Highly symmetric homogeneous Kobayashi-hyperbolic manifolds

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Contents

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Abstract

Kobayashi-hyperbolic manifolds are an important and well-studied class of complex manifolds defined by the property that the Kobayashi pseudodistance is a true distance. Such manifolds that have automorphism group of sufficiently high dimension can be classified up to biholomorphism, and the goal of this thesis is to continue the classification of homogeneous Kobayashi-hyperbolic manifolds started by Alexander Isaev in the early 2000s. We settle the classification of such manifolds with automorphism group dimensions $n^2 - 7$ and $n^2 - 8$, where n is the dimension of the manifold. We do so by analysing the Lie algebra of the automorphism group of a Siegel domain of the second kind corresponding to a homogeneous Kobayashi-hyperbolic manifold of a given automorphism group dimension.

Chapter 1

Introduction

1.1 Background and context

Complex geometry is concerned with the study of complex manifolds, mathematical objects that locally resemble complex Euclidean space. Such manifolds can be thought of as differentiable manifolds equipped with a complex structure, and it is this additional structure that ensures more rigidity than typical geometric structures found in real differential geometry. The topic has been widely studied throughout the last century, and has many fruitful interactions with other fields such as differential and algebraic geometry. It is a major goal in complex geometry to study, classify and characterise complex manifolds where possible. One way of characterising complex manifolds is by their holomorphic automorphism group, the group of bijective holomorphic mappings from the manifold to itself. Even in low dimensions and after imposing restrictive assumptions such as connectedness and compactness, a complete classification of all complex manifolds by their automorphism group is, at this point, unrealistic. It is therefore standard practice to restrict the objects of study even further, and to pursue a classification of certain classes of complex manifolds. One widely studied class of such manifolds is the class of *Kobayashi-hyperbolic* manifolds. These are complex manifolds which admit a certain distance function (to be defined and discussed in the following chapter). Such manifolds have been extensively studied since their introduction in the late 1960s, and have many attractive and interesting properties (see [24], [25] for details). The open unit disc in $\mathbb{C}$ is the most basic example of a Kobayashi-hyperbolic manifold, while $\mathbb{C}$ itself is not Kobayashi-hyperbolic. The class of Kobayashi-hyperbolic manifolds is quite large, and includes all bounded domains and many unbounded domains in $\mathbb{C}^n$. Such manifolds are seen in many subfields of complex analysis such as complex dynamics and

Riemann surface theory, as well as in closely related areas such as algebraic geometry.

Geometric function theory is a subfield of complex geometry concerned with the study of holomorphic mappings between complex manifolds. A cornerstone of the topic is the Riemann mapping theorem, which states that any simply connected domain in $\mathbb{C}$ (that is not $\mathbb{C}$ itself) is biholomorphic to the open unit disc. One of the ingredients in its proof is an important and influential theorem known as the *Schwarz Lemma*. This lemma states that for any holomorphic mapping from the unit disc to itself that fixes the origin, any point in the unit disc is mapped to a point of lesser or equal modulus. It is often cited as one of the simplest results that illustrates the rigidity of holomorphic functions. The *Schwarz-Pick Lemma*, which is discussed at the beginning of our exposition in the following chapter, generalises this result for holomorphic mappings from the disc to itself that do not necessarily fix the origin. In 1938, Ahlfors showed that the Schwarz-Pick lemma could be interpreted in terms of curvatures of Riemannian metrics, and since then the result has enjoyed a prominent place in complex geometry (see the survey article [29] for a summary of the many generalisations of this lemma). It is possible to construct certain distance functions using the Schwarz-Pick Lemma that have interesting properties due to this rigidity. It is in fact a prominent technique in geometric function theory to place distances on complex manifolds, and we discuss this now.

When given a biholomorphic mapping between manifolds, a standard approach in geometric function theory is to place distances on each manifold such that the distance between points is invariant under the mapping and its inverse. Analysis of the distances themselves may then lead to conclusions regarding the mapping [6]. Some examples of such distances include the Poincaré distance, and the Carathéodory, Kobayashi and Bergman pseudodistances. In fact, all we require in the case of biholomorphic mappings is that the distance function be *non-increasing*, and invariance follows immediately. This property is precisely what is given to us by the Schwarz-Pick Lemma, and this lemma is used in the definitions of the distances and pseudodistances listed above. In the following chapter we will examine two of these in particular, the Poincaré distance and the Kobayashi pseudodistance. In fact, the definitions of both of these involve the Schwarz-Pick Lemma, as we will outline.

Beginning in the late 1990s, Alexander Isaev began a program of classification of Kobayashi-hyperbolic manifolds by their automorphism group. The automorphism group of a Kobayashi-hyperbolic manifold has a largest possible dimension relative to the dimension of the manifold, which we discuss below. Isaev began by considering Kobayashi-hyperbolic manifolds of this largest possible dimension, and 'worked downwards' one dimension at a time until further classification became impossible. We outline below the point at which this occurs, and exactly why further classification is impossible. Isaev completed this classification in 2005, and proceeded to publish the monograph [13] which outlined all of the technical work involved in the classification. Since its publication, specialists working in complex geometry have utilised this classification (see for example [9], [37]) and it would be desirable to extend it beyond the point at which the classification becomes infeasible. It is possible to do so provided we restrict further classification to the subclass of *homogeneous* Kobayashi-hyperbolic manifolds. A complex manifold is homogeneous if the action of its automorphism group on the manifold is transitive. The assumption of homogeneity is a reasonable one, as homogeneous complex manifolds are very important and are widely considered in complex geometry. Making this assumption allows us to reduce the study of an arbitrary Kobayashi-hyperbolic manifold to the study of a comparatively straightforward domain through an important structural result that we outline in the following chapter. The domain in question is known as a *Siegel domain of the second kind*, and can be considered as a higher-dimensional analogue of the upper half plane. An important mathematical structure that features in its definition is that of a homogeneous open convex cone. An open convex cone is an open subset of $\mathbb{R}^n$ that is closed with respect to taking linear combinations of its elements with positive coefficients (a precise definition is given in the following chapter). In analysing each Siegel domain of the second kind, we must consider a small number of homogeneous open convex cones of low dimension.

We now discuss the dimension of the automorphism group and formulate the maximality property of the automorphism group mentioned above. Let M be an n-dimensional Kobayashi-hyperbolic complex manifold, and let $d(M) := \dim \mathrm{Aut}(M)$. Then $d(M) \leq n^2 + 2n$, with equality if and only if M is biholomorphic to the open unit ball $B^n \subset \mathbb{C}^n$ (a proof of this important fact can be found in [25, p. 70]). In a series of articles (see the above-mentioned monograph [13] for a consolidation of the results) Isaev classified up to biholomorphim all Kobayashi-hyperbolic manifolds with automorphism group dimension $n^2 - 1 \leq d(M) \leq n^2 + 2n$. As it turns out, no manifolds in fact satisfy $n^2 + 3 \leq d(M) < n^2 + 2n$. Continuing the classification for an arbitrary Kobayashi-hyperbolic manifold (that is, without the homogeneity assumption) beyond the dimension of $d(M) = n^2 - 1$ is impossible. Consider a generic *Reinhardt domain* in $\mathbb{C}^2$, that is, a domain invariant under the rotations $z_j \mapsto e^{i\varphi_j} z_j$ where $\varphi_j \in \mathbb{R}$ for $j = 1, 2$. Such domains have no automorphisms other than these rotations, and hence have a 2-dimensional automorphism group. In particular, if D is a typical Reinhardt domain in $\mathbb{C}^2$, then $d(D) = 2 = n^2 - 2$. Such domains have uncountably many isomorphism classes, and so cannot be explicitly described.

In 2017, Isaev resolved to continue the classification for homogeneous Kobayashi-hyperbolic manifolds beyond this critical dimension of $d(M) = n^2 - 1$. The classification proved to be viable with the homogeneity assumption, and he was able to explicitly describe all homogeneous Kobayashi-hyperbolic manifolds with dimension $n^2 - 6 \leq d(M) \leq n^2 - 2$ (see articles [15], [14] and [16]). With the above facts in mind, let us now present the results of the thesis.

1.2 Results of the thesis

In this thesis, we continue the classification of homogeneous Kobayashi-hyperbolic manifolds by their automorphism group and settle the next two cases in the classification. We identify all n-dimensional homogeneous Kobayashi-hyperbolic manifolds of automorphism group dimension $d(M) = n^2 - 7$ and $d(M) = n^2 - 8$ up to biholomorphism. We prove the following two theorems:

Main Theorem 1. *Let M be a homogeneous n-dimensional Kobayashi-hyperbolic manifold with $d(M) = n^2 - 7$. Then one of the following holds:*

(i) *$n = 5$ and M is biholomorphic to $B^2 \times T_3$, where T_3 is the tube domain*

$$T_3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : (\mathrm{Im}\, z_1)^2 - (\mathrm{Im}\, z_2)^2 - (\mathrm{Im}\, z_3)^2 > 0, \mathrm{Im}\, z_1 > 0\}.$$

(ii) *$n = 5$ and M is biholomorphic to $B^1 \times T_4$, where T_4 is the tube domain*

$$T_4 = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (\mathrm{Im}\, z_1)^2 - (\mathrm{Im} z_2)^2 - (\mathrm{Im}\, z_3)^2 - (\mathrm{Im}\, z_4)^2 > 0, \\ \mathrm{Im}\, z_1 > 0\}.$$

Main Theorem 2. *Let M be a homogeneous n-dimensional Kobayashi-hyperbolic manifold with $d(M) = n^2 - 8$. Then one of the following holds:*

(i) *$n = 5$ and M is biholomorphic to $B^1 \times B^1 \times B^1 \times B^2$.*

(ii) *$n = 6$ and M is biholomorphic to the tube domain*

$$T_6 = \{(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6 : (\mathrm{Im}\, z_1)^2 - (\mathrm{Im}\, z_2)^2 - (\mathrm{Im}\, z_3)^2 \\ - (\mathrm{Im}\, z_4)^2 - (\mathrm{Im}\, z_5)^2 - (\mathrm{Im}\, z_6)^2 > 0, \mathrm{Im}\, z_1 > 0\}.$$

(iii) *$n = 7$ and M is biholomorphic to $B^1 \times B^1 \times B^5$.*

(iv) $n = 8$ *and* M *is biholomorphic to* $B^2 \times B^6$.

We briefly summarise the technical work involved in proving these two theorems. The study of an arbitrary homogeneous Kobayashi-hyperbolic manifold is reduced to the study of a Siegel domain of the second kind, as outlined above. For a given Siegel domain of the second kind, certain dimensional considerations allow us to consider only a small number of possibilities of such Siegel domains. For each of these cases, it is a matter of analysing the Lie algebra of the automorphism group of the corresponding Siegel domain of the second kind and, in particular, computing its dimension. We then use this information to determine whether a contribution to the classification is made.

Compared with the preceding cases settled by Isaev in [15], [14] and [16], the $d(M) = n^2 - 7$ case presented unexpected difficulties. This is largely due to the fact that, for the first time in the classification, homogeneous open convex cones of dimension five need to be considered. We recall the classification of all homogeneous open convex cones up to linear equivalence in the following chapter, which includes a description of all such five-dimensional cones. There are in fact six homogeneous open convex cones in $\mathbb{R}^5$ up to linear equivalence. Two of these five-dimensional cones are of a considerably more complicated structure than the others. These two cones are known in the specialist literature as the *Vinberg cone* and the *dual Vinberg cone* (listed as Ω_7 and Ω_8 in the classification), named after E. B. Vinberg. It was Vinberg who discovered and proved that these two cones are in fact the simplest examples of homogeneous open convex cones which are *non-symmetric*. Due to the complicated structure of these cones, determining their automorphism groups and associated Lie algebras is not a simple task. In fact, to our knowledge, there is currently no explicit description of either automorphism group in the literature, aside from the preprint [18]. In this thesis, we take great care to determine the automorphism group of the Vinberg cone and its associated Lie algebra, and a full description of the process by which we do this is presented at the end of Chapter 3.

As is evident from the above discussion, the Vinberg and dual Vinberg cones are in fact *dual* to each other (we explain the concept of dual cones in the following chapter). Proofs of this fact are surprisingly hard to come by. Some complete proofs do exist in the literature, but require the rather sophisticated machinery of *T-algebras*, an algebraic framework for the study of homogeneous convex cones first introduced by Vinberg in [38] (for recent summaries on T-algebras, see [5], [19]). In this thesis, we provide an elementary proof of this fact using only basic linear algebra. To our knowledge, this is the first elementary proof of this fact that exists in the literature.

We conclude this section by presenting the classification of homogeneous Kobayashi-hyperbolic manifolds with $d(M) \geq n^2 - 8$ up to biholomorphism. Combined with the classical fact for dimension $d(M) = n^2 + 2n$, the results collected in [13], and the articles [15], [14] and [16], the above results yield the following classification for homogeneous Kobayashi-hyperbolic manifolds:

Theorem 1.2.1. *Let M be a homogeneous n-dimensional Kobayashi-hyperbolic manifold satisfying $n^2 - 8 \leq d(M) \leq n^2 + 2n$. Then M is biholomorphic either to a product of unit balls, a tube domain, a product of a unit ball and a tube domain, or to the domain $\mathcal{D}$ given below. Specifically, M is one of the following manifolds:*

(i) B^n (*here* $d(M) = n^2 + 2n$).

(ii) $B^1 \times B^{n-1}$ (*here* $d(M) = n^2 + 2$).

(iii) $B^1 \times B^1 \times B^1$ (*here* $n = 3$, $d(M) = 9 = n^2$).

(iv) $B^2 \times B^2$ (*here* $n = 4$, $d(M) = 16 = n^2$).

(v) $B^1 \times B^1 \times B^2$ (*here* $n = 4$, $d(M) = 14 = n^2 - 2$).

(vi) $B^2 \times B^3$ (*here* $n = 5$, $d(M) = 23 = n^2 - 2$).

(vii) $B^1 \times B^1 \times B^1 \times B^1$ (*here* $n = 4$, $d(M) = 12 = n^2 - 4$).

(viii) $B^1 \times B^1 \times B^3$ (*here* $n = 5$, $d(M) = 21 = n^2 - 4$).

(ix) $B^2 \times B^4$ (*here* $n = 6$, $d(M) = 32 = n^2 - 4$).

(x) $B^1 \times B^2 \times B^2$ (*here* $n = 5$, $d(M) = 19 = n^2 - 6$).

(xi) $B^3 \times B^3$ (*here* $n = 6$, $d(M) = 30 = n^2 - 6$).

(xii) $B^1 \times B^1 \times B^4$ (*here* $n = 6$, $d(M) = 30 = n^2 - 6$).

(xiii) $B^2 \times B^5$ (*here* $n = 7$, $d(M) = 43 = n^2 - 6$).

(xiv) $B^1 \times B^1 \times B^1 \times B^2$ (*here* $n = 5$, $d(M) = 17 = n^2 - 8$).

(xv) $B^1 \times B^1 \times B^5$ (*here* $n = 7$, $d(M) = 41 = n^2 - 8$).

(xvi) $B^2 \times B^6$ (*here* $n = 8$, $d(M) = 56 = n^2 - 8$).

(xvii) *the tube domain T_3 defined in Main Theorem 1* (*here* $n = 3$, $d(M) = 10 = n^2 + 1$).

(xviii) *the tube domain* T_4 *defined in Main Theorem 1 (here* $n = 4$, $d(M) = 15 = n^2 - 1$*).*

(xix) *the tube domain* T_5 *given by*

$$T_5 = \big\{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : (\operatorname{Im} z_1)^2 - (\operatorname{Im} z_2)^2 - (\operatorname{Im} z_3)^2 \\ - (\operatorname{Im} z_4)^2 - (\operatorname{Im} z_5)^2 > 0, \operatorname{Im} z_1 > 0\big\}.$$

(here $n = 5$, $d(M) = 21 = n^2 - 4$*),*

(xx) *the tube domain* T_6 *defined in Main Theorem 2 (here* $n = 6$, $d(M) = 28 = n^2 - 8$*).*

(xxi) $B^1 \times T_3$ *(here* $n = 4$, $d(M) = 13 = n^2 - 3$*).*

(xxii) $B^2 \times T_3$ *(here* $n = 5$, $d(M) = 18 = n^2 - 7$*).*

(xxiii) $B^1 \times T_4$ *(here* $n = 5$, $d(M) = 18 = n^2 - 7$*).*

(xxiv) *the domain* $\mathcal{D}$ *given by*

$$\mathcal{D} = \big\{(z, w) \in \mathbb{C}^3 \times \mathbb{C} : (\operatorname{Im} z_1 - |w|^2)^2 - (\operatorname{Im} z_2 - |w|^2)^2 - (\operatorname{Im} z_3) > 0, \\ \operatorname{Im} z_1 - |w|^2 > 0\big\}.$$

(here $n = 4$, $d(M) = 10 = n^2 - 6$*).*

Note that $\mathcal{D}$, the final domain listed in the above classification, is linearly equivalent to the well-known example of a non-symmetric bounded homogeneous domain in $\mathbb{C}^4$, discovered by I. Pyatetskii-Shapiro (see [30, pp. 26-28]).

1.3 Further directions

In discussing future directions, it is necessary to refer to some of the technicalities in the proofs of the main theorems in the thesis. We prove Main Theorem 1 in Chapter 3, and begin by noting that certain dimensional considerations allow us to rule out a large number of potential domains we might otherwise have had to consider. We prove a lemma to this end, and subsequently consider eight different Siegel domains of the second kind (corresponding to differing values of k and n in the definition of a Siegel domain of the second kind). In the proof of Main Theorem 1 we consider the following subcases:

1. $k = 2, n \geq 4$.
2. $k = 3, n = 4$.
3. $k = 3, n = 5$.

4. $k = 3, n = 6$.
5. $k = 3, n = 7$.
6. $k = 4, n = 4$.
7. $k = 4, n = 5$.
8. $k = 5, n = 5$.

In proving Main Theorem 2 in Chapter 4, we consider nine different Siegel domains of the second kind, namely the eight subcases listed above and the subcase $k = 6, n = 6$.

The immediate further direction is the next step in the classification, namely of homogeneous Kobayashi-hyperbolic manifolds with automorphism group dimension $d(M) = n^2 - 9$. This will require analysis of the eight subcases listed, the subcase $k = 6, n = 6$, as well as two new subcases $k = 3, n = 8$ and $k = 4, n = 6$. We believe the next step in the classification, the $d(M) = n^2 - 10$ case, is also viable. However, at some point the classification is expected to become too difficult due to the level of complexity becoming too great. We briefly outline the ways in which this is expected to occur.

As the value of $n - k$ increases, we are forced to consider vector-valued Hermitian forms on increasingly higher-dimensional complex spaces. In particular, each component of the Hermitian form will have $(n - k)^2$ terms. For instance, in the subcase $k = 3, n = 8$, each component of the Hermitian form will have 25 terms. At this point, while it may be feasible in principle to continue work on the classification, the amount of work required will in practice be too substantial.

Further to this, at the $d(M) = n^2 - 11$ step in the classification, it will be necessary to consider six-dimensional cones. According to the classification in [20], there are two indecomposable cones of dimension six, both of considerable complexity (see [20, p. 39]). It will also be necessary to consider the relevant decomposable cones, i.e., products of cones of lower dimension (for instance, the product of the Vinberg cone and the half-line). As a result, somewhere in the order of a dozen new cones and their associated Siegel domains will have to be considered. Moreover, if we were to continue the classification by looking at cases $d(M) = n^2 - 12$ and so on, by this point the complexity arising from the size of the Hermitian forms would be too great. Based on this information, it may be possible to conclude the next one to two steps in the classification, at which point further progress will be very difficult.

Chapter 2

Background Material

2.1 Kobayashi-hyperbolic manifolds

In this chapter we define the central object of this thesis, a Kobayashi-hyperbolic manifold. Such manifolds are of considerable interest in complex geometry, and have been actively researched since the introduction of the Kobayashi pseudodistance by Shoshichi Kobayashi in 1967. The Kobayashi pseudodistance is a pseudodistance (a concept to be defined later in this chapter) which is invariant under biholomorphic mappings, a property shared with some other well-known distances and metrics in complex analysis, such as the Bergman metric. Further, this pseudodistance has a certain *distance-decreasing* property under holomorphic mappings, which we discuss. The monograph [6] contains a comprehensive summary of the invariant distances we discuss here, and much of the following exposition is adapted from this work. We begin this chapter by introducing a few central concepts needed for the definition of the Kobayashi pseudodistance. In particular, the Poincaré distance, a distance invariant under biholomorphic mappings from the unit disc to itself, will be introduced. The definition of the Kobayashi pseudodistance relies crucially on the definition of the Poincaré distance. After the notion of a Kobayashi-hyperbolic manifold has been discussed, we will consider the group of holomorphic automorphisms of such a manifold. This group is in fact a *Lie group* when given the appropriate topology. Some important notation concerning this group and its dimension will also be presented. Lastly, we discuss two important theorems concerning the structure of Kobayashi-hyperbolic manifolds, which allow us to reduce the study of such manifolds to the study of Siegel domains of the second kind. We now begin this exposition with a well-known theorem from one-variable complex analysis known as the Schwarz-Pick lemma.

2.1.1 The Schwarz-Pick Lemma

The Schwarz lemma is an important result in complex analysis that illustrates the rigidity of holomorphic functions. For our purposes, it is a key ingredient in the proof of the Schwarz-Pick lemma, and so it is presented here. We omit the proof, which can be found in any standard textbook on complex analysis. In what follows, the symbol $\mathbb{D}$ will denote the unit disc B^1 in the complex plane.

Lemma 2.1.1. (Schwarz Lemma). If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ then

(i) $|f(z)| \le |z|$ for all $z \in \mathbb{D}$; and

(ii) $|f'(0)| \le 1$.

Further, if either $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation; i.e., there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$ for all $z \in \mathbb{D}$.

The Schwarz-Pick lemma is a generalisation of the above lemma that extends to functions f which do not fix the origin. Before proving the Schwarz-Pick lemma, we define the mappings ϕ_a which will be used in the proof. Recall that a *Möbius transformation* is a rational function of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. The mappings ϕ_a are particular Möbius transformations given as follows. For $a \in \mathbb{D}$, we have

$$\phi_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

It is easily shown that the ϕ_a are holomorphic mappings from $\mathbb{D}$ into $\mathbb{D}$. We now state and prove the Schwarz-Pick lemma.

Lemma 2.1.2. (Schwarz-Pick). Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then

(i)
$$\left| \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \right| \le \left| \frac{z - w}{1 - z\overline{w}} \right| \quad \text{for all } z, w \in \mathbb{D}, \text{ and}$$

(ii)
$$|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2} \quad \text{for all } z \in \mathbb{D}.$$

Proof. To show (i), we consider the function $g = \phi_{f(w)} \circ f \circ \phi_{-w}$. Then we have

$$g(z) = \phi_{f(w)}\left(f\left(\frac{z+w}{1+\overline{w}z}\right)\right).$$

Since f, ϕ are holomorphic mappings from $\mathbb{D}$ into $\mathbb{D}$, then so is g. Further, $g(0) = \phi_{f(w)}(f(w)) = 0$. Then the conditions of the Schwarz lemma are satisfied, and so we have $|g(\zeta)| \le |\zeta|$ for all $\zeta \in \mathbb{D}$, and it follows that $|g(\phi_w(z))| \le |\phi_w(z)|$ for all $z, w \in \mathbb{D}$. It is easily checked that the inverse of ϕ_w is ϕ_{-w}, and we therefore have

$$\begin{aligned} g \circ \phi_w &= \phi_{f(w)} \circ f \circ \phi_{-w} \circ \phi_w \\ &= \phi_{f(w)} \circ f. \end{aligned}$$

Then $|\phi_{f(w)}(f(z))| \le |\phi_w(z)|$ for all $z, w \in \mathbb{D}$. That is,

$$\left|\frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}}\right| \le \left|\frac{z-w}{1-z\overline{w}}\right| \text{ for all } z, w \in \mathbb{D}.$$

This shows (i).

We show (ii) using the second conclusion of the Schwarz lemma. We have $|g'(0)| \le 1$, and using the chain rule we see that

$$g'(z) = \phi'_{f(w)}(f(\phi_{-w}(z))) \cdot f'(\phi_{-w}(z)) \cdot \phi'_{-w}(z).$$

Then

$$\begin{aligned} g'(0) &= \phi'_{f(w)}(f(w)) \cdot f'(w) \cdot \phi'_{-w}(0) \\ &= \left(\frac{1}{1-|f(w)|^2}\right) \cdot f'(w) \cdot (1-|w|^2). \end{aligned}$$

So we have

$$\left|\frac{f'(w)(1-|w|^2)}{1-|f(w)|^2}\right| \le 1,$$

and finally

$$|f'(w)| \le \frac{1-|f(w)|^2}{1-|w|^2} \text{ for all } w \in \mathbb{D},$$

which completes the proof. □

If we have equality in either relation, that is, if

$$\left|\frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}}\right| = \left|\frac{z-w}{1-z\overline{w}}\right| \text{ for any pair of points } z \neq w$$

or

$$|f'(z)| = \frac{1 - |f(z)|^2}{1 - |z|^2} \text{ at any point } z,$$

then f is an automorphism of $\mathbb{D}$.

2.1.2 The Poincaré distance

Before presenting the Poincaré distance, we specify the precise meaning of the terms *distance* and *pseudodistance*, and note that these terms are distinct from the terms *metric* and *pseudometric*. The latter will be used to refer to a metric tensor, that is, a smooth family of bilinear forms on each tangent space. We have the following definition, where D denotes a domain in $\mathbb{C}^n$.

Definition 2.1.1. *The function* $\rho : D \times D \to [0, \infty)$ *is a* pseudodistance *on* D *if the following three axioms are satisfied:*

(i) $\rho(z, w) \geq 0$;

(ii) $\rho(z, w) = \rho(w, z)$; *and*

(iii) $\rho(z, w) \leq \rho(z, v) + \rho(v, w)$.

Further, ρ *is a* distance *if it satisfies a fourth axiom, namely*

(iv) $\rho(z, w) = 0 \iff z = w$.

Note that these four axioms are the standard axioms for what is usually called a metric. In the remainder of this thesis, we will use the term *distance* in the sense described above. We now consider the Poincaré distance $\rho_{\mathbb{D}}$ on the unit disc, which is a central object of interest in the study of invariant distances and metrics in complex analysis. We discuss it here because it provides an appropriate starting point for examining other distances and pseudodistances in complex analysis. In fact, this distance is used in the definition of the Kobayashi pseudodistance, which is introduced in the next section.

Definition 2.1.2. *The Poincaré distance function is given by*

$$\rho_{\mathbb{D}}(z, w) = \tanh^{-1} \left| \frac{z - w}{1 - z\overline{w}} \right|$$

for all $z, w \in \mathbb{D}$.

A holomorphic mapping $f : \mathbb{D} \to \mathbb{D}$ is *distance-decreasing* under the Poincaré distance as a consequence of the Schwarz-Pick lemma. That is,

$$\begin{aligned}\rho_{\mathbb{D}}(f(z), f(w)) &= \tanh^{-1}\left|\frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}}\right| \\ &\le \tanh^{-1}\left|\frac{z - w}{1 - z\overline{w}}\right| \\ &= \rho_{\mathbb{D}}(z, w)\end{aligned}$$

where the inequality follows from the Schwarz-Pick lemma and the fact that $\tanh^{-1}$ is strictly increasing on $(-1, 1)$. Further, if f is biholomorphic then it follows easily that the distance $\rho_{\mathbb{D}}$ is invariant, that is,

$$\rho_{\mathbb{D}}(f(z), f(w)) = \rho_{\mathbb{D}}(z, w)$$

for all $z, w \in \mathbb{D}$.

Distances that have such a distance-decreasing property for holomorphic maps are widely studied in complex geometry. We now remark on some additional properties of this distance. Since $\tanh^{-1}x = \frac{1}{2}\ln\frac{1+x}{1-x}$, the Poincaré distance $\rho_{\mathbb{D}}$ is very often seen expressed in the form

$$\rho_{\mathbb{D}}(z, w) = \frac{1}{2}\ln\frac{1 + \left|\frac{z-w}{1-z\overline{w}}\right|}{1 - \left|\frac{z-w}{1-z\overline{w}}\right|}.$$

The unit disc $\mathbb{D}$ equipped with the Poincaré distance $\rho_{\mathbb{D}}$ forms a complete metric space, and the topology generated by $\rho_{\mathbb{D}}$ coincides with the standard topology on $\mathbb{D}$. The Poincaré distance $\rho_{\mathbb{D}}$ is, of course, a distance, and showing that the above axioms (i), (ii) and (iv) hold is straightforward. Showing that the triangle inequality holds is somewhat more involved, and a proof can be found in [6, p. 37]. Note that $\rho_{\mathbb{D}}$ measures the distance between two points in the unit disc $\mathbb{D}$, and we now introduce a pseudodistance on pairs of points of an arbitrary complex manifold, called the Kobayashi pseudodistance.

2.1.3 The Kobayashi pseudodistance

We now introduce the Kobayashi pseudodistance. Let M denote a connected complex manifold (from this point, all complex manifolds are considered to be connected).

Definition 2.1.3. *For two points z, w in M we define a* chain *from z to w as follows: a collection of points $z = p_0, \ldots, p_k = w$, points $a_1, b_1, \ldots, a_k, b_k \in \mathbb{D}$*

and holomorphic mappings $f_1, \dots, f_k$ from $\mathbb{D}$ into M such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for $i = 1, \dots, k$.

The Kobayashi pseudodistance K_M is then defined:

$$K_M(z,w) = \inf \left\{ \sum_{i=1}^{k} \rho_{\mathbb{D}}(a_i, b_i) : \textit{all chains in } M \right\}.$$

It is straightforward to prove that K_M is a pseudodistance (see [6, p. 51] for such a proof). Note that K_M is not in general a true distance. For example, it can be shown (see [6, p. 52] for a precise argument) that the Kobayashi pseudodistance between two points in the complex plane is zero. That is, $K_{\mathbb{C}} = 0$. The Kobayashi pseudodistance in some sense captures the natural geometry of the Poincaré distance on the unit disc. For one thing, the Kobayashi pseudodistance and the Poincaré distance coincide on the unit disc, that is, $K_{\mathbb{D}} = \rho_{\mathbb{D}}$. Further, in common with the Poincaré distance, holomorphic maps are distance-decreasing under the Kobayashi pseudodistance. That is, for any holomorphic map f between two complex manifolds M_1 and M_2, we have

$$K_{M_2}(f(p), f(q)) \leq K_{M_1}(p,q)$$

for all $p, q \in M_1$. It follows that K_M is invariant for biholomorphic mappings. That is, for biholomorphic f we have

$$K_{M_2}(f(p), f(q)) = K_{M_1}(p,q)$$

for all $p, q \in M_1$. Note that there do exist distance functions invariant under biholomorphic mappings that are not necessarily distance-decreasing under arbitrary holomorphic mappings. For example, there are such distances associated to the Bergman and Kähler-Einstein Hermitian metrics (for a comprehensive summary of pseudodistances in complex analysis and geometry, see [17]). We may now define the notion of a Kobayashi-hyperbolic manifold.

