The families Seiberg-Witten invariants of smooth families of Kähler surfaces

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Contents

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Thank you all.

Abstract

The Seiberg-Witten invariant has been an indispensable tool in understanding the topology and smooth structure of 4-manifolds, especially Kähler surfaces, where the mutually interacting symplectic and complex structures often allows for an explicit computation of the invariant. We concern ourselves with a natural generalisation of this setup to smooth families of 4-manifolds with fibres diffeomorphic to a single 4-manifold X, where one may define a generalisation of the Seiberg-Witten invariant known as the families Seiberg-Witten invariants. After introducing the necessary background into Seiberg-Witten theory, we provide an exposition on its generalisation to families of 4-manifolds and proceed to obtain a general formula for the invariants for smooth Kähler families with $b_1(X) = 0$. Following this is a further explicit computation for three classes of example families, these being simple constructions of families of with fibres diffeomorphic to $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ and finally a family with fibres being the blowup of a Kähler surface at a point. We then conclude by looking at the consequences of the computations made, in particular investigating constraints on the cohomology of holomorphic line bundles over smooth Kähler families required for a non-vanishing diffeomorphism invariant, with a particular focus on when the base space of the family is S^2, and further apply these considerations to the example families discussed prior.

Introduction

The classification of smooth manifolds of dimensions between 1 and 3 is well-understood, exhibiting a classification by their geometry and their smooth structure completely determined by their homeomorphism class [Whi61]. Smooth manifolds of dimension 5 and above also lend themselves to a classification via the techniques of surgery theory. However, smooth manifolds of dimension 4 act as a boundary case between these two regimes. Although topological 4-manifolds can also be classified via surgery theory, these techniques fail in the smooth category, allowing 4-manifolds to exhibit exotic behaviour.

One striking example of this is a result of Taubes, where it was shown that there exist uncountably smooth manifolds homeomorphic to but not diffeomorphic to $\mathbb{R}^4$ [Tau96]. The existence of these exotic $\mathbb{R}^4$'s is quite peculiar, in stark contrast to the case of $n \neq 4$ where there are no such exotic $\mathbb{R}^n$'s. The key idea of Taubes' proof is a comparison of Freedman's results on topological 4-manifolds in [Fre82] and Donaldson's diagonalisation theorem [Don83], the latter being a key ingredient in many results involving smooth 4-manifolds.

Donaldson's result is of much importance to the study of smooth 4-manifolds by introducing the techniques of gauge theory. Donaldson's theorem was proven via analysis of the moduli space of anti-self dual Yang-Mills instantons. Although this technique has proven very fruitful the moduli space has some undesirable properties, for instance, it is not generally compact, leaving a fairly difficult problem of how one compactifies this moduli space.

In 1994, Edward Witten introduced the Seiberg-Witten equations to the mathematical community [Wit94], based off work on supersymmetrmetric $N = 2$ gauge theory [SW94]. Just as in Donaldson theory, one can analyse the moduli space of solutions to these equations, but in contrast, the moduli space is compact and much easier to work with. This lends to the simplification

of proofs of results proven via Donaldson theory, as well as new results, such as a proof of the Thom conjecture [KM94]. The key construction from the Seiberg-Witten moduli space in many results is the Seiberg-Witten invariant. For 4-manifolds with $b^+ > 1$ this is a diffeomorphism invariant, but it also useful in the case $b^+ = 1$. The Seiberg-Witten invariant has been most fruitful in the study of symplectic 4-manifolds, especially on Kähler surfaces where a general computation of the invariant can be made when $b_1 = 0$.

It was suggested by Donaldson in [Don96], many of the techniques of Seiberg-Witten theory, such as the Seiberg-Witten invariant could be generalised to families of 4-manifolds, that is a smoothly parametrised family X_b with $b \in B$ where B is some compact parameter space and the X_b are diffeomorphic to some fixed 4-manifold X via an oriented diffeomorphism. These techniques were used in papers such as [Nis02], [Rub98] and [Rub02], involving special cases with low-dimensional family parameter spaces, with a more general account of the families Seiberg-Witten invariant given by [LL01]. One expects that analogous to the unparametrised case, the families Seiberg-Witten invariant should be useful in studying smooth families of Kähler surfaces, consequently, the central focus and primary result of this thesis is a general computation of the families Seiberg-Witten invariant for smooth Kähler families with $b_1(X) = 0$.

