The Classifying Spaces of Some Categories

Kieran O'Loughlin

September 11, 2021

Thesis submitted for the degree of
Masters of Philosophy
in
Pure Mathematics
at The University of Adelaide
Faculty of Engineering, Computer and Mathematical Sciences
School of Mathematical Sciences

Contents

Signed Statement

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint-award of this degree.

I give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library Search and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

I acknowledge the support I have received for my research through the provision of an Australian Government Research Training Program Scholarship

Signed: .... Date: 16/04/2021 ....

Acknowledgements

First and foremost, I would like to thank my principal supervisor Dr. Danny Stevenson. Thank you for the many hours of your time you spent explaining new theory and concepts to me, helping me tackle my research problem when I was stuck, encouraging me when I was doubting myself, and helping me draft and polish my thesis. You have consistently gone over and above as a supervisor, and I cannot thank you enough. I'd also like to than my secondary supervisor Prof. Michael Murray for always being available to give support when I needed it, having the patience to each me topology in my third year of my undergraduate degree, and for proofreading my thesis. Thank you also to Dr. David Roberts for all the helpful tea room conversations about maths, resource recommendations, and for his course on Algebraic Topology.

Next I'd like to thank my parents Anna and Richard O'Loughlin. You have both patiently supported me, and put up with me, while I've completed my undergraduate and postgraduate degrees. I hope I can make you both proud as I move onto the next stage of my life.

Finally, I'd like to thank all my amazing friends. Thank you all for filling my life with laughter, adventure, emotional support, deep conversations, and wild nights. Without you all, I don't know how I would have finished my studies.

Dedication

To my parents, who have supported me throughout all my years of study.

Abstract

In this thesis we describe the theory of simplicial covering spaces and present Segal's construction of the Algebraic K-theory spectrum of a permutative category. Using this theory and construction, we then identify the stable homotopy type of the Algebraic K-theory spectrum of the core of the category of finite sheeted simplicial covering spaces. This result generalises the Barratt-Priddy-Quillen Theorem.

Chapter 0

Introduction

To every category C there is a functorially associated topological space BC called the classifying space of C. The classifying space construction is a very useful concept in pure mathematics. For example, in the case that C is a group G the space BG classifies principal G-bundles. Categories can sometimes come equipped with extra structure that is reflected in their classifying spaces. For example, a category C may come equipped with a tensor product $\otimes : C \times C \to C$ that satisfies associativity, identity and commutativity laws up to natural isomorphism. The category C is then called a symmetric monoidal category. The simplest example of a symmetric monoidal category is the category of finite sets **FinSet** equipped with the tensor product $\sqcup : \mathbf{FinSet} \times \mathbf{FinSet} \to \mathbf{FinSet}$ induced by disjoint union. When C is a symmetric monoidal category its classifying space BC inherits the structure of a commutative monoid (up to homotopy) and is the 0-th space of a spectrum $\mathbf{B}\mathcal{K}C$ associated to C called the Algebraic K-theory spectrum of C.

A spectrum $\mathbf{X}$ is a sequence of a sequence of spaces $\{\mathbf{X}_n\}_{n\in\mathbb{N}}$ such that for each $n \geq 0$ the space $\mathbf{X}_n$ is a closed subspace of the loop space $\Omega\mathbf{X}_{n+1}$ of $\mathbf{X}_{n+1}$. A simple example of a spectrum is the sphere spectrum $\mathbb{S}$ whose n-space is the n-sphere $\mathbb{S}^n$. Spectra are the main objects of interest in stable homotopy theory and are usually studied up to stable homotopy type. It turns out that there is a close relationship between **FinSet** and $\mathbb{S}$, as made precise by the Barratt-Priddy-Quillen Theorem. That is, the Barratt-Priddy-Quillen Theorem states that if $\mathbf{FinSet}^{\simeq}$ is the subcategory of **FinSet** which contains only the maps that are isomorphisms, then the Algebraic K-theory spectrum of $\mathbf{FinSet}^{\simeq}$ has the stable homotopy type of the sphere spectrum $\mathbb{S}$ ([27]). The goal of this thesis will be to generalise the Barratt-Priddy-Quillen Theorem.

To see how we will generalise Barratt-Priddy-Quillen Theorem in this thesis, recall that finite sheeted covering spaces (covering spaces whose fibres are finite sets) over a connected space X can be viewed as collections of finite sets parametrised by the points in X, and let $\mathbf{FinCov}(X)^{\simeq}$ be the category whose objects are finite sheeted covering spaces and maps are isomorphisms between them. Also observe that equipping $\mathbf{FinCov}(X)^{\simeq}$ can be equipped with the structure of symmetric monoidal structure by equipping it with the tensor product induced by disjoint union. The Barratt-Priddy-Quillen Theorem will hence be generalised by identifying the stable homotopy type of the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$ when X is connected. To do this we will first need to accomplish the following 3 tasks:

1. Develop some theory for covering spaces which will aid us in achieving our thesis goal.
2. Define the Algebraic K-theory spectrum of a symmetric monoidal category precisely.
3. Identify the symmetric monoidal category $\mathbf{FinCov}(X)^{\simeq}$ with an equivalent symmetric monoidal category whose Algebraic K-theory spectrum's stable homotopy type is easy to identify.

Task 1 will be completed in Chapters 1 and 2. In Chapter 1 we will review the theory of simplicial sets required to understand Chapters 2–5. Namely, we will review the theory of simplicial sets, simplicial objects, and simplicial homotopy theory, as many of the objects and constructions we will consider in Chapters 2–5 will have underlying or associated simplicial structures. In Chapter 2 we will discuss simplicial analogues of covering spaces called simplicial covering spaces. We will show that they enjoy properties analogous to those enjoyed by covering spaces and, most importantly, show that there is an equivalence of categories

$$\mathbf{Cov}(X) \simeq [\Pi_1(X), \mathbf{Set}].$$

Here $\mathbf{Cov}(X)$ is the category of simplicial coverings over the simplicial set X and $\Pi_1(X)$ is the fundamental groupoid of X. A corollary of this equivalence of categories will aid us when completing task 3 in in Chapter 5.

Task 2 will be completed in Chapters 3–4. In Chapter 3, we will review the theory of H-spaces, spectra, and Γ-spaces. The key result proven in Chapter 3 will be that to every Γ-space X there is an associated spectra $\mathbf{B}X$ whose 0-th space is an H-space. Then, in Chapter 4, after reviewing the basic theory of symmetrical monoidal categories, we will show that to every symmetric monoidal category C there exists an associated Γ-space $\mathcal{K}C$, and hence an associated spectrum $\mathbf{B}\mathcal{K}C$. The spectrum $\mathbf{B}\mathcal{K}C$ will then be defined to be the Algebraic K-theory spectrum of C.

Task 3 will be completed in Chapter 5. By appealing to the aforementioned equivalence of categories, and some group theory, we will be able to identify $\mathbf{FinCov}(X)^{\simeq}$ with the free symmetric monoidal category $S(\mathcal{CC}_{\pi_1(X)})$ on the groupoid $\mathcal{CC}_{\pi_1(X)}$ when X is connected. The groupoid $\mathcal{CC}_{\pi_1(X)}$ will be constructed by considering choosing representatives from the conjugacy classes of $\pi_1(X)$. The stable homotopy type of the Algebraic K-theory spectrum $\mathbf{B}\mathcal{K}S(C)$ of $S(C)$, when C is small, will be able to be easily identified by appealing to some theory developed by Quillen (discussed in Chapter 4) and Barratt and Eccles. In fact, we will show that $\mathbf{B}\mathcal{K}S(C)$ has the same stable homotopy type as the spectrum

$$\Sigma^{\infty}|NC_+|.$$

In Section 5.5, after the previously discussed tasks have been accomplished, we will prove:

Theorem 5.5.14. *If X is a connected simplicial set then the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$ has the same stable homotopy type as the spectrum*

$$\Sigma^{\infty}\left(\coprod_{[H]\in\mathcal{CC}_{\pi_1(X)}} B(N_{\pi_1(X)}(H)/H)\right)_+ .$$

This thesis's goal will hence have been achieved. Theorem 5.5.14 will also successfully act as a generalisation of the Barratt-Priddy-Quillen Theorem, as we will be able to recover the Barratt-Priddy-Quillen Theorem by taking X to be a point.

The author of this thesis will assume the reader has a high degree of familiarity with basic category theory and algebraic topology. Knowledge of group completions of monoids, free groups and monoids, G-sets, CW-complexes, and compactly generated weak Hausdorff spaces, covering spaces, and Hurewicz fibrations and cofibrations will also be assumed. Readers unfamiliar with these topics are referred to [9, 12, 13, 21, 28, 30] for suitable introductions to them.

Chapter 1

Simplicial Sets and Homotopy Theory

A simplicial set, put simply, is a set of gluing instructions for constructing a topological space out of certain building blocks called simplicies. Simplicial sets provide a well-behaved combinatorial model for topological spaces with their own native homotopy theory. Further, the category of simplicial sets has nicer properties than the category of topological spaces, and its corresponding homotopy category (a notion we will make precise in this chapter) is equivalent to the homotopy category of topological spaces. So, questions about the homotopy theory of spaces can be studied via simplicial sets. In this chapter we will introduce the basic theory of simplicial sets and discuss how they model topological spaces. The majority of the theory discussed in this chapter can be found in [8] where the reader can find omitted technical details and discussions.

To begin our discussion on simplicial sets we will introduce the basic definitions, and state some of their basic properties, in Section 1.1. We will also discuss some examples of simplicial sets that will play a key role in this thesis. Then, in Section 1.2, we will explain how to construct a topological space from a simplicial set via geometric realisation, and state some of the useful properties of the geometric realisation functor. The homotopy theory of simplicial sets will then be explored in Section 1.3. In Sections 1.4 and 1.5, we will define the notion of simplicial object in an arbitrary category C, paying particular attention to the case where C is the category of pointed sets. Finally, in Section 1.6, we will introduce model categories, discuss the model categories that will play a key role in this thesis, and make precise the notion that simplicial sets model topological spaces.

The primary goal of this chapter is to provide much of the background theory on simplicial sets required for the remainder of this thesis. Additionally, the author hopes this chapter will serve as an approachable and motivated introduction to simplicial sets for the reader.

1.1 The Simplex Category and Simplicial Sets

In this section we will begin to develop the theory of simplicial sets by stating some of its basic definitions. Using some elementary properties of the categories involved in these definitions, we will be able to give both a categorical and a combinatorial definition of a simplicial set, as well as state some of their elementary properties. We will then give a few important examples of simplicial sets, and discuss how they can be constructed from categories, and in particular groups.

To begin, let us define a simplicial set. To do this we must first define the simplex category.

Definition 1.1.1. Let Δ be the category whose objects are the finite, non-empty, totally ordered sets

$$[n] = \{0 < 1 < ... < n\}$$

and maps are non-decreasing functions. The category Δ is called the *simplex category*.

Definition 1.1.2. A *simplicial set* X is a functor $X : \Delta^{op} \to \mathbf{Set}$. A *simplicial map* $X \to Y$ is a natural transformation of the functors X and Y.

Definition 1.1.3. Let **sSet** be the category whose objects are simplicial sets and maps are simplicial maps.

So, a simplicial set is simply a presheaf on the simplex category, and a simplicial map is a map of presheaves. It is not initially clear how such functors and maps act as combinatorial models for topological spaces, but this will become clear as we develop more theory.

When one constructs a new category, a reasonable first question to ask oneself is '*how well-behaved is this category?*'. In the case of **sSet**, the answer to this question is '*very well-behaved*'. For example, **sSet** has all limits and colimits.

Proposition 1.1.4. *The category* **sSet** *has all limits and all colimits.*

Proof. As **Set** has all (co)limits, and (co)limits in functor categories are computed pointwise, **sSet** has all (co)limits. □

Proposition 1.1.4 actually tells us more than just that **sSet** has all (co)limits. It also tells us how to compute them. That is, Proposition 1.1.4 tells us that (co)limits in **sSet** are computed pointwise in **Set**. For example, the product of two simplicial sets X and Y can be explicitly defined to be the simplicial set $X \times Y$ where for all $n \geq 0$

$$(X \times Y)([n]) := X([n]) \times Y([n]),$$

and for all $f : [n] \to [m]$ in Δ

$$(X \times Y)(f) := X(f) \times Y(f).$$

Simplicial sets can be defined both categorically and combinatorially. Definition 1.1.2 is clearly the categorical definition of simplicial set, so let us now derive an equivalent combinatorial definition. To derive the combinatorial definition of simplicial set, we will need to appeal to a property enjoyed by Δ that says every map in Δ can factored as special types of maps called coface and codegeneracy maps.

Definition 1.1.5. For all $n \geq 0$, and integers i, j such that $0 \leq i \leq n+1$ and $0 \leq j \leq n$, let d^i and s^j be the maps

$$d^i : [n] \to [n+1], \quad d^i(x) := \begin{cases} x, & x < i \\ x+1, & x \geq i \end{cases}$$

and

$$s^j : [n+1] \to [n], \quad s^j(x) := \begin{cases} x, & x \leq j \\ x-1, & x > j. \end{cases}$$

The maps d^i are called the *coface maps* in Δ, and the maps s^j are called the *codegeneracy maps*.

Remark 1.1.6. Observe that the coface and codegeneracy maps satisfy the following relations:

$$d^j d^i = d^i d^{j-1}, \quad i < j \tag{1.1}$$

$$s^j s^i = s^i s^{j+1}, \quad i \leq j \tag{1.2}$$

$$s^j d^i = \begin{cases} 1, & i = j, j+1 \\ d^i s^{j-1}, & i < j \\ d^{i-1} s^j, & i > j+1. \end{cases} \tag{1.3}$$

Equations (1.1)–(1.3) are called the *cosimplicial identities*.

Conventions 1.1.7. Note that we will adopt the following conventions when discussing simplicial sets in the remainder of this thesis:

1. Given a simplicial set X, the set $X([n])$ will be denoted by X_n and will be called the set of *n-simplices* of X. The set of 0-simplices of X will be called the set of *vertices* of X.

2. The image of the coface maps d^i and the codegeneracy maps s^j under X will be denoted d_i and s_j, respectively, and will be called the *face* and *degeneracy* maps of X.

3. If x is a simplex of X such that there exists a simplex y where $s_j(y) = x$ for some j, then x is called a *degeneracy* of y. If there exists a simplex z such that $d_i(z) = x$ for some i, then x is said to be the i-th *face* of z.

Lemma 1.1.8. *Any map $f : [n] \to [m]$ in Δ has a unique factorisation*

$$f = d^{i_1} \circ d^{i_2} \circ ... \circ d^{i_k} \circ s^{j_1} \circ s^{j_2} \circ ... \circ s^{j_h},$$

where $n - h + k = m$ and

$$0 \leq i_k < i_{k-1} < ... < i_1 < m$$

and

$$0 \leq j_1 < j_2 < ... < j_h < n - 1.$$

Proof. See the lemma on page 177 in [17]. □

Construction 1.1.9. By appealing to Lemma 1.1.8 we can give the previously alluded to combinatorial definition of simplicial set and simplicial map, as follows: A simplicial set X is a presheaf on Δ. So, for each $n \geq 0$, there is a set X_n, and for each map $[n] \to [m]$ in Δ, there is a function $X_m \to X_n$. As each map $[n] \to [m]$ in Δ can be uniquely factored as a composition of coface and codegeneracy maps by Lemma 1.1.8, and as functors respect composition, the map $X_m \to X_n$ is given by a composition of face and degeneracy maps. This implies that a mapping from Δ^{op} to **Set** is functorial if and only if it satisfies the following relations (called the simplicial identities):

$$d_i d_j = d_{j-1} d_i, \quad i < j \tag{1.4}$$

$$s_i s_j = s_{j+1} s_i, \quad i \leq j \tag{1.5}$$

$$d_i s_j = \begin{cases} 1, & i = j, j+1 \\ s_{j-1} d_i, & i < j \\ s_j d_{i-1}, & i > j+1. \end{cases} \tag{1.6}$$

Thus, to specify a simplicial set X we only need specify a family of sets $\{X_n\}_{n\in\mathbb{N}}$ and, for each $n \geq 0$, functions $d_i : X_{n+1} \to X_n$ for each integer i such $0 \leq i \leq n+1$, and functions $s_j : X_n \to X_{n+1}$ for each integer j such that $0 \leq j \leq n$. The functions $d_i : X_{n+1} \to X_n$ and $s_i : X_n \to X_{n+1}$ must also satisfy the simplicial identities.

A simplicial map $f : X \to Y$ is a natural transformation of the functors X and Y. That is, f consists of a family of functions $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ such that for each map $[n] \to [m]$ in Δ the diagram

$$\begin{array}{ccc} X_m & \longrightarrow & X_n \\ \downarrow{\scriptstyle f_m} & & \downarrow{\scriptstyle f_n} \\ Y_m & \longrightarrow & Y_n \end{array} \tag{1.7}$$

commutes. But, as maps in Δ can be factored as a composition of coface and codegeneracy maps, the commutativity of diagram (1.7) for all maps $[n] \to [m]$ in Δ is equivalent to the commutativity of the diagrams

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} \\ Y_{n+1} & \xrightarrow{d_i} & Y_n \end{array} \tag{1.8}$$

and

$$\begin{array}{ccc} X_n & \xrightarrow{s_j} & X_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ Y_n & \xrightarrow{s_j} & Y_{n+1}, \end{array} \tag{1.9}$$

for all $n \geq 0$ and i and j where $0 \leq i \leq n+1$ and $0 \leq j \leq n$. Thus, to define a simplicial map it is sufficient to define a family of functions $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ that commute with the face and degeneracy maps of X and Y.

Thus, we have both a categorical and combinatorial definition of simplicial set. These two definitions are clearly equivalent.

Throughout this thesis we will extensively use a few different examples of simplicial sets. We define these examples of simplicial sets in Subsections 1.1.1–1.1.4 below, and make some observations about their properties.

1.1.1 The Standard n-Simplex

The first example of a simplicial set we will extensively use in this thesis is called the standard n-simplex Δ^n.

Definition 1.1.10. If $n \geq 0$ the *standard n-simplex* Δ^n is the simplicial set $\Delta^n : \Delta^{op} \to \mathbf{Set}$ defined as the representable functor $\Delta(-, [n])$.

The simplicial set Δ^n is particularly interesting in the theory of simplicial sets as each n-simplex of a simplicial set X can be understood as a simplicial map $\Delta^n \to X$. This observation is made precise in Lemma 1.1.11 below.

Lemma 1.1.11. *There is a bijection*

$$[\Delta^{op}, \mathbf{Set}](\Delta^n, X) \cong X_n$$

that is natural in $[n]$ *and* X*.*

Proof. The given statement is simply a special case of the Yoneda lemma. However, it remains helpful to review how the bijection is explicitly constructed in the case of simplicial sets. First, we define a map which sends a given simplicial map $\sigma : \Delta^n \to X$ to the n-simplex $\widetilde{\sigma} := \sigma_n(Id_{[n]})$. Then, we define a map which sends n-simplices x of X to the simplicial map $\widehat{x} : \Delta^n \to X$ with $[k]$-component

$$\widehat{x}_k : \hom_\Delta([k], [n]) \to X_k, \quad \widehat{x}_k(f) = X(f)(x).$$

The maps $(\,\widehat{}\,)$ and $(\,\widetilde{}\,)$ are mutually inverse and natural. □

1.1.2 The Boundary and Horns of Δ^n

Our next important example of a simplicial set is called the boundary of Δ^n, and is denoted $\partial\Delta^n$. To construct $\partial\Delta^n$ we need to define what it means to generate a simplicial set X_S from a set S, where $S \subseteq X_n$ for some simplicial set X. Of course, there will be some sort of inclusion of the form $\partial\Delta^n \subseteq \Delta^n$, so we will also define what it means for a simplicial set Z to be a simplicial subset of X.

Definition 1.1.12. A *simplicial subset* of a simplicial set X is a simplicial set Z such that $Z_n \subseteq X_n$ for each n and $X(f)|_{Z_m} = Z(f)$ for all $f : [n] \to [m]$ in Δ.

Definition 1.1.13. Let X be a simplicial set and S a set such that $S \subseteq X_n$ for some n. The simplicial subset of X *generated* by S is the simplicial set whose m-simplices are the m-simplices of X that are in the image of $X(f) : S \to X_m$ for some $f : [m] \to [n]$ in Δ.

Remark 1.1.14. In the literature on the theory of simplicial sets the simplicial subset of X generated by $S \subseteq X_n$ is typically defined as the smallest simplicial subset of X which contains S (see [26], for example). This definition and the definition given in Definition 1.1.13 are equivalent. Why? Let Y be the simplicial subset of X generated by $S \subseteq X_n$ as defined in Definition 1.1.13, and let Y' be the simplicial subset of X generated by S as defined in the literature. Clearly $Y' \subseteq Y$. Now if $x \in Y_m$ for some m, then there exists an $s \in S$ and an $f : [m] \to [n]$ such that $X(f)(s) = x$. Then, as $s \in Y'_n$, and Y' is a simplicial subset of X, the simplex x must also be in Y'_m. Hence, $Y' = Y$.

Definition 1.1.15. Let $\partial\Delta^n$ be the simplicial subset of Δ^n generated by the set $\{d^0, d^1, ..., d^n\} \subseteq \Delta^n_{n-1}$. The simplicial set $\partial\Delta^n$ is called the *boundary of* Δ^n.

Another important example of a simplicial set we can construct using the concepts introduced in Definitions 1.1.12 and 1.1.13 is called the k-th horn of Δ^n.

Definition 1.1.16. For each $n \geq 1$ and $0 \leq k \leq n$ let Λ^n_k be the simplicial subset of Δ^n generated by the set $\{d^0, d^1, ..., d^{k-1}, d^{k+1}, ..., d^n\} \subseteq \Delta^n_{n-1}$. The simplicial set Λ^n_k is called the *k-th horn of* Δ^n.

The horn Λ^n_k will be useful when describing the basics of simplicial homotopy theory in Section 1.3.

1.1.3 The Nerve of a Category

The next simplicial set we will introduce is called the nerve NC of the small category C. The nerve NC of the category C is constructed using only the data of C, and is functorial. We are interested in nerves as many of the interesting simplicial sets we will consider in this thesis will be the nerve of some small category.

Definition 1.1.17. Let C be a small category. The *nerve* NC of C is the simplicial set defined as follows:

$$\begin{aligned}
NC_0 &= \mathrm{ob}(C) \\
NC_1 &= \mathrm{mor}(C) \\
NC_2 &= \{(f_0, f_1) \in \mathrm{mor}(C) \times \mathrm{mor}(C) : t(f_0) = s(f_1)\} \\
&\vdots \\
NC_n &= \{(f_0, f_1, ..., f_{n-1}) \in \mathrm{mor}(C)^n : t(f_{k-1}) = s(f_k)\},
\end{aligned}$$

where $k = 0, ..., n-1$ and s and t are the source and target maps of the maps in C. The face maps d_i and degeneracy maps s_j are the maps defined by

$$d_i : NC_{n+1} \to NC_n,$$

$$d_i\left((f_0, ..., f_n)\right) := \begin{cases} (f_1, ..., f_n), & i = 0 \\ (f_0, ..., f_{i+1} \circ f_i, ..., f_n), & 0 < i < n+1 \\ (f_0, ..., f_{n-1}), & i = n+1 \end{cases}$$

and

$$s_j : NC_n \to NC_{n+1},$$

$$s_j\left((f_0, ..., f_{n-1})\right) := (f_0, ..., \mathrm{Id}_{s(f_j)}, f_j, ..., f_{n-1}).$$

Construction 1.1.18. Let **Cat** denote the category of small categories. Definition 1.1.17 can be extended to define a functor

$$N : \mathbf{Cat} \to \mathbf{sSet},$$

where if $F : C \to D$ is a functor of small categories the simplicial map $NF : NC \to ND$ has n-component

$$NF : NC_n \to ND_n, \quad NF(f_0, ..., f_{n-1}) := (F(f_0), ..., F(f_{n-1})).$$

The nerve of a category can also be described slightly more geometrically than the description given in Definition 1.1.17.

Remark 1.1.19. The simplex category Δ can be embedded inside **Cat** via the inclusion functor $\Delta \hookrightarrow \mathbf{Cat}$ which maps $[n]$ to the category

$$[n] = 0 \to 1 \to ... \to n.$$

Then the nerve NC of the category C can then be described as the functor

$$NC := \mathbf{Cat}(-, C).$$

Example 1.1.20. Observe that the nerve $N[n]$ of the category $[n]$ is equal to the standard n-simplex Δ^n.

It turns out that the nerve functor commutes with limits and coproducts.

Proposition 1.1.21. *The nerve functor* $N : \mathbf{Cat} \to \mathbf{sSet}$ *commutes with limits and coproducts.*

Proof. As the nerve functor N has a left adjoint it commutes with limits (see Example 4.6 in [26] — we will discuss this adjoint in Chapter 2). The nerve functor N commutes with coproducts as, if $F : [n] \to \coprod_{i \in I} C_i$ is a functor, then the image of F is connected. Hence, F factors through some C_j where $j \in I$, and so

$$N\left(\coprod_{i\in I} C_i\right)_n = \mathbf{Cat}([n], \coprod_{i\in I} C_i) = \coprod_{i\in I} \mathbf{Cat}([n], C_i) = \coprod_{i\in I} N(C_i)_n.$$

□

1.1.4 The Classifying Simplicial Set

The final simplicial sets $\mathbf{E}G$ and $\mathbf{B}G$ we will introduce are the nerves of the groupoids EG and BG. The groupoids EG and BG are constructed using only the data of a group G, and the simplicial sets $\mathbf{E}G$ and $\mathbf{B}G$ are related to the theory of principal G-bundles, as we shall mention in Section 1.2.

Construction 1.1.22. Let G be a group. There are two ways we can construct a groupoid from G. The first way is to consider G as a one object groupoid denoted BG. The second way is to construct a groupoid EG whose objects are the elements of G. The maps $(x, y) : x \to y$ in EG are pairs of elements of G. That is, for every pair of elements x and y in G, there is a unique map $x \to y$ in EG.

Definition 1.1.23. Let G be a group. Define $\mathbf{B}G$ to be the simplicial set NBG. More explicitly, let $\mathbf{B}G$ be the simplicial set whose set of n-simplices is the set G^n. The face maps d_i and degeneracy maps s_j of $\mathbf{B}G$ are the maps

$$d_i : G^{n+1} \to G^n, \quad d_i((g_0, ..., g_n)) := \begin{cases} (g_1, ..., g_n), & i = 0 \\ (g_0, ..., g_{i+1}g_i, ..., g_n), & 0 < i < n+1 \\ (g_0, ..., g_{n-1}), & i = n+1 \end{cases}$$

and

$$s_j : G^n \to G^{n+1}, \quad s_j((g_0, ...g_{n-1})) := (g_0, ..., e, g_j, ..., g_{n-1}).$$

The simplicial set $\mathbf{B}G$ is called the *classifying simplicial set* of the group G.

Definition 1.1.24. Let G be a group. Define $\mathbf{E}G$ to be the simplicial set NEG. More explicitly, let $\mathbf{E}G$ be the simplicial set whose set of n-simplices is the set G^{n+1}. The face maps d_i and degeneracy maps s_j of $\mathbf{E}G$ are the maps

$$d_i : G^{n+2} \to G^{n+1}, \quad d_i((g_0, ..., g_{n+1})) = (g_0, ..., \hat{g}_i, ..., g_{n+1})$$

and

$$s_j : G^{n+1} \to G^{n+2}, \quad s_j((g_0, ..., g_n)) = (g_0, ..., g_j, g_j, ..., g_n).$$

There is a functor $EG \to BG$ that induces an interesting simplicial map $\mathbf{E}G \to \mathbf{B}G$ which we will construct now. Why the induced map $\mathbf{E}G \to \mathbf{B}G$ is interesting will become clear in Sections 1.2 and 1.3.

Construction 1.1.25. There is a functor $EG \to BG$ which maps pairs of elements (x, y) of G in EG to the group element $y^{-1}x$ in BG. The induced simplicial map $\mathbf{E}G \to \mathbf{B}G$ then has n-component

$$\mathbf{E}G_n \to \mathbf{B}G_n, \quad (g_0, ..., g_n) \mapsto (g_1^{-1}g_0, ..., g_n^{-1}g_{n-1}).$$

1.2 The Realisation of a Simplicial Set

As simplicial sets act as combinatorial models for topological spaces one can construct a topological space out of the data of a simplicial set. The process of constructing a topological space is known as geometric realisation. In this section we will explain how to construct the geometric realisation of a simplicial set, discuss some of the properties that are enjoyed by the realisation functor, and introduce the classifying space of a category.

Before beginning our discussion of the geometric realisation we note that in this thesis all the topological spaces we will work with will be compactly generated weakly hausdorff (CGWH) spaces. The category of CGWH spaces and continuous maps will be denoted **CGWH**. In general, the inclusion of **CGWH** in Top does not preserve limits and colimits; however, many of the limits and colimits of diagrams in **CGWH** considered in this thesis are preserved by the inclusion. In particular, a number of the operations we will perform on simplicial sets (and later, spaces) will involve limits or colimits of diagrams of spaces in **CGWH**; these limits and colimits will be performed in **Top**, but will land in **CGWH**. It is important to keep this fact in mind as, while both **Top** and **CGWH** are (co)complete, how limits and colimits are computed in **CGWH** is slightly different to how they are computed in **Top**. Readers unfamiliar with **CGWH** and its properties are referred to [28].

Before giving the explicit formula for the geometric realisation of a simplicial set, we will first explain the intuition behind how a space is constructed from the data of a simplicial set. To understand this intuition we need to first define the standard topological n-simplex.

Definition 1.2.1. The *standard topological n-simplex* $|\Delta^n|$ is the set

$$\{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : t_0 + \cdots + t_n = 1, t_i \geq 0\} \subseteq \mathbb{R}^{n+1}.$$

equipped with the subspace topology. Let $|d^i|$ and $|s^j|$ be the maps

$$\left|d^i\right| : |\Delta^n| \to \left|\Delta^{n+1}\right|, \quad \left|d^i\right|(t_0, ..., t_n) = (t_0, ..., t_{i-1}, 0, t_i, ..., t_n)$$

and

$$\left|s^j\right| : \left|\Delta^{n+1}\right| \to |\Delta^n|, \quad \left|s^j\right|(t_0, ..., t_{n+1}) := (t_0, ..., t_i + t_{i+1}, ..., t_n).$$

Now suppose we have been given a simplicial set X. To build a space out of the data of X, identify each n-simplex of X with a topological n-simplex. Next, whenever a simplex x is the i-th face of some other simplex y, glue the topological simplex associated to x to the i-th face of the topological simplex associated to y. Finally, if an n-simplex z of X is degenerate, suppress it by collapsing its associated standard topological n-simplex into the topological $n-1$-simplex associated to the simplex it is a degeneracy of. The resulting space $|X|$ is called the geometric realisation of X. This construction is formalised in Definition 1.2.2.

Definition 1.2.2. Let X be a simplicial set. The *geometric realisation* $|X|$ of X is the topological space

$$\left(\coprod_{n=0}^{\infty} X_n \times |\Delta^n|\right) / \sim,$$

where each X_n is equipped with the discrete topology, and $\sim$ is the equivalence relation generated by the relations:

$$(x, \left|d^i\right|(t)) \sim (d_i(x), t) \quad \text{for } x \in X_{n+1},\ t \in |\Delta^n| \tag{1.10}$$

$$(x, \left|s^j\right|(t)) \sim (s_j(x), t) \quad \text{for } x \in X_n,\ t \in \left|\Delta^{n+1}\right| \tag{1.11}$$

Construction 1.2.3. Definition 1.2.2 can be extended to define a functor

$$|-| : \mathbf{sSet} \to \mathbf{Top},$$

where if $f : X \to Y$ is a simplicial map the continuous map $|f| : |X| \to |Y|$ is the unique map making the diagram

$$\begin{array}{ccc} \coprod_{n=0}^{\infty} X_n \times |\Delta^n| & \xrightarrow{f_n \times \mathrm{Id}} & \coprod_{n=0}^{\infty} Y_n \times |\Delta^n| \\ \downarrow & & \downarrow \\ |X| & \dashrightarrow & |Y| \end{array} \tag{1.12}$$

commute.

Remark 1.2.4. Note that the choice of notation for the standard topological n-simplex $|\Delta^n|$ does not contradict the notation used to denote the realisation of the standard n-simplex Δ^n, as the two spaces are homeomorphic.

When reading about the theory of simplicial sets or its applications (in sources such as [8]), the reader may encounter a more abstract definition of the geometric realisation. We pause to comment on this definition now.

Definition 1.2.5. Let X be a simplicial set. Let Δ/X be the category whose objects are simplicial maps $\sigma : \Delta^n \to X$ and maps are simplicial maps $\theta : \Delta^n \to \Delta^m$ that make the diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\theta} & \Delta^m \\ & {}_{\sigma}\searrow \quad \swarrow_{\tau'} & \\ & X & \end{array} \tag{1.13}$$

commute.