Definition 2.1.4. *A complex manifold M for which the Kobayashi pseudodistance K_M is a distance is called* Kobayashi-hyperbolic *or simply* hyperbolic.

For every complex manifold M, K_M is continuous on $M \times M$ with respect to the standard topology and, if M is Kobayashi-hyperbolic, then K_M induces the standard topology [2]. Some examples of manifolds that are Kobayashi-hyperbolic include the unit disc $\mathbb{D}$ in the complex plane, the unit ball B^n in $\mathbb{C}^n$ and, in fact, any bounded domain in $\mathbb{C}^n$. One example (given in [17]) of an unbounded Kobayashi-hyperbolic domain with no bounded realisation is given by the domain

$$\Omega := \left\{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < \frac{1}{1 - |z|}\right\}.$$

2.1.4 The automorphism group is a Lie group

Recall that a holomorphic automorphism of a complex manifold M is a holomorphic bijection $f : M \to M$ whose inverse is also holomorphic. In contrast to the real case, if f is simply a holomorphic bijection then it follows that its inverse is also holomorphic. Clearly, for any complex manifold M, the set of holomorphic automorphisms equipped with the operation of composition forms a group, denoted $\mathrm{Aut}(M)$. In fact, $\mathrm{Aut}(M)$ is a topological group in the *compact-open topology*. The compact-open topology is a metrisable topology on the set of holomorphic transformations and, in the case of metric spaces, coincides with the topology of uniform convergence on compact sets. We denote the real dimension of the automorphism group $d(M)$. That is,

$$d(M) := \dim \mathrm{Aut}(M).$$

Since K_M is invariant for biholomorphic mappings, it is of course invariant for holomorphic automorphisms, and we say that K_M is $\mathrm{Aut}(M)$-*invariant*. This fact is used to prove a remarkable fact about the automorphism group. In 1935, H. Cartan proved that the automorphism group of any bounded domain in $\mathbb{C}^n$ is a real Lie group in the compact-open topology (see [22]). Over the ensuing decades, this result has been generalised in many directions. One such generalisation is of great importance for our purposes, and we present it now.

Theorem 2.1.3. *The automorphism group of any Kobayashi-hyperbolic manifold M is a real Lie group in the compact-open topology.*

We provide a brief sketch of the proof of this fact. It was shown by Kaup in [21] that the $\mathrm{Aut}(M)$-action on a complex manifold M is proper if M admits a continuous $\mathrm{Aut}(M)$-invariant distance function. Since any Kobayashi-hyperbolic manifold admits such a distance, namely the Kobayashi distance, it follows that that the $\mathrm{Aut}(M)$-action on M is proper. It then follows from the definition of a proper action (see e.g. [1]) that a topological group acting properly on a manifold is locally compact. Finally, due to a classical result of Bochner and Montgomery ([3], [4]), the group $\mathrm{Aut}(M)$ is a Lie group if it is locally compact.

2.1.5 An important structure theorem

We now present a theorem fundamental to this thesis which concerns the objects we intend to study and classify, namely *homogeneous* Kobayashi-hyperbolic manifolds. A complex manifold M is *homogeneous* if the action of $\mathrm{Aut}(M)$ on M is *transitive*. The group $\mathrm{Aut}(M)$ acts transitively on M if for every $p, q \in M$ there exists $f \in \mathrm{Aut}(M)$ such that $f(p) = q$.

Before presenting this important theorem, we describe a result in the same direction that was proved some years earlier. In 1963, Vinberg, Gindikin and Pyatetskii-Shapiro proved that every homogeneous bounded domain in $\mathbb{C}^n$ is biholomorphic to an *affinely homogeneous Siegel domain of the second kind*. We will define a Siegel domain of the second kind and the term 'affinely homogeneous' in Section 3 of this chapter. For the moment, it suffices to consider such a domain as a multidimensional analogue of the upper half-plane in $\mathbb{C}$. Indeed, as with the upper half-plane, every Siegel domain of the second kind is unbounded by definition but has a bounded realisation ([30, p. 23]). Siegel domains of the second kind were introduced by Pyatetskii-Shapiro in the late 1950s in relation to problems in the theory of automorphic functions (see [30]), though their utility in complex geometry was soon realised. The above result allows for the mapping of an arbitrary homogeneous domain to an object whose structure is more concrete, and can be more easily analysed. In fact, the automorphism group of an affinely homogeneous Siegel domain of the second kind was presented in [30, pp. 25-26]. In 1970, an explicit description of its Lie algebra was given by Kaup, Matsushima and Ochiai ([23]). The following seminal result, proved by Nakajima in 1985, provides further illustration of the value of such domains.

Theorem 2.1.4. *Every homogeneous Kobayashi-hyperbolic manifold is biholomorphic to an affinely homogeneous Siegel domain of the second kind.*

We provide a brief summary of the highly technical proof of this result, which can be found in [28]. Let G/K be a homogeneous Kobayashi-hyperbolic manifold, where G is a connected Lie group and K a closed subgroup. Then there exists a closed reductive subgroup S containing K such that G/S is homeomorphic to $\mathbb{R}^n$ and S/K is biholomorphic to a noncompact Hermitian symmetric space. Using this fact, it is shown that there exists a solvable subgroup of G which acts on G/K transitively, and we can therefore assume that G is solvable. It follows that S coincides with K and $G/K = G/S$ is biholomorphic to a homogeneous Siegel domain of the second kind. The proof of this theorem allowed for the affirmative resolution of a then-open problem posed by Kobayashi; namely, whether every homogeneous Kobayashi-hyperbolic manifold is biholomorphic to a homogeneous bounded domain in $\mathbb{C}^n$. The above theorem forms the basis of our work in this thesis. Beginning with an arbitrary n-dimensional Kobayashi-hyperbolic manifold, it allows us to consider the corresponding n-dimensional Siegel domain of the second kind, and analyse this object. We proceed in this endeavour by considering the automorphism group of a Siegel domain of the second kind and analysing its Lie algebra. This automorphism group is described in detail in Section 3 of this chapter.

Returning to the main problem of the thesis, we have the following facts about the dimension of the automorphism group of an n-dimensional hyperbolic complex manifold M. Recalling that $d(M) := \dim \mathrm{Aut}(M)$, it was mentioned in the introduction that $d(M) \leq n^2 + 2n$, with equality if and only if M is biholomorphic to the unit ball $B^n \subset \mathbb{C}^n$. In a series of articles (see the monograph [13] for a consolidation of the results) Isaev classified up to biholomorphim all Kobayashi-hyperbolic manifolds with automorphism group dimension $n^2 - 1 \leq d(M) \leq n^2 + 2n$. There are no such manifolds that satisfy $n^2 + 3 \leq d(M) < n^2 + 2n$. We remark that our classification problem is comparable to that for Riemannian manifolds. In [26], it was shown that for a smooth Riemannian manifold N, the group $\mathrm{Isom}(N)$ of all isometries of N is a Lie group in the compact-open topology. The dimension of the isometry group of a Riemannian manifold exhibits a similar lacunary behaviour (see [12] for a brief summary).

Buried inside the definition of a Siegel domain of the second kind is a mathematical structure known as an *open convex cone*. Consideration of an n-dimensional Siegel domain of the second kind requires us to consider every open convex cone in $\mathbb{R}^k$, where $1 \leq k \leq n$. We therefore require a classification of all real open convex cones up to linear equivalence, as well as the need to understand some of their properties. This will require some facts from the theory of convex cones and some facts from convex analysis, which we collect in the next section.

2.2 The theory of convex cones

2.2.1 Convex cones and their duals

The theory of convex cones is essential to this thesis. In particular, results concerning the linear automorphism group of an open convex cone. Much of the background material presented here is taken from [8, pp. 1–10]. We only include the results we need in the thesis. Throughout the following section, we expand on some of the proofs given in this text.

Let V be a finite dimensional real Euclidean space. Let $\langle \cdot, \cdot \rangle$ denote the associated inner product. A subset C of V is called a *cone* if $x \in C$ implies $\lambda x \in C$ for all $\lambda > 0$. A subset $U \subset V$ is said to be *convex* if $x, y \in U$ implies $\lambda x + (1-\lambda)y \in U$ for all $\lambda \in (0,1)$. It follows that $C \in V$ is a *convex cone* if and only if $x, y \in C$ implies $\lambda x + \mu y \in C$ for $\lambda, \mu > 0$. That is, C is a convex cone if it is closed with respect to taking linear combinations of its elements with positive coefficients.

In the following section, C will denote a non-empty convex cone. The closure of C will be denoted $\bar{C}$ or $\operatorname{Cl} C$, and the interior of C will be denoted C° or $\operatorname{Int} C$. The cone C is *proper* if $\bar{C} \cap (-\bar{C}) = \{0\}$. That is, C is proper when

$$x \in \bar{C} \text{ and } -x \in \bar{C} \implies x = 0,$$

which is equivalent to the condition that $\bar{C}$ does not contain any straight line through the origin. The interior C° is non-empty if and only if C contains a basis of V.

Throughout this thesis, we will only be concerned with the following two situations: that in which $V = \mathbb{R}^n$, where V is equipped with the standard inner product; and that in which $V = S^n(\mathbb{R})$, the vector space of $n \times n$ real symmetric matrices, where V is equipped with the trace inner product, which we describe later in this section. We have the following definition.

Definition 2.2.1. *The* closed dual cone *of any cone C in V is given by*

$$C^\# = \{y \in V : \langle x, y\rangle \geq 0 \text{ for all } x \in C\}.$$

It is easily shown that $C^\#$ is a closed convex cone. It is an important fact that for any closed convex cone, the dual of its dual is the cone itself. We require the concept of a *polar set* in the proof of this fact, and so we define this now, along with an important proposition.

Definition 2.2.2. *The polar set $S^\dagger$ of any set S in V is defined by*

$$S^\dagger = \{y \in V : \langle x, y\rangle \leq 1 \text{ for all } x \in S\}.$$

Proposition 2.2.1. *(Theorem of the second polar). For a closed convex set S containing 0, we have*

$$(S^\dagger)^\dagger = S.$$

Proof. Showing the containment $S \subset (S^\dagger)^\dagger$ is straightforward. For the containment $(S^\dagger)^\dagger \subset S$, we will show the contrapositive. That is, we show that if x_0 does not belong to S, then $\langle x_0, y\rangle > 1$ for some $y \in S^\dagger$. Let x_1 be a point of S from which the distance to x_0 is minimal. That is,

$$\|x - x_0\| \geq \|x_1 - x_0\| \text{ for all } x \in S.$$

For $0 \leq \lambda \leq 1$ and $x \in S$, the convexity of S implies $\lambda x + (1-\lambda)x_1 \in S$, and therefore

$$\begin{aligned} \|\lambda x + (1-\lambda)x_1 - x_0\|^2 &\geq \|x_1 - x_0\|^2 \\ \lambda^2 \|x - x_1\|^2 + 2\lambda\langle x - x_1, x_1 - x_0\rangle + \|x_1 - x_0\|^2 &\geq \|x_1 - x_0\|^2 \\ \lambda^2 \|x - x_1\|^2 + 2\lambda\langle x - x_1, x_1 - x_0\rangle &\geq 0. \end{aligned}$$

Since this holds for all λ such that $0 \leq \lambda \leq 1$, we see that

$$\langle x - x_1, x_1 - x_0\rangle \geq 0,$$

which implies

$$\langle x, x_0 - x_1\rangle \leq \langle x_1, x_0 - x_1\rangle \text{ for all } x \in S.$$

Substituting $x = 0$ into the above shows the right-hand side is non-negative. Noting that $\langle x_0 - x_1, x_0 - x_1\rangle > 0$, we can find $\mu > 0$ such that

$$\langle x_1, x_0 - x_1\rangle < \mu < \langle x_0, x_0 - x_1\rangle,$$

and finally we have the inequality

$$\langle x, x_0 - x_1\rangle \leq \langle x_1, x_0 - x_1\rangle < \mu < \langle x_0, x_0 - x_1\rangle \tag{2.2.1}$$

for all $x \in S$. So taking $y = \frac{1}{\mu}(x_0 - x_1)$, we see that $y \in S^\dagger$ since $\langle x, y\rangle = \frac{1}{\mu}\langle x, x_0 - x_1\rangle \leq 1$ for all $x \in S$ by (2.2.1). Also, we see that $\langle x_0, y\rangle = \frac{1}{\mu}\langle x_0, x_0 - x_1\rangle > 1$ by (2.2.1), which completes the proof. □

We now prove that for a closed convex cone, the dual of its dual is the cone itself.

Theorem 2.2.2. *For a non-empty closed convex cone C, we have*

$$(C^\#)^\# = C.$$

Proof. 2.2.2 For any cone C, by definition $x \in C$ implies $\lambda x \in C$ for all $\lambda > 0$. Therefore

$$C^\dagger = \{y \in V : \langle x, y\rangle \leq 0 \text{ for all } x \in C\},$$

and we see that $C^\# = -C^\dagger$. Since $(C^\#)^\# = -(-C^\dagger)^\dagger = (C^\dagger)^\dagger$, the statement is a special case of the above proposition. □

There are two further propositions we need before arriving at the results on the automorphism group.

Proposition 2.2.3. *For a non-empty closed convex cone C, we have*

$$\operatorname{Int}(C^{\#}) = \{y \in V : \langle x, y\rangle > 0 \text{ for all } x \in C \setminus \{0\}\}\,.$$

Furthermore, the following properties are equivalent:
(i) *C is proper, that is, $\bar{\Omega} \cap (-\bar{\Omega}) = \{0\}$.*
(ii) $\operatorname{Int}(C^{\#}) \neq \emptyset$.

Proof. Set

$$D = \{y \in V : \langle x, y\rangle > 0 \text{ for all } x \in C \setminus \{0\}\} \subset C^{\#}.$$

It is easily seen that

$$D = \{y \in V : \langle x, y\rangle > 0 \text{ for all } x \in C \cap S(V)\}\,,$$

where $S(V)$ denotes the unit sphere in V. Since $C \cap S(V)$ is compact, this shows that D is open. Therefore, $D \subset \operatorname{Int}(C^{\#})$. For the converse, suppose that $y \in \operatorname{Int}(C^{\#})$. Then, if $x \in C \setminus \{0\}$, we have $\langle x, z\rangle \geq 0$ for all z in some neighbourhood of y. That is, $\langle x, y\rangle + \langle x, u\rangle = \langle x, y + u\rangle \geq 0$ for all sufficiently small u. It follows that $\langle x, y\rangle > 0$, showing $\operatorname{Int}(C^{\#}) \subset D$. As for the equivalence of (i) and (ii), we omit the proof, which can be found in [8, p. 3]. $\square$

Definition 2.2.3. *The* open dual cone *of an open convex cone Ω in V is given by*

$$\Omega^* = \left\{y \in V : \langle x, y\rangle > 0 \text{ for all } x \in \bar{\Omega} \setminus \{0\}\right\}.$$

We mention an important result needed for the following proposition. It is a fact from convex analysis (see [31, Theorem 6.3]) that for any convex set $A \subset \mathbb{R}^n$ we have $A^\circ = (\bar{A})^\circ$. That is, the interior of A is the interior of its closure. We see therefore that an open convex cone is the interior of its closure.

Proposition 2.2.4. *For a proper open convex cone Ω, we have $(\Omega^*)^* = \Omega$.*

Proof. By Proposition 2.2.3, Ω^* is the interior of the closed dual cone of $\bar{\Omega}$, and is non-empty if and only if Ω is proper. In this case, we see that

$$\begin{aligned}(\Omega^*)^* &= \operatorname{Int}(\overline{\Omega^*})^{\#} \\ &= \operatorname{Int}(\bar{\Omega}^{\#})^{\#} \\ &= \operatorname{Int}\bar{\Omega} \\ &= \Omega,\end{aligned}$$

where the third equality follows from Proposition 2.2.2. $\square$

An open convex cone is called *self-dual* if $\Omega^* = \Omega$, and we see that all such cones are proper.

2.2.2 The automorphism group of an open convex cone

Definition 2.2.4. *The* linear automorphism group *of an open convex cone Ω is defined by*

$$G(\Omega) = \{g \in \mathrm{GL}(V) : g\Omega = \Omega\}.$$

We will hereafter refer to $G(\Omega)$ as simply the *automorphism group.* An element $g \in \mathrm{GL}(V)$ belongs to $G(\Omega)$ if and only if $g\overline{\Omega} = \overline{\Omega}$ (recall that $\Omega = (\bar{\Omega})^{\circ}$). It follows that $G(\Omega)$ is a closed subgroup of $\mathrm{GL}(V)$, and hence is a Lie group. We denote by $\mathfrak{g}(\Omega) \subset \mathfrak{gl}(V)$ its Lie algebra. The open convex cone Ω is said to be *homogeneous* if $G(\Omega)$ acts on Ω transitively. That is, if for all $x, y \in \Omega$ there exists $g \in G(\Omega)$ such that $gx = y$. Furthermore, Ω is said to be *symmetric* if it is homogeneous and self-dual.

Every open convex cone that occurs in n-dimensional Euclidean space with $n \leq 4$ is symmetric (see [20, pp. 38–41]). The simplest example of a homogeneous convex cone that is not self-dual is known as the *Vinberg cone* (see [38]). This cone is 5-dimensional, and is examined in the following chapter.

The following proposition illustrates the relationship between the automorphism group of a cone and the automorphism group of its dual. In the following discussion, g^* will denote the adjoint of an element $g \in \mathrm{GL}(V)$ with respect to the given inner product on V, and $G(\Omega)^*$ will denote the group consisting of the adjoints of each element in $G(\Omega)$.

Proposition 2.2.5. *For any proper open convex cone Ω we have $G(\Omega^*) = G(\Omega)^*$.*

Proof. Let $g \in G(\Omega)$ and $y \in \Omega^*$ (since Ω is proper, Ω^* is non-empty). Then for all non-zero $x \in \bar{\Omega}$, we have

$$\langle x, g^* y\rangle = \langle gx, y\rangle > 0.$$

This proves that $g^*\Omega^* \subset \Omega^*$. By the same argument, this also holds for g^{-1}, and we see that $(g^{-1})^*\Omega^* \subset \Omega^*$, which implies that $\Omega^* \subset g^*\Omega^*$, thus showing the other containment. So $g^*\Omega^* = \Omega^*$, showing that $g^* \in G(\Omega^*)$. It follows that $G(\Omega)^* \subset G(\Omega^*)$. Applying this to Ω^* instead of Ω, we see that $G(\Omega^*)^* \subset G(\Omega^{**}) = G(\Omega)$, and the proposition follows. $\square$

The proposition shows that if $\Omega^* = \Omega$, then $g \in G(\Omega)$ implies $g^* \in G(\Omega)$. That is, if the cone Ω is self-dual, for any element in its automorphism group, the adjoint of that element is also in the automorphism group.

2.2.3 Two examples of symmetric open convex cones

We now provide an example of a homogeneous convex cone which is self-dual, the n-dimensional *Lorentz cone* $\Lambda_n \subset \mathbb{R}^n$. For $n \geq 2$, Λ_n is given by

$$\Lambda_n = \left\{x \in \mathbb{R}^n : x_1^2 - x_2^2 - \cdots - x_n^2 > 0, x_1 > 0\right\}.$$

In the following proposition, $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$ given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

Proposition 2.2.6. *Λ_n is self-dual. That is, $\Lambda_n = \Lambda_n^*$.*

Proof. We show first the containment $\Lambda_n \subset \Lambda_n^*$. Since Λ_n^* is given by

$$\Lambda_n^* = \left\{y \in V : \langle x, y \rangle > 0 \text{ for all } x \in \overline{\Lambda}_n \setminus \{0\}\right\},$$

we will show that for $y \in \Lambda_n$, we have $\langle x, y \rangle > 0$ for all $x \in \overline{\Lambda}_n \setminus \{0\}$. Note that the Cauchy-Schwarz inequality implies

$$-\sqrt{x_2^2 + \cdots + x_n^2}\sqrt{y_2^2 + \cdots + y_n^2} \leq x_2 y_2 + \cdots + x_n y_n,$$

and so we have

$$\begin{aligned} \langle x, y \rangle &= x_1 y_1 + (x_2 y_2 + \cdots + x_n y_n) \\ &\geq x_1 y_1 - \sqrt{x_2^2 + \cdots + x_n^2}\sqrt{y_2^2 + \cdots + y_n^2} \\ &> 0, \end{aligned}$$

where the second inequality follows from the fact that $x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2}$ and $y_1 > \sqrt{y_2^2 + \cdots + y_n^2}$.

Now, we show the other containment $\Lambda_n^* \subset \Lambda_n$. Consider $y \in \Lambda_n^*$. The fact that $\langle x, y \rangle > 0$ for all $x \in \overline{\Lambda}_n \setminus \{0\}$ implies that we must have $y_1 > 0$. If $y_2 = \cdots = y_n = 0$, then clearly $y \in \Lambda_n$. Otherwise, we define x by

$$x_1 = \sqrt{y_2^2 + \cdots y_n^2}, \;\; x_2 = -y_2, \ldots, x_n = -y_n.$$

Then $x_1^2 - x_2^2 - \cdots - x_n^2 = 0$, but x is non-zero, so we see that $x \in \overline{\Lambda}_n \setminus \{0\}$. Then we have

$$\langle x, y \rangle = y_1 \sqrt{y_2^2 + \cdots y_n^2} - (y_2^2 + \cdots y_n^2) > 0,$$

showing that $y \in \Lambda_n$. □

The n-dimensional Lorentz cone Λ_n is shown to be homogeneous by considering the connected identity component of its automorphism group $G(\Lambda_n)^\circ = \mathbb{R}_+ \times \mathrm{SO}^\circ_{1,n-1}$. Here, $\mathrm{SO}_{1,n-1}$ denotes the group of $n \times n$ matrices g of determinant 1 such that $g^T I_{1,n-1} g = I_{1,n-1}$, where

$$I_{1,n-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix}.$$

The group $\mathrm{SO}^\circ_{1,n-1}$ is its identity component. It is easily shown that for any $x \in \Lambda_n$, $G(\Lambda_n)^\circ$ contains an element which maps e_1 (the first basis vector in $\mathbb{R}^n$) to x. Full details can be found in [8, pp. 7–8].

We now give one further example of a cone that is self-dual, namely the cone of *positive definite symmetric matrices*. This cone is of considerable importance in the following chapter, as we are able to realise a certain open convex cone in $\mathbb{R}^5$ as consisting of certain positive definite matrices. We will describe an inner product on the space of real symmetric matrices, and also discuss some identities which make the proof more streamlined. Recall that $S^n(\mathbb{R})$ denotes the space of real $n \times n$ symmetric matrices, and that $\dim S^n(\mathbb{R}) = \frac{1}{2}n(n+1)$. An inner product on this space is given by

$$\begin{aligned} \langle X, Y \rangle &= \mathrm{Tr}(XY) \\ &= \sum_{i=1}^{n} x_{ii} y_{ii} + \sum_{\substack{i=1 \\ i<j}}^{n} 2 x_{ij} y_{ij}. \end{aligned}$$

Recall that any real symmetric $n \times n$ matrix has an associated quadratic form on $\mathbb{R}^n$. The quadratic form Q associated to the symmetric matrix X is given by

$$\begin{aligned} Q(\xi) &= \xi^T X \xi \\ &= \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{12} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \dots & x_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \\ &= \sum_{i=1}^{n} x_{ii} \xi_i^2 + \sum_{1 \le i < j \le n} 2 x_{ij} \xi_i \xi_j. \end{aligned}$$

Recall that the symmetric matrix X is positive definite if for all $\xi \in \mathbb{R}^n \setminus \{0\}$ we have $Q(\xi) > 0$. The matrix X is positive semi-definite if for all $\xi \in \mathbb{R}^n$

we have $Q(\xi) \geq 0$. Note that

$$\xi\xi^T = \begin{bmatrix} \xi_1\xi_1 & \xi_1\xi_2 & \dots & \xi_1\xi_n \\ \xi_1\xi_2 & \xi_2\xi_2 & \dots & \xi_2\xi_n \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1\xi_n & \xi_2\xi_n & \dots & \xi_n\xi_n \end{bmatrix},$$

so we see that

$$\begin{aligned} \mathrm{Tr}(X \cdot \xi\xi^T) &= \sum_{i=1}^{n} x_{ii}\xi_i^2 + \sum_{1 \leq i < j \leq n} 2x_{ij}\xi_i\xi_j \\ &= Q(\xi). \end{aligned}$$

The identity $Q(\xi) = \mathrm{Tr}(X \cdot \xi\xi^T)$ will be useful in the upcoming proof. There is one more fact that will be useful in the proof. A positive semi-definite quadratic form $Q(\xi)$ can be expressed as a sum of squares of k linear forms, where $1 \leq k \leq n$, that is, in the form

$$Q(\xi) = \sum_{j=1}^{k} \left(\sum_{i=1}^{n} \alpha_{ij}\xi_i \right)^2 .$$

The symmetric matrix X associated to Q is then given by

$$X = \sum_{j=1}^{k} \alpha_j \alpha_j^T$$

where $\alpha_j = (\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{nj})$. Since $Q(\xi)$ is a sum of squares, it is clearly positive semi-definite, and so the associated symmetric matrix X is positive semi-definite. We now show that the cone of positive definite symmetric matrices is a symmetric cone.

Proposition 2.2.7. *The cone Ω of real $n \times n$ positive definite symmetric matrices is self-dual and homogeneous.*

Proof. Let $Y \in \Omega^*$. Then $\langle X, Y \rangle > 0$ for all $X \in \bar{\Omega} \backslash \{0\}$. For any $\xi \in \mathbb{R}^n \backslash \{0\}$, the matrix $X = \xi\xi^T$ belongs to $\bar{\Omega} \setminus \{0\}$. Therefore,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} y_{ij}\xi_i\xi_j = \langle X, Y \rangle > 0,$$

showing that Y is positive definite. So we see that $Y \in \Omega$.

Now we show the other containment. Consider $Y \in \Omega$. Any element $X \in \bar{\Omega} \setminus \{0\}$ can be written

$$X = \sum_{j=1}^{n} \alpha_j \alpha_j^T$$

for α_j of the form described above. Therefore if $Y \in \Omega$, then Y is positive definite, and so

$$\langle X, Y \rangle = \left\langle \sum_{j=1}^{n} \alpha_j \alpha_j^T, Y \right\rangle = \sum_{j=1}^{n} \langle \alpha_j \alpha_j^T, Y \rangle > 0,$$

and so $Y \in \Omega^*$.

Lastly, we show that Ω is homogeneous. For any element $A \in \mathrm{GL}(n, \mathbb{R})$ and any symmetric matrix X, we set

$$C_A(X) = AXA^T.$$

Then C_A is a linear transformation of $S^n(\mathbb{R})$ which belongs to the automorphism group $G(\Omega)$. To see this, note firstly that it is easily seen that AXA^T is symmetric for symmetric X. To see that C_A is positive definite, simply observe that for $y \in \mathbb{R}^n$ we have

$$y^T(AXA^T)y = (A^T y)^T X (A^T y) > 0,$$

by the positive-definiteness of X. A positive definite matrix X is invertible, and can be expressed

$$X = \alpha \alpha^T,$$

for an invertible $n \times n$ matrix α. So we have

$$x = C_\alpha(I_n),$$

where I_n is the $n \times n$ identity matrix, which shows that Ω is homogeneous. $\square$

We conclude this section by proving a useful estimate for the dimension of $\mathfrak{g}(\Omega)$, which will be used in the following chapter.

2.2.4 A useful estimate

Let Ω be a proper open convex cone in the real vector space V. For any point $x \in \Omega$ we define the *stabiliser* (or *isotropy*) subgroup of x in $G(\Omega)$ by

$$G(\Omega)_x = \{g \in G(\Omega) : gx = x\}.$$

We show in the following proposition that this subgroup is compact for all $x \in \Omega$. For fixed $a \in \Omega$, let

$$a - \Omega := \{a - y : y \in \Omega\}.$$

Note that for $g \in G(\Omega)_x$ and $y \in \Omega$ we have $g(x-y) = x - gy \in \Omega$, and so $gy \in x - \Omega$. We see then that $\Omega \cap (x - \Omega)$ is invariant under $g \in G(\Omega)_x$. In the following lemma, we use the fact that $\Omega \cap (x-\Omega)$ is bounded, a proof of which can be found in [8, pp. 3–5].

Lemma 2.2.8. *If Ω is a proper open convex cone, then for all $x \in \Omega$, $G(\Omega)_x$ is compact.*

Proof. The set $\Omega \cap (x - \Omega)$ is bounded, open and non-empty (it contains $\frac{x}{2}$ for example). Further, from the above remarks $G(\Omega)_x$ maps the set $\Omega \cap (x - \Omega)$ to itself. It follows easily (for example, by choosing a basis of V contained in $\Omega \cap (x - \Omega)$) that there exists a constant $C > 0$ such that

$$\frac{1}{C}||y|| \leq ||gy|| \leq C||y||$$

for all $g \in G(\Omega)_x$ and $y \in \Omega$. Hence $G(\Omega)_x$ is relatively compact in $\mathrm{GL}(V)$. Since $g\bar{\Omega} = \bar{\Omega}$ (see the remarks after Definition 2.2.4), we see that $G(\Omega)_x$ is compact. □

We now use the compactness of $G(\Omega)_x$ to prove the estimate.

Proposition 2.2.9. *If $\Omega \subset \mathbb{R}^k$ is a proper open convex cone, then*

$$\dim G(\Omega) \leq \frac{k^2}{2} - \frac{k}{2} + 1.$$

Proof. Since $G(\Omega)_x$ is compact we can assume, by changing variables in $\mathbb{R}^k$ if necessary, that it lies in the orthogonal group $O_k(\mathbb{R})$. The group $O_k(\mathbb{R})$ acts transitively on the sphere of radius $||x||$ in $\mathbb{R}^k$, and the stabiliser subgroup I_x of x under the $O_k(\mathbb{R})$-action is isomorphic to $O_{k-1}(\mathbb{R})$. Since $G(\Omega)_x \subset I_x$, we have

$$\dim G(\Omega)_x \leq \dim I_x = \frac{(k-2)(k-1)}{2} = \frac{k^2}{2} - \frac{3k}{2} + 1,$$

and by the orbit-stabiliser theorem we see that

$$\dim G(\Omega) \leq k + \frac{k^2}{2} - \frac{3k}{2} + 1,$$

from which the result follows. □

2.2.5 The classification of homogeneous open convex cones

Lastly, we provide the classification, up to linear equivalence, of homogeneous proper open convex cones in dimensions $k = 2, 3, 4, 5$ (see [20, pp. 38-41]).