This thesis is comprised of 3 parts. The first involves basic introductory notions required to discuss Seiberg-Witten theory, the first chapter consists of a discussion on smooth Fredholm maps in preparation for rigorously establishing key results, such as the fact that the Seiberg-Witten moduli space is a smooth manifold, or is compact. The second chapter establishes basic results on complex manifolds with an eventual focus towards Kähler manifolds and related notions which appear in the computation of the Seiberg-Witten invariant on Kähler surfaces. The third chapter provides an overview of spin^c structures which are integral to Seiberg-Witten theory and its presence in the theory is one key feature that distinguishes it from the theory of Yang-Mills instantons.

The second part acts as an introduction to Seiberg-Witten theory, with Chapter 4 covering the general theory on 4-manifolds, in particular the transversality, orientability and compactness of the Seiberg-Witten moduli space. Chapter 5 focuses on Seiberg-Witten theory on Kähler surfaces in particular, where the analysis of the Seiberg-Witten moduli space simplifies, leading to techniques that allow a general computation of the Seiberg-Witten invariant

The concluding part is concerned with families Seiberg-Witten theory

and contains the main results of the thesis. Chapter 6 outlines the basics of smooth families of 4-manifolds and Seiberg-Witten theory in this context, this proceeds similarly to the unparametrsed case as in Chapter 4 when establishing the transversality and compactness of the moduli space, although the subtleties of a richer 'chamber structure' are also discussed which causes the parametrised theory to often be more complicated. In this chapter we also define the families Seiberg-Witten invariants which generalise the Seiberg-Witten invariant of the unparametrised theory. Chapter 7 contains a computation of the families invariants for families of Kähler surfaces with $b_1 = 0$, which generalises the known computation of the ordinary Seiberg-Witten invariant for Kähler surfaces with $b_1 = 0$. With this computation in hand, we apply it further to three specific families of Kähler surfaces for which more explicit computations can be made, these are a family of $\mathbb{CP}^2$'s obtained from the projectivisation of a rank 3 complex vector bundle, a family of $\mathbb{CP}^1 \times \mathbb{CP}^1$'s obtained from the projectivisations of two rank 2 complex vector bundles, and finally a blowup family with fibres being the blowup of a Kähler surface X at a point x.

The final chapter of this thesis deals with the consequences of the computations made in Chapter 7. We first obtain constraints on the cohomology of line bundles on the family required for a non-vanishing invariant, with a particular focus on the case when the parameter space for the family is S^2. We then further analyse constraints required to ensure the families invariants are diffeomorphism invariants of the family when the parameter space is S^2 and also apply these considerations to the three example families as discussed in Chapter 7.

Part I

Introductory Geometry

Chapter 1

Smooth Maps Between Manifolds

1.1 Properties of Smooth Fredholm Maps

In this section we quickly review selected properties of smooth Fredholm maps between Banach spaces and more generally Banach manifolds. Maps of these type exhibit many similar properties to their finite dimensional counterparts but allow us to work with many of the infinite dimensional spaces that come up in analysing the Seiberg-Witten equations. These results are standard and also used for Yang-Mills moduli spaces. A more detailed exposition of Fredholm maps can be found in [Sal99][Appendix A,B] or [DK07][Chapter 4 and Appendix I] and transversality in the more general setting of Banach manifolds is also discussed in [Lan02].

Banach manifolds generalise the concept of smooth manifolds and are defined almost identically, with the relevant concepts defined on Banach spaces and maps between them instead of $\mathbb{R}^n$. That is, a Banach manifold is a paracompact Hausdorff topological space, equipped with coordinate charts into an open subset of some Banach space with transition functions required to be infinitely differentiable with continuous derivatives with respect to the Fréchet derivative. We say that a map $f : X \to Y$ between Banach manifolds is smooth if and only if it is smooth with respect to coordinate charts, and that that $y \in Y$ is a regular value if $df(x)$ is surjective and has right inverse for all $x \in f^{-1}(y)$.

