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Systems of matrix Riccati equations, linear fractional transformations, partial integrability and synchronization

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Systems of matrix Riccati equations, linear fractional transformations, partial integrability and synchronization

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We partially integrate a system of rectangular matrix Riccati equations which describe the synchronization behaviour of a nonlinear complex system of N globally connected oscillators. The equations take a restricted form in which the time-dependent matrix coefficients are independent of the node. We use linear fractional transformations to perform the partial integration, resulting in a system of reduced size which is independent of N, generalizing the well-known Watanabe-Strogatz reduction for the Kuramoto model. For square matrices the resulting constants of motion are related to the eigenvalues of matrix cross ratios, which we show satisfy various properties such as symmetry relations. For square matrices the variables can be regarded as elements of a classical Lie group, not necessarily compact, satisfying the matrix Riccati equations. Trajectories lie either within, or on the boundary, of a classical domain and we show by numerical example that complete synchronization can occur even for the mixed case. Provided that certain unitarity conditions are satisfied, we extend the definition of cross ratios to rectangular matrix systems and show that again the eigenvalues are conserved. Special cases are models with real vector unknowns for which trajectories lie on the unit sphere in higher dimensions, with well-known synchronization behaviour, and models with complex vector wavefunctions that describe synchronization in quantum systems, possibly infinite-dimensional.

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I. INTRODUCTION

Synchronization phenomena in physical, biological and other complex systems have been extensively investigated in a wide variety of contexts[1–6], including opinion dynamics[7] with related concepts of consensus[8]. In typical models the complex system consists of a large number of elements interacting nonlinearly over a network of connections, with each node comprising an independent dynamical system. The nodes become correlated in time as synchronization occurs, leading to the evolution of the system as a collective entity. In large systems with N nodes there are a correspondingly large number of differential equations to be analyzed and solved, with behaviours that are sensitive to underlying parameters such as coupling strengths, phase lag angles, time delays of interactions between the nodes, as well as the effects of dynamic network topologies. The investigation and analysis of these systems and their synchronization properties for large N remains an ongoing challenge, particularly if the underlying parameters take distributed values that vary with time.

The widely-studied Kuramoto-type models of synchronization[3,4,6,9], with its many extensions and generalizations[5,10], are useful because they are amenable to both numerical and analytic investigations, while at the same time displaying a diverse range of behaviours. In particular, a special case of the Kuramoto model with identical frequencies ω, but of arbitrary size N, can be partially integrated by means of the Watanabe-Strogatz (WS) transform[11–15], which reduces the number of independent equations from N to only 3. This allows for a much more detailed investigation of synchronization properties as a function of the underlying parameters[16,17].

The purpose of this paper is to generalize such models and their property of partial integrability to systems where the variable Z_i at each node i of the network is a rectangular matrix of arbitrary size, by firstly formulating the defining differential equations as a system of cross-coupled matrix Riccati equations with time-dependent coefficients that are independent of the node. Whereas Kuramoto models and their higher-dimensional extensions generally have cubic nonlinearities, in special cases these can be regarded as quadratic nonlinearities with time-dependent coefficients, and therefore take a Riccati form. These equations can be partially integrated by means of linear fractional transformations, leaving a reduced system of a size that depends only on the matrix dimensions, independent of N. We find that for square matrices and also for unitary rectangular systems, the constants of integration are related to the eigenvalues of matrix cross ratios.

Special cases of interest are firstly square matrix variables Z_i which are elements of a classical Lie group, such as $GL(d, \mathbb{C})$ or the compact unitary and orthogonal subgroups, which have previously been investigated in detail[18–22]. It is known that complete synchronization can occur in these models for identical frequency matrices[18,19], i.e. $Z_i Z_j^{-1}$ approaches the identity matrix exponentially quickly for all $i, j = 1, \ldots N$ as the system evolves, for restricted initial values. The second case of special interest, also extensively investigated, is that in which the variables are real vectors that lie on the unit sphere in any dimension, for which complete synchronization again occurs[20,23–27]. Partial integration has previously been performed for these models[28,29], although the reduced equations differ from those derived here, see Sec. VII C; evidently the method of partial integration is not unique. Partial integration extends to general systems of rectangular matrices under the restricted circumstances mentioned such as identical frequency matrices, and we may then in principal use the reduced equations to derive detailed properties of the trajectories as has been done for the Kuramoto system[17]. Here, we focus on the methods of partial integration and the subsequent reduced equations, along with properties of linear fractional transformations, matrix cross ratios and the constants of motion, and it remains to show how to use the reduced equations to analyze the specific behaviour of the system for various underlying parameters.

A. Matrix Riccati equations

There is an extensive body of literature on Riccati equations for rectangular matrices of arbitrary size $p \times q$, with applications to random processes, optimal control, and diffusion problems, as well as to other engineering science applications such as robust stabilization, and network synthesis[30–34], see also several reviews[35–38]. We refer in particular to Ref. 31 for a comprehensive development with further references, although the applications are in these cases to single matrix Riccati equations, sometimes with constant coefficients.

By contrast, we consider cross-coupled Riccati matrix systems of unlimited size N with coefficients of arbitrary time-dependence and these, it appears, have not previously been investigated, either in general or for the specific application to synchronized systems, except for the Kuramoto model[13,39] and the n-sphere generalizations[28,29]. Although the partial integration that we outline in Sec. III has not been previously described in its generality, many known results for a single matrix Riccati equation using for example linear fractional transformations and matrix cross ratios, together with corresponding group properties and classical domains, apply also to our matrix Riccati systems. This is to be expected in view of the fact that the properties of a single Riccati matrix equation are a special case of those for a system of N equations, since one can always replicate a single matrix equation N times to form a system of N uncoupled equations, with different initial values for each node. Such equations can be solved sequentially, whereas our equations can only be solved simultaneously due to the nonlinear cross couplings. We provide independent derivations of the properties of linear fractional transformations and matrix cross ratios, which in some cases are similar or equivalent to those previously derived for a single matrix Riccati equation, for example those of Levin[40] in 1959. In addition we derive properties that are specific to models of synchronization, such as constants of motion which are constructed from rectangular matrices, as applied to models of quantum synchronization, or to the n-sphere models (see Sec. VII C).

Hermitean and symmetric Riccati differential equations, as defined for example in Chapters 4 and 7 in Ref. 31, have previously been investigated for square matrices, whereas we define unitary and orthogonal Riccati systems in Sec. V A more generally for rectangular matrices. These are relevant to synchronized systems, and for square matrices are related to hermitean and symmetric Riccati systems by means of the Cayley transform. Properties of matrix cross ratios in rectangular unitary systems do not appear to have been previously derived.

The main restriction of our approach is that the coefficients of the Riccati system are uniform across the network, i.e. are independent of the node i, and only the N matrix variables Z_i depend on i as they evolve in time. The coefficient matrices can be constant in time (for example the frequency matrices), or else are constructed as linear combinations of the variables $Z_j, j = 1, \ldots, N$, possibly at a delayed time, and so are generally not fixed functions of time. Our system of equations (9) cannot be formulated as a single matrix Riccati equation unlike, for example, that investigated in Ref. 41.

Linear fractional transformations have previously been used to solve or simplify associated Riccati equations[35,40,42]. Riccati equations appear in Lie group theory as the equations of one parameter flows in a Lie group acting on Grassmann manifolds, where the group action is by means of a partitioned matrix which lies in $GL(p+q)$ and acts by means of linear fractional transformations[34,43]. For square matrix Riccati equations it is known that the eigenvalues of the matrix cross ratios are constants of motion[35,38,40], as we discuss in Sec. IV A. For a single Riccati equation the matrix cross ratio is formed from the solutions generated by distinct initial conditions whereas for the Riccati system that we consider, matrix cross ratios C_{ijkl} are formed from all variables Z_i across all distinct nodes $i, j, k, l = 1, \ldots N$. We show explicitly from the Riccati equations that the eigenvalues of C_{ijkl}, and hence the determinant and trace, are constants of motion by deriving a differential equation satisfied by C_{ijkl}, generalizing a property obtained by Whyburn[44] (Theorem V) as long ago as 1934 for a single matrix Riccati equation, see also Levin[40]. Properties of matrix cross ratios are well-known from the early works of Siegel[45] and Hua[46,47] in 1943 and 1945, and their invariance under linear fractional transformations has been analyzed in the context of Riccati equations[34]. It was noted by Siegel[45] that for two cross ratio matrices with the same eigenvalues there exists a linear fractional transformation belonging to the symplectic group which relates the two cross ratios. We extend the definition of cross ratios for square matrices to rectangular matrix Riccati systems, for rectangular unitary systems as defined in Sec. VII, and again show directly that the eigenvalues of the cross ratio matrices are constants of motion.

Related to properties of linear fractional transformations are those of bounded classical domains, defined in our context as the space of $p \times q$ complex matrices Z_i restricted by the condition that $I_p - Z_i Z_i^\dagger > 0$ (meaning that $I_p - Z_i Z_i^\dagger$ is positive definite) for each $i = 1, \ldots N$. There are four families of classical domains[48] of which type I, which is diffeomorphic to $U(p, q)/U(p) \times U(q)$, is relevant to our models of synchronization. The group $U(p, q)$ acts by means of linear fractional transformations, as we show explicitly in Sec. V D. It remains an open question as to whether the other classical domains also lead to matrix Riccati systems with synchronization properties. We allow trajectories to lie either inside the classical domain, i.e. with $I_p - Z_i Z_i^\dagger > 0$, or else on the boundary with $Z_i Z_i^\dagger = I_p$. Synchronization occurs even in the mixed case, where some nodes i have trajectories inside the classical domain and others are confined to the boundary, as we demonstrate by numerical example in Secs. II A and VII C.

For the Kuramoto model, which we review in Sec. II, the WS transform is equivalent to the Möbius transformation which maps the unit circle to itself[11–15] and also preserves the unit disk, and is an example of a linear fractional transformation[13,39] applied to the complex variables z_i. The classical domain in this case is the unit disk on which $SU(1, 1)$

acts transitively by means of the Möbius transformation, and trajectories lie either on the unit circle or within the unit disk, depending on the initial values. The cross ratios C_{ijkl} are constants of motion for all distinct nodes $i, j, k, l = 1, \ldots N$ and are invariant under linear fractional transformations.

B. Summary

In Sec. II we review the Kuramoto model and the WS transform as the simplest example of the method of partial integration, together with properties of linear fractional transformations (the Möbius map), cross ratios, and the identification of the classical domain as the unit disk. We show by numerical example that the system can synchronize whether trajectories lie either inside the unit disk, or on the boundary, or both. In Sec. III we consider a general Riccati system of N rectangular matrix equations which we partially integrate and although the derivation is elementary (Theorem 2), this leads to a significant reduction to a set of equations which in number is independent of N. In Sec. IV we restrict our attention to square matrix systems of size $d \times d$ for which synchronization is known to occur[19], and develop properties of matrix cross ratios, the eigenvalues of which are constants of motion. The existence of these nontrivial constants of motion explains why the system is partially integrable. In Sec. V the system is furthered restricted to unitary matrices which we relate to the well-studied hermitean Riccati matrix equation. The matrix cross ratios lead in special cases to constants of motion for synchronization models on the unit sphere (Corollary 8). We show directly in Sec. V D that trajectories lie in the classical domain $U(d, d)/U(d) \times U(d)$.

In Sec. VI we further restrict the system to orthogonal matrices, firstly to point out that matrix cross ratios are defined only for even d, and secondly to show that properties of partial integration extend to noncompact orthogonal groups. As an example we consider the Lorentz group $SO(1, 1)$ and the associated hyperbolic Kuramoto equations, which are known to describe synchronization of relativistic systems in Minkowski space[49]. We find a hyperbolic form of the WS transform and derive the relevant classical domain which in this case is unbounded. In Sec. VII we return to the rectangular matrix system, restricted to satisfy unitarity conditions, which are of interest because they include the case of complex vector unknowns which can be regarded as quantum wavefunctions. The matrix equations describe quantum synchronization, which has been well-studied[50–56], and we show in particular in

Sec. VII B that partial integration formally extends to infinite-dimensional equations which constitute a nonlinear system of Schrödinger equations. For real vectors the system has trajectories which lie on the unit sphere or within the unit ball, and we show by numerical example that synchronization can occur even for the mixed case (Sec. VII C). A point of particular interest is that the method of partial integration in Sec. VII C, derived using linear fractional transformations, differs from that derived specifically for the n-sphere models[28]; the relationship between these two methods evidently deserves further investigation. We conclude with a summary and final remarks in Sec. VIII.

II. REVIEW OF THE WS TRANSFORM AND EXTENDED KURAMOTO MODELS

The WS transform was introduced by Watanabe and Strogatz[11,12] in 1993 as a trigonometric substitution in order to solve systems of globally coupled oscillators described by the Kuramoto model, but was later seen to be equivalent to the Möbius group of transformations which map the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto itself[13,15]. We refer to Ref. 5 for a summary of the Watanabe-Strogatz theory, also to Ref. 28, Section 1.1, which outlines recent developments and applications of the theory, and generalizes the WS transform to mappings of the unit ball $\mathbb{B}^d = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| < 1\}$ in any dimension d . The Kuramoto model with identical frequencies takes the well-known form

$$\dot{\theta}_i = \omega + \frac{1}{N}\sum_{j=1}^{N} \lambda_j \sin(\theta_j - \theta_i), \quad i = 1, \dots N, \tag{1}$$

where the frequency ω of oscillation is fixed across all nodes, and where we have included multiplicative parameters λ_j. One could also include arbitrary node-dependent phase lag angles α_j as well as a time delays $\tau_j > 0$ for the interacting variables θ_j. Although the WS transform applies to such general models, it is nevertheless restricted in that, as well as requiring identical frequencies, the coupling constant κ must be uniform across all nodes, at least in magnitude, and the network topology must be complete, i.e. we allow only global connectivity. While these restrictions are severe, they result in a system which is expressible in Riccati form and can be partially integrated and analyzed in detail[16,17].

We perform the WS transform on (1) by introducing N angles $\xi_i(t)$ and a complex variable

$z(t)$ according to

$$e^{i\theta_i} = \frac{z + e^{i\xi_i}}{1 + \overline{z}\, e^{i\xi_i}}, \tag{2}$$

then the N equations for ξ_i which follow by substitution into (1) are linear and can be integrated directly, leaving only 3 real equations which govern the time evolution of the system. The mapping (2) is an automorphism of the unit circle which extends to automorphisms of the unit disk, generating the 3-dimensional Möbius group[13,14]. These transformations generalize in higher dimensions to linear fractional transformations which are automorphisms of the group manifolds of the classical Lie groups.

A. Linear fractional transformations and Riccati equations

We rewrite (1) as a system of Riccati equations for $z_i = e^{i\theta_i}$, but more generally we consider a system in which $z_i \in \mathbb{C}$ evolves from the initial value $z_i(0) = z_i^0$ according to

$$\dot{z}_i = i\,\omega\, z_i + \Gamma_1 - z_i^2\, \Gamma_2, \qquad i = 1, \dots N, \tag{3}$$

where $\omega, \Gamma_1, \Gamma_2$ are complex functions of t, independent of i. The Kuramoto model (1) is obtained by setting $z_i = e^{i\theta_i}$ together with $\Gamma_1 = \overline{\Gamma_2} = \sum_j \lambda_j z_j/(2N)$, as previously observed[13,39]. The system (3) in this case allows solutions that can lie inside the unit disk as well as on the unit circle, depending on the initial values z_i^0. Synchronization is measured by the order parameter $r = |\sum_j z_j/N|$, with $r \to 1$ for complete synchronization, for which we have $|z_i - z_j| \to 0$ as $t \to \infty$. Numerically we find that this indeed occurs for random initial values z_i^0, provided that $\sum_j \lambda_j > 0$ and more generally, complete synchronization occurs provided that $\sum_j \lambda_j \cos\beta_j > 0$, where β_j are phase lag angles[16,17]. Rigorous results have been obtained for the standard Kuramoto model in which all trajectories lie on the unit circle[22,57,58].

If we allow some of the parameters λ_i to be negative with $\sum_j \lambda_j < 0$ then the system can settle into asymptotic configurations consisting either of asynchronous or bipolar states. This is already known to occur for trajectories on the unit circle[17], and here we provide numerical examples to show that this occurs also for trajectories in the unit disk. In Fig. 1 we choose $N = 15$ with $\omega = 0$, and plot 12 trajectories in the unit disk, with the remaining 3 confined to the unit circle. The initial values θ_i^0 and the parameters $\lambda_i \in [-1, 1]$ are generated randomly. The trajectories are shown in red, the initial points in blue, and the final locations for which asymptotic configurations have been attained are marked in black. In (a) we have $\sum_j \lambda_j > 0$ and complete synchronization of all nodes occurs, i.e. $z_i \to \mathrm{e}^{\mathrm{i}\phi}$ for some fixed angle ϕ for all nodes i. For (b) we have $\sum_j \lambda_j < 0$, and in this case synchronization does not occur, but the nodes are phase-locked into an asynchronous configuration, with some nodes remaining well inside the unit circle.

