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Coarse geometry and Callias quantisation

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Coarse geometry and Callias quantisation

Hao Guo,^{*} Peter Hochs[†] and Varghese Mathai[‡]

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Abstract

Consider a proper, isometric action by a unimodular, locally compact group G on a complete Riemannian manifold M . For equivariant elliptic operators that are invertible outside a cocompact subset of M , we show that a localised index in the K -theory of the maximal group C^* -algebra of G is well-defined. The approach is based on the use of maximal versions of equivariant localised Roe algebras, and many of the technical arguments in this paper are used to handle the ways in which they differ from their reduced versions.

By using the maximal group C^* -algebra instead of its reduced counterpart, we can apply the trace given by integration over G to recover an index defined earlier by the last two authors, and developed further by Braverman, in terms of sections invariant under the group action. As a very special case, this allows one to refine numerical obstructions to positive scalar curvature on a noncompact Spin manifold X defined via Callias index theory, to obstructions in the K -theory of the maximal C^* -algebra of the fundamental group $\pi_1(X)$.

As a motivating application in another direction, we prove a version of Guillemin and Sternberg's quantisation commutes with reduction principle for equivariant indices of Spin^c Callias-type operators.

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1 Introduction

Background

Let M be a complete Riemannian manifold, and let D be an elliptic differential operator on a vector bundle $E \rightarrow M$. The *coarse index* [37] of D lies in $K_*(C^*(M))$, the K -theory group of the *Roe algebra* $C^*(M)$ of M . This Roe algebra is the closure in the operator norm of the algebra of locally compact, bounded operators on $L^2(E)$ that enlarge supports of sections by

a finite amount. If M is compact, then $C^*(M)$ is the algebra of compact operators, and the coarse index of D is its Fredholm index. A strength of the coarse index is that it applies very generally, without any assumptions on compactness of M , or on the behaviour of D at infinity. Coarse index theory has a range of applications, for example to Riemannian metrics of positive scalar curvature [39], and to the Novikov conjecture [43, 44]. A central role here is played by the coarse Baum–Connes conjecture [36].

The general applicability of the coarse index can come at the cost of computability. For that reason, it is worth looking for special cases, or variations, where a version of the coarse index is more explicit or computable. One useful approach is Roe’s *localised coarse index* [38]. If D^2 is positive outside a subset $Z \subset M$ in a suitable sense, then Roe constructed a *localised coarse index*

$$\text{index}^Z(D) \in K_*(C^*(Z)).$$

The special case where Z is compact is already of interest: then D is Fredholm, and its localised coarse index generalises the Gromov–Lawson index [11], the Atiyah–Patodi–Singer index on compact manifolds with boundary [2], and the index of Callias-type Dirac operators [1, 7, 25] $D = \tilde{D} + \Phi$, where \tilde{D} is a Dirac operator, and Φ is a vector bundle endomorphism making D invertible at infinity.

The localised coarse index was generalised to an equivariant version in [14], for a proper, isometric action by a unimodular locally compact group G on M , preserving all structure including D . Then, if Z/G is compact, one obtains a *localised equivariant index*

$$\text{index}_{G,\text{red}}^{\text{loc}}(D) \in K_*(C_{\text{red}}^*(G)), \tag{1.1}$$

where $C_{\text{red}}^*(G)$ is the reduced group C^* -algebra of G . The fact that this index lies in $K_*(C_{\text{red}}^*(G))$ is useful, because that K -theory group is independent of M , and it is a very well-studied object that is central to many problems in geometry, topology and group theory. In particular, it is large enough to contain relevant group-theoretic information. And importantly, there is a range of traces and higher cyclic cocycles on subalgebras of $C_{\text{red}}^*(G)$ that allows one to obtain a number from the index (1.1), for which one can then find a topological expression. Examples of such expressions are the equivariant Atiyah–Patodi–Singer index theorems in [8, 24, 42], in the case of manifolds with boundary.

Results

This paper is about the construction and application of a *maximal* localised equivariant coarse index, taking values in the K -theory of the maximal group C^* -algebra $C_{\max}^*(G)$

$$\text{index}_G^{\text{loc}}(D) \in K_*(C_{\max}^*(G)). \quad (1.2)$$

The first result in this paper is that this index is well-defined: see Propositions 3.1 and 3.3.

The index (1.2) has several advantages over (1.1). From a general point of view, the natural map from $C_{\max}^*(G)$ to $C_{\text{red}}^*(G)$ maps the index in $K_*(C_{\max}^*(G))$ to the one in $K_*(C_{\text{red}}^*(G))$, so the former is a more refined invariant. On a more practical level, the integration map $I: L^1(G) \rightarrow \mathbb{C}$ extends to a trace on $C_{\max}^*(G)$ (not on $C_{\text{red}}^*(G)$). That means it can be applied to the index (1.2), to yield the integer

$$I_*(\text{index}_G^{\text{loc}}(D)) \in \mathbb{Z}. \quad (1.3)$$

Morally, applying the integration trace I should correspond to taking the G -invariant part of the equivariant index. The second result in this paper, Theorem 3.9, is that this is indeed the case in a precise sense:

$$I_*(\text{index}_G^{\text{loc}}(D)) = \text{index}(D)^G, \quad (1.4)$$

where the right hand side is the Fredholm index of D restricted to G -invariant sections that are square integrable transversally to orbits in a certain sense. The latter index was defined in [18], and developed further by Braverman [6].

In the example where D_X is an elliptic operator on a possibly noncompact manifold X , invertible outside a compact set, and D is its lift to the universal cover of X , Theorem 3.9 implies that I_* maps the $\pi_1(X)$ -equivariant, localised, maximal index of D to the Fredholm index of D_X . This means that the index (1.2) refines the Gromov–Lawson index, the index of Callias-type operators, as well as the Atiyah–Patodi–Singer index. One application of this fact is that it leads to refinements of obstructions to Riemannian metrics of positive scalar curvature defined through the Gromov–Lawson and Callias indices on Spin manifolds. This is analogous to the way in which the image of D under the analytic assembly map [3] for the maximal group C^* -algebra of $\pi_1(X)$ refines the index of D_X in the case where X is compact. Explicit applications to positive scalar curvature will be explored in future work.

A completely different application that motivates the development of the index (1.2) and Theorem 3.9. is a version of Guillemin and Sternberg's *quantisation commutes with reduction* principle [12] for Callias-type Spin^c -Dirac operators. That principle was initially stated and proved for compact Kähler and symplectic manifolds [31, 32, 34, 40]. This principle was extended in various directions, including results for proper actions by possibly noncompact groups, with possibly noncompact orbit spaces, see [18] for the symplectic case and [19] for Spin^c -manifolds. The index, or quantisation, used in those papers, was defined just in terms of sections invariant under the group action. Furthermore, the index was only well-defined after a suitable order zero term was added to the operator in question. The first of these issues was partially remedied in [23], where the quantisation commutes with reduction principle was proved for an index with values in the completed representation ring of a maximal compact subgroup of G .

Since the work of Paradan and Vergne [35], the quantisation commutes with reduction principle is known to be a general property of equivariant indices of Spin^c -Dirac operators in general, and not just of geometric quantisation in the narrow sense. For a Callias-type operator $D = \tilde{D} + \Phi$, where \tilde{D} is a Spin^c -Dirac operator, the third result in this paper, Theorem 3.11, states that the quantisation commutes with reduction principle holds, in the sense that

$$I_*(\text{index}_G^{\text{loc}}(\tilde{D} + \Phi)) = \text{index}(D_0), \quad (1.5)$$

where D_0 is a Dirac operator on a *reduced space* M_0 , a Spin^c -analogue of a reduced space in symplectic geometry, for high enough powers of the determinant line bundle of the Spin^c -structure. In this setting, the use of the maximal localised coarse index allows us to prove such a result in the setting of noncompact groups and orbit spaces, for a truly equivariant index in $K_0(C^*(G))$, which is defined without the need of an added term.

The equality (1.5) already appears to be new in the case where G is compact. Then $\tilde{D} + \Phi$ is Fredholm, and has an equivariant index in the usual sense. In this case, a version of the shifting trick in symplectic geometry applies to yield information about the multiplicities in that index of all irreducible representations of G .

Techniques used

The key ingredient in the construction of the index (1.2) is the notion of a *maximal localised equivariant Roe algebra* for arbitrary unimodular, locally compact groups. This involves the notion of an *admissible module*, which

was defined in [45] for discrete groups, and in [14] in general. In the non-equivariant, non-localised case, the natural maximal norm for such algebras was shown to be well-defined in [10]. In the equivariant, localised case, this is less clear, and getting around this is a step in the construction of the algebras we need.

The construction of the index (1.2) is very different from the construction of the reduced version (1.1) in [14]. Instead of viewing D as an unbounded operator on $L^2(E)$, we view it as an unbounded operator on a maximal localised equivariant Roe algebra A , viewed as a Hilbert C^* -module over itself. The reason for this is that the localisation results in [38] that make the definition of the localised coarse index possible do not directly carry over to the norm on the maximal Roe algebra. Indeed, it is not even clear if the operators involved lie in the unlocalised maximal Roe algebra, let alone if they localise in a suitable way.

We prove versions of Roe's localisation results for D as an operator on A , thus allowing us to define (1.2). To do this we prove that the functional calculus for such operators on A is well-defined. This was done in [15] for the uniform maximal Roe algebra; in our setting it works for usual maximal Roe algebras due to localisation at a cocompact set.

To prove the equality (1.4), we use various *averaging maps*, which map G -equivariant operators on M to operators on M/G . Comparing such maps for operators on $L^2(E)$ and on A to the integration trace I then leads to a proof of (1.4).

Using (1.4), we see that the left hand side of (1.5) equals a more concrete index in terms of G -invariant sections. For the latter index, we obtain localisation estimates that allow us to show that this index equals the right hand side of (1.5). These localisation estimates build on those in [18, 19, 30, 40], but a fundamental difference is that we now need the key deformation term to go to zero at infinity, rather than grow towards infinity.

Outline of this paper

We start by defining equivariant localised maximal Roe algebras in Section 2. That allows us to state the three results in the paper mentioned above, in Section 3. Well-definedness of the index (1.2) is proved in Section 4. To prepare for the proof of (1.4), we construct several averaging maps in section 5. In Section 6, we use these maps to prove (1.4). We conclude this paper by using (1.4) and some localisation estimates to prove (1.5) in Section 7.

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2 Equivariant localised maximal Roe algebras

2.1 Equivariant $C_0(X)$ -modules

Let (X, d) be a metric space in which all closed balls are compact. Let G be a unimodular, locally compact group, acting properly and isometrically on X . Let dg be a Haar measure on G .

A G -equivariant $C_0(X)$ -module is a Hilbert space \mathcal{H}_X equipped with a unitary representation π of G , and a $*$ -homomorphism $\rho: C_0(X) \rightarrow \mathcal{B}(\mathcal{H}_X)$, such that for all $g \in G$ and $\varphi \in C_0(X)$,

$$\pi(g)\rho(\varphi)\pi(g)^{-1} = \rho(g \cdot \varphi).$$

Here $(g \cdot \varphi)(x) = \varphi(g^{-1}x)$, for all $x \in X$. We will omit the representations π and ρ from the notation, and for example write $\varphi \cdot \xi := \rho(\varphi)\xi$, for $\varphi \in C_0(X)$ and $\xi \in \mathcal{H}_X$.

Fix a G -equivariant $C_0(X)$ -module \mathcal{H}_X . Let $\mathcal{B}(\mathcal{H}_X)^G$ be the algebra of G -equivariant bounded operators on \mathcal{H}_X . An operator $T \in \mathcal{B}(\mathcal{H}_X)$ is said to be *locally compact* if for all $\varphi \in C_0(X)$, the operators φT and $T\varphi$ are compact. And T has *finite propagation* if there is an $r > 0$ such that for all $\varphi, \psi \in C_0(X)$ whose supports are at least a distance r apart,

$$\varphi T \psi = 0.$$

In that case, the infimum of such numbers r is the *propagation* of T . The G -equivariant reduced Roe algebra of X with respect to \mathcal{H}_X is the closure in the operator norm of the algebra of locally compact operators in $\mathcal{B}(\mathcal{H}_X)^G$ with finite propagation. In this paper, we will use an algebra that differs from the equivariant reduced Roe algebra in two ways: we consider a *localised* version, and complete it in a *maximal* norm.

A relevant example of a G -equivariant $C_0(X)$ -module is the space $L^2(E)$ of square integrable sections of a G -equivariant, Hermitian vector bundle $E \rightarrow X$, with respect to a G -invariant measure dx on X . The algebra

$C_0(X)$ acts on $L^2(E)$ by pointwise multiplication, and G acts in the usual way. Consider the vector bundle $\text{Hom}(E) := E \boxtimes E^* \rightarrow X \times X$. Let $C_{\text{ker}}^*(X; L^2(E))^G$ be the algebra of locally compact operators $T \in \mathcal{B}(L^2(E))^G$ with finite propagation, for which there is a bounded, measurable¹ section κ of $\text{Hom}(E)$ such that for all $s \in L^2(E)$ and $x \in X$,

$$(Ts)(x) = \int_X \kappa(x, x')s(x') dx'.$$

We will identify such operators with their kernels κ .

2.2 Admissible modules and the maximal Roe algebra

In Definition 2.2 in [45], the notion of an *admissible* Γ -equivariant $C_0(X)$ -module was introduced, for discrete groups Γ . In Definition 2.4 in [14], this was extended to general unimodular, locally compact groups G . The main difference between the discrete and general group case is the role played by local slices in the sense of Palais [33] in the non-discrete case.

A G -equivariant $C_0(X)$ -module \mathcal{H}_X is defined to be admissible if there is a G -equivariant, unitary isomorphism

$$\mathcal{H}_X \cong L^2(G) \otimes \mathcal{H},$$

for a Hilbert space \mathcal{H} , such that locally compact operators on \mathcal{H}_X are mapped to locally compact operators on $L^2(G) \otimes \mathcal{H}$, and operators with finite propagation are mapped to operators with finite propagation, in both cases with respect to the pointwise action by $C_0(G)$.

The point of using admissible modules is that the resulting equivariant Roe algebras encode the relevant group-theoretic information. It is clear that such information may be lost in the example where X is a point, acted on trivially by a compact group, and one uses the non-admissible module \mathbb{C} .