Definition 1.1.1. Let $f : X \to Y$ be a bounded linear map between Banach spaces, we say it is *Fredholm* if it has finite dimensional kernel and cokernel, and its range is closed.

Given a Fredholm map f we define its *index* to be

$$\mathrm{ind}(f) = \dim(\ker(f)) - \dim(\mathrm{coker}(f)).$$

If f is a map between complex Banach spaces, we refer to the complex index as $\mathrm{ind}_{\mathbb{C}}(f)$ and the real index as $\mathrm{ind}(f)$.

Given a smooth map between Banach manifolds $f : X \to Y$ we also say it is Fredholm if the induced map on tangent spaces $df_x : T_xX \to T_{f(x)}Y$ is Fredholm as a map between Banach spaces.

It is a standard well known fact in functional analysis that the Fredholm property of operators and the value of their index are invariant under compact perturbations. We now state a lemma that will be useful later in analysing the Seiberg-Witten equations.

Lemma 1.1.2. *Let X, Y, Z be Banach spaces and suppose that $D : X \longrightarrow Y$ is a bounded linear operator and that $K : X \longrightarrow Z$ is compact. Further, assume that there is a constant $c > 0$ such that*

$$\|x\|_X \leq c(\|Dx\|_Y + \|Kx\|_Z)$$

for all $x \in X$. Then D has closed range and finite dimensional kernel.

Suppose that X is a smooth Banach manifold and $W \subset X$ and suppose that for each $x \in W$ there exists a chart (U, ψ) centred around x such that $U \cong U_1 \times U_2$ where U_1 and U_2 are open subsets of Banach spaces E_1 and E_2 respectively and we also have

$$\psi(U \cap W) = U_1 \times \{0\}$$

then we say that W is a *submanifold* of X.

Now suppose that $f : X \to Y$ is a smooth map between Banach manifolds, we say that the submanifold W is *transverse* to f if for all $x \in X$ such that $f(x) \in W$, given a submanifold chart (U, ψ) centred at $f(x)$, i.e. we have $\psi : U \to U_1 \times U_2$ as above with

$$\psi(f(x)) = (0, 0), \qquad \psi(W \cap U) = U_1 \times \{0\}$$

then there exists an open neighbourhood V of x such that the composition of maps

$$V \xrightarrow{f} U \xrightarrow{\psi} U_1 \times U_2 \xrightarrow{\pi_2} U_2$$

is a submersion.

The key results we will use throughout the thesis are the implicit function theorem and the Sard-Smale theorem.

Theorem 1.1.3 (Implicit function theorem). *Suppose that $f : X \to Y$ is a smooth Fredholm map between Banach manifolds, Y is connected and $y \in Y$ is a regular value of f. Then $f^{-1}(y)$ is a finite dimensional smooth Banach manifold with*

$$dim(f^{-1}(y)) = ind(f).$$

Theorem 1.1.4 (Sard-Smale theorem). *Suppose that X, Y are separable paracompact Banach manifolds, $U \subset X$ is open and $f : U \to Y$ is a smooth Fredholm map, then the set of regular values of f are of second category, that is, a countable intersection of open and dense sets.*

Staying in the context of a Fredholm map between Banach manifolds $f : X \to Y$, these theorems easily generalise if 'regular value' is replaced with 'transverse to f' since submanifolds transverse to f are locally given by the inverse image of zero via the composite maps defined in terms of submanifold coordinate charts and the map f as defined previously in the chapter. As corollaries, we have the following two results.

Theorem 1.1.5. *Suppose that $f : X \to Y$ is a smooth Fredholm map between Banach manifolds, Y is connected and $W \subset Y$ is a finite dimensional submanifold of Y transverse to f. Then $f^{-1}(W)$ is a finite dimensional smooth Banach manifold with*

$$dim(f^{-1}(y)) = ind(f) + dim(W).$$

Theorem 1.1.6. *Suppose that X, Y are separable paracompact Banach manifolds, $U \subset X$ is open and $f : U \to Y$ is a smooth Fredholm map, then the set of finite dimensional submanifolds W transverse to f are of second category in the space of finite dimensional submanifolds of Y*

Remark 1.1.7. One can more generally talk about the transversality of maps $h : B \to Y$ to the map f, discussing the case of a submanifold $W \subset Y$ as above is simply restricting to the case when the maps h are inclusion maps $\iota : W \hookrightarrow Y$. By endowing the space of such maps with the C^∞ topology the previous theorem implies that one can always approximate a submanifold by a transversal one to arbitrary accuracy in the C^∞ topology, since Banach manifolds are Baire spaces.