$k = 2$: $\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\}$, where the algebra $\mathfrak{g}(\Omega_1)$ consists of all diagonal matrices, hence $\dim \mathfrak{g}(\Omega_1) = 2$,

$k = 3$: (i) $\Omega_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$, where the algebra $\mathfrak{g}(\Omega_2)$ consists of all diagonal matrices, hence $\dim \mathfrak{g}(\Omega_2) = 3$,

(ii) $\Omega_3 := \Lambda_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 - x_2^2 - x_3^2 > 0,\ x_1 > 0\}$, where one has $\mathfrak{g}(\Omega_3) = \mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{so}_{1,2}$, hence $\dim \mathfrak{g}(\Omega_3) = 4$; here for any Lie algebra $\mathfrak{h}$ we denote by $\mathfrak{c}(\mathfrak{h})$ its centre,

$k = 4$: (i) $\Omega_4 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_4 > 0\}$, where the algebra $\mathfrak{g}(\Omega_4)$ consists of all diagonal matrices, hence we have $\dim \mathfrak{g}(\Omega_4) = 4$,

(ii) $\Omega_5 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 - x_2^2 - x_3^2 > 0,\ x_1 > 0,\ x_4 > 0\}$, where the algebra $\mathfrak{g}(\Omega_5) = (\mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{so}_{1,2}) \oplus \mathbb{R}$ consists of block-diagonal matrices with blocks of sizes 3×3 and 1×1 corresponding to the two summands, hence $\dim \mathfrak{g}(\Omega_5) = 5$,

(iii) $\Omega_6 := \Lambda_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 - x_2^2 - x_3^2 - x_4^2 > 0,\ x_1 > 0\}$, where $\mathfrak{g}(\Omega_6) = \mathfrak{c}(\mathfrak{gl}_4(\mathbb{R})) \oplus \mathfrak{so}_{1,3}$, hence $\dim \mathfrak{g}(\Omega_6) = 7$.

$k = 5$: (i) $\Omega_7 := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 > 0,\ x_1 x_2 - x_4^2 > 0,\ x_1 x_2 x_3 - x_3 x_4^2 - x_2 x_5^2 > 0\big\}$, where $\dim \mathfrak{g}(\Omega_7) = 5$, which is proved in the following chapter.

(ii) $\Omega_8 := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 > 0,\ x_1 x_2 - x_4^2 > 0,\ x_1 x_3 - x_5^2 > 0\big\}$, where $\dim \mathfrak{g}(\Omega_8) = 5$, which is proved in the following chapter.

(iii) $\Omega_9 := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_4 > 0,\ x_5 > 0\big\}$, where the algebra $\mathfrak{g}(\Omega_9)$ consists of all diagonal matrices, hence we have $\dim \mathfrak{g}(\Omega_9) = 5$,

(iv) $\Omega_{10} := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1^2 - x_2^2 - x_3^2 > 0,\ x_1 > 0,\ x_4 > 0,\ x_5 > 0\big\}$,
where the algebra $\mathfrak{g}(\Omega_{10}) = (\mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{so}_{1,2}) \oplus \mathbb{R} \oplus \mathbb{R}$ consists of block-diagonal matrices with blocks of sizes 3×3, 1×1 and 1×1 corresponding to the three summands, hence $\dim \mathfrak{g}(\Omega_{10}) = 6$,

(v) $\Omega_{11} := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1^2 - x_2^2 - x_3^2 - x_4^2 > 0,\ x_1 > 0,\ x_5 > 0\big\}$,
where the algebra $\mathfrak{g}(\Omega_{11}) = (\mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{so}_{1,3}) \oplus \mathbb{R}$ consists of block-diagonal matrices with blocks of sizes 4×4 and 1×1 corresponding to the two summands, hence $\dim \mathfrak{g}(\Omega_{11}) = 8$,

(vi) $\Omega_{12} := \Lambda_5 = \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 > 0,\ x_1 > 0\big\}$,
where $\mathfrak{g}(\Omega_{12}) = \mathfrak{c}(\mathfrak{gl}_4(\mathbb{R})) \oplus \mathfrak{so}_{1,4}$, hence $\dim \mathfrak{g}(\Omega_6) = 11$.

Note that we use the nonstandard notation $\mathfrak{c}(\mathfrak{gl}_n(\mathbb{R}))$ to denote scalar multiples of the $n \times n$ identity matrix. We do so in order to conform to the presentation given in Isaev's articles [15], [14] and [16]. As seen in the list above, we provide the dimension of the Lie algebra of the automorphism group of each cone. We will illustrate with some examples the method by which this Lie algebra is determined. Note that in the case of the first two 5-dimensional cones listed, Ω_7 and Ω_8, computation of these Lie algebra dimensions is considerably more complicated, and is presented in detail in the final part of the following chapter. Note also that the cones Ω_5, Ω_{10} and Ω_{11} are *reducible*. That is, they can be decomposed into a direct product of two or more of the other cones on the list, each of which is *irreducible* (see [33, p. 313] for details).

Let us describe how to compute the automorphism group of each cone. To begin with, the automorphism group of the positive orthant in $\mathbb{R}^n$ is given by the group of monomial matrices (also known as generalised permutation matrices) with non-negative entries (see [10, p. 186]). For example, the automorphism group of the cone Ω_1 consists of matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ where $a, d > 0$, and $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ where $b, c > 0$. The identity component of the automorphism group clearly consists of matrices of the first type. It is easily seen that the Lie algebra of such matrices consists of all diagonal matrices. The explicit forms of the automorphism groups of the cones Ω_2, Ω_4 and Ω_9 can be determined in a similar manner.

We now describe the automorphism group of the n-dimensional Lorentz cone Λ_n. As in the previous section, let $\mathrm{SO}_{1,n-1}$ denote the group of $n \times n$

matrices g of determinant 1 such that $g^T I_{1,n-1} g = I_{1,n-1}$, where

$$I_{1,n-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix}.$$

Then $\mathrm{SO}_{1,n-1}$ is seen to map the n-dimensional Lorentz cone to itself. The group $\mathrm{SO}^\circ_{1,n-1}$ is its identity component. It is easily seen that scalar matrices αI for $\alpha > 0$ also map Λ_n to itself, and so we see that the group $\mathrm{CO}^\circ_{1,n-1} = \mathbb{R}_+ \times \mathrm{SO}^\circ_{1,n-1}$ maps Λ_n to itself. This group is in fact the automorphism group of Λ_n (see [8, p. 7]). Now simply observe that $\mathfrak{g}(\Lambda_n) = \mathfrak{co}_{1,n-1} = \mathfrak{c}(\mathfrak{gl}_n(\mathbb{R})) \oplus \mathfrak{so}_{1,n-1}$, and we have the Lie algebra of the automorphism group of the n-dimensional Lorentz cone. With the exception of Ω_7 and Ω_8, each cone is either a positive orthant in some dimension, a Lorentz cone, or a product of such cones, and so a similar process is followed to compute their automorphism group.

2.3 Siegel domains of the second kind

A *Siegel domain* is a multidimensional analogue of the upper half-plane $\operatorname{Im} z > 0$. We will be interested in *Siegel domains of the second kind*, certain unbounded domains in $\mathbb{C}^n$ which were introduced by I. Pyatetskii-Shapiro in the late 1950s in relation to problems in the theory of automorphic functions. It was discovered some years after their introduction that such domains are of considerable utility in complex geometry. As mentioned in the introduction, it was shown by Vinberg, Gindikin and Pyatetskii-Shapiro in [40] that every homogeneous bounded domain in $\mathbb{C}^n$ is biholomorphic to an affinely homogeneous Siegel domain of the second kind. In 1985, this result was improved upon by Nakajima, whose result (stated as Theorem 2.1.4 in this thesis) states that every homogeneous Kobayashi-hyperbolic manifold is biholomorphic to an affinely homogeneous Siegel domain of the second kind. It is this result that we will utilise extensively.

In this section, we will define a Siegel domain of the second kind, identify its group of holomorphic affine automorphisms, and discuss the Lie algebra of this group. We begin with the definition of a Siegel domain of the first kind.

Definition 2.3.1. *A* Siegel domain of the first kind *is a domain of the form*

$$S(\Omega) := \{z \in \mathbb{C}^n : \operatorname{Im} z \in \Omega\} = \mathbb{R}^n + i\Omega,$$

where $\Omega \subset \mathbb{R}^n$ is an open convex cone.

Note that a Siegel domain of the first kind is a special case of a tube domain. We now discuss some preliminaries before defining a Siegel domain of the second kind.

A *Hermitian form* on $\mathbb{C}^m$ is a map $H : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^k$ such that for all $u, v \in \mathbb{C}^m$ we have

(i) $H(\lambda u_1 + \mu u_2, v) = \overline{\lambda} H(u_1, v) + \overline{\mu} H(u_2, v)$ where $\lambda, \mu \in \mathbb{C}$;
(ii) $H(u, v) = \overline{H(v, u)}$; and
(iii) $H(u, u) = 0$ implies $u = 0$.

Further, for an open convex cone $\Omega \subset \mathbb{R}^k$, the form H is called Ω-*Hermitian* if $H(u, u) \in \bar{\Omega} \backslash \{0\}$ for all non-zero $u \in \mathbb{C}^m$. Note that we follow the convention of linearity in the second variable, and conjugate linearity in the first variable.

For H that is Ω-Hermitian, if Ω is proper (that is, does not contain an entire line), then Ω is contained in a half-space, and we can always find a vector $c \in \mathbb{R}^k$ such that $\langle c, H\rangle > 0$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^k$. In other words, there exists a positive definite linear combination of the components of H.

Definition 2.3.2. *A* Siegel domain of the second kind *in $\mathbb{C}^n$ is a domain of the form*

$$S(\Omega, H) := \left\{(z, w) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : \operatorname{Im} z - H(w, w) \in \Omega\right\}$$

for some $1 \leq k \leq n$, some open convex cone $\Omega \subset \mathbb{R}^k$, and some Ω-Hermitian form H on $\mathbb{C}^{n-k}$.

Note that for $k = n$ we have $H = 0$, so in this case $S(\Omega, H)$ is the Siegel domain of the first kind

$$\{z \in \mathbb{C}^n : \operatorname{Im} z \in \Omega\}.$$

At the other extreme, when $k = 1$, the domain $S(\Omega, H)$ is linearly equivalent to

$$\left\{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : \operatorname{Im} z - ||w||^2 > 0\right\},$$

which is an unbounded realization of the unit ball B^n (see [32, p. 31]). We briefly illustrate why this is true. For $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$ and $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that $a_1 \neq 1$, the n-dimensional Cayley transform is the map φ that sends a to (z, w) given by

$$(z, w) = i\frac{e_1 + a}{1 - a_1},$$

where $e_1 = (1, 0, \ldots, 0)$. Elementary computations show that

$$\operatorname{Im} z - ||w||^2 = \frac{1 - |a|^2}{|1 - a_1|^2},$$

and

$$a = \frac{2}{i + z}(z, w) - e_1.$$

Hence, φ is a biholomorphic map of the unit ball onto the domain in question. In fact, the above result holds in more generality. If

$$\Omega = \left\{x \in \mathbb{R}^k : x_1 > 0, \ldots, x_k > 0\right\}$$

and $S(\Omega, H)$ is homogeneous, then $S(\Omega, H)$ is linearly equivalent to a product of k unbounded realisations of unit balls as above, and hence biholomorphic to a product of unit balls (see [20, Theorems A, B, C] and [27]).

It is well known that any homogeneous Siegel domain of the second kind is biholomorphic to a homogeneous bounded domain (see [28, p. 1], [30, pp. 23–24]). Further to this, every bounded domain in $\mathbb{C}^n$ is Kobayashi-hyperbolic. Hence, any homogeneous Siegel domain of the second kind is Kobayashi-hyperbolic. We now describe the group of holomorphic affine automorphisms of a Siegel domain of the second kind (see [30, pp. 25–26]).

Theorem 2.3.1. *Any holomorphic affine automorphism of $S(\Omega, H)$ has the form*

$$(z, w) \mapsto (Az + a + 2iH(b, Bw) + iH(b, b), Bw + b),$$

with $a \in \mathbb{R}^k$, $b \in \mathbb{C}^{n-k}$, $A \in G(\Omega)$, $B \in \mathrm{GL}_{n-k}(\mathbb{C})$, where

$$AH(w, w') = H(Bw, Bw') \tag{2.3.1}$$

for all $w, w' \in \mathbb{C}^{n-k}$.

A domain $S(\Omega, H)$ is called *affinely homogeneous* if the above group, which we denote $\mathrm{Aff}(S(\Omega, H))$, acts on $S(\Omega, H)$ transitively. Denote by $G(\Omega, H)$ the subgroup of $G(\Omega)$ that consists of all transformations $A \in G(\Omega)$ as in Theorem 2.3.1, that is, of all elements $A \in G(\Omega)$ for which there exists $B \in \mathrm{GL}_{n-k}(\mathbb{C})$ such that (2.3.1) holds. By [7, Lemma 1.1], the subgroup $G(\Omega, H)$ is closed in $G(\Omega)$. It can be deduced from Theorem 2.3.1 that if $S(\Omega, H)$ is affinely homogeneous, the action of $G(\Omega, H)$ is transitive on Ω (see, e.g., [23, proof of Theorem 8]), so the cone Ω is homogeneous. Note also that since Ω is connected, the action of the identity component $G(\Omega, H)^\circ$ is also transitive on Ω. Conversely, if $G(\Omega, H)$ acts on Ω transitively, the domain

$S(\Omega, H)$ is affinely homogeneous. The subgroup $G(\Omega, H)$ plays an important role in much of our work in the following two chapters.

After realising a homogeneous Kobayashi-hyperbolic manifold as a homogeneous Siegel domain of the second kind using Theorem 2.1.4, we then consider its automorphism group $\mathrm{Aut}(S(\Omega, H))$, and proceed by analysing the Lie algebra of this group, which we denote $\mathfrak{g}(S(\Omega, H))$. We therefore rely heavily on an explicit description of this Lie algebra (see [23, Theorems 4 and 5]), which is rather involved, and we present this now. This algebra is isomorphic to the (real) Lie algebra of complete holomorphic vector fields on $S(\Omega, H)$ (see [34, pp. 209–210]).

Theorem 2.3.2. *The algebra $\mathfrak{g} = \mathfrak{g}(S(\Omega, H))$ admits a grading*

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

with $\mathfrak{g}_\nu$ being the eigenspace with eigenvalue ν of $\mathrm{ad}\,\partial$, where

$$\partial := z \cdot \frac{\partial}{\partial z} + \frac{1}{2} w \cdot \frac{\partial}{\partial w}.$$

Here

$$\mathfrak{g}_{-1} = \left\{ a \cdot \frac{\partial}{\partial z} : a \in \mathbb{R}^k \right\}, \qquad \dim \mathfrak{g}_{-1} = k,$$

$$\mathfrak{g}_{-1/2} = \left\{ 2iH(b, w) \cdot \frac{\partial}{\partial z} + b \cdot \frac{\partial}{\partial w} : b \in \mathbb{C}^{n-k} \right\}, \quad \dim \mathfrak{g}_{-1/2} = 2(n-k),$$

and $\mathfrak{g}_0$ consists of all vector fields of the form

$$(Az) \cdot \frac{\partial}{\partial z} + (Bw) \cdot \frac{\partial}{\partial w}, \tag{2.3.2}$$

with $A \in \mathfrak{g}(\Omega)$, $B \in \mathfrak{gl}_{n-k}(\mathbb{C})$ and

$$AH(w, w') = H(Bw, w') + H(w, Bw') \tag{2.3.3}$$

for all $w, w' \in \mathbb{C}^{n-k}$. Furthermore, one has

$$\dim \mathfrak{g}_{1/2} \leq 2(n-k), \qquad \dim \mathfrak{g}_1 \leq k. \tag{2.3.4}$$

The subspace $\mathfrak{g}_0$ is in fact a subalgebra of $\mathfrak{g}$ (this is proved in Proposition B.1.1 in Appendix B). The matrices A that appear in (2.3.2) clearly form the Lie algebra of $G(\Omega, H)$ and $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0$ is isomorphic to the Lie algebra of the group $\mathrm{Aff}(S(\Omega, H))$. This can be seen by comparing conditions (2.3.1) and (2.3.3), and noting the following result (from [11, Lemma 4.2.2]). Let V and W be finite-dimensional vector spaces and $\beta : V \times V \to W$ be a bilinear map. For x an endomorphism of V and y and endomorphism of W, the following are equivalent:

(i) $e^{ty}\beta(v,v') = \beta(e^{tx}v, e^{tx}v')$ for all $t \in \mathbb{R}$ and all $v, v' \in V$;
(ii) $y\beta(v,v') = \beta(xv,v') + \beta(v,xv')$ for all $v,v' \in V$.

Following [34], for a pair of matrices A, B satisfying (2.3.3) we say that B is *associated to* A (with respect to H). Let $\mathcal{L}$ be the (real) subspace of $\mathfrak{gl}_{n-k}(\mathbb{C})$ of all matrices associated to the zero matrix in $\mathfrak{g}(\Omega)$, i.e., matrices skew-Hermitian with respect to each component of H. That is,

$$\mathcal{L} = \{B \in \mathfrak{gl}_{n-k}(\mathbb{C}) : H(B\cdot,\cdot) + H(\cdot, B\cdot) = 0\}.$$

We present a short proposition concerning the dimension of $\mathcal{L}$.

Proposition 2.3.3. *The map $f : \mathfrak{g}_0 \to \mathfrak{g}(G(\Omega,H))$ given by $f(A,B) = -A$ induces a Lie algebra isomorphism from $\mathfrak{g}_0/\mathcal{L}$ to $\mathfrak{g}(G(\Omega,H))$. Hence,* $\dim \mathfrak{g}_0 = \dim \mathcal{L} + \dim G(\Omega,H)$.

Proof. Consider the map $f : \mathfrak{g}_0 \to \mathfrak{g}(G(\Omega,H))$ given by $f(A,B) = -A$. As mentioned above, the matrices A that appear in (2.3.2) form the Lie algebra of $G(\Omega,H)$ and so f is surjective. Further, the kernel of this map is $\mathcal{L}$. It is shown in Proposition B.1.2 in Appendix B that f is a Lie algebra homomorphism, and it follows that $\mathcal{L}$ is an ideal in $\mathfrak{g}_0$. So we see that f induces a Lie algebra isomorphism from $\mathfrak{g}_0/\mathcal{L}$ to $\mathfrak{g}(G(\Omega,H))$. □

Setting $s := \dim \mathcal{L}$, we have

$$\dim \mathfrak{g}_0 = s + \dim G(\Omega,H) \le s + \dim \mathfrak{g}(\Omega). \tag{2.3.5}$$

By Theorem 2.3.2 and inequality (2.3.5) one obtains

$$d(S(\Omega,H)) \le k + 2(n-k) + s + \dim \mathfrak{g}(\Omega) + \dim \mathfrak{g}_{1/2} + \dim \mathfrak{g}_1, \tag{2.3.6}$$

which, combined with (2.3.4) leads to

$$d(S(\Omega,H)) \le 2k + 4(n-k) + s + \dim \mathfrak{g}(\Omega). \tag{2.3.7}$$

The subspace $\mathcal{L}$ lies in the Lie algebra of matrices skew-Hermitian with respect to any linear combination $\mathbf{H}$ of the components of the Hermitian form H. Since $\mathbf{H}$ can be chosen to be positive definite,

$$s \le (n-k)^2. \tag{2.3.8}$$

By (2.3.8), inequality (2.3.7) yields

$$d(S(\Omega,H)) \le 2k + 4(n-k) + (n-k)^2 + \dim \mathfrak{g}(\Omega). \tag{2.3.9}$$

Combining (2.3.9) with inequality (2.2.9) we deduce the following useful upper bound:

$$d(S(\Omega, H)) \leq \frac{3k^2}{2} - k\left(2n + \frac{5}{2}\right) + n^2 + 4n + 1. \tag{2.3.10}$$

Explicit descriptions of $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ were first given in [34]. By [34, Chapter V, Proposition 2.1] the component $\mathfrak{g}_{1/2}$ of the Lie algebra $\mathfrak{g} = \mathfrak{g}(S(\Omega, H))$ is described as follows:

Theorem 2.3.4. *The subspace $\mathfrak{g}_{1/2}$ consists of all vector fields of the form*

$$2iH(\Phi(\bar{z}), w) \cdot \frac{\partial}{\partial z} + (\Phi(z) + c(w, w)) \cdot \frac{\partial}{\partial w},$$

where $\Phi : \mathbb{C}^k \to \mathbb{C}^{n-k}$ is a $\mathbb{C}$-linear map such that for every $\mathbf{w} \in \mathbb{C}^{n-k}$ one has

$$\Phi_{\mathbf{w}} := \left[x \mapsto \operatorname{Im} H(\mathbf{w}, \Phi(x)),\ x \in \mathbb{R}^k\right] \in \mathfrak{g}(\Omega), \tag{2.3.11}$$

and $c : \mathbb{C}^{n-k} \times \mathbb{C}^{n-k} \to \mathbb{C}^{n-k}$ is a symmetric $\mathbb{C}$-bilinear form on $\mathbb{C}^{n-k}$ with values in $\mathbb{C}^{n-k}$ satisfying the condition

$$H(w, c(w', w')) = 2iH(\Phi(H(w', w)), w') \tag{2.3.12}$$

for all $w, w' \in \mathbb{C}^{n-k}$.

Further, by [34, Chapter V, Proposition 2.2], the component $\mathfrak{g}_1$ of $\mathfrak{g} = \mathfrak{g}(S(\Omega, H))$ admits the following description.

Theorem 2.3.5. *The subspace $\mathfrak{g}_1$ consists of all vector fields of the form*

$$a(z, z) \cdot \frac{\partial}{\partial z} + b(z, w) \cdot \frac{\partial}{\partial w},$$

where $a : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is a symmetric $\mathbb{R}$-bilinear form on $\mathbb{R}^k$ with values in $\mathbb{R}^k$ (which we extend to a symmetric $\mathbb{C}$-bilinear form on $\mathbb{C}^k$ with values in $\mathbb{C}^k$) such that for every $\mathbf{x} \in \mathbb{R}^k$ one has

$$A_{\mathbf{x}} := \left[x \mapsto a(\mathbf{x}, x),\ x \in \mathbb{R}^k\right] \in \mathfrak{g}(\Omega), \tag{2.3.13}$$

and $b : \mathbb{C}^k \times \mathbb{C}^{n-k} \to \mathbb{C}^{n-k}$ is a $\mathbb{C}$-bilinear map such that, if for $\mathbf{x} \in \mathbb{R}^k$ one sets

$$B_{\mathbf{x}} := \left[w \mapsto \frac{1}{2} b(\mathbf{x}, w),\ w \in \mathbb{C}^{n-k}\right], \tag{2.3.14}$$

the following conditions are satisfied:

(i) *$B_{\mathbf{x}}$ is associated to $A_{\mathbf{x}}$ and $\operatorname{Im} \operatorname{Tr} B_{\mathbf{x}} = 0$ for all $\mathbf{x} \in \mathbb{R}^k$,*

(ii) *for every pair $\mathbf{w}, \mathbf{w}' \in \mathbb{C}^{n-k}$ one has*

$$B_{\mathbf{w},\mathbf{w}'} := \left[x \mapsto \operatorname{Im} H(\mathbf{w}', b(x, \mathbf{w})),\ x \in \mathbb{R}^k\right] \in \mathfrak{g}(\Omega),\ \textit{and}$$

(iii) *$H(w, b(H(w', w''), w'')) = H(b(H(w'', w), w'), w'')$ for all $w, w', w'' \in \mathbb{C}^{n-k}$.*

Chapter 3

The $d(M) = n^2 - 7$ case

We begin this chapter by stating the main theorem describing the contribution to the classification of homogeneous hyperbolic n-dimensional manifolds with automorphism group dimension $n^2 - 7$. We see that, up to biholomorphism, there are two homogeneous Kobayashi-hyperbolic manifolds with the given automorphism group dimension.

Main Theorem 1. *Let M be a homogeneous n-dimensional Kobayashi-hyperbolic manifold with $d(M) = n^2 - 7$. Then one of the following holds:*

(i) *$n = 5$ and M is biholomorphic to $B^2 \times T_3$, where T_3 is the tube domain*

$$T_3 = \big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : (\operatorname{Im} z_1)^2 - (\operatorname{Im} z_2)^2 - (\operatorname{Im} z_3)^2 > 0, \operatorname{Im} z_1 > 0\big\}.$$

(ii) *$n = 5$ and M is biholomorphic to $B^1 \times T_4$, where T_4 is the tube domain*

$$T_4 = \big\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (\operatorname{Im} z_1)^2 - (\operatorname{Im} z_2)^2 - (\operatorname{Im} z_3)^2 - (\operatorname{Im} z_4)^2 > 0, \operatorname{Im} z_1 > 0\big\}.$$

In this chapter, we prove the above theorem. Let M be a homogeneous Kobayashi-hyperbolic manifold of dimension n. By Theorem 2.1.4, the manifold M is biholomorphic to an affinely homogeneous Siegel domain of the second kind $S(\Omega, H)$. Recall that

$$S(\Omega, H) := \big\{(z, w) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : \operatorname{Im} z - H(w, w) \in \Omega\big\},$$

for $1 \leq k \leq n$, where $\Omega \subset \mathbb{R}^k$ is an open convex cone and H is an Ω-Hermitian form on $\mathbb{C}^{n-k}$. Since all homogeneous Kobayashi-hyperbolic manifolds of dimensions 2 and 3 have been classified (see [15, Theorem 2.6]), we take $n \geq 4$. Further, we recall from the remarks after Definition 2.3.2 in the previous chapter that if $k = 1$ then $S(\Omega, H)$ is biholomorphic to B^n, so we assume that $k \geq 2$.

3.1 A useful lemma

We can use the following lemma to rule out a large number of remaining possibilities.

Lemma 3.1.1. *For $n \geq 6$ and $k \geq 4$, we cannot have $d(S(\Omega, H)) = n^2 - 7$. Also, for $n \geq 8$ and $k = 3$, we cannot have $d(S(\Omega, H)) = n^2 - 7$.*

Proof. To prove the lemma, we will show that for $n \geq 6, k \geq 4$, as well as for $n \geq 8, k = 3$, the right-hand side of inequality (2.3.10) given by

$$d(S(\Omega, H)) \leq \frac{3k^2}{2} - \left(2n + \frac{5}{2}\right) k + n^2 + 4n + 1$$

is strictly less that $n^2 - 7$. That is, for these k, n the following holds:

$$\frac{3k^2}{2} - \left(2n + \frac{5}{2}\right) k + 4n + 8 < 0.$$

To see this, consider the quadratic function

$$\varphi(t) := \frac{3t^2}{2} - \left(2n + \frac{5}{2}\right) t + 4n + 8.$$

The discriminant of φ is given by

$$\mathcal{D} := 4n^2 - 14n - \frac{167}{4},$$

which is positive for $n \geq 6$. The zeroes of φ are given by

$$t_1 := \frac{2n + \frac{5}{2} - \sqrt{\mathcal{D}}}{3},$$

$$t_2 := \frac{2n + \frac{5}{2} + \sqrt{\mathcal{D}}}{3}.$$

To prove the lemma, it suffices to show that: (i) $t_2 > n$ for $n \geq 6$, (ii) $t_1 < 4$ for $n \geq 6$, and (iii) $t_1 < 3$ for $n \geq 8$. Beginning with the inequality $t_2 > n$, we have

$$n - \frac{5}{2} < \sqrt{\mathcal{D}},$$

or, equivalently, that

$$n^2 - 3n - 16 > 0,$$

which clearly holds for $n \geq 6$. Now considering $t_1 < 4$, we see that

$$2n - \frac{19}{2} < \sqrt{\mathcal{D}},$$

or, equivalently, that

$$n > \frac{11}{2},$$

which holds for $n \geq 6$. Lastly, the inequality $t_1 < 3$ implies that

$$2n - \frac{13}{2} < \sqrt{\mathcal{D}},$$

or, equivalently, that

$$n > 7,$$

which holds for $n \geq 8$. This completes the proof. □

3.2 Beginning of the proof of the main theorem

By the above lemma, we prove the theorem by considering the following eight cases:

1. $k = 2, n \geq 4$
2. $k = 3, n = 4$
3. $k = 3, n = 5$
4. $k = 3, n = 6$
5. $k = 3, n = 7$
6. $k = 4, n = 4$
7. $k = 4, n = 5$
8. $k = 5, n = 5$.

We now begin by considering each case.

Case 1. Suppose that $k = 2, n \geq 4$. Since $H : \mathbb{C}^{n-k} \times \mathbb{C}^{n-k} \to \mathbb{C}^k$, we have that $H = (H_1, H_2)$ is a pair of Hermitian forms on $\mathbb{C}^{n-2}$. After a linear change of z-variables, we may assume that H_1 is positive-definite. Since this is the case, by applying a linear change of w-variables, we can simultaneously diagonalise H_1 and H_2 (see Appendix B) as

$$H_1(w, w) = ||w||^2, \quad H_2(w, w) = \sum_{j=1}^{n-2} \lambda_j |w_j|^2.$$

If all the eigenvalues of H_2 are equal, $S(\Omega, H)$ is linearly equivalent either to

$$D_1 := \left\{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^{n-2} : \text{Im } z_1 - ||w||^2 > 0, \text{ Im } z_2 > 0\right\}$$

if $\lambda_j = 0$, or to

$$D_2 := \left\{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^{n-2} : \text{Im } z_1 - ||w||^2 > 0, \text{ Im } z_2 - ||w||^2 > 0\right\}$$

if $\lambda_j \neq 0$. The domain D_1 is biholomorphic to $B^{n-1} \times B^1$, hence $d(D_1) = n^2+2 > n^2-7$, which shows that $S(\Omega, H)$ cannot be equivalent to D_1. As for the domain D_2, consider the group $G(\Omega_1, (||w||^2, ||w||^2))$. A straightforward computation shows that $G(\Omega_1, (||w||^2, ||w||^2))$ consists of matrices $\begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}$ where $\rho > 0$, and $\begin{bmatrix} 0 & \eta \\ \eta & 0 \end{bmatrix}$ where $\eta > 0$. It is therefore seen that the action of $G(\Omega_1, (||w||^2, ||w||^2))$ is not transitive on Ω_1. Therefore, $S(\Omega, H)$ cannot be equivalent to D_2 either. It follows that H_2 has at least one pair of distinct eigenvalues.

Next, since $\dim \mathfrak{g}(\Omega) = 2$, inequality (2.3.7) yields

$$s \geq n^2 - 4n - 5. \tag{3.2.1}$$

On the other hand, by inequality (2.3.8),

$$s \leq n^2 - 4n + 4. \tag{3.2.2}$$

The exact value of s is given by

$$s = n^2 - 4n + 4 - 2m,$$

where $m \geq 1$ is the number of pairs of distinct eigenvalues of H_2.

This fact is a consequence of the following lemma, which will be referred to often in this thesis.

Lemma 3.2.1. *Let $\mathcal{H}$ be a Hermitian matrix of size $r \times r$ and $\mathcal{K}$ the real vector space of skew-Hermitian matrices of size $r \times r$ that are at the same time skew-Hermitian with respect to $\mathcal{H}$:*

$$\mathcal{K} := \{B \in \mathfrak{gl}_r(\mathbb{C}) : B + B^* = 0, \ \mathcal{H}B + B^*\mathcal{H} = 0\}.$$

Then $\dim \mathcal{K} = r^2 - 2p$, where p is the number of unordered pairs of distinct eigenvalues of $\mathcal{H}$, counted with multiplicity. Hence, if $\dim \mathcal{K} = r^2$, then $\mathcal{H}$ is a scalar matrix.