Construction 1.2.6. All presheaves can be expressed as a colimit of representable functors, as we now recall. That is, every simplicial set X can be expressed as the colimit

$$X = \varinjlim_{\Delta/X} \Delta^n.$$

The geometric realisation $|X|$ of X can then be defined as the colimit

$$|X| := \varinjlim_{\Delta/X} |\Delta^n|$$

in **Top**. This definition of geometric realisation agrees with the one presented in Definition 1.2.2 up to homeomorphism.

We will now state three useful properties of the realisation functor.

Proposition 1.2.7. *If X is a simplicial set, $|X|$ is a CW-complex.*

Proof. See Proposition 2.3 on page 8 of [8]. □

Thus, the realisation functor lands inside the category of CW-complexes, and hence **CGWH** (CW-complexes are compactly generated and Hausdorff). This fact is used in the next useful property of the realisation functor we will state.

Proposition 1.2.8. *The geometric realisation functor*

$$|-| : \mathbf{sSet} \to \mathbf{CGWH}$$

preserves finite products. That is, if X and Y are simplicial sets then

$$|X \times Y| \cong |X| \times |Y|,$$

where the $\times$ on the right hand side is the product in **CGWH**.

Proof. See Section 3.3 in [7]. □

The final useful property of the realisation functor we will discuss is that it has a right adjoint. We will show that the realisation functor has a right adjoint $S(-)$ by explicitly constructing it, and then constructing a unit $Id \to S(-) \circ |-|$ and counit $|-| \circ S(-) \to Id$.

Definition 1.2.9. Let T be a topological space. The *total singular complex* $S(T)$ of T is the simplicial set whose set of n-simplices is $\hom_{\mathbf{Top}}(|\Delta^n|, T)$. The face maps d_i and degeneracy maps s_j of $S(T)$ are the maps

$$d_i : \hom_{\mathbf{Top}}(\left|\Delta^{n+1}\right|, T) \to \hom_{\mathbf{Top}}(|\Delta^n|, T), \quad d_i(f) := f \circ \left|d^i\right|$$

and

$$s_j : \hom_{\mathbf{Top}}(|\Delta^n|, T) \to \hom_{\mathbf{Top}}(\left|\Delta^{n+1}\right|, T), \quad s_j(f) := f \circ \left|s^j\right|.$$

Construction 1.2.10. Definition 1.2.9 can be extended to define a functor

$$S(-) : \mathbf{Top} \to \mathbf{sSet},$$

where if $f : T \to Z$ is a continuous map of topological spaces the simplicial map $S(f) : S(T) \to S(Z)$ has n-component

$$S(f) : \hom_{\mathbf{Top}}(|\Delta^n|, T) \to \hom_{\mathbf{Top}}(|\Delta^n|, Z), \quad S(f)(g) := f \circ g.$$

Proposition 1.2.11. *The geometric realisation functor* $|-| : \mathbf{sSet} \to \mathbf{Top}$ *is left adjoint to the total singular complex functor* $S(-) : \mathbf{Top} \to \mathbf{sSet}$.

Proof. We sketch a proof by writing down the unit and counit of the adjunction. The X-component of the unit map of the adjunction has n-component

$$X_n \to S(|X|)_n, \quad x \mapsto \phi_x,$$

where

$$\phi_x : |\Delta^n| \to |X|, \quad \phi_x(t) := [(t, x)].$$

The counit map of the adjunction has Y-component

$$|S(Y)| \to Y, \quad [(t, f)] \mapsto f(t).$$

□

We will discuss the $|-| \dashv S(-)$ adjunction further in Section 1.6.

We will conclude this section by briefly introducing the classifying space of a category.

Definition 1.2.12. Let C be a small category. The *classifying space* BC of C is the topological space

$$BC := |NC|,$$

i.e. the geometric realisation of the nerve NC of C.

A natural question one might ask after reading Definition 1.2.12 is: '*Why is the classifying space of a category called its classifying space?*'. The name simply comes from the fact if G is a group, then the classifying space BG of the category G (i.e. the realisation of the classifying simplicial set $\mathbf{B}G$) is a model for the classifying space of the group G in the theory of G-bundles. Furthermore, the induced map $EG \to BG$ of spaces (see Construction 1.1.25) is the universal principal G-bundle (see Sections 5.2 and 5.3 in [8]).

1.3 Simplicial Homotopy Theory

In Section 1.6 the notion that simplicial sets model topological spaces will be made precise by appealing to some abstract homotopy theory. To apply abstract homotopy theory to simplicial sets we obviously first need to understand some of the basic homotopy theory of simplicial sets. We will present this basic theory in this section. Note that we will only outline key definitions and results, as the details involved in the homotopy theory of simplicial sets are very technical. The enthusiastic reader is referred to [8] for these details.

We will begin presenting the homotopy theory of simplicial sets by defining the homotopy groups of a simplicial set, and defining what it means for simplicial sets to be weakly homotopy equivalent.

Definition 1.3.1. Let X be a simplicial set and x_0 a vertex of X. Write x_0 for the corresponding point of $|X|$. Define $\pi_0(X)$ to be the set $\pi_0(|X|)$, and for each $k \geq 1$ define $\pi_k(X, x_0)$ to be the homotopy group $\pi_k(|X|, x_0)$. The set $\pi_0(X)$ is called the *path-components* of X, and $\pi_k(X, x_0)$ is called the *k-th homotopy group* of X.

Definition 1.3.2. A simplicial map $X \to Y$ is called a *weak homotopy equivalence* if the induced map $|X| \to |Y|$ is a weak homotopy equivalence of topological spaces.

Remark 1.3.3. Recall that Whitehead's theorem says that a map of CW-complexes is a homotopy equivalence if and only if it is a weak homotopy equivalence. So, a simplicial map $X \to Y$ is a weak homotopy equivalence if and only if the map $|X| \to |Y|$ is a homotopy equivalence.

Unfortunately, as seen in Definition 1.3.2, to define what it mean for a map $X \to Y$ of simplicial sets to be a weak homotopy equivalence we must appeal to geometric realisation. One would reasonably hope that we could determine whether the map $X \to Y$ was a weak homotopy equivalence using only the combinatorial data of X and Y, else what would be the point of modelling spaces as simplicial sets? Fortunately, there is a subclass of simplicial sets, called Kan complexes, for which this is possible. To define Kan complexes, we need to first define Kan fibrations.

Definition 1.3.4. A simplicial map $X \to Y$ is called a *Kan fibration* if for all $n \geq 1$ and $0 \leq k \leq n$ and every commutative diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \Delta^n & \longrightarrow & Y \end{array} \tag{1.14}$$

in **sSet** there is a map $\Delta^n \to X$ such that the diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \longrightarrow & Y \end{array} \tag{1.15}$$

commutes.

Definition 1.3.5. A simplicial set X is called a *Kan complex* if the canonical map $X \to \Delta^0$ is a Kan fibration. That is, X is a Kan complex if for all $n \geq 1$ and $0 \leq k \leq n$ and every map $\Lambda^n_k \to X$ there exists a map $\Delta^n \to X$ such that the diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array} \tag{1.16}$$

commutes.

Example 1.3.6. For all topological spaces X, the simplicial set $S(X)$ is a Kan complex. See Lemma 3.3 on page 11 of [8] for a proof of this.

To define what it means for Kan complexes to be weakly homotopy equivalent without appealing to geometric realisation, we need to give a definition of the homotopy groups of a Kan complex that also does not appeal to geometric realisation. Recall that the n-th homotopy group $\pi_n(X, x)$ of a space X can be defined to be the set of homotopy classes of maps $|\Delta^n| \to X$ (relative to $|\partial\Delta^n|$) that map $|\partial\Delta^n|$ to the point x. Replacing spaces with simplicial sets, we will define the n-th homotopy group $\pi_n(X, x)$ of a Kan complex X to be the set of homotopy classes of the n-simplices of X (relative to $\partial\Delta^n$) that map $\partial\Delta^n$ to x. Thus, we need to define what it means for simplicial maps to be homotopic.

Definition 1.3.7. Let $g, f : X \to Y$ be simplicial maps and Z be a simplicial subset of X. The map f is homotopic to g if there exists a simplicial map $H : X \times \Delta^1 \to Y$ making the diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle Id\times d^1}\big\downarrow & \overset{f}{\searrow} & \\ X \times \Delta^1 & \xrightarrow{H} & Y \\ {\scriptstyle Id\times d^0}\big\uparrow & \overset{g}{\nearrow} & \\ X & & \end{array} \tag{1.17}$$

commute. If $f|_Z = g|_Z$ we say f is homotopic to g relative to Z if the diagram

$$\begin{array}{ccc} Z \times \Delta^1 & \hookrightarrow & X \times \Delta^1 \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle H} \\ Z & \longrightarrow & Y \end{array} \tag{1.18}$$

commutes.

As we will define the homotopy groups of Kan complexes to be the homotopy classes of their simplices, we need to check that the relation of simplicial maps being homotopic is an equivalence relation.

Proposition 1.3.8. *Let Y be a Kan complex and Z a simplicial subset of X. Then*

1. *the relation of simplicial maps $X \to Y$ being homotopic is an equivalence relation; and*
2. *the relation of simplicial maps $X \to Y$ being homotopic (relative Z) is an equivalence relation.*

Proof. See Corollary 6.2 on page 24 in [8]. □

Definition 1.3.9. Let X be a Kan complex and x_0 a vertex of X. Define $\pi_0(X)$ to be the set of homotopy classes of maps $\Delta^0 \to X$. For all $n \geq 1$ define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $\Delta^n \to X$ (relative $\partial\Delta^n$) making the diagram

$$\begin{array}{ccc} \partial\Delta^n & \hookrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{x_0} & X \end{array} \tag{1.19}$$

commute.

Now we need to know that there is some natural group structure on $\pi_n(X, x)$ for all $n \geq 1$, and that this group structure is abelian when $n \geq 2$. This is the content of the next proposition.

Proposition 1.3.10. *Let X be a Kan complex and x_0 a vertex of X. For all $n \geq 1$ the set $\pi_n(X, x_0)$ can be equipped with the structure of a group. Furthermore, $\pi_n(X, x_0)$ will be abelian if $n \geq 2$.*

Proof. See Theorem 7.2 on page 26 in [8]. □

Construction 1.3.11. Note that if $f : X \to Y$ is a map of Kan complexes, and x_0 is a vertex of X, there is an induced group homomorphism

$$\pi_n(f) : \pi_n(X, x_0) \to \pi_0(Y, f(x_0)), \quad \pi_n(f)([x]) := [f(x)],$$

for all $n \geq 1$. Similarly, there is an induced map of sets

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y).$$

With the homotopy groups of Kan complexes defined, we can finally define what it means for Kan complexes to be weakly homotopy equivalent.

Definition 1.3.12. A simplicial map $f : X \to Y$ of Kan complexes is called a *weak homotopy equivalence* if for all vertices x_0 of X, and for all $n \geq 1$, the induced map $\pi_n(f) : \pi_n(X, x_0) \to \pi_0(Y, f(x_0))$ is a group isomorphism, and the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.

We now need to check that Definitions 1.3.1 and 1.3.9 and Definitions 1.3.2 and 1.3.12 are equivalent.

Proposition 1.3.13. *Let X be a Kan complex. Then for all vertices x_0 of X and for all $n \geq 1$ there is an isomorphism of groups*

$$\pi_n(X, x_0) \cong \pi_n(|X|, x_0),$$

and a bijection

$$\pi_0(X) \cong \pi_0(|X|).$$

Furthermore, a simplicial map $X \to Y$ of Kan complexes is a weak homotopy equivalence if and only if the induced map $|X| \to |Y|$ is a weak homotopy equivalence.

Proof. See Section 1.11 in [8]. □

Let us suppose we want to study the homotopy type of a simplicial set X without appealing to its realisation, but it is not a Kan complex, what can we do? It turns out that there is a functor $Ex^\infty(-) : \mathbf{sSet} \to \mathbf{sSet}$ which maps simplicial sets X to Kan complexes $Ex^\infty(X)$, such that there is a weak homotopy equivalence $X \to Ex^\infty(X)$. We will not provide any details on how this functor is constructed (for this the reader is referred to Section 3.4 in [8]), but we state some of its useful properties in Theorem 1.3.14 below.

Theorem 1.3.14. *There is a functor*

$$Ex^\infty(-) : \mathbf{sSet} \to \mathbf{sSet},$$

which has the following properties:

1. *For every simplicial set X, the simplicial set $Ex^\infty(X)$ is a Kan complex.*
2. *$Ex^\infty(-)$ maps Kan fibrations to Kan fibrations.*
3. *There is a canonical map $X \to Ex^\infty(X)$, natural in X, that is a weak homotopy equivalence.*
4. *$Ex^\infty(-)$ commutes with finite limits.*
5. *$Ex^\infty(X)_0 = X_0$ for every simplicial set X.*

Proof. Statements 1-3 are Theorem 4.8 on page 188 in [8]. Statement 4 is Lemma 17.5.4 in [22]. Statement 5 follows easily from the definition of the functor $Ex^\infty(-)$. □

In Chapter 5 the homotopy 1-type of a simplicial set (a notion made precise in Definition 1.3.15) will play an important role. As is the case for CW-complexes, the homotopy 1-type of a simplicial set depends only on its 2-skeleton.

Definition 1.3.15. A simplicial map $f : X \to Y$ is a homotopy 1-equivalence if for all vertices x_0 of X the induced maps $\pi_0(X) \to \pi_0(Y)$ and $\pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ are isomorphisms. Two simplicial sets X and Y have the same *homotopy 1-type* if they are connected by a chain of homotopy 1-equivalences.

Briefly recall the definition of a simplicial subset generated by a set (Definition 1.1.13).

Definition 1.3.16. Let X be a simplicial set. The 2-skeleton sk_2X of X is the simplicial subset of X generated by the simplices of X of degree less than or equal to 2.

Lemma 1.3.17. *If C is a CW-complex the inclusion map $sk_2C \hookrightarrow C$ is a homotopy 1-equivalence, where sk_2C denotes the 2-skeleton of C.*

Proof. This follows from Corollary 4.12 in [9]. □

Proposition 1.3.18. *If X is a simplicial set then its homotopy 1-type depends only on its 2-skeleton. That is, the inclusion map $sk_2X \hookrightarrow X$ is a homotopy 1-equivalence.*

Proof. From the structure of $|X|$ as a CW-complex, one can show that there is a homeomorphism
$$|sk_2X| \cong sk_2|X|$$
(see Proposition 2.3 on page 8 of [8]). The result then follows from Lemma 1.3.17. □

We will now conclude our discussion of the basic homotopy theory of simplicial sets and Kan complexes by stating some results concerning how the theory interacts with categories and functors via the nerve functor.

Proposition 1.3.19. *The nerve NC of a category C is a Kan complex if and only if C is a groupoid.*

Proof. This follows from the argument made to prove Proposition 1.1.2.2 in [16]. □

Lemma 1.3.20. *Let $F, G : C \to D$ be functors and $\alpha : F \to G$ be a natural transformation. Then NF is homotopic to NG, and BF is homotopic to BG.*

Proof. Note that the natural transformation $\alpha : F \to G$ induces a commutative diagram

$$\begin{array}{ccc} C & & \\ {\scriptstyle Id\times 0}\downarrow & \searrow^{F} & \\ C \times [1] & \longrightarrow & D \\ {\scriptstyle Id\times 1}\uparrow & \nearrow_{G} & \\ C & & \end{array} \tag{1.20}$$

As the nerve N and realisation functors $|-|$ preserve products, and $|\Delta^1| \cong I$ where I denotes the unit interval, there are hence commutative diagrams

$$\begin{array}{ccc} NC & & \\ {\scriptstyle Id\times d^1}\downarrow & \searrow^{NF} & \\ NC \times \Delta^1 & \longrightarrow & ND \\ {\scriptstyle Id\times d^0}\uparrow & \nearrow_{NG} & \\ NC & & \end{array} \tag{1.21}$$

and

$$\begin{array}{ccc} BC & & \\ {\scriptstyle Id\times d^1}\downarrow & \searrow^{BF} & \\ BC \times I & \longrightarrow & BD. \\ {\scriptstyle Id\times d^0}\uparrow & \nearrow_{BG} & \\ BC & & \end{array} \tag{1.22}$$

Hence, NF is homotopic to NG, and BF is homotopic to BG. □

Corollary 1.3.21. *If $F : C \rightleftarrows D : G$ is an adjunction (or an equivalence of categories) then NC and ND are homotopy equivalent simplicial sets, and BC and BD are homotopy equivalent spaces.*

Before we continue our discussions in Section 1.6 on how simplicial sets act as models for topological spaces we will discuss simplicial objects, which are the natural generalisation of simplicial sets.

1.4 Simplicial Objects

Observe that the basic definitions in the theory of simplicial sets do not exploit the properties of **Set** very often. Rather, the properties of the simplex category Δ are used more frequently. This observation motivates the definition of simplicial objects in a category C, which are contravariant functors mapping Δ into C. In this section, we will introduce 5 different types of simplicial object. These simplicial objects will play an important role later in this thesis.

Let us begin by defining simplicial objects categorically and commenting on how, like simplicial sets, a simplicial object can also be described combinatorially.

Definition 1.4.1. Let C be a category. A *simplicial object* in the category C is a functor $X : \Delta^{op} \to C$. A map of simplicial objects in C is a natural transformation $X \to Y$. Let $\mathbf{s}C$ denote the category of simplicial objects in C and maps of simplicial objects in C.

Remark 1.4.2. Using identical arguments to those made in Construction 1.1.9, simplicial objects and maps can be described combinatorially as follows: a simplicial object X in a category C is a family of objects $\{X_n\}_{n\in\mathbb{N}}$ in C and maps in $d_i : X_{n+1} \to X_n$ and $s_j : X_n \to X_{n+1}$ in C for each $n \geq 0$ and i and j such that $0 \leq i \leq n+1$ and $0 \leq j \leq n$. The maps d_i and s_j must also satisfy the simplicial identifies (equations (1.4)–(1.6)). The maps d_i and s_j are called the face and degeneracy maps of X, respectively. A map of simplicial objects in C is a family of maps $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ in C that commute with the face and degeneracy maps of X and Y.

In this thesis, aside from simplicial sets, we will use five different types of simplicial objects corresponding to the categories $\mathbf{Set}_*$ of pointed sets, **Mon** of monoids, **Grp** of groups, **Top** of spaces, and **sSet** of simplicial sets. We define these simplicial objects precisely, and explain how to take their realisations, below.

Definition 1.4.3. A simplicial object $X : \Delta^{op} \to \mathbf{Set}_*$ in $\mathbf{Set}_*$ is called a *pointed simplicial set*.

Definition 1.4.4. A simplicial object $X : \Delta^{op} \to \mathbf{Mon}$ in **Mon** is called a *simplicial monoid*.

Definition 1.4.5. A simplicial object $X : \Delta^{op} \to \mathbf{Grp}$ in **Grp** is called a *simplicial group*.

Remark 1.4.6. To take the geometric realisation $|X|$ of a pointed simplicial set, simplicial monoid or simplicial group X, take the realisation of the underlying simplicial set of X. Observe that the realisation of a pointed simplicial set is canonically a pointed space, the realisation of a simplicial monoid is a topological monoid (understood as a monoid object in **CGWH**), and the realisation of a simplicial group is a topological group (understood as a group object in **CGWH**).

Definition 1.4.7. A simplicial object $X : \Delta^{op} \to \mathbf{Top}$ in **Top** is called a *simplicial space*.

Remark 1.4.8. The simplicial spaces we will work with in this thesis will be valued in **CGWH**.

Definition 1.4.9. Let X be a simplicial space. The *geometric realisation* $|X|$ of X is the topological space

$$\left(\coprod_{n=0}^{\infty} X_n \times |\Delta^n|\right) / \sim,$$

where $\sim$ is the equivalence relation generated by the relations:

$$(x, \left|d^i\right|(t)) \sim (d_i(x), t) \quad \text{for } x \in X_{n+1},\, t \in |\Delta^n| \tag{1.23}$$

$$(x, \left|s^j\right|(t)) \sim (s_j(x), t) \quad \text{for } x \in X_n,\, t \in \left|\Delta^{n+1}\right| \tag{1.24}$$

It turns out that the realisation of a simplicial space can also be constructed sequentially. This sequential construction is outlined in Construction 1.4.10 below.

Construction 1.4.10. Let X be a simplicial space. Let $|X|_0 = X_0$, and for each $n \geq 1$ let $|X|_n$ be the pushout of the diagram

$$\begin{array}{ccc} \partial|\Delta^n| \times X_n \cup |\Delta^n| \times s(X_{n-1}) & \longrightarrow & |X|_{n-1} \\ \downarrow & & \downarrow \\ |\Delta^n| \times X_n & \longrightarrow & |X|_n \end{array} \tag{1.25}$$

in **Top**, where $s(X_{n-1}) = \cup_{i=0}^{n-1} s_i(X_{n-1})$. The geometric realisation $|X|$ of X can then be defined as the sequential colimit of the diagram

$$|X|_0 \longrightarrow |X|_1 \longrightarrow |X|_2 \longrightarrow \dots \longrightarrow |X| \tag{1.26}$$

in **Top**.

Remark 1.4.11. Note that the definitions of realisation of a simplicial space given in Definitions 1.4.9 and Construction 1.4.10 agree up to homeomorphism.

Remark 1.4.12. Note that the colimits in Construction 1.4.10 are computed in **Top**. However, if the simplicial space X is valued in **CGWH**, then each colimit computed in Construction 1.4.10 will land inside **CGWH**. This fact will play a role in the proof of Proposition 1.4.13 below.

Using the sequential construction of the realisation of a simplicial space we can prove that if a simplicial space is valued in **CGWH** then its realisation is valued in **CGWH**.

Proposition 1.4.13. *If X be a simplicial space valued in* **CGWH***, then the space $|X|$ is compactly generated weakly Hausdorff.*

Proof. By Lemma 3.3 in [28] to show $|X|$ is compactly generated weak Hausdorff it is sufficient to show that for each $n \geq 0$ the space $|X|_n$ is compactly generated weak Hausdorff, and the map $|X|_n \to |X|_{n+1}$ is a closed inclusion. Note that $|X|_0$ is compactly generated weak Hausdorff. Hence, proceeding by induction, by Proposition 2.35 in [28], it is sufficient to show that for each n the spaces $\partial\Delta^n \times X_n \cup \Delta^n \times s(X_{n-1})$ and $|\Delta^n| \times X_n$ in diagram (1.25) are compactly generated weak hausdorff, and the inclusion map

$$\partial|\Delta^n| \times X_n \cup |\Delta^n| \times s(X_{n-1}) \to |\Delta^n| \times X_n$$

is a closed inclusion. By Proposition 2.6 in [28] the space $X_n \times |\Delta^n|$ is compactly generated weakly Hausdorff. Now, as the map $d_i s_i : X_{n-1} \to X_{n-1}$ is the identity, by Corollary 2.29 in [28], the space $s_i(X_{n-1})$ is closed in X_n. Thus, as the spaces $|\Delta^n| \times s(X_{n-1})$ and $\partial|\Delta^n| \times X_n$ are closed in $|\Delta^n| \times X_n$, so is the space $|\Delta^n| \times s(X_{n-1}) \cup \partial|\Delta^n| \times X_n$. Thus, by Lemma 2.26 in [28], the space $|\Delta^n| \times s(X_{n-1}) \cup \partial|\Delta^n| \times X_n$ is compactly generated weak hausdorff and the inclusion map

$$\partial|\Delta^n| \times X_n \cup |\Delta^n| \times s(X_{n-1}) \to |\Delta^n| \times X_n$$

is a closed inclusion. □

The simplicial spaces we will consider in this thesis will often arise from bisimplicial sets via geometric realisation. Let us introduce bisimplicial sets and their realisations now.

Definition 1.4.14. A simplicial object $X : \Delta^{op} \to$ **sSet** in **sSet** is called a *bisimplicial set*.

Conventions 1.4.15. Note that we will adopt the following conventions when discussing bisimplicial sets in the remainder of this thesis:

1. Given a bisimplicial set X, the set $X_{n,m}$ will be called the set of *(n,m)-bisimplices* of X.
2. The maps $d_i : X_{m,n} \to X_{m-1,n}$ and $s_j : X_{m,n} \to X_{m+1,n}$ will be called the *horizontal* face and degeneracy maps of X.
3. The maps $d_i : X_{m,n} \to X_{m,n-1}$ and $s_j : X_{m,n} \to X_{m,n+1}$ will be called the *vertical* face and degeneracy maps of X.

Construction 1.4.16. There is a functor

$$d : \mathbf{ssSet} \to \mathbf{sSet}$$

which sends bisimplicial sets X to the simplicial set dX whose set of n-simplices is the set $X_{n,n}$. The functor d is called the *diagonal functor*.

Construction 1.4.17. There are 3 different ways to define the geometric realisation of a bisimplicial set, and they are all equivalent up to natural homeomorphism (see Section 1 in [24] for details). We will briefly outline these three approaches now. To define the realisation $|X|$ of the bisimplicial set X note that if we fix an $[n]$ we have that $X_{n,\bullet}$ is a simplicial set, and thus has a realisation. Thus, there is a simplicial space $[n] \mapsto |X_{n,\bullet}|$. Similarly, there is a simplicial space $[n] \mapsto |X_{\bullet,n}|$. So, the realisation of the bisimplicial set X could be defined to be the realisation of one of these two simplicial spaces. Alternatively, we could first form the diagonal dX of X, take its realisation, and define $|dX|$ to be the realisation of X. It does not matter which of these three definitions of realisation one chooses to use as it turns out there are homeomorphisms

$$|dX| \cong |[n] \mapsto |X_{n,\bullet}|| \cong |[n] \mapsto |X_{\bullet,n}||$$

natural in X.

Bisimplicial sets will make a few appearances in this thesis, and we will only need to know how to take their realisation. Further discussion will be provided on simplicial spaces in Chapter 3, simplicial monoids in Chapter 4, and pointed simplicial sets in the next section. The only fact we will need about simplicial groups is stated in Proposition 1.4.18 below.

Proposition 1.4.18. *The underlying simplicial set of a simplicial group is a Kan complex.*

Proof. See Lemma 3.4 in [8]. □

1.5 Pointed Simplicial Sets

As defined in the previous section, a pointed simplicial set is a functor $X : \Delta^{op} \to \mathbf{Set}_*$. There are notions of suspension and loop space for pointed simplicial sets. They are constructed by replacing all the pointed spaces in the definitions of the suspension and loop space of a space with their pointed simplicial analogues. In this section we will define the suspension and loop space of a pointed simplicial set, and provide some intuition behind their definitions.

Let us begin by giving a less abstract definition of pointed simplicial set.

Definition 1.5.1. A simplicial set X is *pointed* if there exists a distinguished vertex x_0. The point x_0 is called the *basepoint* of (X, x_0). We will write (X, x_0) for a pointed simplicial set or just X if x_0 is understood. A *pointed simplicial map* $f : (X, x_0) \to (Y, y_0)$ is a simplicial map such that $f(x_0) = y_0$. The category of pointed simplicial sets and maps is denoted $\mathbf{sSet}_*$.

Remark 1.5.2. If (X, x_0) is pointed then for each n the set of n-simplices X_n has a canonical basepoint given by $s_0^n(x_0)$.

To every simplicial set X there is a functorially associated pointed simplicial set X_+.

Definition 1.5.3. If X is a simplicial set let $(X_+, *)$ be the simplicial set X equipped with a disjoint basepoint.

Construction 1.5.4. Definition 1.5.3 can be extended to define a functor

$$(-)_+ : \mathbf{sSet} \to \mathbf{sSet}_*.$$

It is easy to show that the functor $(-)_+$ is left adjoint to the forgetful functor

$$U(-) : \mathbf{sSet}_* \to \mathbf{sSet}.$$

Recall that the suspension of a pointed space is its smash product with the 1-sphere $\mathbb{S}^1$. Thus, to define the suspension of a pointed simplicial set by replacing each space in the definition of the suspension of a space with their simplicial analogue, we need to first construct simplicial analogues of the smash product and the 1-sphere.

Recall that the smash product of two pointed spaces is given by their product quotient their wedge. Hence, to define the smash product of two pointed simplicial sets we need to first define their wedge.

Definition 1.5.5. Let (X, x_0) and (Y, y_0) be pointed simplicial sets. Let $X \vee Y$ be the pushout in the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{y_0} & Y \\ \downarrow{\scriptstyle x_0} & \ulcorner & \downarrow \\ X & \longrightarrow & X \vee Y \end{array} \tag{1.27}$$

in **sSet**. The pointed simplicial set $(X \vee Y, [x_0])$ is called the *wedge* of (X, x_0) and (Y, y_0).

Definition 1.5.6. Let (X, x_0) and (Y, y_0) be pointed simplicial sets. Define the simplicial set

$$X \wedge Y := X \times Y / X \vee Y.$$

The pointed simplicial set $(X \wedge Y, [x_0, y_0])$ is called the *smash product* of (X, x_0) and (Y, y_0).

Recall that the 1-sphere can be thought of as the unit interval with the endpoints identified. Taking the simplicial unit interval to be Δ^1, and its endpoints to be $\partial\Delta^1$, we can define the 1-sphere as a simplicial set.

Definition 1.5.7. Let $\mathbb{S}^1$ be the pushout in the diagram

$$\begin{array}{ccc} \partial\Delta^1 & \hookrightarrow & \Delta^1 \\ \downarrow & \ulcorner & \downarrow \\ \Delta^0 & \longrightarrow & \mathbb{S}^1 \end{array} \tag{1.28}$$

in **sSet**.

Remark 1.5.8. Note that the simplicial set $\mathbb{S}^1$ has a unique vertex, and so has a canonical basepoint, denoted 0.

Definition 1.5.9. Let (X, x_0) be a pointed simplicial set. Let ΣX be the simplicial set

$$\Sigma X = X \wedge \mathbb{S}^1.$$

The pointed simplicial set $(\Sigma X, [x_0, 0])$ is called the *suspension* of X.

Let's now define the loop space of a pointed simplicial set. The loop space ΩX of a pointed space X is simply the set of pointed maps from $\mathbb{S}^1$ into X equipped with some topology. So, to define the loop space of a pointed simplicial set X, we need to first construct a simplicial analogue for the space of pointed maps from $\mathbb{S}^1$ into X.

Definition 1.5.10. If X and Y are pointed simplicial sets let $\mathbf{Hom}_*(X, Y)$ be the simplicial set whose set of n-simplices is the set

$$\mathbf{sSet}_*(X \wedge \Delta^n_+, Y).$$

The simplicial set $\mathbf{Hom}_*(X, Y)$ is called the *pointed function complex* of X and Y.

Remark 1.5.11. Note that $\mathbf{Hom}_*(X, Y)$ has a canonical basepoint ω_{y_0} given by the pointed constant map $X \to Y$.

The loop space of a pointed simplicial set X is hence defined as the pointed function complex of $\mathbb{S}^1$ and X equipped with its canonical basepoint.

Definition 1.5.12. If (X, x_0) is a pointed simplicial set, let ΩX be the simplicial set $\mathbf{Hom}_*(\mathbb{S}^1, X)$. The pointed simplicial set $(\Omega X, \omega_{x_0})$ is called the *loop space* of X.

Construction 1.5.13. There are functors

$$\Omega : \mathbf{sSet}_* \to \mathbf{sSet}_*$$

and

$$\Sigma : \mathbf{sSet}_* \to \mathbf{sSet}_*,$$

which will both act on pointed simplicial maps in the obvious way.

Thus, we have defined the loop space and the suspension of a pointed simplicial set. In the case of spaces the loop space and suspension functors are adjoint. This is also true in the case of pointed simplicial sets, as one would hope.

Proposition 1.5.14. *The suspension functor Σ is left adjoint to the loop space functor Ω.*

Proof. There is a bijection

$$\hom_{\mathbf{sSet}_*}(X \wedge Y, Z) \cong \hom_{\mathbf{sSet}_*}(X, \mathbf{Hom}_*(Y, Z)).$$

natural in X, Y and Z (see Section 4.4 in [7]). Taking $Y = \mathbb{S}^1$ gives the result. □

1.6 Model Categories and Model Structures

We will now conclude our discussion on the theory of simplicial sets by making precise the notion that simplicial sets model topological spaces. This notion is made precise through the use of model categories and homotopy categories. In this section we discuss the basics of the theory of model categories and their homotopy categories, give some examples of model categories that will be important in this thesis, and discuss the relationship between the homotopy category of spaces and the homotopy category of simplicial sets. The theory involved in formal discussions of these topics is technical, so we will not go into much detail and will only outline the key results of interest.

Let us begin by defining closed model categories.

Definition 1.6.1. A closed model category is a category $\mathcal{C}$, together with three classes of maps called cofibrations, fibrations, and weak equivalences, which satisfy the following axioms:

CM1: $\mathcal{C}$ has all limits and all colimits.