In (c) we solve the Kuramoto system (1) with distributed frequencies ω_i and for $\sum_j \lambda_j > 0$, and observe that phase-locked synchronization occurs, with the final configuration similar to those well-known for the simple Kuramoto model. The only difference is that here some of the trajectories start inside the unit circle. Although this system is not expressible in the Riccati form (3) and so is not partially integrable, we present this example in order to demonstrate that synchronization can still occur for general extended Kuramoto models whether trajectories start inside or on the unit circle, possibly in mixed form.

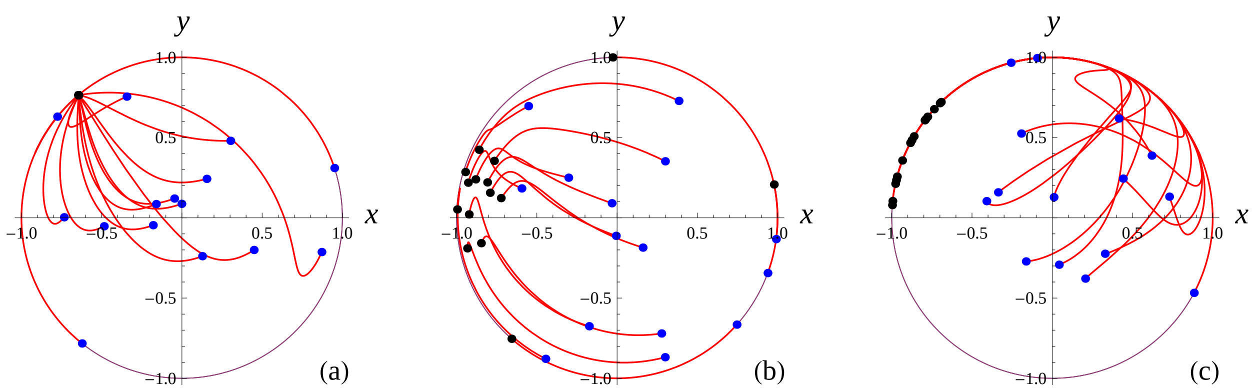

FIG. 1. Trajectories $z_i(t)$ (red) in the unit disk and on the unit circle, with initial values marked in blue, final values in black, for $N = 15, \omega = 0$ with 3 nodes restricted to the unit circle. $\sum_j \lambda_j$ is positive for (a), in which case complete synchronization occurs, and negative for (b). For (c) we select distributed frequencies ω_i with $\sum_j \lambda_j > 0$, and phase-locked synchronization occurs.

Now let us briefly review how the system (3) can be partially integrated[13]. Firstly, we observe that the cross ratios

$$(z_i, z_j, z_k, z_l) = \frac{(z_i - z_k)(z_j - z_l)}{(z_i - z_l)(z_j - z_k)} \tag{4}$$

are conserved functions of t for any distinct indices $i, j, k, l = 1, \ldots N$ and for arbitrary complex functions $\omega(t), \Gamma_1(t), \Gamma_2(t)$, as follows by direct computation using (3). Secondly, these cross ratios are invariant under linear fractional transformations defined by

$$z_i \to g(z_i) = \frac{az_i + b}{cz_i + d}, \quad i = 1, \dots N, \tag{5}$$

where $ad - bc \neq 0$, and where the complex coefficients a, b, c, d, as well as z_i, are time-dependent. We can use these linear fractional transformations therefore to transform the system (3) whilst leaving the constants of motion (4) invariant.

Define

$$\beta = \frac{b}{d}, \quad \gamma = \frac{c}{a}, \quad \zeta_i = \frac{a\, z_i}{d}, \tag{6}$$

(assuming that $ad \neq 0, \beta\gamma \neq 1$), then we rewrite (5) in terms of ζ_i, β, γ and substitute

$$z_i = \frac{\beta + \zeta_i}{1 + \gamma\, \zeta_i} \tag{7}$$

into (3). The resulting equations are solved by:

$$\dot{\beta} = \mathrm{i}\beta\, \omega + \Gamma_1 - \beta^2\Gamma_2, \qquad \dot{\gamma} = -\mathrm{i}\gamma\, \omega + \Gamma_2 - \gamma^2\Gamma_1, \qquad \dot{\zeta}_i = (\mathrm{i}\, \omega + \gamma\Gamma_1 - \beta\Gamma_2)\, \zeta_i,$$

as is verified by direct substitution. The first two of these are themselves Riccati equations and the last, which is linear, is partially integrated by writing $\zeta_i = \zeta_i^0 \exp \mathrm{i}\alpha$, where α is a complex function satisfying $\dot{\alpha} = \omega - \mathrm{i}\, \gamma\Gamma_1 + \mathrm{i}\, \beta\Gamma_2$ with $\alpha(0) = 0$. The initial values β_0, γ_0 for β, γ are arbitrary, and ζ_i^0 is obtained from (7) by imposing $z_i(0) = z_i^0$. The system (3) of N complex equations is therefore reduced to just three equations for the complex variables α, β, γ.

A special case of (3) is when ω is real and $\Gamma_1 = \overline{\Gamma}_2$, which we refer to as a unitary system (since the variables z_i are unitary for all $t > 0$ provided initially they are unitary), and is related to hermitean Riccati systems as discussed in Sec. V A. In this case we can choose $\beta = \overline{\gamma} = z$, and so (7) agrees with (2) upon identifying $\zeta_i = \mathrm{e}^{\mathrm{i}\xi_i}$ and $z_i = \mathrm{e}^{\mathrm{i}\theta_i}$. Such a system is the Kuramoto model for which $\Gamma_1 = \overline{\Gamma}_2 = \sum_j \lambda_j z_j/(2N)$, and in this case α is real.

The transformation (5) is a group transformation in which $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$. For unitary Riccati systems, g preserves the unit circle provided that $|g(z_i)| = 1$ whenever $|z_i| = 1$, which holds provided that

$$|a|^2 - |c|^2 = 1, \quad |d|^2 - |b|^2 = 1, \quad \overline{c}\, d = \overline{a}\, b, \quad \overline{d}\, c = \overline{b}\, a, \tag{8}$$

which in turn implies that $g \in U(1, 1)$. It follows from these relations that for any $g \in U(1, 1)$ we have $|cz_i + d|^2(1 - |g(z_i)|^2) = 1 - |z_i|^2$. Hence, if $|z_i| < 1$ for any node i, then $|g(z_i)| < 1$

and so the transformation $z_i \to g(z_i)$ preserves both the unit disk and the unit circle[48], and (7) is equivalent to the Möbius transformation[13]. Trajectories $z_i(t)$ of the system either lie entirely inside the unit disk or on the unit circle, depending on the initial value z_i^0 for each node i.

From (6) we have $b = dz, c = a\overline{z}$ which implies from (8) that $|a|^2(1-|z|^2) = 1 = |d|^2(1-|z|^2)$. Hence $|z| < 1$, and also $|a| = |d|$, and so from (6) we have $|\zeta_i| = |z_i|$. We therefore have $a = \mathrm{e}^{\mathrm{i}\psi}/\sqrt{1-|z|^2}$ and $d = \mathrm{e}^{\mathrm{i}\varphi}/\sqrt{1-|z|^2}$ for angles ψ, φ, and so $g \in U(1,1)$ factorizes according to

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix},$$

which is the product of an $SU(1,1)$ matrix and an element of $U(1) \times U(1)$. The group $SU(1,1)$ acts transitively on the unit disk, which is diffeomorphic to $U(1,1)/U(1)\times U(1)$, and is an example of a bounded classical domain[48]. Such classical domains generalize to bounded open subsets in $\mathbb{C}^n$ on which the corresponding groups act transitively. The trajectories of the variables in higher-dimensional systems also lie either within the classical domain, or on the boundary. We consider in particular synchronization models for which the classical domain is the set of $p \times q$ complex matrices Z_i such that $I_q - Z_i^\dagger Z_i$ is positive definite for each i, which is diffeomorphic to $U(p,q)/U(p) \times U(q)$, where I_q denotes the $q \times q$ identity matrix.

III. RECTANGULAR MATRIX RICCATI SYSTEMS

We consider a globally connected network of N nodes in which the variable Z_i, a $p \times q$ complex matrix, is located at the ith node and evolves according to the equation:

$$\dot{Z}_i = \Gamma_1 + \mathrm{i}\,\Omega_1 Z_i + \mathrm{i}\,Z_i \Omega_2 - Z_i \Gamma_2 Z_i, \quad i = 1, \dots N, \tag{9}$$

where the matrix dimensions are given by $Z_i : p \times q, \Gamma_1 : p \times q, \Omega_1 : p \times p, \Omega_2 : q \times q, \Gamma_2 : q \times p$. The coefficient matrices $\Gamma_1, \Gamma_2, \Omega_1, \Omega_2$, depend on t, not necessarily as fixed functions of t, but through the time-dependent variables Z_j for $j = 1, \dots N$. The system (9) comprises therefore a set of N cross-coupled nonlinear equations. We specify initial values $Z_i(0) = Z_i^0$, and assume that solutions exist at least locally. Equations (9) reduce to (3) for $p = 1 = q$, with $\omega = \Omega_1 + \Omega_2$.

As examples of coefficient matrices which are used in models of synchronization, Ω_1, Ω_2 are typically constant hermitean matrices, which therefore have real eigenvalues which can be regarded as frequencies of oscillation, and $\Gamma_1 = \sum_{j=1}^{N} a_j Z_j b_j$ where a_j, b_j are any fixed set of $p \times p$ and $q \times q$ complex matrices, respectively, together with $\Gamma_2 = \sum_{j=1}^{N} c_j Z_j^{\dagger} d_j$ where c_j, d_j are also any set of $q \times q$ and $p \times p$ complex matrices, respectively, where $Z_j^{\dagger}$ denotes the hermitean conjugate of Z_j. For the case $p = q = d$ of square matrices an alternative choice is $\Gamma_2 = \sum_{j=1}^{N} c_j Z_j^{-1} d_j$ where the variables Z_j are elements of the general linear group, i.e. invertible square matrices. For the general models considered in Ref. 19 (equation 1.3) we choose $H_i = \Omega_1$ to be independent of i, then we obtain (9) with $\Omega_2 = 0$ and $\Gamma_1 = K \sum_j Z_j/(2N), \Gamma_2 = K \sum_j Z_j^{-1}/(2N)$. Solutions for these particular equations exist locally (Proposition 2.1[19]), and also globally provided that the initial values are suitably restricted. Under these conditions the solutions completely synchronize, meaning that $Z_i(t)Z_j(t)^{-1} \to I_d$ for all $i, j = 1, \dots N$ as $t \to \infty$. (Theorem 4.2[19]).

A. Linear fractional transformations

In order to partially integrate (9) we define the linear fractional transformations

$$Z_i \to g(Z_i) = (AZ_i + B)(CZ_i + D)^{-1}, \tag{10}$$

where the dimensions of the complex matrices A, B, C, D are given by $A : p \times p,\ B : p \times q,\ C : q \times p,\ D : q \times q$. Hence $CZ_i + D$ is a square matrix of dimension $q \times q$, which we assume to be invertible, and $AZ_i + B$ is of dimension $p \times q$. Define the $(p+q) \times (p+q)$ matrix g by

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{11}$$

then successive linear fractional transformations (10), in which $Z_i \to g_1(Z_i) \to g_1(g_2(Z_i))$, are equivalent to a single linear fractional transformation $Z_i \to g_1 g_2(Z_i)$, where $g_1 g_2$ denotes the matrix product for the matrices (11). Hence successive linear fractional transformations correspond to group composition, and so we view (10) as a group transformation with $g \in GL(p+q, \mathbb{C})$, provided that g is invertible. We note the following decomposition of g as a product of lower, diagonal, and upper triangular matrices:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ CA^{-1} & I_q \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I_p & A^{-1}B \\ 0 & I_q \end{pmatrix}, \tag{12}$$

provided that A^{-1} exists. Hence g is invertible provided that $\det g = \det A \det(D - CA^{-1}B)$ is nonzero. The inverse transformation g^{-1} can be deduced from this decomposition, or directly from (10).

Next, we define (generalizing (6)):

$$\zeta_i = AZ_iD^{-1}, \quad \beta = BD^{-1}, \quad \gamma = CA^{-1}, \tag{13}$$

where the complex matrices ζ_i, β, γ are of size $p \times q, p \times q, q \times p$ respectively. The transformation (10) now reads:

$$Z_i \to (\zeta_i + \beta)\,(\gamma\zeta_i + I_q)^{-1}\,. \tag{14}$$

We substitute for Z_i into (9), and therefore replace the N equations for Z_i by $N+2$ equations for the matrices ζ_i, β, γ, which leaves the degrees of freedom residing in the two matrix variables β, γ yet to be fixed; only then is ζ_i determined uniquely for all i. These extra degrees of freedom arise from the group properties of the linear fractional transformations (10) with respect to $GL(p+q, \mathbb{C})$. If we choose $\beta = 0 = \gamma$, we regain Z_i, however we now define β, γ as the solutions of specific equations, from which the equations for ζ_i follow:

Lemma 1. *Equations (9) are satisfied by* $Z_i = (\zeta_i + \beta)\,(\gamma\zeta_i + I_q)^{-1}$ *provided that*

$$\dot{\beta} = \Gamma_1 + \mathrm{i}\,\Omega_1\beta + \mathrm{i}\,\beta\,\Omega_2 - \beta\Gamma_2\beta, \tag{15}$$

$$\dot{\gamma} = \Gamma_2 - \mathrm{i}\,\gamma\,\Omega_1 - \mathrm{i}\,\Omega_2\gamma - \gamma\Gamma_1\gamma, \tag{16}$$

$$\dot{\zeta}_i = (\Gamma_1\gamma + \mathrm{i}\,\Omega_1)\zeta_i - \zeta_i(\Gamma_2\beta - \mathrm{i}\,\Omega_2). \tag{17}$$

Proof. The proof is by direct substitution. We choose the equation for β by requiring that (9) be satisfied to lowest order in ζ_i which, in effect, means we set $\zeta_i \to 0$ in (14) and hence we simply replace $Z_i \to \beta$ in (9), which gives (15). Next, we substitute (14) into (9), and postmultiply both sides by $(\gamma\zeta_i + I_q)$. Inverse matrices are differentiated using the identity $d/dt(A^{-1}) = -A^{-1}\dot{A}A^{-1}$, for any invertible square matrix A. Then we substitute for $\dot{\beta}$ from (15). The resulting equation can be manipulated into the form:

$$\begin{aligned}
\dot{\zeta}_i &= \Gamma_1\,(\gamma\zeta_i + I_q) + \mathrm{i}\,\Omega_1\,(\zeta_i + \beta) - \Gamma_1 - \mathrm{i}\,\Omega_1\,\beta - \mathrm{i}\,\beta\,\Omega_2 + \beta\,\Gamma_2\,\beta \\
&\quad + (\zeta_i + \beta)\,(\gamma\zeta_i + I_q)^{-1}\left[\mathrm{i}\Omega_2\,(\gamma\zeta_i + I_q) + \dot{\gamma}\zeta_i + \gamma\dot{\zeta}_i - \Gamma_2\,(\zeta_i + \beta)\right] \\
&= \Gamma_1\gamma\zeta_i + \mathrm{i}\,\Omega_1\,\zeta_i + \mathrm{i}\,\zeta_i\,\Omega_2 - \zeta_i\Gamma_2\beta \\
&\quad + (\zeta_i + \beta)\,(\gamma\zeta_i + I_q)^{-1}\left[-\mathrm{i}\,\gamma\zeta_i\Omega_2 + \mathrm{i}\Omega_2\gamma\zeta_i + \dot{\gamma}\zeta_i + \gamma\dot{\zeta}_i - \Gamma_2\zeta_i + \gamma\zeta_i\Gamma_2\,\beta\right],
\end{aligned} \tag{18}$$

and we then set the quantity in brackets to zero. The remaining terms in this equation lead directly to (17). On back-substituting for $\dot{\zeta}_i$ into the term in brackets in (18), and setting it to zero, we obtain (16). □

Evidently (15,16) are themselves Riccati equations if we regard the coefficients as having a fixed time-dependence. Similarly, (17) is linear in ζ_i and is sometimes referred to as a system of Sylvester differential equations (in homogeneous form), see Ref. 31, Chapter 1, equation (1.7). The fact that solutions of the matrix Riccati equation (9), for any fixed i, can be associated with a linear matrix system is well-known, and is sometimes known as Radon's Lemma (see Theorem 3.1.1[31]). We emphasize, however, that in applications to models of synchronization the coefficient matrices $\Gamma_1\gamma + \mathrm{i}\,\Omega_1$ and $\Gamma_2\beta - \mathrm{i}\,\Omega_2$ are not actually fixed functions of time, but depend on t through the unknowns Z_j by means of expressions such as $\Gamma_1 = \sum_{j=1}^{N} a_j Z_j b_j$, as described above and hence, upon substituting for Z_j using (14), are in fact nonlinear functions of β, γ, ζ_j. Nevertheless, we may perform N integrations to solve for ζ_i by regarding (17) as a linear system. Firstly, we define the $p \times p$ complex matrix S and the $q \times q$ complex matrix T by means of:

$$\dot{S} = (\Gamma_1\gamma + \mathrm{i}\,\Omega_1)\, S, \quad S(0) = I_p, \quad \dot{T} = (\Gamma_2\beta - \mathrm{i}\,\Omega_2)\, T, \quad T(0) = I_q, \tag{19}$$

which are square matrix equations of size $p \times p$ and $q \times q$ respectively. Hence, if we regard the coefficients as fixed functions of t, then S, T are principal matrix solutions which exist over any interval for which the coefficient matrices exist. Despite the fact that (17) is nonlinear in the variables $\zeta_j, j = 1, \ldots N$, we can now solve for ζ_i for each i in terms of S, T as follows:

Theorem 2. *1. The solution of (17) with $\zeta_i(0) = \zeta_i^0$ is*

$$\zeta_i = S\,\zeta_i^0\, T^{-1}, \tag{20}$$

where S, T are determined by (19).