Proof. Note first that $\mathcal{K}$ is the centraliser of $\mathcal{H}$ in $\mathfrak{u}(r)$. That is,

$$\mathcal{K} = \{B \in \mathfrak{u}(r) : \mathcal{H}B - B\mathcal{H} = 0\}.$$

Since $B \in \mathcal{K}$ commutes with $\mathcal{H}$, it preserves each eigenspace of $\mathcal{H}$, so

$$\mathcal{K} \cong \mathfrak{u}(r_1) \oplus \cdots \oplus \mathfrak{u}(r_k),$$

where $r_1, \ldots, r_k$ are the dimensions of the eigenspaces of $\mathcal{H}$, and k is the number of distinct eigenvalues of $\mathcal{H}$. Hence,

$$\begin{aligned}\dim \mathcal{K} &= r_1^2 + \cdots + r_k^2 \\ &= (r_1 + \cdots + r_k)^2 - 2\sum_{i<j} r_i r_j \\ &= r^2 - 2p,\end{aligned}$$

which completes the proof. □

By (3.2.1) and (3.2.2) above, we see that we must have $1 \leq m \leq 4$, which leads to the following possibilities:

(a) $n = 4$ and $\lambda_1 \neq \lambda_2$ (here $m = 1$ and $s = 2$),
(b) $n = 5$ and, upon permutation of w-variables, $\lambda_1 \neq \lambda_2 = \lambda_3$ (here $m = 2$ and $s = 5$),
(c) $n = 5$ and $\lambda_1, \lambda_2, \lambda_3$ are pairwise distinct (here $m = 3$ and $s = 3$),
(d) $n = 6$ and, upon permutation of w-variables, $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4$ (here $m = 3$ and $s = 10$),
(e) $n = 6$ and, upon permutation of w-variables, $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$ (here $m = 4$ and $s = 8$), or
(f) $n = 7$ and, upon permutation of w-variables, $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5$ (here $m = 4$ and $s = 17$).

We know from the discussion after Definition 2.3.2 in the previous chapter that in this case (when $k = 2, n \geq 4$) $S(\Omega, H)$ is biholomorphic to a product of two unit balls $B^l \times B^{n-l}$ for $1 \leq l \leq n-1$. The dimension of its automorphism group is given by

$$d(B^l \times B^{n-l}) = 2l^2 - 2nl + n^2 + 2n.$$

Since n is limited to the range $4, 5, 6, 7$, we set the right-hand side equal to $n^2 - 7$ and solve for l in each case. For none of the above values of n is l integer-valued, and therefore this case makes no contributions to our classification.

Case 2. Suppose that $k = 3, n = 4$. Then $S(\Omega, H)$ is linearly equivalent to either

$$D_3 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C} : \operatorname{Im} z - v|w|^2 \in \Omega_2\right\},$$

where $v = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$ with non-negative entries, or

$$D_4 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C} : \operatorname{Im} z - v|w|^2 \in \Omega_3\right\},$$

where $v = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$ satisfying $v_1^2 \geq v_2^2 + v_3^2$, $v_1 > 0$. Let us consider each of these cases separately.

Assume that $S(\Omega, H)$ is equivalent to the domain D_3. Since Ω_2 is equivalent to the positive orthant in $\mathbb{R}^3$ then $S(\Omega, H)$ must be biholomorphic to a four-dimensional product of three unit balls, and it is immediate to see that the only possibility is $B^1 \times B^1 \times B^2$. Since $d(B^1 \times B^1 \times B^2) = 3 + 3 + 8 = 14 > 9 = n^2 - 7$, clearly we can rule out this possibility. Therefore, $S(\Omega, H)$ must be equivalent to the domain D_4.

Suppose first that $v_1^2 > v_2^2 + v_3^2$, i.e., that $v \in \Omega_3$. Since the vector v is an eigenvector of every element of $G(\Omega_3, v|w|^2)$, we see that $G(\Omega_3, v|w|^2)$ does not act transitively on Ω_3. Therefore, we have $v_1 = \sqrt{v_2^2 + v_3^2} > 0$, i.e., that $v \in \partial\Omega_3 \setminus \{0\}$. Since the group $G(\Omega_3)^\circ = \mathbb{R}_+ \times \mathrm{SO}_{1,2}^\circ$ acts transitively on $\partial\Omega_3 \setminus \{0\}$, we may suppose that $v = (1, 1, 0.)$

Lemma 3.2.2. *For the Hermitian form $\mathcal{H}(w, w') := (\overline{w}w', \overline{w}w', 0)$, we have*

$$\dim G(\Omega_3, \mathcal{H}) = 3.$$

Proof. A straightforward computation of the Lie algebra of $G(\Omega_3, H)$ will prove the lemma. We momentarily denote this Lie algebra by $\mathfrak{h}$, and note that $\mathfrak{h}$ consists of all elements of $\mathfrak{g}(\Omega_3)$ having $(1, 1, 0)$ as an eigenvector. The Lie algebra $\mathfrak{g}(\Omega_3)$ is given by

$$\mathfrak{g}(\Omega_3) = \mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{o}_{1,2} = \left\{ \begin{bmatrix} \lambda & p & q \\ p & \lambda & r \\ q & -r & \lambda \end{bmatrix} : \lambda, p, q, r \in \mathbb{R} \right\}.$$

Therefore, it follows that

$$\mathfrak{h} = \left\{ \begin{bmatrix} \lambda & p & q \\ p & \lambda & q \\ q & -q & \lambda \end{bmatrix} : \lambda, p, q \in \mathbb{R} \right\},$$

and we see that $\dim \mathfrak{h} = 3$ as required. $\square$

By the above lemma we see that for $\mathfrak{g} = \mathfrak{g}(D_4)$ we have $\dim \mathfrak{g}_0 = 4$ (recall that $s = 1$). We also know (see Appendix A) that for $\mathfrak{g} = \mathfrak{g}(D_4)$, if $v \in \partial\Omega_3 \setminus \{0\}$ we have $\mathfrak{g}_{1/2} = 0$ and $\dim \mathfrak{g}_1 = 1$. So we have

$$d(D_4) = \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-1/2} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1 = 10.$$

Since $d(D_4) = 10 > 9 = n^2 - 7$, we see that $S(\Omega, H)$ is not equivalent to D_4, and so Case 2 contributes nothing to our classification.

Case 3. Suppose that $k = 3, n = 5$. Here, $S(\Omega, H)$ is linearly equivalent either to

$$D_5 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^2 : \operatorname{Im} z - \mathcal{H}(w, w) \in \Omega_2\right\},$$

where $\mathcal{H}$ is an Ω_2-Hermitian form, or to

$$D_6 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^2 : \operatorname{Im} z - \mathcal{H}(w, w) \in \Omega_3\right\},$$

where $\mathcal{H}$ is an Ω_3-Hermitian form.

Assume $S(\Omega, H)$ is equivalent to the domain D_5. Then $S(\Omega, H)$ must be equivalent to either $B^1 \times B^1 \times B^3$ or $B^1 \times B^2 \times B^2$. Since $d(B^1 \times B^1 \times B^3) = 21$ and $d(B^1 \times B^2 \times B^2) = 19$, and neither of these is equal to $18 = n^2 - 7$, we see that consideration of the domain D_5 does not aid our classification.

Suppose then that $S(\Omega, H)$ is equivalent to the domain D_6. With the use of Lemma 3.2.1, we now show that we must have either $s = 1$, $s = 2$ or $s = 4$. Recall that $s := \dim \mathcal{L}$, where

$$\mathcal{L} = \{B \in \mathfrak{gl}_{n-k}(\mathbb{C}) : H(B\cdot, \cdot) + H(\cdot, B\cdot) = 0\},$$

with $H : \mathbb{C}^{n-k} \times \mathbb{C}^{n-k} \to \mathbb{C}^k$. Here, $\mathcal{L}$ consists of matrices $B \in \mathfrak{gl}_2(\mathbb{C})$ such that

$$\begin{aligned} H_1(B\cdot, \cdot) + H_1(\cdot, B\cdot) &= 0, \\ H_2(B\cdot, \cdot) + H_2(\cdot, B\cdot) &= 0, \\ H_3(B\cdot, \cdot) + H_3(\cdot, B\cdot) &= 0. \end{aligned}$$

Writing the above relations in matrix form, and noting that since $H(w, w) \in \bar{\Omega}_3 \setminus \{0\}$ for all non-zero $w \in \mathbb{C}^2$ we may assume that $H_1 = I$, we have

$$\begin{aligned} B + B^* &= 0, \\ H_2 B + B^* H_2 &= 0, \\ H_3 B + B^* H_3 &= 0. \end{aligned}$$

Now consider the two vector spaces given by

$$\begin{aligned} \mathcal{K}_2 &= \{B \in \mathfrak{gl}_2(\mathbb{C}) : B + B^* = 0,\ H_2 B + B^* H_2 = 0\} \\ \mathcal{K}_3 &= \{B \in \mathfrak{gl}_2(\mathbb{C}) : B + B^* = 0,\ H_3 B + B^* H_3 = 0\}. \end{aligned}$$

Then by Lemma 3.2.1 we have either $\dim \mathcal{K}_2 = 2$ or $\dim \mathcal{K}_2 = 4$, and similarly for the vector space $\mathcal{K}_3$. By noting that $s = \dim(\mathcal{K}_2 \cap \mathcal{K}_3)$, we have either $s = 1, s = 2$ or $s = 4$. Finally, the possibility of $s = 0$ is excluded by observing that $iI \in \mathcal{K}_2 \cap K_3$.

In [16], each of these scenarios was dealt with in Sections 5, 4 and 3 respectively. When $s = 4$ we have $d(D_6) = 15 < 18 = n^2 - 7$, and when $s = 2$ the action of $G(\Omega_3, H)$ on Ω_3 is not transitive. When $s = 1$, we see that $d(D_6) \leq 17 < 18 = n^2 - 7$, and so in none of these instances is any contribution made to our classification.

Case 4. Suppose that $k = 3, n = 6$. Here, $S(\Omega, H)$ is linearly equivalent either to

$$D_7 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^3 : \text{Im } z - \mathcal{H}(w, w) \in \Omega_2\right\},$$

where $\mathcal{H}$ is an Ω_2-Hermitian form, or to

$$D_8 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^3 : \text{Im } z - \mathcal{H}(w, w) \in \Omega_3\right\},$$

where $\mathcal{H}$ is an Ω_3-Hermitian form.

Assume $S(\Omega, H)$ is equivalent to D_7. Then as in the previous two cases, $S(\Omega, H)$ must be biholomorphic to a product of three unit balls. The only possibilities are $B^1 \times B^1 \times B^4$, $B^1 \times B^2 \times B^3$ or $B^2 \times B^2 \times B^2$, none of which have automorphism group of dimension $n^2 - 7 = 29$.

So $S(\Omega, H)$ must be equivalent to D_8. By (2.3.7) we have $s + \dim \mathfrak{g}(\Omega) \geq 11$. Since $\dim \mathfrak{g}(\Omega_3) = 4$, we see that $s \geq 7$. On the other hand, by (2.3.8) we have $s \leq 9$. We show now that s cannot equal 7 or 8, and so we must have $s = 9$. By a similar argument to that in the previous case, the two vector spaces $\mathcal{K}_2$ and $\mathcal{K}_3$ are in this instance given by

$$\begin{aligned}
\mathcal{K}_2 &= \{B \in \mathfrak{gl}_3(\mathbb{C}) : B + B^* = 0,\ H_2 B + B^* H_2 = 0\} \\
\mathcal{K}_3 &= \{B \in \mathfrak{gl}_3(\mathbb{C}) : B + B^* = 0,\ H_3 B + B^* H_3 = 0\}.
\end{aligned}$$

By Lemma 3.2.1 we have that $\dim \mathcal{K}_2 = 3, 5$ or 9 (recall that pairs of distinct eigenvalues are counted with multiplicity). Similarly, $\dim \mathcal{K}_3 = 3, 5$ or 9. Noting that $s = \dim(\mathcal{K}_2 \cap \mathcal{K}_3)$, clearly $s = 1, 2, 3, 5$ or 9.

Let $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ and $\mathbf{H}$ be a positive-definite linear combination of $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$. After a linear change of w-variables, we can diagonalise $\mathbf{H}$ as $\mathbf{H}(w, w) = ||w||^2$. Since $s = 9$, by Lemma 3.2.1 each of the $\mathbb{C}$-valued Hermitian forms $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ is proportional to $\mathbf{H}$. Thus we have $\mathcal{H}(w, w) = v||w||^2$, where $v = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$ satisfying $v_1^2 \geq v_2^2 + v_3^2$, $v_1 > 0$.

Suppose first that $v_1^2 > v_2^2 + v_3^2$, i.e., that $v \in \Omega_3$. Since the vector v is an eigenvector of every element of $G(\Omega_3, v|w|^2)$, we see that $G(\Omega_3, v|w|^2)$ does not act transitively on Ω_3. Therefore, we have $v_1 = \sqrt{v_2^2 + v_3^2} > 0$, i.e., that $v \in \partial\Omega_3 \setminus \{0\}$. As the group $G(\Omega_3)^\circ = \mathbb{R}_+ \times \mathrm{SO}_{1,2}^\circ$ acts transitively on $\partial\Omega_3 \setminus \{0\}$, we can suppose that $v = (1, 1, 0)$, so $\mathcal{H}(w, w) = (||w||^2, ||w||^2, 0)$. Here, the domain D_8 coincides with the domain $\widetilde{D}_6$ with $N = 3$ (see Proposition

A.3.1 in Appendix A). Therefore, by Proposition A.3.1 we see that for $\mathfrak{g} = \mathfrak{g}(D_8)$ we have $\mathfrak{g}_{1/2} = 0$, and by Proposition A.4.1 we see that for $\mathfrak{g} = \mathfrak{g}(D_8)$ we have $\dim \mathfrak{g}_1 = 1$. Furthermore, for $w \in \mathbb{C}^3$, the proof of Lemma 3.2.2 gives us

$$\dim G(\Omega_3, (||w||^2, ||w||^2, 0)) = 3,$$

and we see that $\dim \mathfrak{g}_0 = 12$ (since $s = 9$). Therefore, we have

$$d(D_8) = \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-1/2} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{1/2} + \dim \mathfrak{g}_1 = 22 < 29 = n^2 - 7.$$

This shows that $S(\Omega, H)$ cannot be equivalent to D_8, so Case 4 contributes nothing to our classification.

Case 5. Suppose that $k = 3, n = 7$. Here, $S(\Omega, H)$ is linearly equivalent either to

$$D_9 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^4 : \operatorname{Im} z - \mathcal{H}(w, w) \in \Omega_2\right\},$$

where $\mathcal{H}$ is an Ω_2-Hermitian form, or to

$$D_{10} := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^4 : \operatorname{Im} z - \mathcal{H}(w, w) \in \Omega_3\right\},$$

where $\mathcal{H}$ is an Ω_3-Hermitian form.

By (2.3.7) we have $s + \dim \mathfrak{g}(\Omega) \geq 20$. On the other hand, $s \leq 16$ by (2.3.8). Since $\dim \mathfrak{g}(\Omega_2) = 3$ and $\dim \mathfrak{g}(\Omega_3) = 4$, it follows that Ω is linearly equivalent to Ω_3 and $s = 16$. In particular, $S(\Omega, H)$ can only be linearly equivalent to the domain D_{10}.

We proceed in the same manner as the previous case. Let $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ and $\mathbf{H}$ be a positive-definite linear combination of $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$. After a linear change of w-variables, we can diagonalise $\mathbf{H}$ as $\mathbf{H}(w, w) = ||w||^2$. Since $s = 16$, by Lemma 3.2.1 each of the $\mathbb{C}$-valued Hermitian forms $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ is proportional to $\mathbf{H}$. Thus we have $\mathcal{H}(w, w) = v||w||^2$, where $v = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$ satisfying $v_1^2 \geq v_2^2 + v_3^2$, $v_1 > 0$.

We will suppose first that $v_1^2 > v_2^2 + v_3^2$, i.e., that $v \in \Omega_3$. Since the vector v is an eigenvector of every element of $G(\Omega_3, v|w|^2)$, we see that $G(\Omega_3, v|w|^2)$ does not act transitively on Ω_3. Therefore, we have $v_1 = \sqrt{v_2^2 + v_3^2} > 0$, i.e., that $v \in \partial\Omega_3 \setminus \{0\}$. As the group $G(\Omega_3)^\circ = \mathbb{R}_+ \times \mathrm{SO}_{1,2}^\circ$ acts transitively on $\partial\Omega_3 \setminus \{0\}$, we can suppose that $v = (1, 1, 0)$, so $\mathcal{H}(w, w) = (||w||^2, ||w||^2, 0)$. In this case, the domain D_{10} coincides with the domain $\widetilde{D}_6$ with $N = 4$ (see Proposition A.3.1 in Appendix A). As in the previous case, we see that for $\mathfrak{g} = \mathfrak{g}(D_{10})$ we have $\mathfrak{g}_{1/2} = 0$ and $\dim \mathfrak{g}_1 = 1$. Furthermore, for $w \in \mathbb{C}^4$, the proof of Lemma 3.2.2 gives us

$$\dim G(\Omega_3, (||w||^2, ||w||^2, 0)) = 3,$$

and we see that $\dim \mathfrak{g}_0 = 19$ (since $s = 16$). Therefore, we have

$$d(D_8) = \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-1/2} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{1/2} + \dim \mathfrak{g}_1 = 31 < 42 = n^2 - 7.$$

This shows that $S(\Omega, H)$ cannot be equivalent to D_{10}, so Case 5 makes no contributions to the classification.

Case 6. Suppose that $k = 4, n = 4$. In this case, after a linear change of variables, $S(\Omega, H)$ is one of the domains

$$\left\{z \in \mathbb{C}^4 : \text{Im } z \in \Omega_4\right\},$$
$$\left\{z \in \mathbb{C}^4 : \text{Im } z \in \Omega_5\right\},$$
$$\left\{z \in \mathbb{C}^4 : \text{Im } z \in \Omega_6\right\},$$

and therefore is biholomorphic either to $B^1 \times B^1 \times B^1 \times B^1$, or to $B^1 \times T_3$, where T_3 is the domain

$$T_3 = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : (\text{Im } z_1)^2 - (\text{Im } z_2)^2 - (\text{Im } z_3)^2 > 0, \;\; \text{Im } z_1 > 0\right\},$$

or to T_4, where T_4 is the domain

$$T_4 = \Big\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (\text{Im } z_1)^2 - (\text{Im} z_2)^2 - (\text{Im } z_3)^2 - (\text{Im } z_4)^2 > 0, \quad \text{Im } z_1 > 0\Big\}.$$

The dimensions of the respective automorphism groups of these domains are 12, 13 and 15. Each of these numbers is greater than $9 = n^2 - 7$, and so we see that Case 6 contributes nothing to our classification.

Case 7. Suppose that $k = 4, n = 5$. Then $S(\Omega, H)$ is linearly equivalent to either

$$D_{11} := \left\{(z, w) \in \mathbb{C}^4 \times \mathbb{C} : \text{Im } z - v|w|^2 \in \Omega_4\right\},$$

where $v = (v_1, v_2, v_3, v_4)$ is a vector in $\mathbb{R}^4$ with non-negative entries, or

$$D_{12} := \left\{(z, w) \in \mathbb{C}^4 \times \mathbb{C} : \text{Im } z - v|w|^2 \in \Omega_5\right\},$$

where $v = (v_1, v_2, v_3, v_4)$ is a vector in $\mathbb{R}^4$ satisfying $v \in \bar{\Omega}_5 \setminus \{0\}$, or

$$D_{13} := \left\{(z, w) \in \mathbb{C}^4 \times \mathbb{C} : \text{Im } z - v|w|^2 \in \Omega_6\right\},$$

where $v = (v_1, v_2, v_3, v_4)$ is a vector in $\mathbb{R}^4$ satisfying $v \in \bar{\Omega}_6 \setminus \{0\}$, i.e., $v_1^2 \geq v_2^2 + v_3^2 + v_4^2, v_1 > 0$.

Since $s = 1$, by inequality (2.3.7) we see that $\dim \mathfrak{g}(\Omega) \geq 5$. Therefore $S(\Omega, H)$ can only be linearly equivalent to either D_{12} or D_{13}. Let us begin with the second possibility. If $S(\Omega, H)$ is equivalent to D_{13}, then assume firstly that $v_1^2 > v_2^2 + v_3^2 + v_4^2$, i.e., that $v \in \Omega_6$. Since the vector v is an eigenvector of every element of $G(\Omega_6, v|w|^2)$, we see that $G(\Omega_6, v|w|^2)$ does not act transitively on Ω_6. Therefore, we have $v_1 = \sqrt{v_2^2 + v_3^2 + v_4^2} > 0$, i.e., that $v \in \partial\Omega_6 \setminus \{0\}$. As the group $G(\Omega_6)^\circ = \mathbb{R}_+ \times \mathrm{SO}_{1,3}^\circ$ acts transitively on $\partial\Omega_6 \setminus \{0\}$, we can suppose that $v = (1, 1, 0, 0)$, so $v|w|^2 = (|w|^2, |w|^2, 0, 0)$.

Lemma 3.2.3. *For the Hermitian form $\mathcal{H}(w, w') := (\overline{w}w', \overline{w}w', 0, 0)$, we have*

$$\dim G(\Omega_6, \mathcal{H}) = 5.$$

Proof. A straightforward computation of the Lie algebra of $G(\Omega_6, H)$ will prove the lemma. We momentarily denote this Lie algebra by $\mathfrak{h}$, and note that $\mathfrak{h}$ consists of all elements of $\mathfrak{g}(\Omega_6)$ having $(1, 1, 0, 0)$ as an eigenvector. The Lie algebra $\mathfrak{g}(\Omega_6)$ is given by

$$\mathfrak{g}(\Omega_6) = \mathfrak{c}(\mathfrak{gl}_4(\mathbb{R})) \oplus \mathfrak{o}_{1,3} = \left\{ \begin{bmatrix} \lambda & p & q & r \\ p & \lambda & s & t \\ q & -s & \lambda & y \\ r & -t & -y & \lambda \end{bmatrix} : \lambda, p, q, r, s, t, y \in \mathbb{R} \right\}.$$

Therefore, it follows that

$$\mathfrak{h} = \left\{ \begin{bmatrix} \lambda & p & q & r \\ p & \lambda & q & r \\ q & -q & \lambda & y \\ r & -r & -y & \lambda \end{bmatrix} : \lambda, p, q, r, y \in \mathbb{R} \right\},$$

and we see that $\dim \mathfrak{h} = 5$ as required. □

By the above lemma and the equality in (2.3.5) we see that for $\mathfrak{g} = \mathfrak{g}(D_{13})$ we have $\dim \mathfrak{g}_0 = 6$ (recall that $s = 1$). Further, the domain D_{13} coincides with the domain $\widetilde{D}_{13}$ with $N = 1$ (see Proposition A.5.1 in Appendix A). Therefore, by Proposition A.5.1 we see that for $\mathfrak{g} = \mathfrak{g}(D_{13})$ we have $\mathfrak{g}_{1/2} = 0$. Now using these values for $\dim \mathfrak{g}_0$ and $\dim \mathfrak{g}_{1/2}$ along with the second inequality in (2.3.4), we see that

$$d(D_{13}) = \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-1/2} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{1/2} + \dim \mathfrak{g}_1 \leq 16 < 18 = n^2 - 7,$$

showing no contribution to the classification.

Now assume that $S(\Omega, H)$ is equivalent to D_{12}. To begin with, consider the boundary set $\partial\Omega_5 \setminus \{0\}$, which can be described

$$\begin{aligned} \partial\Omega_5 \setminus \{0\} = &\left\{ (v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : v_1^2 \geq v_2^2 + v_3^2,\ v_1 > 0,\ v_4 = 0 \right\} \\ &\cup \left\{ (v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : v_1^2 = v_2^2 + v_3^2,\ v_1 \geq 0,\ v_4 > 0 \right\}. \end{aligned}$$

We can further break up this boundary set into four components, which are invariant under the action of $G(\Omega_5)^\circ$, and which we denote by C_1, C_2, C_3 and C_4. Describing each of these components, we have

$$\begin{aligned}
C_1 &:= \left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : v_1^2 > v_2^2 + v_3^2,\ v_1 > 0,\ v_4 = 0\right\},\\
C_2 &:= \left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : v_1^2 = v_2^2 + v_3^2,\ v_1 > 0,\ v_4 = 0\right\},\\
C_3 &:= \left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : v_1^2 = v_2^2 + v_3^2,\ v_1 > 0,\ v_4 > 0\right\}, \text{ and}\\
C_4 &:= \left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : v_1^2 = v_2^2 + v_3^2,\ v_1 = 0,\ v_4 > 0\right\}.
\end{aligned}$$

Assume first that $v \in C_1$, i.e., that $v \in \Omega_3 \times \{0\}$ (recall that $\Omega_3 := \Lambda_3$, the Lorentz cone in $\mathbb{R}^3$). In this situation, we have the following lemma.

Lemma 3.2.4. *If $v \in \Omega_3 \times \{0\}$, for $\mathfrak{g} = \mathfrak{g}(D_{12})$ we have $\mathfrak{g}_{1/2} = 0$.*

Proof. Since the group $G(\Omega_3)^\circ = \mathbb{R}_+ \times \mathrm{SO}_{1,2}^\circ$ acts transitively on Ω_3, we may suppose that $v = (1, 0, 0, 0)$. We will apply Theorem (2.3.4) to the cone Ω_5 and the Ω_5-Hermitian form

$$\mathcal{H}(w, w') = (\bar{w}w', 0, 0, 0).$$

Let $\Phi : \mathbb{C}^4 \to \mathbb{C}$ be a $\mathbb{C}$-linear map given by

$$\Phi(z_1, z_2, z_3, z_4) = \varphi_1 z_1 + \varphi_2 z_2 + \varphi_3 z_3 + \varphi_4 z_4$$

where $\varphi_j \in \mathbb{C}$. Fixing $\mathbf{w} \in \mathbb{C}$, for $x \in \mathbb{R}^4$ we compute

$$\mathcal{H}(\mathbf{w}, \Phi(x)) = (\bar{\mathbf{w}}(\varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + \varphi_4 x_4), 0, 0, 0).$$

Then from formula (2.3.11) we see

$$\Phi_{\mathbf{w}}(x) = (\mathrm{Im}(\bar{\mathbf{w}}\varphi_1)x_1 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_2)x_2 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_3)x_3 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_4)x_4, 0, 0, 0).$$

Now, since $\mathfrak{g}(\Omega_5) = (\mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{o}_{1,2}) \oplus \mathbb{R}$ consists of all matrices of the form

$$\begin{bmatrix} \lambda & p & q & 0\\ p & \lambda & r & 0\\ q & -r & \lambda & 0\\ 0 & 0 & 0 & \mu \end{bmatrix}, \quad \lambda, \mu, p, q, r \in \mathbb{R}.$$

Therefore, the condition that the map $\Phi_{\mathbf{w}}$ lies in $\mathfrak{g}(\Omega_5)$ for every $\mathbf{w} \in \mathbb{C}$ immediately yields

$$\mathrm{Im}(\bar{\mathbf{w}}\varphi_1) \equiv 0,\ \mathrm{Im}(\bar{\mathbf{w}}\varphi_2) \equiv 0,\ \mathrm{Im}(\bar{\mathbf{w}}\varphi_3) \equiv 0 \ \text{ and } \ \mathrm{Im}(\bar{\mathbf{w}}\varphi_4) \equiv 0,$$

which implies $\Phi = 0$. By formula (2.3.12) we then see that $\mathfrak{g}_{1/2} = 0$ as required. $\square$

Now assume that $v \in C_2$, i.e., that $v \in \partial\Omega_3 \setminus \{0\} \times \{0\}$. In this situation, we have the following lemma, which is analogous to the previous one.

Lemma 3.2.5. *If* $v \in \partial\Omega_3 \setminus \{0\} \times \{0\}$, *for* $\mathfrak{g} = \mathfrak{g}(D_{12})$ *we have* $\mathfrak{g}_{1/2} = 0$.

Proof. Since the group $G(\Omega_3)^\circ = \mathbb{R}_+ \times \mathrm{SO}^\circ_{1,2}$ acts transitively on $\partial\Omega_3 \setminus \{0\}$, we may suppose that $v = (1, 1, 0, 0)$. We will apply Theorem (2.3.4) to the cone Ω_5 and the Ω_5-Hermitian form

$$\mathcal{H}(w, w') = (\bar{w}w', \bar{w}w', 0, 0).$$

Let $\Phi : \mathbb{C}^4 \to \mathbb{C}$ be a $\mathbb{C}$-linear map given by

$$\Phi(z_1, z_2, z_3, z_4) = \varphi_1 z_1 + \varphi_2 z_2 + \varphi_3 z_3 + \varphi_4 z_4$$

where $\varphi_j \in \mathbb{C}$. Fixing $\mathbf{w} \in \mathbb{C}$, for $x \in \mathbb{R}^4$ we compute

$$\begin{aligned}\mathcal{H}(\mathbf{w}, \Phi(x)) = (&\bar{\mathbf{w}}(\varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + \varphi_4 x_4),\\ &\bar{\mathbf{w}}(\varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + \varphi_4 x_4), 0, 0).\end{aligned}$$

Then from formula (2.3.11) we see

$$\begin{aligned}\Phi_{\mathbf{w}}(x) = (&\mathrm{Im}(\bar{\mathbf{w}}\varphi_1)x_1 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_2)x_2 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_3)x_3 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_4)x_4,\\ &\mathrm{Im}(\bar{\mathbf{w}}\varphi_1)x_1 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_2)x_2 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_3)x_3 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_4)x_4, 0, 0).\end{aligned}$$

Now, since $\mathfrak{g}(\Omega_5) = (\mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{o}_{1,2}) \oplus \mathbb{R}$ consists of all matrices of the form

$$\begin{bmatrix} \lambda & p & q & 0 \\ p & \lambda & r & 0 \\ q & -r & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix}, \quad \lambda, \mu, p, q, r \in \mathbb{R}.$$

Therefore, the condition that the map $\Phi_{\mathbf{w}}$ lies in $\mathfrak{g}(\Omega_5)$ for every $\mathbf{w} \in \mathbb{C}$ immediately yields

$$\mathrm{Im}(\bar{\mathbf{w}}\varphi_1) \equiv 0,\ \mathrm{Im}(\bar{\mathbf{w}}\varphi_2) \equiv 0,\ \mathrm{Im}(\bar{\mathbf{w}}\varphi_3) \equiv 0 \ \text{ and } \ \mathrm{Im}(\bar{\mathbf{w}}\varphi_4) \equiv 0,$$

which implies $\Phi = 0$. By formula (2.3.12) we then see that $\mathfrak{g}_{1/2} = 0$ as required. $\square$

Now assume that $v \in C_3$, i.e., that $v \in \partial\Omega_3 \setminus \{0\} \times \mathbb{R}_+$. In this situation, we have the following lemma, which is analogous to the previous two.

