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Hao Guo, Varghese Mathai and Hang Wang
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POSITIVE SCALAR CURVATURE AND POINCARÉ DUALITY FOR PROPER ACTIONS

HAO GUO, VARGHESE MATHAI, AND HANG WANG

ABSTRACT. For G an almost-connected Lie group, we study G-equivariant index theory for proper co-compact actions with various applications, including obstructions to and existence of G-invariant Riemannian metrics of positive scalar curvature. We prove a rigidity result for almost-complex manifolds, generalising Hattori's results, and an analogue of Petrie's conjecture. When G is an almost-connected Lie group or a discrete group, we establish Poincaré duality between G-equivariant K-homology and K-theory, observing that Poincaré duality does not necessarily hold for general G.

1. INTRODUCTION

In this paper, we study G-equivariant index theory for proper co-compact actions with various applications, including obstructions to G-invariant Riemannian metrics of positive scalar curvature. As a corollary, we establish an alternate short proof of a recent result of Weiping Zhang [66] in Section 6. We also prove the existence of G-invariant Riemannian metrics of positive scalar curvature under certain very general hypotheses on G and its action on the manifold (see Theorem 61).

Our G-equivariant index theory is applied to prove rigidity theorems in Section 7.1 for certain Spin^c-Dirac operators on almost-complex manifolds with a proper G-action, generalising results of Hattori [30] (see also Atiyah-Hirzebruch [3] and Hochs-Mathai [37]). In addition, we prove an analogue of Petrie's conjecture [58] in Section 7.2.

We begin in Section 2 by defining the G-equivariant geometric K-homology $K_\bullet^{\mathrm{geo},G}(X)$ for almost-connected G as a natural extension of the original definition of Baum and Douglas [10] and the definition in the compact group case [13]. It is a description of K-homology as a quotient of the equivariant bordism group over X. We extend the result in [13] for compact groups to show that it is isomorphic, via the Baum-Douglas map, to the analytic G-equivariant K-homology $K_\bullet^G(X)$ defined by Kasparov [42, 45].

Part of the technique we develop there, namely induction on analytic K-homology, is then used in Section 3 to prove one of our main theorems: equivariant Poincaré duality for

almost-connected Lie groups and discrete groups. In the almost-connected case, this comes in the form of an isomorphism

$$\mathcal{PD} : K_\bullet^G(C_\tau(X)) \simeq K_G^\bullet(X), \tag{1.1}$$

where $C_\tau(X)$ is an algebra of continuous sections of the Clifford bundle. Our result extends equivariant Poincaré duality for compact groups, which is a consequence of work by Kasparov [42]. Our proof uses a result of Phillips [59] showing that there is an "induction" isomorphism of K-theory groups $K_\bullet^K(C_\tau(Y)) \simeq K_{\bullet+d}^G(C_\tau(X))$, where K is a maximal compact subgroup of G and Y is a K-slice of X. We also establish in this section versions of Poincaré duality when X is either not G-cocompact or has boundary, and relate our results to those obtained by Kasparov in [42]. In Section 3.2, we make use of previous work in [53] and [11] on equivariant K-theory and K-homology for discrete groups to prove equivariant Poincaré duality for discrete groups using the Mayer-Vietoris sequence.

The significance of our results on Poincaré duality can be seen as two-fold. First, they establish that equivariant Poincaré duality holds for a large class of *non-compact* Lie groups, and is an extension to non-compact groups the type of Poincaré duality proved in [42]. Second, the fact that the duality holds for these groups places into context the observation, made by Phillips in [60] and Lück-Oliver in [53] §5, that for an *arbitrary* Lie group G, an equivariant K-theory constructed from finite-dimensional G-vector bundles may not always be a generalised cohomology theory. This means one cannot expect Poincaré duality - as formulated in this way (for another way to formulate it in the case of non-compact group actions see Emerson and Meyer [26] and [25]) - to hold when G is an arbitrary Lie group. A concrete non-example of a group for which it does not hold is the semi-direct product $G = (S^1 \times S^1) \rtimes_\alpha \mathbb{Z}$, where $\alpha(n) = \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix} \right)$, $n \in \mathbb{Z}$, which is a non-linear Lie group. As has been observed by others, in cases such as this one, Phillips' generalisation of equivariant K-theory [60] is not the same as the definition via C^*-algebras.

We observe that one general situation in which they are equivalent is when G has only finitely many connected components (see Lemma 23), and this fact guides our proof of Poincaré duality in the almost-connected group case.

In Section 4 we study the relationship of K-equivariant index theory to G-equivariant index theory, where K is again a maximal compact subgroup of G (compare [55]). The difference between our approach and, for instance, the approach in [33], is that we present a more direct proof, as well as give applications, of the fact that the following diagram relating the compact and non-compact indices commutes:

$$
\begin{array}{ccc}
K_\bullet^K(Y) & \xrightarrow{\text{index}_K} & K_\bullet(C_r^*(K)) \\
\simeq \downarrow \text{K-Ind} & & \simeq \downarrow \text{D-Ind} \\
K_{\bullet+d}^G(X) & \xrightarrow{\text{index}_G} & K_{\bullet+d}(C_r^*(G)).
\end{array}
\tag{1.2}
$$

Here, the left vertical arrow is analytic induction from K to G, and the right vertical arrow is Dirac induction from K to G.

One of the consequences of this result (see Section 5) is an elegant integral trace formula for the special case when G is a connected, semisimple Lie group with finite centre, $\dim G/K$ is even and K has maximal rank. This result builds on the work of Atiyah-Schmid [7] and Lafforgue [48], but combines it with the integral formula for an L^2-index, proved by Wang in [64]. More precisely, we prove that if M is a G-Spin^c-manifold and N a K-slice, then the previous commutative diagram Then the diagram 1.2 fits into a larger commutative diagram involving the von Neumann trace on the K-theory of $C_r^*(G)$, denoted by τ_G, and a formula for the formal degree of discrete series representations in terms of the root systems of K and G (see [48]):

$$
\begin{array}{ccc}
K_0^G(M) & \xrightarrow{\text{index}_G} & K_0(C_r^*(G)) \\
\uparrow \text{\scriptsize K-Ind} & & \uparrow \text{\scriptsize D-Ind} \qquad \searrow^{\tau_G} \\
& & \qquad \qquad \mathbb{R}. \\
K_0^K(N) & \xrightarrow{\text{index}_K} & R(K) \quad \nearrow_{\Pi_K}
\end{array}
$$

Here the quantity $\Pi_K([V_\mu]) := \prod_{\alpha \in \Phi^+} \frac{(\mu + \rho_c, \alpha)}{(\rho, \alpha)}$ (for details of the notation see Section 5). Using the result of Wang [64], we obtain naturally an equality of two integrals of characteristic classes, one on M and the other on the compact slice N.

Contents

2. Equivalence of Analytic and Geometric K-homologies

The goal of this section is to prove Theorem 5, which states the equivalence of two K-homology theories for a large class of groups and spaces, building on the work done in [12], [11] and [13]. Let us give a very brief introduction to these two theories.

Analytic K-homology was first studied by Atiyah [6], who was motivated by the classification of elliptic pseudo-differential operators on a locally compact topological space. Brown-Douglas-Fillmore [16] then understood it for C^*-algebras from the point of view of extension, before Kasparov [45] formulated it in the general setting of KK-theory. It is defined using analytic K-homology cycles.

Definition 1 ([45, 9])**.** Let G be a locally compact group acting properly on a Hausdorff space X. An *analytic K-homology cycle*, or *Kasparov cycle*, is a triple of the form $(\mathcal{H}, \phi, F)$, where

- $\mathcal{H}$ is a $\mathbb{Z}_2$-graded G-Hilbert space,
- $\phi : C_0(X) \to \mathcal{L}(\mathcal{H})$ is an even G-equivariant $*$-homomorphism and
- F is an odd self-adjoint bounded linear operator on $\mathcal{H}$,

where $\mathcal{L}(\mathcal{H})$ is the C^*-algebra of bounded operators on $\mathcal{H}$, such that the elements

$$\phi(a)(F^2 - 1), [\phi(a), F], [g, F]$$

belong to the $*$-subalgebra $\mathcal{K}(\mathcal{H})$ consisting of the compact operators, for any $a \in C_0(X)$ and any $g \in G$. The (even) *G-equivariant analytic K-homology of X*, denoted $K_0^G(X)$, is the abelian group generated by Kasparov cycles subject to certain equivalence relations given by homotopy. There is also an odd part, $K_1^G(X)$, whose elements are represented by cycles $(\mathcal{H}, \phi, F)$ with no imposed $\mathbb{Z}_2$-grading.

Remark 2. When A and B are G-C^*-algebras, a Kasparov (A, B)-cycle is defined similarly as in Definition 1, with the difference that $\mathcal{H}$ is now a Hilbert B-module and $C_0(X)$ is replaced by A (see [45] Definitions 2.2 and 2.3). Equivalence classes of such cycles form an abelian group $KK^G(A, B) := KK_0^G(A, B)$. There also exists an odd part $KK_1^G(A, B)$. KK-theory encompasses analytic K-homology in the sense that

$$KK_\bullet^G(C_0(X), \mathbb{C}) \simeq K_\bullet^G(X).$$

On the other hand, geometric K-homology was introduced by Baum and Douglas [10] from the perspective of Spin geometry and Dirac operators.

Definition 3 ([10, 12])**.** Let X be a G-space. A *geometric K-homology cycle* is a triple of the form (M, E, f), where

- M is a proper G-cocompact manifold with a G-equivariant Spin^c-structure,

- E is a smooth Hermitian G-equivariant vector bundle over M and
- $f : M \to X$ is a continuous G-equivariant map.

The G-*equivariant geometric K-homology*, which we shall write as $K_\bullet^{\mathrm{geo},G}(X)$, where $\bullet = 0$ or 1, is an abelian group generated by geometric cycles (M, E, f) where $\dim M \equiv 1$ or $0 \bmod 2$, subject to an equivalence relation generated by three operations: *direct sum/disjoint union*, *bordism* and *vector bundle modification*. The first relation amounts to the identification

$$(M \sqcup M, E_1 \sqcup E_2, f \sqcup f) \sim (M, E_1 \oplus E_2, f),$$

while the statement of the latter two operations are more involved. For their definitions we refer to pp. 5-6 of [11]. We will follow the definitions contained there, except that G for us is an almost-connected Lie group instead of a discrete group.

Addition in the group is given by

$$(M_1, E_1, f_1) + (M_2, E_2, f_2) = (M_1 \sqcup M_2, E_1 \sqcup E_2, f_1 \sqcup f_2),$$

with the additive inverse of (M, E, f) being given by the cycle $(-M, E, f)$, where by $-M$ we mean the G-Spin^c-manifold with the opposite G-Spin^c-structure to M (see p.5 of [11]).

Implicit in the definition of a bordism W from a G-manifold M_1 to a G-manifold M_2 is that M_1 and M_2 are contained in G-equivariant collar neighbourhoods at the ends of W. The existence of such neighbourhoods for proper actions is proved in [40] Theorem 3.5.

Example 4. Let K be a compact Lie group and Y a compact K-manifold. If Y has a K-equivariant Spin^c-structure, we can define the *fundamental class* of $K_n^{\mathrm{geo},K}(Y)$ to be

$$[Y]_K := [(Y, Y \times \mathbb{C}, \mathrm{id}_Y)] \in K_n^{\mathrm{geo},K}(Y), \qquad n = \dim Y \ (\mathrm{mod}\ 2).$$

Similarly, if G is an almost-connected Lie group acting properly on a G-cocompact manifold X with a G-invariant Spin^c-structure, the fundamental class of $K_n^{\mathrm{geo},G}(X)$ is given by

$$[X]_G := [(X, X \times \mathbb{C}, \mathrm{id}_X)] \in K_n^{\mathrm{geo},G}(X), \qquad n = \dim X \ (\mathrm{mod}\ 2).$$

One of the themes of this section of the paper is that when K is a maximal compact subgroup of G and Y sits suitably inside X as a K-invariant submanifold, there is a one-to-one correspondence between such classes, provided the homogeneous manifold G/K is also G-Spin^c.

Baum and Douglas introduced, originally in the non-equivariant setting, a natural map from geometric K-homology to analytic K-homology [10]; it extends naturally to the equivariant setting. Our version of this map, which is to say in the setting when G is an almost-connected Lie group, will be denoted by

(2.1) $$BD : K_\bullet^{\mathrm{geo},G}(X) \to K_\bullet^G(X),$$

and defined as follows. Let (M, E, f) be a geometric K-homology cycle for X. Since M is G-equivariantly Spin^c by assumption, one can construct a Spin^c-*Dirac operator* on M, which acts on a spinor bundle $S_M \to M$. For reference in later sections, note that S_M is locally constructed by tensoring a Hermitian connection on the determinant line bundle L (associated to the given Spin^c-structure) with the lift of the Levi-Civita connection on TM

to the local spinor bundle. Note also that if M is Spin, the Spin^c-Dirac operator can be realised as the Spin-Dirac operator twisted by a line bundle.

Let D_E be the operator D twisted by the vector bundle E. It can be viewed as an unbounded, densely-defined operator on the Hilbert space $L^2(S_M \otimes E)$. Let m be the representation of $C_0(M)$ on $L^2(S_M \otimes E)$ given by pointwise multiplication of functions on sections. Since D_E is self-adjoint, we can use functional calculus to form the L^2-bounded operator $D_E(1 + D_E^2)^{-\frac{1}{2}}$. Then the triple

$$\left(L^2(S_M \otimes E), m, D_E(1 + D_E^2)^{-\frac{1}{2}} \right)$$

is an analytic K-homology cycle defining a class in $K_0^G(M)$. Call this class $[D_E]$ (we may also denote it as a cap product $[E] \cap [D]$, depending on the context). Let $f' : C_0(X) \to C_0(M)$ be the contravariant map on algebras given by

(2.2) $$(f'(g))(m) = g(f(m)), \qquad g \in C_0(X),\ m \in M.$$

Then the image of $[(M, E, f)]$ under BD is defined to be the class in $K_0^G(X)$ represented by the Kasparov cycle

$$\left(L^2(S_M \otimes E), m \circ f', D_E(1 + D_E^2)^{-\frac{1}{2}} \right).$$

We will also write this class using the pushforward notation $f_*([D_E])$. Notice that

$$f_*([D_E]) = f'^*([D_E]),$$

where $f'^* : K_0^G(M) \to K_0^G(X)$ is the covariant functorial map on analytic K-homology.

It has long been conjectured that equivariant analytic and geometric K-homologies are in general equivalent. This was proved in the case that X is a compact CW-complex without group action in [12], some twenty-five years after the conjecture was posed. Subsequently, the cases of cocompact discrete group action [11] and compact Lie group acting on a compact CW-complex [13] were confirmed. Our main aim in this section is to confirm this conjecture for the case of X a manifold admitting a proper cocompact action of an almost-connected Lie group G. More precisely, we prove:

Theorem 5. *Let G be an almost-connected Lie group acting smoothly, properly and cocompactly on a manifold X. If G/K admits a G-equivariant Spin^c-structure, the Baum-Douglas map relating G-equivariant analytic and geometric K-homologies is an isomorphism*

(2.3) $$K_\bullet^{\mathrm{geo},G}(X) \simeq K_\bullet^G(X).$$

2.1. **Overview of the Proof.** We prove Theorem 5 in several steps. Relevant to us will will be Abels' global slice theorem, which we recall presently.

Theorem 6 (Abels [1]). *Let G be an almost-connected Lie group and K a maximal compact subgroup of G. Then X has a global K-slice, defined by*

$$Y = f^{-1}(eK) \subset X,$$

6

where $f : X \to G/K$ is a G-equivariant smooth map. Y is a K-invariant submanifold and X is diffeomorphic to the associated space

(2.4)
$$G \times_K Y := G \times Y / \{(gh, y) \sim (g, hy), \forall h \in K\}.$$

The associated space (2.4) is a fibre bundle over G/K with fibre the manifold Y. The G-equivariant diffeomorphism $G \times_K Y \to X$ is given by $[(g, y)] \mapsto g \cdot y$, where $[(g, y)]$ denotes the equivalence class of the pair $(g, y) \in G \times Y$ in the quotient $G \times_K Y$.

Remark 7. In the proofs that follow, we will make frequent and essential use of the fact that the slice Y in Abels' theorem is *essentially unique* in a rather strong sense, namely the K-diffeomorphism class of the slice Y depends only upon the G-diffeomorphism class of the G-action on X (for the proof of this fact see Theorem 2.2 in the paper of Abels [1] and the first paragraph of page containing it). In particular, this means that for a fixed G-action on X, the slice Y is unique up to K-diffeomorphism.

Let $d := \dim G/K$. In Section 2.2, we show carefully that that Abels' theorem can be used to write down a well-defined "induction map" on equivariant geometric K-homology,

$$i : K_\bullet^{\mathrm{geo}, K}(Y) \to K_{\bullet + d}^{\mathrm{geo}, G}(X),$$

and show it is an isomorphism by slightly extending Abels' theorem. In Section 2.3 we define a corresponding induction map on analytic K-homology, which we prove is an isomorphism using tools from KK-theory.

$$j : K_\bullet^K(Y) \to K_{\bullet + d}^G(X).$$

Finally, we recall the following key theorem of Baum, Oyono-Oyono, Schick and Walter, which shows that equivariant geometric and analytic K-homologies are equivalent in the case of compact group actions (note that any smooth manifold admits a CW-complex structure):

Theorem 8 ([13])**.** *Let Y be a compact CW-complex with an action of a compact Lie group K. Then the Baum-Douglas map is an isomorphism*

$$K_\bullet^{\mathrm{geo}, K}(Y) \simeq K_\bullet^K(Y).$$

This isomorphism and the isomorphisms i and j fit into the following diagram:

(2.5)
$$
\begin{array}{ccc}
K_\bullet^{\mathrm{geo}, K}(Y) & \xrightarrow{\ BD\ } & K_\bullet^K(Y) \\
{\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\
K_{\bullet + d}^{\mathrm{geo}, G}(X) & \xrightarrow{\ BD\ } & K_{\bullet + d}^G(X),
\end{array}
$$

where the bottom arrow is the G-equivariant Baum-Douglas map we defined previously. The final step is to show that this diagram commutes (Section 2.4). From this Theorem 5 follows.

Remark 9. The assumption that G/K is G-Spin^c is only used in proving Proposition 14.

2.2. Induction on Geometric K-homology.

We define an induction map on equivariant geometric K-homology and show that, when G/K has a G-invariant Spin^c-structure, it is an isomorphism. Before doing so, we need to make an important remark about the relationship between Spin^c-structures on a G-manifold M and a given K-slice N.

