Non-Abelian Electrons: SO(5) Superspin Regimes for Correlated Electrons on a Two-Leg Ladder

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(Received 20 May 1998)

We analyze critical and massive SO(5) superspin regimes for correlated electrons on a two-leg ladder. We identify fundamental low energy excitations, which carry the quantum numbers of a free electron and can be probed in (inverse) photoemission experiments. These excitations do not obey the usual Pauli principle but are governed by specific forms of the so-called non-Abelian exclusion statistics. [S0031-9007(99)08808-0]

PACS numbers: 71.10.Pm, 05.30.–d, 11.25.Hf

It is well known that the quantum statistics of particles in one or two spatial dimensions are not necessarily bosonic or fermionic. In two spatial dimensions all that is needed is a representation of the braid group, and this allows for various alternative possibilities. A particularly intriguing possibility is that of non-Abelian braiding statistics, in which case the braiding of particles is represented by nontrivial matrices acting on multicomponent wave functions. A prototype example is the quasihole excitations over the Pfaffian quantum Hall state [1].

In a recent paper [2], one of us investigated the exclusion statistics properties of the edge quasiholes over the Pfaffian quantum Hall state. A key quantity is the thermodynamic distribution function $n_{\rm qh}(\epsilon)$, which describes the expected occupation of specific one-quasihole states and which generalizes the Fermi-Dirac distribution for fermions. The edge quasihole distribution, which was computed in [2], has the following characteristic features:

$$\begin{aligned} n_{\rm qh}(\epsilon) &\sim 8 \quad \text{for } \epsilon \ll \mu\,, \\ &\sim \sqrt{2}\, e^{-\beta(\epsilon-\mu)} \quad \text{for } \epsilon \gg \mu\,, \end{aligned} \tag{1}$$

with $\beta = (k_B T)^{-1}$ and μ the chemical potential. The nontrivial prefactor $\sqrt{2}$ in the high-energy Boltzmann tail of the distribution $n_{\rm qh}(\epsilon)$ is a direct manifestation of the non-Abelian nature of the braiding statistics of the quasiholes over the Pfaffian quantum Hall state [2].

One may demonstrate that, in general, non-Abelian braiding statistics in a conformal field theory (CFT) lead to nontrivial prefactors in the Boltzmann tails of the corresponding thermodynamic distribution functions [2,3]. We therefore propose the term "non-Abelian exclusion statistics" for all forms of exclusion statistics that exhibit such nontrivial prefactors. In the theoretical literature, a number of examples, other than the Pfaffian, have been discussed [3,4].

In this Letter we focus on situations where excitations with the quantum numbers of free electrons obey specific forms of non-Abelian exclusion statistics. The concrete setting for this is the so-called SO(5) superspin regime for correlated electrons on a two-leg ladder. The excitations, which we call "non-Abelian electrons," can be probed via (inverse) photoemission and, in principle, the non-Abelian nature of their exclusion statistics can be tested in experiments.

Before discussing the SO(5) ladder models and the associated non-Abelian electrons, we briefly discuss the exclusion statistics properties of the spinons and holons of free or weakly interacting electrons in one dimension. In a continuum description, free electrons in one dimension are described by a CFT, with the electron degrees of freedom represented by four real (Majorana) fermion fields

$$(\psi_{(\uparrow,e)}, \psi_{(\downarrow,e)}, \psi_{(\downarrow,-e)}, \psi_{(\uparrow,-e)})\,. \tag{2}$$

These fields transform as a vector under the SO(4) symmetry. Through non-Abelian bosonization, the free fermion CFT is equivalent to an SO(4) Wess-Zumino-Witten theory with level $k = 1$ [$\mathrm{SO}(4)_1$ WZW].

As soon as (weak, repulsive) interactions are introduced in a one-dimensional electron theory, a separation of spin and charge sets in and the fundamental excitations are found to be spinons (carrying spin-$\frac{1}{2}$ and no charge) and holons (carrying charge $\pm e$ and no spin). An explicit example is the $U > 0$ Hubbard model, where one may use the exact Bethe ansatz solution to demonstrate the fundamental role of the spinon and holon excitations [5]. The free electron CFT may equally well be described in terms of spinons and holons, which we write as

$$(\phi_\uparrow, \phi_\downarrow), \qquad (\phi_e, \phi_{-e})\,. \tag{3}$$

Together, the spinon and holon fields form a spinor representation of the SO(4) symmetry. In Fig. 1 we show in an SO(4) weight diagram the quantum numbers of the electron, spinon, and holon fields.