1.2 Sobolev Spaces

If one were to work entirely with smooth maps, many of the spaces involved between mappings when studying the Seiberg-Witten equations would not be complete and the machinery of the previous section would not apply. It is for this reason that we briefly introduce Sobolev completions and some key lemmas and theorems which are integral to rigorously establishing the transversality and compactness results on the Seiberg-Witten moduli spaces. Much of this discussion can be found in [Wel80], [Sal99] and [Nic00].

First assume that $X = \mathbb{R}^n$, for an integer $k \geq 0$ and real number $p \geq 1$ there is a norm on the space of (compactly supported) smooth functions $C^\infty(X)$ given by

$$\|f\|_{p,k}(X) := \left(\sum_{0 \leq |\alpha| \leq k} \int_{\mathbb{R}^n} |D^\alpha f|^p \right)^{\frac{1}{p}}$$

where $\alpha = (\alpha_1, \dots \alpha_{|\alpha|})$ is a multi-index and $D^\alpha = D_1^{\alpha_1} \dots D_{|\alpha|}^{\alpha_{|\alpha|}}$ with $D_j = \partial / \partial x_j$. The *Sobolev spaces* L_k^p are then defined to be the completion of $C^\infty(X)$ equipped with the above norm.

Now suppose that $E \to X$ is a smooth vector bundle of rank m over a compact manifold X of dimension n, $\{U_i, \varphi_i\}$ is a finite trivialising cover of E which induces a trivialising map on sections $\varphi_i^* : \mathbb{C}^\infty(U_i, E) \to C^\infty(U_i)^m$ and ρ_i is a partition of unity subordinate to $\{U_i\}$. For a smooth section $s \in C^\infty(X, E)$ we may define the Sobolev norm by $\|s\|_{p,k} = \sum_i \| \|\varphi_i^* \rho_i s\| \|_{p,k,\mathbb{R}^n}$, i.e. one passes to a local trivialisation where sections may be represented locally by vectors of smooth functions, takes the Sobolev norm and glues them together with the partition of unity, that is

$$\|s\|_{p,k} := \sum_i \left(\sum_{0 \leq |\alpha| \leq k} \int_{U_i} |D^\alpha \|\varphi_i^* \rho_i s\| |^p \right)^{\frac{1}{p}}.$$

The Sobolev space of sections of E, denoted $L_k^p(X, E)$ is then the completion of $C^\infty(X, E)$ with respect to this norm. The norm itself depends on the choice of trivialisation and partition of unity, although any such choice results in equivalent norms and the topology induced on $L_k^p(X, E)$ is the same. The spaces $L_k^p(X, E)$ are Banach spaces and for $p = 2$ in particular they are Hilbert spaces, furthermore $L_0^p(X, E)$ is the familiar space of L^p sections. There are a variety of inclusions of Sobolev spaces in each other, we state the most important embedding theorems for the purpose of studying the Seiberg-Witten moduli space.

Theorem 1.2.1 (Rellich's Theorem)**.** *Let $E \to X$ be a vector bundle over a compact manifold X of dimension n, then if $k > \ell$ and $\frac{1}{p} - \frac{k}{n} < \frac{1}{q} - \frac{\ell}{n}$ then $L^p_k(X, E) \subset L^q_\ell(X, E)$ and the inclusion is compact operator.*

Theorem 1.2.2. *Let X be a vector bundle over a compact manifold X of dimension n, then for $kp > \ell p + n$ there is a continuous embedding*

$$L^p_k(X, E) \subset C^\ell(X, E)$$

where $C^\ell(X, E)$ is the ℓ-times continuously differentiable sections of E.