Remark 10. In [36] Section 2.3, in particular Definition 2.7, a procedure called *stabilisation* is given to define a G-equivariant Spin^c-structure starting from a K-equivariant Spin^c-structure on N, together with a procedure called *destabilisation* going the other way, both based on Plymen's two-out-of-three lemma. Further, Lemma 3.9 of the same paper proves that stabilisation and destabilisation are inverses of one another. Thus there is a one-to-one correspondence between the collection of G-Spin^c-structures on M and the collection of K-Spin^c-structures on N. We note that a similar correspondence holds when M and N have boundary (see also. In the rest of this paper, whenever we speak of a "corresponding" or "compatible" G-Spin^c-structure or K-Spin^c structure given the other, we shall be doing so with this correspondence in mind.

Definition 11. Let G, X, K and Y be as before. The *geometric induction map* is given by

$$i : K_\bullet^{\mathrm{geo},K}(Y) \to K_{\bullet+d}^{\mathrm{geo},G}(X),$$

$$[(N, E, f)] \mapsto [(G \times_K N, G \times_K E, \tilde{f})] =: [(M, \tilde{E}, \tilde{f})].$$

Here $G \times_K N$ is equipped with the corresponding Spin^c-structure (in the sense of the remark above), while the map $f : M = G \times_K N \to G \times_K Y \cong X$ is the natural G-equivariant map determined by the K-equivariant map $f : N \to Y$ on the fibre. The vector bundle $G \times_K E$ can be defined as the pullback bundle $pr_2^* E \to G \times Y$ of E along the projection $pr_2 : G \times Y \to Y$ modulo the K-action on $G \times E$ given by $k(g, e) = (gk^{-1}, ke)$, and equipped with the pull-back Hermitian metric.

Proposition 12. *The map i is a well-defined homomorphism of abelian groups.*

Proof. One sees immediately that $(M, \tilde{E}, \tilde{f})$ satisfies the requirements of a geometric cycle. It is also easy to verify that the map i is well-defined with respect to the disjoint union/direct sum relation. To see that i is well-defined with respect to bordism, suppose that (N_1, E_1, f_1) and (N_2, E_2, f_2) are bordant cycles via a triple (S, E, f). The main point is that $G \times_K N_1$ and $-G \times_K N_2$ are bordant in the following natural way. By hypothesis, we have $\partial S = N_1 \sqcup -N_2$. Since S is a K-manifold with boundary, and K acts by self-diffeomorphisms of S, the K-action separates into two parts: an action on the boundary and an action on the interior of S. Therefore $G \times_K S$ is a G-manifold with boundary $G \times_K \partial S$ and is equal to the disjoint union of $G \times_K N_1$ and $-G \times_K N_2$. The latter two manifolds are bordant via $G \times_K S$. One then verifies that the vector bundle $G \times_K E$ restricts to $G \times_K E_1$ and $G \times_K E_2$ at the ends of the bordism, and similarly that $\tilde{f}$ restricts to $\tilde{f}_1$ and $\tilde{f}_2$.

Next, suppose that (N_1, E_1, f_1) and (N_2, E_2, f_2) are related by a vector bundle modification. In particular N_2 is the total space of the unit sphere sub-bundle of $V \oplus (N_1 \times \mathbb{R}) \to N_1$, for some G-Spin^c-vector bundle V over N_1 with fibres of even dimension $2k$. Denote the

G-Spin^c-structure of V by $P_V \to N_1$. By the definition given on p. 6 of [11], we have

$$E_2 = (P_V \times_{\mathrm{Spin}^c} \beta) \otimes \pi^*(E_1),$$

where β is the Bott generator vector bundle over S^{2k} and $\pi : N_2 \to N_1$ is the projection map for the sphere bundle. Observe that $G \times_K N_2$ is just the unit sphere sub-bundle of $G \times_K V \oplus (G \times_K N_2 \times \mathbb{R})$, which is still a direct sum of an even-rank vector bundle with the trivial real one-dimensional bundle over $G \times_K N_2$. By inspection, the induced vector bundle $G \times_K E_2 = G \times_K (P_V \times_{\mathrm{Spin}^c} \beta)$ over $G \times_K N_2$ is still β when restricted over each spherical fibre of $G \times_K N_2$. It follows that the induced cycles $(G \times_K N_1, G \times_K E_1, \tilde{f}_1)$ and $(G \times_K N_2, G \times_K E_2, \tilde{f}_2)$ are still related by a vector bundle modification, so i is also well-defined with respect to this operation.

Finally, since i preserves disjoint unions, it is a homomorphism of abelian groups. $\qquad\square$

Having seen that it is well-defined, let us show that i is an isomorphism. First we need to establish the following variant of Abels' global slice theorem for manifolds with boundary.

Lemma 13. *Let G, K be as above, and let W be a proper G-manifold with boundary. Then there exists a global K-slice $S \subseteq W$ - that is, a K-submanifold with boundary such that we have a G-equivariant diffeomorphism $W \cong G \times_K S$ with $\partial S = S \cap \partial W$.*

Proof. First we make the claim that there exists a G-equivariant smooth map $f : W \to G/K$. We sketch its proof, mainly noting where it differs from the case proved in [1] p. 8. Let $\pi : W \to W/G$ be the natural projection. It can be shown that there exists for each $w \in W$ a G-invariant neighbourhood U_w that admits a local K-slice. Indeed, if w is in the interior of W, this follows from Proposition 2.2.2. in [57], and if $w \in \partial W$ is a boundary point, we can proceed along the lines of Palais in [57] Section 2 to obtain U_w. Note that the G-map $f_{U_w} : U_w \to G/K$ extends to a slightly larger open set containing $\overline{U}_w$, so we get smooth G-maps $f_{\overline{U}_w} : \overline{U}_w \to G/K$. Define the cover

$$\mathcal{U} := \{U_w : w \in W\}$$

of W constructed out of these U_w. Since W/G is paracompact, there is an open σ-discrete refinement of the cover $\pi(\mathcal{U}) = \{\pi(U) : U \in \mathcal{U}\}$ of $\pi(W)$. That is, there is a sequence $\mathscr{U}_n$, indexed by $n \in \mathbb{N}$, of families of open subsets of W/G such that $\bigcup_{n=1}^{\infty}(\mathscr{U}_n)$ is a cover of W/G that refines $\pi(\mathcal{U})$, and every family $\mathscr{U}_n$ is discrete (see [1] p. 6).

One can then show, as on p.8 of [1], that the set of restricted G-maps $f_{\overline{A}} := f|_{\overline{A}} : \pi^{-1}(\overline{A}) \to G/K$, with A ranging over the elements of $\mathcal{U}$, piece together to give a composite smooth G-map $f : \bigcup_{A \in \mathcal{U}} \pi^{-1}(\overline{A}) \to G/K$. The rest of the construction goes as in [1] and yields the desired map $f : W \to G/K$.

One observes, by Palais' local slice theorem [57] or otherwise, that eK is a regular value for both f and $f|_{\partial W}$. Hence $f^{-1}(eK) = S \subseteq W$ is a submanifold with boundary $\partial S = S \cap \partial W$ (for a proof of this fact see for example [54]). The map $G \times_K S \to W$ given by $[g, s] \mapsto g \cdot s$ provides a G-equivariant diffeomorphism. $\qquad\square$

Proposition 14. *Let G, X, K and Y be as before. If the homogeneous manifold G/K has a G-equivariant* Spin^c*-structure, the map i is an isomorphism.*

Proof. Let (M, E, f) be a geometric cycle for $K_\bullet^{\mathrm{geo}, G}(X)$, where $\bullet = 0$ or 1. Note that M is a G-manifold with a G-equivariant Spin^c-structure. By Remark 10 one can choose a global K-slice N with a compatible K-equivariant Spin^c-structure. The restriction of E and f to the submanifold N is a pre-image $[(N, E|_N, f|_N)]$ of $[(M, E, f)]$. Hence i is surjective.

To show injectivity, let $x_k \in K_0^{\mathrm{geo}, K}(Y)$, $k = 1, 2$, be represented by geometric cycles (N_k, E_k, f_k), such that $i(x_1) = i(x_2)$. That is, we have a relation between cycles

(2.6) $$(G \times_K N_1, G \times_K E_1, \widetilde{f_1}) \sim (G \times_K N_2, G \times_K E_2, \widetilde{f_2}).$$

We show that this induces a relation

(2.7) $$(N_1, E_1, f_1) \sim (N_2, E_2, f_2).$$

Indeed, if (2.6) is a relation by disjoint union/direct sum, it follows that (2.7) is also, once one remembers that the K-slice is essentially unique (see Remark 7 or Theorem 2.2 in [1]).

Now suppose (2.6) is a relation by bordism, so that $G \times_K N_1 \sqcup -G \times_K N_2$ is the boundary of another G-cocompact G-Spin^c-manifold W, which is part of a triple $(W, \tilde{E}, \tilde{f})$ with $\tilde{E}$ and $\tilde{f}$ restricting to $G \times_K E_i$ and $\tilde{f_i}$ at the ends of W. Then by Lemma 13, W has a K-slice S, with boundary $\partial S = \partial W \cap S$. Since the G-action - which by hypothesis is by self-diffeomorphisms of W - preserves the interior and boundary of W, one sees that ∂W has a K-slice ∂W_0 by the usual version of Abels' global slice theorem. Observe that, G-equivariantly,

$$G \times_K \partial S \cong \partial W \cong G \times_K N_1 \sqcup -G \times_K N_2 \cong G \times_K (N_1 \sqcup -N_2),$$

where the first diffeomorphism follows by dimensional considerations. By uniqueness of K-slices up to K-diffeomorphism, this means that, K-equivariantly, $\partial W_0 \cong \partial S \cong N_1 \sqcup -N_2$. We now argue that S can be equipped with a K-Spin^c-structure that is compatible (in the sense of Remark 10) with the G-Spin^c-structure on W. First note that the boundary pieces $G \times_K N_1$ and $G \times_K N_2$ are contained in G-equivariant collar neighbourhoods in W (see the remark above Example 4). This allows one to form the double $\overline{W}$ of W, which is then naturally a G-Spin^c, G-cocompact manifold without boundary. Take a K-slice $\overline{S}$ of $\overline{W}$, observing that $\overline{S}$ is the double of S. By the correspondence mentioned in Remark 10, $\overline{S}$ has a K-Spin^c-structure that induces the G-Spin^c-structure on $\overline{W}$. The manifold S equipped with the restriction of this K-Spin^c-structure to S is now a K-Spin^c-bordism from N_1 to N_2. Hence N_1 and N_2 are bordant. Next, note that the vector bundle $G \times_K E$ by definition restricts to $G \times_K E_1$ and $G \times_K E_2$ on ∂W. Taking the fibre at the identity eK of the fibre bundle $G \times_K E \to G/K$ then gives a K-vector bundle whose restriction to ∂S is isomorphic to $E_1 \sqcup E_2$. One sees that, by construction, $\tilde{f}$ restricts to a K-equivariant map f on S, such that $f|_{\partial S} = f_1 \sqcup f_2$, after identifying ∂S with $N_1 \sqcup N_2$. Thus the two reduced geometric cycles (N_1, E_1, f_1) and (N_2, E_2, f_2) are still related by a bordism operation in geometric K-homology.

Finally, if (2.6) is a relation by a vector bundle modification, then there is a G-Spin^c-vector bundle V over $G \times_K N_1 =: M_1$ with even-dimensional fibres such that $G \times_K N_2 =: M_2$ is the sphere sub-bundle of $(M_1 \times \mathbb{R}) \oplus V$, while $G \times_K E_2$ is (modulo a tensor product with $\pi^*(G \times_K E_1)$, where $\pi : M_2 \to M_1$ is the projection of the sphere bundle) the bundle whose

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fibre over each sphere is the Bott generator vector bundle β. By Remark 10, the bundle $V|_{N_1}$ over the slice N_1 has a compatible K-equivariant Spin^c-structure. Further, the sphere sub-bundle of $(N_1 \times \mathbb{R}) \oplus (V|_{N_1})$ is precisely the restriction of M_2 to the submanifold $N_1 \subseteq M_1$; it is a K-slice of M_2. Thus after restriction, the manifolds N_1 and N_2 in (2.7) are still related by a vector bundle modification. Let us write $G \times_K E_2$ in the form $P_{M_1} \times_{\text{Spin}^c} \beta$, where P_{M_1} is the G-Spin^c-structure on M_1. By the correspondence in [36], P_{M_1} reduces to a compatible Spin^c-structure P_{N_1} on N_1, and the associated bundle reduces to $P_{N_1} \times_{\text{Spin}^c} \beta$, which is still the Bott generator over each spherical fibre of N_2. One verifies also that the function $\tilde{f} \circ \pi$ restricts to $f \circ \check{\pi}$, where $\check{\pi}$ is the projection of the sphere bundle $N_2 \to N_1$. Thus the two geometric cycles (N_1, E_1, f_1) and (N_2, E_2, f_2) are still related by a vector bundle modification.

Hence the map i is also injective and therefore an isomorphism. $\qquad\square$

2.3. **Induction on Analytic K-homology.** As the next step in proving the equivalence of geometric and analytic K-homologies, we now turn to the task of applying Abels' global slice theorem in the context of analytic K-homology. The goal of this subsection is to first show, using KK-theory, that there is have an isomorphism of abelian groups

$$K_\bullet^K(Y) \simeq K_{\bullet+d}^G(X),$$

before providing a natural map at the level of analytic K-cycles that realises this isomorphism. We recall the following definition (see [17] for more details). Let X be a σ-compact G-space and A, B be G-$C_0(X)$-algebras. Then the representable KK-theory $\mathcal{R}KK(X; A, B)$ is defined to be the group of equivalence classes of Kasparov (A, B)-cycles $(\mathcal{E}, \phi, T)$, defined previously, that satisfy the additional condition

$$(fa)eb = ae(fb), \qquad \forall f \in C_0(X), a \in A, b \in B, e \in \mathcal{E},$$

where the equivalence relation is identical to that defining $KK(A, B)$. The proof we give makes use of the following technical result of Kasparov.

Theorem 15 ([42, Theorem 3.4]). *Let X be a σ-compact space on which groups G and Γ act and assume that these actions commute. Suppose also that the Γ-action is proper and free. Then for any G-$C_0(X)$-algebras A, B, the descent map gives rise to an isomorphism*

$$\mathcal{R}KK^{G \times \Gamma}(X; A, B) \simeq \mathcal{R}KK^G(X/\Gamma; A^\Gamma, B^\Gamma).$$

Here, A^Γ is a "fixed-point subalgebra" of A under Γ, defined in [42].

We are now in a position to carry out the proof of the main result of this subsection.

Proposition 16. *Let G be an almost-connected Lie group acting properly on X and K a maximal compact subgroup. For Y a global K-slice of X, we have*

$$K_\bullet^K(Y) \simeq K_{\bullet+d}^G(X).$$

Proof. Note that for the $\mathbb{C}$-algebra $C(Y)$, the definitions of $\mathcal{R}KK$ and KK coincide:

$$K_\bullet^K(Y) \simeq \mathcal{R}KK_\bullet^K(pt; C(Y), \mathbb{C}).$$

Now a manifold is a σ-compact space, and the action of G on itself is proper and free. So from Theorem 15 we obtain (noting that $C_0(G)^G$ is precisely $\mathbb{C}$ in the sense of [42])

$$\mathcal{R}KK_\bullet^K(pt; C(Y), \mathbb{C}) \simeq \mathcal{R}KK_\bullet^{G \times K}(G; C_0(G \times Y), C_0(G)).$$

Applying Theorem 15 again to the right-hand side gives us

$$\mathcal{R}KK_\bullet^{G \times K}(G; C_0(G \times Y), C_0(G)) \simeq \mathcal{R}KK_\bullet^G(G/K; C_0(G \times_K Y), C_0(G/K)).$$

Therefore,

$$(2.8) \quad K_\bullet^K(Y) \simeq \mathcal{R}KK_\bullet^G(G/K; C_0(G \times_K Y), C_0(G/K)) = KK_\bullet^G(C_0(G \times_K Y), C_0(G/K)),$$

noting that G/K is contractible to a point. Because G is almost-connected, it is a *special manifold* in the sense of [42], so that there exist a Dirac element and a Bott element:

$$[\partial_{G/K}] \in KK_d^G(C_0(G/K), \mathbb{C}), \qquad [\beta] \in KK_d^G(\mathbb{C}, C_0(G/K)),$$

such that (denoting by $\otimes$ the Kasparov product between classes, cf. [42])

$$[\partial_{G/K}] \otimes_{\mathbb{C}} [\beta] = 1 \in KK^G(C_0(G/K), C_0(G/K)),$$

$$[\beta] \otimes_{C_0(G/K)} [\partial_{G/K}] = 1 \in KK^G(\mathbb{C}, \mathbb{C}).$$

This leads to the isomorphism

$$KK_\bullet^G(C_0(G \times_K Y), C_0(G/K)) \simeq KK_{\bullet+d}^G(C_0(G \times_K Y), \mathbb{C}).$$

Together with (2.8) we obtain $K_\bullet^K(Y) \simeq K_{\bullet+d}^G(X)$. $\qquad\qquad \square$

Now we give an explicit map for this isomorphism. From the proof of the above Proposition, we find that the image under j of a K-equivariant Kasparov cycle $[(\mathcal{H}, \phi, F)] \in K_\bullet^K(Y)$ is obtained by taking the Kasparov product of the "lifted" cycle

$$(2.9) \qquad\qquad [(C_0(G \times_K \mathcal{H}), \tilde{\phi}, \tilde{F})] \in KK_\bullet^G(C_0(G \times_K Y), C_0(G/K))$$

with the Dirac element $[\partial_{G/K}] \in KK_d^G(C_0(G/K), \mathbb{C})$. Note that $C_0(G \times_K \mathcal{H})$ is a G-Hilbert $C_0(G/K)$-module whose $C_0(G/K)$-valued inner product is given by the fibrewise inner product on $\mathcal{H}$, and $\tilde{\phi}$ is the multiplication action on $C_0(G \times_K \mathcal{H})$ and $\tilde{F}$ is a family of operators indexed by G/K and given on each fibre $\mathcal{H}$ by F.

The proof that j is an isomorphism did not make use of a G-equivariant Spin^c-structure on G/K, but for the remainder of this paper, we shall assume that such a structure on G/K exists. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G and K. There is a Lie algebra $\mathfrak{p}$ such that the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is invariant under the adjoint action of K. Our assumption of a G-equivariant Spin^c-structure on G/K means that $\mathrm{Ad} : K \to SO(\mathfrak{p})$ can be lifted to

$$(2.10) \qquad\qquad\qquad \widetilde{\mathrm{Ad}} : K \to \mathrm{Spin}^c(\mathfrak{p}).$$

Remark 17. Replace G by a double cover $\widetilde{G}$ and consider the diagram

$$\begin{array}{ccc} \widetilde{K} & \xrightarrow{\widetilde{\mathrm{Ad}}} & \mathrm{Spin}^c(\mathfrak{p}) \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle \pi} \\ K & \xrightarrow{\mathrm{Ad}} & \mathrm{SO}(\mathfrak{p}), \end{array}$$

where

$$\widetilde{K} := \{(k,a) \in K \times \mathrm{Spin}^c(\mathfrak{p}); \mathrm{Ad}(k) = \pi(a)\},$$

and the maps π_K and $\widetilde{\mathrm{Ad}}$ are defined by

$$\pi_K(k,a) := k;$$
$$\widetilde{\mathrm{Ad}}(k,a) := a,$$

for $k \in K$ and $a \in \mathrm{Spin}^c(\mathfrak{p})$. Then $\widetilde{G}/\widetilde{K}$ has a G-equivariant Spin^c-structure. Indeed, for all $k \in K$,

$$\pi_K^{-1}(k) \cong \pi^{-1}(\mathrm{Ad}(k)) \cong U(1),$$

so π_K is the projection of a $U(1)$-central extension. Since G/K is contractible, $\widetilde{K}$ is the maximal compact subgroup of a $U(1)$-central extension of G.