By Theorem 2.7 in [14], an example of an admissible module is $L^2(E) \otimes L^2(G)$. Here $E \rightarrow X$ is as at the end of the previous subsection, $C_0(X)$ acts pointwise on the factor $L^2(E)$, and G acts diagonally, with respect to the left regular representation of G in $L^2(G)$. By definition of admissibility, we have an isomorphism

$$L^2(E) \otimes L^2(G) \cong L^2(G) \otimes \mathcal{H} \tag{2.1}$$

¹One can also work with continuous sections, the main reason why we use measurable sections is that the map $\kappa \mapsto \tilde{\kappa}$ in Proposition 4.5 does not preserve continuity.

with the properties above. Let $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ be the algebra of locally compact operators on $L^2(E) \otimes L^2(G)$ with finite propagation, given by bounded, measurable kernels

$$\kappa_G: G \times G \rightarrow \mathcal{K}(\mathcal{H})$$

via the isomorphism (2.1). Explicitly, for such a κ_G , the corresponding operator T is defined by

$$(T(f \otimes \xi))(g) = \int_G f(g') \kappa(g, g') \xi dg',$$

for $f \in L^2(G)$, $\xi \in \mathcal{H}$ and $g \in G$. If X/G is compact, then Theorem 2.11 in [14] states that $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ is isomorphic to a dense subalgebra of $C^*(G) \otimes \mathcal{K}(\mathcal{H})$, where $C^*(G)$ is either the reduced or maximal group C^* -algebra of G . (To be precise, $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ is isomorphic to the convolution algebra of compactly supported, bounded, measurable functions on G with values in the algebra of compact operators on \mathcal{H} .) This implies that the *maximal norm* of an element $\kappa \in C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$,

$$\|\kappa\|_{\max} := \sup_{\eta} \|\eta(\kappa)\|_{\mathcal{B}(\mathcal{H}_{\eta})}, \quad (2.2)$$

where the supremum is over all $*$ -representations

$$\eta: C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G \rightarrow \mathcal{B}(\mathcal{H}_{\eta}),$$

is finite. This norm is equal to the tensor product norm on $C_{\max}^*(G) \otimes \mathcal{K}(\mathcal{H})$. So the completion of $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ in the maximal norm equals

$$C_{\max}^*(X; L^2(E) \otimes L^2(G))^G \cong C_{\max}^*(G) \otimes \mathcal{K}(\mathcal{H}). \quad (2.3)$$

Since this algebra is independent of the admissible module used, we will denote it by

$$C_{\max}^*(X)^G := C_{\max}^*(X; L^2(E) \otimes L^2(G))^G.$$

Remark 2.1. In the case where G is trivial, and X is not assumed to be compact but is only assumed to have bounded geometry, finiteness of the maximal norm (2.2) was proved by Gong, Wang and Yu, see Lemma 3.4 in [10]. See also Lemma 1.10 in [41]. It is our understanding that this generalises directly to free actions by discrete groups, see Lemma 3.16 in [10].

2.3 The map $\oplus 0$ and the maximal norm for non-admissible modules

We will use a completion of $C_{\ker}^*(X; L^2(E))^G$ in a version of the maximal norm. It is unclear a priori if an analogue of the supremum (2.2) is finite, however. We therefore define the norm we use via an embedding of $C_{\ker}^*(X; L^2(E))^G$ into $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$, which has a well-defined maximal norm if X/G is compact, as we saw at the end of the previous subsection.

Let $C_{\text{alg}}^*(X; \mathcal{H}_X)^G$ be the algebra of bounded, G -equivariant, locally compact operators on an equivariant $C_0(X)$ -module \mathcal{H}_X , with finite propagation. In Section 3.2 in [14], a map

$$\oplus 0: C_{\text{alg}}^*(X; L^2(E))^G \rightarrow C_{\text{alg}}^*(X; L^2(E) \otimes L^2(G))^G \quad (2.4)$$

is defined as follows. Let $\chi \in C(X)$ be a function whose support has compact intersections with all G -orbits, and has the property that for all $x \in X$,

$$\int_G \chi(gx)^2 dx = 1. \quad (2.5)$$

(The integrand is compactly supported by properness of the action.) Such a function will be called a *cutoff function*. The map $j: L^2(E) \rightarrow L^2(E) \otimes L^2(G)$, given by

$$(j(s))(x, g) = \chi(g^{-1}x)s(x)$$

for $s \in L^2(E)$, $x \in X$ and $g \in G$, is an isometric, G -equivariant embedding. Let $p: L^2(E) \otimes L^2(G) \rightarrow j(L^2(E))$ be the orthogonal projection. The map

$$\oplus 0: \mathcal{B}(L^2(E)) \rightarrow \mathcal{B}(L^2(E) \otimes L^2(G)) \quad (2.6)$$

that maps $T \in \mathcal{B}(L^2(E))$ to $jTj^{-1}p$ is an injective $*$ -homomorphism, and preserves equivariance, local compactness, and finite propagation. Hence it restricts to an injective $*$ -homomorphism (2.4). (The notation $\oplus 0$ reflects the fact that $T \oplus 0$ equals jTj^{-1} on the image of j , and zero on its orthogonal complement.)

Lemma 2.2. *The map (2.4) maps $C_{\ker}^*(X; L^2(E))^G$ into $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$.*

Proposition 4.5 is a refinement of this lemma. If X/G is compact, then for $\kappa \in C_{\ker}^*(X; L^2(E))^G$, we define its G -maximal norm as

$$\|\kappa\|_{\max, G} := \|\kappa \oplus 0\|_{\max}.$$

We denote the completion of $C_{\ker}^*(X; L^2(E))^G$ in this norm by $C_{\max}^*(X; L^2(E))^G$

Remark 2.3. In the case of reduced Roe algebras, defined with respect to the operator norm for a $C_0(X)$ -module, the algebra $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G$ is dense in $C_{\text{alg}}^*(X; L^2(E) \otimes L^2(G))^G$. See Proposition 5.11 in [14]. In that case, kernels and operators can be used more or less interchangeably, but this is less clear for the maximal completions we use here.

2.4 Localised maximal Roe algebras

Let $Z \subset X$ be a G -invariant subset. Let \mathcal{H}_X be a G -equivariant $C_0(X)$ -module. An operator $T \in \mathcal{B}(\mathcal{H}_X)^G$ is *supported near Z* if there is an $r > 0$ such that for all φ whose support is at least a distance r away from Z , the operators φT and $T\varphi$ are zero. Let $C_{\ker}^*(X; Z, \mathcal{H}_X)^G$ be the algebra of elements of $C_{\ker}^*(X; \mathcal{H}_X)^G$ supported near Z .

For $r \geq 0$ and any subset $Y \subset X$, we write

$$\text{Pen}(Y, r) := \{x \in X; d(x, Y) \leq r\}.$$

Then we have a natural isomorphism

$$C_{\ker}^*(X; Z, \mathcal{H}_X)^G = \varinjlim_r C_{\ker}^*(\text{Pen}(Z, r); \mathcal{H}_X)^G. \quad (2.7)$$

Now suppose that Z/G is compact. The algebra $C_{\ker}^*(X; Z, \mathcal{H}_X)^G$ is then independent of Z , as long as Z/G is compact. For this reason, we write

$$C_{\ker}^*(X; \mathcal{H}_X)_{\text{loc}}^G := C_{\ker}^*(X; Z, \mathcal{H}_X)^G.$$

For every $r > 0$, we have the norm $\|\cdot\|_{\max, G}$ on $C_{\ker}^*(\text{Pen}(Z, r); L^2(E))^G$. Let $\|\cdot\|_{\max, G}$ be the resulting norm on $C_{\ker}^*(X, L^2(E))_{\text{loc}}^G$ via (2.7).

Definition 2.4. The *localised, G -maximal equivariant Roe algebra of X for $L^2(E)$* , denoted by $C_{\max}^*(X; L^2(E))_{\text{loc}}^G$, is the completion of $C_{\ker}^*(X, L^2(E))_{\text{loc}}^G$ in the norm $\|\cdot\|_{\max, G}$.

The *localised, G -maximal equivariant Roe algebra of X* , denoted by $C_{\max}^*(X)_{\text{loc}}^G$, is the completion of $C_{\ker}^*(X; L^2(E) \otimes L^2(G))_{\text{loc}}^G$ in the norm $\|\cdot\|_{\max}$.

By construction, $C_{\max}^*(X; L^2(E))_{\text{loc}}^G$ is isometrically embedded into $C_{\max}^*(X)_{\text{loc}}^G$. By (2.3) and (2.7),

$$C_{\max}^*(X)_{\text{loc}}^G \cong C_{\max}^*(G) \otimes \mathcal{K}(\mathcal{H}).$$

3 Results

Our first result is the fact that a maximal version of the localised equivariant index of [14] is well-defined, see Propositions 3.1 and 3.3 and Definition 3.4. We will show that that index is an equivariant refinement of the index defined in terms of invariant sections in [6, 18, 30], see Theorem 3.9. The quantisation commutes with reduction results for proper, non-cocompact actions in [18, 20] only involved sections invariant under a group action. In Theorem 3.11, we generalise this to the equivariant index of Definition 3.4, in the case of Callias-type Spin^c-Dirac operators.

3.1 The localised maximal equivariant index

From now on, we suppose that $X = M$, a complete Riemannian manifold, and that d is the Riemannian distance corresponding to a G -invariant Riemannian metric. We suppose that $E \rightarrow M$ is a smooth, G -equivariant, Hermitian vector bundle and D a symmetric, first order, elliptic, G -equivariant differential operator on sections of E . Suppose that D has finite propagation speed, i.e. if σ_D is its principal symbol, then

$$\sup\{\|\sigma_D(\xi)\|; \xi \in T^*M, \|\xi\| = 1\} < \infty.$$

Then D is essentially self adjoint as an unbounded operator on $L^2(E)$, see Proposition 10.2.11 in [17].

Let $Z \subset M$ be a closed, cocompact G -invariant subset. Let $C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$ be the algebra of smooth kernels in $C_{\ker}^*(M; L^2(E))_{\text{loc}}^G$. Then D acts on $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$ by

$$(D\kappa)(m, m') := (D \otimes 1_{E_{m'}^*}(\kappa(-, m')))(m).$$

Here we used the fact that for every $m' \in M$, $\kappa(-, m')$ is a smooth section of $E \otimes E_{m'}^*$.

For A a C^* -algebra and \mathcal{M} a Hilbert A -module, we write $\mathcal{L}_A(\mathcal{M})$ and $\mathcal{K}_A(\mathcal{M})$ for the C^* -algebras of bounded adjointable operators and compact operators on \mathcal{M} , respectively. We can view A as a right Hilbert C^* -module over itself, with A -valued inner product.

$$\langle a, b \rangle := a^*b, \tag{3.1}$$

for $a, b \in A$. Then $\mathcal{K}_A(A) \cong A$, with the isomorphism being given by identifying the operator

$$\theta_{a,b} : c \mapsto a\langle b, c \rangle$$

with left multiplication by ab^* . We also have that $\mathcal{L}_A(A)$ is the multiplier algebra of $\mathcal{K}_A(A)$.

To simplify notation, we will from now on use A to denote the G -maximal, localised equivariant Roe algebra $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$. Then A is a Hilbert module over itself. We will use functional calculus for self-adjoint, regular operators on the Hilbert A -module A . (For a uniform version of the maximal Roe algebra, this was developed in [15].) This functional calculus applies to D because of the following result.

Proposition 3.1. *The unbounded operator D on the Hilbert A -module A is essentially self-adjoint and regular.*

This proposition is proved in Subsection 4.4. Because of Proposition 3.1, we can apply the following general result (see [26], [16] Theorem 3.1 and [9] Theorem 1.19) about functional calculus on Hilbert C^* -modules to the self-adjoint closure of D .

Theorem 3.2. *Let B be a C^* -algebra and \mathcal{M} a Hilbert B -module. Let $C(\mathbb{R})$ be the $*$ -algebra of complex-valued continuous functions on \mathbb{R} . For any regular, essentially self-adjoint operator T on \mathcal{M} , there is a $*$ -preserving linear map*

$$\pi_T : C(\mathbb{R}) \rightarrow \mathcal{R}_B(\mathcal{M}),$$

with values in the set $\mathcal{R}_B(\mathcal{M})$ of regular operators on \mathcal{M} , such that:

- (i) π_T restricts to a $*$ -homomorphism $\pi_T : C_b(\mathbb{R}) \rightarrow \mathcal{L}_B(\mathcal{M})$;
- (ii) If $|f(t)| \leq |g(t)|$ for all $t \in \mathbb{R}$, then $\text{dom}(\pi_T(g)) \subseteq \text{dom}(\pi_T(f))$;
- (iii) If $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C(\mathbb{R})$ for which there exists $F \in C(\mathbb{R})$ such that $|f_n(t)| \leq F(t)$ for all $t \in \mathbb{R}$, and if f_n converge to a limit function $f \in C(\mathbb{R})$ uniformly on compact subsets of \mathbb{R} , then $\pi_T(f_n)x \mapsto \pi_T(f)x$ for each $x \in \text{dom}(\pi_T(f))$;
- (iv) $\pi_T(\text{Id}) = T$;
- (v) $\|\pi_T(f)\|_{\mathcal{L}_B(\mathcal{M})} \leq \sup_{\lambda \in \text{spec}_{\mathcal{M}}(T)} |f(\lambda)|$;

In the context of this theorem, we write $f(T) := \pi_T(f)$.

Suppose that there are a G -equivariant, Hermitian vector bundle $F \rightarrow M$, a differential operator $P: \Gamma^\infty(E) \rightarrow \Gamma^\infty(F)$, a G -equivariant vector bundle endomorphism R of E , and a constant $c > 0$ such that

$$D^2 = P^*P + R, \tag{3.2}$$

and $R \geq c^2$, fibrewise outside Z . (The use of c^2 instead of c is a convention here, which implies that $D \geq c$ outside Z in an appropriate sense.)

In this setting, and when G is trivial but without assuming Z to be compact, Roe [38] developed localised index theory with values in the K -theory of a reduced completion of $C_{\text{alg}}^*(M; Z, L^2(E))$. We will use an equivariant version of this index theory for the maximal completion, in terms of admissible modules. The reason for using the maximal completion is that we then obtain an index in the K -theory of $C_{\text{max}}^*(G)$, to which we can apply an integration map to recover the G -invariant index from [18] as a special case, see Theorem 3.9. The construction of the localised index is based on the following analogue of Lemma 2.3 and Theorem 2.4 in [38].

Proposition 3.3. *If $f \in C_c(\mathbb{R})$ is supported in $[-c, c]$, then*

$$f(D) \in A = \mathcal{K}_A(A) \subset \mathcal{L}_A(A).$$

This proposition is proved in Subsection 4.5

Let $b: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, increasing, odd function, such that $b(x) = \pm 1$ for all $x \in \mathbb{R}$ with $|x| \geq c$. Then $b^2 - 1$ has the property of the function f in Proposition 3.3. So, in particular, $b(D) \in \mathcal{L}_A(A)$ is invertible modulo $\mathcal{K}_A(A)$, and hence has an index in

$$K_*(\mathcal{K}_A(A)) = K_*(A).$$

This index lies in even K -theory if D is odd with respect to a G -invariant grading on E , and in odd K -theory otherwise. See for example Definition 3.2 in [14] for details.