Chapter 2

Complex Geometry

This chapter shall provide an overview on basic notions in complex geometry with an eventual focus towards Kähler geometry. We begin by recalling the standard theory of complex manifolds, complex structures, Dolbeault cohomology and a brief overview of Hodge theory on compact oriented complex manifolds. Following this we cover Kähler manifolds and two topics that are of key importance to the study of Seiberg-Witten theory on Kähler surfaces, namely holomorphic vector bundles and divisors, the latter being intimately related to gauge equivalence classes of solutions to the Seiberg-Witten equations on Kähler surfaces. The chapter then concludes with an overview of blowups of complex manifolds which serve as a means to construct a family of Kähler surfaces investigated later in Chapter 7. Standard references for the topics covered in this chapter are [GH94] and [Huy05], although many of the details relevant to Seiberg-Witten theory on Kähler surfaces can also be found in [Sal99] and [Nic00].

2.1 Basic Constructions on Complex Manifolds

The definition of a complex manifold X of complex dimension n is analogous to that of a smooth manifold but in the complex category, that is, a Hausdorff paracompact topological space equipped with an atlas of complex charts $(\{U_\alpha\}, \varphi_\alpha)$, $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ such that the transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are biholomorphisms. Any complex manifold of dimension n is also a smooth manifold of real dimension $2n$. If $x^1, y^1, ..., x^n, y^n$ are real local coordinates for X, define

$$\frac{\partial}{\partial z^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}\right)$$
$$\frac{\partial}{\partial \overline{z}^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right)$$

these form a local basis for $TX \otimes \mathbb{C}$, the complexification of TX and give rise to a splitting

$$TX = T^{1,0}X \oplus T^{0,1}X$$

where locally elements of $T^{0,1}X$ are in the span of the $\partial/\partial z_i$ and elements of $T^{0,1}X$ are locally in the span of $\partial/\partial \overline{z}_i$. The space $T^{1,0}X$ is called the *holomorphic tangent bundle* and similarly $T^{0,1}X$ is the *anti-holomorphic tangent bundle*.

There is a natural endomorphism $J \in \mathrm{End}(TX \otimes \mathbb{C})$ satisfying $J^2 = -\mathbb{1}$ given by

$$J\left(\frac{\partial}{\partial z^j}\right) = i\frac{\partial}{\partial z^j}$$
$$J\left(\frac{\partial}{\partial \overline{z}^i}\right) = -i\frac{\partial}{\partial \overline{z}^j}$$

which induces a real endomorphism on TX by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$$
$$J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}.$$

The endomorphism J is called the *canonical complex structure* on a complex manifold. As a map on the complexified tangent bundle, J is simply obtained as the complexified version of the map on the real tangent bundle TX. Considering this concept in generality leads to the following definition.

Definition 2.1.1. Let X be a smooth manifold of even dimension, an *almost complex structure* is an endomorphism $J \in \mathrm{End}(TX)$ such that $J^2 = -\mathbb{1}_{TX}$.

Any almost complex structure extends to the complexification $TX \otimes \mathbb{C}$ and gives rise to a decomposition $TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$ where $T^{1,0}X$ and $T^{0,1}$ are the $+i$ and $-i$ eigenspaces of $TX \otimes \mathbb{C}$ respectively.

As seen above, any complex manifold induces an almost complex structure via complex coordinates. It is natural to ask the converse question, whether a smooth, even dimensional manifold X can be equipped with the structure of a complex manifold that induces a chosen almost complex structure $J \in \mathrm{End}(TX)$. This is the content of the Newlander-Nirenberg theorem which states that this occurs whenever the *Nijenhuis tensor* N_J vanishes, any such almost complex structure J is called *integrable* and the associated complex structure is in fact unique.

The splitting induced by an almost complex structure J also induces a splitting on the complex differential forms on X. Define

$$\Lambda^{p,q}T^*X := \Lambda^p(T^{1,0}X)^* \otimes_{\mathbb{C}} \Lambda^q(T^{0,1}X)^*$$

then there is a splitting

$$\Lambda^k T^*X \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}T^*X.$$

Suppose X is equipped with both a Riemannian metric g and an integrable complex structure J, we say they are *compatible* if $g(Jv, Jw) = g(v, w)$ for all $v, w \in TX$. When such occurs X is called a *Hermitian manifold.* This compatibility condition implies the existence of a non-degenerate 2-form given by $\omega(v, w) := g(Jv, w)$, the extension of this form to $TX \otimes \mathbb{C}$ via complex-linearity is a $(1, 1)$-form.