Lemma 3.2.6. *If* $v \in \partial\Omega_3 \setminus \{0\} \times \mathbb{R}_+$, *for* $\mathfrak{g} = \mathfrak{g}(D_{12})$ *we have* $\mathfrak{g}_{1/2} = 0$.

Proof. Since the group $G(\Omega_3)^\circ = \mathbb{R}_+ \times \mathrm{SO}^\circ_{1,2}$ acts transitively on $\partial\Omega_3 \setminus \{0\}$, we may suppose that $v = (1, 1, 0, 1)$. We will apply Theorem (2.3.4) to the cone Ω_5 and the Ω_5-Hermitian form

$$\mathcal{H}(w, w') = (\bar{w}w', \bar{w}w', 0, \bar{w}w').$$

Let $\Phi : \mathbb{C}^4 \to \mathbb{C}$ be a $\mathbb{C}$-linear map given by

$$\Phi(z_1, z_2, z_3, z_4) = \varphi_1 z_1 + \varphi_2 z_2 + \varphi_3 z_3 + \varphi_4 z_4$$

where $\varphi_j \in \mathbb{C}$. Fixing $\mathbf{w} \in \mathbb{C}$, for $x \in \mathbb{R}^4$ we compute

$$\begin{aligned}\mathcal{H}(\mathbf{w}, \Phi(x)) = (&\bar{\mathbf{w}}(\varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + \varphi_4 x_4),\\ &\bar{\mathbf{w}}(\varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + \varphi_4 x_4), 0,\\ &\bar{\mathbf{w}}(\varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + \varphi_4 x_4)).\end{aligned}$$

Then from formula (2.3.11) we see

$$\begin{aligned}\Phi_{\mathbf{w}}(x) = (&\mathrm{Im}(\bar{\mathbf{w}}\varphi_1)x_1 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_2)x_2 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_3)x_3 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_4)x_4,\\ &\mathrm{Im}(\bar{\mathbf{w}}\varphi_1)x_1 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_2)x_2 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_3)x_3 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_4)x_4, 0,\\ &\mathrm{Im}(\bar{\mathbf{w}}\varphi_1)x_1 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_2)x_2 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_3)x_3 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_4)x_4).\end{aligned}$$

Now, since $\mathfrak{g}(\Omega_5) = (\mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{o}_{1,2}) \oplus \mathbb{R}$ consists of all matrices of the form

$$\begin{bmatrix} \lambda & p & q & 0 \\ p & \lambda & r & 0 \\ q & -r & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix}, \quad \lambda, \mu, p, q, r \in \mathbb{R}.$$

Therefore, the condition that the map $\Phi_{\mathbf{w}}$ lies in $\mathfrak{g}(\Omega_5)$ for every $\mathbf{w} \in \mathbb{C}$ immediately yields

$$\mathrm{Im}(\bar{\mathbf{w}}\varphi_1) \equiv 0, \ \mathrm{Im}(\bar{\mathbf{w}}\varphi_2) \equiv 0, \ \mathrm{Im}(\bar{\mathbf{w}}\varphi_3) \equiv 0 \ \text{ and } \ \mathrm{Im}(\bar{\mathbf{w}}\varphi_4) \equiv 0,$$

which implies $\Phi = 0$. By formula (2.3.12) we then see that $\mathfrak{g}_{1/2} = 0$ as required. □

We see from the above three lemmas that for the components C_1, C_2 and C_3 we have $\mathfrak{g}_{1/2} = 0$. Then by estimate (2.3.6), the second inequality in (2.3.4) and the above three lemmas, we see that in each of these cases

$$d(D_{12}) \leq 16 < 18 = n^2 - 7$$

(recall that $s = 1$). This shows that in the cases of these components, $S(\Omega, H)$ cannot be equivalent to D_{12}, so no new contributions are made to our classification.

Lastly, let $v \in C_4$, i.e., that $v \in \{(0,0,0)\} \times \mathbb{R}_+$. Since $G(\Omega_5)^\circ$ clearly acts transitively on this set, we may assume that $v = (0,0,0,1)$. Then $S(\Omega, H)$ is equivalent to the domain

$$B^2 \times T_3 = \big\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (\operatorname{Im} z_1)^2 - (\operatorname{Im} z_2)^2 - (\operatorname{Im} z_3)^2 > 0,\ \operatorname{Im} z_1 > 0,\\ \operatorname{Im} z_4 - |w|^2 > 0\big\}.$$

Since $d(B^2 \times T_3) = 10 + 8 = 18 = n^2 - 7$, we see that Case 7 contributes the product $B^2 \times T_3$ to the classification of homogeneous Kobayashi-hyperbolic manifolds with automorphism group dimension $n^2 - 7$.

Case 8. Suppose that $k = 5$ and $n = 5$. Then by inequality (2.3.7) we see that in this situation we have $\dim \mathfrak{g}(\Omega) \geq 8$. Therefore, $S(\Omega, H)$ is equivalent to one of the domains

$$\big\{z \in \mathbb{C}^5 : \operatorname{Im} z \in \Omega_{11}\big\},$$

$$\big\{z \in \mathbb{C}^5 : \operatorname{Im} z \in \Omega_{12}\big\}$$

and therefore is biholomorphic either to $B^1 \times T_4$, where

$$T_4 = \big\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (\operatorname{Im} z_1)^2 - (\operatorname{Im} z_2)^2 - (\operatorname{Im} z_3)^2 - (\operatorname{Im} z_4)^2 > 0,\\ \operatorname{Im} z_1 > 0\big\},$$

or to T_5, where

$$T_5 = \big\{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : (\operatorname{Im} z_1)^2 - (\operatorname{Im} z_2)^2 - (\operatorname{Im} z_3)^2\\ - (\operatorname{Im} z_4)^2 - (\operatorname{Im} z_5)^2 > 0, \operatorname{Im} z_1 > 0\big\}.$$

In the latter case, the dimension of the automorphism group of this domain is $d(T_5) = 21 > 18 = n^2 - 7$, so no contribution is made. However, we see that $d(B^1 \times T_4) = 3 + 15 = 18 = n^2 - 7$, and so Case 4 contributes $B^1 \times T_4$ to the classification of homogeneous hyperbolic manifolds with automorphism group dimension $n^2 - 7$.

In the following three sections of this chapter, we determine the Lie algebras of the automorphism groups of the cones Ω_7 and Ω_8. As for the other homogeneous open convex cones provided in the list at the end of the previous chapter, it is easily seen that they are either a positive orthant in some dimension, a Lorentz cone, or a product of such cones. In this situation, the Lie algebra of the automorphism group of each cone is straightforward to compute. However, the cones Ω_7 and Ω_8 are considerably more complicated, and the task of merely determining their automorphism groups is rather involved. The task is made somewhat easier by the fact that Ω_7 and Ω_8 are in fact dual to each other, and it is this fact that we prove in the following section. We then conclude, using results from the theory of convex cones, that the automorphism groups of Ω_7 and Ω_8 are isomorphic. In the final section, we determine the dimension of their Lie algebras.

3.3 Duality of Ω_7 and Ω_8

According to the classification provided at the end of Section 2.2, Ω_7 and Ω_8 are described as follows:

$$\begin{aligned}\Omega_7 := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 > 0,\ \ x_1x_2 - x_4^2 > 0,& \\ x_1x_2x_3 - x_3x_4^2 - x_2x_5^2 > 0\big\},&\end{aligned}$$

and

$$\Omega_8 := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 > 0,\ \ x_1x_2 - x_4^2 > 0,\ \ x_1x_3 - x_5^2 > 0\big\}\,.$$

From the definition of Ω_7, it is easy to see that $x_2 > 0$ and $x_3 > 0$. So by the third inequality in this definition, we see that $x_2(x_1x_3 - x_5^2) > x_3x_4^2$, and therefore that $x_1x_3 - x_5^2 > 0$. This shows in fact that $\Omega_7 \subset \Omega_8$. We are able to describe each of the cones above in matrix form. Consider the subspace of the vector space V of 3×3 symmetric matrices $S^3(\mathbb{R})$ given by

$$V := \left\{ X = \begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{bmatrix} : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\}.$$

Using $y_1, \ldots, y_5 \in \mathbb{R}$ to identify the variables in Ω_8, we can describe each cone as follows:

$$\Omega_7 = \{X \in V : X \text{ is positive definite}\}\,,$$

and

$$\Omega_8 = \big\{Y \in V : y_1 > 0,\ \ y_1y_2 - y_4^2 > 0,\ \ y_1y_3 - y_5^2 > 0\big\}\,.$$

We now show that Ω_7 and Ω_8 are dual to each other. Recall that the definition of the dual of an open convex cone Ω is given by

$$\Omega^* = \big\{y \in V : \langle x, y\rangle > 0 \text{ for all } x \in \bar{\Omega}\setminus\{0\}\big\}\,.$$

Theorem 3.3.1. *Ω_8 is the dual cone of Ω_7 with respect to the inner product induced by the trace inner product on $S^3(\mathbb{R})$.*

Proof. Following [8], we choose the inner product induced by the trace inner product on $S^3(\mathbb{R})$. That is, for all $x, y \in \mathbb{R}^5$ we have

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + 2x_4y_4 + 2x_5y_5.$$

We begin by showing that $\Omega_8 \subset \Omega_7^*$, that is, that $\langle X, Y\rangle > 0$ is satisfied for $X \in \bar{\Omega}_7 \setminus \{0\}$ and $Y \in \Omega_8$. We do this in two parts. First we prove that $\langle X, Y\rangle > 0$ is satisfied for $X \in \Omega_7$ and $Y \in \Omega_8$, and then show the same inequality holds for $X \in \partial\Omega_7 \setminus \{0\}$ and $Y \in \Omega_8$.

Let $X \in \Omega_7$ be given by

$$X = \begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{bmatrix},$$

for $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$. The positive definiteness condition implies that

$$x_1x_2x_3 - x_3x_4^2 - x_2x_5^2 > 0,$$

which implies

$$\frac{x_4^2}{x_1x_2} < 1 - \frac{x_5^2}{x_1x_3}.$$

This means that for any $X \in \Omega_7$, there exists t such that

$$\frac{x_4^2}{x_1x_2} < t < 1 - \frac{x_5^2}{x_1x_3},$$

and we see that

$$t > \frac{x_4^2}{x_1x_2} \implies tx_1x_2 - x_4^2 > 0 \implies \det\begin{bmatrix} tx_1 & x_4 \\ x_4 & x_2 \end{bmatrix} > 0,$$

and

$$1 - t > \frac{x_5^2}{x_1x_3} \implies (1-t)x_1x_3 - x_5^2 > 0 \implies \det\begin{bmatrix} (1-t)x_1 & x_5 \\ x_5 & x_3 \end{bmatrix} > 0,$$

implying that the matrices $X_1 := \begin{bmatrix} tx_1 & x_4 \\ x_4 & x_2 \end{bmatrix}$ and $X_2 := \begin{bmatrix} (1-t)x_1 & x_5 \\ x_5 & x_3 \end{bmatrix}$ are positive definite. Let $Y \in \Omega_8$ be given by

$$Y = \begin{bmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{bmatrix},$$

for $y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$. The conditions on Y imply that the matrices $Y_1 := \begin{bmatrix} y_1 & y_4 \\ y_4 & y_2 \end{bmatrix}$ and $Y_2 := \begin{bmatrix} y_1 & y_5 \\ y_5 & y_3 \end{bmatrix}$ are positive definite. Using the fact that if $n \times n$ matrices A and B are positive definite (by Lemma 3.3.2 below), then $\mathrm{Tr}(AB) > 0$ and we see that

$$\mathrm{Tr}(X_1Y_1) = \mathrm{Tr}\left(\begin{bmatrix} tx_1 & x_4 \\ x_4 & x_2 \end{bmatrix}\begin{bmatrix} y_1 & y_4 \\ y_4 & y_2 \end{bmatrix}\right) = tx_1y_1 + x_2y_2 + 2x_4y_4 > 0,$$

and

$$\begin{aligned}\mathrm{Tr}(X_2Y_2) &= \mathrm{Tr}\left(\begin{bmatrix} (1-t)x_1 & x_5 \\ x_5 & x_3 \end{bmatrix}\begin{bmatrix} y_1 & y_5 \\ y_5 & y_3 \end{bmatrix}\right) \\ &= (1-t)x_1y_1 + x_3y_3 + 2x_5y_5 > 0.\end{aligned}$$

Adding these two expressions, we see that $\langle X, Y\rangle > 0$ as required.

Now we prove that for $X \in \partial\Omega_7 \setminus \{0\}$, and $Y \in \Omega_8$, we have $\mathrm{Tr}(XY) > 0$. We utilise the following lemma.

Lemma 3.3.2. *For non-zero $n \times n$ symmetric matrices A and B, if A is positive semidefinite and B is positive definite, then* $\mathrm{Tr}(AB) > 0$.

Proof. We employ the usual notation $A = [a_{ij}]$ for matrices. Since A is symmetric it is orthogonally diagonalisable. For orthogonal P, we have $P^TAP = D$ where $d_{kk} > 0$ for some k, since A is non-zero. So we see that

$$\begin{aligned}\mathrm{Tr}(AB) &= \mathrm{Tr}(PDP^TB) \\ &= \mathrm{Tr}(DP^TBP) \\ &= \sum_{i=1}^{n} d_{ii}(P^TBP)_{ii} \\ &\geq d_{kk}(P^TBP)_{kk} \\ &> 0,\end{aligned}$$

where the last inequality follows from the fact that P^TBP is positive definite (since B is positive definite), and so $(P^TBP)_{kk} = e_k^T(P^TBP)e_k > 0$. $\square$

Note that

$$\bar{\Omega}_7 \subset \{X \in V : X \text{ is positive semidefinite}\},$$

which follows from the continuity of the roots of the characteristic polynomial of a square matrix as a function of its entries.

An element of $\partial\Omega_7 \setminus \{0\}$ is given by

$$X = \begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{bmatrix},$$

where X is positive semidefinite. Since X cannot be the zero matrix, it has at least one non-zero eigenvalue. Further, X must have at least one zero eigenvalue, since otherwise it would be positive definite and not in $\partial\Omega_7 \setminus \{0\}$. Therefore, we must have rank $X = 1$ or rank $X = 2$. We consider each of these possibilities.

Case 1. Suppose that rank $X = 1$. Assuming all three columns of X are non-zero, since each column must be a scalar multiple of the others, we see that the zero in the second column forces $x_3 = x_5 = 0$, and the zero in the third column forces $x_2 = x_4 = 0$. Therefore,

$$X = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{with } x_1 \geq 0.$$

In fact $x_1 > 0$, since X is the zero matrix in the case of $x_1 = 0$, which is excluded. We then have

$$\mathrm{Tr}(XY) = \mathrm{Tr}\left(\begin{bmatrix} x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{bmatrix}\right) = x_1 y_1 > 0.$$

Now, assume that X has two non-zero columns. If these are the second and third columns, we have

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}, \quad \text{with } x_2, x_3 \geq 0.$$

Since rank $X = 1$, either $x_2 = 0$ with $x_3 > 0$, or $x_3 = 0$ with $x_2 > 0$. So we see that either

$$\mathrm{Tr}(XY) = \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_3 \end{bmatrix}\begin{bmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{bmatrix}\right) = x_3 y_3 > 0.$$

or

$$\mathrm{Tr}(XY) = \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{bmatrix}\right) = x_2 y_2 > 0.$$

If the two non-zero columns are the first and the second, we have

$$\begin{aligned}\operatorname{Tr}(X_1Y) &= \operatorname{Tr}\left(\begin{bmatrix} x_1 & x_4 & 0 \\ x_4 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{bmatrix}\right) \\ &= \operatorname{Tr}\left(\begin{bmatrix} x_1 & x_4 \\ x_4 & x_2 \end{bmatrix}\begin{bmatrix} y_1 & y_4 \\ y_4 & y_2 \end{bmatrix}\right) \\ &> 0,\end{aligned}$$

where the last inequality follows from the above lemma, since the matrix $\begin{bmatrix} x_1 & x_4 \\ x_4 & x_2 \end{bmatrix}$ is positive semidefinite, and the matrix $\begin{bmatrix} y_1 & y_4 \\ y_4 & y_2 \end{bmatrix}$ is positive definite. A similar argument applies if the two non-zero columns are the first and the third, and similar or identical arguments are made in the case when X has only one non-zero column.

Case 2. Suppose that rank $X = 2$. Suppose first that $x_1 = 0$. Then the inequalities $x_1x_2 - x_4^2 \geq 0$ and $x_1x_3 - x_5^2 \geq 0$ imply that $x_4 = 0$ and $x_5 = 0$. So we see that

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}, \quad \text{with } x_2, x_3 > 0.$$

It follows that

$$\operatorname{Tr}(XY) = \operatorname{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}\begin{bmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{bmatrix}\right) = x_2y_2 + x_3y_3 > 0.$$

Now suppose that $x_1 > 0$. Further, suppose that $x_2 \neq 0$ and $x_3 \neq 0$. Since the second and third columns of X are linearly independent, the first column of X must be a linear combination of these two. That is,

$$\begin{bmatrix} x_1 \\ x_4 \\ x_5 \end{bmatrix} = c_1 \begin{bmatrix} x_4 \\ x_2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} x_5 \\ 0 \\ x_3 \end{bmatrix}$$

for some $c_1, c_2 \in \mathbb{R}$. So we have

$$\begin{aligned} x_1 &= c_1x_4 + c_2x_5 \\ x_4 &= c_1x_2 \\ x_5 &= c_2x_3, \end{aligned}$$

and we see that X then becomes

$$\begin{aligned} X &= \begin{bmatrix} c_1^2x_2 + c_2^2x_3 & c_1x_2 & c_2x_3 \\ c_1x_2 & x_2 & 0 \\ c_2x_3 & 0 & x_3 \end{bmatrix} \\ &= \begin{bmatrix} c_1^2x_2 & c_1x_2 & 0 \\ c_1x_2 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} c_2^2x_3 & 0 & c_2x_3 \\ 0 & 0 & 0 \\ c_2x_3 & 0 & x_3 \end{bmatrix}. \end{aligned}$$

Now, we set

$$X_1 := \begin{bmatrix} c_1^2x_2 & c_1x_2 & 0 \\ c_1x_2 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } X_2 := \begin{bmatrix} c_2^2x_3 & 0 & c_2x_3 \\ 0 & 0 & 0 \\ c_2x_3 & 0 & x_3 \end{bmatrix},$$

and so $X = X_1 + X_2$. Considering X_1 and $Y \in \Omega_8$, we have

$$\begin{aligned} \operatorname{Tr}(X_1Y) &= \operatorname{Tr}\left(\begin{bmatrix} c_1^2x_2 & c_1x_2 & 0 \\ c_1x_2 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{bmatrix} \right) \\ &= \operatorname{Tr}\left(\begin{bmatrix} c_1^2x_2 & c_1x_2 \\ c_1x_2 & x_2 \end{bmatrix} \begin{bmatrix} y_1 & y_4 \\ y_4 & y_2 \end{bmatrix} \right) \\ &> 0, \end{aligned}$$

where the last inequality follows from the above lemma, since the matrix $\begin{bmatrix} c_1^2x_2 & c_1x_2 \\ c_1x_2 & x_2 \end{bmatrix}$ is positive semidefinite, and the matrix $\begin{bmatrix} y_1 & y_4 \\ y_4 & y_2 \end{bmatrix}$ is positive definite. A similar argument is employed in the case of X_2. So we see that

$$\begin{aligned} \operatorname{Tr}(XY) &= \operatorname{Tr}[(X_1 + X_2)Y] \\ &= \operatorname{Tr}(X_1Y + X_2Y) \\ &= \operatorname{Tr}(X_1Y) + \operatorname{Tr}(X_2Y) \\ &> 0. \end{aligned}$$

Now suppose that $x_2 \neq 0$ and $x_3 = 0$. Since again the second and third columns of X are linearly independent, we see that

$$\begin{bmatrix} x_1 \\ x_4 \\ x_5 \end{bmatrix} = c_1 \begin{bmatrix} x_4 \\ x_2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} x_5 \\ 0 \\ 0 \end{bmatrix}$$

for some $c_1, c_2 \in \mathbb{R}$, and we have

$$\begin{aligned} x_1 &= c_1x_4 \\ x_4 &= c_1x_2 \\ x_5 &= 0, \end{aligned}$$

and we see that X then becomes

$$X = \begin{bmatrix} c_1^2 x_2 & c_1 x_2 & 0 \\ c_1 x_2 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = X_1.$$

From the computation above, we know that $\mathrm{Tr}(X_1 Y) > 0$. A similar argument applies in the case of $x_2 = 0$ and $x_3 \neq 0$.

Lastly, consider the situation where $x_2 = 0$ and $x_3 = 0$. Then we see that

$$X = \begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & 0 & 0 \\ x_5 & 0 & 0 \end{bmatrix}.$$

The inequalities $x_1 x_2 - x_4^2 \geq 0$ and $x_1 x_3 - x_5^2 \geq 0$ imply that $x_4 = 0$ and $x_5 = 0$, so we see that

$$X = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and this situation was considered in Case 1. This completes this part of the proof.

Next, we show the other containment, $\Omega_7^* \subset \Omega_8$. That is, if $\langle X, Y \rangle > 0$ for all $X \in \bar{\Omega}_7 \setminus \{0\}$, then $Y \in \Omega_8$. Take

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly $X \in \bar{\Omega}_7$. The requirement $\langle X, Y \rangle > 0$ implies that

$$\mathrm{Tr}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{bmatrix} \right) > 0,$$

which implies that $y_1 > 0$. By a similar argument, we also have $y_2 > 0$ and $y_3 > 0$.

Now, take

$$X = \begin{bmatrix} x_1 & x_4 & 0 \\ x_4 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $x_1, x_2 \geq 0$ and $x_1 x_2 - x_4^2 \geq 0$. Again, $X \in \bar{\Omega}_7$. The requirement $\langle X, Y \rangle > 0$ implies that

$$\mathrm{Tr}\left(\begin{bmatrix} x_1 & x_4 & 0 \\ x_4 & x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & 0 \\ y_5 & 0 & y_3 \end{bmatrix} \right) > 0,$$

which implies that $x_1y_1 + x_2y_2 + 2x_4y_4 > 0$. That is,

$$\mathrm{Tr}\left(\begin{bmatrix} x_1 & x_4 \\ x_4 & x_2 \end{bmatrix}\begin{bmatrix} y_1 & y_4 \\ y_4 & y_2 \end{bmatrix}\right) > 0.$$

We now pose the following question: for symmetric 2×2 matrices X, Y, if $\mathrm{Tr}(XY) > 0$ for all positive semidefinite X, is Y necessarily positive definite? We answer this question in the affirmative with the following argument.

Since Y is symmetric, we can diagonalise Y with an orthogonal matrix A, so that $A^TYA = D$, where $D = \mathrm{diag}(d_1, d_2)$ is a diagonal matrix. Further, we choose X such that

$$A^TXA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the requirement $\langle X, Y\rangle > 0$ implies that

$$\mathrm{Tr}(XY) = \mathrm{Tr}(A^TXA \cdot A^TYA) = \mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\right) > 0,$$

which implies that $d_1 > 0$. Choosing X such that

$$A^TXA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

shows that $d_2 > 0$. Since d_1 and d_2 are precisely the eigenvalues of A^TYA and eigenvalues are preserved under a similarity transformation, we see that both eigenvalues of Y are positive, and Y is therefore positive definite. We see from the above discussion that Y positive definite implies that $y_1y_2 - y_4^2 > 0$. By a similar argument, we conclude that the inequality $y_1y_3 - y_5^2 > 0$ also holds. Since we have shown that $y_1 > 0$, $y_1y_2 - y_4^2 > 0$ and $y_1y_3 - y_5^2 > 0$, we see that $Y \in \Omega_8$ as required. □

We have shown that Ω_8 is the dual cone of Ω_7. By Proposition 2.2.5, the automorphism groups of Ω_7 and Ω_8 are isomorphic. In the sections below, we describe the form of the automorphism group of Ω_7 and compute its Lie algebra. By the same proposition, the dimension of the automorphism group of Ω_8 and its Lie algebra will be equal to the dimension of the automorphism group of Ω_7 and its Lie algebra.

3.4 The automorphism group of Ω_7

Let us again begin with the definition of Ω_7. We have

$$\Omega_7 := \big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 > 0,\ \ x_1x_2 - x_4^2 > 0, \\ x_1x_2x_3 - x_3x_4^2 - x_2x_5^2 > 0\big\}.$$

Let $S^n(\mathbb{R})$ denote the vector space of real symmetric $n \times n$ matrices. We have realised Ω_7 as a certain subset of the cone of positive definite symmetric 3×3 matrices, and we now use known results in the literature to compute its automorphism group. It is well known (see e.g. [8, pp. 8–10], [30, pp. 18–21]) that the space of real positive definite symmetric $n \times n$ matrices, denoted S^n_+, forms a homogeneous open convex cone. For $X \in S^n_+$, the connected identity component of its automorphism group

$$\mathrm{Aut}(S^n_+) = \left\{\mathcal{A} \in \mathrm{GL}(S^n(\mathbb{R})) : \mathcal{A}(S^n_+) = S^n_+\right\}$$

is given by transformations of the form $X \mapsto AXA^T$, where $A \in \mathrm{GL}_n(\mathbb{R})$ (see e.g. [35, pp. 14-15], [39, p. 75]).

Recall the subspace V of $S^3(\mathbb{R})$, given by

$$V = \left\{ X = \begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{bmatrix} : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\}.$$

Consider also the subset of V given by those matrices in V that are positive-definite, which we denote $\mathscr{C}$. That is, $\mathscr{C} = V \cap S^3_+ \subset S^3_+ \subset S^3(\mathbb{R})$. For $X \in \mathscr{C}$, since positive definiteness of a symmetric matrix is equivalent to the positivity of the determinant of each principal submatrix, we see that the positive definiteness of X necessitates that $x_1 > 0, x_1x_2 - x_4^2 > 0$, and $x_1x_2x_3 - x_3x_4^2 - x_2x_5^2 > 0$, and these conditions exactly describe the cone Ω_7. So we see that the cone Ω_7 is mapped bijectively onto $\mathscr{C}$ by the function $f : \Omega_7 \to \mathscr{C}$, which is given by

$$f(x_1, x_2, x_3, x_4, x_5) = \begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{bmatrix}.$$

Following [18], we now define the Lie group

$$H := \left\{ A = \begin{bmatrix} a & b & c \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix} : a, b, c, d, e \in \mathbb{R} \text{ with } a > 0, ei \neq 0 \right\}$$

and let H^+ be the subgroup consisting of matrices in H with positive diagonal entries. Then H^+ is the connected identity component of H. Let $\rho : H \to \mathrm{GL}(V)$ be the representation of H given by

$$\rho(A)X := AXA^T$$

where $A \in H$ and $X \in V$. It is straightforward to show that ρ is a faithful representation. Further, H and H^+ act transitively on the cone $\mathscr{C} \subset V$ by ρ. By a somewhat technical argument, the automorphism group of the cone $\mathscr{C}$, $G(\mathscr{C})$, is given by

$$G(\mathscr{C}) = \rho(H^+) \rtimes G(\mathscr{C})_{I_3}$$

where $G(\mathscr{C})_{I_3}$ is the finite isotropy subgroup of $I_3 \in \mathscr{C}$ (see [18, pp. 4–5]). Since H^+ is five-dimensional, we see that $G(\mathscr{C})$ is five-dimensional, and it is this result that is utilised in our classification.

We have the following commutative diagram:

$$\begin{array}{ccc} \Omega_7 & \xrightarrow{f} & \mathscr{C} \\ {\scriptstyle q\in \mathrm{Aut}\ \Omega_7}\downarrow & & \downarrow{\scriptstyle \rho(A)} \\ \Omega_7 & \xrightarrow{f} & \mathscr{C} \end{array}$$

Since f is a bijection, we see that for any action $\rho(A)$ that preserves the cone $\mathscr{C}$, we have a corresponding $q \in \mathrm{Aut}\ \Omega_7$ given by $q(x) = (f^{-1} \circ \rho(A) \circ f)(x)$ that preserves the cone Ω_7. Let us compute the form of these automorphisms for the matrix

$$A = \begin{bmatrix} a & b & c \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}.$$

For $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, we see that

$$\begin{aligned}
q(x) &= (f^{-1} \circ C_A \circ f)(x_1, x_2, x_3, x_4, x_5) \\
&= (f^{-1} \circ C_A)\left(\begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{bmatrix}\right) \\
&= f^{-1}\left(\begin{bmatrix} a & b & c \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}\begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ b & e & 0 \\ c & 0 & i \end{bmatrix}\right) \\
&= f^{-1}\left(\begin{bmatrix} a^2x_1 + b^2x_2 + c^2x_3 + 2abx_4 + 2acx_5 & aex_4 + bex_2 & aix_5 + cix_3 \\ aex_4 + bex_2 & e^2x_2 & 0 \\ aix_5 + cix_3 & 0 & i^2x_3 \end{bmatrix}\right) \\
&= \begin{pmatrix} a^2x_1 + b^2x_2 + c^2x_3 + 2abx_4 + 2acx_5 \\ e^2x_2 \\ i^2x_3 \\ aex_4 + bex_2 \\ aix_5 + cix_3 \end{pmatrix},
\end{aligned}$$

and so we see the corresponding mappings $q \in \text{Aut } \Omega_7$ are given by linear transformations of the form

$$q = \begin{bmatrix} a^2 & b^2 & c^2 & 2ab & 2ac \\ 0 & e^2 & 0 & 0 & 0 \\ 0 & 0 & i^2 & 0 & 0 \\ 0 & be & 0 & ae & 0 \\ 0 & 0 & ci & 0 & ai \end{bmatrix}, \text{ for } a > 0, ei \neq 0 \text{ and } b, c \in \mathbb{R}.$$

We now see the form of the connected identity component of the automorphism group of the cone Ω_7. Considering Ω_7 as a subset of $\mathbb{R}^5$, this identity component is given by 5×5 real matrices q of the above form.

3.5 Computing the Lie algebra of the automorphism group of Ω_7

To begin with, $\text{Aut}(S^n_+)$ is a closed subgroup of $\text{GL}(S^n(\mathbb{R}))$, and hence a Lie subgroup with Lie algebra $\mathfrak{aut}(S^n_+) \subset \mathfrak{gl}(S^n(\mathbb{R}))$. Consider the surjective Lie group homomorphism given by $\phi : \text{GL}_n(\mathbb{R}) \to \text{Aut}(S^n_+)$ given by $A \mapsto \rho(A)$. Surjectivity of this map follows from [35, pp. 14-15] We now consider the differential of the above map.