This brings us to the statistics of the spinons and holons in a free-electron theory. That these statistics are unusual can be demonstrated by the following argument. We

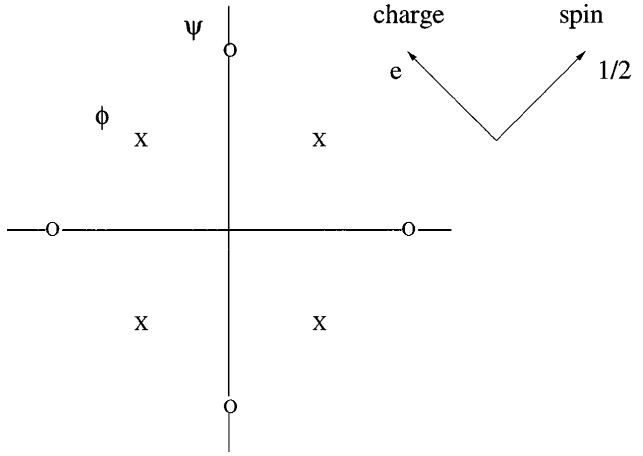

FIG. 1. SO(4) weight diagram, showing the quantum numbers of the vector (electron) representation $\psi_{(\sigma,\pm e)}$ (open dots) and of the spinor (spinon and/or holon) representation $(\phi_\sigma, \phi_{\pm e})$ (crosses).

consider the capacitance of the free-electron gas, i.e., the amount of charge ΔQ pulled from the Fermi sea by an applied voltage V. In the electron picture, this response is generated by the four fermionic fields, Eq.(2), of charge $\pm e$. In the spinon and/or holon picture, the response is generated by the holons alone, i.e., by two fields of charge $\pm e$. Assuming fermionic statistics for the holons thus leads to a mismatch by a factor of 2 between the two approaches. As it turns out, the actual, fractional, statistics of the holons are such that the response to a voltage indeed exhibits an enhancement by a factor of 2 with respect to free fermions (see below).

The appropriate framework for discussing the statistics of spinons and holons is the notion of "fractional exclusion statistics," which was introduced by Haldane in [6] and elaborated on in [7]. The early applications of this new paradigm in many-body theory have hinged on the specifics of exactly solvable models of quantum mechanics with inverse square exchange. In an important step forward [8], it was demonstrated that a fractional statistics assignment can be done as soon as an effective CFT for low-energy behavior has been identified. The method introduced in [8] is based on recursion relations for truncated conformal spectra. In the meantime, a large number of examples of this "statistics from field theory" approach have been worked out [2–4,8,9].

To obtain the exclusion statistics of spinons and holons, we rely on our recent results, presented in [3], for quasiparticle formulations of level-1 WZW models. For the case $SO(4)_1$, we obtained explicit results for the exclusion statistics of quasiparticles transforming in the spinor of SO(4), which consists of the spinon and holon fields combined. The result may be phrased in terms of a quantity $\lambda(x_s^\pm, x_c^\pm)$, which is the approximate partition sum for a single one-particle level in the spectrum. This partition sum depends on the one-particle energy ϵ and on chemical potentials $\mu_s^\pm$, $\mu_c^\pm$ for the various spinor components through the quantities $x_{s,c}^\pm = e^{\beta(\mu_{s,c}^\pm - \epsilon)}$. In [3] we showed that

$$\lambda(x_s^\pm, x_c^\pm) = \tilde\lambda(x_s^+, x_s^-)\tilde\lambda(x_c^+, x_c^-)\,, \tag{4}$$

with the $\tilde\lambda(x^+, x^-)$ related to the single-level partition sum $\mu(x,z)$ for spinons in the $SU(2)_1$ WZW model [8],

$$\begin{aligned}\tilde\lambda(xz, xz^{-1}) &= \mu(x,z) \\ &= 1 + (z^2 + z^{-2})\frac{x^2}{2} \\ &\quad + (z + z^{-1})\frac{x}{2}\sqrt{(z - z^{-1})^2x^2 + 4}\,.\end{aligned} \tag{5}$$