2. The solution of the N matrix equations (9) with initial values $Z_i(0) = Z_i^0$ is

$$Z_i = \left(S\zeta_i^0 + \beta T\right)\left(\gamma S\zeta_i^0 + T\right)^{-1}, \tag{21}$$

where β, γ are determined by (15,16) with arbitrary initial values $\beta_0 = \beta(0), \gamma_0 = \gamma(0)$, and where the initial values ζ_i^0 are given in terms of Z_i^0 by:

$$\zeta_i^0 = -\left(Z_i^0\gamma_0 - I_p\right)^{-1}\left(Z_i^0 - \beta_0\right). \tag{22}$$

Proof. 1. We firstly verify that (20) satisfies (17) with $\zeta_i(0) = \zeta_i^0$, then by uniqueness this is the required solution:

$$\dot{\zeta}_i = \dot{S}\zeta_i^0 T^{-1} - S\zeta_i^0 T^{-1}\dot{T}T^{-1} = \dot{S}S^{-1}\zeta_i - \zeta_i \dot{T}T^{-1} = (\Gamma_1\gamma + \mathrm{i}\,\Omega_1)\zeta_i - \zeta_i(\Gamma_2\beta - \mathrm{i}\,\Omega_2),$$

together with $\zeta_i(0) = \zeta_i^0$, as required.

2. Eq. (21) follows by substituting (20) into (14). The relation between the initial values Z_i^0, ζ_i^0 is determined by setting $t = 0$ in (21), leading to (22).

□

Because we have introduced two extra variables β, γ, the system is not fully determined until we specify initial values $\beta(0) = \beta_0, \gamma(0) = \gamma_0$, which then fixes β, γ as the unique solutions of (15,16). The choice of β_0, γ_0 is arbitrary, reflecting the group transformation properties of the linear fractional transformations (10), although we do not demonstrate this explicitly. A convenient choice is $\beta_0 = 0 = \gamma_0$, for then we have from (22) $\zeta_i^0 = Z_i^0$.

In summary, by partial integration we have reduced the N matrix equations (9) to the four matrix equations (15,16,19) for β, γ, S, T, with associated initial values. For unitary systems, which we define in Secs. V A and VII, we may also choose $\beta = \gamma^\dagger$, which then leaves only three complex matrix equations to be solved. The partial integration is therefore useful for $N \geqslant 4$, but is particularly significant for large N; numerical simulations of synchronization models have been performed, for example, with $N > 25,000$[59]. In terms of matrix elements, (9) comprises pqN complex equations, whereas (15,16,19) comprise $(p+q)^2$ complex equations, independent of N.

In typical models of synchronization Ω_1, Ω_2 are constant matrices, and Γ_1 is a complex linear combination of all elements Z_j, for $j = 1, \dots N$, i.e.

$$\Gamma_1 = \sum_{j=1}^{N} a_j Z_j b_j = \sum_{j=1}^{N} a_j \left(S\zeta_j^0 + \beta T\right)\left(\gamma S\zeta_j^0 + T\right)^{-1} b_j, \tag{23}$$

where a_j, b_j are a fixed set of $p \times p$ and $q \times q$ complex matrices, respectively. This expresses Γ_1 in terms of the four unknowns β, γ, S, T, and similarly for $\Gamma_2 = \sum_j c_j Z_j^\dagger d_j$. Then (15,16,19) together with their corresponding initial values comprises a set of four coupled nonlinear equations for the unknowns β, γ, S, T. Having solved for β, γ, S, T we construct Z_i from (21).

IV. SYSTEMS OF SQUARE MATRICES AND COMPLETE SYNCHRONIZATION

The general results of Section III for rectangular matrices apply in particular to square matrices, and so the system can be partially integrated as before, however we now obtain explicit constants as the eigenvalues of cross ratio matrices, which restrict the possible dynamics of the system. In this section, therefore, we investigate in detail the properties of matrix cross ratios together with linear fractional transformations, which have been well-known since the work of Siegel[45] and Hua[47]. We set $p = q = d$ in (9) so that $Z_i, \Omega_1, \Omega_2, \Gamma_1, \Gamma_2$ are now $d \times d$ complex matrices satisfying

$$\dot{Z}_i = \Gamma_1 + \mathrm{i}\,\Omega_1 Z_i + \mathrm{i}\, Z_i \Omega_2 - Z_i \Gamma_2 Z_i, \quad i = 1, \dots N, \tag{24}$$

with $Z_i(0) = Z_i^0$, and we assume that the unknowns Z_i are invertible at all times.

A specific model of synchronization which is defined for a classical Lie group G, which could be noncompact[19], takes the form:

$$\dot{Z}_i = H_i Z_i + \frac{K}{2N} \sum_{j=1}^{N} \left(Z_j - Z_i Z_j^{-1} Z_i \right), \quad i = 1, \dots N, \tag{25}$$

where $Z_i \in GL(d, \mathbb{C})$, with initial values $Z_i(0) = Z_i^0 \in G$. The matrices H_i are elements of the Lie algebra of G, and therefore are also elements of $GL(d, \mathbb{C})$, and K is a coupling constant. Solutions of (25) exist locally (Proposition 2.1[19]), and global existence also follows provided that some *a priori* conditions are imposed (Proposition 2.2[19]). If G is compact the equations (25) admit a unique smooth global solution for any initial values $Z_i^0 \in G$ (Remark 2.4[19]). Phase-locked states are defined as configurations in which $Z_i(t) Z_j^{-1}(t)$ is constant for all $i, j = 1, \dots, N$, then it is known that asymptotic phase-locking occurs for sufficiently large $K > 0$, and for restricted initial conditions (Theorem 5.1[19]). Further results have recently been obtained for systems with nontrivial topologies[60].

Of particular interest here is the case of identical oscillators in which $H_i = H$ is independent of i, since then the system (25) takes the matrix Riccati form (24) with

$$\mathrm{i}\,\Omega_1 = H, \quad \Omega_2 = 0, \quad \Gamma_1 = \frac{K}{2N} \sum_{j=1}^{N} Z_j, \quad \Gamma_2 = \frac{K}{2N} \sum_{j=1}^{N} Z_j^{-1}. \tag{26}$$

In this case exponential complete entrainment occurs, i.e. $Z_i(t) Z_j^{-1}(t)$ approaches the identity matrix for all $i, j = 1, \dots, N$ exponentially fast as the system evolves (Theorem 4.2[19]).

As outlined in Section III, equations (25) can be partially integrated and so can be reduced to (15,16,17). A specific example is the well-studied Kuramoto model (1) with identical frequencies, which corresponds to (24) with $d = 1$ and $Z_i = \mathrm{e}^{\mathrm{i}\theta_i}$, together with the identifications (26) with $\Omega_1 = \omega$, and can be partially integrated by means of the WS transform as discussed in Sec. II.

A. Matrix cross ratios

For square matrices the partial integration described in Sec. III can be viewed as a consequence of the existence of conserved quantities for the system (24). We define the $d \times d$ matrix cross ratios C_{ijkl} for $i, j, k, l = 1, \ldots, N$, with $i \neq l, j \neq k$ by:

$$C_{ijkl} = (Z_i - Z_k)\,(Z_i - Z_l)^{-1}\,(Z_j - Z_l)\,(Z_j - Z_k)^{-1} = Z_{ik} Z_{il}^{-1} Z_{jl} Z_{jk}^{-1}, \tag{27}$$

where $Z_i, Z_j, Z_k, Z_l \in GL(d, \mathbb{C})$ and $Z_{ij} = Z_i - Z_j = -Z_{ji}$. It is known that matrix cross ratios have simple transformation properties with respect to distinct solutions of a single matrix Riccati equation (Levin[40], Theorem 5), see also Reid[30] (Theorem 12.1) and Zelikin[34] (Chapter 5) for properties of matrix double ratios. Firstly we derive an evolution equation satisfied by C_{ijkl} when the matrices Z_i evolve according to (24), from which it follows that the eigenvalues of C_{ijkl} are constants of motion. We then show that linear fractional transformations result in similarity transformations of C_{ijkl}, which therefore leave invariant the eigenvalues of C_{ijkl}.

Theorem 3. *If the variables Z_i satisfy (24) then the cross ratios (27) evolve according to*

$$\dot{C}_{ijkl} = [C_{ijkl}, Z_k \Gamma_2 - \mathrm{i}\,\Omega_1]. \tag{28}$$

Proof. See Appendix A for the proof by direct calculation. □

Γ_1 does not appear in the commutator on the RHS of (28), as is evident from the defining equations (24) and the fact that the cross ratios depend only on the differences $Z_i - Z_j$. Less obvious is the fact that $\dot{C}_{ijkl}$ is also independent of Ω_2. If we fix the indices i, j, k, l and denote $A = C_{ijkl}, H = \mathrm{i} Z_k \Gamma_2 + \Omega_1$, then (28) reads $\mathrm{i}\dot{A} = [A, H]$ which is the well-known Heisenberg equation of motion for observables A, but in complex form. We may solve the system (28) in the same way as the Heisenberg equations of motion:

Corollary 4. *For each $k = 1, \dots N$ define the $d \times d$ complex matrix S_k by*

$$\dot{S}_k = (\mathrm{i}\,\Omega_1 - Z_k\Gamma_2)\,S_k, \quad S_k(0) = I_d, \tag{29}$$

then the solution of (28) is $C_{ijkl}(t) = S_k(t)\,C^0_{ijkl}S_k^{-1}(t)$ where $C^0_{ijkl} = C_{ijkl}(0)$.

Proof. We can regard S_k as the principal matrix solution of (29) which exists over any interval for which the coefficient matrix $\mathrm{i}\,\Omega_1 - Z_k\Gamma_2$ exists. We have

$$\begin{aligned}\dot{C}_{ijkl} &= \dot{S}_k\,C^0_{ijkl}S_k^{-1} - S_k\,C^0_{ijkl}S_k^{-1}\dot{S}_kS_k^{-1} = \dot{S}_kS_k^{-1}C_{ijkl} - C_{ijkl}\dot{S}_kS_k^{-1}\\ &= [\dot{S}_kS_k^{-1}, C_{ijkl}] = [\mathrm{i}\,\Omega_1 - Z_k\Gamma_2, C_{ijkl}],\end{aligned}$$

together with $C^0_{ijkl} = C_{ijkl}(0)$ as required. □

Hence $C_{ijkl}(t) \sim C^0_{ijkl}$ for all $t > 0$ and so the eigenvalues of C_{ijkl} are constant in time and are therefore determined by the initial values Z^0_i for the system (24). In particular the trace and determinant of C_{ijkl} are constant. For the Kuramoto model the cross ratios reduce to (4), which are constant for all trajectories whether on or inside the unit circle.

The fact that the eigenvalues of C_{ijkl} are constants of motion constrains the possible trajectories of the system, in particular any two nodes which are initially co-located, i.e. $Z^0_i = Z^0_j$ for some i, j, remain co-located for all $t > 0$, despite the nontrivial interactions between nodes, since then we have $C_{ijkl} = 0$ for all $k \neq j, l \neq i$. Conversely, if $Z^0_i \neq Z^0_j$ then $Z_i \neq Z_j$ at all later times $t > 0$, i.e. collisions cannot occur. For general Kuramoto models it is known that at most a finite number of collisions occur[61], see also Ref. 57 (Remark 5.2), and that for identical frequencies there are no finite-time collisions[61], see also Ref. 58 (Lemma 2.5). Evidently this property extends to the matrix systems (24).

Theorem 5. *Under linear fractional transformations $Z_i \to (AZ_i + B)(CZ_i + D)^{-1}$ for $i = 1, \dots, N$ we have $C_{ijkl} \to P_kC_{ijkl}P_k^{-1}$ for all $i, j, k, l = 1, \dots N$, where*

$$P_k = A - (AZ_k + B)(CZ_k + D)^{-1}C. \tag{30}$$

Proof. See Appendix B. This result is similar to that derived by Zelikin[35] (Section III). □

Hence, the eigenvalues of C_{ijkl} are invariant under linear fractional transformations and so can be computed using the expression (21) or by direct evaluation of C_{ijkl} at $t = 0$. A special case of Theorem 5 is for $A = 0 = D, B = I_d = -C$, for which the linear fractional transformation reads $Z_i \to -Z_i^{-1}$ for all $i = 1, \dots N$, in which case $g \in SL(d, \mathbb{C})$ and $P_k = Z_k^{-1}$. We observe that (24) is invariant with respect to $Z_i \to -Z_i^{-1}$ provided that $\Gamma_1 \longleftrightarrow -\Gamma_2$ and $\Omega_1 \longleftrightarrow -\Omega_2$. If all trajectories of the system are confined to the boundary of the classical domain defined by the condition $Z_i^\dagger Z_i = I_d$ for all i, then the transformation $Z_i \to -Z_i^{-1}$ leaves this domain invariant. This property is evident for the Kuramoto model (1) under the transformation $\theta_i \to -\theta_i + \pi$, together with $\omega \to -\omega$.

B. Symmetries and equivalences of the matrix cross ratios

There are $4! = 24$ permutations of the fixed indices i, j, k, l, however the corresponding cross ratios are either similar to C_{ijkl}, or are explicit functions of C_{ijkl}, and hence do not lead to independent constants of motion. The symmetries and similarities that we derive here generalize known properties for the case $d = 1$ of complex cross ratios[13].

Theorem 6. *For the matrix cross ratios C_{ijkl} (27), and for any indices $i \neq l, j \neq k, m \neq k, m \neq l$ with $i, j, k, l, m \in \{1, \dots, N\}$ we have:*

1. $C_{iikl} = I_d = C_{ijkk}$
2. $C_{ijkl} = C_{imkl} C_{mjkl}$
3. $C_{jikl} = C_{ijkl}^{-1}$
4. $C_{ljki} = I_d - C_{ijkl}$
5. $C_{likj} = -C_{ijkl}^{-1} \left(I_d - C_{ijkl} \right)$
6. $C_{jlki} = \left(I_d - C_{ijkl} \right)^{-1}$
7. $C_{ilkj} = -C_{ijkl} \left(I_d - C_{ijkl} \right)^{-1}$

Proof. 1. Set $j = i$ or $l = k$ in (27).

2. $C_{imkl} C_{mjkl} = Z_{ik} Z_{il}^{-1} Z_{ml} Z_{mk}^{-1} Z_{mk} Z_{ml}^{-1} Z_{jl} Z_{jk}^{-1} = Z_{ik} Z_{il}^{-1} Z_{jl} Z_{jk}^{-1} = C_{ijkl}$.