Proposition 3.5.1. *Consider the map $\phi : \text{GL}_n(\mathbb{R}) \to \text{Aut}(S^n_+)$ given by $A \mapsto \rho(A)$, where $\rho(A)X = AXA^T$. Its differential is $\phi_* : \mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{aut}(S^n_+)$ given by $U \mapsto \phi_*(U)$, where $\phi_*(U)(X) = UX + XU^T$ for $X \in S^n(\mathbb{R})$.*

Proof. Let $A'(0)$ for A as above be denoted U. Observe that

$$\begin{aligned} d\phi_I(U) &= \lim_{t\to 0} \frac{\phi(I + tU) - \phi(I)}{t} \\ &= \lim_{t\to 0} \frac{(I + tU)X(I + tU)^T - IXI}{t} \\ &= \lim_{t\to 0} \frac{X + tUX + tXU^T + t^2UXU^T - X}{t} \\ &= UX + XU^T. \end{aligned}$$

After noting that the Lie algebra of $S^n_+(\mathbb{R})$ is $S^n(\mathbb{R})$, the result follows. □

Now let $\mathscr{C} \subset S^n_+(\mathbb{R})$ be as above. Then

$$\text{Aut}(\mathscr{C}) = \left\{ \mathcal{A} \in \text{Aut}(S^n_+(\mathbb{R})) : \mathcal{A}\mathscr{C} = \mathscr{C} \right\} \subset \text{Aut}(S^n_+(\mathbb{R})).$$

We denote the Lie algebra of $\mathrm{Aut}(\mathscr{C})$ by $\mathfrak{aut}(\mathscr{C}) \subset \mathfrak{aut}(S^n_+(\mathbb{R}))$. Let $G := \phi^{-1}(\mathrm{Aut}(\mathscr{C}))$ be the preimage of $\mathrm{Aut}(\mathscr{C})$ under ϕ, and $\mathfrak{g} \in \mathfrak{gl}_n(\mathbb{R})$ be the Lie algebra of G. Then for the Lie group homomorphism $\phi|_G : G \to \mathrm{Aut}(\mathscr{C})$, we have

$$(\phi|_G)_* = \phi_*|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{aut}(\mathscr{C}),$$

and the Lie algebra $\mathfrak{g} = (\phi_*)^{-1}(\mathfrak{aut}(\mathscr{C}))$ is given by

$$\mathfrak{g} = \left\{ U \in \mathfrak{gl}_n(\mathbb{R}) : UX + XU^T \in T_I\mathscr{C} \text{ for all } X \in T_I\mathscr{C} \right\},$$

where $T_I\mathscr{C}$ denotes the tangent space at the identity.

We now use this fact to compute $\mathfrak{aut}(\mathscr{C})$. The elements of $\mathfrak{aut}(\mathscr{C})$ are given by transformations of the form $X \mapsto UX + XU^T$, where $X \in S^n(\mathbb{R})$ and $U \in M_{n\times n}(\mathbb{R})$. We have

$$\begin{aligned} AX + XA^T &= \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{bmatrix} + \begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \delta & \epsilon \\ \delta & \beta & \zeta \\ \epsilon & \zeta & \gamma \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} \alpha &= 2ax_1 + 2bx_4 + 2cx_5, \\ \beta &= 2ex_2 + 2dx_4, \\ \gamma &= 2ix_3 + 2gx_5, \\ \delta &= dx_1 + bx_2 + (a+e)x_4 + fx_5, \\ \epsilon &= gx_1 + cx_3 + hx_4 + (a+i)x_5, \\ \zeta &= hx_2 + fx_3 + gx_4 + dx_5. \end{aligned}$$

We see that this matrix is symmetric, as expected, and so lies in the Lie algebra of S^3_+. Since it also must lie in the Lie algebra of $\mathscr{C}$, we see that we must have $\zeta = hx_2 + fx_3 + gx_4 + dx_5 = 0$. From this condition, we see that $d = f = g = h = 0$, which gives

$$A = \begin{bmatrix} a & b & c \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}, \text{ for } a, e, g, h, i \in \mathbb{R},$$

which accords with the form of the automorphism group of $\mathscr{C}$, determined above. Since the Lie algebra is given by the conjugation action of a matrix A with five non-zero entries on X, we conclude that $\dim \mathfrak{aut}(\mathscr{C}) = 5$.

Since Ω_7 is mapped bijectively to $\mathscr{C}$, we see that the dimension of the Lie algebra of automorphism group of Ω_7 is 5. Since the automorphism groups of Ω_7 and Ω_8 are isomorphic, we see as well that the dimension of the Lie algebra of the automorphism group of Ω_8 is 5.

Chapter 4

The $d(M) = n^2 - 8$ case

We begin this chapter by stating the main theorem describing the contribution to the classification of homogeneous hyperbolic n-dimensional manifolds with automorphism group dimension $n^2 - 8$. We will prove that, up to biholomorphism, there are four such manifolds.

Main Theorem 2. *Let M be a homogeneous n-dimensional Kobayashi-hyperbolic manifold with $d(M) = n^2 - 8$. Then one of the following holds:*

(i) *$n = 5$ and M is biholomorphic to $B^1 \times B^1 \times B^1 \times B^2$.*

(ii) *$n = 6$ and M is biholomorphic to the tube domain*

$$T_6 = \big\{(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6 : (\operatorname{Im} z_1)^2 - (\operatorname{Im} z_2)^2 - (\operatorname{Im} z_3)^2 - (\operatorname{Im} z_4)^2 - (\operatorname{Im} z_5)^2 - (\operatorname{Im} z_6)^2 > 0, \operatorname{Im} z_1 > 0\big\}.$$

(iii) *$n = 7$ and M is biholomorphic to $B^1 \times B^1 \times B^5$.*

(iv) *$n = 8$ and M is biholomorphic to $B^2 \times B^6$.*

In this chapter, we prove the above theorem. Let M be a homogeneous Kobayashi-hyperbolic manifold of dimension n. By Theorem 2.1.4, the manifold M is biholomorphic to an affinely homogeneous Siegel domain of the second kind $S(\Omega, H)$. Recall that

$$S(\Omega, H) := \big\{(z, w) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : \operatorname{Im} z - H(w, w) \in \Omega\big\},$$

where $1 \leq k \leq n$, $\Omega \subset \mathbb{R}^k$ is an open convex cone and H is an Ω-Hermitian form on $\mathbb{C}^{n-k}$. Since all homogeneous Kobayashi-hyperbolic manifolds of dimensions 2 and 3 have been classified (see [15, Theorem 2.6]) and none have automorphism group dimension $n^2 - 8$, we take $n \geq 4$. Further, we recall from the remarks after Definition 2.3.2 in the previous chapter that if $k = 1$ then $S(\Omega, H)$ is biholomorphic to B^n, so we assume that $k \geq 2$.

4.1 A useful lemma

We can use the following lemma to rule out a large number of remaining possibilities.

Lemma 4.1.1. *For the following values of n and k, we cannot have $d(S(\Omega, H)) = n^2 - 8$:*

(i) $n \geq 7, k \geq 4$,
(ii) $n \geq 8, k = 3$,
(iii) $n = 6, k = 4$,
(iv) $n = 6, k = 5$.

Proof. To prove the lemma, we will show that for $n \geq 7, k \geq 4$, as well as for $n \geq 8, k = 3$ and the two cases $n = 6, k = 4$ and $n = 6, k = 5$, the right-hand side of the inequality

$$d(S(\Omega, H)) \leq \frac{3k^2}{2} - \left(2n + \frac{5}{2}\right) k + n^2 + 4n + 1$$

is strictly less that $n^2 - 8$. That is, for these k, n the following holds:

$$\frac{3k^2}{2} - \left(2n + \frac{5}{2}\right) k + 4n + 9 < 0. \tag{4.1.1}$$

To see this, consider the quadratic function

$$\varphi(t) := \frac{3t^2}{2} - \left(2n + \frac{5}{2}\right) t + 4n + 9.$$

The discriminant of φ is given by

$$\mathcal{D} := 4n^2 - 14n - \frac{191}{4},$$

which is positive for $n \geq 6$. The zeroes of φ are given by

$$t_1 := \frac{2n + \frac{5}{2} - \sqrt{\mathcal{D}}}{3},$$

$$t_2 := \frac{2n + \frac{5}{2} + \sqrt{\mathcal{D}}}{3}.$$

To prove the lemma, it suffices to show that: (i) $t_2 > n$ for $n \geq 7$, (ii) $t_1 < 4$ for $n \geq 7$, (iii) $t_1 < 3$ for $n \geq 8$, and lastly (iv) each pair $n = 6, k = 4$

and $n = 6, k = 5$ satisfies (4.1.1). Beginning with the inequality $t_2 > n$, we have

$$n - \frac{5}{2} < \sqrt{\mathcal{D}},$$

or, equivalently, that

$$n^2 - 3n - 18 > 0,$$

which clearly holds for $n \geq 7$. Now considering $t_1 < 4$, we see that

$$2n - \frac{19}{2} < \sqrt{\mathcal{D}},$$

or, equivalently, that

$$n > \frac{23}{4},$$

which holds for $n \geq 7$. Lastly, the inequality $t_1 < 3$ implies that

$$2n - \frac{13}{2} < \sqrt{\mathcal{D}},$$

or, equivalently, that

$$n > \frac{15}{2},$$

which holds for $n \geq 8$.

Finally, the pairs $n = 6, k = 4$ and $n = 6, k = 5$ clearly satisfy (4.1.1). $\square$

4.2 Proof of the main theorem

By the above lemma, we can prove the theorem by considering the following nine cases:

1. $k = 2, n \geq 4$.
2. $k = 3, n = 4$.
3. $k = 3, n = 5$.
4. $k = 3, n = 6$.
5. $k = 3, n = 7$.
6. $k = 4, n = 4$.
7. $k = 4, n = 5$.
8. $k = 5, n = 5$.
9. $k = 6, n = 6$.

We now begin by considering each case.

Case 1. Suppose that $k = 2, n \geq 4$. Recall from the previous chapter that $H = (H_1, H_2)$ is a pair of Hermitian forms on $\mathbb{C}^{n-2}$, where we may take

H_1 to be positive definite. They are simultaneously diagonalised as

$$H_1(w,w) = ||w||^2, \quad H_2(w,w) = \sum_{j=1}^{n-2} \lambda_j |w_j|^2.$$

Recall further that H_2 has at least one pair of distinct eigenvalues, and that $m \geq 1$ denotes the number of pairs of these eigenvalues. Lastly, recall that s denotes the dimension of the real subspace $\mathcal{L}$, given by

$$\mathcal{L} = \{B \in \mathfrak{gl}_{n-k}(\mathbb{C}) : H(B\cdot,\cdot) + H(\cdot, B\cdot) = 0\}.$$

As $\dim \mathfrak{g}(\Omega) = 2$, inequality (2.3.7) yields

$$s \geq n^2 - 4n - 6. \tag{4.2.1}$$

On the other hand, by inequality (2.3.8),

$$s \leq n^2 - 4n + 4. \tag{4.2.2}$$

By Lemma (3.2.1) the exact value of s is given by

$$s = n^2 - 4n + 4 - 2m,$$

which implies $m = 1, 2, 3, 4$ or 5. The values $m = 1, 2, 3, 4$ are treated as in the previous chapter, and contribute no additional domains. However, the possibility of $m = 5$ leads to two additional subcases: (g) where $n = 6$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 = \lambda_4$ where $\lambda_1 \neq \lambda_3$, and (h) where $n = 8$ with $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$. When $k = 2$ and $n \geq 4$, $S(\Omega, H)$ is biholomorphic to a product of two unit balls $B^l \times B^{n-l}$ for $1 \leq l \leq n-1$, and the dimension of its automorphism group is given by

$$d(B^l \times B^{n-l}) = 2l^2 - 2nl + n^2 + 2n.$$

Setting the right-hand side equal to $n^2 - 8$, we see that l is integer-valued only in the case of $n = 8$. In this case, $l = 2$, and so Case 1 contributes the product $B^2 \times B^6$ to the classification, with $d(B^2 \times B^6) = 8 + 48 = 56 = n^2 - 8$.

Case 2. Suppose that $k = 3, n = 4$. Then $S(\Omega, H)$ is equivalent to either

$$D_3 := \left\{(z,w) \in \mathbb{C}^3 \times \mathbb{C} : \text{Im } z - v|w|^2 \in \Omega_2\right\},$$

where $v = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$ with non-negative entries, or

$$D_4 := \left\{(z,w) \in \mathbb{C}^3 \times \mathbb{C} : \text{Im } z - v|w|^2 \in \Omega_3\right\},$$

where $v = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$ satisfying $v_1^2 \geq v_2^2 + v_3^2$, $v_1 > 0$. As in the previous chapter, we begin by assuming that $S(\Omega, H)$ is equivalent to the domain D_3. Then $S(\Omega, H)$ can only be biholomorphic to the product $B^1 \times B^1 \times B^2$. This cannot occur, since $d(B^1 \times B^1 \times B^2) = 14 > 8 = n^2 - 8$.

Therefore, assume $S(\Omega, H)$ is equivalent to D_4. Recall from the previous chapter that if $v \in \Omega_3$, then the vector v is an eigenvector of every element of $G(\Omega_3, v|w|^2)$, from which it follows that $G(\Omega_3, v|w|^2)$ does not act transitively on Ω_3. Therefore, assume that $v \in \partial\Omega_3 \setminus \{0\}$ and recall from the analysis of the $k = 3, n = 4$ case in the previous chapter that in this situation we have $\dim \mathfrak{g}_0 = 4$. In addition (see Appendix A), if $v \in \partial\Omega_3 \setminus \{0\}$ we have $\dim \mathfrak{g}_0 = 0$ and $\dim \mathfrak{g}_1 = 1$. So we see

$$d(D_4) = \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-1/2} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1 = 10.$$

Since $d(D_4) = 10 > 8 = n^2 - 8$, we see that $S(\Omega, H)$ is not equivalent to D_4, and so Case 2 contributes nothing to our classification.

Case 3. Suppose that $k = 3, n = 5$. Here, $S(\Omega, H)$ is linearly equivalent either to

$$D_5 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^2 : \operatorname{Im} z - \mathcal{H}(w, w) \in \Omega_2\right\},$$

where $\mathcal{H}$ is an Ω_2-Hermitian form, or to

$$D_6 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^2 : \operatorname{Im} z - \mathcal{H}(w, w) \in \Omega_3\right\},$$

where $\mathcal{H}$ is an Ω_3-Hermitian form. Consideration of the domain D_5 does not aid our classification since $S(\Omega, H)$ must be biholomorphic to a five-dimensional product of three unit balls, and the only possibilities are $B^1 \times B^1 \times B^3$ and $B^1 \times B^2 \times B^2$. Since neither has automorphism group dimension $17 = n^2 - 8$, we assume then that $S(\Omega, H)$ is equivalent to the domain D_6.

By Lemma 3.2.1 we have either $s = 1$, $s = 2$ or $s = 4$. In [16], each of these scenarios was dealt with in Sections 5, 4 and 3 respectively. When $s = 4$ we have $d(D_6) = 15 < 17 = n^2 - 8$, and when $s = 2$ the action of $G(\Omega_3, \mathcal{H})$ on Ω_3 is not transitive. So consider the situation when $s = 1$. In [16, Lemma 5.1] it was shown that for the domain D_6 with $s = 1$ and $\mathfrak{g} = \mathfrak{g}(D_6)$ we have $\dim \mathfrak{g}_{1/2} \leq 2$. We now prove a stronger result.

Lemma 4.2.1. *For the domain D_6 with $s = 1$ and $\mathfrak{g} = \mathfrak{g}(D_6)$ we have* $\dim \mathfrak{g}_{1/2} = 0$.

Proof. Let us write the Ω_3-Hermitian form $\mathcal{H}$ as

$$\mathcal{H} = u|w_1|^2 + v|w_2|^2 + a\bar{w}_1 w_2 + \bar{a}\,\bar{w}_2 w_1,$$

where $u, v \in \mathbb{R}^3$ and $a \in \mathbb{C}^3$. Choosing $w_1 = 0$ and $w_2 = 0$ shows that $u, v \in \bar{\Omega}_3 \setminus \{0\}$. We will consider two cases.

Case (i). Suppose first that $u \in \Omega_3$. Then, as the cone Ω_3 is homogeneous, we may assume that $u = (1, 0, 0)$. Further, replacing w_1 by $w_1 + a_1 w_2$, we may suppose that $a_1 = 0$. The above steps allow us to reduce $\mathcal{H}$ to the form

$$\mathcal{H} = \begin{bmatrix} |w_1|^2 + v_1 |w_2|^2 \\ v_2 |w_2|^2 + a_2\, \bar{w}_1 w_2 + \bar{a}_2\, \bar{w}_2 w_1 \\ v_3 |w_2|^2 + a_3\, \bar{w}_1 w_2 + \bar{a}_3\, \bar{w}_2 w_1 \end{bmatrix}.$$

Remark 4.2.2. In [16], Isaev further reduced the Hermitian form above by rotating the variables z_2, z_3 by a transformation from O_2, and thus assumed that $\mathcal{H}_3$ has no $|w_2|^2$-term, that is, $v_3 = 0$. We refrain from taking this step and assume the variable v_3 is not necessarily zero.

To utilise Theorem 2.3.4, let $\Phi : \mathbb{C}^3 \to \mathbb{C}^2$ be a $\mathbb{C}$-linear map

$$\Phi(z_1, z_2, z_3) = \left(\varphi_1^1 z_1 + \varphi_2^1 z_2 + \varphi_3^1 z_3, \varphi_1^2 z_1 + \varphi_2^2 z_2 + \varphi_3^2 z_3\right), \tag{4.2.3}$$

where $\varphi_i^j \in \mathbb{C}$. Fixing $\mathbf{w} \in \mathbb{C}^2$, for $x \in \mathbb{R}^3$ we compute

$$\begin{aligned} \mathcal{H}(\mathbf{w}, \Phi(x)) = \big(&\bar{\mathbf{w}}_1(\varphi_1^1 x_1 + \varphi_2^1 x_2 + \varphi_3^1 x_3) + v_1 \bar{\mathbf{w}}_2(\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3), \\ &v_2 \bar{\mathbf{w}}_2(\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3) + a_2 \bar{\mathbf{w}}_1(\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3) \\ &+ \bar{a}_2 \bar{\mathbf{w}}_2(\varphi_1^1 x_1 + \varphi_2^1 x_2 + \varphi_3^1 x_3), v_3 \bar{\mathbf{w}}_2(\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3) \\ &+ a_3 \bar{\mathbf{w}}_1(\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3) + \bar{a}_3 \bar{\mathbf{w}}_2(\varphi_1^1 x_1 + \varphi_2^1 x_2 + \varphi_3^1 x_3)\big) \\ = \big(&(\varphi_1^1 \bar{\mathbf{w}}_1 + v_1 \varphi_1^2 \bar{\mathbf{w}}_2) x_1 + (\varphi_2^1 \bar{\mathbf{w}}_1 + v_1 \varphi_2^2 \bar{\mathbf{w}}_2) x_2 \\ &+ (\varphi_3^1 \bar{\mathbf{w}}_1 + v_1 \varphi_3^2 \bar{\mathbf{w}}_2) x_3, (a_2 \varphi_1^2 \bar{\mathbf{w}}_1 + (\bar{a}_2 \varphi_1^1 + v_2 \varphi_1^2) \bar{\mathbf{w}}_2) x_1 \\ &+ (a_2 \varphi_2^2 \bar{\mathbf{w}}_1 + (\bar{a}_2 \varphi_2^1 + v_2 \varphi_2^2) \bar{\mathbf{w}}_2) x_2 + (a_2 \varphi_3^2 \bar{\mathbf{w}}_1 \\ &+ (\bar{a}_2 \varphi_3^1 + v_2 \varphi_3^2) \bar{\mathbf{w}}_2) x_3, (a_3 \varphi_1^2 \bar{\mathbf{w}}_1 + (\bar{a}_3 \varphi_1^1 + v_3 \varphi_1^2) \bar{\mathbf{w}}_2) x_1 \\ &+ (a_3 \varphi_2^2 \bar{\mathbf{w}}_1 + (\bar{a}_3 \varphi_2^1 + v_3 \varphi_2^2) \bar{\mathbf{w}}_2) x_2 + (a_3 \varphi_3^2 \bar{\mathbf{w}}_1 \\ &+ (\bar{a}_3 \varphi_3^1 + v_3 \varphi_3^2) \bar{\mathbf{w}}_2) x_3\big). \end{aligned}$$

Then from formula (2.3.11) we see

$$\begin{aligned}
\Phi_{\mathbf{w}}(x) = \big(&(\mathrm{Im}(\varphi_1^1\bar{\mathbf{w}}_1) + v_1\,\mathrm{Im}(\varphi_1^2\bar{\mathbf{w}}_2))x_1 + (\mathrm{Im}(\varphi_2^1\bar{\mathbf{w}}_1) + v_1\,\mathrm{Im}(\varphi_2^2\bar{\mathbf{w}}_2))x_2 \\
&+ (\mathrm{Im}(\varphi_3^1\bar{\mathbf{w}}_1) + v_1\,\mathrm{Im}(\varphi_3^2\bar{\mathbf{w}}_2))x_3, (\mathrm{Im}(a_2\varphi_1^2\bar{\mathbf{w}}_1) \\
&+ \mathrm{Im}((\bar{a}_2\varphi_1^1 + v_2\varphi_1^2)\bar{\mathbf{w}}_2))x_1 + (\mathrm{Im}(a_2\varphi_2^2\bar{\mathbf{w}}_1) \\
&+ \mathrm{Im}((\bar{a}_2\varphi_2^1 + v_2\varphi_2^2)\bar{\mathbf{w}}_2))x_2 + (\mathrm{Im}(a_2\varphi_3^2\bar{\mathbf{w}}_1) \\
&+ \mathrm{Im}((\bar{a}_2\varphi_3^1 + v_2\varphi_3^2)\bar{\mathbf{w}}_2))x_3, (\mathrm{Im}(a_3\varphi_1^2\bar{\mathbf{w}}_1) + \mathrm{Im}((\bar{a}_3\varphi_1^1 + v_3\varphi_1^2)\bar{\mathbf{w}}_2))x_1 \\
&+ (\mathrm{Im}(a_3\varphi_2^2\bar{\mathbf{w}}_1) + \mathrm{Im}((\bar{a}_3\varphi_2^1 + v_3\varphi_2^2)\bar{\mathbf{w}}_2))x_2 + (\mathrm{Im}(a_3\varphi_3^2\bar{\mathbf{w}}_1) \\
&\quad + \mathrm{Im}((\bar{a}_3\varphi_3^1 + v_3\varphi_3^2)\bar{\mathbf{w}}_2))x_3\big).
\end{aligned}$$

Using (A.3.3), we then see that the condition that $\Phi_{\mathbf{w}}$ lies in $\mathfrak{g}(\Omega_3)$ for every $\mathbf{w} \in \mathbb{C}^2$ leads to the relations

$$\begin{aligned}
\varphi_1^1 &= a_2\varphi_2^2 = a_3\varphi_3^2 \\
v_1\varphi_1^2 &= \bar{a}_2\varphi_2^1 + v_2\varphi_2^2 = \bar{a}_3\varphi_3^1 + v_3\varphi_3^2 \\
\varphi_2^1 &= a_2\varphi_1^2 \\
v_1\varphi_2^2 &= \bar{a}_2\varphi_1^1 + v_2\varphi_1^2 \\
\varphi_3^1 &= a_3\varphi_1^2 \\
\bar{a}_3\varphi_1^1 + v_3\varphi_1^2 &= v_1\varphi_3^2 \\
a_3\varphi_2^2 &= -a_2\varphi_3^2 \\
\bar{a}_3\varphi_2^1 + v_3\varphi_2^2 &= -\bar{a}_2\varphi_3^1 - v_2\varphi_3^2.
\end{aligned}$$

If $a_2 = 0$, it immediately follows that $\Phi = 0$. Similarly, if $a_3 = 0$, it also immediately follows that $\Phi = 0$. If both $a_2 = 0$ and $a_3 = 0$, a short row echelon computation shows that $\Phi = 0$. Thus by formula (2.3.12) we have $\mathfrak{g}_{1/2} = 0$. Suppose then that $a_2 \neq 0$ and $a_3 \neq 0$. By scaling w_2, we can assume $a_3 = 1$. Then it follows that all $\varphi_i^j = 0$ unless $v_1 = 1, v_2 = 0, v_3 = 0$ and $a_2 = \pm i$. We provide a brief sketch of the argument used to show this, which amounts to a standard row reduction of a large matrix. Writing the ten equations given above in matrix form, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -a_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -\bar{a}_2 & 0 & v_1 & -v_2 & 0 \\ 0 & 0 & -1 & v_1 & 0 & -v_3 \\ 0 & 1 & 0 & -a_2 & 0 & 0 \\ -\bar{a}_2 & 0 & 0 & -v_2 & v_1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -v_3 & 0 & v_1 \\ 0 & 0 & 0 & 0 & 1 & a_2 \\ 0 & 1 & \bar{a}_2 & 0 & v_3 & v_2 \end{bmatrix} \begin{bmatrix} \varphi_1^1 \\ \varphi_2^1 \\ \varphi_3^1 \\ \varphi_1^2 \\ \varphi_2^2 \\ \varphi_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We proceed to row reduce this matrix. After securing pivots in the first three columns, we can focus on the remaining 7×3 matrix. We begin the reduction of this matrix by assuming $v_1 \neq 1$. After securing pivots in the first two columns, by then varying the values of a_2 we can always get a pivot in the third column. We then assume $v_1 = 1$. We see at this stage that if v_3 is non-zero, we get a pivot in every column, and therefore assume $v_3 = 0$. Continuing in this fashion, we eventually see that only when assuming $v_1 = 1, v_2 = 0, v_3 = 0$ and $a_2 = \pm i$ does the matrix fail to be full rank. Therefore, in situations other than this we have $\varphi_i^j = 0$ for all i, j. Then $\Phi = 0$, and by formula (2.3.12) we have $\mathfrak{g}_{1/2} = 0$.

We thus assume that $v_1 = 1, v_2 = 0, v_3 = 0$ and $a_2 = \pm i$. In this situation, the Hermitian form $\mathcal{H}$ is given by

$$\mathcal{H} = (|w_1|^2 + |w_2|^2, \pm i(\bar{w}_1 w_2 - \bar{w}_2 w_1), \bar{w}_1 w_2 + \bar{w}_2 w_1).$$

Changing the w-variables as

$$w_1 \mapsto -\frac{i}{\sqrt{2}}(w_1 + i w_2), \; w_2 \mapsto \frac{1}{\sqrt{2}}(w_1 - i w_2),$$

we can suppose that

$$\mathcal{H} = (|w_1|^2 + |w_2|^2, \mp(|w_1|^2 - |w_2|^2), \bar{w}_1 w_2 + \bar{w}_2 w_1).$$

Further, swapping w_1 and w_2 if necessary, we reduce our considerations to the case where

$$\mathcal{H} = (|w_1|^2 + |w_2|^2, |w_1|^2 - |w_2|^2, \bar{w}_1 w_2 + \bar{w}_2 w_1). \tag{4.2.4}$$

We will now show that for the above Ω_3-Hermitian form $\mathcal{H}$ we have $\mathfrak{g}_{1/2} = 0$. Consider a map $\Phi : \mathbb{C}^3 \to \mathbb{C}^2$ as in (4.2.3), fix $\mathbf{w} \in \mathbb{C}^2$, and for $x \in \mathbb{R}^3$

compute

$$\begin{aligned}\mathcal{H}(\mathbf{w},\Phi(x)) = \big(&\bar{\mathbf{w}}_1(\varphi_1^1x_1+\varphi_2^1x_2+\varphi_3^1x_3)+\bar{\mathbf{w}}_2(\varphi_1^2x_1+\varphi_2^2x_2+\varphi_3^2x_3),\\ &\bar{\mathbf{w}}_1(\varphi_1^1x_1+\varphi_2^1x_2+\varphi_3^1x_3)-\bar{\mathbf{w}}_2(\varphi_1^2x_1+\varphi_2^2x_2+\varphi_3^2x_3),\\ &\bar{\mathbf{w}}_1(\varphi_1^2x_1+\varphi_2^2x_2+\varphi_3^2x_3)+\bar{\mathbf{w}}_2(\varphi_1^1x_1+\varphi_2^1x_2+\varphi_3^1x_3)\big)\\ = \big(&(\varphi_1^1\bar{\mathbf{w}}_1+\varphi_1^2\bar{\mathbf{w}}_2)x_1+(\varphi_2^1\bar{\mathbf{w}}_1+\varphi_2^2\bar{\mathbf{w}}_2)x_2+(\varphi_3^1\bar{\mathbf{w}}_1+\varphi_3^2\bar{\mathbf{w}}_2)x_3,\\ &(\varphi_1^1\bar{\mathbf{w}}_1-\varphi_1^2\bar{\mathbf{w}}_2)x_1+(\varphi_2^1\bar{\mathbf{w}}_1-\varphi_2^2\bar{\mathbf{w}}_2)x_2+(\varphi_3^1\bar{\mathbf{w}}_1-\varphi_3^2\bar{\mathbf{w}}_2)x_3,\\ &(\varphi_1^2\bar{\mathbf{w}}_1+\varphi_1^1\bar{\mathbf{w}}_2)x_1+(\varphi_2^2\bar{\mathbf{w}}_1+\varphi_2^1\bar{\mathbf{w}}_2)x_2+(\varphi_3^2\bar{\mathbf{w}}_1+\varphi_3^1\bar{\mathbf{w}}_2)x_3\big).\end{aligned}$$

Then from formula (2.3.11) we see

$$\begin{aligned}\Phi_{\mathbf{w}}(x) = \big(&(\mathrm{Im}(\varphi_1^1\bar{\mathbf{w}}_1)+\mathrm{Im}(\varphi_1^2\bar{\mathbf{w}}_2))x_1+(\mathrm{Im}(\varphi_2^1\bar{\mathbf{w}}_1)+\mathrm{Im}(\varphi_2^2\bar{\mathbf{w}}_2))x_2\\ &+(\mathrm{Im}(\varphi_3^1\bar{\mathbf{w}}_1)+\mathrm{Im}(\varphi_3^2\bar{\mathbf{w}}_2))x_3, (\mathrm{Im}(\varphi_1^1\bar{\mathbf{w}}_1)-\mathrm{Im}(\varphi_1^2\bar{\mathbf{w}}_2))x_1\\ &+(\mathrm{Im}(\varphi_2^1\bar{\mathbf{w}}_1)-\mathrm{Im}(\varphi_2^2\bar{\mathbf{w}}_2))x_2+(\mathrm{Im}(\varphi_3^1\bar{\mathbf{w}}_1)-\mathrm{Im}(\varphi_3^2\bar{\mathbf{w}}_2))x_3,\\ &(\mathrm{Im}(\varphi_1^2\bar{\mathbf{w}}_1)+\mathrm{Im}(\varphi_1^1\bar{\mathbf{w}}_2))x_1+(\mathrm{Im}(\varphi_2^2\bar{\mathbf{w}}_1)+\mathrm{Im}(\varphi_2^1\bar{\mathbf{w}}_2))x_2\\ &+(\mathrm{Im}(\varphi_3^2\bar{\mathbf{w}}_1)+\mathrm{Im}(\varphi_3^1\bar{\mathbf{w}}_2))x_3)\,.\end{aligned}$$