Explicitly, consider the case where D is odd with respect to a G -invariant grading $E = E_+ \oplus E_-$. Let $C_{\text{ker}}^\infty(X; L^2(E))_{\text{loc}}^G$ be the algebra of kernels in $C_{\text{ker}}^\infty(X; L^2(E))^G$ supported near Z . Let $b(D)_+$ be the restriction of $b(D)$ to kernels in $C_{\text{ker}}^\infty(X; L^2(E))_{\text{loc}}^G$ that are sections of $E_+ \otimes E^*$. Then $b(D)_+$ is invertible modulo $\mathcal{K}_A(A)$, and its inverse is the restriction of $b(D)$ to $E_- \otimes E^*$. Hence this operator defines a class $[b(D)_+] \in K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A))$, and the index of $b(D)$ is defined as

$$\partial[b(D)_+] \in K_0(A), \tag{3.3}$$

where $\partial: K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A)) \rightarrow K_0(\mathcal{K}_A(A))$ is the boundary map in the six-term exact sequence corresponding to the ideal $\mathcal{K}_A(A) \subset \mathcal{L}_A(A)$. For ungraded operators, one uses the projection $\frac{1}{2}(b(D) + 1)$ in $\mathcal{L}_A(A)/\mathcal{K}_A(A)$ and applies the boundary map to its class in even K -theory to obtain the index of $b(D)$ in $K_1(A)$.

Definition 3.4. The *localised, maximal, equivariant index* of D is the image of the index of $b(D)$ in $K_*(A)$ described above under the map

$$\oplus 0: K_*(A) \rightarrow K_*(C_{\max}^*(G)).$$

It is denoted by $\text{index}_G^{\text{loc}}(D)$.

Remark 3.5. One could consider (3.3) (and its analogue in $K_1(A)$ in the non-graded case) as a localised index of D , defined in terms of the non-admissible $C_0(M)$ -module $L^2(E)$. Two advantages of the index in Definition 3.4 over (3.3) are that it takes values in a K -theory group independent of X or E , and that the application of the map $\oplus 0$ on K -theory means that the index of Definition 3.4 captures group-theoretic information that is not encoded in (3.3). This is clear in the example where G is compact, M is a point, and D is the zero operator on $E = V \in \hat{K}$, as discussed in Example 3.8 in [14]. This illustrates why it is useful to use the admissible $C_0(M)$ -module $L^2(E) \otimes L^2(G)$.

Another approach to constructing the index of Definition 3.4 would be to use an extension map $\oplus 1$, extending operators by the identity operator on the orthogonal complement to $j(L^2(E))$, before applying boundary maps. See Definition 3.6 and Lemma 3.7 in [14].

Example 3.6. If D is a Dirac-type operator associated to a Clifford connection ∇ on E , then

$$D^2 = \nabla^* \nabla + R,$$

for a vector bundle endomorphism R of E . (If D is a Spin-Dirac operator, then R is scalar multiplication by a quarter of scalar curvature, by Lichnerowicz' formula.) If $R \geq c^2$ outside Z , then the condition on D holds, with $F = E \otimes T^*M$ and $P = \nabla$. This is the situation considered in [38], for G trivial.

Example 3.7. Let \tilde{D} be a G -equivariant Dirac operator on E , and let Φ be a G -equivariant vector bundle endomorphism of E . Suppose that $\{\tilde{D}, \Phi\} := \tilde{D}\Phi + \Phi\tilde{D}$ is a vector bundle endomorphism of E , and that

$$\{\tilde{D}, \Phi\} + \Phi^2 \geq c^2 \tag{3.4}$$

fibrewise outside Z . Then $D := \tilde{D} + \Phi$ satisfies the conditions on D as above, with $F = E$, $P = \tilde{D}$ and $R = \{\tilde{D}, \Phi\} + \Phi^2$.

This type of operator is a *Callias-type operator*. Indices of Callias-type operators equivariant under proper actions were studied in [13, 14].

The main application of the maximal localised index in this paper, Theorem 3.11, is about the maximal localised index of Callias-type operators.

3.2 The invariant index

Integrating L^1 -functions over G extends to a trace on $C_{\max}^*(G)$. We will see in Theorem 3.9 that applying this trace to the localised index of D recovers an index defined in terms of G -invariant sections in [18]. This fact will be used in the proof of Theorem 3.11. It can also be used to obtain refined index theoretic information on non-compact manifolds; see Remark 3.10.

Let $\chi \in C^\infty(M)$ a function with the property (2.5). Consider the space $\Gamma_{\text{tc}}(E)^G$ of transversally compactly supported sections of E , defined as the space of continuous, G -invariant sections of E whose supports have compact images in M/G under the quotient map. The Hilbert space $L_T^2(E)^G$ of G -invariant, transversally L^2 -sections of E is the completion of $\Gamma_{\text{tc}}(E)^G$ in the inner product

$$(s_1, s_2)_{L_T^2(E)^G} := (\chi s_1, \chi s_2)_{L^2(E)}.$$

The space $L_T^2(E)^G$ is independent of the choice of χ ; see Lemma 4.4 in [18].

Suppose that D is odd with respect to a G -invariant grading $E = E_+ \oplus E_-$. In Proposition 4.7 in [18], it is shown that D defines a Fredholm operator \bar{D} from a suitable Sobolev space inside $L_T^2(E)^G$ into $L_T^2(E)^G$. In Proposition 4.8 in the same paper, it is deduced that the space

$$\ker(D) \cap L_T^2(E)^G$$

is finite-dimensional, and that the index of \bar{D} equals

$$\dim(\ker(D) \cap L_T^2(E_+)^G) - \dim(\ker(D) \cap L_T^2(E_-)^G). \quad (3.5)$$

Definition 3.8. The G -invariant index of D , denoted by $\text{index}(D)^G$, is the number (3.5).

In [6], Braverman further develops the theory of this index, when applied to Dirac operators with an added zero-order term that is relevant to geometric quantisation, and in particular proves that it is invariant under a suitable notion of cobordism.

The map from $L^1(G)$ to \mathbb{C} given by integrating functions over G extends continuously to a $*$ -homomorphism, or a trace

$$I: C_{\max}^*(G) \rightarrow \mathbb{C}.$$

The integer

$$I_*(\text{index}_G^{\text{loc}}(D)) \in K_0(\mathbb{C}) = \mathbb{Z}$$

plays the role of the G -invariant part of the localised index of D_p , and this will be made precise in Theorem 3.9 below.

If M/G is compact, then all smooth sections of E are transversally L^2 . Then the G -invariant index of D equals

$$\dim(\ker(D_+)^G) - \dim(\ker(D_-)^G),$$

where D_\pm is the restriction of D to sections of E_\pm . This index was developed and applied by Mathai and Zhang in [30], with an appendix by Bunke. In Theorem 2.7 and Proposition D.3 in that paper, it is shown that the index can be recovered from the equivariant index of D in $K_0(C_{\max}^*(G))$, defined via the analytic assembly map, if one applies the integration trace I . We will show that this generalises to the index in Definition 3.4 in the non-cocompact case.

Theorem 3.9. *We have*

$$I_*(\text{index}_G^{\text{loc}}(D)) = \text{index}(D)^G \in \mathbb{Z}. \quad (3.6)$$

This theorem is proved in Section 6.

Remark 3.10. Theorem 3.9 allows us to construct a more refined invariant of operators that are invertible at infinity in the non-equivariant case than their Fredholm index. Suppose that M is the universal cover of a manifold \bar{M} and that $G = \Gamma$ is the fundamental group of \bar{M} , acting on M in the natural way. Let \bar{D} be an elliptic operator on \bar{M} that is invertible at infinity in the appropriate sense, so that it lifts to a Γ -equivariant operator on M satisfying the conditions of Theorem 3.9. That theorem then implies that

$$I_*(\text{index}_\Gamma^{\text{loc}}(D)) = \text{index}(\bar{D}).$$

In this sense, $\text{index}_\Gamma^{\text{loc}}(D)$ refines the Fredholm index of \bar{D} , much like the image of D under the analytic assembly map for the maximal group C^* -algebra refines the Fredholm index of \bar{D} if \bar{M} is compact.

This can for example be used to obtain stronger obstructions to Riemannian metrics of positive scalar curvature than classical obstructions from Callias index theory [1] or the Gromov–Lawson index [11]. Applications to positive scalar curvature will be worked out in a future article.

More generally, for any proper, isometric action by a discrete group Γ on M , the quotient $\tilde{M} = M/\Gamma$ is an orbifold, and the $\text{index}_\Gamma^{\text{loc}}(D)$ refines an orbifold version of the index of \tilde{D} .

Proposition 2.4 in [21] shows that the index defined in [22] is another refinement of the invariant index. That index applies to Dirac operators with certain deformation terms added that are relevant to geometric quantisation. It takes values in the completion of the representation ring of a maximal compact subgroup of the group acting.

3.3 Callias quantisation commutes with reduction

In [18, 20], the *quantisation commutes with reduction* principle of Guillemin and Sternberg [12, 31, 32, 34, 40] and its Spin^c -version [35] is generalised to proper actions by possibly noncompact groups, with possibly noncompact orbit spaces, for suitably high powers of the prequantum or determinant line bundle in question. These results in [18, 20] are stated in terms of the invariant index of Definition 3.8. The result in [18] in the symplectic setting generalises the result in [30] from compact to noncompact orbit spaces. This is generalised to the Spin^c -setting in [20].

These were the first results on a version of the quantisation commutes with reduction principle where both the group and orbit space were allowed to be noncompact, but two drawbacks were that the invariant index used

1. was only well-defined after a deformation term (Clifford multiplication by the Kirwan vector field) was added to the relevant Dirac operator;
2. only involved G -invariant sections, and therefore provided no information about the parts of the kernel of D on which G acts nontrivially.

The second point was partially addressed in Theorem 2.13 in [21], a quantisation commutes with reduction result for non-cocompact actions, where quantisation takes values in the completion of the representation ring of a maximal compact subgroup.

We are now able to remedy both points, in the case of Callias-type Spin^c -Dirac operators.

Let $D = D_\Phi$ be a Callias-type operator as in Example 3.7. Suppose that E has a G -invariant $\mathbb{Z}/2$ grading, and that \tilde{D} and Φ , and hence D , are odd for this grading. Suppose that $E = \mathcal{S}$ is the spinor bundle of a G -equivariant Spin^c -structure on M , and let $L \rightarrow M$ be its determinant line bundle. (The assumption that E is $\mathbb{Z}/2$ -graded now means that M is even-dimensional.) Suppose that \tilde{D} is a Spin^c -Dirac operator on \mathcal{S} . The Clifford connection on \mathcal{S} used to define \tilde{D} can be constructed locally from a G -invariant, Hermitian connection ∇^L on L and the connection on the spinor bundle for a local Spin -structure; see e.g. Proposition D.11 in [28]. This also induces a Clifford connection on the spinor bundle $\mathcal{S} \otimes L^p$, for any $p \in \mathbb{Z}_{\geq 0}$. Let \tilde{D}^{L^p} be the corresponding Spin^c -Dirac operator on $\mathcal{S} \otimes L^p$. Set

$$D_p := \tilde{D}^{L^p} + \Phi \otimes 1_{L^p}.$$

We have

$$\{\tilde{D}^{L^p}, \Phi \otimes 1_{L^p}\} = \{\tilde{D}, \Phi\} \otimes 1_{L^p}, \quad (3.7)$$

where $\{-, -\}$ denotes the anticommutator. In what follows, we will omit ‘ $\otimes 1_{L^p}$ ’ from the notation. By (3.4) and (3.7), we have

$$\{\tilde{D}^{L^p}, \Phi\} + \Phi^2 \geq c^2 \quad (3.8)$$

outside Z , for all p . Hence D_p has an index

$$\text{index}_G^{\text{loc}}(D_p) \in K_0(C_{\max}^*(G)). \quad (3.9)$$

The quantisation commutes with reduction principle in general is an equality between the invariant part of the equivariant index of a Spin^c -Dirac operator and the index of a Dirac operator localised at the level set of a *moment map*. The invariant part of the index will now be represented by the image of (3.9) under the integration trace I .

The Spin^c -*moment map* associated to ∇^L is the map

$$\mu: M \rightarrow \mathfrak{g}^*$$

such that for all $X \in \mathfrak{g}$,

$$2\pi i \langle \mu, X \rangle = \mathcal{L}_X - \nabla_{X^M}^L, \quad \in \text{End}(L) = C^\infty(M, \mathbb{C}),$$

where \mathcal{L}_X denotes the Lie derivative with respect to X , and X^M is the vector field induced by X . Our sign convention is that for all $X \in \mathfrak{g}$ and $m \in M$,

$$X^M(m) = \left. \frac{d}{dt} \right|_{t=0} \exp(-tX) \cdot m.$$

Suppose that 0 is a regular value of μ , and that G acts freely on $\mu^{-1}(0)$. Suppose that the *reduced space*

$$M_0 := \mu^{-1}(0)/G$$

is compact. In Lemma 3.3 in [20], a condition is given for M_0 to inherit a Spin^c -structure from M , with determinant line bundle L_0^p , with $L_0 = (L|_{\mu^{-1}(0)})/G \rightarrow M_0$. This is true for example if G is semisimple, see Proposition 3.5 and Example 3.6 in [20]. It is also true in the symplectic setting, where the Spin^c -structure on M is associated to a G -invariant almost complex structure compatible with a G -invariant symplectic form, together with a G -equivariant, Hermitian line bundle on M . From now on, we assume such a Spin^c -structure on M_0 exists. Let $D_{M_0}^{L_0^p}$ be a Spin^c -Dirac operator on M_0 for this Spin^c -structure.

Theorem 3.11 (Callias Spin^c-quantisation commutes with reduction). *There is a $p_0 \in \mathbb{Z}_{\geq 0}$ such that for all $p \geq p_0$,*

$$I_*(\text{index}_G(D_p)) = \text{index}(D_{M_0}^{L^p}) = \int_{M_0} \hat{A}(M_0) e^{\frac{p}{2}c(L_0)}.$$

This theorem is proved in Section 7.