CM2: If the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & C & \end{array} \tag{1.29}$$

commutes in $\mathcal{C}$, and any of the two maps in diagram (1.29) are weak equivalences, then the third map is also a weak equivalence.

CM3: If f is a retract of g and g is a weak equivalence/fibration/cofibration, then f is a weak equivalence/fibration/cofibration.

CM4: If a diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle p} \\ C & \longrightarrow & D \end{array} \tag{1.30}$$

commutes in $\mathcal{C}$ where i is a cofibration and p is a fibration, and if either i or p is also a weak equivalence, then there exists a diagonal filler $C \to B$ in diagram (1.30).

CM5: For all maps $f : A \to B$ in $\mathcal{C}$

(i.) there exists a fibration p and a cofibration i that is also a weak equivalence such that $f = p \circ i$; and

(ii.) there exists a fibration q that is also a weak equivalence and a cofibration j such that $f = q \circ j$.

Remark 1.6.2. For ease of exposition we will now adopt the following conventions: Fibrations (respectively, cofibrations) that are also weak equivalences will be referred to as *trivial fibrations* (respectively, *trivial cofibrations*). In diagrams weak equivalences will be denoted with

$$\xrightarrow{\simeq},$$

fibrations will be denoted with

$$\twoheadrightarrow,$$

and cofibrations will be denoted with

$$\rightarrowtail .$$

The following property is enjoyed by all closed model categories:

Proposition 1.6.3. *Let $\mathcal{C}$ be a closed model category. If $X \to Y$ is a cofibration in $\mathcal{C}$ then the map $Z \to X \cup_Y Z$ in the pushout diagram*

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & X \cup_Y Z \end{array} \tag{1.31}$$

is also a cofibration.

Proof. See Corollary 1.3 on page 68 in [8]. □

In this thesis we will primarily encounter three different model category structures. These model category structures will exist on **sSet**, **CGWH**, and **ssSet**. Let's discuss these model category structures and their properties now.

Theorem 1.6.4. *There is a closed model category structure on* **sSet** *where a simplicial map $p : X \to Y$ is:*

(i) a fibration if it is a Kan fibration;

(ii) a cofibration if it is a monomorphism; and

(iii) a weak equivalence if it is a weak homotopy equivalence.

Proof. See Section 1.11 in [8] □

Definition 1.6.5. Let $f : X \to Y$ be a map in **CGWH**. The map f is called a *Hurewicz fibration* if for every commutative diagram

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ A \times I & \longrightarrow & Y \end{array} \tag{1.32}$$

in **CGWH** there is a map $A \times I \to X$ filling diagram (1.32). The map f is called a *Hurewicz cofibration* if for every homotopy $H : X \times I \to Z$ and each commutative diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{H} & Z \\ \big\downarrow{\scriptstyle f} & \nearrow & \\ Y \times \{0\} & & \end{array} \tag{1.33}$$

in **CGWH** there is a homotopy $\tilde{H} : Y \times I \to Z$ such that the diagram

$$\begin{array}{ccc} Y \times \{0\} \cup_{X \times \{0\}} X \times [0,1] & \longrightarrow & Z \\ \big\downarrow & \nearrow_{\tilde{H}} & \\ Y \times I & & \end{array} \tag{1.34}$$

commutes.

Theorem 1.6.6. *There is a closed model category structure on* **CGWH** *where a continuous map $f : X \to Y$ is:*

(i) a fibration if it is a Hurewicz fibration;

(ii) a cofibration if it is a closed map and a Hurewicz cofibration (a closed Hurewicz cofibration for short); and

(iii) a weak equivalence if it is a homotopy equivalence.

Proof. See Theorem 4.4.4 in [23]. □

Some useful properties enjoyed by the model structure on **CGWH** are stated in Proposition 1.6.7 below.

Proposition 1.6.7. *Consider the closed model category structure on* **CGWH**. *The following statements are true:*

1. *If a map $X \to Y$ in* **CGWH** *is a Hurewicz cofibration then it is a closed inclusion. In particular, $X \to Y$ is a cofibration.*

2. *Let $A \subseteq X$ and $B \subseteq Y$ be cofibrations. If A is closed in X, then $X \times B \cup A \times Y \subseteq X \times Y$ is a cofibration.*

Proof. Statement 1 is Problem 1 on page 48 in [21] and statement 3 is Corollary 1 in [14]. □

Recall that $|-| \dashv S(-)$ adjunction. It turns out that the functors $|-|$ and $S(-)$ respect the respective model structures on **CGWH** and **sSet**. This is an instance of the more general fact that simplicial sets model spaces.

Proposition 1.6.8. *Consider the $|-| \dashv S(-)$ adjunction. The following statements are true:*

1. *For all spaces X the counit map $|S(X)| \to X$ is a weak homotopy equivalence, and for all simplicial sets K the unit map $K \to S(|K|)$ is a weak homotopy equivalence.*

2. *The functor $|-|$ maps fibrations, cofibrations and weak equivalences in* **sSet** *to fibrations, cofibrations and weak equivalences in* **CGWH***, respectively. That is, $|-|$ respects the model structures on* **sSet** *and* **CGWH***.*

3. *The functor $S(-)$ respects the model structures on* **CGWH** *and* **sSet***.*

Proof. Statement 1 is proven in Section 1.11 in [8]. Let's now prove statement 2 and 3 simultaneously. First, recall that the functor $|-|$ preserves weak equivalences (Remark 1.3.3). Now, $|-|$ sends Kan fibrations to Serre fibrations, as stated in Theorem 10.10 on page 57 in [8], and Serre fibrations of CW-complexes are Hurewicz fibrations, so $|-|$ maps fibrations to fibrations. As $S(-)$ clearly maps inclusions to cofibrations, and cofibrations in **CGWH** are closed inclusions, $S(-)$ maps cofibrations to cofibrations. To see that $S(-)$ preserves fibrations note if $f : X \to Y$ is a fibration then the commutative diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & S(X) \\ \big\downarrow & & \big\downarrow{\scriptstyle S(f)} \\ \Delta^n & \longrightarrow & S(Y) \end{array} \tag{1.35}$$

induces a commutative diagram

$$\begin{array}{ccc} |\Lambda^n_k| & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ |\Delta^n| & \longrightarrow & Y. \end{array} \tag{1.36}$$

Thus, as there is a map $|\Delta^n| \to X$ filling diagram (1.36) (f is a fibration), the adjoint map $\Delta^n \to S(X)$ fills diagram (1.35). Hence, $S(f)$ is a fibration, and so $S(-)$ preserves fibrations. If $X \to Y$ is a weak homotopy equivalence of spaces there is a commutative diagram

$$\begin{array}{ccc} |S(X)| & \xrightarrow{\sim} & X \\ \big\downarrow & & \big\downarrow \sim \\ |S(Y)| & \xrightarrow{\sim} & Y, \end{array} \tag{1.37}$$

where $\sim$ denotes a weak homotopy equivalence. Thus, as weak homotopy equivalences satisfy the 2-out-of-3 property, $|S(X)| \to |S(Y)|$ is also a weak homotopy equivalence, and hence a homotopy equivalence. Thus, $S(-)$ preserves weak equivalences, and hence also trivial fibrations. Thus, by adjointness, as $S(-)$ preserves fibrations and trivial fibrations, we also have that $|-|$ preserves cofibrations. Hence, statements 2 and 3 are true. □

Let us now discuss the model structure on **ssSet**.

Theorem 1.6.9. *There is a closed model category structure on* **ssSet** *where a map $p : X \to Y$ of bisimplicial sets is:*

(i) a cofibration if it is a pointwise cofibration; that is, p is a cofibration if for each $n \geq 0$ the map $p_n : X_n \to Y_n$ is a cofibration of simplicial sets;

(ii) a weak equivalence if it is a pointwise weak equivalence; that is, p is a weak if for each $n \geq 0$ the map $p_n : X_n \to Y_n$ is a weak equivalence of simplicial sets; and

(iii) a fibration if it is has the right lifting property (c.f. Definition 2.1.1) with respect to all trivial cofibrations.

Proof. See Section 4.3.2 in [8]. □

Remark 1.6.10. Note that a map $p : X \to Y$ in **ssSet** is a cofibration if and only $p : X_{n,m} \to Y_{n,m}$ is an injective map for all $n, m \geq 0$.

The diagonal functor $d : \mathbf{ssSet} \to \mathbf{sSet}$ turns out to map cofibrations/weak equivalences in **ssSet** to cofibrations/weak equivalences in **sSet**. These facts will be very useful in Chapters 3 and 4.

Proposition 1.6.11. *Consider the closed model category structures on* **sSet** *and* **ssSet**. *The functor* $d : \mathbf{ssSet} \to \mathbf{sSet}$ *maps cofibrations to cofibrations and weak equivalences to weak equivalences.*

Proof. That the functor d maps cofibrations to cofibrations is clear. That d maps weak equivalences to weak equivalences is Proposition 1.7 on page 199 in [8]. □

We are interested in the above model categories structures on **sSet**, **CGWH**, and **ssSet** as much of the work we will do in Chapters 2–5 will take place inside these categories. Their model structures and their properties will be helpful as they will allow us to abstract away technical details.

Previously I claimed that model and homotopy categories make precise the notion that simplicial sets model spaces. Let us begin understanding this claim by first defining the homotopy category of a closed model category $\mathcal{C}$. Essentially, the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is a category in which the weak equivalences of $\mathcal{C}$ are isomorphisms. To construct such a category, we need to define the localisation of a category.

Definition 1.6.12. Let **CAT** denote the category of categories. Let C be a category and Σ a collection of maps in C. Considering Σ as a discrete category, define the category $L(C, \Sigma)$ to be the pushout in the diagram

$$\begin{array}{ccc} \Sigma \times [1] & \longrightarrow & C \\ \downarrow & \ulcorner & \downarrow \\ \Sigma \times G([1]) & \longrightarrow & L(C, \Sigma) \end{array} \tag{1.38}$$

in **CAT**, where $G([1])$ is the category with 2 objects and a unique isomorphism between them. The category $L(C, \Sigma)$ is called the *localisation* of C with respect to Σ.

Proposition 1.6.13. *All the maps in* Σ *are invertible in* $L(C, \Sigma)$. *Furthermore, if* $C \to D$ *is a functor that makes all morphisms of* C *in* Σ *invertible, then there exists a unique functor* $L(C, \Sigma) \to D$ *such that the diagram*

$$\begin{array}{ccc} C & & \\ \downarrow & \searrow & \\ L(C, \Sigma) & \dashrightarrow & D \end{array} \tag{1.39}$$

commutes.

Proof. To prove the first statement of Proposition 1.6.13 let f be a map in Σ. Then f also corresponds to a functor $[1] \to C$. By construction, the composite functor $[1] \to C \to L(C, \Sigma)$ factors through $G([1])$. That is, the image of f in Σ is invertible.

To prove the second part of Proposition 1.6.13 let $F : C \to D$ be a functor that makes the morphisms of Σ invertible. Then let $F' : L(C, \Sigma) \to D$ be a functor that maps objects c in C to $F(c)$. The functor F' maps the maps f in C to $F(f)$, and maps their f^{-1} inverses in $L(C, \Sigma)$) (if they exist) to $(F(f))^{-1}$. The functor F' then makes diagram (1.39) then commute. The functor F' is indeed then unique functor with this property. Why? If a functor $F'' : L(C, \Sigma) \to \mathcal{D}$ makes diagram (1.39) commute then F' and F'' must agree with F on objects, and must agree on maps as functors respect isomorphisms. $\square$

Definition 1.6.14. Let $\mathcal{C}$ be a closed model category. The *homotopy* category $\mathrm{Ho}(\mathcal{C})$ associated to $\mathcal{C}$ is the localisation of $\mathcal{C}$ with respect to the class of weak equivalences of $\mathcal{C}$.

Remark 1.6.15. Note that [8] provides a description in Section 2.1 of $\mathrm{Ho}(\mathcal{C})$ that is more explicit than the one we have given. We shall not discuss this construction here, as we have little utility for it.

Now, there are two different closed model category structures on **CGWH**. The closed model category structure described in Theorem 1.6.6 is called the *Strøm* model structure. The model category structure that we have not described is called the *Quillen* model structure (see Section 2.4 in [10] for a discussion on the Quillen model structure). In the Quillen model structure weak homotopy equivalences are the weak equivalences. The Quillen model structure is interesting as the $S(-) \dashv |-|$ adjunction induces an equivalence between its homotopy category and the homotopy categories of **sSet**. So, when people say (or mathematicians, rather) say that simplicial sets model spaces what they mean precisely is that their homotopy categories are equivalent, as stated in Theorem 1.6.16 below.

Theorem 1.6.16. *The functors $|-|$ and $S(-)$ induce an equivalence of homotopy categories*

$$\mathrm{Ho}(\mathbf{sSet}) \simeq \mathrm{Ho}(\mathbf{CGWH})$$

when **CGWH** *is equipped with the Quillen model structure.*

Proof. This follows from Theorems 2.4.25 and 3.6.7 in [10]. Note that [10] appeals to the notions of 'Quillen adjunction' and 'Quillen equivalence' , which we have not discussed here. □

Intuitively Theorem 1.6.16 says that to understand the homotopy theory of spaces, it is sufficient to understand the homotopy theory of simplicial sets. That is, simplicial sets model spaces!

Chapter 2

Simplicial Covering Theory

Topological covering spaces are interesting objects studied in algebraic topology. Loosely, a topological covering over a base space X is a family of sets parametrised by the points in X. The study of covering spaces with base space X is closely connected with the study of the fundamental group of X. The connection is a consequence of the fact that there is an equivalence of categories $\mathbf{Cov}(X) \simeq [\Pi_1(X), \mathbf{Set}]$, where $\mathbf{Cov}(X)$ is the category of topological covering spaces over a locally path connected space X, and $\Pi_1(X)$ is the fundamental groupoid of X. In particular, topological covering spaces depend only on the homotopy 1-type of their base space. Thus, as the homotopy theory of simplicial sets models the homotopy theory of topological spaces, it is reasonable to assume that there exists simplicial models for topological covering spaces, called simplicial coverings. In this chapter we will define simplicial coverings and describe some of their theory.

We will begin this chapter by defining simplicial coverings, and discussing some of their basic properties, in Section 2.1. In Section 2.2 we will show that all simplicial coverings are Kan fibrations. In Section 2.3, we will discuss the definition of simplicial covering given by Gabriel and Zisman in [7], and show that their definition is equivalent to our own. Sections 2.4 and 2.5 will then be dedicated to proving that the category of simplicial coverings over a simplicial set X, denoted $\mathbf{Cov}(X)$, is equivalent to the functor category $[\Pi_1(X), \mathbf{Set}]$, where $\Pi_1(X)$ is the fundamental groupoid of X. This equivalence will be the simplicial analogue of the equivalence of categories discussed in the paragraph above. This chapter will then be concluded in Section 2.6 where we will show that restricting the equivalence $\mathbf{Cov}(X) \simeq [\Pi_1(X), \mathbf{Set}]$ to the core $\mathbf{FinCov}(X)^{\simeq}$ of the category of simplicial coverings with finite fibres yields an equivalence of categories $\mathbf{FinCov}(X)^{\simeq} \simeq [\Pi_1(X), \mathbf{FinSet}^{\simeq}]$.

The key takeaway from this chapter will be the equivalence of categories $\mathbf{FinCov}(X)^{\simeq} \simeq [\Pi_1(X), \mathbf{FinSet}^{\simeq}]$. This result will be useful in Chapter 5 when we identify the stable homotopy type of the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$.

2.1 Simplicial Coverings

In this section we will define what it means for a simplicial map $p : Y \to X$ to be a simplicial covering over X, and discuss three properties that are enjoyed by simplicial coverings. Comparisons between these properties and analogous properties enjoyed by topological coverings will also be made.

We begin this section with a standard definition.

Definition 2.1.1. Let $p : Y \to X$ and $i : T \to S$ be simplicial maps. The map p has the *(unique) right lifting property* with respect to i if for each commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & Y \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ S & \longrightarrow & X \end{array} \tag{2.1}$$

in **sSet** there exists a (unique) simplicial map $S \to Y$ such that the diagram

$$\begin{array}{ccc} T & \longrightarrow & Y \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle p} \\ S & \longrightarrow & X \end{array} \tag{2.2}$$

commutes.

Definition 2.1.2. Let $n \geq 0$. The simplicial map $0_n : \Delta^0 \to \Delta^n$ induced by the map

$$0_n : [0] \to [n], \quad 0_n(0) := 0$$

in Δ is called the *inital vertex map* of Δ^n. The simplicial map $n_n : \Delta^0 \to \Delta^n$ induced by the map

$$n_n : [0] \to [n], \quad n_n(0) := n$$

in Δ is called the *final vertex map* of Δ^n.

Definition 2.1.3. A simplicial map $p : Y \to X$ is called a *simplicial covering* over X if it has the unique right lifting property with respect to all initial and final vertex maps $0_n : \Delta^0 \to \Delta^n$ and $n_n : \Delta^0 \to \Delta^n$.

Remark 2.1.4. In this chapter we will often consider the vertex $Y(k)(y)$, where $k : [0] \to [n]$ is a map in Δ mapping 0 to k, $n \geq 0$, $0 \leq k \leq n$, and y is an n-simplex of the simplicial set Y. For ease of exposition, for the remainder of this thesis we shall denote such vertices $Y(k)(y)$ with $y(k)$.

Recall that if $p : Y \to X$ is a topological covering space then the fibre $Y|_x$ of $p : Y \to X$ over a point x in X is defined as the pullback in the diagram

$$\begin{array}{ccc} Y|_x & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \bullet & \xrightarrow{x} & X. \end{array} \tag{2.3}$$

By replacing all the spaces in the definition of a fibre of a topological covering with their simplicial analogues, we can define the fibre of a simplicial covering. We can then show that the fibres of simplicial coverings enjoy two properties analogous to those enjoyed by the fibres of covering spaces.

Definition 2.1.5. Let $p : Y \to X$ be a simplicial covering and let $x \in X$ be a vertex. The *fibre* of $p : Y \to X$ over the vertex x is the simplicial set $Y|_x$ defined by the pullback diagram

$$\begin{array}{ccc} Y|_x & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta^0 & \xrightarrow{x} & X. \end{array} \tag{2.4}$$

in **sSet**.

The fibres of a topological covering are by definition discrete. As constant simplicial sets realise to discrete spaces, defining a simplicial set to be discrete if it is isomorphic to a constant simplicial set will allow us to show that fibres of simplicial coverings are also discrete.

Definition 2.1.6. Let S be a simplicial set. The simplicial set S is *discrete* if it isomorphic to a constant simplicial set.

Proposition 2.1.7. *If $p : Y \to X$ is a simplicial covering then the fibres of p are discrete simplicial sets.*

Proof. To prove Proposition 2.1.7 it is sufficient to show:

1. simplicial coverings are stable under pullback; and
2. simplicial coverings over Δ^0 are discrete.

Statement 1 follows immediately from a well known lemma (Lemma 2.1.8), and we will prove statement 2 in Lemma 2.1.9. □

Lemma 2.1.8. *Let $p : Y \to X$ be a simplicial map with the (unique) right lifting property with respect to the simplicial map $T \to S$. Then for all simplicial maps $Z \to X$ the simplicial map $Y \times_X Z \to Z$ in the pullback diagram*

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow p \\ Z & \longrightarrow & X \end{array} \tag{2.5}$$

in **sSet** *has the (unique) right lifting property with respect to $T \to S$.*

Lemma 2.1.9. *If $Y \to \Delta^0$ is a simplicial covering then Y is discrete.*

Proof. We will exhibit an isomorphism between Y and the constant simplicial set Y_0. Given a vertex y of Y note that the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{y} & Y \\ \downarrow 0_n & & \downarrow \\ \Delta^n & \longrightarrow & \Delta^0 \end{array} \tag{2.6}$$

commutes. Hence, as $Y \to \Delta^0$ is a simplicial covering, there exists a unique n-simplex $\phi(y)$ such that the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{y} & Y \\ \downarrow 0_n & \nearrow_{\phi(y)} & \\ \Delta^n & & \end{array} \tag{2.7}$$

commutes. Thus, we can define the function

$$Y_0 \to Y_n, \quad y \mapsto \phi(y).$$

We can also define the function

$$Y_n \to Y_0, \quad x \mapsto x(0).$$

For each $n \geq 0$ the maps $Y_0 \to Y_n$ and $Y_n \to Y_0$ are mutually inverse. Why? If y is a vertex of Y then, by the commutativity of diagram (2.7), we have that $\phi(y)(0) = y$. Similarly, if x is an n-simplex of Y, $\phi(x(0))$ is equal to x, by uniqueness. Hence, the maps $Y_0 \to Y_n$ and $Y_n \to Y_0$ are mutually inverse. Thus, as the family of maps $Y_n \to Y_0$ clearly commute with face and degeneracy maps of Y and Y_0, the simplicial set Y is discrete. □

Remark 2.1.10. By Proposition 2.1.7, as limits in functor categories are computed pointwise, we can explicitly describe the fibre $Y|_x$ of the simplicial covering $p : Y \to X$ over the vertex x as the constant simplicial set associated to the set

$$\{y \in Y_0 : p_0(y) = x\}. \tag{2.8}$$

Recall that topological coverings satisfy the following: if $p : Y \to X$ is a topological covering, and if $\gamma : x \to y$ is a path in X, then the fibres $Y|_x$ and $Y|_y$ are isomorphic. Taking p to be a simplicial covering, an analogous result would be: if there exists a 1-simplex of v of X such that $d_0(v) = y$ and $d_1(v) = x$, then $Y|_x$ and $Y|_y$ are isomorphic. We will prove this result now.

Proposition 2.1.11. *Let $p : Y \to X$ be a simplicial covering. If x and y are vertices of X such that there exists a 1-simplex v with $d_0(v) = y$ and $d_1(v) = x$, then $Y|_x$ and $Y|_y$ are isomorphic.*

Proof. Note that if z' is an element of $Y|_x$ then there exists a unique diagonal filler v' in the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{z'} & Y \\ {\scriptstyle d^1}\downarrow & \overset{v'}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^1 & \xrightarrow{v} & X. \end{array} \tag{2.9}$$

Hence, we can define the function

$$Y|_x \to Y|_y, \quad z' \mapsto d_0(v').$$

Similarly, if z'' is an element of $Y|_y$ then there exists a unique diagonal filler v'' in diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{z''} & Y \\ {\scriptstyle d^0}\downarrow & \overset{v''}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^1 & \xrightarrow{v} & X. \end{array} \tag{2.10}$$

Hence, we can define the function

$$Y|_y \to Y|_x, \quad z'' \mapsto d_1(v'')$$

The functions $Y|_y \to Y|_x$ and $Y|_x \to Y|_y$ are mutually inverse. Why? If z' is an element of $Y|_x$, then note that the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{d_0(v')} & Y \\ {\scriptstyle d^0}\downarrow & \overset{v'}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^1 & \xrightarrow{v} & X. \end{array} \tag{2.11}$$

commutes. Hence, the image of z' under $Y|_x \to Y|_y \to Y|_x$ is $d_1(v')$, which is equal to z' by the commutativity of diagram (2.9). Similarly, if z'' is an element of $Y|_y$, the image of z'' under $Y|_y \to Y|_x \to Y|_y$ is $d_0(v'')$, which is equal to z''. Thus, $Y|_x \to Y|_y$ and $Y|_y \to Y|_x$ are mutually inverse, and hence $Y|_x$ and $Y|_y$ are isomorphic sets. □

Recall that all topological coverings of the unit interval are trivial. That is, all coverings $Y \to I$ of I are isomorphic to the covering $pr_2 : Y|_0 \times I \to I$ over I. Taking Δ^1 to be the simplicial unit interval, we can state an analogous result for simplicial coverings.

Proposition 2.1.12. *Let $Y \to \Delta^1$ be a simplicial covering. There is an isomorphism $S \times \Delta^1 \xrightarrow{\simeq} Y$, where S is some discrete simplicial set, such that the diagram*

$$\begin{array}{ccc} S \times \Delta^1 & \xrightarrow{\quad\simeq\quad} & Y \\ & \searrow \quad \swarrow & \\ & \Delta^1 & \end{array} \tag{2.12}$$

commutes.

Proposition 2.1.12 will be easier to prove once we have developed some more theory on simplicial coverings. We hence delay the proof until the end of Section 2.5.

2.2 Simplicial Coverings are Kan Fibrations

In this section we will show that all simplicial coverings are Kan fibrations. While we shall seldom use this fact, it implies that the simplicial homotopy theory covered in Section 1.3 can be applied to the study of simplicial coverings. To prove that all simplicial coverings are Kan fibrations we will appeal to some facts about anodyne maps and saturated classes. We will not use anodyne maps and saturated classes elsewhere in this thesis, and have not discussed them previously, so the reader is referred to Section 1.4 in [8] for a comprehensive discussion of them.

To show that simplicial coverings are Kan fibrations we will need to find a sufficient condition on saturated classes to ensure they contain the anodyne maps in **sSet**.

Definition 2.2.1. Let $\mathbf{M}$ be a class of monomorphisms in a category C. The class $\mathbf{M}$ is said to satisfy the *right cancellation property* if for all maps $u : a \to b$ and $v : b \to c$ maps in C, u and $v \circ u$ belonging to $\mathbf{M}$ always implies that v is in $\mathbf{M}$.

Lemma 2.2.2 [**Lemma 3.7, Joyal-Tierney** [11]]. *Let* $\mathbf{M}$ *be a saturated class of monomorphisms in* **sSet** *satisfying the right cancellation property. If* $\mathbf{M}$ *contains every coface map* $d^i : \Delta^n \to \Delta^{n+1}$ *then* $\mathbf{M}$ *contains the anodyne maps.*

Proof. See Lemma 3.7 in [11]. □

Lemma 2.2.3. *If* $\mathbf{M}$ *is a saturated class of monomorphisms in* **sSet** *which satisfies the right cancellation property, and contains all the initial and final vertex maps, then* $\mathbf{M}$ *contains all the anodyne maps.*

Proof. By Lemma 2.2.2 it is sufficient to show that $\mathbf{M}$ contains every coface map $d^i : \Delta^n \to \Delta^{n+1}$. Note that if $n = 0$ then $d^0 = 1_1$, and $d^1 = 0_1$. Hence, $\mathbf{M}$ contains the coface maps $d^0 : \Delta^0 \to \Delta^1$ and $d^1 : \Delta^0 \to \Delta^1$. Inducting on n, if $j \neq 0$ then $0_{n+1} = d^j \circ 0_n$, and hence $d^j : \Delta^n \to \Delta^{n+1}$ is in $\mathbf{M}$. Similarly, if $j = 0$ then $n_{n+1} = d^0 \circ n_n$, so $d^0 : \Delta^n \to \Delta^{n+1}$ is also in $\mathbf{M}$. Hence, $\mathbf{M}$ contains every coface map. □

We have thus found the previously alluded to sufficient condition on saturated classes to ensure that they contain the anodyne maps in **sSet**. We will now apply this sufficient condition to show that all simplicial coverings are Kan fibrations.

Proposition 2.2.4. *The class of monomorphisms* $\mathbf{M}_p$ *in* **sSet** *with the unique left lifting property with respect to a fixed simplicial covering* $p : Y \to X$ *contains the anodyne maps.*

Proof. First note as the map $p : Y \to X$ is a simplicial covering, $\mathbf{M}_p$ contains all the initial and final vertex maps. Additionally, note that a class of monomorphisms $\mathbf{M}$ in **sSet** with the unique left lifting property with respect a fixed simplicial map is saturated (this is easy to check). Hence, by Lemma 2.2.3, it is sufficient to show that $\mathbf{M}_p$ satisfies the right cancellation property. Let $u : R \to T$ and $v : T \to S$ be maps in **sSet** such that u and $v \circ u$ are in $\mathbf{M}_p$. Suppose that the diagram

$$\begin{array}{ccc} T & \xrightarrow{x} & Y \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle p} \\ S & \xrightarrow{y} & X \end{array} \tag{2.13}$$

commutes. There then exists a unique map $! : S \to Y$ making the diagram

$$\begin{array}{ccc} R & \xrightarrow{x \circ u} & Y \\ \downarrow{\scriptstyle v \circ u} & \overset{!}{\nearrow} & \downarrow{\scriptstyle p} \\ S & \xrightarrow{y} & X. \end{array} \tag{2.14}$$

commute. Then as the diagrams

$$\begin{array}{ccc} R & \xrightarrow{x \circ u} & Y \\ \downarrow{\scriptstyle u} & \overset{x}{\nearrow} & \downarrow{\scriptstyle p} \\ T & \xrightarrow{y \circ v} & X \end{array} \tag{2.15}$$

and

$$\begin{array}{ccc} R & \xrightarrow{x \circ u} & Y \\ \downarrow{\scriptstyle u} & \overset{! \circ v}{\nearrow} & \downarrow{\scriptstyle p} \\ T & \xrightarrow{y \circ v} & X \end{array} \tag{2.16}$$

commute, by uniqueness, we have $x =! \circ v$. Hence, the diagram

$$\begin{array}{ccc} T & \xrightarrow{x} & Y \\ \downarrow{\scriptstyle v} & \overset{!}{\nearrow} & \downarrow{\scriptstyle p} \\ S & \xrightarrow{y} & X \end{array} \tag{2.17}$$

commutes. Furthermore, the map ! is unique as any map filling diagram (2.17) also fills (2.14), and so is equal to !. Thus, the map v has the unique left lifting property with respect to p. Hence, $\mathbf{M}_p$ satisfies the right cancellation property. □

Proposition 2.2.5. *Every simplicial covering is a Kan fibration.*

Proof. By Proposition 2.2.4 all simplicial coverings have the right lifting property with respect to all anodyne maps. Thus, all simplicial coverings are Kan fibrations. □

Remark 2.2.6. The converse of Proposition 2.2.5 is not true in general. To see this, simply consider a topological space with a non-discrete total singular complex - $\mathbb{S}^1$, for example. Then $S(\mathbb{S}^1) \to \Delta^0$ is a Kan fibration. But if $S(\mathbb{S}^1) \to \Delta^0$ was also a simplicial covering then $S(\mathbb{S}^1)$ would be a discrete simplicial set (Lemma 2.1.9).

2.3 Gabriel and Zisman Simplicial Coverings

In the book 'Calculus of Fractions and Homotopy Theory' ([7]) Gabriel and Zisman give a definition of simplicial covering that is apparently stronger than Definition 2.1.3, and develop some theory using this definition. In this section we will state Gabriel and Zisman's definition of simplicial covering and show that it is equivalent to our own.

Definition 2.3.1. A simplicial map $p : Y \to X$ is called a *Gabriel Zisman simplicial covering* if it has the unique right lifting property with respect to all maps $\Delta^0 \to \Delta^n$.

Proposition 2.3.2. *A simplicial map $p : Y \to X$ is a simplicial covering if and only if it is a Gabriel Zisman simplicial covering.*

Proof. It is clear that all Gabriel Zisman simplicial coverings are simplicial coverings, so we only need to show that all simplicial coverings are Gabriel Zisman simplicial coverings. Let $p : Y \to X$ be a simplicial covering, and suppose the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{y} & Y \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow{\sigma} & X \end{array} \tag{2.18}$$

commutes, where $0 < k < n$. Observe that the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{y} & Y \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle p} \\ \Delta^{n-k} & \xrightarrow[\sigma|_{\Delta^{\{k,\dots,n\}}}]{} & X \end{array} \tag{2.19}$$

also commutes, where $\Delta^{\{k,\dots,n\}}$ is the simplicial subset of Δ^n corresponding to the nerve of the full subcategory of $[n]$ spanned by $k, \dots, n$. Hence, as p is a simplicial covering, there is a unique map $v : \Delta^{n-k} \to Y$ filling diagram (2.19). Similarly, there is a unique diagonal filler $u : \Delta^n \to Y$ in the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{v(n-k)} & Y \\ \downarrow{\scriptstyle n} & \overset{u}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow{\sigma} & X. \end{array} \tag{2.20}$$

Now note that the diagrams

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{v(n-k)} & Y \\ {\scriptstyle n-k}\downarrow & \overset{u'}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^{n-k} & \xrightarrow[\sigma|_{\Delta\{k,\ldots,n\}}]{} & X, \end{array} \tag{2.21}$$

where $u' = u|_{\Delta\{k,\ldots,n\}}$, and

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{v(n-k)} & Y \\ {\scriptstyle n-k}\downarrow & \overset{v}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^{n-k} & \xrightarrow[\sigma|_{\Delta\{k,\ldots,n\}}]{} & X. \end{array} \tag{2.22}$$

commute, as $u(n) = v(n-k)$. Hence, $v = u'$, by uniqueness. Thus, $u(k) = v(0) = y$, and so u is a diagonal filler for diagram (2.18). To see that u is unique, suppose there exists a simplicial map $z : \Delta^n \to X$ filling diagram (2.18). Then $z|_{\Delta\{k,\ldots,n\}}$ fills diagram (2.19), and hence is equal to v, by uniqueness. Thus as $z(n) = z|_{\Delta\{k,\ldots,n\}}(n-k) = v(n-k)$, the map $z : \Delta^n \to Y$ also fills diagram (2.20). The map z is hence equal to u, by uniqueness. Thus, the simplicial covering $p : Y \to X$ has the unique right lifting property with respect to all maps $\Delta^0 \to \Delta^n$ for $0 < k < n$. □

By Proposition 2.3.2, any of the theory developed for Gabriel Zisman simplicial coverings in [7] can also be applied to the simplicial coverings discussed in this thesis.