3. Set $j = i$ in 2., use 1., then relabel $m \to j$.

4. $C_{ijkl} + C_{ljki} = Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} - Z_{kl}Z_{il}^{-1}Z_{ij}Z_{jk}^{-1} = \left(Z_{ik}Z_{il}^{-1}Z_{jl} - Z_{kl}Z_{il}^{-1}Z_{ij}\right)Z_{jk}^{-1}$
$= \left(Z_{ik}Z_{il}^{-1}Z_{jl} - Z_{kl}Z_{il}^{-1}Z_{il} + Z_{kl}Z_{il}^{-1}Z_{jl}\right)Z_{jk}^{-1} = \left(Z_{ik}Z_{il}^{-1}Z_{jl} - Z_{kl} + Z_{kl}Z_{il}^{-1}Z_{jl}\right)Z_{jk}^{-1}$
$= \left(Z_{il}Z_{il}^{-1}Z_{jl} - Z_{kl}\right)Z_{jk}^{-1} = \left(Z_{jl} - Z_{kl}\right)Z_{jk}^{-1} = Z_{jk}Z_{jk}^{-1} = I_d,$
where we have used $Z_{ij} = Z_{il} - Z_{jl}$ and $Z_{ik} + Z_{kl} = Z_{il}$.

5. $C_{likj} + C_{ijkl}^{-1}\left(I_d - C_{ijkl}\right) = C_{likj} + C_{ijkl}^{-1} - I_d = C_{likj} + C_{jikl} - I_d = 0$ using 3. then 4. with $i \leftrightarrow j$.

6. Follows from 3. applied to 4.

7. Follows from 3. applied to 5.

□

Next we show that the 24 matrix cross ratios comprising C_{ijkl} and its permutations split into 6 equivalence classes with respect to similarity, each containing four elements.

Corollary 7. *We have the following similarities:*

1. $C_{ijkl} \sim C_{jilk} \sim C_{klij} \sim C_{lkji}$

2. $C_{jikl} \sim C_{ijlk} \sim C_{lkij} \sim C_{klji}$

3. $C_{ljki} \sim C_{jlik} \sim C_{kilj} \sim C_{ikjl}$

4. $C_{likj} \sim C_{iljk} \sim C_{kjli} \sim C_{jkil}$

5. $C_{jlki} \sim C_{ljik} \sim C_{kijl} \sim C_{iklj}$

6. $C_{ilkj} \sim C_{lijk} \sim C_{kjil} \sim C_{jkli}$

Proof. The equalities 3.–7. in Theorem 6 express each of $C_{jikl}, C_{ljki}, C_{likj}, C_{jlki}, C_{ilkj}$ as explicit functions of C_{ijkl}, and so these particular cross ratios are not similar to C_{ijkl}. It is sufficient to show that the elements listed in item 1 are all similar to C_{ijkl}, and then the similarities for the other equivalence classes follow from the functional relations of 3.–7. in Theorem 6. From (27):

$$C_{jilk} = Z_{jl}Z_{jk}^{-1}Z_{ik}Z_{il}^{-1} = Z_{jl}Z_{jk}^{-1}\left(Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}\right)Z_{jk}Z_{jl}^{-1} = Z_{jl}Z_{jk}^{-1}C_{ijkl}Z_{jk}Z_{jl}^{-1} \sim C_{ijkl}.$$

From $Z_{jl} = Z_{il} + Z_{jk} - Z_{ik}$ we obtain:

$$C_{ijkl} = Z_{ik} Z_{il}^{-1} Z_{jl} Z_{jk}^{-1} = Z_{ik} Z_{jk}^{-1} + Z_{ik} Z_{il}^{-1} - Z_{ik} Z_{il}^{-1} Z_{ik} Z_{jk}^{-1} = P + Q - QP, \tag{31}$$

$$C_{klij} = Z_{ik} Z_{jk}^{-1} Z_{jl} Z_{il}^{-1} = Z_{ik} Z_{jk}^{-1} + Z_{ik} Z_{il}^{-1} - Z_{ik} Z_{jk}^{-1} Z_{ik} Z_{il}^{-1} = P + Q - PQ, \tag{32}$$

where $P = Z_{ik} Z_{jk}^{-1}, Q = Z_{ik} Z_{il}^{-1}$. For any two square matrices A, B we have $AB \sim BA$. On setting $A = I_d - P$ and $B = I_d - Q$, we deduce that $(I_d - P)(I_d - Q) \sim (I_d - Q)(I_d - P)$ which implies that $P + Q - QP \sim P + Q - PQ$, and hence from (31,32) we obtain $C_{ijkl} \sim C_{klij}$. Next, we use the first equivalence $C_{ijkl} \sim C_{jilk}$ applied to C_{klij} to obtain $C_{klij} \sim C_{lkji}$, which completes the proof of item 1. Similarly, we generate the four elements for each of the remaining five equivalence classes. □

It is perhaps useful to observe from items 3.–7. in Theorem 6 that C_{ijkl} commutes with each of $C_{jikl}, C_{ljki}, C_{likj}, C_{jlki}, C_{ilkj}$, but not with any other elements of the corresponding equivalence class.

V. UNITARY GROUP MODELS OF SYNCHRONIZATION

The variables Z_i of the matrix models considered in Section IV are elements of the noncompact group $GL(d, \mathbb{C})$ and as one consequence trajectories on the group manifold are generally unbounded. Despite this, complete synchronization can occur under suitable conditions[19]. For compact subgroups of $GL(d, \mathbb{C})$, however, all trajectories on the group manifold are bounded and exist globally for arbitrary initial values, and synchronization occurs under conditions such as a sufficiently large coupling constant[18–20]. We investigate in this section partial integration of the system for the case where the variables $Z_i = U_i$ are $d \times d$ unitary matrices which evolve according to the equations

$$\mathrm{i}\,\dot{U}_i U_i^{-1} = H - \frac{\mathrm{i}\,\kappa}{2N} \sum_{j=1}^{N} \left(U_i U_j^{\dagger} - U_j U_i^{\dagger} \right), \tag{33}$$

with initial values $U_i(0) = U_i^0$. These models have been extensively investigated[18–21], both numerically and analytically, and have synchronization properties similar to those of the Kuramoto model. We have chosen the coupling constant κ to be uniform across the network and the $d \times d$ hermitean matrix H to be independent of i, in order to allow partial integration of the system. It is usually supposed that H is constant in time, however partial integration can still be performed for any time-dependent H. Provided that the initial matrices U_i^0 are unitary, it follows from (33) that U_i remains unitary as the system evolves, i.e. all trajectories remain on the group manifold[20]. In the application to quantum mechanics U_i can be regarded as the unitary time evolution operator at the ith node, and H is the Hamiltonian at each node of the quantum system, the real eigenvalues of which comprise the energy levels.

We write the system (33) in the form

$$\dot{U}_i = -\mathrm{i}\, H\, U_i + \Gamma - U_i\, \Gamma^\dagger\, U_i, \quad i = 1, \ldots N, \tag{34}$$

with initial values $U_i(0) = U_i^0 \in U(d)$, where $\Gamma = \kappa \sum_j U_j/(2N)$, and where we have set $U_i^\dagger U_i = I_d$. In the form (34) unitarity is still preserved, since it follows from (34) and its hermitean conjugate that

$$\frac{d}{dt}\left(U_i^\dagger U_i - I_d\right) = U_i^\dagger \dot{U}_i + \dot{U}_i^\dagger U_i = -\left(U_i^\dagger U_i - I_d\right)\Gamma^\dagger U_i - U_i^\dagger \Gamma\left(U_i^\dagger U_i - I_d\right), \tag{35}$$

which implies, by uniqueness of solutions, that if $U_i^\dagger U_i = I_d$ at $t = 0$ then $U_i^\dagger U_i = I_d$ for all $t > 0$. The expression for Γ as above can be generalized by choosing, for example, $\Gamma = \sum_j a_j U_j b_j$ for any set of $d \times d$ complex matrices a_j, b_j, which can be regarded as non-Abelian generalizations of the well-known phase lag and multiplicative parameters used in the Kuramoto model. Partial integration can be performed for all such cases, but synchronization properties depend on the explicit form of Γ. The Kuramoto model (1) corresponds to the case $d = 1$ with $U_i = \mathrm{e}^{\mathrm{i}\theta_i}$ and $H = -\omega$, together with real parameters a_j, b_j such that $a_j b_j = \kappa\lambda_j/(2N)$, giving $\Gamma = \kappa \sum_j \lambda_j U_j/(2N)$.

Eq. (34) is a special case of the Riccati system (24) with $\Gamma_1 = \Gamma = \Gamma_2^\dagger, \Omega_1 = -H = \Omega_1^\dagger, \Omega_2 = 0$, and variables $Z_i = U_i$. The partial integration outlined in Section III remains valid, but instead of introducing two independent functions β, γ as in (14), we can choose $\beta = \gamma^\dagger$ and still satisfy the reduced equations (15-17) in Lemma 1. The equations for γ and ζ_i may be integrated as in Theorem 2. The condition $\Omega_2 = 0$ is not necessary in order to partially integrate models of synchronization, see for example Eq. (26) in Ref. 20 and Eq. (2) in Ref. 62 for models with $\Omega_2 \neq 0$, however both Ω_1, Ω_2 must be uniform across the network.

A. Hermitean and unitary Riccati matrix systems

We generalize the properties of the system (34) to define a unitary Riccati matrix system as a square matrix system (24) in which

$$\Gamma_1 = \Gamma_2^\dagger, \quad \Omega_1 = \Omega_1^\dagger. \quad \Omega_2 = \Omega_2^\dagger. \tag{36}$$

These conditions are typical of synchronization models such as the Kuramoto model (1) and more generally (34). A consequence of the defining properties (36) is that U_i evolves as a unitary matrix for all $t > 0$, provided that $U_i(0) = U_i^0$ is unitary. We discuss this more generally in Sec. VII, where we extend (36) to rectangular matrices.

The conditions (36) differ from those which define a hermitean Riccati square matrix equation, as defined for example in Ref. 30 (Eq. (II:2.1)) and Ref. 31 (Chapter 2). For any fixed i our notation for the Riccati equation corresponds to that in Ref. 30, Eq. (II:2.1) according to $Z_i = W, \Gamma_1 = C, \Gamma_2 = B, \mathrm{i}\,\Omega_1 = -D, \mathrm{i}\,\Omega_2 = -A$, and the conditions for a hermitean Riccati square matrix equation are $B = B^\dagger, C = C^\dagger, D = A^\dagger$. Similar conditions define the involutory equation considered in Chapter III[30], and the HRDE equation[31]. For square matrices, unitary systems are related to hermitean Riccati systems by the Cayley transform which maps between unitary and hermitean matrices U_i, X_i respectively, according to

$$U_i = (X_i - \mathrm{i}\, I_d)(X_i + \mathrm{i}\, I_d)^{-1}, \quad X_i = \mathrm{i}\,(I_d + U_i)(I_d - U_i)^{-1}. \tag{37}$$

By substituting for U_i into the unitary Riccati matrix system $\dot{U}_i = \Gamma + \mathrm{i}\,\Omega_1 U_i + \mathrm{i}\, U_i \Omega_2 - U_i \Gamma^\dagger U_i$, in which $\Omega_1 = \Omega_1^\dagger, \Omega_2 = \Omega_2^\dagger$, we obtain:

$$\dot{X}_i = -A^\dagger X_i - X_i A - Q + X_i S X_i, \tag{38}$$

where

$$S = \frac{1}{2}\left(-\mathrm{i}\,\Gamma + \mathrm{i}\,\Gamma^\dagger + \Omega_1 + \Omega_2\right), \quad Q = \frac{1}{2}\left(-\mathrm{i}\,\Gamma + \mathrm{i}\,\Gamma^\dagger - \Omega_1 - \Omega_2\right),$$

and $A = \left(-\Gamma - \Gamma^\dagger + \mathrm{i}\,\Omega_1 - \mathrm{i}\,\Omega_2\right)/2$. Since $Q = Q^\dagger$ and $S = S^\dagger$, (38) constitutes a hermitean Riccati matrix system, and if the coefficient matrices are real then (38) defines a symmetric Riccati system, since the real variables X_i are symmetric provided the initial values are symmetric. We refer to Ref. 31 (Chapter 4) for detailed results about hermitean Riccati matrix equations, leaving open the possibility that these results can be extended in some way to the unitary systems related by means of the Cayley transform (37).

B. Cross ratio matrices and constants of motion

Since the system of Riccati equations (34) is a special case of that for square complex matrices Z_i in Section IV, we can define matrix cross ratios C_{ijkl} as in (27), then the eigenvalues are constants of motion as stated in Corollary 4. Specifically, we have

$$C_{ijkl} = (U_i - U_k)(U_i - U_l)^{-1}(U_j - U_l)(U_j - U_k)^{-1} = U_{ik}U_{il}^{-1}U_{jl}U_{jk}^{-1}, \tag{39}$$

where $U_{ij} = U_i - U_j = -U_{ji}$. The cross ratios evolve according to (28), i.e. $\dot{C}_{ijkl} = [C_{ijkl}, U_k\Gamma^\dagger + \mathrm{i}\, H]$, and it follows that the eigenvalues of C_{ijkl} are constants of motion. The matrices C_{ijkl} are related with respect to permutations of the indices (i,j,k,l) as stated in Theorem 6 and Corollary 7. We also have from (39), using $U_{ij}^\dagger = -U_i^{-1}U_{ij}U_j^{-1}$:

$$\begin{aligned} C_{ijkl}^\dagger &= (U_{jk}^\dagger)^{-1}U_{jl}^\dagger(U_{il}^\dagger)^{-1}U_{ik}^\dagger \\ &= (U_j^{-1}U_{jk}U_k^{-1})^{-1}U_j^{-1}U_{jl}U_l^{-1}(U_i^{-1}U_{il}U_l^{-1})^{-1}U_i^{-1}U_{ik}U_k^{-1} \\ &= U_kU_{jk}^{-1}\left[U_{jl}U_{il}^{-1}U_{ik}U_{jk}^{-1}\right]U_{jk}U_k^{-1} = U_kU_{jk}^{-1}C_{lkji}U_{jk}U_k^{-1} \sim C_{lkji}. \end{aligned} \tag{40}$$

From Corollary 7, item 1. we also have $C_{ijkl} \sim C_{lkji}$, and therefore $C_{ijkl}^\dagger \sim C_{ijkl}$, hence the conserved eigenvalues of C_{ijkl} are either real, or appear as complex conjugate pairs. Both the conserved trace and determinant of C_{ijkl} are therefore real.

C. Trajectories on U(2)

The case $d = 2$ is of particular interest as the simplest example of a non-Abelian model with synchronization properties[18,20,21] that is also partially integrable. We parametrize $U_i \in U(2)$ according to:

$$U_i = \mathrm{e}^{-\mathrm{i}\theta_i}\left(\mathrm{i}\sum_{k=1}^{3} x_i^k\sigma_k + x_i^4 I_2\right) = \mathrm{e}^{-\mathrm{i}\theta_i}\begin{pmatrix} x_i^4 + \mathrm{i}x_i^3 & x_i^2 + \mathrm{i}x_i^1 \\ -x_i^2 + \mathrm{i}x_i^1 & x_i^4 - \mathrm{i}x_i^3 \end{pmatrix}, \tag{41}$$

where σ_k denotes the Pauli matrices, and where $\boldsymbol{x}_i = (x_i^1, x_i^2, x_i^3, x_i^4) \in \mathbb{R}^4$, and in order that $U_i \in U(2)$ we impose the constraint $\boldsymbol{x}_i \cdot \boldsymbol{x}_i = 1$, hence $\boldsymbol{x}_i \in S^3$. Since H is hermitean we have the expansion $H = \nu\, I_2 + \sum_{k=1}^{3}\omega^k\sigma_k$ for frequencies ν and parameters $(\omega^1, \omega^2, \omega^3)$, and we define the antisymmetric matrix Ω by:

$$\Omega = \begin{pmatrix} 0 & -\omega^3 & \omega^2 & -\omega^1 \\ \omega^3 & 0 & -\omega^1 & -\omega^2 \\ -\omega^2 & \omega^1 & 0 & -\omega^3 \\ \omega^1 & \omega^2 & \omega^3 & 0 \end{pmatrix}.$$

The equations of motion (33) or (34) reduce to[20]:

$$\dot{\theta}_i = \nu + \frac{\kappa}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)\, \boldsymbol{x}_i \cdot \boldsymbol{x}_j, \tag{42}$$

$$\dot{\boldsymbol{x}}_i = \Omega\, \boldsymbol{x}_i + \frac{\kappa}{N}\sum_{j=1}^{N} \cos(\theta_j - \theta_i)\,(\boldsymbol{x}_j - \boldsymbol{x}_i \cdot \boldsymbol{x}_j\; \boldsymbol{x}_i)\,, \tag{43}$$

with initial values $\theta_i(0) = \theta_i^0, \boldsymbol{x}_i(0) = \boldsymbol{x}_i^0 \in S^3$. If we choose $\theta_i^0 = 0$ for all i, together with $\nu = 0$, then the unique solution of (42) is $\theta_i = 0$ for all $t > 0$, and so from (41) this implies that $U_i \in SU(2)$. This reduction from $U(2)$ to $SU(2)$ is possible because the RHS of (33) has zero trace for $d = 2$ provided that $\operatorname{tr} H = 0$, which ensures that $\det U_i = 1$ for all i and all $t > 0$, provided that $\det U_i^0 = 1$. That trajectories on $SU(2)$ can be equivalently formulated as trajectories on S^3 is of course due to the fact that the group manifold of $SU(2)$ is diffeomorphic to S^3. Complete synchronization for the system (33) of identical oscillators occurs from any initial configuration[18–22].