Remark 3.12. In the case of Callias-type operators $D = \tilde{D} + \Phi$, as in Example 3.7, an index in $K_0(C^*(G))$ was constructed directly in [13]. Here $C^*(G)$ can be either the reduced or maximal group C^* -algebra. Let us denote this index by

$$\text{index}_G^C(\tilde{D} + \Phi) \in K_0(C^*(G)).$$

Theorem 4.2 in [14] states that, for the reduced group C^* -algebra and Roe algebra, this index of Callias-type operators is a special case of the localised index:

$$\text{index}_G^C(\tilde{D} + \Phi) = \text{index}_G^{\text{loc}}(\tilde{D} + \Phi) \in K_0(C_{\text{red}}^*(G)).$$

Via analogous arguments, one can show that this equality still holds for the maximal group C^* -algebra and Roe algebra. Then Theorem 3.11 implies that, under the conditions in that theorem,

$$I_*(\text{index}_G^C(\tilde{D}_p + \Phi)) = \text{index}(D_{M_0}^{L^p}).$$

Remark 3.13. As far as we are aware, Theorem 3.11 was not known in the case where G is compact, so that D is Fredholm in the classical sense. In that case, by the standard shifting trick (see for example Corollary 1.2 in [31], and [35]), Theorem 3.11 implies expressions for the multiplicities of all irreducible representations of G in the equivariant index of D_p . One can handle cases where a reduced space $\mu^{-1}(\text{Ad}^*(G)\xi)/G$ is not smooth by

1. using orbifold line bundles and indices if ξ is a regular value of μ in an appropriate sense;
2. using reduced spaces at nearby regular values if ξ is a singular value of μ , as in [32, 34, 35]. An alternative approach is developed in [29].

4 Regularity and localisation

In this section, we prove Propositions 3.1 and 3.3, which imply that the localised maximal index of Definition 3.4 is well-defined.

4.1 Maximal operator modules

Our first goal is to make sense of D as an unbounded, regular, essentially self-adjoint operator on certain maximal operator modules that we now introduce.

Let M_1 and M_2 be Riemannian manifolds equipped with proper, isometric G -actions. Let E_1 and E_2 be Hermitian G -vector bundles over M_1 and M_2 respectively. Consider the vector bundle $\text{Hom}(E_2, E_1) := E_1 \boxtimes E_2^* \rightarrow M_1 \times M_2$.

Definition 4.1. Denote by

$$\mathcal{H}_{\text{ker}}^\infty(E_1, E_2)_{\text{loc}}^G \subset \Gamma^\infty(\text{Hom}(E_2, E_1))$$

the \mathbb{C} -vector space of smooth G -invariant, cocompactly supported, finite propagation kernels. Here we say that a kernel κ has cocompact support if there exists cocompact subsets U_1 and U_2 of M_1 and M_2 respectively such that $\text{supp}(\kappa) \subseteq U_1 \times U_2$.

There is no natural product or $*$ -operation on $\mathcal{H}_{\text{ker}}^\infty(E_1, E_2)_{\text{loc}}^G$. However, it admits a natural action of $C_{\text{ker}}^\infty(M_2; L^2(E_2))_{\text{loc}}^G$ from the right, given by composition of kernels. Further, it has a $C_{\text{ker}}^\infty(M_2; L^2(E_2))_{\text{loc}}^G$ -valued inner product given by

$$\langle \kappa, \kappa' \rangle := \kappa^* \kappa',$$

for $\kappa, \kappa' \in \mathcal{H}_{\text{ker}}^\infty(E_1, E_2)_{\text{loc}}^G$, defined through the usual adjoint and multiplication of kernels. This makes $\mathcal{H}_{\text{ker}}^\infty(E_1, E_2)_{\text{loc}}^G$ a pre-Hilbert $C_{\text{ker}}^\infty(M_2; L^2(E_2))_{\text{loc}}^G$ -module.

Now taking the simultaneous completions of $\mathcal{H}_{\text{ker}}^\infty(E_1, E_2)_{\text{loc}}^G$ and $C_{\text{ker}}^\infty(M_2; L^2(E_2))_{\text{loc}}^G$ (see p. 5 of [27]) gives a Hilbert $C_{\text{max}}^*(M_2; L^2(E_2))_{\text{loc}}^G$ -module that we denote by $\mathcal{H}_{\text{max}}(E_1, E_2)_{\text{loc}}^G$.

In the case $M_1 = M_2 = M$ and $E_1 = E_2 = E$, equipped with the same G -action, then $\mathcal{H}_{\text{ker}}^\infty(E, E)_{\text{loc}}^G = C_{\text{ker}}^\infty(M; L^2(E))_{\text{loc}}^G$. In this case $\mathcal{H}_{\text{max}}(E, E)_{\text{loc}}^G$ is the G -maximal equivariant localised Roe algebra $C_{\text{max}}^*(M; L^2(E))_{\text{loc}}^G$ of Definition 2.4.

4.2 Unbounded operators on operator modules

To introduce the idea of an unbounded operator on the operator modules defined in the previous subsections, let us first consider $C_{\text{max}}^*(M; L^2(E))_{\text{loc}}^G$ as a right Hilbert module over itself, as in Subsection 3.1. We will consider D as an unbounded, densely defined operator on this Hilbert module. Note

that D acts naturally on smooth Schwartz kernels via differentiation on the first coordinate, so it defines a map

$$D : C_{\ker}^{\infty}(M; L^2(E))_{\text{loc}}^G \rightarrow C_{\ker}^{\infty}(M; L^2(E))_{\text{loc}}^G.$$

More generally, we may consider the situation when M_1 and M_2 are proper G -manifolds with and E_1, E_2 are Hermitian G -vector bundles over M_1 and M_2 respectively. Let D be a symmetric, G -equivariant differential operator acting on a E_1 . Then D defines an unbounded, symmetric operator

$$D : \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G \rightarrow \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G$$

with initial domain $\mathcal{H}_{\ker}^{\infty}(E_1, E_2)_{\text{loc}}^G$.

4.3 The cocompact case

Proposition 4.2. *Let M be a cocompact G -manifold. Then D is a regular and essentially self-adjoint operator on the Hilbert $C_{\max}^*(M; L^2(E))^G$ -module $C_{\max}^*(M; L^2(E))^G$.*

The proof is based on Lemmas 4.3 and 4.4 and Proposition 4.5. The first lemma is a standard volume estimate for manifolds with bounded Ricci curvature, which is the case here since M is cocompact.

Lemma 4.3. *There exist constants $C_1, C_2 > 0$ such that for any $r \geq 0$, every ball in M of radius r has volume at most $C_1 e^{C_2 r}$.*

The second lemma states that the pointwise norm of the Schwartz kernel of the operator $(D + \mu i)^{-1}$ decays exponentially in μ times the Riemannian distance. More precisely, let κ_{μ} denote the Schwartz kernel of $(D + \mu i)^{-1}$. Then:

Lemma 4.4. *There exists a constant $C_{\mu} > 0$ such that, for all $m, m' \in M$ with $d(m, m') \geq 1$,*

$$\|\kappa_{\mu}(m, m')\| \leq C_{\mu} e^{-\frac{\mu}{2} d(m, m')},$$

where the norm is taken fibrewise in $E \boxtimes E^*$.

Proof. This is Lemma 3.3 in [15]. The proof works because M and E have bounded geometry. \square

Finally, in order to make estimates in the norm of $C_{\max}^*(M; L^2(E))^G$, we need to work with the maximal norm on the Roe algebra of the corresponding admissible module, namely $L^2(E) \otimes L^2(G)$. Thus let j and p be the inclusion and projection maps defined in Subsection 2.3, and consider the map $\oplus 0$ as in (2.6). Let

$$\Psi : L^2(G) \otimes \mathcal{H} \rightarrow L^2(E) \otimes L^2(G)$$

be the G -equivariant unitary isomorphism as in (2.1), defined in (21) in [14]. On a single slice neighbourhood of the form $G \times_K N$ we have $\mathcal{H} = L^2(K \backslash G) \otimes L^2(E|_N)$, and the isomorphism Ψ is induced by a G -equivariant, measure preserving bijection

$$\psi : G \times K \backslash G \times N \rightarrow G \times M.$$

Then for any $T \in \mathcal{B}(L^2(E))$, the operator \tilde{T} on $L^2(G) \otimes \mathcal{H}$ corresponding to $T \oplus 0$ can be written as

$$\tilde{T} := \Psi^{-1}(T \oplus 0)\Psi = \Psi^{-1} \circ j \circ T \circ j^{-1} \circ p \circ \Psi.$$

Proposition 4.5. *Let $\kappa \in \Gamma(\text{Hom}(E))^G$ be a bounded, measurable, G -invariant Schwartz kernel defining a bounded operator T on $L^2(E)$. Then \tilde{T} is given by a G -invariant Schwartz kernel*

$$\tilde{\kappa} : G \times G \rightarrow \mathcal{K}(\mathcal{H}).$$

On any individual slice neighbourhood of the form $G \times_K N$, $\tilde{\kappa}(g, g')$ is the operator defined by taking $\nu \in \mathcal{H} = L^2(K \backslash G) \otimes L^2(E|_N)$ to the element of $L^2(K \backslash G) \otimes L^2(N)$ given by

$$\begin{aligned} (Kh, n) \mapsto & \int_{K \backslash G} \int_N (\psi^* \chi')(g, Kh, n) (\psi^* \kappa)(g, Kh, n, g', Kh', n') \\ & (\psi^* \chi')(g', Kh', n') \nu(Kh', n') dn' d(Kh'), \end{aligned} \tag{4.1}$$

for $h \in G$ and $n \in N$, where

- χ' is defined to be $\chi \otimes 1$;
- $\psi^* \kappa$ is defined to be the pull-back of κ along the Cartesian product with itself of the composition

$$G \times K \backslash G \times N \xrightarrow{\psi} M \times G \xrightarrow{\text{pr}_1} M.$$

Proof. This is a straightforward calculation. \square

Proof of Proposition 4.2. Let $\kappa_0 \in C_{\ker}^\infty(M, L^2(E))^G$. Then for any real number $\mu > 0$, $\kappa := (D + \mu i)^{-1} \kappa_0$ is a smooth, G -invariant Schwartz kernel defining a bounded operator T on $L^2(E)$. Note that κ may not have finite propagation. However, since κ_0 has finite propagation, is G -invariant, and hence is uniformly bounded on $M \times M$, it follows from Lemma 4.4 that $\kappa(m, m')$ decays exponentially in $d(m, m')$, where d is the Riemannian distance on M . We now show that κ in fact defines an element in $C_{\max}^*(G) \otimes \mathcal{K}(\mathcal{H})$.

In order to do this, we view T as a bounded operator \tilde{T} on the admissible module $L^2(G) \otimes \mathcal{H}$, with kernel $\tilde{\kappa}$ given as in Proposition 4.5. By Lemma 5.9 in [14], \tilde{T} is locally compact.

We now show that $\tilde{\kappa}$ corresponds to an element of $L^1(G) \otimes \mathcal{K}(\mathcal{H})$. Since Ψ implements a coarse equivalence, and hence a quasi-isometry since M and G are path length metric spaces, between M and G (see Lemmas 5.6–5.8 in [14]), equation (4.1) and Lemma 4.4 imply that there exist constants $C, C'_\mu > 0$ such that, for all $g, g' \in G$ with $d^G(g, g') \geq C$,

$$\|\tilde{\kappa}(g, g')\| \leq C'_\mu e^{-\frac{\mu}{2}d^G(g, g')},$$

where d^G is a left invariant distance function on G , and the norm is taken in $\mathcal{B}(\mathcal{H})$.

Since G has bounded geometry, we may choose μ sufficiently large that $\tilde{\kappa}(g, g')$ is an element in $L^1(G) \otimes \mathcal{K}(\mathcal{H})$. It follows that κ defines as an element of $C_{\max}^*(M, L^2(E))^G$. \square

The proof given above can be easily extended to give the following result.

Theorem 4.6. *Let M_1 and M_2 be proper G -manifolds with and E_1, E_2 G -vector bundles over M_1 and M_2 respectively. Suppose that M_1 is cocompact, and let D be a symmetric elliptic first-order differential operator acting on E_1 . Then the operator*

$$D : \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G \rightarrow \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G$$

is regular and essentially self-adjoint in the sense of Hilbert $C_{\max}^(M_2, L^2(E_2))_{\text{loc}}^G$ -modules.*

4.4 Proof of Proposition 3.1

We will prove Proposition 3.1 using Proposition 4.2 for the cocompact case.

The following lemma will be used in a few places.

Lemma 4.7. *Let S be a G -equivariant vector bundle homomorphism of E , whose fibrewise norm is bounded. Then $S \in \mathcal{L}_A(A)$, and $\|S\|_{\mathcal{L}_A(A)} \leq \|S\|_\infty$.*

Proof. The endomorphism $\|S\|_\infty 1_E - S$ of E is fibrewise nonnegative. Let $T := (\|S\|_\infty 1_E - S)^{1/2}$ be its fibrewise square root. Then for all $\kappa \in C_{\ker}^*(M; L^2(E))_{\text{loc}}^G$,

$$\langle \kappa, (\|S\|_\infty - S)\kappa \rangle = \langle T\kappa, T\kappa \rangle \geq 0.$$

So $S \leq \|S\|_\infty$ in $\mathcal{L}_A(A)$. \square

Proof of Proposition 3.1. We need to show that the operators $D \pm i$ have ranges that are dense in $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$. We prove the result for $D + i$, with the case of $D - i$ being similar.

Since M is a complete Riemannian manifold, there exists a family $\{a_\varepsilon\}_{\varepsilon \in \mathbb{R}^+}$ of G -invariant, cocompactly supported smooth functions taking values in $[0, 1]$, such that:

- $\text{supp}(a_{\varepsilon_1}) \subseteq \text{supp}(a_{\varepsilon_2})$ whenever $\varepsilon_2 \leq \varepsilon_1$;
- $a_{\varepsilon_1}^{-1}(1) \subseteq a_{\varepsilon_2}^{-1}(1)$ whenever $\varepsilon_2 \leq \varepsilon_1$;
- $\bigcup_\varepsilon \{a_\varepsilon^{-1}(1)\} = M$;
- $\sup_{m \in M} \|da_\varepsilon(m)\| \leq \varepsilon$.

Now let $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. Since κ has cocompact support, there exists ε such that $\text{supp}(\kappa) \subseteq a_\varepsilon^{-1}(1) \times a_\varepsilon^{-1}(1)$. Let U_ε be a G -invariant, relatively cocompact neighbourhood of $\text{supp}(a_\varepsilon)$. Denote the double of its closure \overline{U}_ε by $\overline{U}_\varepsilon^+$, noting that there exists a G -invariant collar neighbourhood of $\partial\overline{U}_\varepsilon$ inside \overline{U}_ε . By restricting the various geometric structures on M to U_ε and extending to $\overline{U}_\varepsilon^+$, we obtain a Dirac operator D_ε acting on a bundle E_ε over the double.