Denote the smooth sections of $\Lambda^k T^*X \otimes \mathbb{C}$ over X by $\Omega^k(X, \mathbb{C})$ and smooth sections of $\Lambda^{p,q}T^*X$ by $\Omega^{p,q}(X)$. The differential extends via $\mathbb{C}$-linearity to a map $d : \Omega^k(X, \mathbb{C}) \to \Omega^{k+1}(X, \mathbb{C})$, we define the following composition of maps

$$\partial : = \pi^{p+1,q} \circ d|_{\Omega^{p,q}(X)} : \Omega^{p,q}(X) \to \Omega^{p+1,q}$$
$$\overline{\partial} : = \pi^{p,q+1} \circ d|_{\Omega^{p,q}(X)} : \Omega^{p,q}(X) \to \Omega^{p,q+1}$$

one equivalent condition to an almost complex structure being integrable is that $d = \partial + \overline{\partial}$, or equivalently $\partial^2 = \overline{\partial}^2 = 0$. It follows that on a complex manifold for each p there is a *Dolbeault complex*

$$\Omega^{p,0}(X) \xrightarrow{\overline{\partial}} \Omega^{p,1}(X) \to \cdots \to \Omega^{p,n}(X)$$

and we define the Dolbeault cohomology groups by

$$H^{p,q}(X) := \frac{\ker(\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1})}{\mathrm{im}(\overline{\partial} : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X))}$$

the following isomorphism between these groups and sheaf cohomology of holomorphic p-forms is a consequence of the Dolbeault lemma and we have

$$H^q(X, \Omega^{p,0}(X)) \cong H^{p,q}(X).$$

2.2 Hodge Theory

Let (X, g) be a compact oriented Hermitian manifold of complex dimension n, then X is equipped with a real metric g on the tangent space which extends to a Hermitian inner product on the complexified space, denoted $\langle \cdot, \cdot \rangle$. We shall take the convention that a Hermitian form is to be complex antilinear in the first argument and linear in the second. Note that the decomposition of complex k-forms into (p, q)-forms is orthogonal with respect to $\langle \cdot, \cdot \rangle$.

Definition 2.2.1. The Hodge star operator can be extended $\mathbb{C}$-linearly to the set of complex valued differential forms, then it satisfies a complex version of its real definition, namely, the complex Hodge star operator is defined by

$$\overline{\alpha} \wedge \star\beta = \langle \alpha, \beta \rangle \,\mathrm{vol}.$$

The Hodge star restricts to a map $\Omega^{p,q} \to \Omega^{n-q,n-p}$ and allows us to define an inner product on $\Omega^*(X, \mathbb{C})$ by

Definition 2.2.2. Let (X, g) be a compact Hermitian manifold, then there is a Hermitian product on $\Omega^*(X, \mathbb{C})$ defined by

$$\langle \alpha, \beta \rangle_{L^2} := \int_X \langle \alpha, \beta \rangle \,\mathrm{vol}$$

Remark 2.2.3. For the above definition we may equivalently write

$$\begin{aligned} \langle \alpha, \beta \rangle_{L^2} &= \int_X \langle \alpha, \beta \rangle \star 1 \\ &= \int_X \overline{\alpha} \wedge \star\beta \end{aligned}$$

It is easily shown via the product rule that the operators d, ∂ and $\overline{\partial}$ have adjoints with respect to this inner product given by $d^* = - \star d \star, \partial^* = - \star \overline{\partial} \star$ and $\overline{\partial}^* = - \star \partial \star$ respectively. We define the Laplacian by

$$\Delta := d^* d + d d^*$$

and Δ_∂ and $\Delta_{\overline{\partial}}$ similarly by replacing d with ∂ or $\overline{\partial}$ respectively. A differential form is then *harmonic* if it lies in the kernel of Δ, when X is compact this is equivalent to lying in both the kernels of d and d^*. The space of real harmonic forms is denoted $\mathcal{H}^*(X, g)$ with the metric dependence dropped when a fixed metric is understood, we may similarly define the complex harmonic forms $\mathcal{H}^*(X, \mathbb{C})$ and ∂ or $\overline{\partial}$ harmonic forms $\mathcal{H}^*_\partial(X)$ and $\mathcal{H}^*_{\overline{\partial}}(X)$ respectively.