Denote by S the K-vector space underlying $\mathfrak{p}$ in the Spin^c-representation (2.10) of K. Fix the normalising function

$$b(x) = \frac{x}{\sqrt{x^2 + 1}}.$$

The Dirac element can be written as

$$[\partial_{G/K}] := \left[\left((L^2(G) \otimes S)^K, m, b(\partial_{G/K}) \right) \right],$$

where m is scalar multiplication of $C_0(G/K)$ on the Hilbert space $(L^2(G) \otimes S)^K$ and $\partial_{G/K}$ is the Spin^c-Dirac operator on G/K. Then the image of $[(\mathcal{H}, \phi, F)]$ under induction is

$$j[(\mathcal{H}, \phi, F)] = [(\mathcal{E}, \tilde{\phi}, \tilde{F} \sharp b(\partial_{G/K}))],$$

where $\mathcal{E}$ is the Hilbert space

$$\mathcal{E} := C_0(G \times_K \mathcal{H}) \otimes_{C_0(G/K)} (L^2(G) \otimes S)^K \simeq (L^2(G) \otimes \mathcal{H} \otimes S)^K,$$

$\tilde{\phi}$ is the representation

$$\tilde{\phi} : C_0(G \times_K Y) \to \mathcal{L}(\mathcal{E})$$

given by the obvious pointwise multiplication determined by $\phi : C(Y) \to \mathcal{L}(\mathcal{H})$, and

$$\tilde{F} \sharp b(\partial_{G/K})$$

means the Kasparov product (of operators) of $\tilde{F}$ and $b(\partial_{G/K}) = \partial_{G/K}(1 + \partial_{G/K}^2)^{-\frac{1}{2}}$. When F is also a Dirac-type operator, the Kasparov product can be explicitly written down, as follows.

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Example 18. Let Y be a K-manifold with a K-equivariant Spin^c-structure and ∂_Y be the associated Spin^c-Dirac operator. Then

$$\mathcal{E} = C_0(G \times_K L^2(Y, S_Y)) \otimes_{C_0(G/K)} (L^2(G) \otimes S)^K \simeq (L^2(G) \otimes L^2(Y, S_Y) \otimes S)^K$$
$$\simeq L^2(G \times_K Y, G \times_K (S_Y \times S)) \simeq L^2(X, S_X).$$

and $b(\partial_X)$ represents the Kasparov product of $b(\tilde{\partial}_Y \otimes 1)$ and $b(1 \otimes \partial_{G/K})$, namely

$$\left(\frac{\tilde{\partial}_Y}{\sqrt{\tilde{\partial}_Y^2 + 1}} \otimes 1 \right) \sharp \left(1 \otimes \frac{\partial_{G/K}}{\sqrt{1 + \partial_{G/K}^2}} \right) = \frac{\partial_X}{\sqrt{1 + \partial_X^2}}.$$

Therefore

(2.11) $$j[\partial_Y] = [\partial_X].$$

This relation also holds for twisted Dirac operators (cf. the proof of Lemma 19).

2.4. Commutativity of the Diagram.

Lemma 19. *Diagram (2.5) commutes.*

Proof. Let x be an element of $K_\bullet^K(Y)$. From [13], there is a geometric cycle (N, E, f) representing an element of $K_\bullet^{\text{geo},K}(Y)$ such that

$$x = f_*([E] \cap [D_N]).$$

By definition, the map i sends the class of geometric cycles $[(N, E, f)]$ to the class of geometric cycles

$$[(M, \tilde{E}, \tilde{f})] \in K_{\bullet+d}^{\text{geo},G}(X),$$

where $M = G \times_K N$, $\tilde{E} = G \times_K E$ and $\tilde{f} : M \to X$ is the lift of $f : N \to Y$ as in Definition 11. Thus, elements in (2.5) are related as follows:

$$
\begin{array}{ccc}
[(N, E, f)] & \xrightarrow{BD} & f_*([E] \cap [D_N]) \\
{\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\
[(M, \tilde{E}, \tilde{f})] & \xrightarrow{BD} & \tilde{f}_*([\tilde{E}] \cap [D_N]),
\end{array}
$$

and commutativity of (2.5) means precisely

$$j(f_*([E] \cap [D_N])) = \tilde{f}_*([\tilde{E}] \cap [D_M]).$$

Writing both sides in terms of cycles, we need to show that the Kasparov $(C_0(X), \mathbb{C})$-cycles

$$((L^2(G) \otimes L^2(S_N \otimes E) \otimes S)^K, \widetilde{m \circ f'}, b(\widetilde{D_{N,E}}) \sharp b(\partial_{G/K}))$$

and

$$(L^2(S_M \otimes \tilde{E}), \tilde{m} \circ \tilde{f}', b(D_{M, \tilde{E}}))$$

give rise to the same element in $K_0^G(X)$. Here S_N stands for the spinor bundle over N and m stands for the pointwise multiplication of the algebra $C_0(N)$ of continuous functions on the Hilbert space. At this point, recall the well-known fact that the K-homology class of a

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Dirac operator is independent of the choice of metric. Given a K-invariant metric on N, let us pick the natural "product-type" metric on $G \times_K N$. Then we have

$$(L^2(G) \otimes L^2(S_N \otimes E) \otimes S)^K \simeq L^2(S_M \otimes \tilde{E}),$$

since $S_M = G \times_K (S_M|_N)$ and the restriction S_M to N splits into a tensor product of S_N and S. Obviously, $\tilde{m} \circ \tilde{f}'$ and $\widetilde{m \circ f'}$ both give the same representation of $C_0(X)$ on $L^2(S_M \otimes \tilde{E})$ by scalar multiplication, compatible with $\tilde{f}' : C_0(X) \to C_0(M)$. Finally, from $D_{N,E}$ to $\widetilde{D_{N,E}}$ we replace the K-equivariant Spin^c-connection ∇^E on N by the G-equivariant connection $\nabla^{\tilde{E}} = G \times_K \nabla^E$ on M. If $\pi : M = G \times_K N \to G/K$ is the projection, the Spin^c-connection $\nabla^{G/K}$ on G/K is pulled back to a G-equivariant Spin^c-connection $\pi^* \nabla^{G/K}$ on M. Let $\{e_i\}$ be a local orthonormal frame in the direction of fibres of $G \times_K N \to G/K$ and $\{f_j\}$ a local orthonormal frame of the base G/K. Then $b(\widetilde{D_{N,E}}) \sharp b(\partial_{G/K})$ looks locally like $b(x)$, where

$$x = \sum_{i=1}^{\dim N} c(e_i) \nabla^{\tilde{E}}_{e_i} + \sum_{j=1}^{d} c(f_j) \pi^* \nabla^{G/K}_{f_j}.$$

With respect to the fibration $N \to M \to G/K$, a G-invariant metric g_M on M determines a metric $g_{G/K}$ on the base G/K and a family of metrics $g_{M/(G/K)}$ on the family of manifolds $\pi : M \to G/K$ parametrised by G/K. Taking the adiabatic limit in the parameter s,

$$g_{M,s} := g_{M/(G/K)} + s^{-2} \pi^* g_{G/K}$$

approaches a product as $s \to 0$, and these metrics are G-invariant (see [14]). Since the K-homology class represented by a Dirac operator is independent of the choice of metrics, the operator $b(\widetilde{D_{N,E}}) \sharp b(\partial_{G/K})$ represents the same K-homology class as $b(D_{M,\tilde{E}})$. The lemma is then proved. $\square$

Remark 20. Let us summarise of the consequences of Theorem 5. Let G be an almost-connected Lie group acting on a manifold X properly and cocompactly. For every element x in the analytic K-homology $K_0^G(X)$, there exists a unique class $[(M, E, f)] \in K_0^{G,geo}(X)$ such that

$$x = f_*([E] \cap [D_M]) = f_*([D_{M,E}]),$$

where $f : M \to X$. Moreover, letting K be a maximal compact subgroup, there exists a K-submanifold N such that:

- M is diffeomorphic to $G \times_K N$,
- N admits a K-equivariant Spin^c-structure compatible with the G-equivariant Spin^c-structure on M and
- $(N, E|_N, f|_N)$ is a geometric cycle for $K_d^{K,geo}(Y)$,

and there exists a unique element $(f|_N)_*([D_{N,E|_N}]) \in K_d^K(Y)$ satisfying

(2.12) $\qquad K_d^K(Y) \to K_0^G(X), \qquad (f|_N)_*([D_{N,E|_N}]) \mapsto f_*([D_{M,E}]) = x.$

Here $Y = f(N) \subset X$ is a K-slice of X and d is the (mod 2) dimension of G/K.

Remark 21. In Theorem 4.6 in [33] and Theorem 4.5 in [34], a map

$$\text{K-Ind}_K^G \colon K_\bullet^K(Y) \to K_{\bullet+d}^G(X)$$

is constructed by a different method. In Section 6 of [33], it is shown that the K-homology class of a Spin^c-Dirac operator on Y, associated to a connection ∇^Y on the determinant line bundle of a Spin^c-structure, is mapped to the class of a Spin^c-Dirac operator on X associated to a connection ∇^X induced by ∇^Y on the determinant line bundle of the induced Spin^c-structure, by the map K-Ind$_K^G$:

$$\text{K-Ind}_K^G[\partial_Y] = [\partial_X].$$

Since we have shown that the two K-homology theories are isomorphic and that the diagram (2.5) commutes, the two induction isomorphisms i and j can be thought of as being the same map. In view of Remark 21, we shall denote both i and j by K-Ind$_K^G$, which we refer to as *induction on K-homology*. (2.12) can now be formulated in the following way:

Proposition 22. *Any $x \in K_0^G(X)$ can be represented by a G-equivariant* Spin^c-*Dirac operator on M twisted by a G-vector bundle E, where $f : M \to X$ is a continuous G-equivariant map such that*

$$x = f_*([D_{M,E}]) = \text{K-Ind}_K^G((f|_N)_*([D_{N,E|_N}])).$$

3. Poincaré Duality

3.1. **Almost-connected Groups.** We begin by remarking that Phillips' [59] generalisation of equivariant K-theory, denoted $\bar{K}_G^\bullet(X)$, which is defined using finite-dimensional equivariant vector bundles over X, is not necessarily the same as the definition via C^*-algebras, namely $K_G^\bullet(X) := K_\bullet(C_0(X) \rtimes G)$. However, in the case of almost-connected Lie groups, these groups are isomorphic, as will be argued presently.

Lemma 23. *Let G be an almost-connected Lie group acting properly and cocompactly on a smooth manifold X. Then*

(3.1) $$\bar{K}_G^\bullet(X) \simeq K_G^\bullet(X) = K_\bullet(C_0(X) \rtimes G).$$

Proof. Without loss of generality, assume $\bullet = 0$. Under the hypotheses of the lemma, $C_0(X) \rtimes G$ and $C_0(Y) \rtimes K$ are strongly Morita-equivalent, by Rieffel-Green [61], where Y is a global slice given by Abels' theorem [1]. By Green-Julg [38], one has

$$K_0(C_0(Y) \rtimes K) \simeq K_K^0(Y).$$

By definition,

$$K_G^0(X) = K_0(C_0(X) \rtimes G),$$

therefore

$$K_G^0(X) \simeq K_K^0(Y).$$

But Phillips' [59] proves that $\bar{K}_G^0(X) \simeq K_K^0(Y)$, so we conclude. $\qquad\square$

Remark 24. We do not need to specify whether or not the crossed product $C_0(X) \rtimes G$ is reduced, as the action of G on X is proper. See also Remark 39.

Remark 25. Concretely, the isomorphism (3.1) can be expressed as the composition

$$\bar{K}_G^0(X) \to KK^G(C_0(X), C_0(X)) \to KK(C_0(X) \rtimes G, C_0(X) \rtimes G) \to K_0(C_0(X) \rtimes G),$$

where the first map takes a G-equivariant vector bundle to its continuous sections, the second is the descent map (4.3) in KK-theory and the last is left multiplication via KK-product by the canonical projection (4.1) in $C_0(X) \rtimes G$. Given a G-equivariant vector bundle V over X, the images in the above sequence of maps are

$$[V] \mapsto [(\Gamma(V), 0)] \mapsto [(\Gamma(V) \rtimes G, 0)] \mapsto [p \cdot (\Gamma(V) \rtimes G)].$$

The induction on $\bar{K}^0$ is simpler than the induction on K^0. In fact, if E is a K-equivariant vector bundle over Y and p^K, p^G are the canonical projections in $C(Y) \rtimes K$, $C_0(X) \rtimes G$ respectively, then in the diagram

(3.2)
$$\begin{array}{ccc}
\bar{K}_K^0(Y) & \longrightarrow & K_K^0(Y) \\
\downarrow & & \downarrow \\
\bar{K}_G^0(X) & \longrightarrow & K_G^0(X)
\end{array}$$

we have

(3.3)
$$\begin{array}{ccc}
[V] & \longrightarrow & [p^K \cdot (\Gamma(V) \rtimes K)] \\
\downarrow & & \downarrow \\
[G \times_K V] & \longrightarrow & [p^G \cdot (\Gamma(G \times_K V) \rtimes G)].
\end{array}$$

In particular, if V is the rank 1 trivial vector bundle over Y, then $G \times_K V$ is also a rank 1 vector bundle over X and they correspond to the canonical projections $[p^K] \in K_K^0(Y)$ and $[p^G] \in K_G^0(X)$ respectively. The induction map $K_K^0(Y) \to K_G^0(X)$ can also be understood using the isomorphism

(3.4)
$$K_0(C_0(X) \rtimes G) \simeq K_d(C_0(X) \rtimes K)$$

proved in [27] and the fact that X is K-homeomorphic to $Y \times \mathbb{R}^d$, where K acts on $\mathbb{R}^d$ via the diffeomorphism $\mathbb{R}^d \cong G/K$ (see Remark 5.19 in [27]). Finally, the induction maps (3.2)-(3.3) imply that both $K_G^0(X)$ and $\bar{K}_G^0(X)$ are $R(K)$-modules and that (3.1) is an isomorphism of $R(K)$-modules.

We now prove an equivariant Poincaré duality under the same hypotheses as above.

Theorem 26 (Poincaré duality: cocompact case). *Let G be an almost-connected Lie group acting properly and cocompactly on a smooth manifold X. Then there are isomorphisms*

(3.5)
$$\mathcal{PD}_{X*} : K_\bullet^G(C_\tau(X)) \simeq K_\bullet^G(X);$$

(3.6)
$$\mathcal{PD}_X^* : K_G^\bullet(C_\tau(X)) \simeq K_G^\bullet(X),$$

where $C_\tau(X)$ is the algebra of continuous sections, tending to 0 at ∞, of the complex Clifford bundle associated with the tangent bundle TX of X.

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Proof. We use Abels' global slice theorem to see that X is diffeomorphic to $G \times_K Y$, where K is a maximal compact subgroup of G and Y is a smooth compact manifold. Using Morita equivalence of $C_\tau(X)$ and $C_0(TX)$ [41, Theorem 2.7], the decomposition

$$(3.7) \qquad\qquad TX = G \times_K [TY \oplus \mathfrak{p}],$$

where $\mathfrak{p} \oplus \mathfrak{k} = \mathfrak{g}$, and Phillips' result [60, 59] proving that induction from K to G in K-theory is an isomorphism, we obtain

$$(3.8) \qquad\qquad K_\bullet^K(C_\tau(Y)) \simeq K_{\bullet+d}^G(C_\tau(X)).$$

Then, (3.8) using equivariant Poincaré duality in the compact case (see [42]),

$$\mathcal{PD} : K_\bullet^K(C_\tau(Y)) \simeq K_\bullet^K(Y),$$

together with analytic induction from K to G, which is an isomorphism

$$K_\bullet^K(Y) \simeq K_{\bullet+d}^G(X),$$

we deduce the isomorphism (3.5). The second isomorphism is proved analogously. $\qquad\square$

We note that the above proof only used induction on the *analytic* version of K-homology. However, the equivalence between the two K-homologies can be used to give the following geometric interpretation of Poincaré duality.

Corollary 27. *Let G be an almost-connected Lie group acting properly and cocompactly on a manifold X. If X and G/K admit G-equivariant Spin^c-structures, we have*

$$(3.9) \qquad \mathcal{PD}_X : K_G^0(X) \to K_a^{G,geo}(X), \quad [E] \mapsto [X] \cap [E] := [X, E \otimes \mathbb{C}, \mathrm{id}_X],$$

where E is a finite-rank G-equivariant vector bundle over X and $a = \dim X \ (mod\ 2)$.

Proof. Since X is a proper G-cocompact Spin^c-manifold, $C_\tau(X)$ is Morita-equivalent to $C_0(X)$. Thus we may write

$$(3.10) \qquad\qquad \mathcal{PD}_X : K_G^0(X) \to K_a^G(X).$$

Then the claim follows by the equivalence between geometric and analytic K-homologies given in Theorem 5. $\qquad\square$

Remark 28. When X is a proper and cocompact G-manifold, not necessarily Spin^c, the first isomorphism (3.5) is equivalent to

$$(3.11) \qquad\qquad K_\bullet^G(X) \simeq K_\bullet^G(C_0(TX)) \simeq K_\bullet(C_0(TX) \rtimes G).$$

This isomorphism is closely related to a generalisation of the Atiyah-Singer index formula, since it is an operator-to-symbol map. In fact, recall that for a K-slice Y of X, K-invariant pseudo-differential operators represent classes in $K_\bullet^K(Y)$, while their symbols give rise to classes in $K_\bullet^K(C_0(TY))$ (cf. [5, Section 5]). Then, for a class of G-invariant pseudo-differential operators, which we shall denote by $[D_X] \in K_\bullet^G(X)$, with symbol class $[\sigma(D_X)] \in KK_\bullet^G(C_0(X), C_0(TX))$ (cf. [41]), the map

$$[D_X] \mapsto [p] \otimes_{C_0(X) \rtimes G} j^G[\sigma(D_X)]$$

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realises the isomorphism (3.11). Note that using Theorem 5 and the commutative diagram

$$\begin{array}{ccc} K_0^G(M) & \longrightarrow & K_0^G(C_0(T^*M)) \\ f_* \downarrow & & f_* \downarrow \\ K_0^G(X) & \longrightarrow & K_0^G(C_0(T^*X)), \end{array}$$

we see that every element of $K_\bullet^G(X)$ is represented by a G-invariant Spin^c-Dirac operator.