As in Subsection 4.2, D_ε defines an operator on $C_{\ker}^\infty(\overline{U}_\varepsilon^+, L^2(E_\varepsilon))^G$ that extends to an unbounded operator on the maximal completion $C_{\max}^*(\overline{U}_\varepsilon^+, L^2(E_\varepsilon))^G$, whose norm we will denote by $\|\cdot\|_{\varepsilon, \max, G}$. By Proposition 4.2, D_ε is regular, so there exists a sequence $\{e_{\varepsilon, j}\}_{j \in \mathbb{N}}$ in $C_{\ker}^\infty(\overline{U}_\varepsilon^+, L^2(E_\varepsilon))^G$ such that

$$(D_\varepsilon + i)e_{\varepsilon, j} \rightarrow \kappa. \tag{4.2}$$

in $\|\cdot\|_{\varepsilon, \max, G}$. Since the action of the operator $(D_\varepsilon + i)^{-1}$ on κ preserves support in the second coordinate, we may assume that

$$\text{pr}_2(\text{supp}(e_{\varepsilon, j})) \subseteq \text{pr}_2(\text{supp}(\kappa)),$$

where $\text{pr}_2 : \overline{U}_\varepsilon^+ \times \overline{U}_\varepsilon^+ \rightarrow \overline{U}_\varepsilon^+$ is the projection onto the second factor, so that $a_\varepsilon e_{\varepsilon, j}$ lies in the domain of D . For each j ,

$$\begin{aligned} (D + i)(a_\varepsilon e_{\varepsilon, j}) - \kappa &= (D_\varepsilon + i)(a_\varepsilon e_{\varepsilon, j}) - \kappa \\ &= a_\varepsilon((D_\varepsilon + i)e_{\varepsilon, j} - \kappa) + \sigma_D(da_\varepsilon)e_{\varepsilon, j}, \end{aligned} \quad (4.3)$$

acting again on the first coordinate.

Because D has finite propagation speed, there is a $C > 0$ such that $\sigma_D(\xi) \leq C\|\xi\|$ for all $\xi \in T^*M$. By Lemma 4.7, this implies that $\|\sigma_D(da_\varepsilon)\|_{\mathcal{L}_A(A)} \leq C\|\nabla a_\varepsilon\|_\infty$. It follows that

$$\|\sigma_D(da_\varepsilon)\kappa\|_{\max, G} \leq C\|da_\varepsilon\|_\infty \|\kappa\|_{\max, G}.$$

Similarly,

$$\|a_\varepsilon\kappa\|_{\max, G} \leq \|\kappa\|_{\max, G}.$$

Thus $\sigma_D(da_\varepsilon)$ and a_ε define bounded multipliers of $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$ with norms bounded above by their supremum norms. Combining this with (4.3) gives

$$\begin{aligned} \|(D + i)(a_\varepsilon e_{\varepsilon, j}) - \kappa\|_{\max, G} &= \|a_\varepsilon((D_\varepsilon + i)e_{\varepsilon, j} - \kappa) + \sigma_D(da_\varepsilon)e_{\varepsilon, j}\|_{\max, G} \\ &\leq \|(D_\varepsilon + i)e_{\varepsilon, j} - \kappa\|_{\max, G} + C\|da_\varepsilon\|_\infty \|e_{\varepsilon, j}\|_{\max, G}. \end{aligned}$$

The algebra $C_{\max}^*(\overline{U}_\varepsilon; L^2(E|_{\overline{U}_\varepsilon}))^G$ is a subalgebra of the admissible Roe algebra $C_{\max}^*(\overline{U}_\varepsilon; L^2(E|_{\overline{U}_\varepsilon}) \otimes L^2(G))^G$, which is itself a common subalgebra of both $C_{\max}^*(M)_{\text{loc}}^G$ and $C_{\max}^*(\overline{U}_\varepsilon^+; L^2(E_\varepsilon) \otimes L^2(G))^G$. This implies that for any kernel $\kappa' \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$ supported on $U_\varepsilon \times U_\varepsilon$,

$$\|\kappa'\|_{\max, G} = \|\kappa'\|_{\varepsilon, \max, G},$$

as both sides are equal to the norm of the image of κ' in the algebra $C_{\max}^*(\overline{U}_\varepsilon; L^2(E|_{\overline{U}_\varepsilon}) \otimes L^2(G))^G$.

Also note that (4.2) implies that there exists j_0 such that for all $j > j_0$, we have $\|e_{\varepsilon, j}\|_{\varepsilon, \max, G} \leq 2\|\kappa\|_{\varepsilon, \max, G}$. Continuing the computation with these facts in mind gives

$$\begin{aligned} \|(D + i)(a_\varepsilon e_{\varepsilon, j}) - \kappa\|_{\max, G} &\leq \|(D_\varepsilon + i)e_{\varepsilon, j} - \kappa\|_{\varepsilon, \max, G} + C\|da_\varepsilon\|_\infty \|e_{\varepsilon, j}\|_{\varepsilon, \max, G} \\ &\leq \|(D_\varepsilon + i)e_{\varepsilon, j} - \kappa\|_{\varepsilon, \max, G} + C\|da_\varepsilon\|_\infty \|\kappa\|_{\max, G}. \end{aligned}$$

Take a sequence $\varepsilon_k \rightarrow 0$. Then as $j, k \rightarrow \infty$, one sees that $(D+i)(a_{\varepsilon_k} e_{\varepsilon_k}) \rightarrow \kappa$ in $\|\cdot\|_{\max, G}$. □

A straightforward adaptation of the above proof, together with Theorem 4.6, gives the following result in the more general situation involving two different bundles and manifolds.

Theorem 4.8. *Let M_1 and M_2 be proper, isometric G -manifolds with and E_1, E_2 Hermitian G -vector bundles over M_1 and M_2 respectively. Suppose that M_1 , and let D be a symmetric, G -equivariant, first order differential operator acting on a E_1 , with finite propagation speed. Then the unbounded operator*

$$D : \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G \rightarrow \mathcal{H}_{\max}(E_1, E_2)_{\text{loc}}^G$$

is regular and essentially self-adjoint.

4.5 Generalised Fredholmness

We will prove Proposition 3.3, by adapting the method in [38] to the Hilbert A -module A .

We begin by establishing a useful property of the wave operator group associated to an essentially self-adjoint regular operator, following Proposition 3.4 of [16].

Lemma 4.9. *Let D be an essentially self-adjoint regular operator on A . Then the wave operator group $\{e^{itD}\}_{t \in \mathbb{R}}$ satisfies the wave equation: for $\kappa \in C_{\text{ker}}^\infty(M; L^2(E))_{\text{loc}}^G$,*

$$\frac{d}{dt} e^{itD} \kappa = i D e^{itD} \kappa.$$

Moreover, each operator e^{itD} has propagation at most $|t|$, in the sense that it does not increase the propagation of κ by more than $|t|$.

Proof. The function $s \mapsto e^{its}$ is in $C_b(\mathbb{R})$. Thus for each $t \in \mathbb{R}$, the operator e^{itD} is bounded adjointable and unitary. For each $t \in \mathbb{R}$, the difference quotient

$$\frac{e^{i(t+h)s} - e^{its}}{h} \rightarrow i s e^{its}$$

as $h \rightarrow 0$, uniformly in s in compact sets. Furthermore, this difference quotient is bounded uniformly by $|1 + s|$. The wave equation property then follows from the third point in Theorem 3.2. The finite propagation property can be proved exactly as in Proposition 3.4 in [16]. □

Corollary 4.10. *Let K be a cocompact subset of M_1 . Let $r > 0$. Let D_1 and D_2 be essentially self-adjoint differential operators on M_1 that are equal on $\text{Pen}(K, r)$. Then for a kernel $\kappa \in \mathcal{H}_{\ker}^\infty(E_1, E_2)_{\text{loc}}^G$ supported on $K \times M_2$,*

$$e^{itD_1}\kappa = e^{itD_2}\kappa$$

if $-r \leq t \leq r$.

Now suppose that D is as in Subsection 3.1, and in particular that it satisfies (3.2). We will use Corollary 4.10 to establish a norm estimate for $f(D)$ in $\mathcal{L}_A(A)$ when f has compactly supported Fourier transform.

Lemma 4.11. *The operator D on the Hilbert module $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$ satisfies*

$$D^2 \geq c^2$$

with respect to the Hilbert module inner product.

Proof. For any $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$,

$$\langle D^2\kappa, \kappa \rangle = \langle (P^*P + R)\kappa, \kappa \rangle = \langle P\kappa, P\kappa \rangle + \langle R\kappa, \kappa \rangle. \quad (4.4)$$

As in the proof of Lemma 4.7, we find that $\langle R\kappa, \kappa \rangle \geq c^2\langle \kappa, \kappa \rangle$. So the right hand side of (4.4) is at least equal to $c^2\langle \kappa, \kappa \rangle$. \square

Lemma 4.11 is the place where we use form (3.2) of D^2 , rather than a slightly milder positivity condition on D^2 outside Z .

Lemma 4.12. *Let $r > 0$. Suppose $f \in \mathcal{S}(\mathbb{R})$ is a function with Fourier transform \hat{f} supported in $[-r, r]$. Let φ be a smooth, bounded, G -invariant function with support disjoint from $\text{Pen}(Z, 2r)$. Then*

$$\|f(D)\varphi\|_{\mathcal{L}_A(A)} \leq \|\varphi\|_\infty \cdot \sup\{|f(\lambda)|; |\lambda| \geq c\}.$$

The same estimate applies to $\varphi f(D)$.

Proof. For $n = 1, 2$, let $U_n = \{m \in M; d(m, Z) > nr\}$. Let \overline{U}_1^+ denote the double of \overline{U}_1 . By extending the various geometric structures on U_1 to \overline{U}_1^+ , we may extend the Dirac operator $D|_{U_1}$ to an operator \tilde{D} on \overline{U}_1^+ acting on the extension $\tilde{E} \rightarrow \overline{U}_1^+$ of $E|_{\overline{U}_1}$. Then \tilde{D} is an unbounded symmetric operator on $\mathcal{H}_{\max}(\overline{U}_1^+, M)_{\text{loc}}^G$ with initial domain $\mathcal{H}_{\ker}^\infty(\overline{U}_1^+, M)_{\text{loc}}^G$. By Lemma 4.11, we have $\tilde{D}^2 \geq c^2$. Since \overline{U}_1^+ is complete, Theorem 4.8 implies that \tilde{D} is essentially self-adjoint and regular.

Now for any $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$ with support contained in $U_2 \times M$, Corollary 4.10 implies that

$$e^{itD}\kappa = e^{it\tilde{D}}\kappa$$

for all $-r \leq t \leq r$. Together with the equality

$$f(D) = \frac{1}{\pi} \int_{-r}^r \hat{f}(t) e^{itD} dt,$$

this implies that $f(D)\varphi = f(\tilde{D})\varphi$. The bound $\tilde{D}^2 \geq c^2$ implies, by the fifth point of Theorem 3.2, that

$$\|f(\tilde{D})\|_{\mathcal{L}_A(A)} \leq \sup\{|f(\lambda)|; |\lambda| \geq c\}.$$

Together with the fact that φ defines an element of $\mathcal{L}_A(A)$ with norm at most $\|\varphi\|_\infty$ (a special case of Lemma 4.7), this gives

$$\|f(D)\varphi\|_{\mathcal{L}_A(A)} \leq \|\varphi\|_\infty \cdot \sup\{|f(\lambda)|; |\lambda| \geq c\}.$$

□

With these preparations, we can now finish the proof of Proposition 3.3. Again, we follow the idea of Roe [38].

Proof of Proposition 3.3. Let $f \in C_c(\mathbb{R})$ be supported in $[-c, c]$, and let $\varepsilon > 0$. There exists a smooth function g with compactly supported Fourier transform such that

$$\sup\{|g(\lambda) - f(\lambda)|; \lambda \in \mathbb{R}\} < \varepsilon.$$

This implies that $|g(\lambda)| < \varepsilon$ when $|\lambda| > c$. Suppose that $\text{supp}(\hat{g}) \subseteq [-r, r]$ for some $r > 0$. Let $\psi : M \rightarrow [0, 1]$ be a smooth G -invariant function such that

$$\psi(m) = \begin{cases} 1 & \text{for } m \in \text{Pen}(Z, 2r) \\ 0 & \text{for } m \in M \setminus \text{Pen}(Z, 3r). \end{cases}$$

We can write

$$f(D) = \psi g(D) \psi + (1 - \psi) g(D) \psi + g(D) (1 - \psi) + (f(D) - g(D)).$$

Now the first term on the right hand side is a G -invariant cocompactly supported smooth kernel. The second and third terms each have maximal norm bounded by ε by Lemma 4.12, while the maximal norm of the last term is bounded by ε by the fifth point in Theorem 3.2. Thus, for any $\varepsilon > 0$, $f(D)$ lies within 3ε of a G -invariant cocompactly supported smooth kernel. Thus $f(D)$ is in the completion $C_{\max}^*(M; L^2(E))_{\text{loc}}^G$. □

5 Averaging maps

In Subsections 5.1–5.3, we return to the general setting of Subsection 2.1, of a metric space (X, d) rather than a Riemannian manifold.

The main tools in the proof of Theorem 3.9 are several *averaging maps*, which map G -equivariant operators on X to operators on X/G . In this section, we introduce these maps, and prove their properties that we need. We then use these maps in Section 6 to prove Theorem 3.9.

5.1 Averaging kernels

Consider the setting of Subsection 2.1. Consider the action by $G \times G$ on $\Gamma(\text{Hom}(E))$ given by

$$((g, g') \cdot \kappa)(x, x') := g\kappa(g^{-1}x, g'x')g', \quad (5.1)$$

for $g, g' \in G$, $x, x' \in X$ and $\kappa \in \Gamma(\text{Hom}(E))$. Let $\Gamma(\text{Hom}(E))^{G \times G} \subset \Gamma(\text{Hom}(E))$ be the subspace of sections invariant under this action.

Let d_G be the metric on X/G induced by d :

$$d_G(Gx, Gx') := \inf_{g \in G} d(gx, x'),$$

for $x, x' \in X$. Consider the measure $d(Gx)$ on X/G such that for all $\varphi \in C_c(X)$,

$$\int_X \varphi(x) dx = \int_{X/G} \int_G \varphi(gx) dg d(Gx). \quad (5.2)$$

(See for example [4], Chapter VII, Section 2.2, Proposition 4b.)

Consider the Hilbert space $L_T^2(E)^G$ defined in Subsection 3.2. We view it as a $C_0(X/G)$ -module by pointwise multiplication after pullback along the quotient map. Let $C_{\text{ker}}^*(X/G; L_T^2(E)^G)$ be the subalgebra of $\mathcal{B}(L_T^2(E)^G)$ of locally compact operators T with finite propagation, given by a continuous kernel $\kappa \in \Gamma(\text{Hom}(E))^{G \times G}$ via

$$(Ts)(x) = \int_{X/G} \kappa(x, x')s(x') d(Gx'),$$

for $x \in X$ and $s \in L_T^2(E)^G$. The integral is independent of the Borel section $X/G \rightarrow X$ used implicitly, by G -invariance of s and $G \times G$ -invariance of κ . We will identify operators in $C_{\text{ker}}^*(X/G; L_T^2(E)^G)$ with their kernels.