From (A.3.3) we then see that the condition that $\Phi_{\mathbf{w}}$ lies in $\mathfrak{g}(\Omega_3)$ for every $\mathbf{w}\in\mathbb{C}^2$ leads to the relations

$$\varphi_1^1=\varphi_2^1=\varphi_3^2,\ \varphi_1^2=-\varphi_2^2=\varphi_3^1, \tag{4.2.5}$$

Further, let c be a symmetric $\mathbb{C}$-bilinear form on $\mathbb{C}^2$ with values in $\mathbb{C}^2$:

$$c(w,w)=\left(c_{11}^1w_1^2+2c_{12}^1w_1w_2+c_{22}^1w_2^2, c_{11}^2w_1^2+2c_{12}^2w_1w_2+c_{22}^2w_2^2\right),$$

where $c_{ij}^\ell\in\mathbb{C}$. Then for $w,w'\in\mathbb{C}^2$ using (4.2.4) we calculate

$$\begin{aligned}\mathcal{H}(w,c(w',w')) = \big(&\bar{w}_1(c_{11}^1(w_1')^2+2c_{12}^1w_1'w_2'+c_{22}^1(w_2')^2)+\bar{w}_2(c_{11}^2(w_1')^2\\ &+2c_{12}^2w_1'w_2'+c_{22}^2(w_2')^2), \bar{w}_1(c_{11}^1(w_1')^2+2c_{12}^1w_1'w_2'\\ &+c_{22}^1(w_2')^2)-w_2(c_{11}^2(w_1')^2+2c_{12}^2w_1'w_2'+c_{22}^2(w_2')^2),\\ &\bar{w}_1(c_{11}^2(w_1')^2+2c_{12}^2w_1'w_2'+c_{22}^2(w_2')^2)\\ &+\bar{w}_2(c_{11}^1(w_1')^2+2c_{12}^1w_1'w_2'+c_{22}^1(w_2')^2)\big).\end{aligned} \tag{4.2.6}$$

On the other hand, we have

$$\begin{aligned}\Phi(\mathcal{H}(w',w)) &= \big(\varphi_1^1(\bar w_1' w_1 + \bar w_2' w_2) + \varphi_2^1(\bar w_1' w_1 - \bar w_2' w_2) + \varphi_3^1(\bar w_1' w_2 + \bar w_2' w_1),\\ &\qquad \varphi_1^2(\bar w_1' w_1 + \bar w_2' w_2) + \varphi_2^2(\bar w_1' w_1 - \bar w_2' w_2) + \varphi_3^2(\bar w_1' w_2 + \bar w_2' w_1)\big)\\ &= \big((\varphi_1^1 + \varphi_2^1)\bar w_1' w_1 + (\varphi_1^1 - \varphi_2^1)\bar w_2' w_2 + \varphi_3^1(\bar w_1' w_2 + \bar w_2' w_1),\\ &\qquad (\varphi_1^2 + \varphi_2^2)\bar w_1' w_1 + (\varphi_1^2 - \varphi_2^2)\bar w_2' w_2 + \varphi_3^2(\bar w_1' w_2 + \bar w_2' w_1)\big)\end{aligned}$$

Therefore

$$\begin{aligned}&2i\mathcal{H}(\Phi(\mathcal{H}(w',w)),w') =\\ &\quad 2i\,(w_1'\,((\bar\varphi_1^1 + \bar\varphi_2^1)w_1'\bar w_1 + (\bar\varphi_1^1 - \bar\varphi_2^1)w_2'\bar w_2 + \bar\varphi_3^1(w_1'\bar w_2 + w_2'\bar w_1))\\ &\qquad +w_2'\,((\bar\varphi_1^2 + \bar\varphi_2^2)w_1'\bar w_1 + (\bar\varphi_1^2 - \bar\varphi_2^2)w_2'\bar w_2 + \bar\varphi_3^2(w_1'\bar w_2 + w_2'\bar w_1))\,,\\ &\qquad w_1'\,((\bar\varphi_1^1 + \bar\varphi_2^1)w_1'\bar w_1 + (\bar\varphi_1^1 - \bar\varphi_2^1)w_2'\bar w_2 + \bar\varphi_3^1(w_1'\bar w_2 + w_2'\bar w_1)) \qquad (4.2.7)\\ &\qquad -w_2'\,((\bar\varphi_1^2 + \bar\varphi_2^2)w_1'\bar w_1 + (\bar\varphi_1^2 - \bar\varphi_2^2)w_2'\bar w_2 + \bar\varphi_3^2(w_1'\bar w_2 + w_2'\bar w_1))\,,\\ &\qquad w_1'\,((\bar\varphi_1^2 + \bar\varphi_2^2)w_1'\bar w_1 + (\bar\varphi_1^2 - \bar\varphi_2^2)w_2'\bar w_2 + \bar\varphi_3^2(w_1'\bar w_2 + w_2'\bar w_1))\\ &\qquad +w_2'\,((\bar\varphi_1^1 + \bar\varphi_2^1)w_1'\bar w_1 + (\bar\varphi_1^1 - \bar\varphi_2^1)w_2'\bar w_2 + \bar\varphi_3^1(w_1'\bar w_2 + w_2'\bar w_1)))\,.\end{aligned}$$

Let us now compare expressions (4.2.6) and (4.2.7) as required by condition (2.3.12). Specifically, looking at the coefficients of $(w_2')^2\bar w_1$ and $(w_1')^2\bar w_2$ in the first and second components of these expressions, we obtain the identities:

$$c_{22}^1 = 2i\bar\varphi_3^2, \qquad c_{22}^1 = -2i\bar\varphi_3^2, \qquad c_{11}^2 = 2i\bar\varphi_3^1, \qquad -c_{11}^2 = 2i\bar\varphi_3^1,$$

which imply $\varphi_3^1 = 0$, $\varphi_3^2 = 0$. Taken together with (4.2.5), these conditions yield $\Phi = 0$, hence $\mathfrak{g}_{1/2} = 0$ as required.

Case (ii). Suppose now that $u \in \partial\Omega_3\backslash\{0\}$. In this situation, as the group $G(\Omega_3)^\circ = \mathbb{R}_+ \times \mathrm{SO}_{1,2}^\circ$ acts transitively on $\partial\Omega_3 \setminus \{0\}$, we may assume that $u = (1,1,0)$. Further, replacing w_1 by $w_1 + a_1 w_2$, we may suppose that $a_1 = 0$. The above steps allow us to reduce $\mathcal{H}$ to the form

$$\mathcal{H} = \begin{bmatrix} |w_1|^2 + v_1|w_2|^2 \\ |w_1|^2 + v_2|w_2|^2 + a_2\bar w_1 w_2 + \bar a_2\,\bar w_2 w_1 \\ v_3|w_2|^2 + a_3\bar w_1 w_2 + \bar a_3\,\bar w_2 w_1 \end{bmatrix}.$$

Let $\Phi : \mathbb{C}^3 \to \mathbb{C}^2$ be a $\mathbb{C}$-linear map as in (4.2.3). Fixing $\mathbf{w} \in \mathbb{C}^2$, for

$x \in \mathbb{R}^3$ we compute

$$\begin{aligned}
\mathcal{H}(\mathbf{w}, \Phi(x)) = \big(&\bar{\mathbf{w}}_1(\varphi_1^1 x_1 + \varphi_2^1 x_2 + \varphi_3^1 x_3) + v_1 \bar{\mathbf{w}}_2(\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3),\\
&\quad \bar{\mathbf{w}}_1(\varphi_1^1 x_1 + \varphi_2^1 x_2 + \varphi_3^1 x_3) + v_2 \bar{\mathbf{w}}_2(\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3)\\
&+ a_2 \bar{\mathbf{w}}_1(\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3) + \bar{a}_2 \bar{\mathbf{w}}_2(\varphi_1^1 x_1 + \varphi_2^1 x_2 + \varphi_3^1 x_3),\\
&\quad (\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3) + a_3 \bar{\mathbf{w}}_1(\varphi_1^2 x_1 + \varphi_2^2 x_2 + \varphi_3^2 x_3)\\
&+ \bar{a}_3 \bar{\mathbf{w}}_2(\varphi_1^1 x_1 + \varphi_2^1 x_2 + \varphi_3^1 x_3)\big)\\
= \big(&(\varphi_1^1 \bar{\mathbf{w}}_1 + v_1 \varphi_1^2 \bar{\mathbf{w}}_2) x_1 + (\varphi_2^1 \bar{\mathbf{w}}_1 + v_1 \varphi_2^2 \bar{\mathbf{w}}_2) x_2\\
&+ (\varphi_3^1 \bar{\mathbf{w}}_1 + v_1 \varphi_3^2 \bar{\mathbf{w}}_2) x_3, ((\varphi_1^1 + a_2 \varphi_1^2) \bar{\mathbf{w}}_1 + (\bar{a}_2 \varphi_1^1 + v_2 \varphi_1^2) \bar{\mathbf{w}}_2) x_1\\
&+ ((\varphi_2^1 + a_2 \varphi_2^2) \bar{\mathbf{w}}_1 + (\bar{a}_2 \varphi_2^1 + v_2 \varphi_2^2) \bar{\mathbf{w}}_2) x_2 + ((\varphi_3^1 + a_2 \varphi_3^2) \bar{\mathbf{w}}_1\\
&+ (\bar{a}_2 \varphi_3^1 + v_2 \varphi_3^2) \bar{\mathbf{w}}_2) x_3, (a_3 \varphi_1^2 \bar{\mathbf{w}}_1 + (\bar{a}_3 \varphi_1^1 + v_3 \varphi_1^2) \bar{\mathbf{w}}_2) x_1\\
&+ (a_3 \varphi_2^2 \bar{\mathbf{w}}_1 + (\bar{a}_3 \varphi_2^1 + v_3 \varphi_2^2) \bar{\mathbf{w}}_2) x_2 + (a_3 \varphi_3^2 \bar{\mathbf{w}}_1\\
&\qquad + (\bar{a}_3 \varphi_3^1 + v_3 \varphi_3^2) \bar{\mathbf{w}}_2) x_3\big)
\end{aligned}$$

Then from formula (2.3.11) we see

$$\begin{aligned}
\Phi_{\mathbf{w}}(x) = \big(&(\operatorname{Im}(\varphi_1^1 \bar{\mathbf{w}}_1) + v_1 \operatorname{Im}(\varphi_1^2 \bar{\mathbf{w}}_2)) x_1 + (\operatorname{Im}(\varphi_2^1 \bar{\mathbf{w}}_1) + v_1 \operatorname{Im}(\varphi_2^2 \bar{\mathbf{w}}_2)) x_2\\
&+ (\operatorname{Im}(\varphi_3^1 \bar{\mathbf{w}}_1) + v_1 \operatorname{Im}(\varphi_3^2 \bar{\mathbf{w}}_2)) x_3, (\operatorname{Im}((\varphi_1^1 + a_2 \varphi_1^2) \bar{\mathbf{w}}_1)\\
&+ \operatorname{Im}((\bar{a}_2 \varphi_1^1 + v_2 \varphi_1^2) \bar{\mathbf{w}}_2)) x_1 + (\operatorname{Im}((\varphi_2^1 + a_2 \varphi_2^2) \bar{\mathbf{w}}_1)\\
&+ \operatorname{Im}((\bar{a}_2 \varphi_2^1 + v_2 \varphi_2^2) \bar{\mathbf{w}}_2)) x_2 + (\operatorname{Im}((\varphi_3^1 + a_2 \varphi_3^2) \bar{\mathbf{w}}_1)\\
&+ \operatorname{Im}((\bar{a}_2 \varphi_3^1 + v_2 \varphi_3^2) \bar{\mathbf{w}}_2)) x_3, (\operatorname{Im}(a_3 \varphi_1^2 \bar{\mathbf{w}}_1) + \operatorname{Im}((\bar{a}_3 \varphi_1^1 + v_3 \varphi_1^2) \bar{\mathbf{w}}_2)) x_1\\
&+ (\operatorname{Im}(a_3 \varphi_2^2 \bar{\mathbf{w}}_1) + \operatorname{Im}((\bar{a}_3 \varphi_2^1 + v_3 \varphi_2^2) \bar{\mathbf{w}}_2)) x_2 + (\operatorname{Im}(a_3 \varphi_3^2 \bar{\mathbf{w}}_1)\\
&\qquad + \operatorname{Im}((\bar{a}_3 \varphi_3^1 + v_3 \varphi_3^2) \bar{\mathbf{w}}_2)) x_3\big).
\end{aligned}$$

Using (A.3.3), we then see that the condition that $\Phi_{\mathbf{w}}$ lies in $\mathfrak{g}(\Omega_3)$ for every $\mathbf{w} \in \mathbb{C}^2$ leads to the relations

$$\begin{aligned}
\varphi_1^1 &= \varphi_2^1 + a_2\varphi_2^2 = a_3\varphi_3^2 \\
v_1\varphi_1^2 &= \bar{a}_2\varphi_2^1 + v_2\varphi_2^2 = \bar{a}_3\varphi_3^1 + v_3\varphi_3^2 \\
\varphi_2^1 &= \varphi_1^1 + a_2\varphi_1^2 \\
v_1\varphi_2^2 &= \bar{a}_2\varphi_1^1 + v_2\varphi_1^2 \\
\varphi_3^1 &= a_3\varphi_1^2 \\
\bar{a}_3\varphi_1^1 + v_3\varphi_1^2 &= v_1\varphi_3^2 \\
a_3\varphi_2^2 &= -\varphi_3^1 - a_2\varphi_3^2 \\
\bar{a}_3\varphi_2^1 + v_3\varphi_2^2 &= -\bar{a}_2\varphi_3^1 - v_2\varphi_3^2.
\end{aligned}$$

It easily follows that if $a_3 = 0$, then $\Phi = 0$, so by formula (2.3.12) we have $\mathfrak{g}_{1/2} = 0$. If $a_3 \neq 0$, then, by scaling w_2, we can assume that $a_3 = 1$. Similarly to case (i), we consider the above ten equations in matrix form and row reduce. We have

$$\begin{bmatrix}
1 & -1 & 0 & 0 & -a_2 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 \\
0 & -\bar{a}_2 & 0 & v_1 & -v_2 & 0 \\
0 & 0 & -1 & v_1 & 0 & -v_3 \\
-1 & 1 & 0 & -a_2 & 0 & 0 \\
-\bar{a}_2 & 0 & 0 & -v_2 & v_1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & v_3 & 0 & -v_1 \\
0 & 0 & 1 & 0 & 1 & a_2 \\
0 & 1 & \bar{a}_2 & 0 & v_3 & v_2
\end{bmatrix}
\begin{bmatrix}
\varphi_1^1 \\ \varphi_2^1 \\ \varphi_3^1 \\ \varphi_1^2 \\ \varphi_2^2 \\ \varphi_3^2
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}.$$

After easily securing pivots in the first four columns, we can focus on the resulting 6×2 matrix. We begin the row reduction of this matrix by assuming $a_2 \neq 0$. Then by varying the values of v_1, v_2 and v_3 we see that this matrix is always full rank. Therefore, assume $a_2 = 0$. By again varying the values of v_1, v_2 and v_3 we find the only situation in which the matrix is not full rank is when $a_2 = 0, v_1 = 1, v_2 = -1$, and $v_3 = 0$. Therefore, in situations other than this we have $\varphi_i^j = 0$ for all i, j. Then $\Phi = 0$, and by formula (2.3.12) we have $\mathfrak{g}_{1/2} = 0$. Finally, notice that for the above values of v_1, v_2, v_3, a_2 the form $\mathcal{H}$ coincides with the right-hand side of (4.2.4), for which we have already shown that $\mathfrak{g}_{1/2} = 0$. □

Now, Lemma 4.2.1 together with (2.3.6) and the second inequality in (2.3.4) yields $d(D_6) \leq 15 < 17 = n^2 - 8$. Thus, we have shown that Case (3) makes no contributions to the classification.

Case 4. Suppose that $k = 3, n = 6$. Here, $S(\Omega, H)$ is linearly equivalent either to

$$D_7 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^3 : \mathrm{Im}\ z - \mathcal{H}(w, w) \in \Omega_2\right\},$$

where $\mathcal{H}$ is an Ω_2-Hermitian form, or to

$$D_8 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^3 : \mathrm{Im}\ z - \mathcal{H}(w, w) \in \Omega_3\right\},$$

where $\mathcal{H}$ is an Ω_3-Hermitian form.

Assume $S(\Omega, H)$ is equivalent to D_7. Then $S(\Omega, H)$ must be biholomorphic to a product of three unit balls. The only possibilities are $B^1 \times B^1 \times B^4$, $B^1 \times B^2 \times B^3$ or $B^2 \times B^2 \times B^2$, none of which have automorphism group of dimension $n^2 - 8 = 28$.

So $S(\Omega, H)$ must be equivalent to D_8. By (2.3.7) we have $s + \dim \mathfrak{g}(\Omega) \geq 10$. Since $\dim \mathfrak{g}(\Omega_3) = 4$, we see that $s \geq 6$. On the other hand, by (2.3.8) we have $s \leq 9$. Now recall from the corresponding case in the previous chapter that we must have $s = 1, 2, 3, 5$ or 9. Therefore we see that $s = 9$ and the argument proceeds in the same way, showing that $S(\Omega, H)$ cannot be equivalent to D_8 and no contribution to the classification is made.

Case 5. Suppose that $k = 3, n = 7$. Here, $S(\Omega, H)$ is linearly equivalent either to

$$D_9 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^4 : \mathrm{Im}\ z - \mathcal{H}(w, w) \in \Omega_2\right\},$$

where $\mathcal{H}$ is an Ω_2-Hermitian form, or to

$$D_{10} := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^4 : \mathrm{Im}\ z - \mathcal{H}(w, w) \in \Omega_3\right\},$$

where $\mathcal{H}$ is an Ω_3-Hermitian form. Inequality (2.3.7) implies $s + \dim \mathfrak{g}(\Omega) \geq 19$, and so we must consider the possibility that $s = 16$ and Ω is linearly equivalent to Ω_2. Then $S(\Omega, H)$ is equivalent to the domain D_9, which yields the product $B^1 \times B^1 \times B^5$, since $d(B^1 \times B^1 \times B^5) = 3+3+35 = 41 = n^2 - 8$. It follows from the analysis of the $k = 3, n = 7$ case in the previous chapter that if Ω is linearly equivalent to Ω_3, then Case 5 provides no further contributions to the classification.

Case 6. Suppose that $k = 4, n = 4$. In this case, after a linear change of variables, $S(\Omega, H)$ is one of the domains

$$\left\{z \in \mathbb{C}^4 : \mathrm{Im}\ z \in \Omega_4\right\},$$

$$\left\{z \in \mathbb{C}^4 : \mathrm{Im}\ z \in \Omega_5\right\},$$

$$\left\{z \in \mathbb{C}^4 : \operatorname{Im} z \in \Omega_6\right\},$$

and therefore is biholomorphic either to $B^1 \times B^1 \times B^1 \times B^1$, or to $B^1 \times T_3$, or to T_4, as in the case of automorphism group dimension $d(M) = n^2 - 7$. The dimensions of the respective automorphism groups of these domains are 12, 13 and 15. Each of these numbers is greater than $8 = n^2 - 8$, and so we see that Case 6 contributes nothing to our classification.

Case 7. Suppose that $k = 4, n = 5$. Then $S(\Omega, H)$ is linearly equivalent to either

$$D_{11} := \left\{(z, w) \in \mathbb{C}^4 \times \mathbb{C} : \operatorname{Im} z - v|w|^2 \in \Omega_4\right\},$$

where $v = (v_1, v_2, v_3, v_4)$ is a vector in $\mathbb{R}^4$ with non-negative entries, or

$$D_{12} := \left\{(z, w) \in \mathbb{C}^4 \times \mathbb{C} : \operatorname{Im} z - v|w|^2 \in \Omega_5\right\},$$

where $v = (v_1, v_2, v_3, v_4)$ is a vector in $\mathbb{R}^4$ satisfying $v \in \bar{\Omega}_5 \setminus \{0\}$, or

$$D_{13} := \left\{(z, w) \in \mathbb{C}^4 \times \mathbb{C} : \operatorname{Im} z - v|w|^2 \in \Omega_6\right\},$$

where $v = (v_1, v_2, v_3, v_4)$ is a vector in $\mathbb{R}^4$ satisfying $v \in \bar{\Omega}_6 \setminus \{0\}$, i.e., $v_1^2 \geq v_2^2 + v_3^2 + v_4^2, v_1 > 0$.

Since $s = 1$, by inequality (2.3.7) we see that $\dim \mathfrak{g}(\Omega) \geq 4$. Therefore, in contrast to the corresponding case in the previous chapter, we must also consider the domain D_{11}. Let us begin with this possibility, and assume $S(\Omega, H)$ is equivalent to D_{11}. Then $S(\Omega, H)$ is biholomorphic to the product of unit balls given by $B^1 \times B^1 \times B^1 \times B^2$, since $d(B^1 \times B^1 \times B^1 \times B^2) = 3 + 3 + 3 + 8 = 17 = n^2 - 8$.

Next, assume that $S(\Omega, H)$ is equivalent to D_{12}. As in the analysis of the same case in the previous chapter, by Theorem (2.3.4) we see that in the cases of the boundary components C_1, C_2 and C_3, for $\mathfrak{g} = \mathfrak{g}(D_{12})$ we have $\mathfrak{g}_{1/2} = 0$. Then by estimate (2.3.6), the second inequality in (2.3.4) and Lemmas 3.2.4, 3.2.5 and 3.2.6 we see that in each of these cases

$$d(D_{12}) \leq 16 < 18 = n^2 - 7$$

(recall that $s = 1$). This shows that in the cases of these components, $S(\Omega, H)$ cannot be equivalent to D_{12}, so no new contributions are made to our classification.

Lastly, assume that $S(\Omega, H)$ is equivalent to D_{13}. Recall from the analysis of the same case in the previous chapter that for $\mathfrak{g} = \mathfrak{g}(D_{13})$ we have $\dim \mathfrak{g}_0 = 6$. Then using the second inequality in (2.3.4) and Proposition A.5.1 for $N = 1$, we see that

$$d(D_{13}) = \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-1/2} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{1/2} + \dim \mathfrak{g}_1 \leq 16 < 17 = n^2 - 8,$$

showing no contribution to the classification. Therefore, the product $B^1 \times B^1 \times B^1 \times B^2$ is the only contribution made to the classification by Case 7.

Case 8. Suppose that $k = 5$ and $n = 5$. Then by inequality (2.3.7) we see that in this situation we have $\dim \mathfrak{g}(\Omega) \geq 7$. Therefore, after a linear change of variables $S(\Omega, H)$ turns into one of the domains

$$\left\{z \in \mathbb{C}^5 : \operatorname{Im} z \in \Omega_{11}\right\},$$

$$\left\{z \in \mathbb{C}^5 : \operatorname{Im} z \in \Omega_{12}\right\}$$

and therefore is biholomorphic either to $B^1 \times T_4$, or to T_5, as in the case of automorphism group dimension $d(M) = n^2 - 7$. The dimensions of the respective automorphism groups of these domains are 18 and 21. Each of these numbers is greater than $17 = n^2 - 8$, and so we see that Case 8 makes no contribution to the classification.

Case 9. Suppose that $k = 6, n = 6$. Consider the following lemma from [16], which we state without proof.

Lemma 4.2.3. *If for $k \geq 3$ we set*

$$K := \frac{(k-2)(k-3)}{2} + k + 1,$$

then the inequality $\dim \mathfrak{g}(\Omega) \geq K$ implies that Ω is linearly equivalent to Λ_k.

In this case, inequality (2.3.7) implies that $\dim \mathfrak{g} \geq 16 > 13 = K$, and so by the above lemma Ω is linearly equivalent to the Lorentz cone Λ_6. Therefore, after a linear change of variables, $S(\Omega, H)$ turns into the domain

$$\left\{z \in \mathbb{C}^6 : \operatorname{Im} z \in \Lambda_6\right\},$$

which is the tube domain T_6, where

$$T_6 = \big\{(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6 : (\operatorname{Im} z_1)^2 - (\operatorname{Im} z_2)^2 - (\operatorname{Im} z_3)^2 \\ - (\operatorname{Im} z_4)^2 - (\operatorname{Im} z_5)^2 - (\operatorname{Im} z_6)^2 > 0, \operatorname{Im} z_1 > 0\big\}.$$

Note that $d(T_6) = 28 = n^2 - 8$, so Case 9 contributes T_6 to the classification of homogeneous Kobayashi-hyperbolic manifolds with automorphism group dimension $n^2 - 8$.

Appendix A

A.1 Determination of $\dim \mathfrak{g}_{1/2}$ for $\mathfrak{g} = \mathfrak{g}(D_4)$ when $v \in \partial\Omega_3 \setminus \{0\}$

For the convenience of the reader, we reproduce in this appendix Isaev's results [15, Lemma 3.8 and Proposition A.3] and [16, Lemmas 3.4, 3.5 and 4.3] and his proofs as they appear in these papers.

Consider the domain

$$D_4 := \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C} : \mathrm{Im}\ z - v|w|^2 \in \Omega_3\right\},$$

where $v = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$ satisfying $v_1^2 \geq v_2^2 + v_3^2$, $v_1 > 0$. We have the following proposition.

Proposition A.1.1. *If $v \in \partial\Omega_3 \setminus \{0\}$, for $\mathfrak{g} = \mathfrak{g}(D_4)$ we have $\mathfrak{g}_{1/2} = 0$.*

Proof. As the group $G(\Omega_3)^\circ = \mathbb{R}_+ \times \mathrm{SO}^\circ_{1,2}$ acts transitively on $\partial\Omega_3 \setminus \{0\}$, we suppose that $v = (1, 1, 0)$. We will apply Theorem 2.3.4 to the cone Ω_3 and the Ω_3-Hermitian form

$$\mathcal{H}(w, w') := (\bar{w}w', \bar{w}w', 0). \tag{A.1.1}$$

Let $\Phi : \mathbb{C}^3 \to \mathbb{C}$ be a $\mathbb{C}$-linear map:

$$\Phi(z_1, z_2, z_3) = \varphi_1 z_1 + \varphi_2 z_2 + \varphi_3 z_3,$$

where $\varphi_j \in \mathbb{C}$. Fixing $\mathbf{w} \in \mathbb{C}$, for $x \in \mathbb{R}^3$ we compute

$$\mathcal{H}(\mathbf{w}, \Phi(x)) = (\bar{\mathbf{w}}(\varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3), \bar{\mathbf{w}}(\varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3), 0).$$

Then from formula (2.3.11) we see

$$\Phi_{\mathbf{w}}(x) = (\mathrm{Im}(\bar{\mathbf{w}}\varphi_1)x_1 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_2)x_2 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_3)x_3,$$
$$\mathrm{Im}(\bar{\mathbf{w}}\varphi_1)x_1 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_2)x_2 + \mathrm{Im}(\bar{\mathbf{w}}\varphi_3)x_3, 0).$$

Now, $\mathfrak{g}(\Omega_3) = \mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{o}_{1,2}$ consists of all matrices of the form

$$\begin{pmatrix} \lambda & p & q \\ p & \lambda & r \\ q & -r & \lambda \end{pmatrix}, \quad \lambda, p, q, r \in \mathbb{R}. \tag{A.1.2}$$

Therefore, the condition that the map $\Phi_{\mathbf{w}}$ lies in $\mathfrak{g}(\Omega_3)$ for every $\mathbf{w} \in \mathbb{C}$ immediately yields

$$\mathrm{Im}(\bar{\mathbf{w}}\varphi_1) \equiv 0, \ \mathrm{Im}(\bar{\mathbf{w}}\varphi_2) \equiv 0, \ \mathrm{Im}(\bar{\mathbf{w}}\varphi_3) \equiv 0,$$

which implies $\Phi = 0$. Hence, by formula (2.3.12) we have $\mathfrak{g}_{1/2} = 0$ as required. $\square$

A.2 Determination of $\dim \mathfrak{g}_1$ for $\mathfrak{g} = \mathfrak{g}(D_4)$ when $v \in \partial\Omega_3 \setminus \{0\}$

Proposition A.2.1. *If $v \in \partial\Omega_3 \setminus \{0\}$, for $\mathfrak{g} = \mathfrak{g}(D_4)$ we have* $\dim \mathfrak{g}_1 = 1$.

Proof. As in the proof of Proposition A.1.1, we assume that $v = (1, 1, 0)$. We will utilise Theorem 2.3.5 for the cone Ω_3 and the Ω_3-Hermitian form $\mathcal{H}$ defined in (A.1.1).

Let $b : \mathbb{C}^3 \times \mathbb{C} \to \mathbb{C}$ be a $\mathbb{C}$-bilinear map:

$$b(z, w) = (b_1 z_1 + b_2 z_2 + b_3 z_3)w,$$

where $b_j \in \mathbb{C}$. For every fixed pair $\mathbf{w}, \mathbf{w}' \in \mathbb{C}$ we compute

$$\mathcal{H}(\mathbf{w}', b(x, \mathbf{w})) = (\bar{\mathbf{w}}'\mathbf{w}(b_1x_1 + b_2x_2 + b_3x_3), \bar{\mathbf{w}}'\mathbf{w}(b_1x_1 + b_2x_2 + b_3x_3), 0),$$

with $x \in \mathbb{R}^3$. Then from (ii) of Theorem 2.3.5 we obtain

$$\begin{aligned} B_{\mathbf{w},\mathbf{w}'}(x) = (&\mathrm{Im}(b_1\bar{\mathbf{w}}'\mathbf{w})x_1 + \mathrm{Im}(b_2\bar{\mathbf{w}}'\mathbf{w})x_2 + \mathrm{Im}(b_3\bar{\mathbf{w}}'\mathbf{w})x_3, \\ &\mathrm{Im}(b_1\bar{\mathbf{w}}'\mathbf{w})x_1 + \mathrm{Im}(b_2\bar{\mathbf{w}}'\mathbf{w})x_2 + \mathrm{Im}(b_3\bar{\mathbf{w}}'\mathbf{w})x_3, 0). \end{aligned}$$

By (A.1.2), the condition that this map lies in $\mathfrak{g}(\Omega_3)$ for all $\mathbf{w}, \mathbf{w}' \in \mathbb{C}$ immediately yields

$$\mathrm{Im}(b_1\bar{\mathbf{w}}'\mathbf{w}) \equiv 0, \quad \mathrm{Im}(b_2\bar{\mathbf{w}}'\mathbf{w}) \equiv 0, \quad \mathrm{Im}(b_3\bar{\mathbf{w}}'\mathbf{w}) \equiv 0,$$

hence $b = 0$.

