobi identity* [9],

$$\theta'(0, \tau) = \pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)$$

holds.

The theta functions satisfy the the following transformation laws (cf. [9]),

$$(3.23) \quad \theta(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta(v, \tau), \quad \theta(v, -1/\tau) = \frac{1}{\sqrt{-1}} \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta(\tau v, \tau);$$

$$(3.24) \quad \theta_1(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta_1(v, \tau), \quad \theta_1(v, -1/\tau) = \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_2(\tau v, \tau);$$

$$(3.25) \quad \theta_2(v, \tau + 1) = \theta_3(v, \tau), \quad \theta_2(v, -1/\tau) = \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_1(\tau v, \tau);$$

$$(3.26) \quad \theta_3(v, \tau + 1) = \theta_2(v, \tau), \quad \theta_3(v, -1/\tau) = \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_3(\tau v, \tau).$$

Let  $\Gamma$  be a subgroup of  $SL_2(\mathbb{Z})$ . A modular form over  $\Gamma$  is a holomorphic function  $f(\tau)$  on  $\mathbb{H} \cup \{\infty\}$  such that for any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

the following property holds

$$f(g\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^k f(\tau),$$

where  $\chi : \Gamma \rightarrow \mathbf{C}^*$  is a character of  $\Gamma$  and  $k$  is called the weight of  $f$ .

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