The parameter x is related to a chemical potential for spinons and z keeps track of their SU(2) quantum number S^z. The factorization (4) is a direct consequence of the identity $SO(4)_1 = SU(2)_1 \times SU(2)_1$ at the CFT level [10]. The exclusion statistics encoded in $\mu(x,z)$ agree with Haldane's fractional exclusion statistics, with the statistics matrix given by

$$G = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

From Eqs. (4) and (5), the spinon and holon thermodynamics are easily derived. The thermal response follows from $\mu(x, z = 1) = (1 + x)^2$, leading to a CFT central charge $c^{\mathrm{CFT}} = 1$ for both spinons and holons. The response to a field follows from

$$z\partial_z \log[\mu(x,z)] = \frac{2x(z - z^{-1})}{\sqrt{(z - z^{-1})^2x^2 + 4}}\,. \tag{6}$$

The above-mentioned factor of 2 in the holon response to a voltage is established by comparing the integral expressions (with $\mu_c^\pm = \pm eV$ and ρ the density of states)

$$\begin{aligned}\Delta Q_{\mathrm{fermion}}(\beta, V) &= \rho e\int_0^\infty d\epsilon\,\frac{e^{-\beta(\epsilon - eV)} - e^{-\beta(\epsilon + eV)}}{(1 + e^{-\beta(\epsilon - eV)})(1 + e^{-\beta(\epsilon + eV)})} = \rho e^2 V\,, \\ \Delta Q_{\mathrm{holon}}(\beta, V) &= \rho e\int_0^\infty d\epsilon\,\frac{2(e^{-\beta(\epsilon - eV)} - e^{-\beta(\epsilon + eV)})}{\sqrt{(e^{\beta eV} - e^{-\beta eV})^2 e^{-2\beta\epsilon} + 4}} = 2\rho e^2 V\end{aligned} \tag{7}$$

for a single charged fermion and a holon, respectively.

The low-temperature behavior of a one-dimensional electron gas (away from half filling) in the presence of weak repulsive interactions is described by the Luttinger liquid, which is parametrized by a single interaction parameter K_c

and two velocities u_c and u_s for gapless excitations. The above discussion of the spinon and holon statistics is easily generalized to the general Luttinger liquid. The absence of a nontrivial parameter K_s makes it clear that the spinon statistics in a Luttinger liquid are identical to those in the free-electron theory, while a nontrivial value $K_c \neq 1$ will lead to a modification of the holon statistics.

We now turn to a specific system of strongly correlated electrons, in which the fundamental degrees of freedom belong to the spinor representation of SO(5) and are radically different from the spinons and holons in the weakly interacting case. SO(5) symmetry has recently been proposed as a structure that unifies antiferromagnetic (AF) ordering and d-wave superconductivity (dSC). As such, it may be invoked in attempts to sort out the competition between AF and dSC ordering in the cuprate superconductors [11]. As a laboratory for studying the implications of SO(5) symmetry, Scalapino, Zhang, and Hanke (SZH) [12] have introduced a class of two-leg ladder models with exact SO(5) symmetry [13]. In their paper, SZH give a strong-coupling phase diagram for these models, as a function of the couplings U, V within a single rung. Among others, they identify a so-called SO(5) superspin phase (called E_1), where the electron degrees of freedom on a single rung reduce to an SO(5) vector $n_a(x)$, $a = 1, \ldots, 5$, composed of electron bilinears that represent the AF and dSC order parameters,

$$n_a = \left(\frac{\Delta^\dagger + \Delta}{2}, N_x, N_y, N_z, \frac{\Delta^\dagger - \Delta}{2i} \right). \tag{8}$$

In the superspin phase, interrung interactions lead to an effective SO(5) spin chain, with the fundamental "spins" in the vector representation of SO(5). This may be compared to an SO(3) spin chain with spins in the vector representation, which is an $S = 1$ spin chain. On the basis of this analogy, one expects that generic interrung couplings in an SO(5) ladder lead to a "Haldane gap." A particular example of a gapped phase (the so-called Affleck-Kennedy-Lieb-Tasaki point) was studied in [12]. One also expects a single critical point, which avoids the Haldane gap, and which is analogous to the integrable $S = 1$ chain. It will be represented by interrung interactions

$$\sum_j [\alpha_1 (L_j^{ab} L_{j+1}^{ab}) + \alpha_2 (L_j^{ab} L_{j+1}^{ab})^2] \tag{9}$$

with L^{ab} the ten generators of SO(5) and the ratio α_2/α_1 tuned to a critical value. Deviations from the critical α_2/α_1 will be relevant and will destroy the critical behavior.