Remark 29. When X is a proper and cocompact G-manifold, not necessarily Spin^c, the second isomorphism (3.6) maps a G-equivariant vector bundle $[E] \in K_G^0(X)$ to

$$[d_{X,E}] = [d_X] \cap [E] \in K_G^0(C_\tau(X)),$$

where $[d_X] \in K_G^0(C_\tau(X))$ is the Dirac element defined using the de Rham operator on X in [42]. This can be easily verified using induction:

$$\begin{array}{ccc} K_0^K(Y) & \longrightarrow & K_K^0(C_\tau(Y)) \\ \downarrow & & \downarrow \\ K_0^G(X) & \longrightarrow & K_G^0(C_\tau(X)), \end{array}$$

given by

$$\begin{array}{ccc} [E|_Y] & \longrightarrow & [d_Y] \cap [E|_Y] \\ \downarrow & & \downarrow \\ [E] & \longrightarrow & [d_X] \cap [E], \end{array}$$

with the help of the fact that $[d_X]$ is mapped to $[d_Y]$ under K-homology induction adapted to Clifford algebras.

Remark 30. From [59, Theorem 4.1], for any proper G-space X (not necessarily cocompact), there is an isomorphism

$$K_G^\bullet(X) \simeq K_K^\bullet(X \times \mathfrak{g}/\mathfrak{k}) \simeq K_K^{\bullet+d}(X).$$

This is the same map as (3.4). Replacing X by TX and using KK-equivalence of $C_0(TX)$ and $C_\tau(X)$ and applying Theorem 4.11 of [42] and Theorem 26, we obtain the dual version of Phillips' isomorphism for a proper *cocompact* G-space X:

$$K_\bullet^G(X) \simeq RK_{\bullet+d}^K(X) \simeq RK_\bullet^K(X \times \mathfrak{g}/\mathfrak{k}).$$

For compact manifolds $\bar{Y}$ with boundary $\partial\bar{Y} \neq \emptyset$, Poincaré duality is proved in [42]:

Theorem 31 (Poincaré duality for K-compact manifolds with boundary). *Assume that $\bar{Y}$ is a smooth compact manifold with boundary $\partial\bar{Y}$ and that a compact group K acts on $\bar{Y}$ smoothly. Set $Y = \bar{Y} \setminus \partial\bar{Y}$. Then there are isomorphisms*

$$\mathcal{PD}_{Y*} : K_\bullet^K(C_\tau(Y)) \simeq K_\bullet^K(\bar{Y});$$
$$\mathcal{PD}_Y^* : K_K^\bullet(C_\tau(Y)) \simeq K_K^\bullet(\bar{Y}).$$

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The proof of the following theorem is similar to that of Theorem 26 but using instead Theorem 31 and will be omitted. Observe that by Abels' global slice theorem, $\bar{X} = G \times_K \bar{Y}$, $\partial \bar{X} = G \times_K \partial \bar{Y}$, and $X = G \times_K Y$.

Theorem 32 (Poincaré duality for G-cocompact manifolds with boundary)**.** *Assume that $\bar{X}$ is a smooth G-cocompact manifold with boundary $\partial \bar{X}$ and that an almost-connected Lie group G acts on $\bar{X}$ smoothly. Set $X = \bar{X} \setminus \partial \bar{X}$. Then there are isomorphisms*

$$\mathcal{PD}_{X*} : K_\bullet^G(C_\tau(X)) \simeq K_\bullet^G(\bar{X});$$

$$\mathcal{PD}_X^* : K_G^\bullet(C_\tau(X)) \simeq K_G^\bullet(\bar{X}).$$

We now generalise Poincaré duality to the case when X is not necessarily G-cocompact.

Recall that the representable equivariant K-theory of a G-space X, as defined by Fredholm complexes in [62], denoted by $RK_G^\bullet(X)$, is equal to $K_\bullet^G(C_0(X))$ when X is G-cocompact, and is defined as the direct limit

$$RK_G^\bullet(X) := \lim_{Z \subset X} K_\bullet^G(C_0(Z))$$

over the inductive system of all cocompact G-subsets $Z \subset X$.

Theorem 32 allows us to prove Poincaré duality for non-cocompact manifolds, the main result of this section:

Theorem 33 (Poincaré duality for non-cocompact manifolds)**.** *Assume that X is a complete Riemannian manifold on which an almost-connected Lie group G acts isometrically. Then one has isomorphisms*

(3.12) $$\mathcal{PD}_{X*} : K_\bullet^G(C_\tau(X)) \simeq RK_\bullet^G(X);$$

(3.13) $$\mathcal{PD}_X^* : K_G^\bullet(C_\tau(X)) \simeq RK_G^\bullet(X),$$

where the right-hand side denotes the representable versions of equivariant K-homology and K-theory.

Proof. We sketch the proof here. Consider an exhaustive increasing sequence of cocompact G-manifolds with boundary $\bar{X}_j$ such that $X = \bigcup \bar{X}_j$. Theorem 32 gives us a coherent system of isomorphisms:

$$\mathcal{PD}_{j*} : K_\bullet^G(C_\tau(X_j)) \simeq K_\bullet^G(\bar{X}_j);$$

$$\mathcal{PD}_j^* : K_G^\bullet(C_\tau(X_j)) \simeq K_G^\bullet(\bar{X}_j).$$

The isomorphism $\mathcal{PD}_{X*}$ is obtained as the direct limit isomorphism of the first coherent system of isomorphisms above, and $\mathcal{PD}_X^*$ is Milnor's $\lim^1$ inverse limit of the second coherent system of isomorphisms above. $\qquad\square$

Our result is related to the Poincaré duality in [42, Section 4] as follows.

Remark 34. Theorem 33 is a generalisation of Corollary 4.11 in [42]. Kasparov's first Poincaré duality ([42, Theorem 4.9]) states that for any locally compact group G,

$$RKK^G(Y \times X; A, B) \simeq RKK^G(Y; A \hat{\otimes} C_\tau(X), B),$$

where X is a complete Riemmanian manifold with an isometric G-action, Y is a σ-compact G-space and A, B are separable G-C^*-algebras. Here,

$$RKK(X; A, B) := \mathcal{R}KK(X; A(X), B(X))$$

(cf. [42, Section 2.19]). When $A = B = \mathbb{C}$ and Y is a point, this isomorphism reduces to the second isomorphism of Theorem 33.

We end this section by generalising the Atiyah-Segal completion theorem [4] using induction on K-homology and Poincaré duality.

Remark 35. Let K be a compact Lie group and A, B be G-C^*-algebras. $KK^K(A, B)$ is an $R(K)$-module. Denote by $KK^K(A, B)^\wedge$ the $I(K)$-adic completion of $KK^K(A, B)$ in the sense of [4]. In [2, Theorem 3.19], a generalisation of the Atiyah-Segal completion theorem is proved:

$$KK^K(A, B)^\wedge \simeq RKK^K(EK; A, B),$$

where $KK^K(A, B)$ is a finite $R(K)$-module. Note that choosing finite $R(K)$-modules $A = \mathbb{C}$ and $B = C(Y)$ for a compact K-manifold Y, we obtain

$$RKK^K(EK; \mathbb{C}, C(Y)) \simeq RK^0(EK \times_K Y),$$

and the Atiyah-Segal completion theorem [4] is recovered.

Now assume A, B to be G-algebras. Note that $EG = G \times_K EK$ and that there is the following induction isomorphism, from Section 3.6 of [42]:

$$RKK^K(EK; A, B) \simeq RKK^G(G \times_K EK; A, B).$$

(1) Letting $A = \mathbb{C}$ and $B = C(X)$, we have

$$KK^K_\bullet(\mathbb{C}, C_0(X)) \simeq K^\bullet_G(X)$$

as $R(K)$-modules. So its $I(K)$-adic completion is isomorphic to $RKK^G(EG; \mathbb{C}, C(X))$, and

$$K^\bullet_G(X)^\wedge \simeq RKK^G_\bullet(EG; \mathbb{C}, C_0(X)) \simeq RK^\bullet(EG \times_G X),$$

whence we recover Phillips' result [59, Theorem 5.3].

(2) Letting $A = C(X)$ and $B = \mathbb{C}$, we have from induction on K-homology that

$$KK^K_\bullet(C_0(X), \mathbb{C}) \simeq K^G_\bullet(X)$$

as $R(K)$-modules. So its $I(K)$-adic completion is isomorphic to $RKK^G(EG; C(X), \mathbb{C})$, and

$$K^G_\bullet(X)^\wedge \simeq RKK^G_\bullet(EG; C_0(X), \mathbb{C}) \simeq KK_\bullet(C_0(EG \times_G X), C_0(BG)),$$

whence we obtain a new Atiyah-Segal completion result for G-equivariant K-homology.

Note that if X is Spin^c, (1) and (2) are related by Poincaré duality (cf. [42, Theorem 4.10]).

3.2. **Discrete Groups.** Let G be a locally compact group and X a proper G-manifold where X/G is compact. As mentioned previously, the K-theory $K_G^*(X)$ fails to be a generalised cohomology in general. In [60], Phillips constructed a cohomology theory using G-vector bundles with Hilbert space fibres and shows in an example that finite-dimensional vector bundles are not enough. The example is a semidirect product of $\mathbb{Z}^2$ by the 4-torus, which is neither discrete nor linear. However, it has been verified that finite-dimensionality is enough if G is either discrete (Theorem 3.2 in [53]) or linear ([59]), and moreover that $K_G^*(X)$ is a generalised cohomology theory in these cases.

Assume G to be a discrete or linear group and X a proper G-manifold with compact quotient. Every element of K-theory $K_G^*(X)$ is represented by a compact G-equivariant vector bundle E over X [53, 59] for $* = 0$. (For $* = 1$ a K-theory element is given by G-vector bundle over the suspension of X.) Let D be the de Rham operator on X representing the Dirac element $[D]$ in the equivariant K-homology $K_G^*(C_\tau(X))$ in the sense of [42] Definition-lemma 4.2. Twisting E with the de Rham operator D gives rise to a twisted Dirac element and hence the homomorphism below:

$$(3.14) \qquad \mathcal{PD} : K_G^*(X) \to K_G^*(C_\tau(X)) \qquad [E] \mapsto [D_E].$$

Theorem 36. *Let G be a discrete group acting properly on a G manifold X with compact quotient. The Poincaré homomorphism (3.14) is an isomorphism, defining the Poincaré duality $\mathcal{PD} : K_G^\bullet(X) \simeq K_G^\bullet(C_\tau(X))$.*

Every proper compact G-manifold X is covered by G-slices of the form $G \times_H U$ where H is a compact subgroup of G and U is an H-space:

$$X = \cup_{i=1}^N G \times_{H_i} U_i.$$

We may assume that all U_is and their nonempty "intersections" are homeomorphic to $\mathbb{R}^m$ for some m. Here, U_{ij} is an nonempty intersection of U_i and U_j means that

$$[G \times_{H_i} U_i] \cap [G \times_{H_j} U_j] = G \times_{H_{ij}} U_{ij}$$

for some compact subgroup H_{ij} of G.

A cover $\{G \times_{H_i} U_i\}_{i=1}^N$ satisfying the above conditions is called a *good cover* of X.

As one would expect, we will apply Mayer-Vietoris sequence and the Five Lemma to prove Poincaré duality. This means that we need to extend the Poincaré duality map to non-G-cocompact manifolds. See [15] for the proof Poincaré duality for ordinary (co)-homology and [32] 11.8.11 for the K-theory analogue for compact manifolds in the nonequivariant case.

For every open proper G-manifold U, denote by $RK_G^\bullet(C_\tau(U))$ the equivariant K-homology with compact support defined by the direct limit over all G-compact submanifolds L in U with respect to the restriction map $C_\tau(U) \to C_\tau(L)$:

$$RK_G^\bullet(C_\tau(U)) = \lim_{L \subset M} K_G^\bullet(C_\tau(L)).$$

The equivariant K-theory of U, by definition, is represented by a G-vector bundle E that is trivial outside a G-compact subset $C \subset U$. Thus, the Poincaré duality map (3.14) is

extended to a non G-compact manifold $U \subset M$ by

(3.15) $\mathcal{PD} : K_G^\bullet(U) \to RK_G^\bullet(C_\tau(U))$ $[E] \mapsto \{[D_E|_{C \cup L}]\}_{L \subset U, L/G \text{ compact}},$

where the inductive limit is taken over G-compact submanifolds $C \cup L \subset U$.

Proof of Theorem 36. Because G is discrete, it follows from [53] that the equivariant K-theory $K_G^*(X)$ is a generalised cohomology theory. The (analytic) equivariant homology $RK_G^\bullet(C_\tau(X))$ is known to be a generalised homology theory from the work of Kasparov. Therefore for two open G-submanifolds U and V of X, we have the Mayer-Vietoris sequences for both K-theory and K-homology, with the corresponding terms related (vertically in the following diagram) by Poincaré duality maps (3.15):

$$\begin{array}{ccccc}
\to K_G^j(U \cap V) & \longrightarrow & K_G^j(U) \oplus K_G^j(V) & \longrightarrow & K_G^j(U \cup V) \to \\
\downarrow{\scriptstyle\mathcal{PD}} & & \downarrow{\scriptstyle\mathcal{PD}} & & \downarrow{\scriptstyle\mathcal{PD}} \\
\to RK_G^j(C_\tau(U \cap V)) & \longrightarrow & RK_G^j(C_\tau(U)) \oplus RK_G^j(C_\tau(V)) & \longrightarrow & RK_G^j(C_\tau(U \cup V)) \to
\end{array}$$

where $j = 0, 1$. This diagram commutes.

Choose a good open cover $\{G \times_{H_i} U_i\}$ for X. Using the Five Lemma and induction, it suffices to show that (3.15) is an isomorphism when U is homeomorphic to $G \times_H \mathbb{R}^m$ for some m and a finite subgroup H of G acting on $\mathbb{R}^m$: that is, show that

(3.16) $\mathcal{PD} : K_G^\bullet(G \times_H \mathbb{R}^m) \to RK_G^\bullet(C_\tau(G \times_H \mathbb{R}^m))$

is an isomorphism. Recall the following induction isomorphisms for discrete groups:

(3.17) $K_H^\bullet(\mathbb{R}^m) \simeq K_G^\bullet(G \times_H \mathbb{R}^m);$

(3.18) $RK_\bullet^H(\mathbb{R}^{2m}) \simeq RK_\bullet^G(G \times_H \mathbb{R}^{2m}).$

For first isomorphism see [53] and for the second see [11]. The latter isomorphism implies

$$RK_H^\bullet(C_\tau(\mathbb{R}^m)) \simeq RK_\bullet^H(T\mathbb{R}^m) \simeq RK_\bullet^G(G \times_H T\mathbb{R}^m) \simeq$$
$$RK_\bullet^G(T(G \times_H \mathbb{R}^m)) \simeq RK_G^\bullet(C_\tau(G \times_H \mathbb{R}^m)),$$

where the third isomorphism follows from G being discrete. Thus to prove (3.16) it only remains to show that

$$K_H^\bullet(\mathbb{R}^m) \simeq RK_H^\bullet(C_\tau(\mathbb{R}^m)).$$

But because $\mathbb{R}^m$ admits an H-Spin^c structure, we have

$$RK_\bullet^H(C_\tau(\mathbb{R}^m)) \simeq RK_H^{\bullet+m}(\mathbb{R}^m), \qquad K_{\bullet+m}^H(\mathbb{R}^m) \simeq K_H^\bullet(C_\tau(\mathbb{R}^m))$$

and so we need only to show $K_H^\bullet(C_\tau(\mathbb{R}^m)) \simeq RK_H^\bullet(\mathbb{R}^m)$. This follows from Corollary 4.11 in [42] or (4.14), which completes the proof. $\qquad\square$

Remark 37. Our Poincaré duality between equivariant K-theory and K-homology for proper actions of an almost-connected group or a discrete group is optimal in the following sense. There are two reasons preventing us from generalising the Poincaré duality to a setting beyond discrete groups and almost connected groups. One is that equivariant K-theory given by equivariant vector bundles may not be a generalised cohomology theory and

so the Mayer-Vietoris sequence cannot be applied. The other is that the induction homo-morphism (3.18) on K-homology from an arbitrary compact subgroup H to G is not an isomorphism in general.

4. Index Theory of K-homology Classes

Let G be an almost-connected Lie group acting properly and cocompactly on a manifold X. Recall that elements of $K_0^G(X)$ are represented by abstract elliptic operators. In this section, we study index theory associated to each element in $K_0^G(X)$ and its relation to induction from a maximal compact subgroup.

4.1. **Higher Index.** Let G and X be as above and K a maximal compact subgroup of G. Let Y be a global K-slice of X. For a proper cocompact action there exists a *cut-off* function, a nonnegative function $c \in C_c^\infty(X)$ whose integral over every orbit is 1:

$$\int_G c(g^{-1}x)dg = 1, \qquad g \in G, \, x \in X.$$

The function c gives rise to an idempotent p in $C_0(X) \rtimes G$, satisfying

(4.1) $$(p(g))(x) := \sqrt{\mu(g^{-1})c(g^{-1}x)c(x)}, \qquad g \in G, \, x \in X.$$

Here μ is the modular function on G; that is, if dg is the left Haar measure on G and $s \in G$, then μ satisfies $d(gs) = \mu(s)dg$. Note that the K-theory class $[p]$ of p in $K_0(C_0(X) \rtimes G)$ is independent of the choice of c.

Definition 38 ([44]). The higher index map $\mathrm{index}_G : K_\bullet^G(X) \to K_\bullet(C_r^*(G))$ is given by

(4.2) $$\mathrm{index}_G(x) = [p] \otimes_{C_0(X) \rtimes G} j_r^G(x),$$

where $[p] \in K_0(C_0(X) \rtimes G)$ and j_r^G is the descent homomorphism

(4.3) $$j_r^G : KK_\bullet^G(A, B) \to KK_\bullet(A \rtimes_r G, B \rtimes_r G)$$

for $A = C_0(X)$ and $B = \mathbb{C}$.

Remark 39. The reduced group C^*-algebra $C_r^*(G)$ is isomorphic to the reduced crossed product $\mathbb{C} \rtimes_r G$. The higher index map (4.2) is defined using the reduced, rather than the full, group C^*-algebra, with the idempotent $[p] \in K_0(C_0(X) \rtimes_r G)$. However, we have

$$C_0(X) \rtimes G \simeq C_0(X) \rtimes_r G$$

whenever G acts properly (cf. [42, Theorem 3.13]). Thus we shall write $C_0(X) \rtimes G$ in place of $C_0(X) \rtimes_r G$. Note also that since a compact group K acts properly on a point, we have that $C_r^*(K) \simeq C^*(K)$, whereas this does not hold for a general non-compact group G. We say that G is *amenable* when $C_r^*(G) \simeq C^*(G)$.

Remark 40. If X is a classifying space for proper actions, index_G can be used to define the *analytic assembly map* in the Baum–Connes [8, 9] and Novikov [42] conjectures. For a compact group K, $K_0(C_r^*(K))$ can be identified with the representation ring $R(K)$, while $K_1(C_r^*(K)) = 0$, and index_K is the usual equivariant index.