In order to explicitly evaluate the constants of motion, namely the determinant and trace of C_{ijkl} for any $i, j, k, l = 1, \ldots N$, we calculate from (41):

$$\det(U_i - U_j) = 2\mathrm{e}^{-\mathrm{i}(\theta_i+\theta_j)}\left[\cos(\theta_i - \theta_j) - \boldsymbol{x}_i \cdot \boldsymbol{x}_j\right],$$

where we have used $\boldsymbol{x}_i, \boldsymbol{x}_j \in S^3$. From (39) we obtain:

$$\det C_{ijkl} = \frac{\left[\cos(\theta_i - \theta_k) - \boldsymbol{x}_i \cdot \boldsymbol{x}_k\right]\left[\cos(\theta_j - \theta_l) - \boldsymbol{x}_j \cdot \boldsymbol{x}_l\right]}{\left[\cos(\theta_i - \theta_l) - \boldsymbol{x}_i \cdot \boldsymbol{x}_l\right]\left[\cos(\theta_j - \theta_k) - \boldsymbol{x}_j \cdot \boldsymbol{x}_k\right]}, \tag{44}$$

and we may also evaluate the conserved quantities $\operatorname{tr} C_{ijkl}$. These are not independent constants, but are related to $\det C_{ijkl}$ as follows from the identity $\operatorname{tr} A = 1 + \det A - \det(I_2 - A)$ which is valid for all 2×2 matrices A. On setting $A = C_{ijkl}$ and using Theorem 6, item 4., we obtain $\operatorname{tr} C_{ijkl} = 1 + \det C_{ijkl} - \det C_{ljki}$.

It is evident that the system (42,43) can be extended to any dimension n with trajectories on $S^{n-1} \times S^1$, by allowing $\boldsymbol{x}_i \in S^{n-1} \subset \mathbb{R}^n$, and so we deduce:

Corollary 8. *For the system (42,43) in which $\boldsymbol{x}_i \in S^{n-1} \subset \mathbb{R}^n$, where Ω is any $n \times n$ antisymmetric matrix, the parameters $\lambda_{ijkl} = \det C_{ijkl}$ defined by (44) are constants of motion for any $i \neq l, j \neq k$.*

For $\theta_i = 0$ these constants λ_{ijkl} are precisely those found for the unit sphere models, which are partially integrable by means of transformations which preserve the unit sphere[28]. Symmetry properties of λ_{ijkl} can be deduced from Theorem 6 and Corollary 7, in particular we have $\lambda_{ijkl} = \lambda_{imkl}\lambda_{mjkl}$ and $\lambda_{jikl} = 1/\lambda_{ijkl}$, and the similarities in Corollary 7 become equalities.

D. Linear fractional transformations, unitarity and classical domains

We partially integrate the system (34) in any dimension d as in Theorem 1, and hence we substitute (setting $\beta = \gamma^\dagger$)

$$U_i = \left(\zeta_i + \gamma^\dagger\right)\left(\gamma\zeta_i + I_d\right)^{-1} \tag{45}$$

into (34) which is satisfied provided that

$$\dot{\gamma} = \Gamma^\dagger + \mathrm{i}\,\gamma H - \gamma\Gamma\gamma, \quad \dot{\zeta}_i = (\Gamma\gamma - \mathrm{i}\,H)\zeta_i - \zeta_i\Gamma^\dagger\gamma^\dagger. \tag{46}$$

The equations for ζ_i can be integrated as shown in (20), giving $\zeta_i = S\,\zeta_i^0\,T^{-1}$, and so we obtain $U_i = \left(S\zeta_i^0 + \gamma^\dagger T\right)\left(\gamma S\zeta_i^0 + T\right)^{-1}$. For the choice

$$\Gamma = \frac{\kappa}{2N}\sum_{j=1}^{N} U_j = \frac{\kappa}{2N}\sum_{j=1}^{N}\left(S\zeta_j^0 + \gamma^\dagger T\right)\left(\gamma S\zeta_j^0 + T\right)^{-1},$$

the three matrix equations, namely (46) for γ and (19) for S and T, together with the initial conditions $\zeta_i^0 = U_i^0, \gamma_0 = 0, S(0) = I_d, T(0) = I_d$ comprises a system of three nonlinear matrix equations which determine γ, S, T, and hence the unknowns U_i for all $i = 1, \dots N$ and any $t > 0$, which can be solved by standard numerical methods.

It is known[48,63] that linear fractional transformations leave invariant the bounded classical domain defined by complex $d \times d$ matrices U_i satisfying $U_i^\dagger U_i < I_d$. For models of synchronization the trajectories are usually confined to the boundary of the classical domain but, as the example of the Kuramoto model in Sec. II A shows, synchronization can still occur in the mixed case, where trajectories lie inside or on the boundary of the classical domain. So we consider now explicitly the linear fractional transformation (45) and determine the conditions which ensure that U_i is unitary, and hence the conditions which are satisfied by the trajectories of the variable γ which appears in (45). We firstly return to the definition (10), $U_i \to g(U_i) = (AU_i + B)(CU_i + D)^{-1}$ and consider conditions on the $d \times d$ coefficient matrices A, B, C, D which ensure that $g(U_i)$ is unitary. As is well-known[63], unitarity is satisfied provided that

$$A^\dagger A - C^\dagger C = I_d, \quad D^\dagger D - B^\dagger B = I_d, \quad C^\dagger D = A^\dagger B, \quad D^\dagger C = B^\dagger A, \tag{47}$$

in which case the $2d \times 2d$ matrix g defined by (11) satisfies $g^\dagger J g = J$, where

$$J = \begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix}. \tag{48}$$

Hence $g \in U(d,d)$ and the mapping $U_i \to g(U_i) = (AU_i + B)(CU_i + D)^{-1}$ preserves unitarity for any square matrices A, B, C, D satisfying (47). It also follows from (47) that

$$I_d - g(U_i)^\dagger g(U_i) = (CU_i + D)^{-1\dagger}(I_d - U_i^\dagger U_i)(CU_i + D)^{-1},$$

hence if $I_d - U_i^\dagger U_i$ is positive definite, then $g(U_i)$ is also positive definite, i.e. g preserves the bounded classical domain defined by the condition $U_i^\dagger U_i < I_d$.

Next, we define as in (13), $\zeta_i = AU_iD^{-1}, \gamma = CA^{-1}$, then it follows from (47) that $\beta = BD^{-1} = \gamma^\dagger$. Directly from (45), unitarity $U_i^{-1} = U_i^\dagger$ requires

$$\zeta_i^\dagger \left(I_d - \gamma^\dagger \gamma\right) \zeta_i = I_d - \gamma\gamma^\dagger. \tag{49}$$

Hence, the $d \times d$ matrices ζ_i are not themselves unitary unless $d = 1$. We suppose that $I_d - \gamma^\dagger\gamma$ and $I_d - \gamma\gamma^\dagger$, which are each hermitean, are positive definite matrices; if for example we choose $\gamma(0) = 0$, then both $I_d - \gamma^\dagger\gamma > 0$ and $I_d - \gamma\gamma^\dagger > 0$ hold for all $t > 0$, and so the square roots of these matrices are well-defined. Then $u_i = (I_d - \gamma^\dagger\gamma)^{1/2}\,\zeta_i\,(I_d - \gamma\gamma^\dagger)^{-1/2}$ is unitary as a consequence of (49). Define also the $d \times d$ matrices V, W according to

$$V = (I_d - \gamma^\dagger\gamma)^{1/2}\, A, \qquad W = (I_d - \gamma\gamma^\dagger)^{1/2}\, D, \tag{50}$$

then from (47) both V and W are unitary, and these expressions represent polar decompositions of A, D. Also from (50) we have:

$$B = \gamma^\dagger D = \gamma^\dagger (I_d - \gamma\gamma^\dagger)^{-1/2} W, \qquad C = \gamma A = \gamma (I_d - \gamma^\dagger\gamma)^{-1/2} V. \tag{51}$$

From the definition of g we find

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & \gamma^\dagger D \\ \gamma A & D \end{pmatrix} = \begin{pmatrix} (I_d - \gamma^\dagger\gamma)^{-1/2} & \gamma^\dagger (I_d - \gamma\gamma^\dagger)^{-1/2} \\ \gamma (I_d - \gamma^\dagger\gamma)^{-1/2} & (I_d - \gamma\gamma^\dagger)^{-1/2} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} = g_\gamma \, g_{VW}, \tag{52}$$

which defines matrices g_γ and g_{VW}. Evidently $g_{VW} \in U(d) \times U(d)$. By factorizing g_γ according to:

$$g_\gamma = \begin{pmatrix} I_d & 0 \\ \gamma & I_d \end{pmatrix} \begin{pmatrix} I_d & 0 \\ 0 & I_d - \gamma\gamma^\dagger \end{pmatrix} \begin{pmatrix} I_d & \gamma^\dagger \\ 0 & I_d \end{pmatrix} \begin{pmatrix} (I_d - \gamma^\dagger\gamma)^{-1/2} & 0 \\ 0 & (I_d - \gamma\gamma^\dagger)^{-1/2} \end{pmatrix},$$

we deduce that $\det g_\gamma = \det(I_d - \gamma\gamma^\dagger)^{1/2} / \det(I_d - \gamma^\dagger\gamma)^{1/2} = 1$, since the positive definite matrices $I_d - \gamma\gamma^\dagger$ and $I_d - \gamma^\dagger\gamma$ are similar and so have the same positive eigenvalues. Hence $g_\gamma \in SU(d,d)$. The trajectory γ satisfies $\gamma^\dagger\gamma < I_d$ and so is an element of the classical domain which by means of the mapping $\gamma \to g_\gamma$ defined in (52) is diffeomorphic to the homogeneous space $U(d,d)/U(d) \times U(d)$. This holds whether the matrices U_i lie on the boundary or within the classical domain.

As a specific example, for $d = 1$ denote $\gamma = \overline{z}$, and set $V = \mathrm{e}^{\mathrm{i}\psi}, W = \mathrm{e}^{\mathrm{i}\phi}$, then from (50,51) we have:

$$A = \frac{\mathrm{e}^{\mathrm{i}\psi}}{\sqrt{1-|z|^2}}, \quad B = \frac{z\,\mathrm{e}^{\mathrm{i}\phi}}{\sqrt{1-|z|^2}}, \quad C = \frac{\overline{z}\,\mathrm{e}^{\mathrm{i}\psi}}{\sqrt{1-|z|^2}}, \quad D = \frac{\mathrm{e}^{\mathrm{i}\phi}}{\sqrt{1-|z|^2}}.$$

Since $I_d - \gamma^\dagger\gamma = I_d - \gamma\gamma^\dagger = 1 - |z|^2$ is positive definite, the trajectories $z(t)$ lie within the unit disk, and the group element is

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi} & z\,\mathrm{e}^{\mathrm{i}\phi} \\ \overline{z}\,\mathrm{e}^{\mathrm{i}\psi} & \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix},$$

which satisfies $g^\dagger J g = J$, and so $g \in U(1,1)$. The matrix

$$g_z = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix}$$

is an element of $SU(1,1)$ and acts transitively on the unit disk, which is precisely the classical domain for the Kuramoto model. By means of the matrix g_z therefore we can transform any initial value $z_0 = \overline{\gamma(0)}$ to lie at the origin. The substitution (45), on setting $U_i = \mathrm{e}^{\mathrm{i}\theta_i}, \zeta_i = \mathrm{e}^{\mathrm{i}\xi_i}$, reduces to the Möbius transformation (2).

VI. ORTHOGONAL GROUP MODELS OF SYNCHRONIZATION

The square matrix models considered in Section IV are also of particular interest for the group $SO(d)$, since it is known from numerical and analytic investigations[19,20] that synchronization occurs in such models. The variables R_i in this case are $d \times d$ real orthogonal matrices with unit determinant which evolve according to (33) upon replacing $U_i = R_i, H = \mathrm{i}\,\Omega$, where Ω is a $d \times d$ real antisymmetric matrix, with initial values $R_i(0) = R_i^0 \in SO(d)$. The case $d = 2$ reduces to the Kuramoto model. From the general considerations of Sect. III, this system can be partially integrated by the substitution $R_i = \left(\zeta_i + \gamma^{\mathsf{T}}\right)\left(\gamma\zeta_i + I_d\right)^{-1}$, where γ, ζ_i are real $d \times d$ matrices, having set $\beta = \gamma^{\mathsf{T}}$.

Matrix cross ratios C_{ijkl} are defined as in Section IV A for complex square matrices, by replacing $Z_i \to R_i$, except that now the dimension d is restricted to be even, since otherwise $R_i - R_j$ is singular for any two elements $R_i, R_j \in SO(d)$. This follows from:

$$\det(R_i - R_j) = \det R_i \det(I_d - R_i^{\mathsf{T}} R_j) = \det(I_d - R_i^{\mathsf{T}} R_j) = \det(I_d - R_j^{\mathsf{T}} R_i), \tag{53}$$

using $\det A = \det A^{\mathsf{T}}$ for any square matrix A, and also

$$\det(R_i - R_j) = \det R_j \det(R_j^{\mathsf{T}} R_i - I_d) = \det(R_j^{\mathsf{T}} R_i - I_d) = (-1)^d \det(I_d - R_j^{\mathsf{T}} R_i). \tag{54}$$

For odd d this implies $\det(R_i - R_j) = 0$, and so the matrix cross ratios are singular. For even d, however, the matrices C_{ijkl} evolve as in Theorem 3, and the eigenvalues are constants of motion.

A. Noncompact orthogonal group models

The unitary groups considered in Sect. V are compact and so the trajectories $U_i(t)$ are bounded, and synchronization occurs under conditions which generalize those for the Kuramoto model. Synchronization can also occur for noncompact groups in which case trajectories are unbounded, although in some cases solutions exist only locally[19]. We demonstrate here that partial integrability and properties of linear fractional transformations and matrix cross ratios extend to the noncompact case by considering the noncompact group $SO(m, n)$, with $m + n = d$, for which the $d \times d$ real matrices $M_i \in SO(m, n)$ satisfy $M_i^{\mathsf{T}} K M_i = K$, where

$$K = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}.$$

A special case is $n = 0$ for which K is the identity matrix, when we regain the $SO(d)$ model.

On replacing $Z_i \to M_i$ with $M_i^{-1} = KM_i^{\mathsf{T}}K$, we write (25) in the Riccati form

$$\dot{M}_i = \Omega M_i + \Gamma - M_i K \Gamma^{\mathsf{T}} K M_i, \tag{55}$$

where the $d \times d$ real matrix Ω is an element of the Lie algebra of $SO(m, n)$, and hence satisfies $\Omega^{\mathsf{T}} K + K\Omega = 0$, where we have set $H_i = \Omega$ in (25). We choose $\Gamma = \kappa \sum_j \lambda_j M_j/(2N)$ with multiplicative real constants λ_j. We obtain the Riccati system (24) by identifying $\mathrm{i}\,\Omega_1 = \Omega, \Omega_2 = 0, \Gamma_1 = \Gamma, \Gamma_2 = K\Gamma^{\mathsf{T}}K$, and the initial values are $M_i(0) = M_i^0$ where $M_i^0 \in SO(m, n)$. Trajectories M_i are confined to the group manifold for all $t > 0$, and complete synchronization occurs under certain sufficient conditions[19].

The system (55) can be partially integrated as described in Section III A, where now we set $\beta = K\gamma^{\mathsf{T}}K$ and substitute

$$M_i = \left(\zeta_i + K\gamma^{\mathsf{T}}K\right)\left(\gamma\zeta_i + I_d\right)^{-1}, \tag{56}$$

into (55) to obtain:

$$\dot{\gamma} = -\gamma\Omega + K\Gamma^{\mathsf{T}}K - \gamma\Gamma\gamma, \quad \dot{\zeta}_i = (\Gamma\gamma + \Omega)\zeta_i - \zeta_i K\Gamma^{\mathsf{T}}\gamma^{\mathsf{T}}K. \tag{57}$$

The equations for ζ_i may be partially integrated as in Theorem 2. Let us consider this now in detail for $d = 2$, for which there are applications to synchronization in relativistic dynamical systems.