These qualitative expectations are in agreement with the perturbative renormalization group analysis of [14], which has identified two SO(5)-invariant fixed points: a massless point, described by the $SO(5)_1$ WZW theory and a massive point corresponding to the so-called SO(5) Gross-Neveu (GN) field theory.

In what follows, we shall first focus on the critical $SO(5)_1$ WZW theory, where we establish the existence of non-Abelian electrons and give full quantitative results for their exclusion statistics. After that, we consider the massive SO(5) GN theory and argue that also in this massive phase non-Abelian electrons are present. We shall also explain how these excitations can be probed in photoemission-type experiments.

Through non-Abelian bosonization, the critical $SO(5)_1$ WZW theory is equivalent to a theory of five free fermions ψ_a. These fermions carry the quantum numbers of the SO(5) vector n_a, Eq. (8), which was formed as a bilinear in the original electron operators on the ladder.

In close analogy to the spinon and/or holon formulation of the free electron CFT, we here propose a formulation of the $SO(5)_1$ WZW theory in terms of fundamental quasiparticles ϕ that transform in the (four-dimensional) spinor representation of SO(5). Inspecting the quantum numbers for spin and charge, one finds (see Fig. 2) that the four spinor components carry the quantum numbers $S^z = \pm\frac{1}{2}$, $q = \pm e$ of a free electron:

$$(\phi_{(\uparrow,e)}, \phi_{(\downarrow,e)}, \phi_{(\downarrow,-e)}, \phi_{(\uparrow,-e)}). \tag{10}$$

The exclusion statistics of spinor quasiparticles in the $SO(5)_1$ WZW theory have been obtained in our paper [3]. An important aspect of this analysis has been the fact that, group theoretically, a product of two SO(5) spinors contains both the vector representation and the singlet. This means that, for example, a state constructed from the vacuum by exciting two ϕ-quanta can be either an SO(5) singlet (i.e., a neutral, spinless excitation) or an SO(5) vector carrying the quantum numbers of the SO(5) order parameter n_a. If we consider a 4-ϕ state and insist that it forms an SO(5) singlet, we have contributions from two distinct "fusion channels" depending on the nature of the intermediate state with the first two ϕ-quanta in place. When building the full spectrum in terms of ϕ quanta, one has to specify a fusion channel for each multi-ϕ state, and this gives a degeneracy factor of 2 for every two ϕ-quanta that are added. The geometric average of these

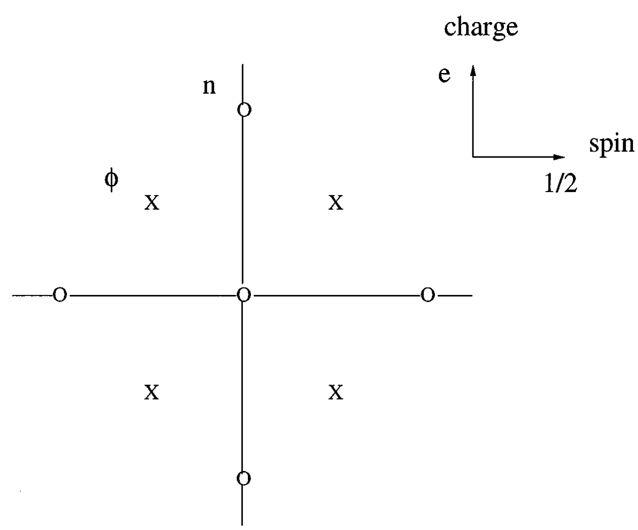

FIG. 2. SO(5) weight diagram, showing the quantum numbers of the vector (order parameter) representation n_a (open dots) and of the spinor representation $\phi_{(\sigma,\pm e)}$ (crosses).

degeneracy factors equals $\sqrt{2}$ per ϕ quantum and this factor ends up in the Boltzmann tail of the generalized distribution for these quasiparticles.