2.4 The Fundamental Groupoid

When one studies the homotopy theory of topological spaces one quickly encounters the concept of the 'fundamental groupoid' of a space. There is of course an analogous concept for simplicial sets which yields an equivalence of groupoids when passing to realisation (c.f. Proposition 2.4.10). In this section we will define the fundamental groupoid of a simplicial set X in two different ways, without appealing to the realisation of X, and show that they are equivalent as groupoids. Both such definitions will appeal to a construction known as the 'free groupoid' on a category.

As both of our definitions of the fundamental groupoid of a simplicial set will appeal to the free groupoid on a category, let us begin by defining the free groupoid on a category.

Definition 2.4.1. Let C be a category. Let $G(C)$ be the category whose objects are the objects of C. The maps in $G(C)$ are the maps of C and their formal inverses. The maps in $G(C)$ are composed by concatenation. The category $G(C)$ is called the *free groupoid* on C.

The reader may wonder why $G(C)$ is called the *free* groupoid on X. The reason for this is as follows: Recall that if some 'forgetful' functor has a left adjoint, then the left adjoint is often called a free functor. Thus, as the induced functor $G(-) : \mathbf{Cat} \to \mathbf{Gpd}$ is left adjoint to the forgetful functor $i : \mathbf{Gpd} \hookrightarrow \mathbf{Cat}$ (see Construction 2.4.2), it is a free functor.

Construction 2.4.2. The free groupoid on a category C can be characterised as a left adjoint. Why? Clearly Definition 2.4.1 can be extended to define a functor

$$G(-) : \mathbf{Cat} \to \mathbf{Gpd}.$$

Note that there is also functor

$$i : \mathbf{Gpd} \hookrightarrow \mathbf{Cat}$$

which forgets that a category is a groupoid and includes it into the category of categories. There is then a unit $Id \to i \circ G$ which has C-component

$$C \to G(C), \quad C \hookrightarrow G(C),$$

and a counit $G \circ i \to Id$ whose C'-component

$$G(C') \to C'$$

acts as the identity on C' and sends the formal inverses of maps f in $G(C')$ to the inverse f^{-1} of f in C'. That is, there is an adjunction $G \dashv i$. Note that the counit of this adjunction is a natural isomorphism which reflects the fact that i is fully faithful. Hence, the functor $G(-)$ can be characterised as the left adjoint of the inclusion of groupoids into categories.

Remark 2.4.3. It is interesting to observe also that the functor $i : \mathbf{Gpd} \hookrightarrow \mathbf{Cat}$ also has a right adjoint $(-)^{\simeq} : \mathbf{Cat} \to \mathbf{Gpd}$. The functor $(-)^{\simeq}$ maps a category C to the category $C^{\simeq}$ which contains only the objects and isomorphisms of C. The category $C^{\simeq}$ is called the *core* of C.

Recall that the fundamental groupoid $\Pi_1(X)$ of a space X encapsulates its homotopy 1-type. Also recall from Chapter 1 that the homotopy 1-type of a simplicial set X depends only on its 2-skeleton. Hence, to give our first definition of the fundamental groupoid of a simplicial set X, following Goerss and Jardine in [8], we first define the path category P_*X, which is constructed using only the 2-skeleton of X. The fundamental groupoid of X will then be defined as the free groupoid on P_*X.

Definition 2.4.4. Let X be a simplicial set. Let P_*X be the category whose objects are the vertices of X. Maps in P_*X are strings of 1-simplices of X

$$d_1(v_0) \xrightarrow{v_0} d_0(v_0) = d_1(v_1) \xrightarrow{v_1} \dots \xrightarrow{v_{n-1}} d_0(v_{n-1})$$

subject to the relation: for each 2-simplex σ of X, the diagram

$$\begin{array}{ccccc} & & v_1 & & \\ & \overset{d_2\sigma}{\nearrow} & & \overset{d_0\sigma}{\searrow} & \\ v_0 & & \xrightarrow{\quad d_1\sigma \quad} & & v_2 \end{array} \tag{2.23}$$

commutes in P_*X. The category P_*X is called the *path category* of X.

Remark 2.4.5. Recall that in Section 1.1.3 we mentioned the nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ has a left adjoint. The left adjoint turns out to be the functor

$$P_*(-) : \mathbf{sSet} \to \mathbf{Cat}$$

induced by mapping simplicial sets X to P_*X.

Definition 2.4.6. Let X be a simplicial set. Let $\Pi_1(X)$ be the category $G(P_*X)$. The category $\Pi_1(X)$ is called the *fundamental groupoid* of X.

Our second definition of the fundamental groupoid will utilise the simplex category Δ/X of X (recall Definition 1.2.5), and hence will be constructed using all the simplices of X, as well as the free groupoid construction.

Definition 2.4.7. Let X be a simplicial set. Let $\Pi_1(X)$ be the category $G(\Delta/X)$.

The following proposition shows that the two definitions of $\Pi_1(X)$ we have given are interchangeable up to groupoid equivalence:

Proposition 2.4.8. *Let X be a simplicial set. There is an equivalence of groupoids*

$$G(P_*X) \simeq G(\Delta/X).$$

Proof. See Theorem 1.1 on page 140 in [8]. □

Remark 2.4.9. Which definition of $\Pi_1(X)$ one should use depends on their context. We note that $G(\Delta/X)$ is best suited to the development of theory, and $G(P_*X)$ is best used when doing explicit calculations.

As simplicial sets act as combinatorial models for topological spaces it should be the case that the groupoids $\Pi_1(X)$ and $\Pi_1(|X|)$ are equivalent for each simplicial set X. This is the content of the following proposition:

Proposition 2.4.10. *Let X be a simplicial set. There is an equivalence of categories*

$$\Pi_1(X) \simeq \Pi_1(|X|).$$

Proof. See Theorem 1.1 on page 140 in [8]. □

2.5 An Equivalence of Categories

Recall the well known result that for all locally path connected spaces X there is an equivalence of categories

$$\mathbf{Cov}(X) \simeq [\Pi_1(X), \mathbf{Set}]. \tag{2.24}$$

In this section we shall prove an analogous result for all simplicial sets X. The meta-theorem afforded by this result is that the simplicial coverings over X depend only on the homotopy 1-type of X, as is the case for spaces.

To prove a result for simplicial sets analogous to equation (2.24) we will find an equivalence of categories between the category $\mathbf{sSet}/X$, which contains a category of simplicial coverings as a full subcategory, and the presheaf category on Δ/X. Then, after restricting this equivalence to the subcategory of $\mathbf{sSet}/X$ consisting of the simplicial coverings, and finding a relationship between $\Pi_1(X)$ and a localisation of Δ/X, the result will follow.

Definition 2.5.1. Let X be a simplicial set. Let $\mathbf{sSet}/X$ denote the slice category of $\mathbf{sSet}$ over X, i.e. let $\mathbf{sSet}/X$ be the category whose objects are pairs (Y, σ) where Y is a simplicial set and $\sigma : Y \to X$ is a simplicial map. Maps in $\mathbf{sSet}/X$ are simplicial maps $f : Y \to Y'$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\sigma'} & \\ & X & \end{array} \tag{2.25}$$

commutes.

Proposition 2.5.2. *Let X be a simplicial set. There is an equivalence of categories*

$$\mathbf{sSet}/X \simeq [(\Delta/X)^{op}, \mathbf{Set}].$$

Proof. To prove Proposition 2.5.2 we will explicitly construct an equivalence of categories. We will begin constructing an equivalence by constructing a functor $X_{(-)} : [(\Delta/X)^{op}, \mathbf{Set}] \to \mathbf{sSet}/X$. Given a functor F in $[(\Delta/X)^{\mathrm{op}}, \mathbf{Set}]$, let X_F be the simplicial set whose set of n-simplices is the set

$$(X_F)_n := \coprod_{x \in X_n} F(x).$$

The face maps d_i and degeneracy maps s_j of X_F are the maps

$$d_i : (X_F)_{n+1} \to (X_F)_n, \quad d_i((y, x)) := \left(F(d^i)(y), d_i(x)\right),$$

and

$$s_j : (X_F)_n \to (X_F)_{n+1}, \quad s_j((y,x)) := \left(F(s^j)(y), s_j(x)\right),$$

where $x \in X_n$ and $y \in F(x)$. Also let $\sigma_F : X_F \to X$ be the simplicial map with n-component

$$\sigma_F : (X_F)_n \to X_n, \quad \sigma\left((y,x)\right) := x.$$

Now, if $\alpha : F \to G$ is a map in $[(\Delta/X)^{op}, \mathbf{Set}]$, define the simplicial map $X_\alpha : X_F \to X_G$ with n-component

$$X_\alpha : (X_F)_n \to (X_G)_n, \quad X_\alpha(y,x) := (\alpha_x(y), x),$$

where α_x denotes the x-component $\alpha_x : F(x) \to G(x)$ of the map $\alpha : F \to G$. The map $X_\alpha : X_F \to X_G$ is a map in $\mathbf{sSet}/X$ as the diagram

$$\begin{array}{ccc} X_F & \xrightarrow{X_\alpha} & X_G \\ & {\scriptstyle \sigma_F}\searrow \quad \swarrow{\scriptstyle \sigma_G} & \\ & X & \end{array} \tag{2.26}$$

commutes. Hence, let $X_{(-)} : [(\Delta/X)^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{sSet}/X$ be the functor which sends functors F in $[(\Delta/X)^{\mathrm{op}}, \mathbf{Set}]$ to pairs (X_F, σ_F), and natural transformations $\alpha : F \to G$ in $[(\Delta/X)^{\mathrm{op}}, \mathbf{Set}]$ to maps $X_\alpha : X_F \to X_G$ in $\mathbf{sSet}/X$.

We will now construct a functor $F_{(-,-)} : \mathbf{sSet}/X \to [(\Delta/X)^{op}, \mathbf{Set}]$ that we will show is a pseudo-inverse of $X_{(-)}$. Given an object (Y, σ) in $\mathbf{sSet}/X$, define the functor $F_{(Y,\sigma)} : (\Delta/X)^{op} \to \mathbf{Set}$ which maps n-simplices x of X to the sets

$$F_{(Y,\sigma)}(x) := \{y \in Y_n : \sigma(y) = x\}.$$

The functor $F_{(Y,\sigma)}$ sends simplicial maps $\theta : x \to y$ in Δ/X to the functions $Y(\theta)$. If $f : (Y,\sigma) \to (Y', \sigma')$ is a map in $\mathbf{sSet}/X$ define the natural transformation $F_f : F_{(Y,\sigma)} \to F_{(Y',\sigma')}$ whose component at $x : \Delta^n \to X$ in Δ/X is defined by

$$F_f : F_{(Y,\sigma)}(x) \to F_{(Y',\sigma')}(x), \quad F_f(y) := f(y).$$

Hence, let $F_{(-,-)} : \mathbf{sSet}/X \to [(\Delta/X)^{op}, \mathbf{Set}]$ be the functor which sends pairs (Y, σ) to the functors $F_{(Y,\sigma)} : (\Delta/X)^{op} \to \mathbf{Set}$, and sends the maps $f : (Y,\sigma) \to (Y', \sigma')$ in $\mathbf{sSet}/X$ to the natural transformations $F_f : F_{(Y,\sigma)} \to F_{(Y',\sigma')}$.

Let's now show that the functors $F_{(-,-)} : \mathbf{sSet}/X \to [(\Delta/X)^{op}, \mathbf{Set}]$ and $X_{(-)} : [(\Delta/X)^{op}, \mathbf{Set}] \to \mathbf{sSet}/X$ are pseudo-inverses. Let $F : (\Delta/X)^{op} \to \mathbf{Set}$ be a functor. Observe that if x is an n-simplex of X then

$$\begin{aligned} F_{(X_F,\sigma_F)}(x) &= \{y \in (X_F)_n : \sigma_F(y) = x\} \\ &= \{y \in \coprod_{z \in X_n} F(z) : \sigma_F((y,z)) = x\} \\ &\cong F(x), \end{aligned}$$

where the last isomorphism is natural in x. Now let (Y, σ) be a pair in $\mathbf{sSet}/X$. Then

$$\begin{aligned} (X_{F_{(Y,\sigma)}})_n &= \coprod_{x \in X_n} F_{(Y,\sigma)}(x) \\ &= \coprod_{x \in X_n} \{y \in Y_n : \sigma(y) = x\} \\ &\cong Y_n, \end{aligned}$$

where again the last isomorphism is natural. Thus, the functors $F_{(-,-)}$ and $X_{(-)}$ are pseudo-inverses. □

Thus, we have that the categories $\mathbf{sSet}/X$ and $[(\Delta/X)^{op}, \mathbf{Set}]$ are equivalent. Using this result, we will now relate the full subcategory of $\mathbf{sSet}/X$ that only contains simplicial coverings to a full subcategory of $[(\Delta/X)^{op}, \mathbf{Set}]$.

Definition 2.5.3. Let $\mathbf{Cov}(X)$ be the full subcategory of $\mathbf{sSet}/X$ spanned by simplicial coverings $p : Y \to X$.

Definition 2.5.4. Let $[(\Delta/X)^{op}, \mathbf{Set}]^*$ be the full subcategory of $[(\Delta/X)^{op}, \mathbf{Set}]$ spanned by functors which send all initial and final vertex maps to isomorphisms. In other words, $F : (\Delta/X)^{op} \to \mathbf{Set}$ belongs to the subcategory $[(\Delta/X)^{op}, \mathbf{Set}]^*$ if and only if for all $n \geq 0$ the maps $F(0_n : \Delta^0 \to \Delta^n)$ and $F(n_n : \Delta^0 \to \Delta^n)$ are isomorphisms.

Proposition 2.5.5. *The category* $[(\Delta/X)^{op}, \mathbf{Set}]^*$ *is equivalent to* $\mathbf{Cov}(X)$*.*

Proof. By Proposition 2.5.2 it is sufficient to show:

1. The image of the functor $F_{(-,-)} : \mathbf{sSet}/X \to [(\Delta/X)^{op}, \mathbf{Set}]$ restricted to $\mathbf{Cov}(X)$ is contained in $[(\Delta/X)^{op}, \mathbf{Set}]^*$.

2. The image of the functor $X_{(-)} : [(\Delta/X)^{op}, \mathbf{Set}] \to \mathbf{sSet}/X$ restricted to $[(\Delta/X)^{op}, \mathbf{Set}]^*$ is contained in $\mathbf{Cov}(X)$.

We prove 1. Let $p : Y \to X$ be a simplicial covering and consider $F_{(Y,p)} : (\Delta/X)^{op} \to \mathbf{Set}$. Suppose $0_n : x \to y$ is an initial vertex map in Δ/X — i.e. suppose y is an n-simplex of X such that $y(0) = x$. Then if z is in $F_{(Y,p)}(x)$ the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{z} & Y \\ \downarrow{\scriptstyle 0_n} & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow{y} & X \end{array} \tag{2.27}$$

commutes. Hence, there is a unique diagonal filler $z' : \Delta^n \to Y$ in diagram (2.27). The mapping

$$z \mapsto z'$$

then defines a function mutually inverse to $F_{(Y,p)}(0_n)$, as $p : Y \to X$ is a simplicial covering. Hence, the function $F_{(Y,p)}(0_n)$ is an isomorphism. An identical argument can be made to show that all final vertex maps in Δ/X are mapped to isomorphisms.

We prove 2. Let $F : (\Delta/X)^{op} \to \mathbf{Set}$ be a functor in $[(\Delta/X)^{op}, \mathbf{Set}]^*$ and consider (X_F, σ_F). Now suppose the diagram

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{x_F} & X_F \\ \downarrow{\scriptstyle 0_n} & & \downarrow{\scriptstyle \sigma_F} \\ \Delta^n & \xrightarrow{x} & X \end{array} \tag{2.28}$$

commutes, and let $\phi : F(x(0)) \to F(x)$ be the mutual inverse of $F(0_n) : F(x) \to F(x(0))$. Note that the commutativity of diagram (2.28) implies that x_F is in $F(x(0))$. Hence $\phi(x_F) : \Delta^n \to X_F$ is a unique diagonal filler for the diagram (2.28). Thus, $\sigma_F : X_F \to X$ is a simplicial covering. □

We will now relate the fundamental groupoid $\Pi_1(X)$ of a simplicial set X to Δ/X by appealing to a certain localisation (recall Definition 1.6.12) of Δ/X.

Remark 2.5.6. In the following discussion we will want to refer to the localisation of Δ/X with respect to the initial and final vertex maps $0_n : \Delta^0 \to \Delta^n$ and $n_n : \Delta^0 \to \Delta^n$ (for all $n \geq 0$) in Δ/X. For simplicity, this category will be denoted $L(\Delta/X, \{0_n, n_n\})$.

Proposition 2.5.7. *Let X be a simplicial set. Then*

$$L(\Delta/X, \{0_n, n_n\}) = G(\Delta/X).$$

Proof. Clearly $L\left(\Delta/X, \{0_n, n_n\}\right)$ and $G(\Delta/X)$ have the same objects. Then, as all the maps in Δ/X can be factored into coface and codegeneracy maps d^i and s^j (Lemma 1.1.8), to prove Proposition 2.5.7 it is sufficient to show that all the maps d^i and s^j are invertible in $L(\Delta/X, \{0_n, n_n\})$. But this follows from the following three equations:

$$n_{n+1} = d^0 \circ n_n \tag{2.29}$$

$$0_{n+1} = d^i \circ 0_n \quad \text{for } i > 0 \tag{2.30}$$

$$0_n = s^j \circ 0_{n+1}. \tag{2.31}$$

□

Remark 2.5.8. We have hence found a new way of describing the fundamental groupoid $\Pi_1(X)$ of X. That is, $\Pi_1(X)$ can be taken to be the category $L(\Delta/X, \{0_n, n_n\})$. For the remainder of this thesis, we will adopt this description of $\Pi_1(X)$.

Finding the above precise relationship between $\Pi_1(X)$ and Δ/X puts us in a position to prove a simplicial analogue to equation (2.24).

Theorem 2.5.9. *Let X be a simplicial set. There is an equivalence of categories*

$$\mathbf{Cov}(X) \simeq [\Pi_1(X), \mathbf{Set}].$$

Proof. By Proposition 2.5.5 we have the equivalence of categories

$$\mathbf{Cov}(X) \simeq [(\Delta/X)^{op}, \mathbf{Set}]^*.$$

Using Proposition 1.6.13 is easy to see that there is an isomorphism

$$[(\Delta/X)^{op}, \mathbf{Set}]^* \cong [L(\Delta/X, \{0_n, n_n\})^{op}, \mathbf{Set}].$$

Hence, $\mathbf{Cov}(X) \simeq [L(\Delta/X, \{0_n, n_n\})^{op}, \mathbf{Set}]$. Then, by Proposition 2.5.7, we have an equivalence $\mathbf{Cov}(X) \simeq [G(\Delta/X)^{op}, \mathbf{Set}]$. But as $G(\Delta/X)$ is a groupoid we have $G(\Delta/X)^{op} \cong G(\Delta/X)$, and hence $\mathbf{Cov}(X) \simeq [G(\Delta/X), \mathbf{Set}]$. □

In proving Theorem 2.5.9 two functors $F_{(-,-)} : \mathbf{Cov}(X) \to [\Pi_1(X), \mathbf{Set}]$ and $X_{(-)} : [\Pi_1(X), \mathbf{Set}] \to \mathbf{Cov}(X)$ were constructed. Understanding the actions of these functors in some detail will be helpful in Section 2.6, and in proving Proposition 2.1.12. Let's unwind the actions of the functors $F_{(Y,p)}$ and $X_{(-)}$ now.

Construction 2.5.10. To prove Theorem 2.5.9 we constructed two functors $F_{(-,-)} : \mathbf{Cov}(X) \to [\Pi_1(X), \mathbf{Set}]$ and $X_{(-)} : [\Pi_1(X), \mathbf{Set}] \to \mathbf{Cov}(X)$ and showed that they are pseudo-inverses. The functor $F_{(-,-)} : \mathbf{Cov}(X) \to [\Pi_1(X), \mathbf{Set}]$ maps the simplicial covering (Y, p) over X to the functor $F_{(Y,p)} : \Pi_1(X) \to \mathbf{Set}$. The functor $F_{(Y,p)}$ maps the n-simplices x of X to the set

$$F_{(Y,p)}(x) = \{y \in Y_n : p(y) = x\}.$$

How the functor $F_{(Y,p)}$ acts on maps in $\Pi_1(X)$ is difficult to describe. Thankfully, we shall seldom need such a concrete description in this thesis, so we will not give one here.

The functor $X_{(-)} : [\Pi_1(X), \mathbf{Set}] \to \mathbf{Cov}(X)$ maps functors F to simplicial coverings (X_F, σ_F). The simplicial set X_F has the set of n-simplices

$$(X_F)_n = \bigsqcup_{x \in X_n} F(x),$$

and sends the maps $f : [n] \to [m]$ in Δ to the functions

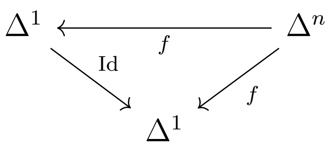
$$X_F(f) : X_m \to X_n, \quad X_F(f)(x, z) := (F(f^{-1})(x), X(f)(z)),$$

where f^{-1} is a map in $L(\Delta/X, \{0_n, n_n\})$. The simplicial map $\sigma_F : X_F \to X$ has n-component

$$\sigma_F : (X_F)_n \to X_n, \quad \sigma_F(x, z) := z.$$

We will now conclude this section by proving Proposition 2.1.12 using Theorem 2.5.9.

Proof. As $\Delta^1_{(-)} : [\Pi_1(\Delta^1), \mathbf{Set}] \to \mathbf{Cov}(\Delta^1)$ is essentially surjective, to prove Proposition 2.1.12 it is sufficient to show that for all functors $F : \Pi_1(\Delta^1) \to \mathbf{Set}$ the simplicial covering Δ^1_F is isomorphic to $\Delta^1 \times F(Id_{[1]})$ in $\mathbf{Cov}(\Delta^1)$. Note that if $f : [n] \to [1]$ is a map in Δ then the diagram



(2.32)

commutes in the simplex category Δ/Δ^1 of Δ^1, and hence in $\Pi_1(\Delta^1)$. Hence, there is an isomorphism of sets $F(f) : F(f) \to F(Id_{[1]})$. Thus, let $\Delta^1_F \to \Delta^1 \times F(Id_{[1]})$ be the simplicial map with n-component

$$(\Delta^1_F)_n \to \Delta^1_n \times F(Id_{[1]}), \quad (x, f) \mapsto (f, F(f)(x)).$$

The simplicial map $\Delta^1_F \to \Delta^1 \times F(Id_{[1]})$ is mutually inverse to the map $\Delta^1 \times F(Id_{[1]}) \to \Delta^1_F$ with n-component

$$\Delta^1_n \times F(Id_{[1]}) \to (\Delta^1_F)_n, \quad (f, x) \mapsto (F(f)^{-1}(x), f),$$

and hence is an isomorphism of simplicial sets. Thus, as the diagram

$$\begin{array}{ccc} \Delta^1_F & \xrightarrow{\cong} & \Delta^1 \times F(Id_{[1]}) \\ & \searrow \quad \swarrow & \\ & \Delta^1 & \end{array} \tag{2.33}$$

commutes, we have $\Delta^1_F \cong \Delta^1 \times F(Id_{[1]})$ in $\mathbf{Cov}(\Delta^1)$. $\square$

2.6 Finite Sheeted Covering Spaces

In this section we will define what it means for a simplicial covering to be finite sheeted. We will then consider how restricting to the core of the full subcategory of $\mathbf{Cov}(X)$ spanned by finite sheeted covering spaces changes Theorem 2.5.9.

Definition 2.6.1. A simplicial covering $p : Y \to X$ is called a *finite sheeted* simplicial covering if the fibres of $p : Y \to X$ are finite sets.

Definition 2.6.2. Let $\mathbf{FinCov}(X)$ be the full subcategory of $\mathbf{Cov}(X)$ spanned by finite sheeted covering spaces.

Restricting $\mathbf{Cov}(X)$ to $\mathbf{FinCov}(X)$ changes Theorem 2.5.9 in the obvious way (as stated in Corollary 2.6.3), as does restricting to $\mathbf{FinCov}(X)^{\simeq}$ (stated in Corollary 2.6.4).

Corollary 2.6.3. *Let X be a simplicial set. There is an equivalence of categories*

$$\mathbf{FinCov}(X) \simeq [\Pi_1(X), \mathbf{FinSet}].$$

Proof. By Theorem 2.5.9 it is sufficient to show:

1. The image of the functor $\mathbf{Cov}(X) \to [\Pi_1(X), \mathbf{Set}]$ restricted to $\mathbf{FinCov}(X)$ only contains functors which are valued in the category $\mathbf{FinSet}$.
2. The image of the functor $[\Pi_1(X), \mathbf{Set}] \to \mathbf{Cov}(X)$ restricted to functors which are valued in the category $\mathbf{FinSet}$ is contained in $\mathbf{FinCov}(X)$.

1. Let $\sigma : Y \to X$ be a finite sheeted simplicial covering. Let x be a 0-simplex of X and note that $F_{(Y,\sigma)}(x)$ is equal to the finite set $Y|_x$. Hence, if y is an n-simplex of X, as $F_{(Y,\sigma)}\left((d^0)^n\right) : F_{(Y,\sigma)}\left((d_0)^n(y)\right) \to F_{(Y,\sigma)}\left(y\right)$ is an isomorphism, $F_{(Y,\sigma)}(y)$ is finite. Hence, $F_{(Y,\sigma)}$ is a functor in $[\Pi_1(X), \mathbf{FinSet}]$.

2. Let $F : \Pi_1(X) \to \mathbf{FinSet}$ be a functor, and let x be a vertex of X. Then the set $X_F|_x$ is equal to the finite set $F(x)$. Hence, X_F is a finite sheeted simplicial covering. □

Corollary 2.6.4. *Let X be a simplicial set. There is an equivalence of categories*

$$\mathbf{FinCov}(X)^{\simeq} \simeq [\Pi_1(X), \mathbf{FinSet}^{\simeq}].$$

Proof. The result follows from Corollary 2.6.3 and Lemma 2.6.5 (stated below). □

Lemma 2.6.5. *If C is a category and G is a groupoid then* $[G, C]^{\simeq} = [G, C^{\simeq}]$.

Proof. Observe that the objects of $[G, C]^{\simeq}$ and $[G, C^{\simeq}]$ are the same. Thus, as a natural transformation $\alpha : F \to G$ in $[G, C]$ is an isomorphism if and only if the map $\alpha_g : F(g) \to G(g)$ is an isomorphism for all g in G, we have $[G, C]^{\simeq} = [G, C^{\simeq}]$. □

Remark 2.6.6. Recall that a category is said to be *essentially small* if it is equivalent to small category. Hence, an immediate corollary of Corollary 2.6.4 is that the category $\mathbf{FinCov}(X)^{\simeq}$ is essentially small. This is because $\Pi_1(X)$ is small, $\mathbf{FinSet}^{\simeq}$ is equivalent to the small category $\coprod_{n \geq 0} \Sigma_n$, where Σ_n is the group of automorphisms on $\mathbf{n}$, and functor categories between small categories are small.

The equivalence of categories

$$\mathbf{FinCov}(X)^{\simeq} \simeq [\Pi_1(X), \mathbf{FinSet}^{\simeq}]$$

lays the foundation of the work to be done in Chapter 5 where the stable homotopy type of the stable homotopy type of the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$ will be identified, once a nice assumption is placed on the simplicial set X.

Chapter 3

Algebraic K-theory: H-spaces, Spectra, and Γ-Spaces

In his seminal 1974 paper 'Categories and Cohomology Theories' Segal introduced the notion of the Algebraic K-theory spectrum of a small permutative category. Put simply, a permutative category is a category C equipped with a composition which obeys associativity, identity, and commutativity laws. The Algebraic K-theory spectrum of C is a sequence of spaces $\mathbf{B}\mathcal{K}C = \{\mathbf{B}\mathcal{K}C_n\}_{n\in\mathbb{N}}$ that is constructed using only the data of C. Each space is related via a closed inclusion $\mathbf{B}\mathcal{K}C_n \hookrightarrow \Omega\mathbf{B}\mathcal{K}C_{n+1}$ that is a homotopy equivalence if $n \geq 1$. Segal's method of associating the Algebraic K-theory spectrum $\mathbf{B}\mathcal{K}C$ to C is of course functorial. In the next 2 chapters we will give a presentation of Segal's theory. In doing so we will also allow for the Algebraic K-theory spectra of symmetric monoidal categories (a class of categories broader than the class of permutative categories) to be defined (following Mandell in [18]).

To understand how to construct the Algebraic K-theory spectrum of a symmetric monoidal category an understanding of three notions is required. These three notions are the notions of H-spaces, spectra, and Γ-spaces. We will study H-spaces and their group completions in Sections 3.1 and 3.2. Spectra will be studied in Section 3.3, and in Section 3.4 we will discuss a special class of spectra called infinite loop spaces. We will discuss Γ-spaces in Section 3.6, but only once we have rectified some issues concerning the realisation of simplicial spaces in Section 3.5. Finally, in Section 3.7, we will combine much of the theory discussed in Sections 3.1–3.6 to functorially associate a spectrum with some nice properties to every Γ-space.

The key takeaways of this chapter will be an understanding of H-spaces, spectra, Γ-spaces, and the functor that sends Γ-spaces to spectra. These takeaways will be built on in the next chapter.

3.1 H-Spaces and their Group Completions

In this section we will review some of the basic theory of H-spaces. In particular, we will discuss what it means to group complete a homotopy commutative H-space, and how it relates to the group completion of a monoid. Note that in this section, and in Section 3.2, we shall assume all spaces, maps, and products live inside **CGWH**.

We begin by defining what it means for a space to be an H-space.

Definition 3.1.1. Let X be space. If there exists a point e in X and a map

$$m : X \times X \to X$$

such that the diagrams

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times Id} & X \times X \\ \downarrow{\scriptstyle Id \times m} & & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow{m} & X \end{array} \tag{3.1}$$

and

$$\begin{array}{ccccc} X & \xrightarrow{Id \times e} & X \times X & \xleftarrow{e \times Id} & X \\ & {\scriptstyle Id}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle Id} & \\ & & X & & \end{array} \tag{3.2}$$

commute up to homotopy, then X is called an *H-space*. The point e is called the *identity* of the H-space X, and m is called the *multiplication map* of X. If the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \times X \\ & {\scriptstyle m}\searrow & \downarrow{\scriptstyle m} \\ & & X \end{array} \tag{3.3}$$

also commutes up to homotopy, where μ is the map

$$\mu : X \times X \to X \times X, \quad \mu(x, y) := (y, x)$$

then M is called a *homotopy commutative* H-space. If there also exists a map $\nu : M \to M$ such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{Id\times\nu} & X\times X & \xleftarrow{\nu\times Id} & X \\ & {\scriptstyle e}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle e} & \\ & & X & & \end{array} \tag{3.4}$$

commutes up to homotopy, then X is said to have a *homotopy inverse.*

Essentially, an H-space is a topological monoid where the associativity and identity laws hold up to homotopy. Such spaces will arise quite naturally when studying the Algebraic K-theory spectrum of a symmetric monoidal category, as we shall see later.

Thinking of an H-space as a topological monoid up to homotopy gives an obvious notion of H-space map and H-space equivalence.

Definition 3.1.2. Let X and X' be homotopy commutative H-spaces. A map $f : X \to X'$ is an *H-space map* if the diagrams

$$\begin{array}{ccc} X\times X & \xrightarrow{m} & X \\ \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} \\ X'\times X' & \xrightarrow{m'} & X' \end{array} \tag{3.5}$$

and

$$\begin{array}{ccc} \bullet & \xrightarrow{e} & X \\ & {\scriptstyle e'}\searrow & \downarrow{\scriptstyle f} \\ & & X' \end{array} \tag{3.6}$$

commute up to homotopy. An H-space map that is also a homotopy equivalence is called an *H-space equivalence.*

To define what it means to group complete a homotopy commutative H-space X we will want to draw upon our understanding of what it means to group complete an abelian monoid. This is most easily done by functorially associating an abelian monoid to X.