B. The Lorentz group SO(1,1) and partial integrability

For $d = 2$ with $n = 1 = m$ the elements M_i of the system are $SO(1, 1)$ matrices which evolve according to (55), and complete synchronization occurs[19] in the sense that $M_i M_j^{-1} \to I_2$ as $t \to \infty$. This system describes the synchronization of a relativistic cluster of particles in Minkowski space-time[49], for distributed matrices Ω_i. We consider here in particular identical matrices Ω for which the Riccati system (55) can be partially integrated. We parametrize elements of $SO(1,1)$ according to

$$M_i = \mathrm{e}^{\alpha_i J} = \begin{pmatrix} \cosh \alpha_i & \sinh \alpha_i \\ \sinh \alpha_i & \cosh \alpha_i \end{pmatrix}, \tag{58}$$

for hyperbolic angles α_i, hence $M_i \in SO^+(1,1)$ (the connected component of the identity) and then $M_i^\mathsf{T} K M_i = K$ where

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with $JK + KJ = 0$. Define $\Omega = \omega J$ for some real parameter ω, then Ω is an element of the Lie algebra of $SO(1,1)$ as required. The evolution equations (55), in which we choose $\Gamma = \kappa \sum_j \lambda_j M_j/(2N)$, reduce to:

$$\dot{\alpha_i} = \omega + \frac{\kappa}{N} \sum_{j=1}^{N} \lambda_j \sinh(\alpha_j - \alpha_i), \quad i = 1, \dots N, \tag{59}$$

which evidently is a hyperbolic form of the Kuramoto model (1). This model has been discussed in the context of relativistic mechanics in $1+1$ dimensions[49], in which the particle coordinates (x_i^0, x_i^1) in Minkowski space-time are given by $\lambda_i(\sinh \alpha_i, \cosh \alpha_i)$, where α_i is the rapidity of the particle of unit mass at the ith node. For identical parameters $\omega_i = \omega$ and with $\kappa_i = \kappa/\lambda_i$, complete synchronization occurs exponentially quickly, with $\alpha_i - \alpha_j \to 0$ as $t \to \infty$.

The system (59) may be partially integrated by substituting for M_i as in (56). Because $SO(1,1)$ is abelian we have $\zeta_i \in SO(1,1)$ and hence we parametrize $\zeta_i = \mathrm{e}^{\beta_i J}$ for hyperbolic angles β_i, similar to (58). Since γ commutes with ζ_i for all i we can parametrize γ according to $\gamma = v_1 I_2 - v_2 J$ for time-dependent real coordinates v_1, v_2, and hence γ is a symmetric matrix. The formula (56) reduces to

$$\mathrm{e}^{\alpha_i} = \frac{v_1 + v_2 + \mathrm{e}^{\beta_i}}{1 + (v_1 - v_2)\,\mathrm{e}^{\beta_i}}, \tag{60}$$

which is a hyperbolic form of the WS transform (2).

The trajectory of the vector $(v_1, v_2) \in \mathbb{R}^2$ is restricted to a region in $\mathbb{R}^2$ as determined by the positivity of the LHS of (60). By taking the cases in which e^{β_i} is either arbitrarily small or arbitrarily large, positivity requires both $v_1 + v_2 \geqslant 0$ and $v_1 - v_2 \geqslant 0$. The inverse mapping to (60) is given by

$$\mathrm{e}^{\beta_i} = \frac{-(v_1 + v_2) + \mathrm{e}^{\alpha_i}}{1 - (v_1 - v_2)\,\mathrm{e}^{\alpha_i}}, \tag{61}$$

and is of the same form as (60), but exists only if $1 - v_1^2 + v_2^2 \neq 0$. Hence the trajectories (v_1, v_2) are restricted to the region R in the plane defined by

$$R = \{(v_1, v_2) : |v_2| \leqslant v_1, v_1^2 - v_2^2 < 1\}, \tag{62}$$

which is the region bounded by the lines $v_1 = \pm v_2$ for $v_1 \geqslant 0$, and the right branch of the hyperbola $v_1^2 - v_2^2 = 1$, provided that the initial value $(v_1(0), v_2(0))$ lies in R. The mapping (60) does not give all possible values for e^{α_i} as (v_1, v_2) evolves, rather trajectories α_i are such that, from (61), $v_1 + v_2 < \mathrm{e}^{\alpha_i} < 1/(v_1 - v_2)$ for all i.

Partial integration of the system (59) is achieved by substituting for α_i using (60) to obtain equations for v_1, v_2, β_i, as may also be derived directly from (57). The N equations for β_i can be integrated directly to obtain $\beta_i(t) = \beta_i(0) + b(t)$ for some function $b(t)$ with $b(0) = 0$. If we choose the initial values $v_1(0) = 0 = v_2(0)$ then from (60) $\beta_i(0) = \alpha_i(0)$, which are the given initial values for (59). Hence we obtain a closed system of 3 equations for v_1, v_2, b, which determine α_i for all $t > 0$

The matrix cross ratio C_{ijkl} given by (27) with $Z_i = M_i = \mathrm{e}^{\alpha_i J}$ reduces to $C_{ijkl} = \lambda_{ijkl} I_2$, where

$$\lambda_{ijkl} = \frac{\sinh \frac{1}{2}(\alpha_i - \alpha_k) \sinh \frac{1}{2}(\alpha_j - \alpha_l)}{\sinh \frac{1}{2}(\alpha_i - \alpha_l) \sinh \frac{1}{2}(\alpha_j - \alpha_k)}.$$

These cross ratios are conserved with respect to the evolution equations (59), as may be verified by direct calculation, and satisfy symmetries as derived in Sec. IV B.

For general d, the linear fractional transformation $M_i \to (AM_i + B)(CM_i + D)^{-1}$, where A, B, C, D are $d \times d$ real matrices with $m + n = d$, and with $M_i \in SO(m, n)$, preserves orthogonality, i.e. the relation $M_i^\mathsf{T} K M_i = K$ is satisfied provided that the $2d \times 2d$ matrix g as defined in (11) is an element of $O(d, d)$. We can factorize $g = g_\gamma \, g_{VW}$, similar to (52) for the unitary group, but now with $g_{VW} \in O(m, n) \times O(m, n)$. For $d = 2$ and $m = 1 = n$ we use the fact that

$$I_2 - K\gamma^\mathsf{T} K\gamma = I_2 - \gamma K \gamma^\mathsf{T} K = (1 - v_1^2 + v_2^2) I_2,$$

(setting $\gamma = v_1 I_2 - v_2 J$) is a positive definite matrix, since $(v_1, v_2) \in R$. The 4×4 matrix g_γ in the factorization $g = g_\gamma \, g_{VW}$ is given explicitly by

$$g_\gamma = \frac{1}{\sqrt{1 - v_1^2 + v_2^2}} \begin{pmatrix} I_2 & K\gamma^\mathsf{T} K \\ \gamma & I_2 \end{pmatrix} = \frac{1}{\sqrt{1 - v_1^2 + v_2^2}} \begin{pmatrix} 1 & 0 & v_1 & v_2 \\ 0 & 1 & v_2 & v_1 \\ v_1 & -v_2 & 1 & 0 \\ -v_2 & v_1 & 0 & 1 \end{pmatrix}, \tag{63}$$

and satisfies $g_\gamma^{\mathsf{T}} K' g_\gamma = K'$, as well as $\det g_\gamma = 1$, where $K' = \mathrm{diag}[1,-1,-1,1]$, and so we deduce that $g_\gamma \in SO(2,2)$. Trajectories (v_1, v_2) are therefore constructed on the homogeneous space $O(2,2)/O(1,1)\times O(1,1)$ which is diffeomorphic to the unbounded classical domain R defined in (62) via the mapping $\gamma \to g_\gamma$.

VII. RECTANGULAR UNITARY MATRIX RICCATI SYSTEMS

We return now to the general Riccati system of equations (9) for rectangular $p\times q$ matrices Z_i with dimensions as in Sec. III, but with coefficients restricted according to

$$\Gamma_1 = \Gamma_2^\dagger, \quad \Omega_1 = \Omega_1^\dagger, \quad \Omega_2 = \Omega_2^\dagger, \tag{64}$$

which extend the conditions for square matrices that define a unitary Riccati system in (36); hence we refer to this as a rectangular unitary matrix Riccati system. For square matrices these conditions are related to those which define a hermitean Riccati matrix system by means of the Cayley transform, as discussed in Sec. V A. The case of rectangular matrices is of considerable interest in the formulation of synchronization models, particularly the case $q = 1$ for which the unknowns Z_i are column vectors of length p, either real or complex. For the real case these vectors are confined to the unit sphere S^{p-1}, leading to synchronization models which have been extensively investigated[20,23–27], and for $p = 2$ reduce to the Kuramoto model. For the complex case we obtain models of quantum synchronization[50–55,64].

Writing $\Gamma_1 = \Gamma$, we consider therefore the system

$$\dot{Z}_i = \Gamma + \mathrm{i}\,\Omega_1 Z_i + \mathrm{i}\, Z_i \Omega_2 - Z_i \Gamma^\dagger Z_i, \quad i = 1, \dots N, \tag{65}$$

with $Z_i(0) = Z_i^0$, where typically $\Gamma = \sum_{j=1}^N a_j Z_j b_j$, where a_j, b_j are any set of $p\times p$ and $q\times q$ complex matrices, respectively, together with constant hermitean matrices $\Omega_1 = \Omega_1^\dagger, \Omega_2 = \Omega_2^\dagger$. Eqs. (65) can be partially integrated as described in Section III with $\beta = \gamma^\dagger$, and hence we substitute

$$Z_i = \left(\zeta_i + \gamma^\dagger\right)\left(\gamma\zeta_i + I_q\right)^{-1} \tag{66}$$

into (65), which is satisfied provided that

$$\dot{\gamma} = \Gamma^\dagger - \mathrm{i}\,\gamma\,\Omega_1 - \mathrm{i}\,\Omega_2\gamma - \gamma\Gamma\gamma, \quad \dot{\zeta}_i = (\Gamma\gamma + \mathrm{i}\,\Omega_1)\zeta_i - \zeta_i(\Gamma^\dagger\gamma^\dagger - \mathrm{i}\,\Omega_2). \tag{67}$$

The latter N equations can be integrated as in (20) to give $\zeta_i = S\,\zeta_i^0\,T^{-1}$, and so we obtain $Z_i = \left(S\zeta_i^0 + \gamma^\dagger T\right)\left(\gamma S\zeta_i^0 + T\right)^{-1}$. The three matrix equations, (67) for γ and (19) for S with

$\dot{T} = \left(\Gamma^\dagger \gamma^\dagger - \mathrm{i}\,\Omega_2\right) T$, together with the initial conditions $\zeta_i^0 = Z_i^0, \gamma_0 = 0, S(0) = I_p, T(0) = I_q$ comprise a system of nonlinear matrix equations, independent of N, which determine γ, S, T, and hence the unknowns Z_i for all $i = 1, \ldots N$ and any $t > 0$.

It follows from (65), as shown in (C2,C3) by setting $i = j$, that

$$\frac{d}{dt}\left(I_q - Z_i^\dagger Z_i\right) = -\mathrm{i}\left[\Omega_2, I_q - Z_i^\dagger Z_i\right] - Z_i^\dagger \Gamma \left(I_q - Z_i^\dagger Z_i\right) - \left(I_q - Z_i^\dagger Z_i\right)\Gamma^\dagger Z_i, \tag{68}$$

$$\frac{d}{dt}\left(I_p - Z_i Z_i^\dagger\right) = \mathrm{i}\left[\Omega_1, I_p - Z_i Z_i^\dagger\right] - Z_i \Gamma^\dagger \left(I_p - Z_i Z_i^\dagger\right) - \left(I_p - Z_i Z_i^\dagger\right)\Gamma Z_i^\dagger. \tag{69}$$

If we set $Z_i^\dagger Z_i = I_q$ initially, then we must have $Z_i^\dagger Z_i = I_q$ for all $t > 0$, and similarly for $Z_i Z_i^\dagger = I_p$, by uniqueness of solutions. For square matrices this means that if Z_i is initially chosen to be unitary for all i, then this unitarity is preserved as the system evolves, as discussed in Sec. V. For $q = 1$ with Z_i a real column p-vector, if we specify $Z_i^\dagger Z_i = 1$ at $t = 0$ then Z_i remains a unit vector for all $t > 0$, i.e. the trajectory of the ith node is restricted to the unit sphere.

On the other hand, if we choose $I_q - Z_i^\dagger Z_i$ to be positive definite at $t = 0$, then it also follows from (68) that $I_q - Z_i^\dagger Z_i$ remains a positive definite matrix for all $t > 0$, since trajectories cannot cross the boundary. The space of $p \times q$ matrices Z_i which satisfy $I_q - Z_i^\dagger Z_i > 0$ constitutes a bounded classical domain which is diffeomorphic to $U(p,q)/U(p) \times U(q)$, with a group action on the elements Z_i by means of linear fractional transformations[48,63]. The example of the extended Kuramoto model in Sec. II A shows that trajectories can lie within the classical domain for some nodes, or else on the boundary for others, and synchronization can still occur for this mixed case.

A. Matrix cross ratios

The partial integrability of the system of square matrix Riccati equations discussed in Sec. IV can be viewed as a consequence of the existence of conserved quantities, namely the eigenvalues of the cross ratio matrices C_{ijkl} defined in (27). Although the expression for C_{ijkl} requires that the matrices Z_i be square, we can manipulate C_{ijkl} into a form which, upon replacing the inverse Z_i^{-1} by the hermitean conjugate $Z_i^\dagger$, is also well-defined for rectangular matrices, and satisfies conservation properties with respect to the system (65), similar to those of C_{ijkl}. Define therefore the $p \times p$ matrix D_{ijkl}^p for any $i, j, k, l = 1, \ldots N$:

$$D_{ijkl}^p = \left(I_p - Z_k Z_i^\dagger\right)\left(I_p - Z_l Z_i^\dagger\right)^{-1}\left(I_p - Z_l Z_j^\dagger\right)\left(I_p - Z_k Z_j^\dagger\right)^{-1}, \tag{70}$$

and the $q \times q$ matrix

$$D^q_{ijkl} = \left(I_q - Z_k^\dagger Z_i\right)\left(I_q - Z_l^\dagger Z_i\right)^{-1}\left(I_q - Z_l^\dagger Z_j\right)\left(I_q - Z_k^\dagger Z_j\right)^{-1}, \tag{71}$$

For $d \times d$ unitary matrices $Z_i = U_i$ as discussed in Sec. V B, we have $D^{p=d}_{ijkl} = C_{ijkl} = U_k D^{q=d}_{ijkl} U_k^{-1}$, where C_{ijkl} is defined in (39).

Lemma 9. *The matrix cross ratios D^p_{ijkl}, D^q_{ijkl} evolve according to:*

$$\dot{D}^p_{ijkl} = \left[D^p_{ijkl}, Z_k\Gamma^\dagger - \mathrm{i}\,\Omega_1\right], \qquad \dot{D}^q_{ijkl} = \left[D^q_{ijkl}, Z_k^\dagger\Gamma + \mathrm{i}\,\Omega_2\right],$$

for any $i, j, k, l = 1, \ldots N$.

Proof. See Appendix C. □

It follows as in Corollary 4 that the eigenvalues of D^p_{ijkl}, D^q_{ijkl} are constants of motion. Of the relations and symmetries for C_{ijkl} listed in Theorem 6, those for items 1-3 remain valid for D^p_{ijkl}, D^q_{ijkl}, and numerical evaluations indicate that the similarities in Corollary 7 also hold.

We consider next the special case $q = 1$ with $p = d$ for which Z_i is a real or complex d-vector. General properties such as partial integrability and the existence of constants of motion remain valid for $q = 1$, but we consider this particular case here for several reasons; firstly, complex vectors Z_i are relevant to quantum synchronization where they can be viewed as wavefunctions which evolve according to finite-dimensional nonlinear Schrödinger equations[64]. The system of equations can be extended to infinite dimensions in which the wavefunctions are elements of a Hilbert space in which the Hamiltonian at each node acts as a self-adjoint operator. It is known[50–56] for models with identical potentials that solutions exist globally, and that complete synchronization occurs under specified conditions[56]. We show that partial integration extends to this infinite-dimensional case, with a fixed number of nodes N. Secondly, we consider in Sec. VII C the formulas for partial integration for models with real d-vectors lying on the unit sphere which have been extensively investigated, mostly without using the properties of partial integration. As we will see, the method of partial integration using linear fractional transformations is not unique, and we briefly compare two alternative schemes.