Summarizing, we see that the spinor quasiparticles in the $SO(5)_1$ theory, which carry the quantum numbers of a free electron, obey a form of non-Abelian exclusion statistics and are properly called non-Abelian electrons.

In our paper [3], we obtained quantitative results for the exclusion statistics of $SO(5)_1$ spinor quasiparticles. These results can be phrased in terms of a single-level partition sum $\lambda(x)$, with x keeping track of the various chemical potentials and energy. For the case where one gives equal weight to all four components, $x = e^{\beta(\mu-\epsilon)}$, with μ an overall chemical potential, $\lambda(x)$ is a solution of

$$\lambda^{\frac{3}{2}} - (2 + 3x^2)\lambda + (3x^2 - 1)(x^2 - 1)\lambda^{\frac{1}{2}} - x^2(x^2 - 1)^2 = 0. \quad (11)$$

The generalized Fermi-Dirac distribution is given by

$$n_\phi(\epsilon) = [x\partial_x \log \lambda(x)](x = e^{\beta(\mu-\epsilon)}). \quad (12)$$

From (11) one immediately derives the properties

$$\lambda(x) \sim x^4 \quad \text{for } x \gg 1, \quad (13)$$

which implies a maximum $n_\phi^{\max} = 1$ for the occupation per level of each of the ϕ quanta, and

$$\lambda(x) = 1 + 4\sqrt{2}\,x + \mathcal{O}(x^2) \quad \text{for } x \ll 1, \quad (14)$$

which implies the prefactor $\sqrt{2}$ in the Boltzmann tail of $n_\phi(\epsilon)$.

Comparing the central charge c^{CFT} and the spin and charge susceptibilities χ^s and χ^c to those of standard free electrons, Eq. (2), one finds

$$c_\phi^{\mathrm{CFT}}/c_e^{\mathrm{CFT}} = 5/4, \qquad \chi_\phi^s/\chi_e^s = \chi_\phi^c/\chi_e^c = 2. \quad (15)$$

These results are readily checked in a picture with five fundamental fermions. Note that the deviations from the free-electron results are not Fermi-liquid effects but manifestations of the unusual statistics of the non-Abelian electrons.

In the massive SO(5) GN model, the SO(5) vector and spinor excitations exist in the form of massive particles with mass ratio [15]

$$M_\psi = \sqrt{3}\,M_\phi. \quad (16)$$

The ϕ particles, which are the lightest particles in the theory, may be viewed as "kinks" that carry a topological charge related to the SO(5) group theory [15]. In principle, one may derive the thermodynamics of the ϕ particles by starting from an exact scattering matrix and applying the thermodynamic Bethe ansatz. To our knowledge this procedure has not been carried out, but it is clear that also in this massive theory a Boltzmann tail prefactor equal to $\sqrt{2}$ will arise. (The detailed form of the distribution n_ϕ will differ between the critical and the massive cases.) We conclude that also in the massive case the terminology non-Abelian electrons is justified. We remark that in the massive SO(8) phase of two-leg ladder models [14] there are no excitations with non-Abelian statistics.

The physical processes by which non-Abelian electrons can be excited inside the SO(5) ladders, both in the critical and in the massive phase, are easily described and can possibly be realized in experiments. The physical setup is a photoemission process, where a high energy photon ejects an electron from the ladder. After such a process, or its inverse, one of the rungs in the ladder violates the SO(5) superspin condition, and the ladder finds itself in an excited state with the quantum numbers of a free electron. As we discussed, such excited states correspond to the ϕ quanta, i.e., to the non-Abelian electrons.

With regards to the prospect of experimental detection of effects of non-Abelian statistics, there is the complication that, to establish either the critical $SO(5)_1$ WZW phase or the massive SO(5) GN phase, one needs a fine-tuning of parameters which is easily destroyed by relevant perturbations. One may argue, however, that some of the qualitative features, possibly including the effects of non-Abelian statistics, of the GN phase persist in other massive phases [16]. Clearly, this important issue deserves further study.

The authors thank A. de Visser, S. Guruswamy, R. Konik, B. Nachtergaele, S. Sachdev, J. Zaanen, and F. Wilczek for discussions. This research was supported in part by the Australian Research Council and by the foundation FOM of the Netherlands.

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