Remark 41. When G is an almost-connected Lie group, a classifying space for proper G-actions is G/K. Assume that G/K is Spin. For every proper cocompact G-space X, there is a continuous G-equivariant proper map $p\colon X \to G/K$ (see Theorem 6). The map p_* induced on K-homology relates the equivariant indices on X and G/K via the diagram

(4.4)
$$K_\bullet^G(X) \xrightarrow{\ \mathrm{index}_G\ } K_\bullet(C_r^*(G)).$$
$$p_* \downarrow \quad \overset{\cong}{\underset{\mathrm{index}_G}{\nearrow}}$$
$$K_\bullet^G(G/K)$$

Since the Baum–Connes conjecture is true for almost-connected groups by Theorem 1.1 in [19], the equivariant index on G/K defines an isomorphism

(4.5)
$$\mathrm{index}_G : K_\bullet^G(G/K) \cong K_\bullet(C_r^*(G)).$$

4.2. **Dirac Induction.** We will assume throughout this subsection and the next that G/K is Spin. We first recall the definition of the *Dirac induction* map, which is a special case of a G-index map when $X = G/K$. First, note that we have an isomorphism

$$R(K) \simeq K_d^G(G/K), \qquad d = \dim G/K.$$

If (ρ, V) is an irreducible representation of K, its image under this isomorphism is represented by the K-homology cycle of the Spin^c-Dirac operator $D_{G/K}$ on G/K coupled with V. Taking the index of this operator then defines the *Dirac induction* of the class of (ρ, V):

(4.6) $\text{D-Ind}_K^G : R(K) \to K_d(C_r^*(G)), \qquad \text{D-Ind}_K^G([V]) := [p] \otimes_{C_0(G/K) \rtimes G} j_r^G([D_{G/K}^V]).$

This map is an isomorphism of abelian groups by the Connes–Kasparov conjecture [20, 43, 19], proved for almost-connected groups in [19] based on important earlier results in [65, 49].

Equivalently, D-Ind_K^G can be constructed as follows. Let

$$[\partial_{G/K}] \in KK_d^G(C_0(G/K), \mathbb{C})$$

denote the Dirac element on G/K. Then, as we show below, Dirac induction $R(K) \to K_d(C_r^*(G))$ can also be defined by

(4.7) $\text{D-Ind}_K^G([\rho]) = [\rho] \otimes_{C_0(G/K) \rtimes G} j_r^G([\partial_{G/K}]), \qquad [\rho] \in R(K) \simeq K_0(C_0(G/K) \rtimes G).$

The last isomorphism follows from the fact that $C_r^*(K)$ and $C_0(G/K) \rtimes G$ are Morita-equivalent. To see that definitions (4.6) and (4.7) coincide, the following lemma relating $C_r^*(K)$ and $C_0(G/K) \rtimes G$ on the level of K-theory is needed.

Lemma 42. *Let V be a finitely generated projective module over $C_r^*(K)$. Then V corresponds to the projective $C_0(G/K) \rtimes G$-module $p \cdot (\Gamma(G \times_K V) \rtimes G)$ under the Morita equivalence of $C_r^*(K)$ and $C_0(G/K) \rtimes G$. That is,*

$$[V] \mapsto [p \cdot (\Gamma(G \times_K V) \rtimes G)]$$

in the isomorphism

(4.8)
$$K_0(C_r^*(K)) \simeq K_0(C_0(G/K) \rtimes G).$$

Proof. From [24], we know that the $C_0(G/K) \rtimes G$-module $p \cdot (C_0(G/K) \rtimes G)$ and the $C_r^*(K)$-module $\mathbb{C}$ correspond under Morita equivalence. In fact, $C_c(G)$ can be regarded as a right $C_c(K)$-module and a left $C_c(G, C_c(G/K))$-module. A cut-off function c on G/K with respect to the G-action can be lifted to an element in $C_c(G)$ satisfying

$$\langle c, c \rangle_{C_c(G,C_c(G/K))} = p, \qquad \langle c, c \rangle_{C_c(K)} = 1.$$

So the projection 1 in $C_r^*(K)$ and the projection p in $C_0(G/K) \rtimes G$ correspond under Morita equivalence. In fact, this follows from:

$$\mathcal{K}(p \cdot (C_0(G/K) \rtimes G)) \simeq p \cdot (C_0(G/K) \rtimes G) \cdot p \simeq \mathbb{C},$$

where $\mathcal{K}$ is the space of compact operators on the Hilbert $C_0(G/K) \rtimes G$-module. See details in [24, Example 5.2]. Hence, the module $1 \cdot C_r^*(K)$ corresponds to $p \cdot (C_0(G/K) \rtimes G)$ under Morita equivalence:

$$1 \cdot C_r^*(K) \sim_{M.E.} p \cdot (C_0(G/K) \rtimes G).$$

Here $\cdot$ means module multiplication given by convolution with respect to K or G. One directly calculates that $1 \cdot C_r^*(K) \simeq \mathbb{C}$, so that we have

(4.9) $$\mathbb{C} \sim_{M.E.} p \cdot (C_0(G/K) \rtimes G).$$

In particular, $a \in \mathbb{C}$ is identified as an element $\tilde{a} := p \cdot f_a$ in $p \cdot (C_0(G/K) \rtimes G)$ where $(f_a(g))(x) = a$ for $g \in G$ and $x \in G/K$. This means that the Lemma is true when $V = \mathbb{C}$ is the trivial representation. The proof for general V then follows by observing that (4.8) is an isomorphism of $R(K)$-modules. In fact, (4.9) implies the following module isomorphisms:

(4.10) $$p \cdot (\Gamma(G \times_K \mathbb{C}) \rtimes G) \otimes_{C_c(G,C_0(G/K))} C_c(G) \simeq \mathbb{C};$$

(4.11) $$C_c(G) \otimes_{C_c(K)} \mathbb{C} \simeq p \cdot (\Gamma(G \times_K \mathbb{C}) \rtimes G).$$

Applying the $C_r^*(K)$-module V on the left in (4.10) or on the right in (4.11), we obtain versions of the isomorphisms (4.10)-(4.11) with $\mathbb{C}$ replaced by V. Equivalently, we have:

$$V \sim_{M.E.} p \cdot (\Gamma(G \times_K V) \rtimes G).$$

Here, every $v \in V$ is identified as an element $\tilde{v} := p \cdot f_v$ in $p \cdot (\Gamma(G \times_K V) \rtimes G)$, where

$$(f_v(g))(hK) = h[(1, v)], \qquad g, h \in G, \ hK \in G/K, \ [(1, v)] \in G \times_K V.$$

This completes the proof. $\qquad \square$

Remark 43. Under the Morita equivalence $C_r^*(K) \simeq C_0(G/K) \rtimes G$, the projection p in $C_0(G/K) \rtimes G$ corresponds the constant function 1 on K. If ρ_0 is the trivial representation of K, then under the isomorphism

$$R(K) \simeq K_0(C_r^*(K)) \simeq K_0(C_0(G/K) \rtimes G),$$

we have

$$R(K) \ni [\rho_0] \leftrightarrow [p] \in K_0(C_0(G/K) \rtimes G).$$

Remark 44. The isomorphism $K_0(C_r^*(K)) \to K_0(C_0(G/K) \rtimes G)$ given by $[V] \mapsto [p \cdot (\Gamma(G \times_K V) \rtimes G)]$ is the composition of the following maps:

$$j : R(K) \to KK^G(C_0(G/K), C_0(G/K))$$
$$[V] \mapsto [(\Gamma(G \times_K V), 0)];$$
$$j^G : KK^G(C_0(G/K), C_0(G/K)) \to KK(C_0(G/K) \rtimes G, C_0(G/K) \rtimes G)$$
$$[(\Gamma(G \times_K V), 0)] \mapsto [(\Gamma(G \times_K V) \rtimes G, 0)];$$
$$[p] \otimes_{C_0(G/K) \rtimes G} : KK(C_0(G/K) \rtimes G, C_0(G/K) \rtimes G) \to K_0(C_0(G/K) \rtimes G)$$
$$[(\Gamma(G \times_K V) \rtimes G, 0)] \mapsto [p \cdot (\Gamma(G \times_K V) \rtimes G)].$$

This description helps us prove the equivalence of definitions (4.6) and (4.7).

Lemma 45. *Let* $[V] \in R(K)$*. Then definitions (4.6) and (4.7) coincide:*

$$\mathrm{index}_G([D_{G/K}^V]) = [p] \otimes_{C_0(G/K) \rtimes G} j_r^G([D_{G/K}^V]) = [V] \otimes_{C_0(G/K) \rtimes G} j_r^G([\partial_{G/K}]).$$

Proof. Observe by comparing KK-cycles that we have

$$j_r^G([D_{G/K}^V]) = j^G \circ j([V]) \otimes_{C_0(G/K) \rtimes G} j_r^G([\partial_{G/K}]).$$

Thus,

$$[p] \otimes_{C_0(G/K) \rtimes G} j_r^G([D_{G/K}^V]) = [p] \otimes_{C_0(G/K)} j^G \circ j([V]) \otimes_{C_0(G/K) \rtimes G} j_r^G([\partial_{G/K}]).$$

The Lemma follows from Remark 44. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.3. **Higher Index Commutes with Induction.** The main theorem of this section is the following:

Theorem 46 (Higher index commutes with induction)**.** *The following diagram commutes:*

(4.12)
$$\begin{array}{ccc} K_\bullet^K(Y) & \xrightarrow{\mathrm{index}_K} & K_\bullet(C_r^*(K)) \\ \text{K-Ind}_K^G \downarrow & & \text{D-Ind}_K^G \downarrow \\ K_{\bullet+d}^G(X) & \xrightarrow{\mathrm{index}_G} & K_{\bullet+d}(C_r^*(G)). \end{array}$$

Proof. We only need to give the proof for $\bullet = 0$. Consider the G-equivariant maps $\lambda : Y \to pt$, $\tilde\lambda : X = G \times_K Y \to G/K$ and the contravariant maps on algebras

$$\lambda' : \mathbb{C} \to C_0(Y), \qquad \tilde\lambda' : C_0(G/K) \to C_0(X).$$

Let $(\mathcal{H}, f, F)$ be a Kasparov cycle for $K_0^K(Y)$. Then the commutativity of

$$\begin{array}{ccc} f : C_0(Y) \to \mathcal{L}(\mathcal{H}) & \longrightarrow & f \circ \lambda' \\ \downarrow & & \downarrow \\ \tilde f : C_0(X) \to \mathcal{L}(G \times_K \mathcal{H}) & \longrightarrow & \tilde f \circ \tilde\lambda' = \widetilde{f \circ \lambda'} \end{array}$$

implies the commutativity of

$$(4.13) \qquad \begin{array}{ccc} K_0^K(Y) & \xrightarrow{\ \lambda_*\ } & K_0^K(pt) \\ \simeq \downarrow & & \simeq \downarrow \\ KK^G(C_0(X), C_0(G/K)) & \xrightarrow{\ \tilde{\lambda}_*\ } & KK^G(C_0(G/K), C_0(G/K)). \end{array}$$

Every element in $K_0^K(pt) \simeq KK^K(\mathbb{C}, \mathbb{C})$ can be represented by a K-vector space $[V]$, or a finite-dimensional representation of K. Regarding V as a $C_r^*(K)$-module, the higher index is the identity map

$$K_0^K(pt) \to K_0(C_r^*(K)), \qquad \text{index}_K([V]) = [V].$$

So in the diagram

$$(4.14) \qquad \begin{array}{ccc} KK^K(\mathbb{C}, \mathbb{C}) & \xrightarrow{\ \text{index}_K\ } & K_0(C_r^*(K)) \\ \simeq \downarrow & & \simeq \downarrow \\ KK^G(C_0(G/K), C_0(G/K)) & \xrightarrow{\ \text{index}_G\ } & K_0(C_0(G/K) \rtimes G), \end{array}$$

the elements are mapped as

$$\begin{array}{ccc} [(V,0)] & \xrightarrow{\ \text{index}_K\ } & [(V,0)] \\ \downarrow & & \downarrow \\ [(\Gamma(G \times_K V), 0)] & \xrightarrow{\ \text{index}_G\ } & [(p \cdot (\Gamma(G \times_K V) \rtimes G), 0)]. \end{array}$$

Thus, diagram (4.14) commutes if V and $p \cdot (\Gamma(G \times_K V) \rtimes G)$ are Morita-equivalent as modules; but this has been proved in Lemma 42.

Noting that the higher index map factors through the higher index map for classifying spaces (cf. (4.4)), commutativity of (4.13)-(4.14) implies that the following diagram commutes (cf. (2.8) for the left isomorphism):

$$(4.15) \qquad \begin{array}{ccc} K_0^K(Y) & \xrightarrow{\ \text{index}_K\ } & K_0(C_r^*(K)) \\ \simeq \downarrow & & \simeq \downarrow \\ KK^G(C_0(X), C_0(G/K)) & \xrightarrow{\ \text{index}_G\ } & K_0(C_0(G/K) \rtimes G). \end{array}$$

Finally, note that for $x \in KK^G(C_0(X), C_0(G/K))$, we have

$$\begin{aligned} \text{index}_G(x) \otimes_{C_0(G/K) \rtimes G} j_r^G([\partial_{G/K}]) &= \big([p] \otimes_{C_0(X) \rtimes G} j^G(x)\big) \otimes_{C_0(G/K) \rtimes G} j_r^G([\partial_{G/K}]) \\ &= [p] \otimes_{C_0(X) \rtimes G} \big(j^G(x) \otimes_{C_0(G/K) \rtimes G} j_r^G([\partial_{G/K}])\big) \\ &= [p] \otimes_{C_0(X) \rtimes G} j_r^G(x \otimes_{C_0(G/K)} [\partial_{G/K}]) \\ &= \text{index}_G(x \otimes_{C_0(G/K)} [\partial_{G/K}]). \end{aligned}$$

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In other words, the following diagram commutes:

$$(4.16) \quad \begin{array}{ccc} KK^G(C_0(X), C_0(G/K)) & \xrightarrow{\text{index}_G} & K_0(C_0(G/K) \rtimes G)) \\ \downarrow {\scriptstyle \otimes_{C_0(G/K)} [\partial_{G/K}]} & & \downarrow {\scriptstyle \otimes_{C_0(G/K) \rtimes G} j_r^G([\partial_{G/K}])} \\ K_d^G(X) & \xrightarrow{\text{index}_G} & K_d(C_r^*(G)). \end{array}$$

Putting together (4.15) and (4.16) completes the proof. $\qquad \square$

Remark 47. The commutative diagram in Theorem 46 is closely related to the *quantisation commutes with induction* results of [33, 34]. We acknowledge that Theorem 46 was proved in [33] by more involved diagram chasing. Here we have given a more direct proof by making use of the properties of KK-theory.

We end this subsection by stating an implication of Remark 20 and Theorem 46.

Corollary 48. *For every $x \in K_\bullet^G(X)$, there exists a geometric cycle (M, E, f) representing an element in $K_\bullet^{G, geo}(X)$ such that $x = f_*([D_{M,E}])$, and upon choosing a K-slice N of M with a compatible K-equivariant Spin^c-structure, we have*

$$\text{index}_G \, x = \text{index}_G \, f_*([D_{M,E}]) = \text{D-Ind}_K^G(\text{index}_K(f|_N)_*([D_{N,E|_N}]))) \in K_\bullet(C_r^*(G)).$$

5. A Trace Theorem for Equivariant Index

We now focus on the case when G is a connected, semisimple Lie group with finite centre and K a maximal compact subgroup, such that G/K is Spin. We further assume that $d = \dim G/K$ is even and rank $G = $ rank K, in order to allow G to have discrete series representations. Let M be a G-equivariant Spin^c-manifold with Dirac operator D_M and N a K-slice of G with Dirac operator D_N associated with a compatible (see Remark 10) K-equivariant Spin^c-structure on N. We shall use the commutativity result in Section 4 together with known representation-theoretic properties of the group $K_0(C_r^*(G))$ for such G to deduce a formula for the L^2-index of D_M in terms of the L^2-index of D_N. Since twisted Spin^c-Dirac operators exhaust $K_0^G(M)$, this allows us to relate the L^2-index of any operator representing a class in $K_0^G(M)$ to the L^2-index of K-equivariant Dirac operators in $K_0^K(N)$.

Remark 49. For simplicity, we have assumed that M has a G-equivariant Spin^c-structure. Of course, since the commutativity result of Section 4 did not require this, all of the statements in this section apply in an appropriate form to operators on any proper G-cocompact manifold X, not necessarily having a Spin^c-structure, and a compact K-slice Y.

Let $\mathcal{R}_G \subseteq \mathcal{B}(L^2(G))$ be the commutant of the right-regular representation R of G. Then $\mathcal{R}_G$ is a von Neumann algebra with a faithful, normal, semi-finite trace τ determined by

$$\tau(R(f)^* R(f)) = \int_G |f(g)|^2 \, dg,$$

for all $f \in L^2(G)$ such that $R(f)$ is a bounded operator on $L^2(G)$. Let E be a $\mathbb{Z}_2$-graded G-equivariant vector bundle over M, and let $E \cong G \times_K (E|_N)$ be a fixed choice of decomposition. Denote by $\mathcal{R}_G(E)$ the K-invariant elements of

$$\mathcal{R}_G \otimes \mathcal{B}(L^2(N, E|_N)).$$

Then $\mathcal{R}_G(E)$ is equipped with a natural trace, which we will also denote by τ, given by combining τ on $\mathcal{R}_G$ with the trace on bounded operators.

Let $D = D^+ \oplus D^-$ be an odd-graded first-order G-invariant elliptic operator acting on E and defining a class in $K_0^G(M)$. By definition, the L^2-*index* of D is (see Connes-Moscovici [22] or Wang [64])

$$L^2\text{-index}(D) := \tau(\mathrm{pr}_{\ker D^+}) - \tau(\mathrm{pr}_{\ker D^-}),$$

where pr_L denotes the projection of $L^2(M, E)$ onto a G-invariant subspace L. It was shown in [64] that the L^2-index of D can be calculated via the composition of maps

$$K_0^G(M) \xrightarrow{\mathrm{index}_G} K_0(C_r^*(G)) \xrightarrow{\tau_G} \mathbb{R},$$

where τ_G is the von Neumann trace on $C_r^*(G)$, defined first on the dense subspace $C_c(G)$ by evaluation at the identity $e \in G$:

$$\tau_G : C_c(G) \to \mathbb{C}, \qquad f \mapsto f(e),$$

and extended to $C_r^*(G)$. The same paper provides an integral formula for the L^2-index:

(5.1) $$L^2\text{-index}(D) = \int_{TM} (c \circ \pi)(\hat{A}(M))^2 \cup \mathrm{ch}(\sigma_D),$$

where $\pi : TM \to M$ is the natural projection and c is a compactly supported cut-off function on M, with the property that $\int_G c(g^{-1}x)\, dg = 1$ for any $x \in M$. Since we have assumed that the manifold M is Spin^c, this integral is equal to

$$\int_M c(x) e^{\frac{1}{2}c_1(L_M)} \hat{A}(M),$$

where L_M is the line bundle defining the G-equivariant Spin^c-structure on M.