Proposition 3.1.3. *If X is a homotopy commutative H-space with multiplication map m, then $\pi_0(X)$ has the structure of an abelian monoid, where $\pi_0(X)$ is the set of path components of X. Furthermore, if X has a homotopy inverse then $\pi_0(X)$ is an abelian group.*

Proof. There is an induced multiplication map on $\pi_0(X)$

$$m : \pi_0(X) \times \pi_0(X) \to \pi_0(X), \quad [x][y] \mapsto [m(x, y)].$$

Using the homotopies that make diagrams (3.1) – (3.3) commute, one can show that this map is unital, associative, and commutative. For example, to see that m has an identity in $\pi_0(X)$ recall that there is a homotopy $H : I \times X \to X$ from $m(e, -)$ to Id. Fixing an x gives us a path from $m(e, x)$ to x. Hence, $[m(e, x)] = [x]$. Similarly, the homotopy that makes diagram (3.4) commute shows that if X has a homotopy inverse then the induced map

$$\nu : \pi_0(X) \to \pi_0(X)$$

gives $\pi_0(X)$ the structure of a group. □

We want to define the group completion of a homotopy commutative H-space X to be an H-space map $X \to Y$ such that the induced map $\pi_0(X) \to \pi_0(Y)$ is a group completion. However, for technical reasons which we will not discuss here, to ensure the group completion of an H-space has useful properties we require it satisfies an additional axiom on homology.

Definition 3.1.4. Let X be a homotopy commutative H-space. A *group completion* of X is an H-space map $X \to Y$ such that:

1. the induced map $\pi_0(X) \to \pi_0(Y)$ is a group completion of $\pi_0(X)$; and
2. the homology ring $H_*(Y; R)$ is isomorphic to the localised homology ring $H_*(X; R)[\pi_0(X)^{-1}]$, for all commutative rings R.

There is a general method for constructing group completions of homotopy commutative topological monoids, which we will briefly outline now (see Section 15 of [20] for details). This general construction will be useful later when discussing the Algebraic K-theory space of a symmetric monoidal category.

Construction 3.1.5. Let X be a homotopy commutative topological monoid. Let NX be the simplicial space whose space of n-simplicies is the space X^n. The face and degeneracy maps of NX are the maps

$$d_i : X^{n+1} \to X^n, \quad d_i((x_0, ..., x_n)) := \begin{cases} (x_1, ..., x_n), & i = 0 \\ (x_0, ..., x_{i+1}x_i, ..., x_n), & 0 < i < n+1 \\ (x_0, ..., x_{n-1}), & i = n+1 \end{cases}$$

and

$$s_j : X^n \to X^{n+1}, \quad s_j((x_0, ...x_{n-1})) := (x_0, ..., e, x_j, ..., x_{n-1}).$$

Now recall Construction 1.4.10. Consider the diagram (1.25) for NX when $n = 1$. As NX_0 is a point so is $s(NX_0)$. So, the pushout diagram defining $|NX|_1$ is the same as the pushout diagram defining the suspension ΣX of X. It follows that $|NX|$ is isomorphic to ΣX. Therefore the canonical map $|NX|_1 \to BX := |NX|$ induces a map $X \to \Omega BX$ by adjointness.

Theorem 3.1.6. *Let X be a homotopy commutative topological monoid. The canonical map $X \to \Omega BX$ is a group completion of X.*

Proof. See Theorem 15.1 in [20]. □

3.2 Numerably Contractible H-spaces

Observe that in Section 3.1 it was never asserted that if $X \to Y$ is a group completion of an H-space X, then Y would have some group-like structure. In particular, it is not required that Y have a homotopy inverse. This seems at odds with the analogy made with the group completions of monoids. Hence, in the following section we will use some of the theory of numerably contractible spaces and fibre homotopy equivalences to give a sufficient condition on H-spaces to ensure that they have homotopy inverses, amplifying a note made by Segal in [27].

Let us begin this section by reviewing the definitions of numerably contractible space and fibre homotopy equivalence.

Definition 3.2.1. Let X be a space. The space X is called *numerably contractible* if

1. it has an open cover $\{U_i\}_{i \in I}$ such that the inclusion maps $U_i \hookrightarrow X$ are nulhomotopic; and
2. there is a partition of unity $\{\rho_i\}_{i \in I}$ subordinate to $\{U_i\}_{i \in I}$.

Definition 3.2.2. Let $p : E \to X$ and $q : E' \to X$ be maps of spaces. A homotopy $H : E \times I \to E'$ is called a *homotopy over* X from H_0 to H_1 if for each t the map H_t makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{H_t} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array} \tag{3.7}$$

commute. A map $f : E \to E'$ making diagram (3.7) commute is called a *fibre homotopy equivalence* if there exists a map $g : E' \to E$ making diagram (3.7) commute and homotopies over X from $f \circ g$ to $Id_{E'}$ and from Id_E to $g \circ f$.

A homotopy equivalence $f : E \to E'$ making diagram (3.7) commute which is also a homotopy equivalence when restricted to the fibres of p is not necessarily a fibre homotopy equivalence. However, if X is numerably contractible, and p and q are Hurewicz fibrations, then not only is this the case, but it is a necessary and sufficient condition.

Proposition 3.2.3 [**Theorem 6.3, Dold** [4]]**.** *Let X be a numerably contractible space and let $p : E \to X$ and $q : E' \to X$ be Hurewicz fibrations. A map $f : E \to E'$ making the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array} \tag{3.8}$$

commute is a fibre homotopy equivalence if and only if for all x in X the restricted map

$$f|_x : p^{-1}\{x\} \to q^{-1}\{x\}$$

is a homotopy equivalence.

Proof. This follows immediately from Theorem 6.3 in [4] as Hurewicz fibrations have the covering homotopy property, and hence the weak homotopy covering property. □

Proposition 3.2.3 then allows us to give the previously alluded to sufficient condition on H-spaces to ensure that they have homotopy inverses.

Corollary 3.2.4. *Let X be a homotopy commutative H-space with multiplication map m and identity e. If X is numerably contractible and its monoid of path components $\pi_0(X)$ is a group, then X has a homotopy inverse.*

Proof. First define the map

$$\varphi : X \times X \to X \times X, \quad \varphi(x, y) := (x, m(x, y)).$$

We will show that the map φ is a fibre homotopy equivalence. Now, as the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\quad \varphi \quad} & X \times X \\ & {\scriptstyle pr_1}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & X & \end{array} \tag{3.9}$$

commutes, and as the projection map $pr_1 : X \times X \to X$ is a Hurewicz fibration, by Proposition 3.2.3, to show φ is a fibre homotopy equivalence it is sufficient to show that for each x in X the map

$$\varphi|_x : X \to X, \quad \varphi|_x(y) := m(x, y)$$

is a homotopy equivalence. As $\pi_0(X)$ is a group, for all $[x]$ in $\pi_0(X)$ there is an $[x^*]$ in $\pi_0(X)$ such that $[x][x^*]$ is the identity in $\pi_0(X)$. That is, there is a path $\gamma : m(x^*, x) \rightsquigarrow e$ in X. Hence, there is a homotopy

$$X \times I \to X, \quad (z, t) \mapsto m(\gamma(t), z)$$

from the map $m(m(x^*, x), -)$ to $m(e, -)$. Thus,

$$\varphi|_x \circ \varphi|_{x^*} = m(x^*, m(x, -)) \sim m(m(x^*, x), -) \sim m(e, -) \sim Id.$$

An identical argument will show that $\varphi|_{x^*} \circ \varphi|_x \sim Id$. Hence, for each x in X the map $\varphi|_x$ is a homotopy equivalence with homotopy inverse $\varphi|_{x^*}$, and so φ is a fibre homotopy equivalence.

Let

$$\phi : X \times X \to X \times X, \quad \phi(x, y) := (\phi_1(x, y), \phi_2(x, y))$$

be a fibre homotopy inverse of φ, and let $H : I \times X \times X \to X \times X$ be a homotopy over X from $\varphi \circ \phi$ to $Id_X \times Id_X$. As H_t must make the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{H_t} & X \times X \\ & {\scriptstyle pr_1} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & X & \end{array} \tag{3.10}$$

commute for all t, if we consider $t = 1$, we must have that $\phi_1(x, y) = x$. Hence, we have that the map

$$pr_2 \circ H|_e : X \times I \to X,$$

where

$$H|_e : X \times I \to X \times X, \quad H|_e(x, t) := H(x, e, t),$$

is a homotopy from $m(-, \phi_2(-, e)) : X \to X$ to the constant map $e : X \to X$. Thus, the diagram

$$\begin{array}{ccc} X & \xrightarrow{Id \times \phi_2(-, e)} & X \times X \\ & {\scriptstyle e} \searrow & \downarrow {\scriptstyle m} \\ & & X \end{array} \tag{3.11}$$

commutes up to homotopy. As the multiplication map m is homotopy commutative, the diagram

$$\begin{array}{ccc} X \times X & \xleftarrow{\phi_2(-, e) \times Id} & X \\ {\scriptstyle m} \downarrow & \swarrow {\scriptstyle e} & \\ X & & \end{array} \tag{3.12}$$

then also commutes up to homotopy. That is, X has a homotopy inverse. $\square$

3.3 The Category of Spectra

In this section we will review the basic theory of spectra. A spectrum is essentially a sequence of spaces $\{\mathbf{X}_n\}_{n\in\mathbb{N}}$ such that each space $\mathbf{X}_n$ is a closed subspace of $\Omega\mathbf{X}_{n+1}$. Spectra are the basic objects of study in stable homotopy theory. As its name suggests, the Algebraic K-theory spectrum of a symmetric monoidal category is an example of a spectrum.

Remark 3.3.1. In this section, and in Section 3.4, *all* the (pointed) spaces we will consider will be CGWH. As per usual, all (co)limits we will discuss will be computed in **Top** (or $\mathbf{Top}_*$) but will land in **CGWH** (or $\mathbf{CGWH}_*$).

Definition 3.3.2. A *spectrum* $\mathbf{X}$ is a sequence of pointed topological spaces $\{\mathbf{X}_n\}_{n\in\mathbb{N}}$ and pointed closed inclusions

$$\sigma_n : \mathbf{X}_n \to \Omega\mathbf{X}_{n+1}.$$

The maps σ_n are called the *structure maps* of the spectrum $\mathbf{X}$. A *map of spectra* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a sequence of pointed maps

$$\mathbf{f}_n : \mathbf{X}_n \to \mathbf{Y}_n$$

such that for each $n \geq 0$ the diagram

$$\begin{array}{ccc} \mathbf{X}_n & \longrightarrow & \Omega\mathbf{X}_{n+1} \\ \downarrow{\scriptstyle \mathbf{f}_n} & & \downarrow{\scriptstyle \Omega\mathbf{f}_n} \\ \mathbf{Y}_n & \longrightarrow & \Omega\mathbf{Y}_{n+1} \end{array} \tag{3.13}$$

commutes. Let **Spec** denote the category of spectra and maps of spectra.

Remark 3.3.3. As the spectra we are considering are sequences of CGWH spaces, to specify the structure maps of a spectrum, by the $\Sigma \dashv \Omega$ adjunction on spaces, it is sufficient to specify a closed inclusion

$$\overline{\sigma_n} : \Sigma\mathbf{X}_n \to \mathbf{X}_{n+1}.$$

The map $\sigma_n : \mathbf{X}_n \to \Omega\mathbf{X}_{n+1}$ adjoint to $\overline{\sigma_n}$ will be a closed inclusion by Corollary 5.11 in [28].

Let's now describe two examples of spectra that will arise in our discussions of Algebraic K-theory spectra.

Example 3.3.4. The suspension spectrum $\Sigma^\infty X$ of a pointed space X is the spectrum with $(\Sigma^\infty X)_n = \Sigma^n X$, and whose structure maps

$$\Sigma^n X \to \Omega\Sigma^{n+1} X$$

are the transposes of the identity maps $\Sigma^{n+1} X \to \Sigma^{n+1} X$.

Construction 3.3.5. Example 3.3.4 can be extended to define a functor

$$\Sigma^\infty : \mathbf{CGWH}_* \to \mathbf{Spec}.$$

The functor Σ^∞ has a right adjoint

$$(-)_0 : \mathbf{Spec} \to \mathbf{CGWH}_*$$

which maps a spectrum $\mathbf{X}$ to its 0-th space $\mathbf{X}_0$. The unit and counit map of $\Sigma^\infty \dashv (-)_0$ adjunction are induced by identity map and the $\Sigma \dashv \Omega$ adjunction, respectively.

Example 3.3.6. Recall that $\Sigma\mathbb{S}^n = \mathbb{S}^{n+1}$ for all $n \geq 0$. The sphere spectrum $\mathbb{S}$ is a special case of the suspension spectrum where $X = \mathbb{S}^0$. That is, the sphere spectrum $\mathbb{S}$ is the spectrum with $\mathbb{S}_n = \mathbb{S}^n$, and whose structure maps are the transposes of the identity maps

$$\mathbb{S}^{n+1} \to \mathbb{S}^{n+1}.$$

Let X be a pointed CW-complex. Observe that the canonical map $X \to \Omega\Sigma X$ induces a diagram

$$\pi_k(X) \longrightarrow \pi_{k+1}(\Sigma X) \longrightarrow \pi_{k+2}(\Sigma^2 X) \longrightarrow \ldots \tag{3.14}$$

The Freudenthal Suspension Theorem states that if X is n-connected, then the induced map

$$\pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k < 2n+1$. In fact it can be shown that diagram (3.14) stabilises in the sense that all but finitely many of its maps are isomorphisms. These observations motivate the definition of the stable homotopy groups of a pointed space

$$\pi_k^S(X) = \varinjlim \pi_{k+n}(\Sigma^n X).$$

The notion of stable homotopy groups of pointed spaces can be generalised to spectra.

Definition 3.3.7. Let $\mathbf{X}$ be a spectrum and m an integer. If $m \geq 0$ let $\pi_m(\mathbf{X})$ be the colimit of the diagram

$$\pi_m(\mathbf{X}_0) \longrightarrow \pi_{m+1}(\mathbf{X}_1) \longrightarrow \pi_{m+2}(\mathbf{X}_2) \longrightarrow \ldots \tag{3.15}$$

in the category **AbGrp** of abelian groups, where the map $\pi_{m+n}(\mathbf{X}_n) \to \pi_{m+n+1}(\mathbf{X}_{n+1})$ is induced by the structure map $\mathbf{X}_n \to \Omega\mathbf{X}_{n+1}$. If $m < 0$ let $\pi_m(\mathbf{X})$ be the colimit of the diagram

$$\pi_0(\mathbf{X}_{|m|}) \longrightarrow \pi_1(\mathbf{X}_{|m|+1}) \longrightarrow \pi_2(\mathbf{X}_{|m|+2}) \longrightarrow \ldots \tag{3.16}$$

in **AbGrp**. The group $\pi_m(\mathbf{X})$ is called the *m-th stable homotopy group* of the spectrum $\mathbf{X}$.

Remark 3.3.8. Definition 3.3.7 carries with it a small technical issue. For an integer m and spectrum $\mathbf{X}$ the diagram defining the m-th stable homotopy group $\pi_m(\mathbf{X})$ may contain objects that are not abelian groups. When this occurs to compute $\pi_m(\mathbf{X})$ simply truncate the diagram defining $\pi_m(\mathbf{X})$ such that it only contains abelian groups. Then compute the colimit of the resulting diagram. Computing $\pi_m(\mathbf{X})$ in this way does not influence any of the results that we will discuss in the this thesis, and so we will henceforth ignore this issue.

Remark 3.3.9. Note that the stable homotopy groups of the suspension spectrum $\Sigma^\infty X$ of a pointed space X are precisely the stable homotopy groups of X.

Definition 3.3.10. A spectrum $\mathbf{X}$ is called *connective* if for each $n \leq -1$ the group $\pi_n(\mathbf{X})$ is trivial. Let $\mathbf{Spec}_{\geq 0}$ be the full subcategory of $\mathbf{Spec}$ spanned by connective spectra.

Example 3.3.11. The functor

$$\Sigma^\infty : \mathbf{CGWH}_* \to \mathbf{Spec}$$

introduced in Construction 3.3.5 lands inside $\mathbf{Spec}_{\geq 0}$, as taking the suspension of a space increases its connectivity.

Definition 3.3.12. A map $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of spectra is called a *stable weak equivalence* if for each $m \in \mathbb{Z}$ the induced map

$$\pi_m(\mathbf{X}) \to \pi_m(\mathbf{Y})$$

is an isomorphism. Two spectra are said to have the same *stable homotopy type* if there is a zigzag of stable weak equivalences between them.

As the stable homotopy groups $\pi_m(\mathbf{X})$ of a spectrum $\mathbf{X}$ are defined as sequential colimits they can be computed by considering the tail behaviour of certain diagrams. By making this observation precise we can give a sufficient condition on maps of connective spectra to ensure they are stable weak equivalences.

Lemma 3.3.13. *Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ be a map of connective spectra. If there exists an m such that for all $n \geq m$ the map $\mathbf{f}_n : \mathbf{X}_n \to \mathbf{Y}_n$ is a weak homotopy equivalence then $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a stable weak equivalence.*

Proof. We have that all $i \geq 0$ and for all $n \geq m$ the map $\pi_i(\mathbf{X}_n) \to \pi_i(\mathbf{Y}_n)$ is an isomorphism. Fixing an $n \geq m$, note that for all $p \geq 0$ the colimit of the diagram

$$\pi_{p+n}(\mathbf{X}_n) \longrightarrow \pi_{p+1}(\mathbf{X}_{n+1}) \longrightarrow \pi_{p+2}(\mathbf{X}_{n+2}) \longrightarrow \ldots \tag{3.17}$$

is $\pi_p(\mathbf{X})$. Thus, the induced map $\pi_p(\mathbf{X}) \to \pi_p(\mathbf{Y})$ is the map making the diagram

$$\begin{array}{ccccccccc} \pi_{p+n}(\mathbf{X}_n) & \longrightarrow & \pi_{p+n+1}(\mathbf{X}_{n+1}) & \longrightarrow & \pi_{p+n+2}(\mathbf{X}_{n+2}) & \longrightarrow & \ldots & \longrightarrow & \pi_p(\mathbf{X}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & & & \downarrow \\ \pi_{p+n}(\mathbf{Y}_n) & \longrightarrow & \pi_{p+n+1}(\mathbf{Y}_{n+1}) & \longrightarrow & \pi_{p+n+2}(\mathbf{Y}_{n+2}) & \longrightarrow & \ldots & \longrightarrow & \pi_p(\mathbf{Y}). \end{array} \tag{3.18}$$

commute. Thus, inverting each vertical map $\pi_{p+n+j}(\mathbf{X}_{n+j}) \to \pi_{p+n+j}(\mathbf{Y}_{n+j})$ in diagram (3.18) induces a map $\pi_p(\mathbf{Y}) \to \pi_p(\mathbf{X})$ that is mutually inverse to the map $\pi_p(\mathbf{X}) \to \pi_p(\mathbf{Y})$. Thus, the groups $\pi_p(\mathbf{X})$ and $\pi_p(\mathbf{Y})$ are isomorphic. □

We will now introduce the concept of Ω-spectra.

Definition 3.3.14. Let $\mathbf{X}$ be a spectrum. The spectrum $\mathbf{X}$ is called an Ω-spectrum if for each $n \geq 0$ the structure map

$$\sigma_n : \mathbf{X}_n \to \Omega\mathbf{X}_{n+1}$$

is a homotopy equivalence. Let $\mathbf{Spec}^{\Omega}$ be the full subcategory of $\mathbf{Spec}$ spanned by Ω-spectra.

Remark 3.3.15. If $\mathbf{X}$ is an Ω-spectrum and $m \geq 0$, then the diagram defining $\pi_m(\mathbf{X})$ can be rewritten as the diagram

$$\pi_m(\mathbf{X}_0) \longrightarrow \pi_m(\mathbf{X}_0) \longrightarrow \pi_m(\mathbf{X}_0) \longrightarrow \ldots \tag{3.19}$$

Hence, $\pi_m(\mathbf{X}) = \pi_m(\mathbf{X}_0)$.

Corollary 3.3.16. *Let* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ *be a map of connective* Ω*-spectra. The map* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ *is a stable weak equivalence if and only if* $\mathbf{f}_0 : \mathbf{X}_0 \to \mathbf{Y}_0$ *is a weak homotopy equivalence.*

Proof. This is an immediate corollary of Remark 3.3.15. □

Example 3.3.17. The sphere spectrum $\mathbb{S}$ is clearly not an Ω-spectrum. For example, consider the structure map

$$\mathbb{S}^0 \to \Omega\mathbb{S}^1.$$

As $\pi_0(\mathbb{S}^0)$ is equal to a set of two points, and $\pi_0(\Omega\mathbb{S}^1)$ is equal to the integers, the structure map $\mathbb{S}^0 \to \Omega\mathbb{S}^1$ can't be a homotopy equivalence.

Example 3.3.18. Let A be a discrete abelian group. The classifying space BA of A also has the structure of an abelian group, and so also has a classifying space B^2A. Furthermore, the map $A \to \Omega BA$ adjoint to the natural closed inclusion $\Sigma A \to BA$ is a homotopy equivalence. Thus, there is an Ω-spectrum $\mathbf{B}A$ with $\mathbf{B}A_n = B^nA$ called the *Eilenberg–MacLane spectrum* for the group A. See Sections 16.5 and 22.1 in [21] for details.

To every spectrum $\mathbf{X}$ there is an associated stably equivalent Ω-spectrum $Q\mathbf{X}$.

Construction 3.3.19. Let $\mathbf{X}$ be a spectrum. For each $n \geq 0$ let $Q\mathbf{X}_n$ be the colimit of the diagram

$$\mathbf{X}_n \longrightarrow \Omega\mathbf{X}_{n+1} \longrightarrow \Omega^2\mathbf{X}_{n+2} \longrightarrow \dots \tag{3.20}$$

in $\mathbf{Top}_*$. Note that, by Lemma 5.9 in [28], $\Omega Q(\mathbf{X}_n) \cong Q(\Omega\mathbf{X}_n)$ for each $n \geq 0$. Hence, as the diagram defining $\Omega Q(\mathbf{X}_{n+1})$ can be written as

$$\Omega\mathbf{X}_{n+1} \longrightarrow \Omega^2\mathbf{X}_{n+2} \longrightarrow \Omega^3\mathbf{X}_{n+3} \longrightarrow \dots, \tag{3.21}$$

there is a homeomorphism $Q(\mathbf{X}_n) \cong \Omega Q(\mathbf{X}_{n+1})$ for each n. Hence, there is a spectrum $Q\mathbf{X}$ whose n-th space is $Q\mathbf{X}_n$ and whose structure maps $Q\mathbf{X}_n \to \Omega Q\mathbf{X}_{n+1}$ are homeomorphisms. This construction extends to define a functor

$$Q : \mathbf{Spec} \to \mathbf{Spec}^{\Omega},$$

by the functoriality of colimits.

Proposition 3.3.20. *If* $\mathbf{X}$ *is a spectrum there is a map of spectra* $\mathbf{X} \to Q\mathbf{X}$ *that is a stable weak equivalence.*

Proof. This is discussed in Section 25.7 in [21]. □

3.4 Infinite Loop Spaces

In this section we will discuss some of the theory of infinite loop spaces. In particular, we will discuss free infinite loop spaces. Such infinite loop spaces will make an appearance when identifying the stable homotopy types of the Algebraic K-theory spectra of some categories in Chapter 5.

Definition 3.4.1. A space X is said to be an *infinite loop space* if there is sequence of pointed topological spaces $\{X_n\}_{n\in\mathbb{N}}$ such that $X_0 = X$ and there are homeomorphisms

$$X_n \cong \Omega X_{n+1}.$$

An *infinite loop map* $f : X \to Y$ is a sequence of maps $f_k : X_k \to Y_k$ such that the diagram

$$\begin{array}{ccc} X_k & \xrightarrow{\cong} & \Omega X_{k+1} \\ \big\downarrow{\scriptstyle f_k} & & \big\downarrow{\scriptstyle \Omega f_k} \\ Y_k & \xrightarrow{\cong} & \Omega Y_{k+1} \end{array} \tag{3.22}$$

commutes. Let $\mathbf{Inf}\,\Omega$ be the category of infinite loop spaces and infinite loop maps.

Remark 3.4.2. Note that an infinite loop space is equivalently a spectrum where each structure map is a homeomorphism. Consequently, we may regard $\mathbf{Inf}\,\Omega$ as a full subcategory of $\mathbf{Spec}^{\Omega}$.

To each pointed space X we can functorially associate an infinite loop space QX known as its free infinite loop space.

Definition 3.4.3. Let X be a pointed topological space. The *free infinite loop space* QX on X is the 0-th space of the Ω-spectrum $Q(\Sigma^\infty X)$.

Remark 3.4.4. Note that there has been an abuse of notation between Construction 3.3.19 and Definition 3.4.3. There will be no risk of confusion as from Section 3.5 onwards the author will always use $Q(\Sigma^\infty X)_0$ to denote the free infinite loop space on X.

Remark 3.4.5. The free infinite loop space QX is defined as the colimit of the diagram

$$X \longrightarrow \Omega\Sigma X \longrightarrow \Omega^2\Sigma^2 X \longrightarrow \ldots \tag{3.23}$$

in $\mathbf{Top}_*$. As the maps $\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1} X$ in diagram (3.23) are closed inclusions (see Corollary 5.12 in [28]), the free infinite loop space QX can be described more concretely as the set

$$QX = \bigcup_{n\geq 0} \Omega^n\Sigma^n X$$

equipped with the topology of the union.

Construction 3.4.6. The composite functor

$$\mathbf{CGWH}_* \xrightarrow{\Sigma^\infty} \mathbf{Spec}_{\geq 0} \xrightarrow{Q} \mathbf{Spec}^{\Omega}_{\geq 0} \xrightarrow{(-)_0} \mathbf{Inf}\,\Omega.$$

extends Definition 3.4.3 to a functor

$$Q : \mathbf{CGWH}_* \to \mathbf{Inf}\,\Omega.$$

The infinite loop space QX is called the *free* infinite loop space on X as the functor $Q : \mathbf{CGWH}_* \to \mathbf{Inf}\,\Omega$ is left adjoint to a forgetful functor (Proposition 3.4.7).

Proposition 3.4.7. *The functor $Q : \mathbf{CGWH}_* \to \mathbf{Inf}\,\Omega$ is left adjoint to the functor $(-)_0 : \mathbf{Inf}\,\Omega \to \mathbf{CGWH}_*$ induced by mapping an infinite loop space X to it's* 0*-th space.*

Proof. See Proposition 1 in [19]. □

3.5 Good Simplicial Spaces

Recall that in Section 1.4 we introduced simplicial spaces, and discussed how to take their realisation. Unfortunately, unlike the realisation of simplicial sets, the realisation of simplicial spaces is not always well-behaved. Thankfully, we can replace each simplicial space X with a levelwise weakly homotopy equivalent 'good' simplicial space TX whose realisation $|TX|$ is well-behaved. In this section we will discuss good simplicial spaces and the good simplicial space TX.

The realisation of simplicial spaces, as defined in Definition 1.4.9, is not well-behaved. For example, let X be a simplicial space. The following properties need *not* be true in general:

1. If X_n has the homotopy type of a CW-complex for all $n \geq 0$, then $|X|$ has the homotopy type of a CW-complex.

2. If $X \to Y$ is a map of simplicial spaces such that $X_n \to Y_n$ is a homotopy equivalence for all $n \geq 0$, then the induced map $|X| \to |Y|$ is a homotopy equivalence.

However, we can replace X with a levelwise weakly homotopy equivalent good simplicial space TX whose realisation $|TX|$ is well-behaved. That is, there is a good simplicial space TX and a map $TX \to X$ such that for each $n \geq 0$ the map $TX_n \to X_n$ is a weak homotopy equivalence. The realisation $|TX|$ of TX also satisfies the two above properties (c.f. Proposition 3.5.7). Before we explain how to construct the good simplicial space TX, let us first define good simplicial space.

Remark 3.5.1. In this section, all the simplicial spaces we will consider will be valued in **CGWH**.

Definition 3.5.2 [**Definition A.4, Segal** [27]]**.** A simplicial space X is called *good* if for each $n \geq 0$ and $0 \leq i \leq n$ the inclusion map

$$s_i(X_n) \hookrightarrow X_{n+1}$$

is a cofibration. Let $\mathbf{sCGWH}^*$ be the full subcategory of **sCGWH** spanned by good simplicial spaces.

Two properties of good simplicial spaces are discussed in Remark 3.5.3 and Proposition 3.5.4 below.

Remark 3.5.3. Recall Construction 1.4.10 and the proof of Proposition 1.4.13. If a simplicial space X is good, by statement 3 of Proposition 1.6.7, the map

$$\partial|\Delta^n| \times X_n \cup |\Delta^n| \times s(X_{n-1}) \to |\Delta^n| \times X_n$$

is a cofibration. Thus, as cofibrations are stable under pushouts, the maps $|X|_{n-1} \to |X|_n$, as constructed in Construction 1.4.10, are cofibrations. Hence, the maps $|X|_n \to |X|$ are also cofibrations.

Proposition 3.5.4. *If X is a simplicial space such that X_n is a CW-complex for all $n \geq 0$, then X is good.*

Proof. For each i the map $s_i d_i : X_{n+1} \to s_i(X_n)$ is a retraction. Hence, by Corollary 2.4 (a) and Lemma 3.1 (a) in [15], the inclusion map $s_i(X_n) \hookrightarrow X_{n+1}$ is a cofibration. □

Let us now explain how to functorially associate to every simplicial space X the previously alluded to good simplicial space TX.

Construction 3.5.5. There is a functor

$$T : \mathbf{sCGWH} \to \mathbf{sCGWH}^*$$

which maps a simplicial space X to the good simplicial space TX. The space of n-simplices of TX is the space $|S(X_n)|$. The simplicial space TX is hence good by Proposition 3.5.4 as the geometric realisations of simplicial sets are CW-complexes (Proposition 1.2.7).

Remark 3.5.6. Recall Construction 1.4.17. Observe that $|TX|$ is the realisation of the bisimplicial set $S(X)$ whose set of (n, m)-bisimplices is $S(X_n)_m$. Thus, $|TX| = |dS(X)|$, and if $f : X \to Y$ is a map of simplicial spaces, then the map $|Tf| : |TX| \to |TY|$ is the realisation of the simplicial map $df : dS(X) \to dS(Y)$.

As mentioned previously, the realisation of TX is well-behaved. The well-behavedness of $|TX|$ is explicitly spelled out in Proposition 3.5.7.

Proposition 3.5.7. *The following statements are true:*

1. *The space $|TX|$ is a CW-complex for every simplicial space X.*
2. *If the map $X \to Y$ of simplicial spaces is a level-wise weak homotopy equivalence then $|TX| \to |TY|$ is a homotopy equivalence.*
3. *The functor $|T(-)|$ preserves finite products. I.e. $|T(X \times Y)| = |T(X)| \times |T(Y)|$.*

4. *There is map $TX \to X$ of simplicial spaces that is a levelwise weak homotopy equivalence. If X is good then the induced map $|TX| \to |X|$ is a homotopy equivalence.*

Proof. Statement 1 is true as $|TX| = |dS(X)|$ (Remark 3.5.6), and the realisation of a simplicial set is always a CW-complex (Proposition 1.2.7). Statement 2 is true as for each $n \geq 0$ the map $S(X_n) \to S(Y_n)$ is a weak homotopy equivalence. That is, $S(X) \to S(Y)$ is a weak equivalence of bisimplicial sets. Thus, as the functors d and $|-|$ preserve weak equivalences (Propositions 1.6.8 and 1.6.11), $|TX| \to |TY|$ is a homotopy equivalence. Statement 3 immediately follows the fact that $S(-)$ and $|-|$ preserve finite products. Finally, defining a map $TX \to X$ whose n-component is the counit map $|S(X_n)| \to X_n$ of the $|-| \dashv S(-)$ adjunction proves the first part of statement 4. The second part of statement 4 is discussed in Appendix A of [27]. □

There is more we can say about the functor T. It turns out that T is an example of a comonad in **sCGWH**. Let us conclude this section by discussing this fact, and some of its implications.

Definition 3.5.8. A *comonad* $< L, \lambda, \delta >$ in a category C consists of a functor $L : C \to C$ and two natural transformations $\epsilon : L \to Id$ and $\delta : L \to L^2$ which make the diagrams

$$\begin{array}{ccc} L & \xrightarrow{\delta} & L^2 \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle L\delta} \\ L^2 & \xrightarrow{\delta L} & L^3 \end{array} \tag{3.24}$$

and

$$\begin{array}{ccccc} & & L & & \\ & \swarrow^{Id} & \downarrow{\scriptstyle \delta} & \searrow^{Id} & \\ L & \xleftarrow[\epsilon L]{} & L^2 & \xrightarrow[L\epsilon]{} & L \end{array} \tag{3.25}$$

commute. We say L is a comonad in C if ϵ and δ are understood.