For $\kappa \in C_{\text{ker}}^*(X; L^2(E))^G$ and $x, x' \in X$, set

$$\text{Av}(\kappa)(x, x') := \int_G g\kappa(g^{-1}x, x') dg.$$

The integrand is bounded, measurable and compactly supported, because κ has finite propagation and the action is proper.

Lemma 5.1. *For every $\kappa \in C_{\ker}^*(X; L^2(E))^G$, $\text{Av}(\kappa)$ is an element of $C_{\ker}^*(X/G; L_T^2(E))^G$. This defines a surjective $*$ -homomorphism*

$$\text{Av}: C_{\ker}^*(X; L^2(E))^G \rightarrow C_{\ker}^*(X/G; L_T^2(E))^G.$$

Proof. Let $\kappa \in C_{\ker}^*(X; L^2(E))^G$. It follows from a computation involving G -equivariance of κ and left and right invariance of dg that $\text{Av}(\kappa)$ is $G \times G$ -invariant. Furthermore, the propagation of $\text{Av}(\kappa)$ in X/G equals the propagation of κ in X . So indeed $\text{Av}(\kappa) \in C_{\ker}^*(X/G; L_T^2(E))^G$.

Using the fact that the Haar measure dg is invariant under inversion, one computes directly that for all $\kappa \in C_{\ker}^*(X; L^2(E))^G$,

$$\text{Av}(\kappa^*) = \text{Av}(\kappa)^*.$$

Let $\kappa, \kappa' \in C_{\ker}^*(X; L^2(E))^G$ be given. Then a computation involving (5.2) shows that

$$\text{Av}(\kappa\kappa') = \text{Av}(\kappa)\text{Av}(\kappa').$$

For surjectivity, let χ be a cutoff function as in (2.5). Choose this function such that, in addition to its other properties,

$$d(x, x') \leq d_G(Gx, Gx') + 1 \tag{5.3}$$

for all $x, x' \in \text{supp}(\chi)$. In other words, the support of χ mainly extends transversally to G -orbits. (See Lemma 5.2 below.)

Suppose that $\kappa_G \in C_{\ker}^*(X/G; L_T^2(E))^G$, and let r be its propagation in X/G . Define $\kappa \in \Gamma(\text{Hom}(E))$ by

$$\kappa(x, x') = \int_G \chi(gx)^2 \chi(gx')^2 dg \cdot \kappa_G(x, x'),$$

for $x, x' \in X$.

We first claim that κ has finite propagation. Indeed, let $x, x' \in X$ be such that $d(x, x') > r + 1$. If $d_G(Gx, Gx') > r$, then $\kappa_G(x, x') = 0$, so $\kappa(x, x') = 0$. So suppose that $d_G(Gx, Gx') \leq r$. If $g \in G$, and $\chi(gx)^2 \chi(gx')^2$ is nonzero, then (5.3) implies that

$$d(x, x') = d(gx, gx') \leq r + 1,$$

a contradiction. So $\chi(gx)^2 \chi(gx')^2 = 0$ for all $g \in G$, and hence $\kappa(x, x') = 0$.

Right invariance of dg implies that for all $x, x' \in X$ and $g \in G$,

$$g^{-1}\kappa(gx, gx')g = \kappa(x, x'),$$

i.e. κ is invariant under the restriction of the action (5.1) to the diagonal. This implies that the operator on $L^2(E)$ defined by κ is G -equivariant.

The property (2.5) of χ and left invariance of dg imply that

$$\text{Av}(\kappa) = \kappa_G,$$

so Av is surjective. □

Lemma 5.2. *There is a cutoff function χ , satisfying (2.5), such that for all $x, x' \in \text{supp}(\chi)$,*

$$d(x, x') \leq d_G(Gx, Gx') + 1.$$

Proof. Let $Y \subset X$ be a subset intersecting all G -orbits, such that for all $y, y' \in Y$,

$$d(y, y') \leq d_G(Gy, Gy') + 1/2.$$

Let U be the open set of all points in X closer than $1/8$ to Y . Then the intersection of U with any G -orbit is open in that orbit. For all $u, u' \in U$, choose $y, y' \in Y$ such that $d(u, y) < 1/8$ and $d(u', y') < 1/8$. Then

$$d(u, u') < d(y, y') + 1/4 \leq d_G(Gy, Gy') + 3/4 < d_G(Gu, Gu') + 1.$$

Let $\tilde{\chi}$ be any nonnegative continuous function on X such that the interior of its support is U . Then the function χ , given by

$$\chi(x) := \frac{\tilde{\chi}(x)}{\left(\int_G \tilde{\chi}(gx)^2 dg\right)^{1/2}}$$

for $x \in X$, has the desired properties. □

5.2 Averaging operators on Hilbert spaces

We will use an extension of the homomorphism Av to more general bounded operators on $L^2(E)$, not necessarily given by integrable kernels.

We choose a partition of unity $\{\psi_j\}_{j=1}^\infty$ on X , which restricts to a compactly supported partition of unity on every orbit, such that there is an $r > 0$ such that for all j , the set of k for which $d(\text{supp } \psi_j, \text{supp } \psi_k) \leq r$ is finite.

Let $\mathcal{B}_{\text{fp}}(L^2(E))^G$ be the algebra of G -equivariant, bounded operators on $L^2(E)$ with finite propagation. Given $T \in \mathcal{B}_{\text{fp}}(L^2(E))^G$ and $s \in L^2_T(E)^G$, note that $\psi_j s \in L^2(E)$. Hence $T(\psi_j s)$ is well-defined, and for $x \in X$, we set

$$(\text{Av}_{L^2}(T)s)(x) := \sum_j (T(\psi_j s))(x). \quad (5.4)$$

Lemma 5.3. *The sum (5.4) converges for all $x \in X$, and the result is independent of the choice of the partition of unity $\{\psi_j\}$.*

Proof. Fix $x \in X$. Let $f \in C_c(X)$ be a function such that $f(x) = 1$. Let r be greater than both the diameter of $\text{supp } f$ and the propagation of T , and such that

$$d(\text{supp } \psi_j, \text{supp } \psi_k) \leq 2r$$

for only finitely many k , for any fixed j . Then there are only finitely many j for which

$$d(\text{supp } f, \text{supp } \psi_j) \leq r.$$

Hence $fT\psi_j$ is nonzero for only finitely many j , and we see that

$$(\text{Av}_{L^2}(T)s)(x) = (f \text{Av}(T)s)(x) = \sum_j (fT(\psi_j s))(x)$$

is a finite sum, and hence converges.

Let $\{\psi'_j\}$ be another partition of unity on X , with the same properties as $\{\psi_j\}$. We will write

$$\text{Av}_{L^2}^{\{\psi_j\}}(T) \quad \text{and} \quad \text{Av}_{L^2}^{\{\psi'_j\}}(T)$$

for the operators defined by (5.4) using these two partitions of unity. As above, let $J \subset \mathbb{N}$ be a finite set such that $fT\psi_j$ and $fT\psi'_j$ are zero if $j \notin J$. Then

$$fT \sum_{j \in J} \psi_j = fT = fT \sum_{j \in J} \psi'_j.$$

Since the finite sum over J commutes with T , we conclude that

$$\begin{aligned} (\text{Av}_{L^2}^{\{\psi_j\}}(T)s)(x) &= (f \text{Av}_{L^2}^{\{\psi_j\}}(T)s)(x) = \sum_{j \in J} (fT(\psi_j s))(x) \\ &= \sum_{j \in J} (fT(\psi'_j s))(x) = (\text{Av}_{L^2}^{\{\psi'_j\}}(T)s)(x). \end{aligned}$$

□

Lemma 5.4. *The construction (5.4) defines a *-homomorphism*

$$\text{Av}_{L^2}: \mathcal{B}_{\text{fp}}(L^2(E))^G \rightarrow \mathcal{B}(L_T^2(E))^G.$$

Proof. Let $T \in \mathcal{B}_{\text{fp}}(L^2(E))^G$ and $s \in L_T^2(E)^G$. We first claim that $\text{Av}_{L^2}(T)s \in L_T^2(E)^G$. Indeed, the properties of the functions ψ_j , and finite propagation of T imply that $\chi \text{Av}(T)s \in L^2(E)$. And for all $g \in G$ and $x \in X$, one checks, using G -equivariance of T and G -invariance of s , that

$$(g \cdot (\text{Av}(T)s))(x) = \sum_j T((g \cdot \psi_j)s)(x).$$

Since $\{g \cdot \psi_j\}_{j=1}^\infty$ is a partition of unity with the same properties as $\{\psi_j\}_{j=1}^\infty$, the second part of Lemma 5.3 implies that the right hand side equals $(\text{Av}_{L^2}(T)s)(x)$.

Boundedness of the operator $\text{Av}_{L^2}(T)$ on $L_T^2(E)^G$ follows from boundedness and finite propagation of T , via the fact that the sum $\chi \sum_j T\psi_j$ is finite.

If T' is an other operator in $\mathcal{B}_{\text{fp}}(L^2(E))^G$, then for all $x \in X$,

$$\begin{aligned} (\text{Av}_{L^2}(TT')s)(x) &= \sum_j (TT'\psi_j s)(x) \\ &= \sum_{j,k} (T\psi_k T'\psi_j s)(x) = (\text{Av}_{L^2}(T) \text{Av}_{L^2}(T')s)(x). \end{aligned}$$

Here we used the fact that the sum over k is finite for each j , since T' has finite propagation. One checks directly that Av_{L^2} preserves *-operations. \square

Lemma 5.5. *For an operator $T \in C_{\text{ker}}^*(X, L^2(E))^G$, with kernel κ , the operator $\text{Av}_{L^2}(T)$ is an element of $C_{\text{ker}}^*(X/G; L_T^2(E))^G$, and its kernel is $\text{Av}(\kappa)$.*

Proof. This follows from a direct computation involving (5.2). \square

If s_1 and s_2 are sections of E such that their pointwise inner product $(s_1, s_2)_E$ is in $L^1(X)$, we will say that the inner product $(s_1, s_2)_{L^2(E)}$ converges, and define it as the integral of $(s_1, s_2)_E$ over X .

Lemma 5.6. *For all $T \in \mathcal{B}_{\text{fp}}(L^2(E))^G$, $s \in \Gamma_{\text{tc}}(E)^G$ and $\sigma \in \Gamma_c(E)$,*

$$|(\text{Av}_{L^2}(T)s, \sigma)_{L^2(E)}| \leq \|T\| |(s, \sigma)_{L^2(E)}|.$$

In particular, this inner product converges.

Proof. Let $J \subset \mathbb{N}$ be a finite subset (depending on T , s and σ) such that $\sum_{j \in J} \psi_j = 1$ on $\text{supp}(\sigma)$, and for all $j \in \mathbb{N} \setminus J$, $(T\psi_j s, \sigma)_E = 0$. Then

$$\begin{aligned} |(\text{Av}(T)s, \sigma)_{L^2(E)}| &= |(T \sum_{j \in J} \psi_j s, \sigma)_{L^2(E)}| \\ &\leq \|T\| |(\sum_{j \in J} \psi_j s, \sigma)_{L^2(E)}| = \|T\| |(s, \sigma)_{L^2(E)}|. \end{aligned}$$

□

5.3 Relation to the integration trace

In this subsection, we suppose that X/G is compact. Consider the map $\oplus 0$ in (2.4).

Lemma 5.7. *For all $\kappa \in C_{\ker}^*(X; L^2(E))^G$,*

$$\|\text{Av}(\kappa)\|_{\mathcal{B}(L_T^2(E))^G} = \|(I \otimes 1)(\kappa \oplus 0)\|_{\mathcal{B}(\mathcal{H})}.$$

Lemma 5.8. *If X/G is compact, then Av has a unique extension to a *-homomorphism*

$$\text{Av}: C_{\max}^*(X; L^2(E))^G \rightarrow \mathcal{K}(L_T^2(E))^G.$$

Proof. If X/G is compact, then $C_{\ker}^*(X/G; L_T^2(E))^G \subset \mathcal{K}(L_T^2(E))^G$. So the claim is that Av is continuous with respect to the norm $\|\cdot\|_{\max, G}$ and the operator norm on $\mathcal{K}(L_T^2(E))^G$. And Lemma 5.7 implies that for all $\kappa \in C_{\ker}^*(M; L^2(E))^G$,

$$\|\text{Av}(\kappa)\|_{\mathcal{B}(L_T^2(E))^G} \leq \|\kappa \oplus 0\|_{\max} = \|\kappa\|_{\max, G},$$

because $I \otimes 1$ is a *-representation of $C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G \subset L^1(G) \otimes \mathcal{K}(\mathcal{H})$. □

Proposition 5.9. *The following diagram commutes:*

$$\begin{array}{ccc} K_0(C_{\max}^*(X; L^2(E))^G) & \xrightarrow{\oplus 0} & K_0(C_{\max}^*(X)^G) \\ \text{Av} \downarrow & & \cong \downarrow \\ K_0(\mathcal{K}) = \mathbb{Z} & \xleftarrow{I_*} & K_0(C_{\max}^*(G)). \end{array}$$

As in Section 4.4 of [24], consider the map

$$\widetilde{\text{TR}}: C_{\text{ker}}^*(X; L^2(E))^G \rightarrow L^1(G) \otimes \mathcal{K}(L^2(E))$$

such that for $\kappa \in C_{\text{ker}}^*(X; L^2(E))^G$ and $g \in G$, $\widetilde{\text{TR}}(\kappa)(g)$ is the operator with compactly supported Schwartz kernel given by

$$(\widetilde{\text{TR}}(\kappa)(g))(x, x') = \chi(x)\chi(x')g\kappa(g^{-1}x, x'),$$

for $x, x' \in X$. We will use the following fact in the proofs of Lemma 5.7 and Proposition 5.9.

Proposition 5.10. *There is a unitary isomorphism $\eta: \mathcal{H} \xrightarrow{\cong} L^2(E)$ such that for every conjugation-invariant map $\tau: L^1(G) \rightarrow \mathbb{C}$, the following diagram commutes:*

$$\begin{array}{ccc} C_{\text{ker}}^*(X; L^2(E))^G & \xrightarrow{\oplus 0} & C_{\text{ker}}^*(X; L^2(E) \otimes L^2(G))^G \\ \widetilde{\text{TR}} \downarrow & & \cong \downarrow \\ L^1(G) \otimes \mathcal{K}(L^2(E)) & & L^1(G) \otimes \mathcal{K}(\mathcal{H}) \\ \tau \otimes 1 \downarrow & & \tau \otimes 1 \downarrow \\ \mathcal{K}(L^2(E)) & \xleftarrow{\eta^*} & \mathcal{K}(\mathcal{H}) \end{array}$$

This is essentially Proposition 4.10 in [24]. There a specific map τ is used, but its only property used in the proof of this proposition is conjugation invariance.