On the other hand, it is known that, for our chosen class of Lie groups G, the discrete series representations of G can be identified with elements of $K_0(C_r^*(G))$ (although not necessarily all of them) in the following way. Let (H, π) be a discrete series representation of G. Let d_H be the *formal degree* of (H, π) - equal to its mass in the Plancherel measure - and c_v be a *matrix coefficient* of the representation, given by

$$c_v : G \to \mathbb{C}, \qquad c_v(g) := \langle v, \pi(g)v \rangle,$$

for some $v \in H, ||v|| = 1$ (it is independent of the choice of v). It can be verified that $d_H c_v$ is a projection in the C^*-algebra $C_r^*(G)$ and hence defines an element in its K-theory. It is also known that the von Neumann trace τ_G (defined above) of the class $[d_H c_v] \in K_0(C_r^*(G))$ is equal to $(-1)^{d/2}$ times the formal degree (see [48]).

For a more extensive discussion we refer to [48]. It is shown there that the map taking (H, π) to $[d_H c_v] \in K_0(C_r^*(G))$ is an injection from the set of discrete series representations to a set of generating elements for $K_0(C_r^*(G))$ [48].

It is interesting to note that for connected semisimple Lie groups having discrete series, including $G = SL(2, \mathbb{R})$, the elements of $K_0(C_r^*(G))$ can be given *entirely*, in the aforesaid way, as representations from the discrete series and limits of discrete series of G. See [65]. This motivates us to understand $K_0(C_r^*(G))$, as well as G-equivariant index theory, from the point of view of representation theory of G. By comparison, the theory of representations of compact groups already plays an important role in index theory. The goal of this section is to observe an explicit representation-theoretic relationship between a G-equivariant index and its corresponding K-equivariant index, in the sense of the previous section.

The elements of $K_0(C_r^*(G))$, considered as representations of G, are in bijection with the irreducible representations of K via the Dirac induction map D-Ind (see [9]). On the other hand, there exists a formula for the formal degree of a discrete series representation of G in terms of the highest weight of a corresponding K-representation together with information about the root systems of G and K (see [7] or [49]).
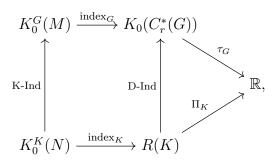
To state this formula precisely, let us first fix the following notation. Let T be a common maximal torus of K and G, and choose a Weyl chamber C for the root sytem Φ of $\mathfrak{g}$, determining a set Φ^+ of positive roots. Pick the Weyl chamber for $\mathfrak{k}$ that contains C, and let Φ_c^+ be the system of compact roots. Let $\Phi_n^+ := \Phi^+ \backslash \Phi_c^+$, ρ_c and ρ_n be the half-sums of the positive-compact and non-compact roots respectively, and let $\rho := \rho_c + \rho_n$.

We can relate the formula for the formal degree proved in [48] to the commutativity result Section 4, as follows. This gives an interpretation of a result of Atiyah and Schmid (p. 25 of [7]) in terms of equivariant index theory.

Proposition 50. *Let G and K be as above, with G/K Spin. Let μ be the highest weight of an irreducible representation V_μ of K. Suppose $\mu + \rho_c$ is the Harish-Chandra parameter of a discrete series representation of (H, π) of G. Then the formal degree of (H, π) is given by*

$$d_H = (-1)^{d/2} \tau_G([\text{D-Ind}([V_\mu])]) = \prod_{\alpha \in \Phi^+} \frac{(\mu + \rho_c, \alpha)}{(\rho, \alpha)},$$

with both sides vanishing when the class $\text{D-Ind}[V_\mu] \in K_0(C_r^*(G))$ *is not given by a discrete series representation. We have a well-defined commutative diagram:*

$$
\begin{array}{ccc}
K_0^G(M) & \xrightarrow{\text{index}_G} & K_0(C_r^*(G)) \\
\Big\uparrow {\scriptstyle \text{K-Ind}} & & \Big\uparrow {\scriptstyle \text{D-Ind}} \quad \searrow^{\tau_G} \\
& & \qquad\qquad \mathbb{R}, \\
& & \qquad \nearrow_{\Pi_K} \\
K_0^K(N) & \xrightarrow{\text{index}_K} & R(K)
\end{array}
$$

where $\Pi_K([V_\mu]) := (-1)^{d/2} \prod_{\alpha \in \Phi^+} \frac{(\mu + \rho_c, \alpha)}{(\rho, \alpha)}$. *Here the Haar measure on G is normalised by*

$$\text{vol } K = \text{vol } M_1/K_1 = 1,$$

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where M_1 is a maximal compact subgroup of the universal complexification $G^{\mathbb{C}}$ of G and $K_1 < G_1$ a maximal compact subgroup of a real form G_1 of $G^{\mathbb{C}}$ (see [7] for more details).

Remark 51. We have $\tau_G(\text{D-Ind}[V_\mu]) = 0$ when $\text{D-Ind}[V_\mu] \in K_0(C_r^*(G))$ is not given by a discrete series representation of G, since the Harish-Chandra parameter is then singular, hence the right-hand side of the formula above vanishes.

We can state this result in a more suggestive way that makes clear the relationship between the von Neumann traces τ_G on $K_0(C_r^*(G))$ and τ_K on $R(K) \cong K_0(C_r^*(K))$.

Proposition 52. *With the notation above, we have:*

$$\tau_G([\text{D-Ind}([V_\mu])]) = (-1)^{d/2} \left(\frac{\prod_{\alpha \in \Phi_n^+} (\mu + \rho_c, \alpha) \prod_{\alpha \in \Phi_c^+} (\rho_c, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha)} \right) \tau_K([V_\mu]).$$

Proof. By the Weyl dimension formula for V_μ and the Proposition above,

$$\begin{aligned}
\tau_G([\text{D-Ind}([V_\mu])]) &= (-1)^{d/2} \, d_H \\
&= (-1)^{d/2} \prod_{\alpha \in \Phi^+} \frac{(\mu + \rho_c, \alpha)}{(\rho, \alpha)} \\
&= (-1)^{d/2} \prod_{\alpha \in \Phi_c^+} \frac{(\mu + \rho_c, \alpha)}{(\rho_c, \alpha)} \left(\frac{\prod_{\alpha \in \Phi_n^+} (\mu + \rho_c, \alpha) \prod_{\alpha \in \Phi_c^+} (\rho_c, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha)} \right) \\
&= (-1)^{d/2} \dim V_\mu \left(\frac{\prod_{\alpha \in \Phi_n^+} (\mu + \rho_c, \alpha) \prod_{\alpha \in \Phi_c^+} (\rho_c, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha)} \right).
\end{aligned}$$

It can be verified that the trace τ_K applied to a general element $[V] \in R(K) \cong K_0(C_r^*(K))$ returns the dimension of the representation space V, which concludes the proof. $\quad\square$

Thus τ_G and τ_K are related by a scalar factor that, up to a sign, depends only upon the highest weight μ of the representation $[V_\mu] \in R(K)$ and the chosen root systems of $\mathfrak{g}$ and $\mathfrak{k}$. In particular, as one sees by direct computation in the example $G = SL(2, \mathbb{R})$ and $K = SO(2)$, this scalar factor varies significantly depending on the weight μ (see [9] for more details on the calculation in this case), reflecting the fact that Dirac induction plays a significant role in relating the two L^2-indices for G and K.

Finally, as was mentioned in earlier, the von Neumann traces τ_G and τ_K give rise to L^2-indices when applied to the G and K-equivariant indices of operators on M and N respectively. Using Wang's formula for the L^2-index (5.1), the result above becomes an equality of integrals involving characteristic classes on the non-compact manifold M and the compact manifold N, as follows.

Corollary 53. *Let D_M and D_N be Spin^c-Dirac operators on M and N for compatible equivariant Spin^c-structures, which are defined by line bundles L_M and L_N respectively,*

where $\mathrm{index}_G(D_M) \in K_0(C_r^*(G))$ *corresponds to a discrete series representation with Harish-Chandra parameter* $\mu + \rho_c$. *Then the L^2-indices of D_M and D_N are related by*

$$
\begin{aligned}
L^2\text{-index}(D_M) &= \int_M c(x) e^{\frac{1}{2}c_1(L_M)} \hat{A}(M) \\
&= (-1)^{d/2} \left(\frac{\prod_{\alpha \in \Phi_n^+} (\mu + \rho_c, \alpha) \prod_{\alpha \in \Phi_c^+} (\rho_c, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha)} \right) \int_N e^{\frac{1}{2}c_1(L_N)} \hat{A}(N) \\
&= (-1)^{d/2} \left(\frac{\prod_{\alpha \in \Phi_n^+} (\mu + \rho_c, \alpha) \prod_{\alpha \in \Phi_c^+} (\rho_c, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha)} \right) L^2\text{-index}(D_N).
\end{aligned}
$$

6. Positive Scalar Curvature for Proper Cocompact Actions

We now apply equivariant index theory to study obstructions to the existence of invariant metrics of positive scalar curvature on proper cocompact manifolds, before studying the existence of such metrics in the next subsection.

6.1. **Obstructions.** Suppose that G is a non-compact Lie group acting properly and cocompactly on a smooth even-dimensional, G-equivariantly spin complete Riemannian manifold M. Given a G-invariant Riemannian metric, the usual Lichnerowicz formula [52] for the square of the Dirac operator $\slashed{\partial}$ is

$$
\slashed{\partial}^2 = (\nabla^S)^\dagger \nabla^S + \kappa_M/4
$$

where κ_M denotes the scalar curvature of M and ∇^S denotes the connection on the spinor bundle that is induced by the Levi-Civita connection on M. Suppose that M has positive scalar curvature. Since M/G is compact, we have $\inf(\kappa_M) > 0$ and thus $\slashed{\partial}^2$ is a strictly positive operator from the Sobolev space $L^{2,2}(S)$ to the Hilbert space $L^2(S)$ of L^2 sections, with a bounded inverse

$$
(\slashed{\partial}^2)^{-1} : L^2(S) \to L^{2,2}(S)
$$

(see Theorem 2.11 in [29]). Let $C^*(G)$ be the maximal group C^*-algebra of G. The space $C_c^\infty(S)$ of smooth cocompactly supported sections of S has the structure of a right pre-Hilbert module over $C_c(G)$ prescribed by the formulas (for more details see [41] Section 5):

$$
(s \cdot b)(x) = \int_G g(s)(x) \cdot b(g^{-1}) \, dg \in C_c^\infty(S),
$$

$$
(s_1, s_2)(g) = \int_M (s_1(x), g(s_2)(x)) \, d\mu \in C_c^\infty(G),
$$

for $s, s_1, s_2 \in C_c^\infty(S)$ and $b \in C_c^\infty(G)$. Let us denote by $\mathcal{E}$ the Hilbert $C^*(G)$-module completion $\mathcal{E}$ of $C_c^\infty(S)$. Theorem 5.8 of [41] shows that the pseudodifferential operator $F = \slashed{\partial}(\slashed{\partial}^2 + 1)^{-\frac{1}{2}} : \Gamma(S) \to \Gamma(S)$ defines an element, which we will also call F, in the bounded adjointable operators $\mathcal{L}(\mathcal{E})$. Indeed, the same Theorem shows that F is an invertible element in the Calkin algebra $\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})$ and so defines an element $\mathrm{index}_G \in K_0(C^*(G))$.

One of the main results of this paper is the following theorem on the vanishing of the equivariant index.

Theorem 54 (Vanishing Theorem 1)**.** *Let M, G, S and $\not{\partial}$ be as in the previous paragraph, with M having a Riemannian metric of pointwise-positive scalar curvature. Then*

$$\mathrm{index}_G(\not{\partial}) = 0 \in K_0(C^*(G)).$$

Proof. The operator $\not{\partial} : C_c^\infty(S) \to C_c^\infty(S)$ gives a densely-defined (and a priori unbounded) operator on the Hilbert $C^*(G)$-module $\mathcal{E}$. By G-invariance, one can verify that $\not{\partial}$ is symmetric: that is, we have $C_c^\infty(S) \subseteq \mathrm{dom}(\not{\partial}^*)$, where

$$\mathrm{dom}(\not{\partial}^*) := \{y \in \mathcal{E} : \exists z \in \mathcal{E} \text{ with } \langle \not{\partial}x, y \rangle_\mathcal{E} = \langle x, z \rangle_\mathcal{E} \ \forall x \in C_c^\infty(S)\}.$$

Hence $\not{\partial}$ is a closable operator, and we shall write $\mathrm{dom}(\not{\partial})$ for the domain of its closure, a closed Hilbert $C^*(G)$-submodule of $\mathcal{E}$. In what follows we will be concerned with the closed operator $\not{\partial} : \mathrm{dom}(\not{\partial}) \to \mathcal{E}$. Define the *graph norm* $|| \cdot ||_{\mathrm{dom}(\not{\partial})}$ on $\mathrm{dom}(\not{\partial})$ to be the norm induced by the $C^*(G)$-valued inner product

$$\langle u, v \rangle_{\mathrm{dom}(\not{\partial})} := \langle u, v \rangle_\mathcal{E} + \langle \not{\partial}u, \not{\partial}v \rangle_\mathcal{E}.$$

With this norm, $\not{\partial}$ is a bounded adjointable operator $\mathrm{dom}(\not{\partial}) \to \mathcal{E}$. Since $\not{\partial}$ is a first-order elliptic differential operator on a G-cocompact manifold, Theorem 5.8 of [41] implies that both operators $\not{\partial} \pm i$ have dense range as operators on $\mathcal{E}$. Thus $\not{\partial}$ is a self-adjoint regular operator on the Hilbert $C^*(G)$-module $\mathcal{E}$ [50].

Our main task is to show that $\not{\partial}$ has an inverse $\not{\partial}^{-1} : \mathcal{E} \to \mathrm{dom}(\not{\partial})$ that is bounded and adjointable. First we show that its square $\not{\partial}^2 : \mathrm{dom}(\not{\partial}^2) \to \mathcal{E}$ is invertible in the bounded adjointable sense. Here the domain

$$\mathrm{dom}(\not{\partial}^2) := \{u \in \mathrm{dom}(\not{\partial}) : \not{\partial}u \in \mathrm{dom}(\not{\partial})\}$$

is equipped with the graph norm induced by the inner product

$$\langle u, v \rangle_{\mathrm{dom}(\not{\partial}^2)} := \langle u, v \rangle_\mathcal{E} + \langle \not{\partial}u, \not{\partial}v \rangle_\mathcal{E} + \langle \not{\partial}^2 u, \not{\partial}^2 v \rangle_\mathcal{E}.$$

By Proposition 1.20 of [23], $\not{\partial}^2$ is a densely-defined, closed, self-adjoint regular operator on $\mathcal{E}$. We proceed along the lines of [18] Proposition 4.9. By regularity, $\not{\partial}^2 + \mu^2$ is surjective for every positive number μ^2 (see Chapter 9 of [50]). Further, since $\not{\partial}^2$ is strictly positive, $\not{\partial}^2 + \mu^2$ is injective. By the open mapping theorem, its inverse $(\not{\partial}^2 + \mu^2)^{-1}$ is bounded. It remains to show that $(\not{\partial}^2 + \mu^2)^{-1}$ is adjointable. Write for short $B := (\not{\partial}^2 + \mu^2)^{-1}$. Note that B is self-adjoint as a bounded operator $\mathcal{E} \to \mathcal{E}$ (defined by composing B with the bounded inclusion $\mathrm{dom}(\not{\partial}^2) \hookrightarrow \mathcal{E}$), which follows from Lemma 4.1 in [50] and the estimate

$$\langle u, Bu \rangle_\mathcal{E} = \langle (\not{\partial} + \mu^2)Bu, Bu \rangle_\mathcal{E} \geq \mu^2 \langle Bu, Bu \rangle_\mathcal{E} \geq 0.$$

Next, for any $w \in \mathcal{E}$ and $u \in \text{dom}(\eth^2)$, we have

$$\langle Bu, w \rangle_{\text{dom}(\eth^2)} = \langle \eth^2 Bu, \eth^2 w \rangle_{\mathcal{E}} + \langle \eth Bu, \eth w \rangle_{\mathcal{E}} + \langle Bu, w \rangle_{\mathcal{E}}$$
$$= \langle (\eth^2 + \mu^2) Bu, \eth^2 w \rangle_{\mathcal{E}} + (1 - \mu^2) \langle Bu, \eth^2 w \rangle_{\mathcal{E}} + \langle u, Bw \rangle_{\mathcal{E}}$$
$$= \langle u, \eth^2 w \rangle_{\mathcal{E}} + (1 - \mu^2) \langle u, B\eth^2 w \rangle_{\mathcal{E}} + \langle u, Bw \rangle_{\mathcal{E}}$$
$$= \langle u, (\eth^2 + (1 - \mu^2) B\eth^2 + B) w \rangle_{\mathcal{E}},$$

where we used symmetry of $\eth$ in $\mathcal{E}$ self-adjointness of B as shown above. This shows that $(\eth^2 + \mu^2)^{-1} \in \mathcal{L}(\mathcal{E}, \text{dom}(\eth^2))$.