Proposition 3.5.9. *If $F : C \rightleftarrows D : G$ is an adjunction with unit $\eta : Id \to G \circ F$ and counit $\epsilon : F \circ G \to Id$ then $< F \circ G, \epsilon, F\eta G >$ is a comonad in D.*

Proof. See page 139 in [17]. □

Corollary 3.5.10. *The functor T :* **sCGWH** $\to$ **sCGWH** *is a comonad in* **sCWGH**. *In particular, there is a comultiplication natural transformation $T \to T^2$ whose X-component is the map*

$$|S(X)| \to |S|S(X)||$$

which is the realisation of the unit of the $|-| \dashv S(-)$ *adjunction on* $S(X)$*.*

An interesting fact about the comultiplication map $T \to T^2$ is that it induces cofibrations. This fact will help us when constructing spectra in Section 3.7.

Lemma 3.5.11. *If* K *is a simplicial set then there is a section of the canonical map* $T|K| \to K$ *which is a cofibration.*

Proof. The canonical map $T|K| \to |K|$ is the component $|S(|K|)| \to |K|$ of the counit map the $|-| \dashv S(-)$ adjunction at $|K|$. By the triangle identities, the geometric realization $|K| \to |S(|K|)|$ of the unit map $K \to S(|K|)$ of the same adjunction is a right inverse to the canonical map $T|K| \to |K|$. In particular, $|K| \to |S(|K|)|$ is injective. But any simplicial map whose geometric realization is injective is itself injective (see (ii) of Proposition 4.4.3 of [6]). Therefore $|K| \to |S(|K|)|$ is the geometric realization of an injective map and hence is a cofibration. □

Lemma 3.5.12. *For any simplicial space* X *the map* $|TX| \to |T^2X|$ *is a cofibration.*

Proof. First observe that, by the triangle identities for the unit and counit of an adjunction, for each $n \geq 0$ the map $S(X_n) \to S|S(X_n)|$ is a cofibration in **sSet**. That is, the map $S(X) \to S|S(X)|$ of the underlying bisimplicial sets of $|TX|$ and $|T^2X|$ is a cofibration in **ssSet**. Thus, as the functors d and $|-|$ preserve cofibrations, the map $|T(X)| \to |T^2(X)|$ is a cofibration. □

Corollary 3.5.13. *Let* X *be a simplicial space such that* X_0 *is a point. There is a closed inclusion*

$$\Sigma TX_1 \to T\big|T^2X\big|.$$

Proof. Note that as X_0 is a point X_1 can be equipped with a canonical basepoint given by the image of the map $X_0 \to X_1$. Now, as X_0 is a point, the space $|TX|_1$ (as in Construction 1.4.10) is equal to ΣTX_1 (see Construction 3.1.5). Thus, by Remark 3.5.3 there is a cofibration $\Sigma TX_1 \to |TX|$. Hence, the composite map

$$\Sigma TX_1 \to |TX| \to \big|T^2X\big|$$

is a cofibration (statement 1 in Proposition 1.6.7). In Remark 3.5.6 we observed that $|T^2X|$ is the geometric realization of a simplicial set (namely $|dS(TX)|$). Therefore, by Lemma 3.5.12 there is a cofibration $|T^2X| \to T|T^2X|$. The composite map

$$\Sigma TX_1 \to |TX| \to \big|T^2X\big| \to T\big|T^2X\big|$$

is hence a cofibration. Thus, as cofibrations in **CGWH** are closed inclusions, we are done. □

3.6 Γ-Spaces and Categories

In this section we will discuss Γ-spaces and Γ-categories, which are clever gadgets first defined by Segal in [27]. We are interested in Γ-categories, as they can be constructed from symmetric monoidal categories, and they induce Γ-spaces. We are interested in Γ-spaces as they give rise to spectra.

We will begin this section by defining the category Γ^{op}. We will then define Γ-spaces and Γ-categories.

Definition 3.6.1. Let Γ^{op} be the category of finite pointed sets and pointed maps. If $n \geq 0$ is a natural number write $\mathbf{n}$ for the finite pointed set $\mathbf{n} = \{0, 1, ..., n\}$ with basepoint 0.

Remark 3.6.2. Note that Γ^{op} has finite products; in particular, if $S \in \Gamma^{op}$ then there is a functor

$$S \times - : \Gamma^{op} \to \Gamma^{op}, \quad T \mapsto S \times T.$$

Definition 3.6.3. A functor $X : \Gamma^{op} \to \mathbf{CGWH}$ is called a Γ-*space* if:

1. $X(\mathbf{0})$ is contractible; and
2. the map
$$X(\mathbf{n}) \to \underbrace{X(\mathbf{1}) \times \cdots \times X(\mathbf{1})}_{n-\text{times}}$$
induced by the maps
$$p_i : \mathbf{n} \to \mathbf{1}, \quad p_i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise,} \end{cases}$$
in Γ^{op} for $i = 1, ..., n$ is a homotopy equivalence.

A Γ-space is called *reduced* if $X(\mathbf{0})$ is a point. Let $\Gamma\mathbf{CGWH}$ denote the category of Γ-spaces, and let $\Gamma\mathbf{CGWH}_0$ denote the full subcategory of $\Gamma\mathbf{CGWH}$ spanned by reduced Γ-spaces.

Definition 3.6.4. A functor $X : \Gamma^{op} \to \mathbf{Cat}$ is called a Γ-*category* if:

1. $X(\mathbf{0})$ is equivalent to the terminal category; and
2. the functor
$$X(\mathbf{n}) \to \underbrace{X(\mathbf{1}) \times \cdots \times X(\mathbf{1})}_{n-\text{times}}$$

induced by the maps

$$p_i : \mathbf{n} \to \mathbf{1}, \quad p_i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise,} \end{cases}$$

in Γ^{op} for $i = 1, ..., n$ is an equivalence of categories.

Let $\Gamma\mathbf{Cat}$ be the category of Γ-categories and natural transformations of Γ-categories.

Construction 3.6.5. Note that given a Γ-category X we can always construct a Γ-space $|NX|$ by composing X with the nerve functor N and the realisation functor $|-|$. The functor $|NX|$ satisfies axioms 1 and 2 of Definition 3.6.3 as the composition of the nerve N and realisation $|-|$ functors sends equivalences of categories to homotopy equivalences (Corollary 1.3.21).

Every reduced Γ-space X carries the structure of a commutative H-space. The structure of this H-space will help us identify the stable homotopy types of the Algebraic K-theory spectra of some categories later.

Proposition 3.6.6. *If X is a reduced Γ-space then $X(\mathbf{1})$ has the structure of a commutative H-space.*

Proof. The multiplication map m on $X(\mathbf{1})$ is given by the map

$$X(\mathbf{1}) \times X(\mathbf{1}) \xrightarrow{\simeq} X(\mathbf{2}) \xrightarrow{X(m')} X(\mathbf{1}),$$

where m' is the map

$$m' : \mathbf{2} \to \mathbf{1}, \quad m'(j) := \begin{cases} 0 & \text{if } j = 0 \\ 1 & \text{otherwise} \end{cases}$$

in Γ^{op}, and the unit e is the image of the map $X(\mathbf{0}) \to X(\mathbf{1})$. Showing that m and e equip $X(\mathbf{1})$ with the structure of a commutative H-space quickly turns into a long diagram chase. We will not include it here as it is not at all enlightening. □

To every Γ-space X there is an associated simplicial space. Appealing to the simplicial space associated to X will allow us to define what it means for X to be good, and define the realisation $|X|$ of X.

Construction 3.6.7. There is a functor

$$\Delta^{op} \to \Gamma^{op}$$

which maps objects $[n]$ in Δ to $\mathbf{n}$ and maps $f : [n] \to [m]$ to the pointed map $f' : \mathbf{m} \to \mathbf{n}$ such that

$$(f')^{-1}(j) = \{i \in \mathbf{m} : f(j-1) < i \leq f(j)\},$$

for $1 \leq j \leq n$. Thus, to every Γ-space X there is an associated simplicial space defined by the restriction of X along $\Delta^{op} \to \Gamma^{op}$.

Definition 3.6.8. If the associated simplicial space of a Γ-space X is good, then X is called a *good* Γ-space. Let $\Gamma\mathbf{CGWH}^*$ be the category of good Γ-spaces.

To ensure a well-behaved notion of realisation for Γ-spaces we will appeal to the functor T constructed in Construction 3.5.5.

Proposition 3.6.9. *There is a functor*

$$T : \Gamma\mathbf{CGWH} \to \Gamma\mathbf{CGWH}^*$$

such that for each Γ-space X there is a map $TX \to X$ of Γ-spaces whose S-component $TX(S) \to X(S)$ is a weak homotopy equivalence for each $S \in \Gamma^{op}$.

Proof. Suppose $X : \Gamma^{op} \to \mathbf{CGWH}$ is a Γ-space. Let $TX : \Gamma^{op} \to \mathbf{CGWH}$ be the composite functor

$$\Gamma^{op} \xrightarrow{X} \mathbf{CGWH} \xrightarrow{S(-)} \mathbf{sSets} \xrightarrow{|-|} \mathbf{CGWH}.$$

The functor TX is a Γ-space as both the functors $S(-)$ and $|-|$ preserve homotopy equivalences and finite products. The mapping $X \mapsto TX$ is clearly functorial on $\Gamma\mathbf{CGWH}$. Furthermore, the counit map of the $|-| \dashv S(-)$ adjunction induces a map $TX \to X$ whose S-component $TX(S) \to X(S)$ is a weak homotopy equivalence for each $S \in \Gamma^{op}$. □

Remark 3.6.10. Note that if a Γ-space X is reduced, then so is the Γ-space TX.

Definition 3.6.11. Let X be a Γ-space. The *geometric realisation* $|X|$ of the Γ-space X is defined to be the geometric realisation of the simplicial space associated to the good Γ-space TX.

Remark 3.6.12. When introducing Γ-spaces in [27], Segal discussed a few different ways the realisation of a simplicial space, and hence a Γ-space, can be defined such that it is well-behaved. Which definition Segal employs, and how it interacts with the theory he develops, is not always crystal clear. The author hopes their chosen definition of the realisation of Γ-spaces yields a clear treatment of the Algebraic K-theory spectrum of a symmetric monoidal category.

3.7 Spectra from Γ-Spaces

In this section we will discuss how to construct spectra from Γ-spaces. To do this we will associate to every Γ-space X a new Γ-space BX called the classifying space of X. We will then able to associate a sequence of spaces

$$TX(\mathbf{1}),\quad TBTX(\mathbf{1}),\quad TBTBTX(\mathbf{1}),\quad ...$$

to every reduced Γ-space X. We will see that the spaces in this sequence can be related by closed inclusions $(TB)^nTX(\mathbf{1}) \to \Omega(TB)^{n+1}TX(\mathbf{1})$. Hence, the sequence will be able to be equipped with the structure of a spectrum $\mathbf{B}X$. The process of constructing spectra from Γ-spaces will of course be functorial.

To define the classifying space of a Γ-space we must first make the following definition (following Segal):

Definition 3.7.1 [**Segal** [27]]**.** Let X be a Γ-space. Fixing an S in Γ^{op}, let $X(S\times -) : \Gamma^{op} \to \mathbf{CGWH}$ be the Γ-space defined on objects by

$$X(S\times -)(T) = X(S\times T).$$

If $f : T \to Z$ is a map in Γ^{op} then

$$X(S\times -)(f) = X(Id_S \times f).$$

Remark 3.7.2. Note that if $f : T \to Z$ is a map in Γ^{op}, there is a map

$$X(f\times -) : X(T\times -) \to X(Z\times -)$$

of Γ-spaces.

Definition 3.7.3 [**Segal** [27]]**.** Let X be a Γ-space. Let $BX : \Gamma^{op} \to$ **CGWH** be the functor such that

$$BX(S) = |X(S\times -)|,$$

where $|X(S\times -)|$ is the realisation of the Γ-space $X(S\times -)$ as in Definition 3.6.11. If $f : T \to Z$ is a map in Γ^{op},

$$BX(f) = |X(f\times -)|.$$

The functor BX is called the *classifying space* of X.

Remark 3.7.4. Note that $BX(\mathbf{1})$ is the realisation of the Γ-space X; hence $BX(\mathbf{1})$ is the realisation of the simplicial space associated to the Γ-space TX.

Remark 3.7.5. For every Γ-space X the functor BX is a Γ-space. Why? Observe that $X(\mathbf{0} \times -)$ is the constant Γ-space $X(\mathbf{0})$. Thus, as $X(\mathbf{0})$ is contractible, by statement 2 in Proposition 3.5.7, we have the realisation of the constant Γ-space $X(\mathbf{0})$ is contractible. Hence, $BX(\mathbf{0})$ is contractible. Furthermore, for each $n \geq 1$ there is a homotopy equivalence $X(\mathbf{n}) \simeq X(\mathbf{1})^n$ induced by the maps $p_i : \mathbf{n} \to \mathbf{1}$ in Γ^{op}. Thus, by statements 2 and 3 in Proposition 3.5.7, we have

$$BX(\mathbf{n}) = |X(\mathbf{n} \times -)| \simeq |X(-)^n| = |X(-)|^n = BX(\mathbf{1})^n.$$

That is, for each $n \geq 1$ there is a homotopy equivalence $BX(\mathbf{n}) \simeq BX(\mathbf{1})^n$ induced by the maps $p_i : \mathbf{n} \to \mathbf{1}$ in Γ^{op}. Hence, BX is a Γ-space.

Remark 3.7.6. It is clear from the Definition 3.7.3 that if X is reduced, then so is BX. Furthermore, as BX is a Γ-space, the classifying space B^2X of BX is defined, and so on.

The observations made in Remarks 3.7.4-3.7.6 allow us to construct a spectrum from a Γ-space.

Construction 3.7.7. Let X be a reduced Γ-space. Note that each space $X(\mathbf{n})$ has a canonical basepoint given by the image of the map $X(\mathbf{0}) \to X(\mathbf{n})$. Now, to associate a spectrum to X, Segal in [27] considers the sequence of spaces

$$X(\mathbf{1}), \quad BX(\mathbf{1}), \quad B^2X(\mathbf{1}), \quad ...$$

and shows that this sequence forms a spectrum. However, it is a little difficult to check that Segal's choice of maps $B^nX(\mathbf{1}) \to \Omega B^{n+1}X(\mathbf{1})$ are indeed closed inclusions for each $n \geq 0$. To address this issue we will consider the sequence of spaces

$$TX(\mathbf{1}), \quad TBTX(\mathbf{1}), \quad TBTBTX(\mathbf{1}), \quad ...$$

That is, we will modify Segal's sequence by replacing each Γ-space B^nX with the good Γ-space $(TB)^nTX$ via the functor $T(-)$. To see that this sequence of spaces can be equipped with the structure of a spectrum $\mathbf{B}X$ we need to show that there exists a family of closed inclusions

$$(TB)^nTX(\mathbf{1}) \to \Omega(TB)^{n+1}TX(\mathbf{1}).$$

Note that this is the claim that there is a closed inclusion

$$TX_1 \to \Omega T\big|T^2X\big|$$

when $n = 0$, that there is a closed inclusion

$$TBTX_1 \to \Omega T\big|T^2BTX\big|,$$

when $n = 1$, and so on. Hence, it suffices to construct a closed inclusion $TX_1 \to \Omega T|T^2X|$ when X is a reduced simplicial space. Thus, by Remark 3.3.3, it is sufficient to construct a closed inclusion

$$\Sigma TX_1 \to T\big|T^2X\big|.$$

But we have already constructed such a closed inclusion in Corollary 3.5.13.

Construction 3.7.7 can be extended to define a functor.

Proposition 3.7.8 [**Segal** [27]]**.** *There is a functor*

$$\mathbf{B} : \Gamma\mathbf{CGWH}_0 \to \mathbf{Spec}$$

mapping reduced Γ*-spaces to spectra.*

Proof. We need to show that given a map $X \to Y$ of reduced Γ-spaces, then there is an associated map $\mathbf{B}X \to \mathbf{B}Y$ of spectra. It is hence sufficient to check that if $X \to Y$ is a map of reduced simplicial spaces, then there is a commutative diagram

$$\begin{array}{ccccccc} \Sigma TX_1 & \longrightarrow & |TX| & \longrightarrow & |T^2X| & \longrightarrow & T|T^2X| \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma TY_1 & \longrightarrow & |TY| & \longrightarrow & |T^2Y| & \longrightarrow & T|T^2Y|, \end{array} \tag{3.26}$$

where the horizontal maps are the maps constructed in the proof of Corollary 3.5.13. First note that, by Construction 1.4.10, there is a commutative diagram

$$\begin{array}{ccccccc} |TX|_0 & \longrightarrow & |TX|_1 & \longrightarrow & \cdots & \longrightarrow & |TX| \\ \downarrow & & \downarrow & & & & \downarrow \\ |TY|_0 & \longrightarrow & |TY|_1 & \longrightarrow & \cdots & \longrightarrow & |TY| \end{array} \tag{3.27}$$

and hence, as $\Sigma TX_1 = |TX|_1$ when X is reduced, an induced commutative diagram

$$\begin{array}{ccc} \Sigma TX_1 & \longrightarrow & |TX| \\ \downarrow & & \downarrow \\ \Sigma TY_1 & \longrightarrow & |TY|. \end{array} \tag{3.28}$$

The map $X \to Y$ then also induces a commutative diagram

$$\begin{array}{ccc} |TX| & \longrightarrow & |T^2X| \\ \downarrow & & \downarrow \\ |TY| & \longrightarrow & |T^2Y|, \end{array} \tag{3.29}$$

as the comultiplication map $T \to T^2$ is natural. Finally, there is a commutative diagram

$$\begin{array}{ccc} |T^2X| & \longrightarrow & T|T^2X| \\ \downarrow & & \downarrow \\ |T^2Y| & \longrightarrow & T|T^2Y| \end{array} \tag{3.30}$$

as the maps $|T^2X| \to T|T^2X|$ and $|T^2Y| \to T|T^2Y|$ are induced by the unit of the $|-| \dashv S(-)$ adjunction, and are hence natural. Thus, pasting diagrams (3.28), (3.29) and (3.30) together gives the required commutative diagram. □

Thus, we have accomplished what we set out to do: To every Γ-space X we have associated a spectrum $\mathbf{B}X$. The spectrum $\mathbf{B}X$ turns out to have some nice properties, as stated below in Proposition 3.7.9.

Proposition 3.7.9 [**Segal** [27]]**.** *The spectrum $\mathbf{B}X$ associated to a reduced Γ-space X is connective and is an Ω-spectrum after the 0-th term. Furthermore, if the H-space $X(\mathbf{1})$ has a homotopy inverse, then $\mathbf{B}X$ is an Ω-spectrum.*

Proof. This is a restatement of Propositions 1.4 and 3.4(a) in [27]. □

To conclude this section, let us observe that the functor $\mathbf{B}$ maps Γ-spaces that are levelwise weakly homotopy equivalent to stably equivalent spectra.

Proposition 3.7.10. *Let $X \to Y$ be a map of reduced Γ-spaces such that for each $n \geq 0$ the map $X(\mathbf{n}) \to Y(\mathbf{n})$ is a weak homotopy equivalence. The induced map of spectra $\mathbf{B}X \to \mathbf{B}Y$ is a stable weak equivalence.*

Proof. Note that it is sufficient to show that if $X \to Y$ is a map of reduced simplicial spaces that is a levelwise weak equivalence then $TX_1 \to TY_1$ and $T|T^2X| \to T|T^2Y|$ are weak homotopy equivalences. But this follows as $S(-)$ and $|-|$ preserve weak equivalences and by statement 2 in Proposition 3.5.7. □

Chapter 4

Algebraic K-theory: The Algebraic K-theory Spectrum

In this chapter we will appeal to the theory discussed in Chapter 3 to define the Algebraic K-theory spectrum $\mathbf{B}\mathcal{K}C$ of a small symmetric monoidal category C.

To define the Algebraic K-theory spectrum of a symmetric monoidal category we first need to review symmetric monoidal categories, and discuss how to associate a Γ-space to every small symmetric monoidal category. We will do this in Section 4.1. We will then define the Algebraic K-theory spectrum $\mathbf{B}\mathcal{K}C$ and Algebraic K-theory space of a small symmetric monoidal category C in Section 4.2. We will also discuss how Algebraic K-theory spaces can help identify the stable homotopy types of the Algebraic K-theory spectra of some symmetric monoidal categories. In Section 4.3 we will discuss how a category can be equipped with a symmetric monoidal structure if it has coproducts and an initial object, and how the core respects the structure of a symmetric monoidal category. Finally, in Section 4.4, we will discuss how good simplicial monoids and their group completions can be used to help us identify a category's Algebraic K-theory space.

4.1 Symmetric Monoidal Categories to Γ-Spaces

In Chapter 3 we proved that to every Γ-space X we can functorially associate a spectrum $\mathbf{B}X$. In this section we shall see that special types of categories called symmetric monoidal categories come equipped with enough structure such that a Γ-category, and hence a Γ-space, can be functorially associated to them.

Let us begin by reviewing the definitions of symmetric monoidal category and symmetric monoidal functor.

Definition 4.1.1. Let C be a category. The category C is said to be *monoidal* if there exists a functor $\otimes : C \times C \to C$, an object e, and three natural isomorphisms

$$\alpha : \otimes \circ (Id_C \times \otimes) \to \otimes \circ (\otimes \times Id_C),$$

$$\beta : e \otimes - \to Id_C,$$

and

$$\gamma : - \otimes e \to Id_C$$

such that:

1. $\beta_e = \gamma_e$; and
2. the diagrams

$$\begin{array}{ccccc} a \otimes (b \otimes (c \otimes d)) & \xrightarrow{\alpha} & (a \otimes b) \otimes (c \otimes d) & \xrightarrow{\alpha} & ((a \otimes b) \otimes c) \otimes d \\ \downarrow{\scriptstyle Id_C \otimes \alpha} & & & & \uparrow{\scriptstyle \alpha \otimes Id_C} \\ a \otimes ((b \otimes c) \otimes d) & & \xrightarrow{\qquad\qquad \alpha \qquad\qquad} & & (a \otimes (b \otimes c)) \otimes d \end{array} \tag{4.1}$$

and

$$\begin{array}{ccc} a \otimes (e \otimes c) & \xrightarrow{\alpha} & (a \otimes e) \otimes c \\ & {\scriptstyle Id_C \otimes \beta} \searrow \quad \swarrow {\scriptstyle \gamma \otimes Id_C} & \\ & a \otimes c & \end{array} \tag{4.2}$$

commute.

The functor $\otimes$ is called the *tensor product* of the monoidal category C, the object e is called the *identity*, and the natural isomorphisms α, β, and γ are called the *associator*, *left unitor*, and *right unitor*, respectively. If C is a monoidal category we write $< C, \otimes, e, \alpha, \beta, \gamma >$, or just C if $\otimes$, e, α, β and γ are understood.

Definition 4.1.2. Let $< C, \otimes, e, \alpha, \beta, \gamma >$ be a monoidal category. The category C is said to be *symmetric monoidal* if there exists a family of isomorphisms

$$B_{a,b} : a \otimes b \to b \otimes a$$

in C, natural in a and b, such that:

1. For all objects $a, b \in C$,

$$B_{a,b} \circ B_{b,a} = 1_{b \otimes a} \quad \text{and} \quad \beta_b \circ B_{b,e} = \gamma_b;$$

   and

2. For all objects $a, b, c \in C$, the diagram

$$\begin{array}{ccccc} a \otimes (b \otimes c) & \xrightarrow{\alpha} & (a \otimes b) \otimes c & \xrightarrow{B} & c \otimes (a \otimes b) \\ \downarrow{\scriptstyle B} & & & & \downarrow{\scriptstyle \alpha} \\ (b \otimes c) \otimes a & \xrightarrow{\alpha^{-1}} & b \otimes (c \otimes a) & \xrightarrow{B} & (c \otimes a) \otimes b \end{array} \tag{4.3}$$

   commutes.

The family of maps $(B_{a,b})_{(a,b) \in C \times C}$ is called a *braiding* on the monoidal category C. If C is a symmetric monoidal category we write $< C, \otimes, e, \alpha, \beta, \gamma, B >$, or just C if $\otimes$, e, α, β, γ and B are understood.

Example 4.1.3. Let **FinSet** be the category of finite sets. Equipping **FinSet** with the tensor product

$$\mathbf{FinSet} \times \mathbf{FinSet} \to \mathbf{FinSet}, \quad (S, T) \mapsto S \sqcup T$$

equips **FinSet** with the structure of a symmetric monoidal category, where the unit of **FinSet** is the empty set $\emptyset$. The associator, braiding, left unitor and right unitor of **FinSet** are induced by the canonical maps arising from the universal property of the coproduct. This construction is an instance of a more general method of constructing symmetric monoidal categories that we will discuss in Section 4.3.

Example 4.1.4. Let $\mathbf{Vect}_{\mathbb{F}}$ be the category of vector spaces over a fixed field $\mathbb{F}$. The functor

$$\otimes : \mathbf{Vect}_{\mathbb{F}} \times \mathbf{Vect}_{\mathbb{F}}, \quad (V, W) \mapsto V \otimes W,$$

mapping the vector spaces V and W to their tensor product $V \otimes W$ equips $\mathbf{Vect}_{\mathbb{F}}$ with the structure of a symmetric monoidal category. The unit of $\mathbf{Vect}_{\mathbb{F}}$ is the vector space $\mathbb{F}$ over $\mathbb{F}$. The the associator, braiding, left unitor and right unitor of $\mathbf{Vect}_{\mathbb{F}}$ are induced by the well-known isomorphisms

$$V \otimes (W \otimes Z) \cong (V \otimes W) \otimes Z, \quad V \otimes W \cong W \otimes V, \quad V \otimes 0 \cong 0 \otimes V \cong V.$$

Definition 4.1.5. Let $< C, \otimes, e, \alpha, \beta, \gamma, B >$ and $< C', \otimes', e', \alpha', \beta', \gamma', B' >$ be symmetric monoidal categories. A functor $F : C \to C'$ is called a *symmetric monoidal functor* if there exists a natural family of maps

$$F_2(a, b) : F(a \otimes b) \to F(a) \otimes' F(b)$$

in C', a map

$$F_0 : F(e) \to e'$$

in C', and the diagrams

$$\begin{array}{ccc} F(a \otimes (b \otimes c)) & \xrightarrow{F_2} & F(a) \otimes' F(b \otimes c) \\ \downarrow{\scriptstyle F(\alpha)} & & \downarrow{\scriptstyle Id \otimes' F_2} \\ F((a \otimes b) \otimes c) & & F(a) \otimes' (F(b) \otimes' F(c)) \\ \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle \alpha'} \\ F(a \otimes b) \otimes' F(c) & \xrightarrow{F_2 \otimes' Id} & (F(a) \otimes' F(b)) \otimes' F(c), \end{array} \tag{4.4}$$

$$\begin{array}{ccc} F(b \otimes e) & \xrightarrow{F_2} & F(b) \otimes' F(e) \\ \downarrow{\scriptstyle F(\gamma)} & & \downarrow{\scriptstyle Id \otimes' F_0} \\ F(b) & \xleftarrow[\gamma']{} & F(b) \otimes' e', \end{array} \tag{4.5}$$

$$\begin{array}{ccc} F(e \otimes b) & \xrightarrow{F_2} & F(e) \otimes' F(b) \\ \downarrow{\scriptstyle F(\beta)} & & \downarrow{\scriptstyle F_0 \otimes' Id} \\ F(b) & \xleftarrow[\beta']{} & e' \otimes' F(b), \end{array} \tag{4.6}$$

and

$$\begin{array}{ccc} F(a \otimes b) & \xrightarrow{F(B)} & F(b \otimes a) \\ \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_2} \\ F(a) \otimes' F(b) & \xrightarrow{B'} & F(b) \otimes' F(a) \end{array} \tag{4.7}$$

commute. If F_2 and F_0 are natural isomorphisms then F is called a *strong* monoidal functor. If the map F_0 is the identity map then F is called *strictly unital*. If the maps F_2 and F_0 are identity maps then F is called *strict*. If F is a strictly unital strong monoidal functor that is also an equivalence of categories then we say F is a *monoidal equivalence* and C and C' are *monoidally equivalent*.

Definition 4.1.6. Let **SymMCat** be the category whose objects are small symmetric monoidal categories and maps are strictly unital symmetric monoidal functors.

Remark 4.1.7. For ease of exposition, we will henceforth refer to strictly unital symmetric monoidal functors as monoidal functors.

Essentially, a symmetric monoidal category is a category with a multiplication which obeys associativity, commutativity and identity laws, up to isomorphism. A monoidal functor is then a functor between symmetric monoidal categories which respects the respective tensor products, up to isomorphism. That is, symmetric monoidal categories can be viewed as a 'categorification' of abelian monoids, and monoidal functors as a categorification of homomorphisms of abelian monoids.

When working with **SymMCat** a natural question one might ask is: '*How is the extra structure of a symmetric monoidal category reflected in its classifying space?*'. This question is answered in the Proposition 4.1.8.

Proposition 4.1.8. *Let C be a small symmetric monoidal category. The classifying space BC of C is a commutative H-space. Furthermore, if $C \to C'$ is a monoidal equivalence then the induced map $BC \to BC'$ is an equivalence of H-spaces.*

Proof. Recall that if $F, G : C \to C'$ are functors, and $\alpha : F \to G$ is a natural transformation, then there is a homotopy $H : I \times BC \to BC'$ from BF to BG. Thus, if C is a symmetric monoidal category, the diagrams

$$\begin{array}{ccc} BC \times BC \times BC & \xrightarrow{B\otimes \times Id} & BC \times BC \\ \downarrow{\scriptstyle Id \times B\otimes} & & \downarrow{\scriptstyle B\otimes} \\ BC \times BC & \xrightarrow{B\otimes} & BC, \end{array} \tag{4.8}$$

$$\begin{array}{ccccc} BC & \xrightarrow{Id \times Be} & BC \times BC & \xleftarrow{Be \times Id} & BC \\ & {\scriptstyle Id}\searrow & \downarrow{\scriptstyle B\otimes} & \swarrow{\scriptstyle Id} & \\ & & BC, & & \end{array} \tag{4.9}$$

and

$$\begin{array}{ccc} BC \times BC & \xrightarrow{\mu} & BC \times BC \\ & {\scriptstyle B\otimes}\searrow & \downarrow{\scriptstyle B\otimes} \\ & & BC \end{array} \tag{4.10}$$

commute up to homotopy. Thus, BC is an H-space. Now suppose that there is a monoidal equivalence $F : C \to C'$. Then the induced map $BF : BC \to BC'$ is a homotopy equivalence. Furthermore, $BF([e]) = [e']$, and the diagram

$$\begin{array}{ccc} BC \times BC & \xrightarrow{BF \times BF} & BC' \times BC' \\ \downarrow{\scriptstyle B\otimes'} & & \downarrow{\scriptstyle B\otimes} \\ BC & \xrightarrow{BF} & BC' \end{array} \tag{4.11}$$

commutes up to homotopy. Thus, BF is an equivalence of H-spaces. □

Remark 4.1.9. Proposition 4.1.8 is an instance of Proposition 3.6.6, as we will see in Proposition 4.1.14.

At the beginning of this section we claimed that to every symmetric monoidal category C one can functorially associate a Γ-category. That is, we claimed there is a functor

$$K : \mathbf{SymMCat} \to \Gamma\mathbf{Cat}.$$

We will now construct this functor, following Mandell in [18].

Definition 4.1.10 [Construction 3.1, Mandell [18]]. Let $< C, \otimes, e, \alpha, \beta, \gamma, B >$ be a small symmetric monoidal category. Let $KC(\mathbf{0})$ be the terminal category. For each $\mathbf{n}$, where $n > 0$, let $KC(\mathbf{n})$ be the category whose objects are collections $(x_I, f_{I,J})$, where I is a subset of $\underline{n} = \{1, ..., n\}$, x_I is an object of C, and for each pair of disjoint subsets I and J of $\underline{n}$

$$f_{I,J} : x_{I \cup J} \to x_I \otimes x_J$$

is a map in C. We also require that each collection $(x_I, f_{I,J})$ satisfies the following properties:

1. $x_\emptyset = e$, and $f_{\emptyset,J} = \beta_{x_J}^{-1}$ and $f_{J,\emptyset} = \gamma_{x_J}^{-1}$.
2. $f_{I,J} = B \circ f_{J,I}$.
3. If I_1, I_2, and I_3 are mutually disjoint then the diagram

$$\begin{array}{ccc} x_{I_1 \cup I_2 \cup I_3} & \xrightarrow{f_{I_1, I_2 \cup I_3}} & x_{I_1} \otimes x_{I_2 \cup I_3} \\ \downarrow{\scriptstyle f_{I_1 \cup I_2, I_3}} & & \\ x_{I_1 \cup I_2} \otimes x_{I_3} & & \downarrow{\scriptstyle Id \otimes f_{I_2, I_3}} \\ \downarrow{\scriptstyle f_{I_1, I_2} \otimes Id} & & \\ (x_{I_1} \otimes x_{I_2}) \otimes x_{I_3} & \xrightarrow{\alpha^{-1}} & x_{I_1} \otimes (x_{I_2} \otimes x_{I_3}) \end{array} \tag{4.12}$$

commutes.