Next, consider a symmetric $\mathbb{R}$-bilinear form on $\mathbb{R}^3$ with values in $\mathbb{R}^3$:

$$\begin{aligned} a(x,x) = (&a^1_{11}x_1^2 + 2a^1_{12}x_1x_2 + 2a^1_{13}x_1x_3 + a^1_{22}x_2^2 + 2a^1_{23}x_2x_3 + a^1_{33}x_3^2,\\ &a^2_{11}x_1^2 + 2a^2_{12}x_1x_2 + 2a^2_{13}x_1x_3 + a^2_{22}x_2^2 + 2a^2_{23}x_2x_3 + a^2_{33}x_3^2,\\ &a^3_{11}x_1^2 + 2a^3_{12}x_1x_2 + 2a^3_{13}x_1x_3 + a^3_{22}x_2^2 + 2a^3_{23}x_2x_3 + a^3_{33}x_3^2)\,, \end{aligned}$$

where $a^k_{ij} \in \mathbb{R}$. Then for a fixed $\mathbf{x} \in \mathbb{R}^3$ from (2.3.13) we compute

$$\begin{aligned} A_{\mathbf{x}}(x) = (&a^1_{11}\mathbf{x}_1x_1 + a^1_{12}\mathbf{x}_1x_2 + a^1_{12}\mathbf{x}_2x_1 + a^1_{13}\mathbf{x}_1x_3 + a^1_{13}\mathbf{x}_3x_1 + a^1_{22}\mathbf{x}_2x_2\\ &+ a^1_{23}\mathbf{x}_2x_3 + a^1_{23}\mathbf{x}_3x_2 + a^1_{33}\mathbf{x}_3x_3, a^2_{11}\mathbf{x}_1x_1 + a^2_{12}\mathbf{x}_1x_2 + a^2_{12}\mathbf{x}_2x_1 + a^2_{13}\mathbf{x}_1x_3\\ &+ a^2_{13}\mathbf{x}_3x_1 + a^2_{22}\mathbf{x}_2x_2 + a^2_{23}\mathbf{x}_2x_3 + a^2_{23}\mathbf{x}_3x_2 + a^2_{33}\mathbf{x}_3x_3, a^3_{11}\mathbf{x}_1x_1 + a^3_{12}\mathbf{x}_1x_2\\ &+ a^3_{12}\mathbf{x}_2x_1 + a^3_{13}\mathbf{x}_1x_3 + a^3_{13}\mathbf{x}_3x_1 + a^3_{22}\mathbf{x}_2x_2 + a^3_{23}\mathbf{x}_2x_3 + a^3_{23}\mathbf{x}_3x_2 + a^3_{33}\mathbf{x}_3x_3)\\ =\ &((a^1_{11}\mathbf{x}_1 + a^1_{12}\mathbf{x}_2 + a^1_{13}\mathbf{x}_3)x_1 + (a^1_{12}\mathbf{x}_1 + a^1_{22}\mathbf{x}_2 + a^1_{23}\mathbf{x}_3)x_2 + (a^1_{13}\mathbf{x}_1\\ &+a^1_{23}\mathbf{x}_2 + a^1_{33}\mathbf{x}_3)x_3, (a^2_{11}\mathbf{x}_1 + a^2_{12}\mathbf{x}_2 + a^2_{13}\mathbf{x}_3)x_1 + (a^2_{12}\mathbf{x}_1 + a^2_{22}\mathbf{x}_2 + a^2_{23}\mathbf{x}_3)x_2\\ &+(a^2_{13}\mathbf{x}_1 + a^2_{23}\mathbf{x}_2 + a^2_{33}\mathbf{x}_3)x_3, (a^3_{11}\mathbf{x}_1 + a^3_{12}\mathbf{x}_2 + a^3_{13}\mathbf{x}_3)x_1 + (a^3_{12}\mathbf{x}_1 + a^3_{22}\mathbf{x}_2\\ &+a^3_{23}\mathbf{x}_3)x_2 + (a^3_{13}\mathbf{x}_1 + a^3_{23}\mathbf{x}_2 + a^3_{33}\mathbf{x}_3)x_3)\,, \end{aligned}$$

where $x \in \mathbb{R}^3$. By (A.1.2), the condition that this map lies in $\mathfrak{g}(\Omega_3)$ for every $\mathbf{x} \in \mathbb{R}^3$ is equivalent to

$$\begin{aligned} &a^1_{11}\mathbf{x}_1 + a^1_{12}\mathbf{x}_2 + a^1_{13}\mathbf{x}_3 \equiv a^2_{12}\mathbf{x}_1 + a^2_{22}\mathbf{x}_2 + a^2_{23}\mathbf{x}_3 \equiv\\ &\qquad\qquad a^3_{13}\mathbf{x}_1 + a^3_{23}\mathbf{x}_2 + a^3_{33}\mathbf{x}_3,\\ &a^1_{12}\mathbf{x}_1 + a^1_{22}\mathbf{x}_2 + a^1_{23}\mathbf{x}_3 \equiv a^2_{11}\mathbf{x}_1 + a^2_{12}\mathbf{x}_2 + a^2_{13}\mathbf{x}_3,\\ &a^1_{13}\mathbf{x}_1 + a^1_{23}\mathbf{x}_2 + a^1_{33}\mathbf{x}_3 \equiv a^3_{11}\mathbf{x}_1 + a^3_{12}\mathbf{x}_2 + a^3_{13}\mathbf{x}_3,\\ &a^2_{13}\mathbf{x}_1 + a^2_{23}\mathbf{x}_2 + a^2_{33}\mathbf{x}_3 \equiv -(a^3_{12}\mathbf{x}_1 + a^3_{22}\mathbf{x}_2 + a^3_{23}\mathbf{x}_3). \end{aligned} \tag{A.2.1}$$

Further, recalling that any map $b : \mathbb{C}^3 \times \mathbb{C} \to$ as above is zero, we will utilise the condition that the zero matrix is associated to $A_{\mathbf{x}}$ for every $\mathbf{x} \in \mathbb{R}^3$ as in (i) of Theorem 2.3.5. This condition means

$$\begin{aligned} &a^1_{12}\mathbf{x}_1 + a^1_{22}\mathbf{x}_2 + a^1_{23}\mathbf{x}_3 \equiv -(a^1_{11}\mathbf{x}_1 + a^1_{12}\mathbf{x}_2 + a^1_{13}\mathbf{x}_3),\\ &a^2_{13}\mathbf{x}_1 + a^2_{23}\mathbf{x}_2 + a^2_{33}\mathbf{x}_3 \equiv a^3_{11}\mathbf{x}_1 + a^3_{12}\mathbf{x}_2 + a^3_{13}\mathbf{x}_3. \end{aligned} \tag{A.2.2}$$

Combining identities (A.2.1) and (A.2.2), we obtain the following relations for the coefficients of the form a:

$$\begin{aligned} &a^1_{11} = a^2_{12} = a^3_{13},\ a^1_{12} = a^2_{22} = a^3_{23},\ a^1_{13} = a^2_{23} = a^3_{33},\ a^1_{12} = a^2_{11},\ a^1_{22} = a^2_{12},\\ &a^1_{23} = a^2_{13}, a^1_{13} = a^3_{11},\ a^1_{23} = a^3_{12},\ a^1_{33} = a^3_{13},\ a^2_{13} = -a^3_{12},\ a^2_{23} = -a^3_{22},\\ &a^2_{33} = -a^3_{23}, a^1_{12} = -a^1_{11}, a^1_{22} = -a^1_{12},\ a^1_{23} = -a^1_{13},\ a^2_{13} = a^3_{11},\ a^2_{23} = a^3_{12},\\ &a^2_{33} = a^3_{13}. \end{aligned}$$

By the above relations, each coefficient of a either is zero or is equal to $\pm a^1_{11}$ as follows:

$$a^1_{12} = -a^1_{11},\ a^1_{13} = 0,\ a^1_{22} = a^1_{11},\ a^1_{23} = 0,\ a^1_{33} = a^1_{11},\ a^2_{11} = -a^1_{11}$$
$$a^2_{12} = a^1_{11},\ a^2_{13} = 0,\ a^2_{22} = -a^1_{11},\ a^2_{23} = 0,\ a^2_{33} = a^1_{11},$$
$$a^3_{11} = 0,\ a^3_{12} = 0,\ a^3_{13} = a^1_{11},\ a^3_{22} = 0,\ a^3_{23} = -a^1_{11},\ a^3_{33} = 0.$$

Therefore

$$a(x,x) = a^1_{11}\left((x_1 - x_2)^2 + x_3^2, -(x_1 - x_2)^2 + x_3^2, 2(x_1 - x_2)x_3\right).$$

This shows that $\dim \mathfrak{g}_1 = 1$ as required. □

A.3 Determination of $\dim \mathfrak{g}_{1/2}$ for $\mathfrak{g} = \mathfrak{g}(\widetilde{D}_6)$

Let $N \geq 1$, and $\widetilde{\mathcal{H}}$ be an Ω_3-Hermitian form on $\mathbb{C}^N$ defined as

$$\widetilde{\mathcal{H}}(w, w') := \left(\sum_{j=1}^{N} \bar{w}_j w'_j, \sum_{j=1}^{N} \bar{w}_j w'_j, 0\right). \tag{A.3.1}$$

Consider the domain

$$\widetilde{D}_6 := \left\{(z, w) \in \times\mathbb{C}^3 \times \mathbb{C}^N : \operatorname{Im} z - \widetilde{\mathcal{H}}(w, w) \in \Omega_3\right\}. \tag{A.3.2}$$

We have the following proposition.

Proposition A.3.1. *For $\mathfrak{g} = \mathfrak{g}(\widetilde{D}_6)$ we have $\mathfrak{g}_{1/2} = 0$.*

Proof. We will apply Theorem 2.3.4 to the cone Ω_3 and the Ω_3-Hermitian form $\widetilde{\mathcal{H}}$. Let $\Phi : \mathbb{C}^3 \to \mathbb{C}^N$ be a $\mathbb{C}$-linear map given by a matrix (φ^j_i), with $j = 1, \ldots, N$, $i = 1, 2, 3$. Fixing $\mathbf{w} \in \mathbb{C}^N$, for $x \in \mathbb{R}^3$ we compute

$$\widetilde{\mathcal{H}}(\mathbf{w}, \Phi(x)) = \left(\sum_{j=1}^{N} \bar{\mathbf{w}}_j(\varphi^j_1 x_1 + \varphi^j_2 x_2 + \varphi^j_3 x_3), \sum_{j=1}^{N} \bar{\mathbf{w}}_j(\varphi^j_1 x_1 + \varphi^j_2 x_2 + \varphi^j_3 x_3), 0\right)$$

$$= \Bigg(x_1 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j\varphi^j_1 + x_2 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j\varphi^j_2 + x_3 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j\varphi^j_3,$$
$$x_1 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j\varphi^j_1 + x_2 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j\varphi^j_2 + x_3 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j\varphi^j_3, 0\Bigg).$$

Then from formula (2.3.11) we see

$$\Phi_{\mathbf{w}}(x) = \Bigg(x_1 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_1^j) + x_2 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_2^j) + x_3 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_3^j),$$
$$x_1 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_1^j) + x_2 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_2^j) + x_3 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_3^j), 0 \Bigg).$$

Recall now that

$$\mathfrak{g}(\Omega_3) = \mathfrak{c}(\mathfrak{gl}_3(\mathbb{R})) \oplus \mathfrak{o}_{1,2} = \left\{ \begin{pmatrix} \lambda & p & q \\ p & \lambda & r \\ q & -r & \lambda \end{pmatrix}, \ \lambda, p, q, r \in \mathbb{R} \right\}. \qquad \text{(A.3.3)}$$

It is then clear that the condition that $\Phi_{\mathbf{w}}$ lies in $\mathfrak{g}(\Omega_3)$ for every $\mathbf{w} \in \mathbb{C}^2$ leads to the relations

$$\sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_i^j) \equiv 0, \ \ i = 1, 2, 3,$$

which yield $\Phi = 0$. By formula (2.3.12) we then see that $\mathfrak{g}_{1/2} = 0$ as required. □

A.4 Determination of $\dim \mathfrak{g}_1$ for $\mathfrak{g} = \mathfrak{g}(\widetilde{D}_6)$

Proposition A.4.1. *For $\mathfrak{g} = \mathfrak{g}(\tilde{D}_6)$ we have $\dim \mathfrak{g}_1 = 1$.*

Proof. We will utilise Theorem 2.3.5 for the cone Ω_3 and the Ω_3-Hermitian form $\widetilde{\mathcal{H}}$ given by (A.3.1). Consider a symmetric $\mathbb{R}$-bilinear form on $\mathbb{R}^3$ with values in $\mathbb{R}^3$:

$$a(x, x) = (a_{11}^1 x_1^2 + a_{22}^1 x_2^2 + a_{33}^1 x_3^2 + 2a_{12}^1 x_1 x_2 + 2a_{13}^1 x_1 x_3 + 2a_{23}^1 x_2 x_3,$$
$$a_{11}^2 x_1^2 + a_{22}^2 x_2^2 + a_{33}^2 x_3^2 + 2a_{12}^2 x_1 x_2 + 2a_{13}^2 x_1 x_3 + 2a_{23}^2 x_2 x_3,$$
$$a_{11}^3 x_1^2 + a_{22}^3 x_2^2 + a_{33}^3 x_3^2 + 2a_{12}^3 x_1 x_2 + 2a_{13}^3 x_1 x_3 + 2a_{23}^3 x_2 x_3),$$

where $a^\ell_{ij} \in \mathbb{R}$. Then for a fixed $\mathbf{x} \in \mathbb{R}^3$, from (2.3.13) we compute

$$\begin{aligned}
A_{\mathbf{x}}(x) = (&a^1_{11}\mathbf{x}_1x_1 + a^1_{22}\mathbf{x}_2x_2 + a^1_{33}\mathbf{x}_3x_3 + a^1_{12}\mathbf{x}_1x_2 + a^1_{12}\mathbf{x}_2x_1 + a^1_{13}\mathbf{x}_1x_3 \\
&+ a^1_{13}\mathbf{x}_3x_1 + a^1_{23}\mathbf{x}_2x_3 + a^1_{23}\mathbf{x}_3x_2, a^2_{11}\mathbf{x}_1x_1 + a^2_{22}\mathbf{x}_2x_2 + a^2_{33}\mathbf{x}_3x_3 + a^2_{12}\mathbf{x}_1x_2 \\
&+ a^2_{12}\mathbf{x}_2x_1 + a^2_{13}\mathbf{x}_1x_3 + a^2_{13}\mathbf{x}_3x_1 + a^2_{23}\mathbf{x}_2x_3 + a^2_{23}\mathbf{x}_3x_2, a^3_{11}\mathbf{x}_1x_1 + a^3_{22}\mathbf{x}_2x_2 \\
&+ a^3_{33}\mathbf{x}_3x_3 + a^3_{12}\mathbf{x}_1x_2 + a^3_{12}\mathbf{x}_2x_1 + a^3_{13}\mathbf{x}_1x_3 + a^3_{13}\mathbf{x}_3x_1 + a^3_{23}\mathbf{x}_2x_3 + a^3_{23}\mathbf{x}_3x_2) \\
&= ((a^1_{11}\mathbf{x}_1 + a^1_{12}\mathbf{x}_2 + a^1_{13}\mathbf{x}_3)x_1 + (a^1_{12}\mathbf{x}_1 + a^1_{22}\mathbf{x}_2 + a^1_{23}\mathbf{x}_3)x_2 + (a^1_{13}\mathbf{x}_1 \\
&+a^1_{23}\mathbf{x}_2 + a^1_{33}\mathbf{x}_3)x_3, (a^2_{11}\mathbf{x}_1 + a^2_{12}\mathbf{x}_2 + a^2_{13}\mathbf{x}_3)x_1 + (a^2_{12}\mathbf{x}_1 + a^2_{22}\mathbf{x}_2 \\
&+a^2_{23}\mathbf{x}_3)x_2 + (a^2_{13}\mathbf{x}_1 + a^2_{23}\mathbf{x}_2 + a^2_{33}\mathbf{x}_3)x_3, (a^3_{11}\mathbf{x}_1 + a^3_{12}\mathbf{x}_2 + a^3_{13}\mathbf{x}_3)x_1 \\
&+(a^3_{12}\mathbf{x}_1 + a^3_{22}\mathbf{x}_2 + a^3_{23}\mathbf{x}_3)x_2 + (a^3_{13}\mathbf{x}_1 + a^3_{23}\mathbf{x}_2 + a^3_{33}\mathbf{x}_3)x_3),
\end{aligned}$$

where $x \in \mathbb{R}^3$. By (A.3.3), the condition that this map lies in $\mathfrak{g}(\Omega_3)$ for every $\mathbf{x} \in \mathbb{R}^3$ is equivalent to the relations

$$\begin{aligned}
&a^1_{11} = a^2_{12} = a^3_{13},\ a^1_{12} = a^2_{22} = a^3_{23},\ a^1_{13} = a^2_{23} = a^3_{33}, \\
&a^2_{13} = -a^3_{12},\ a^2_{23} = -a^3_{22},\ a^2_{33} = -a^3_{23},\ a^1_{13} = a^3_{11}, \\
&a^1_{23} = a^3_{12},\ a^1_{33} = a^3_{13},\ a^1_{12} = a^2_{11},\ a^1_{22} = a^2_{12},\ a^1_{23} = a^2_{13}.
\end{aligned} \tag{A.4.1}$$

Next, let $b : \mathbb{C}^3 \times \mathbb{C}^N \to \mathbb{C}^N$ be a $\mathbb{C}$-bilinear map with the jth component given by a matrix $(b^j_{i\ell})$, $j, \ell = 1, \dots, N$, $i = 1, 2, 3$. For every fixed pair of vectors $\mathbf{w}, \mathbf{w}' \in \mathbb{C}^N$ we then compute

$$\begin{aligned}
\widetilde{\mathcal{H}}(\mathbf{w}', b(x, \mathbf{w})) = \Bigg(&x_1 \cdot \sum_{j,\ell=1}^N b^j_{1\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell + x_2 \cdot \sum_{j,\ell=1}^N b^j_{2\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell + x_3 \cdot \sum_{j,\ell=1}^N b^j_{3\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell, \\
&x_1 \cdot \sum_{j,\ell=1}^N b^j_{1\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell + x_2 \cdot \sum_{j,\ell=1}^N b^j_{2\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell + x_3 \cdot \sum_{j,\ell=1}^N b^j_{3\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell, 0\Bigg).
\end{aligned}$$

Then from (ii) of Theorem 2.3.5 we obtain

$$\begin{aligned}
B_{\mathbf{w},\mathbf{w}'}(x) = \Bigg(&x_1 \cdot \sum_{j,\ell=1}^N \mathrm{Im}(b^j_{1\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell) + x_2 \cdot \sum_{j,\ell=1}^N \mathrm{Im}(b^j_{2\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell) \\
&+ x_3 \cdot \sum_{j,\ell=1}^N \mathrm{Im}(b^j_{3\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell),\ x_1 \cdot \sum_{j,\ell=1}^N \mathrm{Im}(b^j_{1\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell) \\
&+ x_2 \cdot \sum_{j,\ell=1}^N \mathrm{Im}(b^j_{2\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell) + x_3 \cdot \sum_{j,\ell=1}^N \mathrm{Im}(b^j_{3\ell}\bar{\mathbf{w}}'_j\mathbf{w}_\ell), 0\Bigg).
\end{aligned}$$

Now, the condition that this map lies in $\mathfrak{g}(\Omega_3)$ for all $\mathbf{w}, \mathbf{w}' \in \mathbb{C}^N$ is easily seen to be equivalent to $b = 0$. Hence $B_{\mathbf{x}} = 0$ for every $\mathbf{x} \in \mathbb{R}^3$.

We will now utilise the requirement that $B_{\mathbf{x}} = 0$ is associated to $A_{\mathbf{x}}$ with respect to $\mathcal{H}$ for every $\mathbf{x} \in \mathbb{R}^3$ (see condition (i) in Theorem 2.3.5). This requirement is immediately seen to be equivalent to the relations

$$\begin{aligned}
&a^1_{12} = -a^1_{11},\ a^1_{22} = -a^1_{12},\ a^1_{23} = -a^1_{13},\\
&a^2_{12} = -a^2_{11},\ a^2_{22} = -a^2_{12},\ a^2_{23} = -a^2_{13},\\
&a^3_{12} = -a^3_{11},\ a^3_{22} = -a^3_{12},\ a^3_{23} = -a^3_{13}.
\end{aligned}$$

Together with (A.4.1), these relations imply that each a^ℓ_{ij} is either zero or equal to $\pm a^1_{11}$ as follows:

$$\begin{aligned}
&a^1_{22} = a^1_{11},\ a^1_{33} = a^1_{11},\ a^1_{12} = -a^1_{11},\ a^1_{13} = 0,\\
&a^1_{23} = 0,\ a^2_{11} = -a^1_{11},\ a^2_{22} = -a^1_{11},\ a^2_{33} = a^1_{11},\\
&a^2_{12} = a^1_{11},\ a^2_{13} = 0,\ a^2_{23} = 0,\ a^3_{11} = 0,\ a^3_{22} = 0,\\
&a^3_{33} = 0,\ a^3_{12} = 0,\ a^3_{13} = a^1_{11},\ a^3_{23} = -a^1_{11}.
\end{aligned}$$

Therefore,

$$a(x,x) = a^1_{11}((x_1 - x_2)^2 + x_3^2, -(x_1 - x_2)^2 + x_3^2, 2(x_1 - x_2)x_3).$$

This shows that $\dim \mathfrak{g}_1 = 1$ as required. □

A.5 Determination of $\dim \mathfrak{g}_{1/2}$ for $\mathfrak{g} = \mathfrak{g}(\widetilde{D}_{13})$

Set

$$\widetilde{D}_{13} := \left\{ (z, w) \in \times\mathbb{C}^4 \times \mathbb{C}^N : \operatorname{Im} z - \hat{\mathcal{H}}(w, w) \in \Omega_6 \right\},$$

where $N \geq 1$ and $\hat{\mathcal{H}}$ is the Ω_6-Hermitian form analogous to the one introduced in (A.3.1):

$$\hat{\mathcal{H}}(w, w') := \left(\sum_{j=1}^N \bar{w}_j w'_j, \sum_{j=1}^N \bar{w}_j w'_j, 0, 0 \right).$$

Proposition A.5.1. *For* $\mathfrak{g} = \mathfrak{g}(\widetilde{D}_{13})$ *one has* $\mathfrak{g}_{1/2} = 0$.

Proof. We will apply Theorem 2.3.4 to the cone Ω_6 and the Ω_6-Hermitian form $\hat{\mathcal{H}}$. Let $\Phi : \mathbb{C}^4 \to \mathbb{C}^N$ be a $\mathbb{C}$-linear map given by a matrix (φ^j_i), with

$j = 1, \ldots, N$, $i = 1, 2, 3, 4$. Fixing $\mathbf{w} \in \mathbb{C}^N$, for $x \in \mathbb{R}^4$ we compute

$$\begin{aligned}
\mathcal{H}(\mathbf{w}, \Phi(x)) = \Bigg(& \sum_{j=1}^{N} \bar{\mathbf{w}}_j(\varphi_1^j x_1 + \varphi_2^j x_2 + \varphi_3^j x_3 + \varphi_4^j x_4), \\
& \qquad \sum_{j=1}^{N} \bar{\mathbf{w}}_j(\varphi_1^j x_1 + \varphi_2^j x_2 + \varphi_3^j x_3 + \varphi_4^j x_4), 0, 0 \Bigg) \\
= \Bigg(& x_1 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j \varphi_1^j + x_2 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j \varphi_2^j + x_3 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j \varphi_3^j + x_4 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j \varphi_4^j, \\
& x_1 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j \varphi_1^j + x_2 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j \varphi_2^j + x_3 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j \varphi_3^j + x_4 \cdot \sum_{j=1}^{N} \bar{\mathbf{w}}_j \varphi_4^j, 0, 0 \Bigg).
\end{aligned}$$

Then from formula (2.3.11) we see

$$\begin{aligned}
\Phi_{\mathbf{w}}(x) = \Bigg(& x_1 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_1^j) + x_2 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_2^j) + x_3 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_3^j) \\
& + x_4 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_4^j),\ x_1 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_1^j) + x_2 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_2^j) \\
& + x_3 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_3^j) + x_4 \cdot \sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_4^j), 0, 0 \Bigg).
\end{aligned}$$

Recall now that

$$\mathfrak{g}(\Omega_6) = \mathfrak{c}(\mathfrak{gl}_4(\mathbb{R})) \oplus \mathfrak{o}_{1,3} = \left\{ \begin{pmatrix} \lambda & p & q & r \\ p & \lambda & s & t \\ q & -s & \lambda & y \\ r & -t & -y & \lambda \end{pmatrix},\ \lambda, p, q, r, s, t, y \in \mathbb{R} \right\}.$$

It is then clear that the condition that $\Phi_{\mathbf{w}}$ lies in $\mathfrak{g}(\Omega_4)$ for every $\mathbf{w} \in \mathbb{C}$ leads to the relations

$$\sum_{j=1}^{N} \mathrm{Im}(\bar{\mathbf{w}}_j \varphi_i^j) \equiv 0,\ i = 1, 2, 3, 4,$$

which yield $\Phi = 0$. By formula (2.3.12) we then see that $\mathfrak{g}_{1/2} = 0$ as required. $\square$

Appendix B

B.1 Proof that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}(G(\Omega, H))$

For completeness, we include here a proof of the following result.

Proposition B.1.1. *The subspace $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g} = \mathfrak{g}(S(\Omega, H))$.*

Proof. Let

$$V = Az \cdot \frac{\partial}{\partial z} + Bw \cdot \frac{\partial}{\partial w} \text{ and } W = Cz \cdot \frac{\partial}{\partial z} + Dw \cdot \frac{\partial}{\partial w}.$$

Then

$$\begin{aligned}[V, W] &= \left[Az \cdot \frac{\partial}{\partial z} + Bw \cdot \frac{\partial}{\partial w}, Cz \cdot \frac{\partial}{\partial z} + Dw \cdot \frac{\partial}{\partial w}\right] \\ &= \left[Az \cdot \frac{\partial}{\partial z}, Cz \cdot \frac{\partial}{\partial z}\right] + \left[Bw \cdot \frac{\partial}{\partial w}, Dw \cdot \frac{\partial}{\partial w}\right].\end{aligned}$$

In order to simplify the computation of the above, let us momentarily consider only the first term in this expression. To further simplify the presentation, we utilise the Einstein summation convention. Let

$$A = \begin{bmatrix} a_1^1 & \dots & a_k^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \dots & a_k^k \end{bmatrix} \text{ and } z = \begin{bmatrix} z^1 \\ \vdots \\ z^k \end{bmatrix},$$

and similarly for matrices B, C, D and variable w.

We have

$$\begin{aligned}\left[Az\cdot\frac{\partial}{\partial z}, Cz\cdot\frac{\partial}{\partial z}\right] &= \left[a^i_j z^j\frac{\partial}{\partial z^i}, c^m_n z^n\frac{\partial}{\partial z^m}\right]\\ &= a^i_j c^m_n\left[z^j\frac{\partial}{\partial z^i}, z^n\frac{\partial}{\partial z^m}\right]\\ &= a^i_j c^m_n\left(z^j\frac{\partial}{\partial z^i}\left(z^n\frac{\partial}{\partial z^m}\right) - z^n\frac{\partial}{\partial z^m}\left(z^j\frac{\partial}{\partial z^i}\right)\right)\\ &= a^i_j c^m_n\left(z^j\delta^n_i\frac{\partial}{\partial z^m} - z^n\delta^j_m\frac{\partial}{\partial z^i}\right)\\ &= c^m_i a^i_j z^j\frac{\partial}{\partial z_m} - a^i_m c^m_n z^n\frac{\partial}{\partial z^i}\\ &= CAz\cdot\frac{\partial}{\partial z} - ACz\cdot\frac{\partial}{\partial z}\\ &= [C,A]\,z\cdot\frac{\partial}{\partial z}.\end{aligned}$$

Note the absence of second derivative terms, since we are dealing with a Lie algebra of vector fields. By an analogous computation, we also have

$$\left[Bw\cdot\frac{\partial}{\partial w}, Dw\cdot\frac{\partial}{\partial w}\right] = [D,B]\,w\cdot\frac{\partial}{\partial w},$$

allowing us to conclude that

$$[V,W] = [C,A]\,z\cdot\frac{\partial}{\partial z} + [D,B]\,w\cdot\frac{\partial}{\partial w}.$$

Clearly $[C,A]\in\mathfrak{g}(\Omega)$ and $[D,B]\in\mathfrak{gl}_{n-k}(\mathbb{C})$, and so we see that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. □

Further, we show that the function f given in Proposition 2.3.3 is a Lie algebra homomorphism.

Proposition B.1.2. *The function $f:\mathfrak{g}_0\to\mathfrak{g}(G(\Omega,H))$ given by $f(A,B) = -A$ is a Lie algebra homomorphism.*

Proof. We know from Proposition B.1.1 that

$$[V,W] = [C,A]\,z\cdot\frac{\partial}{\partial z} + [D,B]\,w\cdot\frac{\partial}{\partial w}.$$

So we see that

$$\begin{aligned} f([V,W]) &= f\left([C,A]\,z\cdot\frac{\partial}{\partial z} + [D,B]\,w\cdot\frac{\partial}{\partial w}\right) \\ &= -\,[C,A] \\ &= AC - CA \\ &= [-A,-C] \\ &= [f(V), f(W)] \end{aligned}$$

as required. $\square$

B.2 Simultaneous diagonalisation of Hermitian forms

Lastly, for the reader's convenience, we review here some well-known linear algebra. The following is reproduced from [36, pp. 126–127].

Let H_1 and H_2 be Hermitian quadratic forms on an n-dimensional vector space V, where H_1 is also positive definite. Then we can write $H_1(x,x) = x^*Ax$, and $H_2(x,x) = x^*Bx$, where A and B are Hermitian matrices, with A positive definite. Further, since H_1 is positive definite, we have $x^*Ax > 0$ for all non-zero $x \in V$.

To begin with, consider the Hermitian matrix A, which is also positive definite. There exists an invertible P such that $P^*AP = I$. To see this, simply note that since there exists a unitary U such that $U^*AU = D$ with the entries of D real and positive, we have

$$D^{-1/2}U^*AUD^{-1/2} = I$$

and, therefore, that

$$(UD^{-1/2})^*A(UD^{-1/2}) = I.$$

So take $P = UD^{-1/2}$, which is invertible.

Now consider the Hermitian matrix B. Since B is Hermitian, the matrix $C := P^*BP$ is also Hermitian, and we say that the matrices B and C are $*$-congruent. There exists a unitary Q such that $Q^*CQ = \Lambda$, where Λ is a diagonal matrix whose entries consist of the (real) eigenvalues of C.

If we now apply the transformation $x = PQy$ (where PQ is invertible, since both P and Q are invertible), we obtain

$$\begin{aligned} x^*Ax &= (PQy)^*A(PQy) \\ &= y^*Q^*P^*APQy \\ &= y^*Q^*IQy \\ &= y^*y \\ &= ||y||^2 \end{aligned}$$

where the penultimate equality follows from the fact that Q is unitary. We also have

$$\begin{aligned} x^*Bx &= (PQy)^*B(PQy) \\ &= y^*Q^*P^*BPQy \\ &= y^*Q^*CQy \\ &= y^*\Lambda y \\ &= \lambda_1|y_1|^2 + \lambda_2|y_2|^2 + \cdots + \lambda_n|y_n|^2 \end{aligned}$$

where the λ_i are the eigenvalues of C.

By Sylvester's law of inertia, since the matrices B and C are $*$-congruent, they have the same number of positive, negative and zero eigenvalues.

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