Consider the isometric embedding $j_\chi: L_T^2(E)^G \rightarrow L^2(E)$ given by point-wise multiplication by χ . It induces $(j_\chi)_*: \mathcal{K}(L_T^2(E)^G) \rightarrow \mathcal{K}(L^2(E))$, given by

$$(j_\chi)_*(T)s = \begin{cases} \chi T \sigma & \text{if } s = \chi \sigma \text{ for } \sigma \in L_T^2(E)^G; \\ 0 & \text{if } s \in (\chi L_T^2(E)^G)^\perp, \end{cases}$$

for all $T \in \mathcal{K}(L_T^2(E)^G)$ and $s \in L^2(E)$. Consider the map

$$I \otimes 1: L^1(G) \otimes \mathcal{K}(L^2(E)) \rightarrow \mathcal{K}(L^2(E)).$$

Lemma 5.11. *For all $\kappa \in C_{\text{ker}}^*(M; L^2(E))^G$,*

$$(I \otimes 1) \circ \widetilde{\text{TR}}(\kappa) = (j_\chi)_*(\text{Av}(T_\kappa)).$$

Proof. Let $\kappa \in C_{\ker}^*(X; L^2(E))^G$, and write

$$T := (I \otimes 1)(\widetilde{\text{TR}}(\kappa)).$$

Write

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition $L^2(E) = \chi L_T^2(E)^G \oplus (\chi L_T^2(E)^G)^\perp$.

If $s \in L^2(E)$ and $x \in X$, then it follows from the definitions that

$$(Ts)(x) = \chi(x) \int_X \int_G \chi(x') g \kappa(g^{-1}x, x') s(x') dg dx'. \quad (5.5)$$

For $x \in X$, set

$$\sigma(x) := \int_X \int_G \chi(x') g \kappa(g^{-1}x, x') s(x') dg dx'.$$

It follows from left invariance of dg that σ is a G -invariant section of E . And $\chi\sigma \in L^2(E)$, so $\sigma \in L_T^2(E)^G$. So the image of T lies inside $\chi L_T^2(E)^G$, which means that $c = d = 0$.

Because I and $\widetilde{\text{TR}}$ are $*$ -homomorphisms,

$$\begin{pmatrix} a^* & 0 \\ b^* & 0 \end{pmatrix} = T^* = (I \otimes 1)(\widetilde{\text{TR}}(\kappa^*)).$$

Since the image of $(I \otimes 1)(\widetilde{\text{TR}}(\kappa^*))$ lies inside $\chi L_T^2(E)^G$ by the same argument as for κ , we find that $b^* = 0$, so $b = 0$.

If $\sigma \in L_T^2(E)^G$, then it follows from (5.5), G -invariance of κ , σ and dm , and from (2.5) that

$$T\chi\sigma = \chi \text{Av}(\kappa)\sigma.$$

So $a = j_\chi \circ \text{Av}(\kappa) \circ j_\chi^{-1}$. □

Proof of Lemma 5.7 and Proposition 5.9. Proposition 5.10, with $\tau = I$, and Lemma 5.11 imply that the diagram

$$\begin{array}{ccc} C_{\ker}^*(X; L^2(E))^G & \xrightarrow{\oplus 0} & C_{\ker}^*(X; L^2(E) \otimes L^2(G))^G \\ \text{Av} \downarrow & & \downarrow \\ \mathcal{K}(L_T^2(E)^G) & & L^1(G) \otimes \mathcal{K}(\mathcal{H}) \\ (j_\chi)_* \downarrow & & I \otimes 1 \downarrow \\ \mathcal{K}(L^2(E)) & \xleftarrow{\eta_*} & \mathcal{K}(\mathcal{H}) \end{array}$$

commutes. This implies both Lemma 5.7 and Proposition 5.9. □

5.4 Averaging operators on Hilbert C^* -modules

As in Subsection 3.1, we write $A := C_{\max}^*(X; L^2(E))_{\text{loc}}^G$. By (2.7) and Lemmas 5.1 and 5.8, we obtain a surjective $*$ -homomorphism

$$\text{Av}: A \rightarrow \mathcal{K}(L_T^2(E))^G.$$

This extends uniquely to multiplier algebras, giving

$$\widetilde{\text{Av}}: \mathcal{L}_A(A) \rightarrow \mathcal{B}(L_T^2(E))^G. \quad (5.6)$$

We return to the setting where $X = M$ is a Riemannian manifold, as in Subsection 3.1.

For clarity, we will use subscripts A and $L^2(E)$ to denote functional calculus of operators on the Hilbert module A and on the Hilbert space $L^2(E)$, as in the following lemma.

Lemma 5.12. *Let $f \in C_b(\mathbb{R})$ such that $f \in C_0(\mathbb{R})$, or $f(x) = \mathcal{O}(x)$ as $x \rightarrow 0$. By functional calculus on the Hilbert A -module A , we can form the operator*

$$f(D)_A \in \mathcal{L}_A(A).$$

Via the usual functional calculus, we can form the operator

$$f(D)_{L^2(E)} \in \mathcal{B}(L^2(E))^G.$$

Let $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. Let $T_\kappa \in \mathcal{B}(L^2(E))^G$ be the operator with Schwartz kernel κ . Then the operator $f(D)_{L^2(E)} \circ T_\kappa$ has a smooth kernel, and so does

$$f(D)_A(\kappa).$$

These two smooth kernels are equal.

Proof. For an operator $S \in \mathcal{B}(L^2(E))^G$ with a smooth kernel with finite propagation, and for all $n \in \mathbb{N}$, the operator $D^n f(D)S = f(D)D^n S$ is a bounded operator on $L^2(E)$. (Indeed, $D^n S$ has a smooth, G -invariant kernel with finite propagation, so it defines a bounded operator since M/G is compact.) Since D is elliptic, it follows that the image of $f(D)S$ lies in the smooth sections, so that this operator also has a smooth kernel. Let $f(D)_{L^2(E)}\kappa$ be the smooth kernel of $f(D)_{L^2(E)} \circ T_\kappa$.

Next, suppose that $f(x) = (x \pm i)^{-1}$. The unbounded operator $D \pm i$ on A is given by applying $D \pm i$ to the first coordinate of a smooth kernel. The element $f(D)_A(\kappa)$ lies in the domain of this operator, and

$$(D \pm i)(f(D)_A(\kappa)) = \kappa = (D \pm i)(f(D)_{L^2(E)}\kappa).$$

So $f(D)_A(\kappa) = f(D)_{L^2(E)}\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. Since the functions $x \mapsto (x \pm i)^{-1}$ generate $C_0(\mathbb{R})$, the claim follows for all $f \in C_0(\mathbb{R})$.

Now suppose that $f \in C_b(\mathbb{R})$, and $f(x) = \mathcal{O}(x)$ as $x \rightarrow 0$. Then $g(x) = f(x)/x$ defines a function $g \in C_0(\mathbb{R})$. The preceding arguments imply that

$$f(D)_A(\kappa) = Dg(D)_A(\kappa) = Dg(D)_{L^2(E)}\kappa = f(D)_{L^2(E)}\kappa.$$

□

We will use the following relation between the averaging maps in (5.4) and (5.6) in the proof of Theorem 3.9.

Lemma 5.13. *Let $b \in C_b(\mathbb{R})$ be such that $b(x) = xg(x)$ for all $x \in \mathbb{R}$, where $g \in C_0(\mathbb{R})$ has compactly supported Fourier transform. Then*

$$\widetilde{\text{Av}}(b(D)_A) = \text{Av}_{L^2}(b(D)_{L^2(E)}) \in \mathcal{K}(L_T^2(E)^G).$$

Proof. The map (5.6) is uniquely determined by the property that for all $\kappa \in A$ and all $S \in \mathcal{L}_A(A)$,

$$\text{Av}(\kappa S) = \text{Av}(\kappa) \widetilde{\text{Av}}(S) \quad \text{and} \quad \text{Av}(S\kappa) = \widetilde{\text{Av}}(S) \text{Av}(\kappa).$$

In fact, $\widetilde{\text{Av}}$ is already determined by these properties for κ in the dense subalgebra $C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. So the claim is that for all $\kappa \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$,

$$\begin{aligned} \text{Av}(\kappa b(D)_A) &= \text{Av}(\kappa) \text{Av}_{L^2}(b(D)_{L^2}); \\ \text{Av}(b(D)_A \kappa) &= \text{Av}_{L^2}(b(D)_{L^2}) \text{Av}(\kappa). \end{aligned} \tag{5.7}$$

The second equality in (5.7) is true, because by Lemmas 5.5 and 5.12,

$$\text{Av}_{L^2}(b(D)_{L^2}) \text{Av}(\kappa) = \text{Av}(b(D)_{L^2} \kappa) = \text{Av}(b(D)_A \kappa).$$

The element $\kappa b(D)_A$ is defined as

$$L_\kappa \circ b(D)_A \in \mathcal{K}_A(A) \cong A.$$

Here L_κ is left composition with κ . Lemma 5.12 implies that for all $\kappa, \kappa' \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$, the element $\kappa b(D)_A \kappa'$ has a smooth kernel, equal to the composition of the kernels of κ and of $b(D)_{L^2} \kappa'$. By associativity, that equals the composition of the smooth kernels of $\kappa b(D)_{L^2}$ and κ' . So $\kappa b(D)_A \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$, and its kernel is the smooth kernel of $\kappa b(D)_{L^2}$. Hence, by Lemma 5.5,

$$\text{Av}(\kappa b(D)_A) = \text{Av}(\kappa b(D)_{L^2}) = \text{Av}(\kappa) \text{Av}_{L^2}(b(D)_{L^2}).$$

□

6 The invariant index

In this section, we use the averaging maps from Section 5 to prove Theorem 3.9.

6.1 The index of $\widetilde{\text{Av}}(b(D))$

Let b be a normalising function as below Proposition 3.3. That proposition implies that

$$\widetilde{\text{Av}}(b(D))^2 - 1 = \text{Av}(b(D)^2 - 1) \in \mathcal{K}(L_T^2(E)^G).$$

So $\widetilde{\text{Av}}(b(D))$ is a Fredholm operator.

We will prove Theorem 3.9 by proving Propositions 6.1 and 6.2 below. Here, to be precise, by the index of the odd-graded operator $\widetilde{\text{Av}}(b(D))$ on $L_T^2(E)^G$, we mean the index in the graded sense; i.e. the index of its restriction $\widetilde{\text{Av}}(b(D))_+$ to even-graded sections.

Proposition 6.1.

$$\text{index}(\widetilde{\text{Av}}(b(D))) = \text{index}(D)^G.$$

Proposition 6.2.

$$\text{index}(\widetilde{\text{Av}}(b(D))) = I_*(\text{index}_G^{\text{loc}}(D)).$$

Proof. Consider the boundary maps

$$\begin{aligned} \partial_{\mathcal{B}}: K_1(\mathcal{B}(L_T^2(E)^G)/\mathcal{K}(L_T^2(E)^G)) &\rightarrow K_0(\mathcal{K}(L_T^2(E)^G)); \\ \partial_A: K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A)) &\rightarrow K_0(\mathcal{K}_A(A)). \end{aligned}$$

Naturality of boundary maps with respect to $*$ -homomorphisms implies that

$$\text{index}(\widetilde{\text{Av}}(b(D))) = \partial_{\mathcal{B}}[\widetilde{\text{Av}}(b(D))_+] = \text{Av}(\partial_A[b(D)_+])$$

Here we used the fact that $\widetilde{\text{Av}}(b(D))_+ = \widetilde{\text{Av}}(b(D)_+)$. Proposition 5.9 and (2.7) now imply that the right hand side equals

$$I_*(\partial_A[b(D)_+] \oplus 0) = I_*(\text{index}_G^{\text{loc}}(D)).$$

□

To prove Theorem 3.9, it remains to prove Proposition 6.1.

6.2 The index of $\widetilde{\text{Av}}(b(D))$ and the invariant index

In this subsection, we prove Proposition 6.1. The main point of the proof is dealing with the fact that sections in $L_T^2(E)^G$ are not square integrable in general.

We may choose the normalising function b so that $b(t) = \mathcal{O}(t)$ as $t \rightarrow 0$. Then the operator

$$S := \frac{b(D)}{D}$$

on $L^2(E)$ is bounded.

Lemma 6.3. *For all $\sigma \in \Gamma_c^\infty(E)$ and $s \in \Gamma^\infty(E) \cap L_T^2(E)^G$, the inner product $(Ds, S\sigma)_{L^2(E)}$ is well-defined and equals*

$$(\widetilde{\text{Av}}(b(D))s, \sigma)_{L^2(E)}.$$

Proof. As in Subsection 5.4, we use subscripts A and L^2 to distinguish functional calculus of operators on the Hilbert A -module A and on $L^2(E)$.

Let $(b_j)_{j=1}^\infty$ be a sequence in $C_b(\mathbb{R})$ converging to b in the sup-norm, such that for each j , and all $x \in \mathbb{R}$, $b_j(x) = xg_j(x)$, where $g_j \in C_0(\mathbb{R})$ has compactly supported Fourier transform. Then $g_j(D)_{L^2}$, and hence $b_j(D)_{L^2}$, has finite propagation. So by Lemma 5.6, for all $s \in \Gamma_{tc}^G$ and $\sigma \in \Gamma_c(E)$,

$$|(\text{Av}_{L^2}(b_j(D)_{L^2})s, \sigma)_{L^2(E)}| \leq \|b_j(D)_{L^2}\| |(s, \sigma)_{L^2(E)}|. \quad (6.1)$$

We also have

$$\|b_j(D)_{L^2}\| \leq \|b_j(D)_A\|_{\mathcal{L}_A(A)} \quad (6.2)$$

To see that this is true, note that by Lemma 5.12,

$$b_j(D)_{L^2} = b_j(D)_A =: \kappa \in C_{\text{ker}}^*(M; L^2(E))_{\text{loc}}^G.$$

And

$$\|\kappa\|_{\mathcal{B}(L^2(E))} = \|\kappa \oplus 0\|_{\mathcal{B}(L^2(E) \otimes L^2(G))} \leq \|\kappa \oplus 0\|_{\text{max}} = \|\kappa\|_A = \|\kappa\|_{\mathcal{L}_A(A)}.$$

The inequality is true, because the defining representation of $\mathcal{B}(L^2(E) \otimes L^2(G))$ in $L^2(E) \otimes L^2(G)$ trivially restricts to a $*$ -representation of $C_{\text{ker}}^*(M; L^2(E))_{\text{loc}}^G$. So (6.2) follows.