A map $(h_I) : (x_I, f_{I,J}) \to (x'_I, f'_{I,J})$ in $KC(\mathbf{n})$ is a collection of maps $h_I : x_I \to x'_I$ in C for all I such that $h_\emptyset = Id$, and the diagram

$$\begin{array}{ccc} x_{I\cup J} & \xrightarrow{h_{I\cup J}} & x'_{I\cup J} \\ \Big\downarrow{\scriptstyle f_{I,J}} & & \Big\downarrow{\scriptstyle f'_{I,J}} \\ x_I \otimes x_J & \xrightarrow{h_I \otimes h_J} & x'_I \otimes x'_J \end{array} \tag{4.13}$$

commutes for all disjoint I and J. If $\phi : \mathbf{m} \to \mathbf{n}$ is a map in Γ^{op} then define the functor

$$K\phi : KC(\mathbf{m}) \to KC(\mathbf{n})$$

which maps objects $(x_I, f_{I,J})$ in $KC(\mathbf{m})$ to $(y_I, g_{I,J})$ in $KC(\mathbf{n})$, where

$$y_I = x_{\phi^{-1}(I)}$$

and

$$g_{I,J} = f_{\phi^{-1}(I),\phi^{-1}(J)}.$$

The functor $K\phi$ maps the maps (h_I) in $KC(\mathbf{m})$ to (h'_I) in $KC(\mathbf{n})$, where

$$h'_I = h_{\phi^{-1}(I)}.$$

Remark 4.1.11. The assignment

$$KC : \Gamma^{op} \to \mathbf{Cat}$$

which maps objects $\mathbf{n}$ in Γ^{op} to $KC(\mathbf{n})$, and maps $\phi : \mathbf{n} \to \mathbf{m}$ to $KC(\phi) : KC(\mathbf{n}) \to KC(\mathbf{m})$, is a functor. In fact KC is a Γ-category (see Section 3 in [18] for details).

Proposition 4.1.12. *Definition* 4.1.10 *can be extended to define a functor*

$$K : \mathbf{SymMCat} \to \Gamma\mathbf{Cat}.$$

Proof. Given a strictly unital symmetric monoidal functor $F : C \to C'$, for each $n \geq 0$ define the functor

$$KF_{\mathbf{n}} : KC(\mathbf{n}) \to KC'(\mathbf{n})$$

which maps the objects $(x_I, f_{I,J})$ in $KC(\mathbf{n})$ to $(F(x_I), F_2 \circ F(f_{I,J}))$ in $KC'(\mathbf{n})$, and the maps (h_I) to $(F(h_I))$. The family of functors $(KF_{\mathbf{n}})_{n\geq 0}$ is natural in $\mathbf{n}$. □

Thus, to every symmetric monoidal category C we can associate a Γ-category KC, and hence associate a Γ-space $\mathcal{K}C$ (Construction 3.6.5). Further, it turns out that the induced functor $\mathcal{K}$ has some nice properties.

Definition 4.1.13. Let $\mathcal{K} : \mathbf{SymMCat} \to [\Gamma^{op}, \mathbf{CGWH}]$ be the functor induced by the post-composing the functor K with the nerve N and geometric realisation $|-|$ functors.

Proposition 4.1.14 [**Mandell** [18]]**.** *Let C be a small symmetric monoidal category. The following statements are true:*

1. *The functor $\mathcal{K}C$ is a good reduced Γ-space.*
2. *The space $\mathcal{K}C(\mathbf{1})$ is the classifying space of the category C.*
3. *If $C \overset{\simeq}{\to} C'$ is a monoidal equivalence then for each $n \geq 0$ the induced map $\mathcal{K}C(\mathbf{n}) \to \mathcal{K}C'(\mathbf{n})$ is an homotopy equivalence.*

Proof. Statement 1 follows from Proposition 3.5.4 and Construction 3.6.5. Statement 2 and 3 follow from arguments made in Section 3 in [18]. □

4.2 The Algebraic K-theory Spectrum of a Symmetric Monoidal Category

In Sections 3.1–4.1 we proved two key results. The first key result we proved is that to every Γ-space X there is an associated connective spectrum $\mathbf{B}X$ which is an Ω-spectrum above the 0-term. The second result we proved is that to every small symmetric monoidal category C there is an associated Γ-space $\mathcal{K}C$, where $\mathcal{K}C(\mathbf{1})$ is the classifying space of C. Combining these two results we can functorially associate a spectrum $\mathbf{B}\mathcal{K}C$ to every symmetric monoidal category C which enjoys some nice properties. In this section, we will define the spectrum $\mathbf{B}\mathcal{K}C$ to be the Algebraic K-theory spectrum of C, and discuss the Algebraic K-theory space of C.

Let us begin by combining Proposition 3.7.8 and Proposition 4.1.14 to construct a functor which will send small symmetric monoidal categories to spectra. This functor will then be used to define the Algebraic K-theory spectrum of a small symmetric monoidal category C.

Construction 4.2.1. Consider the composite functor

$$\mathbf{SymMCat} \xrightarrow{\mathcal{K}} \Gamma\mathbf{CGWH}_0^* \xrightarrow{\mathbf{B}} \mathbf{Spec}_{\geq 0},$$

where $\Gamma\mathbf{CGWH}_0^*$ is the category of good reduced Γ-spaces. Observe that for every small symmetric monoidal category C the spectrum $\mathbf{B}\mathcal{K}C$ is an Ω-spectrum after the 0-th term (Proposition 3.7.9). Also observe that $\mathbf{B}\mathcal{K}C_0 = |S(BC)| \simeq BC$.

Definition 4.2.2. Let C be a small symmetric monoidal category. The *Algebraic K-theory spectrum* of C is the spectrum $\mathbf{B}\mathcal{K}C$.

As spectra are typically understood up to their stable homotopy type, for the remainder of this thesis when studying the Algebraic K-theory spectrum of a symmetric monoidal category C we will be aiming to identify its stable homotopy type. By Proposition 4.2.3, it will be sufficient to identify the stable homotopy type of the Algebraic K-theory spectrum of a monoidally equivalent symmetric monoidal category.

Proposition 4.2.3. *If $C \xrightarrow{\simeq} C'$ is a monoidal equivalence of small symmetric monoidal categories, then the induced map $\mathbf{B}\mathcal{K}C \to \mathbf{B}\mathcal{K}C'$ is a stable weak equivalence.*

Proof. This follows immediately from statement 3 in Proposition 4.1.14 and Proposition 3.7.10. □

Remark 4.2.4. By Proposition 4.2.3 we can also define the Algebraic K-theory spectra of essentially small symmetric monoidal categories, up to stable homotopy equivalence. That is, if C is an essentially small symmetric monoidal category such that there is a monoidal equivalence $C \xrightarrow{\simeq} C'$, where C' is a small symmetric monoidal category, then let the spectrum $\mathbf{B}\mathcal{K}C'$ be the Algebraic K-theory spectrum of C.

When attempting to identify the stable homotopy type of a Algebraic K-theory spectrum of a symmetric monoidal category of interest, it turns out it is sufficient to identify the homotopy type of its Algebraic K-theory space with an infinite loop space. Let us expand upon this now.

Definition 4.2.5. Let C be a small symmetric monoidal category. The *Algebraic K-theory space* of C is the space $\Omega\mathbf{B}\mathcal{K}C_1$.

Lemma 4.2.6. *Let $\mathbf{B}\mathcal{K}C$ be the Algebraic K-theory spectrum of a symmetric monoidal category C. There is a stably equivalent Ω-spectrum $\mathbf{E}C$ whose 0-th space is $\Omega\mathbf{B}\mathcal{K}C_1$.*

Proof. Let $\mathbf{E}C$ be the spectrum

$$\Omega\mathbf{B}\mathcal{K}C_1, \quad \mathbf{B}\mathcal{K}C_1, \quad \mathbf{B}\mathcal{K}C_2, \quad ...$$

with the obvious structure maps. The structure maps $\mathbf{E}C_n \to \Omega\mathbf{E}C_{n+1}$ are closed inclusions for each $n \geq 0$ as the maps $\mathbf{B}\mathcal{K}C_n \to \Omega\mathbf{B}\mathcal{K}C_{n+1}$ are closed inclusions. The spectrum $\mathbf{E}C$ is connective as $\mathbf{B}\mathcal{K}C$ is connective, and is an Ω-spectrum as $\mathbf{B}\mathcal{K}C$ is an Ω-spectrum above the 0-th term. There is an obvious map of spectra $\mathbf{B}\mathcal{K}C \to \mathbf{E}C$ which is the identity map $\mathbf{B}\mathcal{K}C_n \to \mathbf{B}\mathcal{K}C_n$ for all $n \geq 1$. Thus, by Lemma 3.3.13, the spectra $\mathbf{B}\mathcal{K}C$ and $\mathbf{E}C$ are stably equivalent. □

Construction 4.2.7. Let C be a small symmetric monoidal category and suppose there is a homotopy equivalence $Q(\Sigma^\infty X)_0 \simeq \Omega\mathbf{B}\mathcal{K}C_1$, where X is some pointed CGWH space. By Proposition 3.3.20, there is a stable weak equivalence of spectra $\mathbf{E}C \to Q\mathbf{E}C$ which is a weak homotopy equivalence at the 0-th space $\mathbf{E}C_0 \to Q\mathbf{E}C_0$ (Corollary 3.3.16). Thus, there is a weak equivalence of spaces $Q(\Sigma^\infty X)_0 \to Q\mathbf{E}C_0$. Now, the map $X \to Q(\Sigma^\infty X)_0 \to \mathbf{E}C_0$ induces a map $\Sigma^\infty X \to \mathbf{E}C$ of spectra by the $\Sigma^\infty \dashv (-)_0$ adjunction (Construction 3.3.5). Thus, there is a map $Q(\Sigma^\infty X) \to Q\mathbf{E}C$ of spectra whose 0-th map is the weak homotopy equivalence $Q(\Sigma^\infty X)_0 \to Q\mathbf{E}C_0$. Hence, we have a map of spectra $Q(\Sigma^\infty X) \to Q\mathbf{E}C$ which is a stable weak equivalence (Corollary 3.3.16). Thus, we have the zigzag of stable weak equivalences

$$\Sigma^\infty X \to Q(\Sigma^\infty X) \to Q\mathbf{E}C \leftarrow \mathbf{E}C \leftarrow \mathbf{B}\mathcal{K}C,$$

and so $\mathbf{B}\mathcal{K}C$ has the same stable homotopy type as $\Sigma^\infty X$.

4.3 The Symmetric Monoidal Structure Induced by Coproducts

In this section we will discuss how a category C with finite coproducts and an initial object can be equipped with the structure of a symmetric monoidal category. We also briefly discuss why one might restrict to a category's core when studying its Algebraic K-theory spectrum.

Let us begin by showing that a category C with finite coproducts and an initial object can be equipped with the structure of a symmetric monoidal category.

Proposition 4.3.1. *Let C be a category with finite coproducts $\sqcup$ and an initial object e. There is a symmetric monoidal structure on C induced by $\sqcup$ and e.*

Proof. The category C can be equipped with a symmetric monoidal structure $< C, \sqcup, e, \alpha, \beta, \gamma, B >$, where the tensor product on C is induced by the mapping

$$C \times C \to C, \quad (a, b) \mapsto a \sqcup b.$$

The associator, left and right unitors, and braiding of C are then induced by the canonical maps arising from the universal property of the coproduct. For example, the associator of C is induced by the canonical maps

$$a \sqcup (b \sqcup c) \to (a \sqcup b) \sqcup c$$

for each $a, b, c \in C$. The universal property of the coproduct forces all needed diagrams in Definitions 4.1.1 and 4.1.2 to commute, and forces all needed identities to be satisfied. □

Corollary 4.3.2. *Let C be a category with finite coproducts $\sqcup$ and an initial object e. If D is a category then there is an induced symmetric monoidal structure on the category of functors $[D, C]$ mapping D into C.*

Proof. As (co)limits are computed in functor categories pointwise, the category $[D, C]$ has finite coproducts and an initial object. Hence, by Proposition 4.3.1, there is a symmetric monoidal structure on $[D, C]$. □

Recall that if $F : C \to C'$ is an equivalence of categories, then F commutes with all limits and colimits (this is made precise in Theorem 4.3.3). Hence, one would reasonably expect that if C and C' are symmetric monoidal categories whose symmetric monoidal structures are induced by coproducts and initial objects, then F would be a monoidal equivalence. We prove that this is indeed the case in Corollary 4.3.4.

Theorem 4.3.3. *Let $F : C \to C'$ be an equivalence of categories. The functor F preserves all colimits. That is,*

$$F\left(\varinjlim_I D\right) \cong \varinjlim_I F \circ D,$$

where $D : I \to C$ is a diagram in C.

Proof. The functor $F : C \to C'$ has a pseudo-inverse $G : C' \to C$, and hence has a right adjoint. Thus, F preserves colimits. $\square$

Corollary 4.3.4. *Let C and D be categories with finite coproducts $\sqcup$ and $\coprod$ and initial objects e and e', respectively. If $F : C \to D$ is an equivalence of categories such that $F(e) = e'$, then it is also a monoidal equivalence with respect to the induced symmetric monoidal structures on C and D.*

Proof. There is natural family of isomorphisms

$$F(a) \coprod F(b) \to F(a \sqcup b)$$

in D as in Theorem 4.3.3. Diagrams (4.4)–(4.7) then commute by the universal property of the coproduct. $\square$

Despite coproducts and initial objects providing categories with symmetric monoidal structures, it turns out the Algebraic K-theory spectra of such categories are boring to study. Why? Observe that if a category C has an initial object e then there is a natural bijection

$$C(e, c) \cong \mathcal{T}(\bullet, \bullet),$$

where $\mathcal{T}$ is the terminal category. That is, there is an adjunction $\mathcal{T} \rightleftarrows C$. Thus, by Corollary 1.3.21, the classifying space of C is homotopy equivalent to a point. Hence, so is $\mathbf{B}\mathcal{K}C_0$. Thus, $\mathbf{B}\mathcal{K}C_0$ has a homotopy inverse by Corollary 3.2.4, and so the spectrum $\mathbf{B}\mathcal{K}C$ is an Ω-spectrum (Proposition 3.7.9) whose 0-th space is homotopy equivalent to a point. That is, the Algebraic K-theory spectrum $\mathbf{B}\mathcal{K}C$ of C has the stable homotopy type of the spectra consisting of single points only. So, to ensure the Algebraic K-theory spectrum of a symmetric monoidal category C whose structure has been induced by coproducts and an initial object is interesting to study, we need a way to remove some of the maps from the collections $C(e, c)$, without losing the structure of the symmetric monoidal category on C. By Remark 4.3.5, his can be achieved by restricting to the core of the category.

Remark 4.3.5. Observe that if C is a symmetric monoidal category then its core $C^{\simeq}$ (recall Remark 2.4.3) is too. Furthermore, if $F : C \to C'$ is a monoidal equivalence then so is its restriction to $C^{\simeq}$.

4.4 Permutative Categories and Good Simplicial Monoids

Permutative categories are symmetric monoidal categories where the unitors and associator are identity natural transformations. Permutative categories will be useful for us as their Algebraic K-theory spaces, in suitably nice cases, have the same homotopy type as the realisations of the group completions of their nerves. We will discuss this idea in this section.

We will begin this section by defining permutative categories, and showing that their Algebraic K-theory spaces are homotopy equivalent to a group completion of their classifying spaces.

Definition 4.4.1. If C is a symmetric monoidal category such that the associator, left unitor, and right unitor are identity natural transformations, then C is called a *permutative category.* Let **PermCat** be the category whose objects are small permutative categories and maps are strict symmetric monoidal functors.

Remark 4.4.2. Note that the nerve NP of a permutative category P is a simplicial monoid, and its classifying space BP is a topological monoid (recall Remark 1.4.6).

Lemma 4.4.3. *Let X be a reduced Γ-space such that $X(\mathbf{1})$ is a topological monoid. There is a map of simplicial spaces $X \to NX(\mathbf{1})$ from the associated simplicial space of X into the nerve $NX(\mathbf{1})$ of $X(\mathbf{1})$ that is a levelwise homotopy equivalence.*

Proof. For each $n \geq 0$ the map

$$X(p_1) \times \cdots \times X(p_n) : X(\mathbf{n}) \to X(\mathbf{1})^n$$

is a homotopy equivalence, where $X(p_i) : X(\mathbf{n}) \to X(\mathbf{1})$ is induced by the map

$$p_i : \mathbf{n} \to \mathbf{1}, \quad p_i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

in Γ^{op}, as X is a Γ-space. This family of maps commutes with the face and degeneracy maps of $NX(\mathbf{1})$ and the simplicial space associated to X. $\square$

Proposition 4.4.4. *Let P be a permutative category. There is a homotopy equivalence*

$$\Omega BBP \simeq \Omega \mathbf{B}\mathcal{K}P_1.$$

Proof. Recall that $\mathbf{B}\mathcal{K}P_1$ is the space $T|T^2\mathcal{K}P|$ (Construction 3.7.7). There is hence a weak homotopy equivalence $\mathbf{B}\mathcal{K}P_1 \to |T^2\mathcal{K}P|$ between $\mathbf{B}\mathcal{K}P_1$ the realisation of the simplicial space associated to the Γ-space $T^2\mathcal{K}P$ (statement 1 in Proposition 1.6.8). But this weak homotopy equivalence is a homotopy equivalence by Whitehead's theorem and statement 1 in Proposition 3.5.7 . Now, as $\mathcal{K}P$ is a reduced Γ-space and $\mathcal{K}P(\mathbf{1}) = BP$ is a topological monoid, by Lemma 4.4.3, there is a map of simplicial spaces $\mathcal{K}P \to NBP$ that is a levelwise homotopy equivalence. Hence, by statement 4 in Proposition 3.5.7, there is a map

$$T\mathcal{K}P \to \mathcal{K}P$$

of simplicial spaces that is a levelwise weak equivalence. Thus, the composite map

$$T\mathcal{K}P \to \mathcal{K}P \to NBP$$

induces a homotopy equivalence $|T^2\mathcal{K}P| \to |TNBP|$, by statement 2 in Proposition 3.5.7. But then as NBP is good (Proposition 3.5.4), by statement 4 in Proposition 3.5.7, there is a homotopy equivalence $|TNBP| \to |NBP| = BBP$. Thus, there is a homotopy equivalence

$$\mathbf{B}\mathcal{K}P_1 \xrightarrow{\simeq} \left|T^2\mathcal{K}P\right| \xrightarrow{\simeq} |TNBP| \xrightarrow{\simeq} BBP,$$

and hence $\mathbf{B}\mathcal{K}P_1$ and BBP have the same homotopy type. Hence, so do $\Omega\mathbf{B}\mathcal{K}P_1$ and ΩBBP. □

Thus, the Algebraic K-theory space of a permutative category P has the homotopy type of a group completion of its classifying space (recall Theorem 3.1.6). So one could identify the Algebraic K-theory space of a permutative category P by identifying the homotopy type of ΩBBP. In the case where NP is a 'good' simplicial monoid, this identification can be made by group completing NP. Let us discuss this now.

Definition 4.4.5. Let M be a simplicial monoid. Let $\overline{M}$ be the simplicial group such that $\overline{M}_n$ is the group completion of M_n for each $n \geq 0$. The canonical map of simplicial monoids $M \to \overline{M}$ is called the *group completion* of M.

Definition 4.4.6. Let M be a simplicial monoid. Let $N(M)$ be the bisimplicial set whose set of (m, n)-bisimplices is $(M_m)^n$. The horizontal face and degeneracy maps of $N(M)$ are given by

$$d_i : NM_{m+1,n} \to NM_{m,n}, \quad d_i(x_1, ..., x_n) := (d_i(x_1), ..., d_i(x_n))$$

and

$$s_j : NM_{m,n} \to NM_{m+1,n}, \quad s_j(x_1, ..., x_n) := (s_j(x_1), ..., s_j(x_n)).$$

The vertical face and degeneracy maps of $N(M)$ are given by

$$d_i : NM_{m,n+1} \to NM_{m,n},$$

$$d_i(x_1, ..., x_{n+1}) := \begin{cases} (x_2, ..., x_{n+1}) & \text{if } i = 0 \\ (x_1, ..., x_{i+1}x_i, ..., x_{n+1}) & \text{if } 0 < i < n+1 \\ (x_1, ..., x_n) & \text{if } i = n+1 \end{cases}$$

and

$$s_j : NM_{m,n} \to NM_{m,n+1}, \quad s_j(x_1, ..., x_n) := (x_1, ..., e, x_j, x_{j+1}, ..., x_n).$$

The bisimplicial set NM is called the *nerve* of M. The simplicial set $dN(M)$, denoted BM, is called the *classifying space* of M.

Remark 4.4.7. The motivation behind calling the simplicial set BM the classifying space of M is clear when one recalls that

$$|dBM| \cong |[n] \mapsto |M|^n|,$$

as in Construction 1.4.17. That is, the realisation of BM corresponds to the classifying space of the topological monoid $|M|$.

Construction 4.4.8. Definition 4.4.6 can be extended to define a functor

$$B : \mathbf{sMon} \to \mathbf{sSet}.$$

Using the functoriality of the classifying space of a simplicial monoid we can define what it means for a simplicial monoid to be good.

Definition 4.4.9 [Quillen [25]]. Let M be a simplicial monoid. The simplicial monoid M is called *good* if the canonical map $M \to \overline{M}$ induces a weak equivalence $BM \to B\overline{M}$ of simplicial sets.

As mentioned above, if the nerve NP of a permutative category P is good $NP \to \overline{NP}$ provides us with a group completion of BP.

Corollary 4.4.10 [Theorem Q.4, Quillen [25]]. *Let P be a permutative category such that NP is good. Then $BP \to \left|\overline{NP}\right|$ is a group completion.*

Proof. This follows from Theorem Q.4 in [25]. □

Not all group completions of an H-space are equivalent. However, it turns out that $\left|\overline{NP}\right|$ has the same homotopy type as the group completion ΩBBP.

Corollary 4.4.11 [Quillen [25]]. *Let P be a permutative category such that NP is good. There is a homotopy equivalence $\left|\overline{NP}\right| \to \Omega BBP$.*

Proof. This follows from a remark Quillen makes on page 96 of [25]. □

Thus, the Algebraic K-theory space of a permutative category P is homotopy equivalent to $\left|\overline{NP}\right|$ when NP is good.

We will now conclude this section by stating a sufficient condition on simplicial monoids for them to be good.

Definition 4.4.12. Let M be a simplicial monoid. If M_n is a free monoid for each $n \geq 0$, then M is called *free.*

Proposition 4.4.13 [Quillen [25]]. *If M is a free simplicial monoid then M is good.*

Proof. This follows from Propositions Q.1 and Q.2 in [25]. □

Chapter 5

The Algebraic K-theory Spectrum of $\mathbf{FinCov}(X)^{\simeq}$

In this chapter we will identify the stable homotopy type of the Algebraic K-theory spectrum $\mathbf{B}\mathcal{K}\mathbf{FinCov}(X)^{\simeq}$ of $\mathbf{FinCov}(X)^{\simeq}$, when X is a connected simplicial set. This identification will serve as a generalisation of the Barratt-Priddy-Quillen Theorem.

To discuss the Algebraic K-theory spectrum $\mathbf{B}\mathcal{K}\mathbf{FinCov}(X)^{\simeq}$ of the category $\mathbf{FinCov}(X)^{\simeq}$, we must first equip it with the structure of a symmetric monoidal category. We will do this using Proposition 4.3.1 in Section 5.1. Then, to aid us in identifying the stable homotopy type of $\mathbf{B}\mathcal{K}\mathbf{FinCov}(X)^{\simeq}$, we will define a permutative category $S(C)$ called the free symmetric monoidal category on the category C in Section 5.2. Using some work by Barratt and Eccles (reviewed in Section 5.3), and some of the results discussed in Chapter 4, we will be able to identify the stable homotopy type of $\mathbf{B}\mathcal{K}S(C)$ in Section 5.4. Then, in Section 5.5, using Corollary 2.6.4, we will be able to relate $\mathbf{FinCov}(X)^{\simeq}$ to the free symmetric monoidal category $S(\mathcal{CC}_G)$ via a zigzag of monoidal equivalences, where $\mathcal{CC}_G$ is a groupoid is constructed using choices of representatives from certain conjugacy classes of subgroups of $\pi_1(X)$. We will then be able to identify the stable homotopy type of $\mathbf{B}\mathcal{K}\mathbf{FinCov}(X)^{\simeq}$.

This chapter will be the final chapter of this thesis. The work done in Chapters 1–4, and in this chapter, will culminate in Theorem 5.5.14. Theorem 5.5.14 will tell us that the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$ can be identified with the suspension spectrum on the disjoint union of the classifying spaces of groups constructed from certain conjugacy classes of $\pi_1(X)$ attached with a disjoint basepoint. We will demonstrate that Theorem 5.5.14 generalises the Barratt-Priddy-Quillen Theorem by recovering the

Barratt-Priddy-Quillen Theorem from it.

5.1 The Symmetric Monoidal Structure on $\mathbf{FinCov}(X)^{\simeq}$

To discuss the Algebraic K-theory spectrum of the category $\mathbf{FinCov}(X)^{\simeq}$, we must first equip it with the structure of a symmetric monoidal category. Recall that in Section 4.3 we showed that there are induced symmetric monoidal structures on categories which have binary coproducts and an initial object. Furthermore, we observed that passing to cores respects symmetric monoidal structures. In this section we will equip $\mathbf{FinCov}(X)^{\simeq}$ with a symmetric monoidal structure by showing that $\mathbf{FinCov}(X)$ has binary coproducts and an initial object.

Proposition 5.1.1. *The category* $\mathbf{FinCov}(X)$ *has binary coproducts.*

Proof. By Proposition 1.1.4, the category $\mathbf{sSet}$ has all coproducts. In particular, if Y and Z are simplicial sets then their coproduct $Y \sqcup Z$ is the simplicial set with n-simplices

$$(Y \sqcup Z)_n := Y_n \sqcup Z_n.$$

Hence the category $\mathbf{sSet}/X$ also has all coproducts, by general abstract nonsense. Thus, to prove the given statement, we will show that the coproducts $\sqcup$ in $\mathbf{sSet}/X$ restrict to define binary coproducts in the full subcategory $\mathbf{FinCov}(X)$.

We will first show that $\sqcup$ is the coproduct in $\mathbf{Cov}(X)$. As $\mathbf{Cov}(X)$ is a full subcategory of $\mathbf{sSet}/X$, it is sufficient to check that $\mathbf{Cov}(X)$ is closed under $\sqcup$. Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be simplicial covers of X, and suppose the diagram

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & Y_1 \sqcup Y_2 \\ \downarrow{\scriptstyle 0_n} & & \downarrow{\scriptstyle p_1 \sqcup p_2} \\ \Delta^n & \longrightarrow & X \end{array} \tag{5.1}$$

commutes. Then the image of the map $\Delta^0 \to Y_1 \sqcup Y_2$ is contained entirely within either Y_1 or Y_2. Without loss of generality, suppose that $\Delta^0 \to Y_1 \sqcup Y_2$ is contained within Y_1. There is then a unique map $\Delta^n \to Y_1$ making the diagram

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & Y_1 \\ {\scriptstyle 0_n}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p_1} \\ \Delta^n & \longrightarrow & X \end{array} \tag{5.2}$$

commute. The composite map $f : \Delta^n \to Y_1 \hookrightarrow Y_1 \sqcup Y_2$ then makes diagram (5.1) commute. Observe also that the map $f : \Delta^n \to Y_1 \sqcup Y_2$ is the unique map making diagram (5.1) commute. Why? Suppose there is another map $f' : \Delta^n \to Y_1 \sqcup Y_2$ making diagram (5.1) commute. Then the image of f' is also contained within Y_1, as $f'(0) \in Y_1$. Hence, the map f' also makes diagram (5.2) commute, and hence is equal to $f : \Delta^n \to Y_1 \sqcup Y_2$ by uniqueness. Thus, $p_1 \sqcup p_2 : Y_1 \sqcup Y_2 \to X$ has the unique right lifting property with respect to all initial vertex inclusions. An identical argument shows that $p_1 \sqcup p_2 : Y_1 \sqcup Y_2 \to X$ also has the unique right lifting property with respect to all final vertex inclusions. Thus, $p_1 \sqcup p_2 : Y_1 \sqcup Y_2 \to X$ is a simplicial covering of X. Hence, $\mathbf{Cov}(X)$ is closed under $\sqcup$.

To conclude that $\sqcup$ is the coproduct in $\mathbf{FinCov}(X)$, it is sufficient to show that $\mathbf{FinCov}(X)$ is closed under $\sqcup$. Observe that if Y_1 and Y_2 are finite sheeted covering spaces then for all vertices x of X, by Remark 2.1.10, the fibre $Y_1 \sqcup Y_2|_x$ is equal to the set

$$\{y \in (Y_1)_0 : p_1(y) = x\} \sqcup \{x \in (Y_2)_0 : p_2(y) = x\}.$$

This set is clearly finite. Hence, $\mathbf{FinCov}(X)$ is closed under $\sqcup$. □

Using the observations made in Section 4.3, Proposition 5.1.1 allows us to easily equip $\mathbf{FinCov}(X)^{\simeq}$ with the structure of a symmetric monoidal category.

Corollary 5.1.2. *There is the structure of a symmetric monoidal category on* $\mathbf{FinCov}(X)^{\simeq}$ *with tensor product* $\sqcup$*.*

Proof. Recall from Proposition 4.3.1 that a category with binary coproducts and an initial object has an induced symmetric monoidal structure. Also recall that if C is a symmetric monoidal category then so is $C^{\simeq}$ with the induced tensor product (Remark 4.3.5). Thus, as we have shown that $\mathbf{FinCov}(X)$ has binary coproducts, and as the map $\emptyset \to X$ is clearly an initial object in $\mathbf{FinCov}(X)$, the category $\mathbf{FinCov}(X)^{\simeq}$ has the structure of a symmetric monoidal category. □

5.2 Free Symmetric Monoidal Categories

To identify the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$ it will be helpful to first relate it to a certain free symmetric monoidal category via a zigzag of monoidal equivalences. This is because, as we will see later, the Algebraic K-theory spectra of free symmetric monoidal categories are easy to understand. In this section we will define free symmetric monoidal categories and explain in what sense they are 'free'.

To define the free symmetric monoidal category on a category C we must first define the k-th symmetrized power of C, following Baez in [1].

Definition 5.2.1. Let C be a category. Let

$$\frac{C^k}{k!}$$

be the category whose objects are k-tuples $(c_1, ..., c_k)$ of objects of C. Maps

$$((f_1, ..., f_k), \sigma) : (c_1, ..., c_k) \to (d_1, ..., d_k)$$

in $C^k/k!$ consist of k-tuples $(f_1, ..., f_k)$ of maps of C equipped with a permutation σ on k-elements such that $f_i : c_{\sigma(i)} \to d_i$ in C for each $i = 1, ..., k$. Composition in $C^k/k!$ sends the pair of maps

$$(c_1, ..., c_k) \xrightarrow{((f_1,...,f_n),\sigma)} (d_1, ..., d_k) \xrightarrow{((g_1,...,g_n),\tau)} (e_1, ..., e_k)$$

to the map

$$\big(g_1 \circ f_{\tau(1)}, ..., g_n \circ f_{\tau(n)}, \tau \circ \sigma\big) : (c_1, ..., c_k) \to (e_1, ..., e_k).$$

The category $C^k/k!$ is called the *k-th symmetrized power* of C.

Remark 5.2.2. We will take $C^0/0!$ to be the category with one object and one map.

The free symmetric monoidal category on a category C is then defined as follows:

Definition 5.2.3. Let C be a category. The category

$$S(C) := \coprod_{k \geq 0} \frac{C^k}{k!},$$

where $\coprod$ is the coproduct in **CAT**, is called the *free symmetric monoidal category* on C.

We obviously want to check that the category $S(C)$ is indeed a symmetric monoidal category, else its name would be very misleading. It turns out that $S(C)$ is a permutative category, as shown in Proposition 5.2.4.