We claim that $\eth^2$ is invertible. For write it as $(1 - \mu^2(\eth^2 + \mu^2)^{-1})(\eth^2 + \mu^2)$. Since $\eth^2$ is a strictly positive operator, there exists $C > 0$ such that for all $s \in \text{dom}(\eth^2)$, we have $\langle \eth^2 s, s \rangle_{\mathcal{E}} \geq C \langle s, s \rangle_{\mathcal{E}}$. It follows from the Cauchy-Schwarz inequality for Hilbert modules that for any element $t \in \mathcal{E}$,

$$\|\mu^2(\eth^2 + \mu^2)^{-1} t\|_{\mathcal{E}} \leq \frac{\mu^2}{\mu^2 + C} \|(\eth^2 + \mu^2) Bt\|_{\mathcal{E}} = \frac{\mu^2}{\mu^2 + C} \|t\|_{\mathcal{E}}.$$

Hence $(1 - \mu^2(\eth^2 + \mu^2)^{-1})$ has an adjointable inverse given by a Neumann series. It follows that $\eth^2$ has a bounded adjointable inverse

$$(\eth^2)^{-1} = B(1 - \mu^2(\eth + \mu^2)^{-1})^{-1} : \mathcal{E} \to \text{dom}(\eth^2).$$

One verifies that $\eth(\eth^2)^{-1} : \mathcal{E} \to \text{dom}(\eth)$ is a two-sided inverse for $\eth$. Thus $\eth + \lambda$ is invertible for all λ in a ball around $0 \in \mathbb{C}$. Thus, by the functional calculus for self-adjoint regular operators (see Theorem 1.19 of [23] for a list of its properties), we may define the operator

$$\frac{\sqrt{\eth^2 + 1}}{\eth} \in \mathcal{L}(\mathcal{E}),$$

which is the inverse of $F = \frac{\eth}{\sqrt{\eth^2 + 1}} \in \mathcal{L}(\mathcal{E})$ by the homomorphism property of the functional calculus. Recall that the G-equivariant index of $\eth$ can be computed by taking trace of the idempotent element

$$\begin{bmatrix} S_0^2 & S_0(1 + S_0)Q \\ S_1 F & 1 - S_1^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \text{Mat}_\infty(\mathcal{K}(\mathcal{E})),$$

where Q is any parametrix of F and $S_0 = 1 - QF$, $S_1 = 1 - FQ$ (see for example the first display on p. 353 of [21]). In our setting we can take $Q = F^{-1}$, hence $S_0 = S_1 = 0$. It then follows that the G-equivariant index of $\eth$ vanishes, after making the canonical identification $K_0(\mathcal{K}(\mathcal{E})) \cong K_0(C^*(G))$. $\qquad \square$

Now, as defined in [55], integration $\int_G$ on $L^1(G)$ induces a tracial map on $C^*(G)$ and also on its K-theory. By the Appendix in [55], we have $\int_G (\text{index}_G(\eth)) = \text{ind}_G(\eth)$ where the right-hand side is the G-invariant index of [55]. Thus we immediately obtain as a corollary the following recent result of Weiping Zhang [66]:

Corollary 55 ([66])**.** *Let M, G and $\eth$ be as in Theorem 54. Then one has $\text{ind}_G(\eth) = 0$.*

Using the canonical projection from $C^*(G)$ to $C_r^*(G)$, Theorem 54 also gives a vanishing result for the reduced equivariant index:

Theorem 56 (Vanishing Theorem 2). *Let M, G and $\eth$ be as in Theorem 54. Then*

$$\text{index}_G(\eth) = 0 \in K_0(C_r^*(G))$$

Upon applying the von Neumann trace τ to this index, we conclude that $\text{index}_{L^2}(\eth) = \tau(\text{index}_G(\eth)) = 0$, where the L^2-index is as defined in [64]. Let $c \in C_c^\infty(M)$ be a cut-off function, that is to say a non-negative function satisfying

$$\int_G c(g^{-1}m)\,dg = 1$$

for all $m \in M$, for a fixed left Haar measure dg on G. The *averaged $\hat{A}$-genus* of the action by G on M is then defined to be (see also [28])

$$\hat{A}_G(M) := \int_M c\hat{A}_M.$$

We have as a consequence of Vanishing Theorem 2, together with the L^2-index theorem of Wang [64], the following result proved by Fukumoto (Corollary B in [28]):

Corollary 57 ([28]). *Let M and G be as in Theorem 54, and assume further that G is unimodular. Then $\hat{A}_G(M) = 0$.*

6.2. **Existence.** We now turn to some existence results. Throughout this subsection, we suppose that G is almost-connected, while M is still a proper G-cocompact manifold. By Abels' slice theorem [1], M is G-equivariantly diffeomorphic to $G \times_K N$, where K is a maximal compact subgroup of G, N is a compact manifold with an action of K. One of our main theorems is the following.

Theorem 58. *Let G be an almost-connected Lie group and that K is a maximal compact subgroup of G. If N is a compact manifold with a K-invariant Riemannian metric of positive scalar curvature, then $M = G \times_K N$ has a G-invariant Riemannian metric of pointwise-positive scalar curvature.*

To prove it, recall the following theorem of Vilms:

Theorem 59 ([63]). *Let $\pi \colon M \to B$ be a fibre bundle with fibre N and structure group K. Let g_B be a Riemannian metric on B and g_N be a K-invariant Riemannian metric on N. Then there is a Riemannian metric g_M on M such that π is a Riemannian submersion with totally geodesic fibres.*

We prove an equivariant version this result, namely:

Theorem 60. *Let $\pi \colon M \to B$ be a fibre bundle with compact fibre N and structure group K. Suppose that M and B both have proper G-actions making π G-equivariant. Let g_N be a K-invariant Riemannian metric on N. Then there is a G-invariant Riemannian metric g_M on M such that π is a G-equivariant Riemannian submersion with totally geodesic fibres.*

Proof. We adapt the proof of Theorem 3.5 in [63]. Let us choose an Ehresmann connection H on M (with the horizontal lifting property), so that $TM = H \oplus V$, where V is the vertical subbundle $\ker \pi_*$. By a result of Palais [57] there exists a G-invariant metric g_B on B. Using the map $\pi_*|_H$, this induces a G-invariant metric g_H on H. Also g_N is K-invariant and so can be transferred to a metric g_V on V. Define the metric $g_M = \langle,\rangle_M$ as the direct sum $g_H \oplus g_V$. One verifies that π is a Riemannian submersion. In what follows, let the Levi-Civita connections on M and N be denoted D and D^N respectively.

Note that parallel transport along a curve in B gives a K-isomorphism of fibres of M. Since the metric on N is K-invariant, the above construction of g_M means that parallel transport is furthermore an isometry of the fibres.

To show that the fibres of M are totally geodesic, it suffices to show that D and D^N act in the same way on tangent vectors to curves in any given fibre. Thus take any curve $\alpha : I \to M_b$, where M_b is the fibre over a base point $b \in B$, and parameterise it proportionally to arc-length. Let $v(0)$ be a given horizontal vector at $\alpha(0)$, and define $\sigma(t, s)$ to be the parallel translation of $\alpha(t)$ along some curve in B with initial vector $\pi_* v(0)$. Define the horizontal vector field

$$v(t) := \frac{\partial}{\partial s} \sigma(t, 0)$$

along $\alpha(t)$. For each s, let $f(s)$ denote the arc-length of the curve $\sigma(s, t)$ from $t = 0$ to 1. Since parallel transport is an isometry, the arc-length $f(s)$ is a constant with respect to s. In particular, we have $f'(0) = 0$. By the formula for the first variation of the arc-length [31], we have

$$f'(0) = \int_0^1 \frac{1}{||\alpha'(0)||} \langle D_t \alpha'(t), v(t) \rangle_M \, dt = \int_0^1 \frac{1}{f(0)} \langle D_t \alpha'(t), v(t) \rangle_M \, dt = 0,$$

whence by continuity, there must be $t_1 \in I$ such that $\langle D_t \alpha'(t_1), v(t_1) \rangle_M = 0$. Next let $\alpha_1(t) := \alpha(\frac{t}{2})$, $0 \le t \le 1$, and apply the same procedure. Since $D_t \alpha_1'(t)$ and $D_t \alpha'(t)$ are proportional, we have $\langle D_t \alpha'(t_2), v(t_2) \rangle_M = 0$ for some $0 \le t_2 \le \frac{1}{2}$. Repeating this procedure gives a sequence $t_i \to 0$ for which the quantity $\langle D_t \alpha'(t), v(t) \rangle_M$ vanishes, which implies that $\langle D_t \alpha'(0), v(0) \rangle_M = 0$.

As $v(0)$ was arbitrarily chosen in H_b, we conclude that $D_t \alpha'(0)$ is vertical and so equal to $D_t^N \alpha'(0)$. This concludes the proof that the fibres are totally geodesic. $\square$

Proof of Theorem 58. Let $\kappa_{G/K}$ denote the scalar curvature of the G-invariant Riemannian metric $g_{G/K}$ on the base. Note that since G/K is a homogeneous space, $-\infty < \kappa_{G/K} < 0$ is a negative constant. Let $H \subseteq TM$ be an Ehresmann connection. Then as in the proof of Theorem 60 above, we may lift $g_{G/K}$ to a G-invariant metric g_H on H, as well as lift the K-invariant Riemannian metric g_N on N to a metric on the vertical subbundle $V \subseteq TM$. Define a G-invariant metric on M by $g_M := g_H \oplus g_V$.

Since N is compact by hypothesis, its scalar curvature κ_N satisfies $\inf\{\kappa_N\} =: \kappa_0 > 0$. Now let T and A denote the O'Neill tensors of the submersion π (their definitions can be found in [56]). By Theorem 60 above, the fibres of M are totally geodesic, so $T = 0$. Pick an orthonormal basis of horizontal vector fields $\{X_i\}$. By G-invariance, we have that for any

$p \in M$,

$$\sum_{i,j} ||A_{X_i}(X_j)||_p = \sum_{i,j} ||A_{X_i}(X_j)||_{gp}$$

for all group elements $g \in G$. This means $\sup_{p \in X}\{\sum_{i,j} ||A_{X_i}(X_j)||_p\} =: A_0 < \infty$, as M/G is compact. Now by a result of Kramer ([47] p. 596), we can relate the scalar curvatures by

$$\kappa_M(p) = \kappa_{G/K} + \kappa_N(p) - \sum_{i,j} ||A_{X_i}(X_j)||_p.$$

Upon scaling the fibre metric on N by a positive factor t, we obtain

$$\kappa_M(p) \geq \kappa_{G/K} + t^{-2}\kappa_0 - A_0 > 0 \qquad \text{whenever} \quad 0 < t < \sqrt{\frac{\kappa_0}{-\kappa_{G/K} + A_0}}.$$

Thus g_M is a G-invariant metric of positive scalar curvature on M. $\qquad\square$

This enables us to establish the following existence theorem for PSC metrics:

Theorem 61. *Let G be an almost-connected Lie group acting properly and co-compactly on M and let K be a maximal compact subgroup of G such that the identity component of K is non-abelian. If there is a global slice such that K acts effectively on it, then M has a G-invariant Riemannian metric of positive scalar curvature.*

Proof. This theorem follows from Theorem 58 and the following theorem of Lawson and Yau:

Theorem 62 ([51]). *Let K be a compact Lie group whose identity component is nonabelian. If N is a compact manifold with an effective action of K, then N admits a K-invariant Riemannian metric of positive scalar curvature.*

$\qquad\square$

7. Two Applications of the Induction Principle

The induction principle exhibited in the earlier parts of this paper allows us to generalise a few interesting results involving compact group actions. We owe inspiration to the paper of Hochs-Mathai [37], where the theorem of Atiyah and Hirzebruch is generalised to the non-compact setting.

7.1. **Hattori's Vanishing Theorem.**
In 1978, Hattori ([30], Theorem 1 and Lemma 3.1) proved the following interesting result:

Theorem 63. *Let Y be a compact, connected, almost-complex manifold of dimension greater than 2 on which S^1 acts smoothly and non-trivially, preserving the almost-complex structure. Suppose that the first Betti number of Y vanishes and that the first Chern class is*

$$c_1(Y) = k_0 x,$$

where $k_0 \in \mathbb{N}$ and $x \in H^2(Y, \mathbb{Z})$. Let L be a line bundle with $c_1(L) = kx$, for an integer k satisfying $|k| < k_0$ and $k = k_0 \pmod 2$. Then

$$\text{index}_{S^1}(\partial_Y^L) = 0 \quad \in R(S^1),$$

where ∂_Y^L is the equivariant Spin^c-Dirac operator on Y twisted by L.

Hattori's result was inspired by the vanishing theorem of Atiyah and Hirzebruch [3] for non-trivial circle actions on compact Spin-manifolds. We first mildly generalise Theorem 63 from non-trivial circle actions to non-trivial actions of compact, connected Lie groups.

Theorem 64. *Let Y be a compact, connected, almost-complex manifold of dimension greater than 2 and K a compact, connected Lie group acting smoothly and non-trivially on Y, preserving the almost-complex structure. Suppose also that the first Betti number of Y vanishes and that the first Chern class is*

$$c_1(Y) = k_0 x,$$

where $k_0 \in \mathbb{N}$ and $x \in H^2(Y, \mathbb{Z})$. Let L be a line bundle with $c_1(L) = kx$, for an integer k satisfying $|k| < k_0$ and $k = k_0$ (mod 2). Then

$$\mathrm{index}_K(\partial_Y^L) = 0 \quad \in R(K),$$

where ∂_Y^L is the equivariant Spin^c-Dirac operator on Y twisted by L.

Proof. For any $g \in K$, there is a maximal torus T of K containing g, whence

$$\mathrm{index}_K(\partial_Y^L)(g) = \mathrm{index}_T(\partial_Y^L)(g).$$

Since by assumption K is connected and acts on Y non-trivially, the action T of on Y given by restriction must also be non-trivial. In the special case $T = S^1 \times S^1$, this means that at most two circles in T can act trivially on Y. Thus the circles in T that act non-trivially on Y form a dense subset of T. For each t in such a circle $S < T$, we have that

$$\mathrm{index}_T(\partial_Y^L)(t) = \mathrm{index}_S(\partial_Y^L)(t) = 0$$

by Hattori's Theorem 63. Since the character of a K-representation is continuous in K, $\mathrm{index}_T(\partial_Y^L)(t) = 0$ for all $t \in T$. This completes the argument for $T = S^1 \times S^1$.

The argument extends to the general case by induction on the dimension of T. One can show that for an l-dimensional torus T acting non-trivially on Y, no more than $l-1$ circles can act trivially. From this it follows that, under the hypotheses of the theorem, $\mathrm{index}_K(\partial_Y^L) = 0 \in R(K)$. $\qquad\square$

Our goal in the following subsection is to extend Theorem 64 to the non-compact setting. The result is Theorem 66, which can be stated equivalently in the form of Theorem 67. Let X be a manifold on which a connected Lie group G acts properly and isometrically. Suppose that the action is cocompact and that X has a G-equivariant Spin^c-structure. Let

$$\mathrm{index}_G(\partial_X^L) \in K_\bullet(C_r^*(G))$$

be the equivariant index of the associated Spin^c-Dirac operator.

Let $K < G$ be a maximal compact subgroup, and suppose G/K has an almost-complex structure (this is always true for a double cover of G, as pointed out in Remark 17).

Definition 65 ([37])**.** The action of G on X is *properly trivial* if all stabilisers are maximal compact subgroups of G. For a proper action, the stabilisers cannot be larger. The action is called *properly non-trivial* if it is not properly trivial.

Hattori's Theorem 64 generalises as follows.

Theorem 66. *As above, let G be a connected Lie group with a maximal compact subgroup K, and assume that G/K has an almost-complex structure. Suppose that G acts properly and cocompactly on a connected almost-complex manifold X and that G preserves the almost-complex structure. Suppose also that the first Betti number of X vanishes and that the first Chern class is*

$$c_1(X) = k_0 x,$$

where $k_0 \in \mathbb{N}$ and $x \in H^2(X, \mathbb{Z})$. Assume that the G-action on X is properly non-trivial. Let L be a G-equivariant line bundle with $c_1(L) = kx$, for an integer k satisfying $|k| < k_0$ and $k = k_0 \pmod 2$. Then

$$\mathrm{index}_G(\partial_X^L) = 0 \quad \in K_\bullet(C_r^*(G)),$$

where ∂_X^L is the equivariant Spin^c-Dirac operator on X twisted by L.

Theorem 46 (*higher index commutes with induction*) allows us to deduce Theorem 66 from Hattori's Theorem 64. This is based on the fact that the Dirac induction map (4.6) relates the equivariant indices of the Spin^c-Dirac operators ∂_Y^L on Y and ∂_X^L on X, associated to the Spin^c-structures P_Y and P_X respectively, to one another. (See Corollary 48, also [36, Theorem 5.7].)

Proof of Theorem 66. Let $Y \subset X$ be as in Abels' Theorem (Theorem 6). Using the decomposition (3.7) of the tangent bundle of X, the almost-complex structures on X and on G/K (hence on $\mathfrak{p}$) give rise to an almost-complex structure on $G \times_K TY$. Since G preserves the almost-complex structure, we obtain an almost-complex structure on Y preserved by the K-action. Note that a manifold with an almost-complex structure preserved under a group action has an equivariant Spin^c-structure. By Corollary 48, we have

(7.1) $$\mathrm{index}_G(\partial_X^L) = \text{D-Ind}_K^G\big(\mathrm{index}_K(\partial_Y^{L|_Y})\big).$$

Let $X_{(K)}$ be the set of points in X with stabilisers conjugate to K. By Lemma 9 of [37], the fixed point set Y^K of the action by K on Y is related to the action of G on X by

$$X_{(K)} = G \cdot Y^K \cong G/K \times Y^K.$$

The stabiliser of a point $m \in X$ is a maximal compact subgroup of G if and only if $m \in X_{(K)}$. Thus, the condition on the stabilisers of the action of G on X is equivalent to the requirement that the action of K on Y be non-trivial. Moreover, because G/K is contractible, we have

$$H^*(G \times_K Y, \mathbb{Z}) \simeq H^*(Y, \mathbb{Z}),$$

while $T(G/K)$ is trivial. Hence, $X = G \times_K Y$ and Y have the same first Betti number. In a similar fashion, we have

$$c_1(G \times_K TY) = c_1(TY), \qquad c_1(L) = c_1(G \times_K L|_Y) = c_1(L|_Y).$$

Theorem 64 now implies that, under our hypotheses,

$$\text{index}_K(\partial_Y^{L|_Y}) = 0.$$

The theorem then follows from relation (7.1). □

The Theorem we have just proved has the following equivalent statement.

Theorem 67. *In the setting of Theorem 66,* $\text{index}_G(\partial_X^L) \neq 0$ *if and only if there is a compact* Spin^c*-manifold* Y *with* $e^{kx/2}\hat{A}(Y) \neq 0$*, and a* G*-equivariant diffeomorphism*

$$X \simeq G/K \times Y,$$

where G *acts trivially on* Y*.*

Proof. Because the Dirac induction map D-Ind$_K^G$ in (7.1) is an isomorphism,

$$\text{index}_G(\partial_X^L) \neq 0 \quad \Leftrightarrow \quad \text{index}_K(\partial_Y^{L|_Y}) \neq 0.$$

Moreover, we have an equivalence

$$\text{index}_K(\partial_Y^{L|_Y}) \neq 0 \quad \Leftrightarrow \quad K \text{ acts trivially on } Y \text{ and } e^{kx}\hat{A}(Y) \neq 0,$$

which follows from Theorem 64, since if K acts trivially on Y, then $\text{index}_K(\partial_Y^{L|_Y}) \in R(K)$ equals $\text{index}(\partial_Y^{L|_Y}) = e^{kx}\hat{A}(Y)$ copies of the trivial representation. Since K acts trivially on Y if and only if $X = (G/K) \times Y$, the claim follows. □

Remark 68. The proofs of Theorems 66 and 67 are inspired by the proofs of Theorems 2 and 3 in [37]. Moreover (see also Remark 7 in [37]), the non-vanishing of $\text{index}_G(\partial_X^L)$ in Theorems 66 and 67 can be replaced by the non-vanishing of the class $p_*[\partial_X^L]$ in (4.4) because of the Baum-Connes isomorphism (4.5).