B. Quantum synchronization and partial integration

We first consider models of quantum synchronization[64] in which a quantum oscillator such as a quantum particle of fixed spin is located at each node i of the network with a corresponding wavefunction $|\psi_i\rangle$. Each d-dimensional wavefunction evolves according to

$$\mathrm{i}\frac{\partial}{\partial t}|\psi_i\rangle = H|\psi_i\rangle + \frac{\mathrm{i}\,\kappa}{2N}\sum_{j=1}^{N}\Big[|\psi_j\rangle - |\psi_i\rangle\langle\psi_j|\psi_i\rangle\Big], \quad i = 1, \ldots, N \tag{72}$$

where we have set $\hbar = 1$ and $a_{ij} = 1$ (for all-to-all coupling), and where the Hamiltonian $H = H_i$ is a prescribed $d \times d$ hermitean matrix, independent of i. Each wavefunction $|\psi_i\rangle$ is normalized to unity, $\langle\psi_i|\psi_i\rangle = 1$. Although we regard (72) as a system of finite-dimensional nonlinear evolution equations for spin $(d-1)/2$ particles which interact nonlinearly over a quantum network, (72) can be extended to infinite dimensions in which N remains fixed and finite, but $|\psi_i\rangle$ becomes an element of a Hilbert space $\mathcal{H}$ in n spatial dimensions, and is therefore a complex function $\psi_i(x,t)$ of both $x \in \mathbb{R}^n$ and t. The Hamiltonian H, which is a sum of kinetic and potential terms, acts in $\mathcal{H}$ as a self-adjoint operator with an identical potential for each node, and the time-dependent scalar product $\langle\psi_j|\psi_i\rangle$ is defined in the usual way as a Hilbert space inner product:

$$\langle\psi_j|\psi_i\rangle = \int_{\mathbb{R}^n} \overline{\psi_j(x,t)}\,\psi_i(x,t)\,dx.$$

Equations (72) therefore comprise a system of N coupled partial differential-integral equations. It is known[50–56] that complete synchronization can occur in both finite and infinite dimensions.

We write (72) in the form

$$\mathrm{i}\frac{\partial}{\partial t}|\psi_i\rangle = H|\psi_i\rangle + \mathrm{i}\,|\Phi\rangle - \mathrm{i}\,\langle\Phi|\psi_i\rangle|\psi_i\rangle, \tag{73}$$

where the averaged wavefunction $|\Phi\rangle$ is defined by $|\Phi\rangle = \kappa\sum_j|\psi_j\rangle/(2N)$. In finite dimensions we identify $Z_i = |\psi_i\rangle, \Omega_1 = -H, \Omega_2 = 0, \Gamma = |\Phi\rangle$, and these equations take the form of a unitary Riccati system (65). Partial integration proceeds by means of the linear fractional transformations (66), which in turn leads to the system (67) which may be partially integrated. This procedure formally extends to infinite dimensions in which (73) is regarded as an infinite-dimensional Riccati system. Hence, denoting $\zeta_i = |\zeta_i\rangle$ and $\gamma^\dagger = |\beta\rangle$, which are now elements of a Hilbert space $\mathcal{H}$ and are therefore functions of $x \in \mathbb{R}^n$ and t, we substitute

$$|\psi_i\rangle = \frac{|\zeta_i\rangle + |\beta\rangle}{1 + \langle\beta|\zeta_i\rangle} \tag{74}$$

into (65), which is satisfied provided that

$$\mathrm{i}\frac{\partial}{\partial t}|\beta\rangle = H\,|\beta\rangle + \mathrm{i}\,|\Phi\rangle - \mathrm{i}\,\langle\Phi|\beta\rangle\,|\beta\rangle, \quad \mathrm{i}\frac{\partial}{\partial t}|\zeta_i\rangle = H\,|\zeta_i\rangle - \mathrm{i}\,\langle\Phi|\beta\rangle\,|\zeta_i\rangle + \mathrm{i}\,\langle\beta|\zeta_i\rangle\,|\Phi\rangle. \tag{75}$$

The equations for $|\zeta_i\rangle$ are satisfied by the wavefunctions $|\zeta_i\rangle = M\,|\zeta_i^0\rangle$, where $|\zeta_i^0\rangle = |\zeta_i\rangle_{t=0}$, and M is a (nonunitary) operator in $\mathcal{H}$ satisfying $\mathrm{i}\,\dot{M} = H\,M - \mathrm{i}\,\langle\Phi|\beta\rangle M + \mathrm{i}|\Phi\rangle\langle\beta|\,M$, with $M_{t=0}$ equal to the identity operator. The system is reduced therefore to partial differential-integral equations for the operator M and for the wavefunction $|\beta\rangle \in \mathcal{H}$. The cross ratios $D_{ijkl}^{q=1}$ in (71) are constant in time, as follows from Lemma 9, with the explicit form:

$$D_{ijkl}^{q=1} = \frac{(1 - \langle\psi_i|\psi_k\rangle)\,(1 - \langle\psi_j|\psi_l\rangle)}{(1 - \langle\psi_i|\psi_l\rangle)\,(1 - \langle\psi_j|\psi_k\rangle)}. \tag{76}$$

That these cross ratios are conserved has been independently observed in recent work[56], where various synchronization and stability properties of the system are also derived but without using the partially integrated (reduced) system (75). Again, our aim in this section is to demonstrate that partial integration extends formally to infinite-dimensional systems.

C. Vector models on the unit sphere

Models of synchronization on the unit sphere S^{d-1} have been extensively developed, including proofs of synchronization properties together with various applications[20,23–27,65–67], including consensus properties in opinion dynamics[68–71]. It has been previously observed that the system with identical frequency matrices can be partially integrated in any dimension d using the vector transform[28], and also for S^3 by using quaternionic variables[29]. We show here that the general formalism applied to these vector models leads to known properties such as conserved cross ratios, but in particular we observe that the details of partial integration differ from those in Ref. 28. Evidently partial integration can be performed in several distinct ways. Here, we use linear fractional transformations which differ from the vector transform, which algebraically maps the unit sphere to itself in any dimension and also preserves the unit ball.

We choose $q = 1, p = d$ and denote the real column d-vector $Z_i = \boldsymbol{x}_i$, then $\boldsymbol{x}_i \in \mathbb{R}^d$ satisfies (65) which reads:

$$\dot{\boldsymbol{x}}_i = \Omega\, \boldsymbol{x}_i + \boldsymbol{X} - \boldsymbol{x}_i \boldsymbol{X}^{\mathsf{T}} \boldsymbol{x}_i, \tag{77}$$

where we have set $\Omega_2 = 0$ and have defined the real antisymmetric $d \times d$ matrix $\Omega = \mathrm{i}\,\Omega_1$, and have denoted $\Gamma = \boldsymbol{X}$, a real column d-vector with $\Gamma^\dagger = \boldsymbol{X}^{\mathsf{T}}$. A typical choice could be $\boldsymbol{X} = \sum_j \lambda_j \boldsymbol{x}_j/(2N)$ where λ_j are multiplicative real parameters, but more generally could be any set of constant rotation matrices in $SO(d)$ which implement rotational phase lag. We choose initial values $\boldsymbol{x}_i(0) = \boldsymbol{x}_i^0 \in S^{d-1}$ which ensures that $\boldsymbol{x}_i \in S^{d-1}$ for all $t > 0$, but we also allow $\boldsymbol{x}_i^0 \in \mathbb{B}^d$ for some or all nodes i, in which case the corresponding trajectories are strictly confined to the unit ball $\mathbb{B}^d$. In order to partially integrate (77) by means of the linear fractional transformation (66), we substitute for $\boldsymbol{x}_i$ according to:

$$\boldsymbol{x}_i = \frac{\boldsymbol{w} + \boldsymbol{\zeta}_i}{1 + \boldsymbol{w} \cdot \boldsymbol{\zeta}_i}, \tag{78}$$

where $\boldsymbol{w} = \gamma^{\mathsf{T}}$ and $\boldsymbol{\zeta}_i = \zeta_i$ are real column d-vectors. The equations for $\boldsymbol{w}, \boldsymbol{\zeta}_i$ which solve (77) are given by (67), which reads:

$$\dot{\boldsymbol{w}} = \Omega\, \boldsymbol{w} + \boldsymbol{X} - (\boldsymbol{w} \cdot \boldsymbol{X})\, \boldsymbol{w}, \quad \dot{\boldsymbol{\zeta}}_i = \Omega\, \boldsymbol{\zeta}_i + \boldsymbol{X}\, \boldsymbol{w} \cdot \boldsymbol{\zeta}_i - (\boldsymbol{w} \cdot \boldsymbol{X})\, \boldsymbol{\zeta}_i = M \boldsymbol{\zeta}_i, \tag{79}$$

where the $d \times d$ matrix M is defined by $M = \Omega + \boldsymbol{X}\boldsymbol{w}^{\mathsf{T}} - (\boldsymbol{w} \cdot \boldsymbol{X}) I_d$, and so is independent of i. The equations for $\boldsymbol{\zeta}_i$ can be integrated as before, and we obtain $\boldsymbol{\zeta}_i = S\boldsymbol{\zeta}_i^0$ where the initial values are $\boldsymbol{\zeta}_i(0) = \boldsymbol{\zeta}_i^0$, and S is the principal matrix solution of $\dot{S} = M\, S$ with $S(0) = I_d$. Since S has d^2 independent elements, there are $d(d+1)$ coupled nonlinear equations for $\boldsymbol{w}, S$ which remain to be solved.

By contrast, the vector transform proceeds[28] by means of the mapping $\boldsymbol{u}_i \to \boldsymbol{x}_i$ defined by:

$$\boldsymbol{x}_i = \boldsymbol{v} + \frac{(\boldsymbol{u}_i + \boldsymbol{v})(1 - \|\boldsymbol{v}\|^2)}{\|\boldsymbol{v}\|^2 + 2\boldsymbol{u}_i \cdot \boldsymbol{v} + 1}, \tag{80}$$

which maps the unit sphere to itself and also preserves the unit ball for any $\boldsymbol{v} \in \mathbb{R}^d$. The defining equations (77) are partially integrated by specifying equations to be satisfied by $\boldsymbol{v}$ and R, where $\boldsymbol{u}_i = R\boldsymbol{u}_i^0$; in this case the reduced equations are $d(d+1)/2$ in number because $R \in SO(d)$. The mapping (80) differs from (78), which does not algebraically preserve the unit sphere since $\boldsymbol{\zeta}_i$ is not a unit vector for arbitrary $\boldsymbol{w} \in \mathbb{R}^d$, whereas (80) preserves the unit sphere for any $\boldsymbol{v} \in \mathbb{R}^d$. Nevertheless, because the equation $\boldsymbol{x}_i \cdot \dot{\boldsymbol{x}}_i = (1 - \boldsymbol{x}_i \cdot \boldsymbol{x}_i)\boldsymbol{X} \cdot \boldsymbol{x}_i$

is satisfied, $\boldsymbol{x}_i$ remains a unit vector as the system evolves, whether we use (78) or (80), provided that $\boldsymbol{x}_i^0$ is a unit vector. Similarly, if we choose $\boldsymbol{x}_i^0 \in \mathbb{B}^d$, then $\boldsymbol{x}_i$ remains inside $\mathbb{B}^d$ for all $t > 0$ for that particular node i.

The cross ratios defined in (71) take the explicit form:

$$D_{ijkl}^{q=1} = \frac{(1 - \boldsymbol{x}_i \cdot \boldsymbol{x}_k)(1 - \boldsymbol{x}_j \cdot \boldsymbol{x}_l)}{(1 - \boldsymbol{x}_i \cdot \boldsymbol{x}_l)(1 - \boldsymbol{x}_j \cdot \boldsymbol{x}_k)}, \tag{81}$$

and are constants of motion as follows from Lemma 9, in agreement with Ref. 28 (equation (27)). These are a special case of the cross ratios discussed in Sec. V C, see in particular the cross ratios (44) which are constants of motion for the system (42,43) as stated in Corollary 8, which holds for any time-dependent coefficient $\boldsymbol{X}(t)$.

Let us briefly discuss synchronization properties of the system (77) as determined numerically for $d = 3$, choosing $\boldsymbol{X} = \sum_j \lambda_j \boldsymbol{x}_j/(2N)$ where λ_j are real parameters. In particular we demonstrate that synchronization can occur even if some trajectories are confined to the unit ball, and others lie on the sphere. For $d = 2$ it is known[28] that complete synchronization occurs whenever $\sum_j \lambda_j > 0$, but for $\sum_j \lambda_j < 0$ either asynchronous states appear in which all asymptotic positions $\boldsymbol{x}_i$ are distinct, or else $(N-1, 1)$ states, sometimes known as bipolar states[22], occur in which all nodes except one approach the same position. Bipolar states also occur in models of quantum synchronization[56] and are unstable if $\lambda_i = 1$ for all i. Our numerical studies indicate that similar behaviour occurs for $d > 2$, and that these properties are maintained whether trajectories lie inside the unit ball or on the unit sphere. Such behaviour occurs for $d = 2$ as shown in Fig. 1 (a,b).

In Fig. 2 with $d = 3$ we choose $N = 15, \Omega = 0$, with randomly generated values for each $\lambda_j \in [-1, 1]$, together with randomly generated initial values $\boldsymbol{x}_i^0$ that lie either inside or on S^2. We plot trajectories (shown in red) in the unit ball $\mathbb{B}^3$, with just one node confined to the surface S^2. The initial points are shown in blue, except for the single node on S^2 in green, and the asymptotic (final) locations are marked in black. In (a) we have $\sum_j \lambda_j > 0$ and complete synchronization occurs with all trajectories converging to a single point which, in the limit, lies on S^2. For (b) we have $\sum_j \lambda_j < 0$, and in this case asynchronous states appear in which the asymptotic positions $\boldsymbol{x}_i$ are distinct, with all nodes except one on S^2 (in green) remaining well inside the unit ball. The plots Fig. 2 (a,b) are generalizations to $d = 3$ of those in Fig. 1 (a,b) for $d = 2$. In (c), for which again $\sum_j \lambda_j < 0$, we obtain a bipolar state in which all nodes except one completely synchronize to a final position which is arbitrarily close to

S^2. The single node which lies exactly on S^2, for this example, is diametrically opposite the remaining asymptotic nodes. We have obtained these configurations numerically by solving both the original equations (77) and, as a consistency check, the reduced equations (79). The numerical accuracy can be estimated in each case by evaluating the constants of motion (81). The classical domain is the unit ball $\mathbb{B}^3$ and so synchronization occurs whether trajectories are confined to the sphere S^2 and/or remain inside $\mathbb{B}^3$.

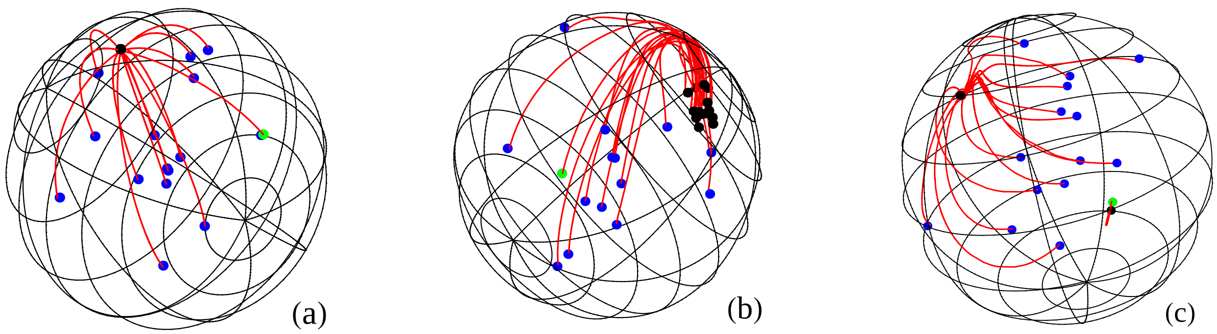

FIG. 2. Trajectories $z_i(t)$ (red) in the unit ball and on the unit sphere, with initial values marked in blue, final values in black, for $N = 15$ nodes with the frequency matrix $\Omega = 0$. $\sum_j \lambda_j$ is positive for (a), in which complete synchronization occurs, and negative for (b,c) for which asynchronous states and bipolar states appear asymptotically, respectively.

VIII. CONCLUSION

We have partially integrated a system of rectangular matrix Riccati equations of arbitrary size which in special cases is known to describe the synchronization behaviour of a nonlinear complex system of N globally connected oscillators. The Riccati equations are of a special form in which the time-dependent matrix coefficients are independent of the node, the simplest example being the Kuramoto model with identical frequencies. Partial integration reduces the system to a size independent of N, generalizing the well-known Watanabe-Strogatz reduction for the Kuramoto model. We have developed properties of the system which generalize those already known for a single matrix Riccati equation, in particular we have used linear fractional transformations to convert the equations to a form which can be partially integrated, and for square matrices have shown that the resulting constants of motion are related to the eigenvalues of matrix cross ratios. We have derived symmetries and other properties of these matrix cross ratios and have shown how to extend the definition to rectangular matrices, for the special case in which the Riccati system satisfies unitarity conditions.