Proposition 5.2.4. *The free symmetric monoidal category $S(C)$ on a category C is a permutative category.*

Proof. Let $\otimes : S(C) \times S(C) \to S(C)$ be the functor which maps pairs of objects

$$((x_1, ..., x_n), (y_1, ..., y_m))$$

to the object

$$(x_1, ..., x_n, y_1, ..., y_m).$$

The functor $\otimes$ sends the pair of maps

$$((f_1, ..., f_n), \sigma) : (x_1, ..., x_n) \to (x'_1, ..., x'_n)$$

and

$$((g_1, ..., g_m), \tau) : (y_1, ..., y_m) \to (y'_1, ..., y'_m)$$

to the map

$$((f_1, ..., f_n, g_1, ..., g_m), \sigma + \tau) : (x_1, ..., x_n, y_1, ..., y_m) \to (x'_1, ..., x'_n, y'_1, ..., y'_m).$$

Equipping the category $S(C)$ with the functor $\otimes$ gives it the structure of a strict monoidal category. There is a braiding on $(S(C), \otimes)$ whose $(x_1, ..., x_n, y_1, ..., y_m)$-component is an $n+m$ tuple of identity maps equipped with the permutation which swaps the block $\{1, ..., n\}$ of n letters with the block $\{n+1, ...n+m\}$ of m letters. □

So we have that $S(C)$ is a symmetric monoidal category, but in what sense is it free? As with the free infinite loop space and free groupoid constructions, the category $S(C)$ is free in the sense that Definition 5.2.3 can be extended to define a functor that is left adjoint to a forgetful functor.

Remark 5.2.5. Let **PermCat** be the category of small permutative categories and monoidal functors. Definition 5.2.3 extends to a functor

$$S(-) : \mathbf{Cat} \to \mathbf{PermCat}.$$

There is also a functor

$$U : \mathbf{PermCat} \to \mathbf{Cat}$$

which forgets the permutative structure of a permutative category.

Theorem 5.2.6. *The functors* $S : \mathbf{Cat} \rightleftarrows \mathbf{PermCat} : U$ *are adjoint.*

Proof. See Lemma 4.1 in [5]. □

To conclude our discussion of free symmetric monoidal categories observe that equivalences of categories induce monoidal equivalences.

Proposition 5.2.7. *Let* C *and* D *be categories and* $F : C \xrightarrow{\simeq} D$ *be an equivalence of categories. The induced functor* $S(F) : S(C) \to S(D)$ *is a monoidal equivalence.*

Proof. If $(d_1, ..., d_k)$ is a tuple of objects in $S(D)$ then for each d_i there is an object c_i in C such that there is an isomorphism $f_i : F(c_i) \to d_i$ in D. Thus, the map

$$((f_1, ..., f_k), Id) : (F(c_1), ..., F(c_k)) \to (d_1, ..., d_k)$$

is an isomorphism in $S(D)$, and so $S(F)$ is essentially surjective. Now suppose

$$((g_1, ..., g_k), \sigma) : (F(x_1), ..., F(x_k)) \to (F(y_1), ..., F(y_k))$$

is a map in $S(D)$. For each g_i in D there is a unique map h_i in C such that $F(h_i) = g_i$. Thus, the map $((h_1, ..., h_k), \sigma)$ in $S(C)$ is sent to $((g_1, ..., g_k), \sigma)$ by $S(F)$. Furthermore, $((h_1, ..., h_k), \sigma)$ is the unique map sent to $((g_1, ..., g_k), \sigma)$ by $S(F)$. Hence, the functor $S(F)$ is fully faithful. So $S(F)$ is an equivalence of categories, and, in particular, is a monoidal equivalence. □

5.3 Group Completions of Certain Free Simplicial Monoids

As previously mentioned, we want to identify the Algebraic K-theory spectra of free symmetric monoidal categories. Identifying the Algebraic K-theory spectra of free symmetric monoidal categories will be easy, as their classifying spaces can identified with the realisations of certain free simplicial monoids whose group completions have the homotopy type of a free infinite loop space. In this section, we will describe these certain free simplicial monoids and their group completions. All the theory we will discuss was first published in the paper [2] by Barratt and Eccles.

To define the free simplicial monoids we will be interested in we first need to define some useful maps. These maps are defined in Definitions 5.3.2, 5.3.3, 5.3.4, and 5.3.6 below.

Definition 5.3.1. Let $\underline{n} = \{1, ..., n\}$. For all positive integers $\underline{m}$ and $\underline{n}$, let C^n_m be the set of strictly increasing maps $\underline{m} \to \underline{n}$.

Definition 5.3.2. For positive integers $\underline{k}$ and $\underline{n}$ and $1 \leq i \leq k$ define the function

$$\lambda_i : \underline{n} \to \underline{kn}, \quad \lambda_i(j) := (i-1)n + j$$

in C^{kn}_n.

Definition 5.3.3. Let $\alpha \in C^n_m$ and X be a set. Define the function

$$\alpha^* : X^n \to X^m, \quad \alpha^*(x_1, ..., x_n) := (x_{\alpha(1)}, ..., x_{\alpha(n)}).$$

If the set X has a base point, and the components of $\boldsymbol{x} := (x_1, ..., x_n)$ that α^* omits are base points only, then α is called *entire* for $\boldsymbol{x}$.

Definition 5.3.4. If α is a map in C^n_m define the group monomorphism

$$\alpha_* : \Sigma_m \to \Sigma_n, \quad \sigma \mapsto \alpha_*(\sigma),$$

where

$$\begin{aligned} \alpha_*(\sigma)(j) &= j && \text{if } j \in \underline{n} \setminus \alpha(\underline{m}) \\ \alpha_*(\sigma)(\alpha(j)) &= \alpha(\sigma(j)) && \text{otherwise.} \end{aligned}$$

The definition of our next map will rely on the observation that there is a canonical bijection between C^n_m and the set of subsets of $\underline{n}$ of cardinality m.

Construction 5.3.5. Note that the set C^n_m is canonically isomorphic to the set of subsets of $\underline{n}$ of cardinality m. The bijection is explicitly given by the function

$$C^n_m \to \{A \in \mathcal{P}(\underline{n}) : |A| = \underline{m}\}, \quad \alpha \mapsto \{\alpha(1), ..., \alpha(m)\}.$$

Now suppose σ is an element in Σ_n. The image of the function $\sigma \circ \alpha$ is a subset of $\underline{n}$ of cardinality m, and thus, by the identification above, corresponds to a map in C^n_m, which will be denoted $\sigma_*(\alpha)$. There is then a unique map $\alpha^*(\sigma)$ in Σ_m making the diagram

$$\begin{array}{ccc} \mathbf{m} & \xrightarrow{\alpha} & \mathbf{n} \\ {\scriptstyle \alpha^*(\sigma)}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \mathbf{m} & \xrightarrow{\sigma_*(\alpha)} & \mathbf{n} \end{array} \tag{5.3}$$

commute.

Definition 5.3.6. If α is a map in C^n_m define the map

$$\alpha^* : \Sigma_n \to \Sigma_m, \quad \sigma \mapsto \alpha^*(\sigma),$$

where the map $\alpha^*(\sigma)$ is the unique permutation making diagram (5.3) commute.

Remark 5.3.7. Note that there has been an abuse of notation between Definitions 5.3.3 and 5.3.6. However, there will be no risk of confusion in this thesis, as whether α^* acts on tuples of a set X, or whether α^* acts on permutations, will be clear from context.

Using the maps defined above we can now construct the previously alluded to free simplicial monoids. To construct these free simplicial monoids $\Gamma^+(X)$ we will define their underlying simplicial sets, and then state a result that says they can be equipped with the structures of free simplicial monoids. As we shall seldom explicitly use the multiplication rule on $\Gamma^+(X)$ we will not describe it in this thesis.

Definition 5.3.8 [Definition 3.1, Barratt and Eccles [2]]. Suppose X is a pointed simplicial set. Let $\mathcal{U}(X)$ be the simplicial set

$$\coprod_{n \geq 0} X^n \times \mathbf{E}\Sigma_n,$$

where $\mathbf{E}\Sigma_n$ was defined in Definition 1.1.24. Let $\Gamma^+(X)$ be the simplicial set $\mathcal{U}(X)$ mod the equivalence relations generated by the relations:

$$((x_1, ..., x_n), \sigma) \sim ((x_{\omega(1)}, ..., x_{\omega(n)}), \omega \circ \sigma) \tag{5.4}$$

$$((x_1, ..., x_n), \sigma) \sim (\alpha^*((x_1, ..., x_n)), \alpha^*(\sigma)) \tag{5.5}$$

where $\sigma \in \mathbf{E}\Sigma_n$, $(x_1, ..., x_n) \in X^n$, $\omega \in \Sigma_n$, $\alpha \in C^n_m$ and is entire for $(x_1, ..., x_n)$, and $n, m \geq 0$. Note that $\Gamma^+(X)$ can be equipped with the canonical basepoint $[(*, Id)]$.

Remark 5.3.9. Note that in Definition 5.3.8 the notation being employed suppresses which set of n-simplices an element $((x_1, ..., x_n), \sigma)$ of $\mathcal{U}(X)$ belongs to, but specifies the summand. Also note that the operations $\omega \circ \sigma$ and $\alpha^*(\tau)$ are done componentwise.

Proposition 5.3.10 [Proposition 3.11, Barratt and Eccles [2]]. *If X is a pointed simplicial set then Γ^+X is a free simplicial monoid.*

Proof. See Section 3 in [2]. □

Thanks to Barratt and Eccles, simplicial models for the group completions of the free simplicial monoids $\Gamma^+(X)$ are well understood up to homotopy equivalence. Let us construct these simplicial models now.

Definition 5.3.11. Let $\Gamma^+(X) \to \Gamma X$ be the group completion of the simplicial monoid Γ^+X.

Construction 5.3.12. Recall from Section 1.5 that there is a natural bijection

$$\hom_{\mathbf{sSet}_*}(X \wedge Y, Z) \cong \hom_{\mathbf{sSet}_*}(X, \mathbf{Hom}_*(Y, Z)).$$

Let $\Sigma X := X \wedge Ex^\infty(\mathbb{S}^1)$ and $\Omega X := \mathbf{Hom}_*(Ex^\infty(\mathbb{S}^1), X)$. For each $n \geq 0$ there is a map $\Sigma^n X \to \Omega\Sigma^{n+1} X$ adjoint to the identity map $\Sigma^{n+1} X \to \Sigma^{n+1} X$. Hence, applying the induced Ω functor n-times, we have a map

$$\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1} X.$$

Thus, define $\Omega^\infty\Sigma^\infty X$ as the colimit of the diagram

$$X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to ... \tag{5.6}$$

in $\mathbf{sSet}_*$.

Theorem 5.3.13 [Theorem 4.10, Barratt and Eccles, [2]]. *If X is a pointed Kan complex then ΓX and $\Omega^\infty\Sigma^\infty X$ are homotopy equivalent as simplicial sets.*

Proof. This is Theorem 4.10 in [2]. □

Observe that $\Omega^\infty\Sigma^\infty X$ is constructed analogously to the way the free infinite loop space on a pointed space is constructed (see Remark 3.4.5). That is, $\Gamma(X)$ is homotopy equivalent to a simplicial analogue of the free infinite loop space on a based space. It turns out that that this equivalence is respected when passing to realisation. That is, $|\Gamma(X)|$ is homotopy equivalent to the free infinite loop space on $|X|$.

Proposition 5.3.14. *If X is a pointed simplicial set then the spaces $|\Gamma(X)|$ and $Q(\Sigma^\infty|X|)_0$ are homotopy equivalent.*

Proof. This is Theorem 7.4.1(b) in [3]. □

5.4 The Algebraic K-theory Spectrum of $S(C)$

In this section we will identify the stable homotopy type of the Algebraic K-theory spectrum of the free symmetric monoidal category $S(C)$ on C, when C is small. To identify the Algebraic K-theory spectrum of $S(C)$ we will appeal to some results from Chapter 4, the work of Barratt and Eccles reviewed in Section 5.3, and another result which we will state.

To discuss the Algebraic K-theory spectrum of free symmetric monoidal category $S(C)$ on C we require that $S(C)$ be small. But observe that if C is small, then clearly so is $S(C)$. So let us identify the Algebraic K-theory spectrum of $S(C)$, when C is small.

Proposition 5.4.1. *For all small categories C there is an isomorphism of topological monoids*

$$BS(C) \cong \left|\Gamma^+(NC_+)\right|.$$

Proof. See Section 2 of [29]. □

Theorem 5.4.2. *Let C be a small category. The Algebraic K-theory spectrum of $S(C)$ has the same stable homotopy type as the spectrum $\Sigma^\infty|NC_+|$.*

Proof. As $S(C)$ is a permutative category (Proposition 5.2.4), there is a homotopy equivalence $\Omega BBS(C) \simeq \Omega\mathbf{B}\mathcal{K}S(C)_1$ (Proposition 4.4.4). Now as there is an isomorphism of topological monoids $BS(C) \cong |\Gamma^+(NC_+)|$ (Proposition 5.4.1), and $\Gamma^+(NC_+)$ is good (Propositions 4.4.13 and 5.3.10), there is a homotopy equivalence $|\Gamma(NC_+)| \simeq \Omega BBS(C)$ (Corollary 4.4.11). Hence, by Proposition 5.3.14, there are homotopy equivalences

$$Q(\Sigma^\infty|NC_+|)_0 \simeq |\Gamma(NC_+)| \simeq \Omega BBS(C) \simeq \Omega\mathbf{B}\mathcal{K}S(C)_1.$$

Thus, by Construction 4.2.7, the Algebraic K-theory spectrum of $S(C)$ has the same stable homotopy type as the spectrum $\Sigma^\infty|NC_+|$. □

5.5 Identifying the Algebraic K-theory Spectrum of $\mathbf{FinCov}(X)^{\simeq}$

In this section we will identify the stable homotopy type of the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$, where X is a connected simplicial set. We will do this by exhibiting a zigzag of monoidal equivalences between $\mathbf{FinCov}(X)^{\simeq}$ and the free symmetric monoidal category $S(\mathcal{CC}_G)$, where $\mathcal{CC}_G$ is a groupoid constructed using choices of representatives from certain conjugacy classes of subgroups of the fundamental group $\pi_1(X)$ of X. The stable homotopy type of the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$ will then be identified using Propositions 4.2.3 and Theorem 5.4.2.

First, lets make precise what we mean by connected simplicial set.

Definition 5.5.1. A simplicial set X is called connected if its fundamental groupoid $\Pi_1(X)$ is a connected groupoid.

Remark 5.5.2. If a simplicial set X is connected then its fundamental groupoid $\Pi_1(X)$ is equivalent (as a groupoid) to a group, namely its fundamental group $\pi_1(X)$. Hence, when assuming a simplicial set is connected, we will take its fundamental groupoid to be $\pi_1(X)$.

Remark 5.5.3. If a simplicial set X is connected, then so is its realisation $|X|$.

Let us now relate $\mathbf{FinCov}(X)^{\simeq}$ to a free symmetric monoidal category. Recall from Section 2.6 we have an equivalence of categories

$$\mathbf{FinCov}(X)^{\simeq} \xrightarrow{\simeq} [\Pi_1(X), \mathbf{FinSet}^{\simeq}], \tag{5.7}$$

for each simplicial set X. This equivalence is monoidal as there is a symmetric monoidal structure on $[\Pi_1(X), \mathbf{FinSet}]$ is induced by coproducts and an initial object (see Corollarys 4.3.2 and 4.3.4), and restricting to cores respects monoidal equivalences. Taking X to be connected, the above monoidal equivalence becomes

$$\mathbf{FinCov}(X)^{\simeq} \xrightarrow{\simeq} [\pi_1(X), \mathbf{FinSet}^{\simeq}]. \tag{5.8}$$

It then turns out that $[\pi_1(X), \mathbf{FinSet}^{\simeq}]$ is monoidally equivalent to the core of the category of finite left $\pi_1(X)$-sets.

Proposition 5.5.4. *Let G be a group, and let $G - \mathbf{FinSet}$ be the category of left G-sets with finitely many elements. There is an isomorphism of categories*

$$[G, \mathbf{FinSet}] \cong G - \mathbf{FinSet}.$$

Proof. If $X : G \to \mathbf{FinSet}$ is a functor, the finite set $X := X(\bullet)$ has the structure of a left G-set when equipped with the map

$$g \cdot x := X(g)(x).$$

If $\alpha : X \to Y$ is a natural transformation in $[G, \mathbf{FinSet}]$, then the diagram

$$\begin{array}{ccc} X & \xrightarrow{X(g)} & X \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow{X(g)} & Y \end{array} \tag{5.9}$$

commutes for all g in G. Thus, the map $\alpha : X \to Y$ is also a map of G-sets. Hence, mapping functors $X : G \to \mathbf{FinSet}$ to the G-sets X and mapping the natural transformations α to the G-set maps α defines a functor $[G, \mathbf{FinSet}] \to G - \mathbf{FinSet}$. This functor has an obvious inverse. □

Corollary 5.5.5. *If* $G - \mathbf{FinSet}$ *is equipped with the structure of a symmetric monoidal category induced by coproducts and an initial object, the isomorphism of categories*

$$[G, \mathbf{FinSet}^{\simeq}] \cong (G - \mathbf{FinSet})^{\simeq}$$

is a monoidal equivalence.

Proof. This follows immediately from Proposition 5.5.4, Corollary 4.3.4, and Remark 4.3.5. □

We thus have the monoidal equivalence

$$\mathbf{FinCov}(X)^{\simeq} \xrightarrow{\simeq} (\pi_1(X) - \mathbf{FinSet})^{\simeq}. \tag{5.10}$$

We will now see that $(\pi_1(X) - \mathbf{FinSet})^{\simeq}$ is monoidally equivalent to the free symmetric monoidal category on the core of the category of finite transitive $\pi_1(X)$-sets. To construct this monoidal equivalence it will be helpful to first make some observations about finite $\pi_1(X)$-sets.

Construction 5.5.6. Let X be a finite left G-set. Recall that any G-set can be written as the disjoint union of its orbits. That is,

$$X = \coprod_{[x] \in X/G} G \cdot x,$$

where $G \cdot x$ denotes the orbit of an element x. Now, let G_x denote the stabiliser subgroup of x in G. Recall that $G \cdot x$ and G/G_x are isomorphic as transitive G-sets, where G/G_x is the set of left cosets of G_x. Thus, we have

$$X = \coprod_{[x] \in X/G} G/G_x.$$

Now suppose that $y \in X$ such that $y = g \cdot x$. Then we have that

$$\begin{aligned} h \in G_y &\iff h \cdot y = y \\ &\iff g^{-1}hg \cdot x = x \\ &\iff g^{-1}hg \in G_x \\ &\iff h \in gG_xg^{-1}. \end{aligned}$$

That is, if y is in the orbit of x, then G_x and G_y are conjugate subgroups of G. Thus, as conjugate subgroups induce isomorphisms on the sets of cosets (i.e. if G_x and G_y are conjugate subgroups then $G/G_x \cong G/G_y$ as G-sets), the G-set X is entirely determined by n conjugacy classes $[G_x]$ of subgroups of G, where $n = |X/G|$.

Proposition 5.5.7. *Let* $G - \mathbf{FinSet}^T$ *be the category of finite transitive* G*-sets. There is a monoidal equivalence*

$$S\left((G - \mathbf{FinSet}^T)^{\simeq}\right) \xrightarrow{\simeq} (G - \mathbf{FinSet})^{\simeq}.$$

Proof. Let

$$S\left((G - \mathbf{FinSet}^T)^{\simeq}\right) \to (G - \mathbf{FinSet})^{\simeq}. \tag{5.11}$$

be the functor that maps the k-tuple $(X_1, ..., X_k)$ of finite transitive G-sets to the G-set

$$\coprod_{n=1}^{k} X_k.$$

If

$$((f_1, ..., f_k), \sigma) : (X_1, ..., X_k) \to (Y_1, ..., Y_k)$$

is a map in $S\left((G - \mathbf{FinSet}^T)^{\simeq}\right)$ the functor (5.11) sends it to the map

$$f : \coprod_{n=1}^{k} X_n \to \coprod_{n=1}^{k} Y_n,$$

where $f|_{X_{\sigma(i)}} = f_i$. As the monoidal structure on $(G - \mathbf{FinSet})^{\simeq}$ is induced by coproducts, the functor (5.11) is a monoidal functor. The functor (5.11) is also essentially surjective as all finite G-sets can be written as a finite disjoint union of finite transitive G-sets of the form G/H, where H is a subgroup of G. Now if $f : \coprod_{n=1}^{k} X_n \to \coprod_{n=1}^{k} Y_n$ is an isomorphism of G-sets that are in the image of the functor (5.11), for each i there exists a j such that the map $f|_{X_j} : X_j \to Y_i$ is an isomorphism of G-sets. Thus, let σ be the permutation where $\sigma(i) = j$. The functor (5.11) then sends the map

$$\left(\left(f|_{X_{\sigma(1)}}, ..., f|_{X_{\sigma(k)}}\right), \sigma\right) : (X_1, ..., X_k) \to (Y_1, .., Y_k)$$

to f. Thus, the functor (5.11) is full. Furthermore, such a map $f : \coprod_{n=1}^{k} X_n \to \coprod_{n=1}^{k} Y_n$ is uniquely determined by its k-restrictions $f|_{X_j}$. Thus, the functor (5.11) is also faithful. □

Hence, we now have zigzag of monoidal equivalences

$$\mathbf{FinCov}(X)^{\simeq} \xrightarrow{\simeq} (\pi_1(X) - \mathbf{FinSet})^{\simeq} \xleftarrow{\simeq} S\left((\pi_1(X) - \mathbf{FinSet}^T)^{\simeq}\right). \quad (5.12)$$

We have hence related $\mathbf{FinCov}(X)^{\simeq}$ to a free symmetric monoid category. However, the category $S\left((\pi_1(X) - \mathbf{FinSet}^T)^{\simeq}\right)$ and its Algebraic K-theory spectrum are difficult to understand. Thus, we will now relate $S\left((\pi_1(X) - \mathbf{FinSet}^T)^{\simeq}\right)$ to a monoidally equivalent free symmetric monoidal category that can be more easily understood. To do this we will show that for every group G the groupoid $(G - \mathbf{FinSet}^T)^{\simeq}$ is equivalent to a groupoid constructed using representatives from certain conjugacy classes of subgroups of G.

Definition 5.5.8. Let G be a group, and let $N_G(H)$ denote the normaliser subgroup of H in G. Let $\mathcal{CC}_G$ be the discrete groupoid whose objects are finite G-sets of the form G/H, where H is a representative from a conjugacy class $[H]$ of subgroups in G. The automorphism group of each object G/H in $\mathcal{CC}_G$ is the group $N_G(H)/H$.

Remark 5.5.9. Let CC_G denote the set of conjugacy classes of subgroups $[H]$ of G such that G/H is finite. Note that $\mathcal{CC}_G$ can be thought of as the disjoint union of groupoids

$$\coprod_{[H] \in CC_G} N_G(H)/H.$$

To show that $\mathcal{CC}_G$ and $(G - \mathbf{FinSet}^T)^{\simeq}$ are equivalent it will be helpful to first prove that the group of G-set automorphisms mapping G/H to itself and $N_G(H)/H$ are isomorphic as groups.

Lemma 5.5.10. *Let G be a group, H be a subgroup of G, and let $Map(G/H, G/H)$ be the group of G-set automorphisms mapping the G-set G/H to itself. There is an isomorphism of groups*

$$N_G(H)/H \cong Map(G/H, G/H).$$

Proof. Let f be a G-set automorphism in $Map(G/H, G/H)$. Note that for all g in G

$$f(gH) = f(g \cdot eH) = gf(eH),$$

where e is the identity of G. Then, as f maps cosets of H to cosets of H, we can write

$$gf(eH) = gf(e)H.$$

That is, f is determined entirely by its action on the identity e of G. Also note that $f(e)H$ is an element of $N_G(H)/H$. Why? If h is in H then $f(hH) = hf(e)H$. But $f(hH) = f(H) = f(e)H$. Hence, $f(e)H = hf(e)H$, and so $f(e)^{-1}hf(e)$ is an element of H. We can hence define the maps

$$\phi : N_G(H)/H \to Map(G/H, G/H), \quad \phi(xH) := f_x,$$

where $f_x(gH) = gxH$, and

$$\psi : Map(G/H, G/H) \to N_G(H)/H, \quad \psi(f) := f(e)H.$$

It is easy to check that ϕ and ψ are well-defined group homomorphisms, and are mutually inverse. Thus, $N_G(H)/H$ and $Map(G/H, G/H)$ are isomorphic groups. □

Proposition 5.5.11. *There is an equivalence of categories*

$$\mathcal{CC}_G \xrightarrow{\simeq} (G - \mathbf{FinSet}^T)^{\simeq}.$$

Proof. Let

$$\mathcal{CC}_G \to (G - \mathbf{FinSet}^T)^{\simeq} \tag{5.13}$$

be the functor that maps a finite G-set G/H to the transitive G set G/H. The functor $\mathcal{CC}_G \to (G - \mathbf{FinSet}^T)^{\simeq}$ maps elements x of the group $N_G(H)/H$ to the map f_x, as in Lemma 5.5.10. This functor is an equivalence of categories, as it is essentially surjective (by Construction 5.5.6) and fully faithful (by Lemma 5.5.10). □

Corollary 5.5.12. *There is a monoidal equivalence*

$$S\left(\mathcal{CC}_G\right) \xrightarrow{\simeq} S\left((G - \mathbf{FinSet}^T)^{\simeq}\right).$$

Proof. The statement immediately follows from Propositions 5.2.7 and 5.5.11. □

By equation (5.12) and Corollary 5.5.12 we thus have:

Corollary 5.5.13. *There is a zigzag of monoidal equivalences*

$$\mathbf{FinCov}(X)^{\simeq} \xrightarrow{\simeq} (\pi_1(X) - \mathbf{FinSet})^{\simeq} \xleftarrow{\simeq} S(\mathcal{CC}_{\pi_1(X)}).$$

We have hence related $\mathbf{FinCov}(X)^{\simeq}$ to a free symmetric monoidal category on a groupoid which is constructed using choices of representatives from certain conjugacy classes of subgroups of $\pi_1(X)$. Using this result, we can finally identify the stable homotopy type of the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$.

Theorem 5.5.14. *If X is a connected simplicial set then the Algebraic K-theory spectrum of* $\mathbf{FinCov}(X)^{\simeq}$ *has the same stable homotopy type as the spectrum*

$$\Sigma^{\infty}\left(\coprod_{[H]\in CC_{\pi_1(X)}} B(N_{\pi_1(X)}(H)/H)\right)_+ .$$

Proof. As there is a zigzag of monoidal equivalences

$$\mathbf{FinCov}(X)^{\simeq} \xrightarrow{\simeq} (\pi_1(X) - \mathbf{FinSet})^{\simeq} \xleftarrow{\simeq} S(\mathcal{CC}_{\pi_1(X)}),$$

by Proposition 4.2.3, $\mathbf{B}\mathcal{K}\mathbf{FinCov}(X)^{\simeq}$ and $\mathbf{B}\mathcal{K}S(\mathcal{CC}_{\pi_1(X)})$ have the same stable homotopy type. Thus, by Theorem 5.4.2, $\mathbf{B}\mathcal{K}\mathbf{FinCov}(X)^{\simeq}$ has the same stable homotopy type as $\Sigma^{\infty}\big|N\mathcal{CC}_{\pi_1(X)+}\big|$. Hence, writing

$$\mathcal{CC}_{\pi_1(X)} = \coprod_{[H]\in CC_{\pi_1(X)}} N_{\pi_1(X)}(H)/H$$

(Remark 5.5.10), and commuting N and $|-|$ with coproducts, gives the result. □

That is, the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$ has the same stable homotopy type as the suspension spectrum on the disjoint union of the classifying spaces of groups constructed from certain conjugacy classes of $\pi_1(X)$ attached with a disjoint basepoint.

In the Introduction of this thesis it was stated that the goal of this thesis was to generalise the Barratt-Priddy-Quillen theorem by identifying the stable homotopy type of the Algebraic K-theory spectrum of $\mathbf{FinCov}(X)^{\simeq}$. To demonstrate that proving Theorem 5.5.14 successfully completed this goal, we will now use it to recover the Barratt-Priddy-Quillen theorem.

Corollary 5.5.15 [**Barratt-Priddy-Quillen**]**.** *The Algebraic K-theory spectrum of Σ_∞, where*

$$\Sigma_\infty = \coprod_{n\geq 0} \Sigma_n,$$

has the same stable homotopy type as the sphere spectrum $\mathbb{S}$.

Proof. As simplicial coverings of Δ^0 are discrete simplicial sets (Lemma 2.1.9), taking $X = \Delta^0$, we have $\mathbf{FinCov}(\Delta^0)^{\simeq} \cong \mathbf{FinSet}^{\simeq} \simeq \Sigma_\infty$. Hence, by Theorem 5.5.14, as $\pi_1(\Delta^0)$ is the trivial group, the Algebraic K-theory spectrum of Σ_∞ is stably equivalent to the spectrum $\Sigma^\infty \mathbb{S}^0 = \mathbb{S}$. □

Bibliography

[1] J. Baez. *Random Permutations (Part 12).* Jan 11, 2020. https://math.ucr.edu/home/baez/permutations/permutations_12.html.

[2] M. G. Barratt and P. J. Eccles. Γ^+-Structures-I: A Free Group Functor For Stable Homotopy Theory. *Topology*, 13:25–45, 1974.

[3] G. Carlsson and R. Milgram. Stable homotopy and iterated loop spaces. In *Handbook of Algebraic Topology*, pages 505–583. Elsevier Science, North Holland, 1995.

[4] A. Dold. Partitions of Unity in the Theory of Fibrations. *Annals of Mathematics*, 78:223–255, 1963.

[5] A. D. Elmendorf and M. A. Mandell. Permutative categories, multicategories and algebraic K–theory. *Algebraic and Geometric Topology*, 9:2391–441, 2009.

[6] R. Fritsch and R. A. Piccinini. *Cellular structures in Topology*. Cambridge Advanced Studies in Mathematics, Cambridge University Press, Cambridge, 1990.

[7] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*. A Series of Modern Surveys in Mathematics, Springer, New York, 1967.

[8] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*. Volume 174 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.

[9] A. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2000.

[10] M. Hovey. *Model Categories.* Volume 63 of *Mathematical Surveys and Monographs*, American Mathematical Society, 2007.

[11] A. Joyal and M. Tierney. Quasi-categories vs Segal spaces. *Contemporary Mathematics*, 431:277—326, 2007.

[12] S. Lang. *Algebra.* Volume 211 of *Graduate Texts in Mathematics*, Springer, New York, 2002.

[13] T. Leinster. *Basic Category Theory.* Volume 143 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2014.

[14] J. Lellig. A union theorem for cofibrations. *Archiv der Mathematik*, 24:410—415, 1973.

[15] L. G. Lewis. When is the natural map $X \to \Omega\Sigma X$ a cofibration? *Transactions of the American Mathematical Society*, 273:147–155, 1982.

[16] J. Lurie. *Higher Topos Theory.* Volume 170 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, 2009.

[17] S. Mac Lane. *Categories for the Working Mathematician.* Volume 5 of *Graduate Texts in Mathematics*, Springer, New York, 1971.

[18] M. A. Mandell. An Inverse K-theory Functor. *Documenta Mathematica*, 15:765-–791, 2010.

[19] J. P. May. Categories of spectra and infinite loop spaces. In *Category Theory, Homology Theory and Their Applications III*, pages 448–479. Springer, New York, 1968.

[20] J. P. May. *Classifying Spaces and Fibrations* No. 155 of *Memoirs of the American Mathematical Society*, American Mathematical Society, Providence, 1975.

[21] J. P. May. *A Concise Course in Algebraic Topology.* Chicago Lectures in Mathematics, University of Chicago Press, Chicago, 1999.

[22] J. P. May and K. Ponto. *More Concise Algebraic Topology: Localization, Completion, and Model Categories.* Chicago Lectures in Mathematics Series, University of Chicago Press, Chicago, 2012.

[23] J. P. May. and J. Sigurdsson. *Parametrised Homotopy Theory.* Volume 132 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, 2006.

[24] D. Quillen. Higher Algebraic K-theory: I. In *Higher K-Theories*, pages 85–147. Lecture Notes in Mathematics, Vol. 341, Springer, Berlin, 1973.

[25] D. Quillen. On the group completion of a simplicial monoid. Published as an appendix in E. M. Friedlander and B.Mazur, Filtrations on the homology of algebraic varieties, *Memoirs of the AMS*, 529:89–105, 1994.

[26] E. Riehl. *A Leisurely Introduction to Simplicial Sets.* 2011. Available from the author's webpage at http://www.math.jhu.edu/ eriehl/s-sets.pdf.

[27] G. Segal. Categories and Cohomology Theories. *Topology*, 13:293–312, 1974.

[28] N. P. Strickland. *The Category of CGWH Spaces.* 2009. Available from the author's webpage at https://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf

[29] R. W. Thomason. First Quadrant Spectral Sequences in Algebraic K-theory via Homotopy Colimits. *Communications in Algebra*, 10:1589–1668, 1982.

[30] C. Weibel. *The K-book: An Introduction to Algebraic K-theory*. Volume 145 of *Graduate Studies in Mathematics*, American Mathematical Society, 2013.