7.2. **An Analogue of Petrie's Conjecture.** The Pontryagin class of a closed oriented manifold is not usually a homotopy invariant. However, for the Kähler manifolds $\mathbb{C}P^n$, Petrie [58] has an interesting conjecture, motivated by the question of whether or not manifolds of a given homotopy type admit non-trivial circle actions. Recall that the total Pontryagin class of $\mathbb{C}P^n$ is

$$p(\mathbb{C}P^n) = (1 + x^2)^{n+1}, \qquad x \in H^2(\mathbb{C}P^n).$$

There is a natural action of S^1 on $\mathbb{C}P^n$ given by

$$(\lambda, [z_0 : \ldots : z_n]) \mapsto [\lambda^{a_0} z_0 : \ldots : \lambda^{a_n} z_n], \qquad a_i \in \mathbb{Z}.$$

Conjecture 69 (Petrie's Conjecture [58]). *If a closed oriented manifold* Y *of dimension* $2n$ *admits an orientation-preserving homotopy equivalence* $f : Y \to \mathbb{C}P^n$ *and also a non-trivial circle action, then its total Pontryagin class is given by*

$$p(Y) = f^*p(\mathbb{C}P^n).$$

It is a theorem of Hattori [30] that this conjecture holds if Y has an almost-complex structure preserved under the S^1-action and if $c_1(Y) = \pm(n + 1)x$, where $x \in H^2(Y, \mathbb{Z})$ is the generator.

Recall that the $\mathcal{L}$-class is given by the multiplicative sequence of polynomials belonging to the power series $\frac{\sqrt{t}}{\tanh\sqrt{t}}$ and that the signature of a smooth compact oriented manifold Y^{4k} is equal to the $\mathcal{L}$-genus $\mathcal{L}[Y^{4k}]$. We have the following lemma, inspired by a result of Kaminker [39]:

Lemma 70. *Let Y be a compact* Spin^c*-manifold. Let $D_Y, D_{\mathbb{C}P^n}$ denote signature operators on Y and $\mathbb{C}P^n$ respectively. Then $\mathcal{L}(Y) = f^*\mathcal{L}(\mathbb{C}P^n)$ if and only if*

$$f_*[D_Y] = [D_{\mathbb{C}P^n}] \in K_\bullet^{geo}(\mathbb{C}P^n).$$

Proof. Assume without loss of generality that $\dim Y$ is even. Poincaré duality for geometric K-homology is given by taking cap product with the fundamental class $[Y]$ (cf. Example 4 and Corollary 27):

$$\mathcal{PD}_K : K^0(Y) \to K_0^{geo}(Y) \qquad [E] \to [Y] \cap [E] := [(Y, E \otimes \mathbb{C}_Y, \mathrm{id}_Y)].$$

Recall that there is a Chern character for K-homology,

$$\mathrm{ch} : K_0(Y) \to H_{\mathrm{ev}}(Y) \otimes \mathbb{Q} \qquad [(Y, E, f)] \mapsto f_*(\mathcal{PD}_H(\mathrm{ch}(E) \cup \mathrm{td}(Y))),$$

that makes the following diagram commutative:

(7.2)
$$
\begin{array}{ccc}
K^0(Y) & \xrightarrow[\simeq]{\mathrm{ch}(-)\cup\mathrm{td}(Y)} & H^{\mathrm{ev}}(Y) \otimes \mathbb{Q} \\
\mathcal{PD}_K \downarrow \simeq & & \simeq \downarrow mathcalPD_H \\
K_0(Y) & \xrightarrow[\mathrm{ch}(-)]{} & H_{\mathrm{ev}}(Y) \otimes \mathbb{Q}.
\end{array}
$$

In this diagram, $\mathrm{ch}(-) \cup \mathrm{td}(Y)$ is the Atiyah-Singer integrand and $\mathcal{PD}_H$ is the Poincaré duality map on homology and cohomology. Note that if D_Y is the signature operator on Y, then $\mathrm{ch}[\sigma(D_Y)] \cup \mathrm{td}(Y) = \mathcal{L}(Y)$. Together with the commutative diagram (7.2) we have

$$\mathcal{PD}_H(\mathcal{L}(Y)) = \mathcal{L}(Y) \cap [Y] = \mathrm{ch}(\mathcal{PD}_K([\sigma(D_Y)])).$$

The proof is then completed by naturality of the Chern character (note that $\mathcal{L}(Y) = f^*\mathcal{L}(\mathbb{C}P^n)$ if and only if the intersection products are equal, if and only if $f_*(\mathcal{L}(Y) \cap [Y]) = \mathcal{L}(\mathbb{C}P^n) \cap [\mathbb{C}P^n])$. $\square$

Using induction on K-homology, we are able to generalise a weaker version of Hattori's result on Petrie's Conjecture, as follows.

Theorem 71 (Analogue of Petrie's Conjecture)**.** *Let X be a connected manifold admitting an almost-complex structure and a properly non-trivial $SU(1,1)$-action preserving the almost-complex structure. If there is an $SU(1,1)$-equivariant homotopy equivalence*

$$f : X \to SU(1,1) \times_{U(1)} \mathbb{C}P^n,$$

and $c_1(X) = \pm(n+1)x$ *with* $x \in H^2(X, \mathbb{Z})$ *being the generator, then*

$$f_*[D_X] = [D_{SU(1,1)\times_{U(1)}\mathbb{C}P^n}] \in K_\bullet^{SU(1,1)}(SU(1,1) \times_{U(1)} \mathbb{C}P^n) \simeq K_\bullet^{U(1)}(\mathbb{C}P^n).$$

Proof. Let Y be a $U(1)$-slice of X. There exist $SU(1,1)$-equivariant maps

$$f: SU(1,1) \times_{U(1)} Y \to SU(1,1) \times_{U(1)} \mathbb{C}P^n, \qquad g: SU(1,1) \times_{U(1)} \mathbb{C}P^n \to SU(1,1) \times_{U(1)} Y,$$

such that $g \circ f$ and $f \circ g$ are $SU(1,1)$-equivariantly homotopic to the respective identity maps. These maps induce $U(1)$-equivariant homotopies from $f|_Y \circ g|_{\mathbb{C}P^n}$ and $g|_{\mathbb{C}P^n} \circ f|_Y$ to the identity maps. Thus, Y is $U(1)$-homotopic to $\mathbb{C}P^n$. Hattori's result now implies that $\mathcal{L}(Y) = f^*\mathcal{L}(\mathbb{C}P^n)$. Finally, by Lemma 70 and induction on K-homology, the conclusion follows. $\qquad\square$

REFERENCES

[1] H. Abels, *Parallelizability of proper actions, global K-slices and maximal compact subgroups*, Math. Ann. 212 (1974), 1–19. 6, 7, 2.2, 2.2, 3.1, 6.2

[2] Y. Arano, Y. Kubota, *A categorical perspective on the Atiyah-Segal completion theorem in KK-theory* ArXiv: 1508.06815. 35

[3] M. Atiyah, F. Hirzebruch, *Spin-manifolds and group actions*, 1970 Essays on Topology and Related Topics (Mémoires dédiés Georges de Rham) Springer, New York, 18–28. 1, 7.1

[4] M. Atiyah, G.B. Segal, *Equivariant K-theory and completion*, J. Differential Geometry 3 (1969), 1–18. 3.1, 35

[5] M. F. Atiyah, I. M. Singer, *The index of elliptic operators. I.* Ann. of Math. (2) 87 1968 484–530. 28

[6] M. F. Atiyah, *Global theory of elliptic operators.* In Proc. of the International Symposium on Functional Analysis, Tokyo, 1970. University of Tokyo Press. 1. 2

[7] M. F. Atiyah, W. Schmid, *A geometric construction of the discrete series for semisimple Lie groups* Inventiones Mathematicae 42 1977 1–62. 1, 5, 50

[8] P. Baum, A. Connes, *Geometric K-theory for Lie groups and foliations*, Enseign. Math. (2), Vol 46, (2000), No. 1-2, 3–42. 40

[9] P. Baum, A. Connes, N. Higson, *Classifying space for proper actions and K-theory of group C*-algebras*, C*-algebras: 1943-1993 (San Antonio, TX, 1993), Contemp. Math., 167, Amer. Math. Soc., Providence, RI (1994), 240–291. 1, 40, 5, 5

[10] P. Baum, R. Douglas, *K-homology and index theory.* In R. Kadison, editor, Operator Algebras and Applications, volume 38 of Proceedings of Symposia in Pure Mathematics, pages 117–173, Providence, RI, 1982. American Mathematical Society. 1, 2, 3, 2

[11] P. Baum, N. Higson, T. Schick, *A geometric description of equivariant K-homology for proper actions.* Quanta of maths, 1-22, Clay Math. Proc., 11, Amer. Math. Soc., Providence, RI, 2010. 1, 2, 3, 2, 2.2, 3.2

[12] P. Baum, N. Higson, T. Schick, *On the equivalence of geometric and analytic K-homology.* Pure Appl. Math. Q. 3 (2007), no. 1, part 3, 1–24. 2, 3, 2

[13] P. Baum, H. Oyono-Oyono, T. Schick, M. Walter, *Equivariant geometric K-homology for compact Lie group actions.* Abh. Math. Semin. Univ. Hambg. 80 (2010), no. 2, 149–173. 1, 2, 2, 8, 2.4

[14] J.M. Bismut, *The Atiyah-Singer index theorem for families of Dirac operators: two heat equation proofs.* Invent. Math. 83 (1986), no. 1, 91–151. 2.4

[15] R. Bott and L. Tu, Differential forms in algebraic topology, GTM **82**. 3.2

[16] L. Brown, R. Douglas, and P. Fillmore, *Extensions of C*-algebras and K-homology.* Annals of Math., 105: 265–324, 1977. 2

[17] B. Blackadar, *K-theory for operator algebras (2nd ed.).* Mathematical Sciences Research Institute Publications, Cambridge University Press, Cambridge (1998) xx+300. 2.3

[18] S. Cecchini, *Callias-type operators in C-algebras and positive scalar curvature on noncompact manifolds.* Preprint, https://arxiv.org/abs/1611.01800. 6.1

[19] J. Chabert, S. Echterhoff and R. Nest, *The Connes–Kasparov conjecture for almost-connected groups and for linear p-adic groups*, Publ. Math. Inst. Hautes Études Sci. 97 (2003), 239–278. 41, 4.2

[20] A. Connes, *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of* **R**, Adv. in Math., Vol 39, (1981), No. 1, 31–55. 4.2

[21] A. Connes and H. Moscovici, *Cyclic cohomology, the Novikov conjecture and hyperbolic groups.* Topology Vol. 29 No. 3 345-388, (1990). 6.1

[22] A. Connes and H. Moscovici, *L^2-index theorem for homogeneous spaces of Lie groups.* Ann. Math. Vol. **115**, No. 2, 291–330 (1982). 5

[23] J. Ebert, *Index theory in spaces of noncompact manifolds I: Analytical foundations.* Preprint, https://arxiv.org/abs/1608.01699. 6.1

[24] S. Echterhoff, H. Emerson, *Structure and K-theory of crossed products by proper actions.* Expo. Math. 29 (2011), no. 3, 300–344. 4.2

[25] H. Emerson, R. Meyer, *Bivariant K-theory via correspondences.* Adv. Math. 225 (2010), no. 5, 2883–2919. 1

[26] H. Emerson, R. Meyer, *Equivariant Lefschetz maps for simplicial complexes and smooth manifolds.* Math. Ann. 345 (2009), no. 3, 599–630. 1

[27] J. Fox, P. Haskell, I. Raeburn, *Kasparov products, KK-equivalence, and proper actions of connected reductive Lie groups*, J. Operator Theory 22 (1989), 3–29. 25

[28] Y. Fukumoto, *On the Strong Novikov Conjecture of Locally Compact Groups for Low Degree Cohomology Classes.* Preprint, https://arxiv.org/abs/1604.00464. 6.1, 57

[29] M. Gromov and H. B. Lawson, Positive Scalar Curvature and the Dirac Operator on Complete Riemannian Manifolds. 83-196. Publications Mathmatiques de l'Institut des Hautes Scientifiques, 58, 1, (1983). 6.1

[30] A. Hattori, *$Spin^c$-structures and S^1-actions*, Inven. Math. **48** (1978) 7-31. 1, 7.1, 7.2

[31] R. Hermann, *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle*, Proc. Amer. Math. Soc., **11** (1960) 236–242. 6.2

[32] N. Higson and J. Roe, Analytic K-homology. Oxford Mathematical Monographs. Oxford Science Publications. Oxford University Press, Oxford, 2000. xviii+405 pp. 3.2

[33] P. Hochs, *Quantisation commutes with reduction at discrete series representations of semisimple groups*, Adv. Math. 222 (2009), no. 3, 862–919. 1, 21, 47

[34] P. Hochs, *Quantisation of presymplectic manifolds, K-theory and group representations*, Proc. Amer. Math. Soc. 143 (2015), no. 6, 2675-2692. 21, 47

[35] P. Hochs, V. Mathai, *Geometric quantization and families of inner products*, Adv. Math., **282** (2015) 362-426. ArXiv:1309.6760

[36] P. Hochs, V. Mathai, *Quantising proper actions on* $Spin^c$-*manifolds*, Asian J. Math. (to appear) 62 pages, ArXiv:1408.0085. 10, 2.2, 7.1

[37] P. Hochs, V. Mathai, Spin-*structures and proper group actions*, Adv. Math., **292** (2016) 1-10. ArXiv:1411.0781. 1, 7, 65, 7.1, 68

[38] P. Julg, *K-théorie équivariante et produits croisés.* (French. English summary) [Equivariant K-theory and crossed products] C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 13, 629–632. 3.1

[39] J. Kaminker, J.G. Miller, *Homotopy invariance of the analytic index of signature operators over C^*-algebras.* J. Operator Theory 14 (1985), no. 1, 113–127. 7.2

[40] M. Kankaanrinta, *Equivariant collaring, tubular neighbourhood and gluing theorems for proper Lie group actions.* Algebr. Geom. Topol. 7 (2007), 1–27. 2

[41] G. G. Kasparov, *Elliptic and transversally elliptic index theory from the viewpoint of KK-theory.* J. Noncommut. Geom. 10 (2016) 4, 1303–1378. 3.1, 28, 6.1, 6.1

[42] G. G. Kasparov, *Equivariant KK-theory and the Novikov conjecture.* Invent. Math. **91** (1988), no. 1, 147-201. 1, 1, 15, 2.3, 2.3, 3.1, 29, 30, 3.1, 3.1, 34, 35, 35, 3.2, 3.2, 39, 40

[43] G. G. Kasparov, *Operator K-theory and its applications: elliptic operators, group representations, higher signatures, C^*-extensions*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 987–1000, (1984). 4.2

[44] G. G. Kasparov, *The index of invariant elliptic operators, K-theory, and Lie group representations.* Dokl. Akad. Nauk SSSR, 268 (1983), 533–537; translation: Soviet Mathematics-Doklady, 27 (1983), 105–109. 38

[45] G. G. Kasparov, *Topological invariants of elliptic operators I: K- homology.* Math. USSR Izvestija, 9: 751–792, 1975. 1, 2, 1, 2

[46] G. G. Kasparov, G. Skandalis, *Groups acting properly on "bolic" spaces and the Novikov conjecture.* Ann. of Math. (2) 158 (2003), no. 1, 165-206.

[47] W. Kramer, The scalar curvature on totally geodesic fiberings. Ann. Global Anal. Geom. 18 (2000), no. 6, 589-600. 6.2

[48] V. Lafforgue, *Banach KK-theory and the Baum-Connes conjecture*, Banach KK-theory and the Baum-Connes conjecture, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 795–812 Vol 149, (2002), No. 1, 1–95. 1, 5

[49] V. Lafforgue, *K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes*, Invent. Math., Vol 149, (2002), No. 1, 1–95. 4.2, 5

[50] E. C. Lance, *Hilbert C^*-modules, a toolkit for operator algebraists*, Cambridge University Press, Cambridge, 1995, x+130. 6.1

[51] B. Lawson and S. T. Yau, Scalar curvature, non-abelian group actions, and the degree of symmetry of exotic spheres. Comment. Math. Helv. 49 (1974), 232-244. 62

[52] A. Lichnerowicz, Spineurs harmoniques, C. R. Acad. Sci. Paris. SérieA,257 (1963),7-9. 6.1

[53] W. Lück and B. Oliver, The completion theorem in K-theory for proper actions of a discrete group. Topology 40 (2001), no. 3, 585-616. 1, 3.2, 3.2, 3.2

[54] J. W. Milnor *Topology from the differentiable viewpoint, based on notes by David W. Weaver.* The University Press of Virginia, Charlottesville, Va. (1965) ix+65. 2.2

[55] V. Mathai and W. Zhang, *Geometric quantization for proper actions* (with an appendix by U. Bunke), Adv. Math. 225 (2010), no. 3, 1224-1247. ArXiv:0806.3138. 1, 6.1

[56] B. O'Neill, The fundamental equations of a submersion. Michigan Math. J. 13 (1966) 459-469. 6.2

[57] R. Palais, On the existence of slices for actions of non-compact Lie groups. Ann. of Math. (2) 73 (1966) 295-323. 2.2, 6.2

[58] T. Petrie, *Smooth S^1 actions on homotopy complex projective spaces and related topics.* Bull. Amer. Math. Soc. 78 (1972), 105–153. 1, 7.2, 69

[59] C. N. Phillips, *Equivariant K-theory for proper actions. II. Some cases in which finite-dimensional bundles suffice.* Index theory of elliptic operators, foliations, and operator algebras, 205-227, Contemp. Math., 70, Amer. Math. Soc., Providence, RI, 1988. 1, 3.1, 3.1, 3.1, 30, 1, 3.2

[60] C. N. Phillips, *Equivariant K-theory for proper actions and C*-algebras.* Index theory of elliptic operators, foliations, and operator algebras, 175-204, Contemp. Math., 70, Amer. Math. Soc., Providence, RI, 1988. 1, 3.1, 3.2

[61] M.A. Rieffel, *Applications of strong Morita equivalence to transformation group C^*-algebras.* Operator algebras and applications, Part I (Kingston, Ont., 1980), pp. 299–310, Proc. Sympos. Pure Math., **38**, Amer. Math. Soc., Providence, R.I. (1982) 3.1

[62] G. Segal, *Fredholm complexes*, Quart. J. Math. 21 (1970), 385–402. 3.1

[63] J. Vilms, Totally geodesic maps. J. Differential Geometry 4 (1970) 73-79. 59, 6.2

[64] H. Wang, *L^2-index formula for proper cocompact group actions.* J. Noncommut. Geom. 8 (2014), no.2, 393–432. 1, 5, 6.1

[65] A. Wassermann, *Une démonstration de la conjecture de Connes-Kasparov pour les groupes de Lie linéaires connexes réductifs.* C. R. Acad. Sci. Paris Sér. I Math., Vol. 304, (1987) No. 18, 559–562. 4.2, 5

[66] W. Zhang, A note on the Lichnerowicz vanishing theorem for proper actions, J. Noncomm. Geom. 11 (2017) 823 1, 6.1, 55

School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005 Australia.
E-mail address: oug.oah@gmail.com

School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005 Australia.
E-mail address: mathai.varghese@adelaide.edu.au

School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005 Australia.
E-mail address: hang.wang01@adelaide.edu.au