The matrix unknowns can be regarded as elements of classical domains which are invariant under the group of linear fractional transformations. Numerical examples show that trajectories can lie either within or on the boundary of the classical domain, and yet complete synchronization can still occur for this mixed case. We have considered several special cases for which synchronization has been proved to occur, including the hyperbolic Kuramoto model for which the group is $SO^+(1,1)$, and the non-Abelian model for $SU(2)$.

It remains to use the reduced equations to derive explicit synchronization properties for general models, including those with rectangular matrices, and determine conditions under which complete synchronization occurs as a function of the underlying parameters. Rectangular matrix models have been studied only for the vector and square matrix cases, although there have been some recent extensions[72]. Having developed properties of these restricted models (assuming global coupling, for example), it is straightforward to formulate more general models of synchronization in which the coefficients of the Riccati system depend on the node, and so have cubic nonlinearities. It would be of interest to determine the effect of nontrivial network couplings or phase lag parameters, for example, for models in which partial integrability no longer holds. It is also an open question as to whether there are models of synchronization corresponding to each of the four classical domains[63] as classified by Cartan[73]. We have considered only those of type I which have an explicit realization as the space of $p \times q$ matrices Z_i satisfying $I_p > Z_i Z_i^\dagger$, or $I_p = Z_i Z_i^\dagger$.

Appendix A: Time evolution of the matrix cross ratios

We prove here Theorem 3. From (27) $C_{ijkl} = Z_{ik} Z_{il}^{-1} Z_{jl} Z_{jk}^{-1}$, and so

$$
\begin{aligned}
\dot{Z}_{ij} = \dot{Z}_i - \dot{Z}_j &= \Gamma_1 + \mathrm{i}\Omega_1 Z_i + \mathrm{i} Z_i \Omega_2 - Z_i \Gamma_2 Z_i - (\Gamma_1 + \mathrm{i}\Omega_1 Z_j + \mathrm{i} Z_j \Omega_2 - Z_j \Gamma_2 Z_j) \\
&= \mathrm{i}\Omega_1 Z_{ij} + \mathrm{i} Z_{ij} \Omega_2 - Z_i \Gamma_2 Z_i + Z_j \Gamma_2 Z_j,
\end{aligned}
$$

leading to:

$$\begin{aligned}\dot{C}_{ijkl} &= \dot{Z}_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} - Z_{ik}Z_{il}^{-1}\dot{Z}_{il}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} + Z_{ik}Z_{il}^{-1}\dot{Z}_{jl}Z_{jk}^{-1} - Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}\dot{Z}_{jk}Z_{jk}^{-1} \\ &= \mathrm{i}\Omega_1 Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} - Z_i\Gamma_2 Z_i Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} + Z_k\Gamma_2 Z_k Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} \\ &\quad + Z_{ik}Z_{il}^{-1}Z_i\Gamma_2 Z_i Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} - Z_{ik}Z_{il}^{-1}Z_l\Gamma_2 Z_l Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} \\ &\quad - Z_{ik}Z_{il}^{-1}Z_j\Gamma_2 Z_j Z_{jk}^{-1} + Z_{ik}Z_{il}^{-1}Z_l\Gamma_2 Z_l Z_{jk}^{-1} \\ &\quad - \mathrm{i}Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}\Omega_1 + Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}Z_j\Gamma_2 Z_j Z_{jk}^{-1} - Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}Z_k\Gamma_2 Z_k Z_{jk}^{-1},\end{aligned}$$

where we have cancelled multiplicative factors and additive terms wherever possible. In particular, the terms involving Ω_2 cancel out completely. Collecting all terms, as well as subtracting and adding $Z_{ik}Z_{il}^{-1}Z_l\Gamma_2 Z_{jl}Z_{jk}^{-1}$, we obtain:

$$\begin{aligned}\dot{C}_{ijkl} &= [C_{ijkl}, -\mathrm{i}\Omega_1] \\ &\quad + \left(-Z_i\Gamma_2 Z_i + Z_k\Gamma_2 Z_k + Z_{ik}Z_{il}^{-1}Z_i\Gamma_2 Z_i - Z_{ik}Z_{il}^{-1}Z_l\Gamma_2 Z_l - Z_{ik}Z_{il}^{-1}Z_l\Gamma_2 Z_{il}\right) Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} \\ &\quad + Z_{ik}Z_{il}^{-1}\left(-Z_j\Gamma_2 Z_j + Z_l\Gamma_2 Z_l + Z_{jl}Z_{jk}^{-1}Z_j\Gamma_2 Z_j - Z_{jl}Z_{jk}^{-1}Z_k\Gamma_2 Z_k + Z_l\Gamma_2 Z_{jl}\right) Z_{jk}^{-1}.\end{aligned} \tag{A1}$$

We also have (using $Z_{il} = Z_i - Z_l, Z_{ik} = Z_i - Z_k, Z_{jk} = Z_j - Z_k$):

$$\begin{aligned}-Z_i\Gamma_2 Z_i + Z_k\Gamma_2 Z_k + Z_{ik}Z_{il}^{-1}Z_i\Gamma_2 Z_i - Z_{ik}Z_{il}^{-1}Z_l\Gamma_2 Z_l - Z_{ik}Z_{il}^{-1}Z_l\Gamma_2 Z_{il} &= -Z_k\Gamma_2 Z_{ik}, \\ -Z_j\Gamma_2 Z_j + Z_l\Gamma_2 Z_l + Z_{jl}Z_{jk}^{-1}Z_j\Gamma_2 Z_j - Z_{jl}Z_{jk}^{-1}Z_k\Gamma_2 Z_k + Z_l\Gamma_2 Z_{jl} &= Z_{jl}Z_{jk}^{-1}Z_k\Gamma_2 Z_{jk},\end{aligned}$$

which, after substitution into (A1), leads to the required result $\dot{C}_{ijkl} = [C_{ijkl}, Z_k\Gamma_2 - \mathrm{i}\,\Omega_1]$.

Appendix B: Transformation of matrix cross ratios

We prove here Theorem 5. We decompose g as shown in (12) and consider separately the effect on the matrix cross ratio of upper, lower and diagonal matrix transformations. For $A = I_d = D$ and $C = 0$ (10) reads $Z_i \to Z_i + B$ and so Z_{ij} is unchanged, as is C_{ijkl}. For $A = I_d$ and $B = 0 = C$ (10) reads $Z_i \to Z_i D^{-1}$ and so $Z_{ij} \to Z_{ij}D^{-1}$ in which case

$$C_{ijkl} = Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} \to Z_{ik}D^{-1}DZ_{il}^{-1}Z_{jl}D^{-1}DZ_{jk}^{-1} = C_{ijkl}, \tag{B1}$$

and so C_{ijkl} again remains invariant. For $B = 0 = C$ and $D = I_d$ (10) reads $Z_i \to AZ_i$, in which case $Z_{ij} \to AZ_{ij}$ and

$$C_{ijkl} \to AZ_{ik}Z_{il}^{-1}A^{-1}AZ_{jl}Z_{jk}^{-1}A^{-1} = AC_{ijkl}A^{-1}. \tag{B2}$$

Finally, for $A = I_d = D$ and $B = 0$ (10) reads $Z_i \to Z_i(CZ_i+I_d)^{-1} = C^{-1}-C^{-1}(CZ_i+I_d)^{-1}$, and so

$$
\begin{aligned}
Z_{ij} = Z_i - Z_j &\to C^{-1} - C^{-1}(CZ_i + I_d)^{-1} - C^{-1} + C^{-1}(CZ_j + I_d)^{-1} \\
&= C^{-1}(CZ_i + I_d)^{-1}C\ Z_{ij}\ (CZ_j + I_d)^{-1},
\end{aligned}
\tag{B3}
$$

which gives

$$
\begin{aligned}
C_{ijkl} &= Z_{ki}Z_{li}^{-1}Z_{lj}Z_{kj}^{-1} \to C^{-1}(CZ_k + I_d)^{-1}CZ_{ki}(CZ_i + I_d)^{-1}(CZ_i + I_d)Z_{li}^{-1}C^{-1} \\
&\quad\times(CZ_l + I_d)CC^{-1}(CZ_l + I_d)^{-1}CZ_{lj}(CZ_j + I_d)^{-1}(CZ_j + I_d)Z_{kj}^{-1}C^{-1}(CZ_k + I_d)C \\
&= C^{-1}(CZ_k + I_d)^{-1}CZ_{ki}Z_{li}^{-1}Z_{lj}Z_{kj}^{-1}C^{-1}(CZ_k + I_d)C \\
&= (Z_kC + I_d)^{-1}C_{ijkl}\ (Z_kC + I_d).
\end{aligned}
\tag{B4}
$$

From the decomposition (12), by applying successive factors from the right, we obtain:

$$
\begin{aligned}
Z_k &\to Z_k' = Z_k + A^{-1}B \to Z_k'' = Z_k'(D - CA^{-1}B)^{-1} = (Z_k + A^{-1}B)(D - CA^{-1}B)^{-1} \\
&\to Z_k''' = AZ_k'' \to Z_k'''' = Z_k'''(CA^{-1}Z_k''' + I_d)^{-1} = AZ_k''(CZ_k'' + I_d)^{-1},
\end{aligned}
\tag{B5}
$$

which, on substituting for Z_k'', correctly reproduces (10), while for C_{ijkl} we find:

$$
\begin{aligned}
C_{ijkl} &\to C_{ijkl}' = C_{ijkl} \to C_{ijkl}'' = C_{ijkl} \to C_{ijkl}''' = AC_{ijkl}A^{-1} \\
&\to C_{ijkl}'''' = (Z_k'''CA^{-1} + I_d)^{-1}C_{ijkl}'''\ (Z_k'''CA^{-1} + I_d) \\
&= (AZ_k''CA^{-1} + I_d)^{-1}A\,C_{ijkl}\,A^{-1}(AZ_k''CA^{-1} + I_d) = P_kC_{ijkl}P_k^{-1},
\end{aligned}
\tag{B6}
$$

where

$$
P_k^{-1} = Z_k''CA^{-1} + A^{-1} = (Z_k + A^{-1}B)(D - CA^{-1}B)^{-1}CA^{-1} + A^{-1}. \tag{B7}
$$

We can rearrange this expression by means of the Sherman-Morrison-Woodbury formula[74], which reads:

$$
(A - UV)^{-1} = A^{-1} + A^{-1}U(I_d - VA^{-1}U)^{-1}VA^{-1}.
$$

We set $UV = (AZ_k + B)(CZ_k + D)^{-1}C$ in this formula to obtain (30).

Appendix C: Rectangular matrix cross ratios

We prove here Lemma 9. From (65) we obtain

$$
\dot{Z}_i^\dagger = \Gamma^\dagger - \mathrm{i}\, Z_i^\dagger\Omega_1 - \mathrm{i}\,\Omega_2 Z_i^\dagger - Z_i^\dagger\Gamma Z_i^\dagger. \tag{C1}
$$

Define the $q \times q$ matrix $Z_{ij} = I_q - Z_j^\dagger Z_i = Z_{ji}^\dagger$ then from (65) and (C1) we obtain

$$\begin{aligned}
\dot{Z}_{ij} &= -\frac{d}{dt}\left(Z_j^\dagger Z_i\right) = -\dot{Z}_j^\dagger Z_i - Z_j^\dagger \dot{Z}_i \\
&= -\Gamma^\dagger Z_i + \mathrm{i}\, Z_j^\dagger \Omega_1 Z_i + \mathrm{i}\, \Omega_2 Z_j^\dagger Z_i + Z_j^\dagger \Gamma Z_j^\dagger Z_i - Z_j^\dagger \Gamma - \mathrm{i}\, Z_j^\dagger \Omega_1 Z_i - \mathrm{i}\, Z_j^\dagger Z_i \Omega_2 + Z_j^\dagger Z_i \Gamma^\dagger Z_i \\
&= -\mathrm{i}\, \Omega_2 Z_{ij} + \mathrm{i}\, Z_{ij}\Omega_2 - Z_j^\dagger \Gamma Z_{ij} - Z_{ij}\Gamma^\dagger Z_i.
\end{aligned} \tag{C2}$$

From the definition (71), $D^q_{ijkl} = Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}$, and by cancelling multiplicative factors and additive terms wherever possible, we obtain:

$$\begin{aligned}
\dot{D}^q_{ijkl} &= \dot{Z}_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} - Z_{ik}Z_{il}^{-1}\dot{Z}_{il}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} + Z_{ik}Z_{il}^{-1}\dot{Z}_{jl}Z_{jk}^{-1} - Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}\dot{Z}_{jk}Z_{jk}^{-1} \\
&= -\mathrm{i}\, \Omega_2 Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} + \mathrm{i}\, Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}\Omega_2 - Z_k^\dagger \Gamma Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} + Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}Z_k^\dagger \Gamma \\
&= -\mathrm{i}\, \Omega_2 D^q_{ijkl} + \mathrm{i}\, D^q_{ijkl}\Omega_2 - Z_k^\dagger \Gamma D^q_{ijkl} + D^q_{ijkl} Z_k^\dagger \Gamma = [D^q_{ijkl}, Z_k^\dagger \Gamma + \mathrm{i}\, \Omega_2].
\end{aligned}$$

Similarly, with the definition $Z_{ij} = I_p - Z_j Z_i^\dagger = Z_{ji}^\dagger$ we obtain

$$\begin{aligned}
\dot{Z}_{ij} &= -\frac{d}{dt}\left(Z_j Z_i^\dagger\right) = -\dot{Z}_j Z_i^\dagger - Z_j \dot{Z}_i^\dagger \\
&= -\Gamma Z_i^\dagger - \mathrm{i}\, \Omega_1 Z_j Z_i^\dagger - \mathrm{i}\, Z_j \Omega_2 Z_i^\dagger + Z_j \Gamma^\dagger Z_j Z_i^\dagger - Z_j \Gamma^\dagger + \mathrm{i}\, Z_j Z_i^\dagger \Omega_1 + \mathrm{i}\, Z_j \Omega_2 Z_i^\dagger + Z_j Z_i^\dagger \Gamma Z_i^\dagger \\
&= \mathrm{i}\, \Omega_1 Z_{ij} - \mathrm{i}\, Z_{ij}\Omega_1 - Z_j \Gamma^\dagger Z_{ij} - Z_{ij}\Gamma Z_i^\dagger.
\end{aligned} \tag{C3}$$

Then, using $D^p_{ijkl} = Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}$ we obtain:

$$\begin{aligned}
\dot{D}^p_{ijkl} &= \dot{Z}_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} - Z_{ik}Z_{il}^{-1}\dot{Z}_{il}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} + Z_{ik}Z_{il}^{-1}\dot{Z}_{jl}Z_{jk}^{-1} - Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}\dot{Z}_{jk}Z_{jk}^{-1} \\
&= (\mathrm{i}\, \Omega_1 Z_{ik} - \mathrm{i}\, Z_{ik}\Omega_1 - Z_k \Gamma^\dagger Z_{ik} - Z_{ik}\Gamma Z_i^\dagger) Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} \\
&\quad - Z_{ik}Z_{il}^{-1}(\mathrm{i}\, \Omega_1 Z_{il} - \mathrm{i}\, Z_{il}\Omega_1 - Z_l \Gamma^\dagger Z_{il} - Z_{il}\Gamma Z_i^\dagger) Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} \\
&\quad + Z_{ik}Z_{il}^{-1}(\mathrm{i}\, \Omega_1 Z_{jl} - \mathrm{i}\, Z_{jl}\Omega_1 - Z_l \Gamma^\dagger Z_{jl} - Z_{jl}\Gamma Z_j^\dagger) Z_{jk}^{-1} \\
&\quad - Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}(\mathrm{i}\, \Omega_1 Z_{jk} - \mathrm{i}\, Z_{jk}\Omega_1 - Z_k \Gamma^\dagger Z_{jk} - Z_{jk}\Gamma Z_j^\dagger) Z_{jk}^{-1} \\
&= \mathrm{i}\, \Omega_1 Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} - Z_k \Gamma^\dagger Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1} - \mathrm{i}\, Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}\Omega_1 + Z_{ik}Z_{il}^{-1}Z_{jl}Z_{jk}^{-1}Z_k \Gamma^\dagger \\
&= \mathrm{i}\, \Omega_1 D^p_{ijkl} - Z_k \Gamma^\dagger D^p_{ijkl} - \mathrm{i}\, D^p_{ijkl}\Omega_1 + D^p_{ijkl} Z_k \Gamma^\dagger = [D^p_{ijkl}, Z_k \Gamma^\dagger - \mathrm{i}\, \Omega_1],
\end{aligned}$$

as required.

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