By the first point of Theorem 3.2,

$$b_j(D) \rightarrow b(D) \in \mathcal{L}_A(A).$$

By Lemma 5.13,

$$\widetilde{\text{Av}}(b_j(D)) = \text{Av}_{L^2}(b_j(D)) \in \mathcal{B}(L_T^2(E))^G.$$

This equality, together with (6.1) and (6.2) implies that

$$|(\widetilde{\text{Av}}(b_j(D))s, \sigma)_{L^2(E)}| \leq \|b_j(D)\|_{\mathcal{L}_A(A)} |(s, \sigma)_{L^2(E)}|$$

and hence

$$(\widetilde{\text{Av}}(b(D))s, \sigma)_{L^2(E)} = \lim_{j \rightarrow \infty} (\widetilde{\text{Av}}(b_j(D))s, \sigma)_{L^2(E)}.$$

This means it suffices to prove the claim for each finite-propagation approximant $b_j(D)$, namely that

$$(\widetilde{\text{Av}}(b_j(D))s, \sigma)_{L^2(E)} = (\text{Av}_{L^2}(b_j(D))s, \sigma)_{L^2(E)} = (Ds, S_j\sigma)_{L^2(E)}.$$

To prove the latter equality, let $r_j > 0$ be the propagation of $b_j(D)$, and let the partition of unity $\{\psi_k\}_{k=1}^\infty$ be as in (5.4). Suppose these functions are real-valued. For $\sigma \in \Gamma_c^\infty(E)$, only finitely many of the functions ψ_k have supports closer than r_j to $\text{supp}(\sigma)$. Let $K_\sigma \subset \mathbb{N}$ be the set of the corresponding indices k . Then for $s \in \Gamma^\infty(E) \cap L_T^2(E)^G$, since $b_j(D)$ commutes with finite sums,

$$\begin{aligned} (\text{Av}(b_j(D))s, \sigma)_{L^2(E)} &= (b_j(D) \sum_{k \in K_\sigma} \psi_k s, \sigma)_{L^2(E)} \\ &= (s, \sum_{k \in K_\sigma} \psi_k b_j(D)\sigma)_{L^2(E)} = (s, b_j(D)\sigma)_{L^2(E)} = (Ds, S_j\sigma)_{L^2(E)}. \end{aligned}$$

□

Lemma 6.4.

$$\ker(\widetilde{\text{Av}}(b(D))) \subset \Gamma^\infty(E)$$

Proof. Let $s \in \ker(\widetilde{\text{Av}}(b(D)))$. Then

$$s = \text{Av}(1 - b(D)^2)s.$$

Here we used the fact that $1 - b(D)^2 \in C_{\ker}^\infty(M; L^2(E))_{\text{loc}}^G$. That also implies that the right hand side is smooth, and hence so is s . □

Lemma 6.5. *The space $S(\Gamma_c^\infty(E))$ is dense in $L^2(E)$.*

Proof. Let $s \in L^2(E)$, and let $(s_j)_{j=1}^\infty$ be a sequence in $\Gamma_c^\infty(E)$ converging to s in L^2 -norm. Then

$$\|Ss - Ss_j\|_{L^2(E)} \rightarrow 0.$$

So $S(\Gamma_c^\infty(E))$ is dense in $\text{im}(S)$. Now if $t \in \Gamma_c^\infty(E)$, then $t \in \text{dom}(S^{-1})$. Hence $t = S(S^{-1}t) \in \text{im}(S)$. So $\text{im}(S) \subset L^2(E)$ is dense, which completes the proof. \square

Proof of Proposition 6.1. By elliptic regularity, $\ker(D) \subset \Gamma^\infty(E)$. So Lemma 6.3 implies that

$$\ker(D) \cap L_T^2(E)^G \subset \ker(\widetilde{\text{Av}}(b(D))).$$

We claim that for all $s \in \ker(\widetilde{\text{Av}}(b(D))) \cap \Gamma^\infty(E)$ and $\sigma \in \Gamma_c^\infty(E)$,

$$(Ds, \sigma)_{L^2(E)} = 0. \quad (6.3)$$

Indeed, Let $\sigma \in \Gamma_c^\infty(E)$. By Lemma 6.5, there is a sequence $(s_j)_{j=1}^\infty$ in $\Gamma_c^\infty(E)$ such that $(Ss_j)_{j=1}^\infty$ converges to σ in L^2 -norm. For all $s \in \ker \widetilde{\text{Av}}(b(D)) \cap \Gamma^\infty(E)$, Lemma 6.3 implies that $(Ds, Ss_j)_{L^2(E)} = 0$. So (6.3) follows.

By Lemma 6.4, (6.3) implies that

$$\ker(\widetilde{\text{Av}}(b(D))) \subset \ker(D) \cap L_T^2(E)^G.$$

So

$$\ker(\widetilde{\text{Av}}(b(D))) = \ker(D) \cap L_T^2(E)^G,$$

including gradings. \square

7 Quantisation commutes with reduction

In this section, we use Theorem 3.9 to prove Theorem 3.11.

7.1 A localisation estimate

Let U be a relatively cocompact, G -invariant neighbourhood of $\mu^{-1}(0)$. Since U is relatively cocompact, we can enlarge Z , outside which the estimate (3.4) holds, if needed so that its interior contains the closure of U .

Fix a G -invariant metric on $M \times \mathfrak{g} \rightarrow M$, where G acts on \mathfrak{g} via the adjoint action. Let $\|\mu\|^2$ be the norm-squared of the Spin^c -moment map μ with respect to this metric. Let v^μ be the vector field on M induced by the

map $M \rightarrow \mathfrak{g}$ dual to μ with respect to this metric. Explicitly, if $m \in M$ and $X \in \mathfrak{g}$ is dual to $\mu(m) \in \mathfrak{g}^*$ for the metric at m , then

$$v^\mu(m) = X^M(m).$$

By G -invariance of the metric on $M \times \mathfrak{g}$, this vector field is G -invariant. Let $f \in C^\infty(M)^G$ be a nonnegative function with cocompact support,² such that $f \equiv 1$ on Z . For any $h \in \mathbb{R}$, and any $p \in \mathbb{Z}_{\geq 0}$, consider the operator

$$D_{p,h} := \tilde{D}^{L^p} + h\Phi - ifc(v^\mu)$$

on $\Gamma^\infty(\mathcal{S} \otimes L^p)$.

Let $\chi \in C^\infty(M)$ be a cutoff function for the action by G on M , as in (2.5). The proof of Theorem 3.11 is based on the following localisation property. This is an analogue of Theorem 2.1 in [40]. Let c be as in (3.4).

Proposition 7.1. *There are a constant $h \geq 1$ and a $p_0 \in \mathbb{Z}_{\geq 0}$, such that for all $p \geq p_0$ and all $s \in \Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$ supported outside U ,*

$$\|\chi D_{p,h}s\|_{L^2(\mathcal{S} \otimes L^p)}^2 \geq \min(c^2/2, 1) \|\chi s\|_{L^2(\mathcal{S} \otimes L^p)}^2. \quad (7.1)$$

Similarly to [18, 20, 30, 40], the proof of Proposition 7.1 is based on an expression for squares of deformed operators, Proposition 7.2. This expression is deduced from an expression in [20].

For $p \in \mathbb{Z}_{\geq 0}$ and $h \in \mathbb{R}$, let $\hat{D}_{p,h}$ be the operator on $\chi\Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$ defined by

$$\hat{D}_{p,h}(\chi s) = \chi D_{p,h}s,$$

for all $s \in \Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G$. Note that

$$\chi\Gamma_{\text{tc}}^\infty(\mathcal{S} \otimes L^p)^G \subset \Gamma_c^\infty(\mathcal{S} \otimes L^p).$$

Let $\hat{D}_{p,h}^*$ be the formal adjoint of $\hat{D}_{p,h}$ with respect to the L^2 -inner product.

Proposition 7.2. *There is a G -equivariant vector bundle endomorphism B of $\mathcal{S} \otimes L^p$, which vanishes at points where f and df vanish, such that, on G -invariant smooth sections,*

$$\hat{D}_{p,h}^* \hat{D}_{p,h} = (\tilde{D}^{L^p} + h\Phi)^2 + B + (2p+1)2\pi f \|\mu\|^2. \quad (7.2)$$

²In earlier work on deformed Dirac operators of the form $D + ifc(v^\mu)$ on non-cocompact manifolds [5, 6, 18, 20, 22], the function f was required to grow at infinity in a suitable way. In our setting, we actually need f to vanish outside a cocompact set (this is used in the proof of Proposition 7.1). This is possible, because invertibility at infinity is guaranteed by the term Φ .

Proof. This follows from Corollary 8.5 in [20]. Compared to that corollary, there is an additional term Φ added to the Dirac operator \tilde{D}^{L^p} . This leads to an extra term $\{\Phi, -ifc(v^\mu)\}$, which we absorb into B . We have also absorbed a term proportional to $\|v^\mu\|^2$ that is present in Corollary 8.5 in [20] into B . \square

Proof of Proposition 7.1. On the cocompact set $Z \setminus U$, the positive function $\|\mu\|^2$ is bounded below by a positive constant. Also, $\|B\|$ is bounded on Z . Choose $p_0 \in \mathbb{Z}_{\geq 0}$ such that, on $Z \setminus U$,

$$(2p_0 + 1)2\pi\|\mu\|^2 \geq \|B\| + 1. \quad (7.3)$$

Let $h \geq 1$ be such that, on the cocompact set $\text{supp}(f)$,

$$hc^2/2 \geq \|B\|. \quad (7.4)$$

We claim that the estimate (7.1) holds for these choices of p_0 and h .

Because of (3.7),

$$(\tilde{D}^{L^p} + h\Phi)^2 = (\tilde{D}^{L^p})^2 + h^2\Phi^2 + h\{\tilde{D}, \Phi\}.$$

(Here, as in (3.8), we omit ' $\otimes 1_{L^p}$ '.) Hence Proposition 7.2 implies that, on G -invariant sections,

$$\hat{D}_{p,h}^* \hat{D}_{p,h} = (\tilde{D}^{L^p})^2 + h^2\Phi^2 + h\{\tilde{D}, \Phi\} + B + (2p + 1)2\pi f\|\mu\|^2.$$

On $M \setminus \text{supp}(f)$, we have $B = 0$, so

$$\hat{D}_{p,h}^* \hat{D}_{p,h} = (\tilde{D}^{L^p})^2 + h^2\Phi^2 + h\{\tilde{D}, \Phi\}$$

And, since $h \geq 1$,

$$h^2\Phi^2 + h\{\tilde{D}, \Phi\} \geq c^2.$$

On $\text{supp}(f) \setminus Z$, we similarly have

$$h^2\Phi^2 + h\{\tilde{D}, \Phi\} + B + (2p+1)2\pi f\|\mu\|^2 \geq h\Phi^2 + \{\tilde{D}, \Phi\} + h^{-1}B \geq c^2 - h^{-1}\|B\| \geq c^2/2.$$

Finally, on $Z \setminus U$, the last two terms on the right hand side of (7.2) satisfy

$$B + (2p + 1)2\pi f\|\mu\|^2 = B + (2p + 1)2\pi\|\mu\|^2 \geq 1$$

if $p \geq p_0$ as in (7.3). \square

7.2 Proof of Theorem 3.11

Lemma 7.3. *Let D be any operator as in Subsection 3.1, where E is $\mathbb{Z}/2$ -graded. Let $S \in \text{End}(E)^G$ be an odd, fibrewise self-adjoint vector bundle endomorphism which is zero outside a cocompact set. Then $(D + S)^2$ has a uniform lower bound outside a cocompact set, and*

$$\text{index}_G^{\text{loc}}(D + S) = \text{index}_G^{\text{loc}}(D) \in K_0(C_{\max}^*(G)).$$

Proof. The operator $D + S$ is elliptic, and $(D + S)^2$ has a positive lower bound outside $Z \cup \text{supp}(S)$. Hence its index is well-defined. Since $\text{supp}(S)$ is cocompact, S is a bounded operator on $L^2(E)$. Hence the path of operators

$$t \mapsto b(D + tS)$$

is continuous in the operator norm, where b is a normalising function as in Subsection 3.1. This defines an operator homotopy showing that

$$[b(D + tS)_+] \in K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A))$$

is independent of t . □

Lemma 7.4. *For all $h \geq 1$,*

$$\text{index}_G^{\text{loc}}(\tilde{D} + \Phi) = \text{index}_G^{\text{loc}}(\tilde{D} + h\Phi).$$

Proof. Set

$$\tilde{D}_t := \tilde{D} + (1 - t + th)\Phi$$

For all t , we have $(1 - t + th) \geq 1$, so that

$$\tilde{D}_t^2 = \tilde{D}^2 + (1 - t + th)(\tilde{D}\Phi + \Phi\tilde{D}) + (1 - t + th)^2\Phi^2 \geq (D + \Phi)^2.$$

This has a positive lower bound outside a fixed cocompact set. So, for a suitable normalising function b , we have an invertible element

$$b(D_t)_+ \in \mathcal{L}_A(A)/\mathcal{K}_A(A)$$

for all $t \in [0, 1]$. Because $\|\Phi\|$ is bounded, Lemma 4.7 implies that this path of operators is continuous in the operator norm. Hence the class

$$[b(D_t)_+] \in K_1(\mathcal{L}_A(A)/\mathcal{K}_A(A))$$

is independent of t . □

Proof of Theorem 3.11. Let μ be as in Subsection 7.1. Let h and p_0 be as in Proposition 7.1, and fix $p \geq p_0$. By Lemma 7.4,

$$\text{index}_G^{\text{loc}}(D_p) = \text{index}_G^{\text{loc}}(\tilde{D}^{L^p} + h\Phi).$$

And by Lemma 7.3 and the fact that f has compact support,

$$\text{index}_G^{\text{loc}}(\tilde{D}^{L^p} + h\Phi) = \text{index}_G^{\text{loc}}(D_{p,h}).$$

So by Theorem 3.9,

$$I_*(\text{index}_G^{\text{loc}}(D_p)) = \text{index}(D_{p,h})^G.$$

Proposition 7.1 implies that the index on the right hand side localises near $\mu^{-1}(0)$ in the sense of Proposition 3.1 in [30], Proposition 6.3 in [18] and Proposition 8.2 in [20]. Hence the result follows as in Section 7 of [18]. The only difference is the addition of the vector bundle endomorphism Φ , but this does not affect the index on M_0 . Indeed, Lemma 7.3 implies that we may replace Φ by an endomorphism that is zero near $\mu^{-1}(0)$ and equal to Φ outside a cocompact set, without changing the index. \square

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