The linearity of Δ and the bidegree composition of differential forms gives rise to the following decomposition

Proposition 2.2.4. *Let (X, g) be a Hermitian manifold then*

$$\mathcal{H}^k_{\partial}(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\partial}(X)$$

and

$$\mathcal{H}^k_{\overline{\partial}}(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\overline{\partial}}(X)$$

importantly we obtain a decomposition of differential forms, namely the *Hodge decomposition.*

Theorem 2.2.5 (Hodge decomposition theorem)**.** *Let X be a compact Hermitian manifold, then there are the following orthogonal decompositions*

$$\Omega^{p,q}(X, \mathbb{C}) = \partial\Omega^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\partial}(X) \oplus \partial^*\Omega^{p+1,q}(X)$$

and

$$\Omega^{p,q}(X, \mathbb{C}) = \overline{\partial}\Omega^{p,q-1}(X) \oplus \mathcal{H}^{p,q}_{\overline{\partial}}(X) \oplus \overline{\partial}^*\Omega^{p,q+1}(X)$$

and the spaces $\mathcal{H}^{p,q}_{\partial}(X)$ and $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$ are finite dimensional.

Since an element of $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$ is necessarily $\overline{\partial}$ closed there is a canonical map to the Dolbeault cohomology groups $\mathcal{H}^{p,q}_{\overline{\partial}}(X) \to H^{p,q}(X)$. The Hodge decomposition of Theorem 2.2.5 gives that this is an isomorphism as a corollary.

Corollary 2.2.6. *Let X be a compact Hermitian manifold, the map $\mathcal{H}^{p,q}_{\overline{\partial}}(X) \to H^{p,q}(X)$ given by $\alpha \mapsto [\alpha]$ is an isomorphism.*

Remark 2.2.7. There is an analogous decomposition and isomorphism result to the above for the d-operator involving deRham cohomology as opposed to Dolbeault cohomology on a compact Riemannian manifold.

2.3 Holomorphic Vector Bundles

Suppose that (X, J) is an almost complex manifold and $E \longrightarrow X$ a complex vector bundle, then then the bidegree decomposition extends to E-valued forms on X as follows.

$$\Omega^k(X, E) = \bigoplus_{p+q=k} \Omega^{p,q}(X, E)$$

Definition 2.3.1. A *Cauchy-Riemann operator* on E is a linear map $D'' : C^\infty(X,E) \longrightarrow \Omega^{1,0}(X,E)$ which satisfies

$$D''(fs) = \overline{\partial} f \otimes s + f D'' s$$

$s \in C^\infty(X,E)$ and $f \in C^\infty(X,\mathbb{C})$.

This extends to an operator $D'' : \Omega^{p,q}(X,E) \longrightarrow \Omega^{p,q+1}(X,E)$ by the Leibniz rule

$$D''(\tau \otimes s) = \overline{\partial}\tau \otimes s + (-1)^{\deg(\tau)} \tau \wedge D'' s$$

where $s \in C^\infty(X,E)$ and $\tau \in \Omega^{p,q}(X,E)$.

A Cauchy-Riemann operator is closely tied to the notion of a Hermitian connection when E has a Hermitian structure. Suppose that $E \longrightarrow X$ is a Hermitian vector bundle and that d_B is a Hermitian connection, denote the complex linear and complex anti-linear parts, that is the projections onto the corresponding spaces in the bidegree decomposition by

$$\partial_B : C^\infty(X,E) \longrightarrow \Omega^{1,0}(X,E)$$
$$\overline{\partial}_B : C^\infty(X,E) \longrightarrow \Omega^{0,1}(X,E)$$

There are explicit formulae given by

$$\partial_B s := \frac{1}{2}\left(d_B s + i \mathrm{d}_B s \circ J\right)$$
$$\overline{\partial}_B s := \frac{1}{2}\left(d_B s - i \mathrm{d}_B s \circ J\right)$$

and these extend to operators on $\Omega^{p,q}(X,E)$ by the usual extension onto $\Omega^k(X,E)$, restricting to $\Omega^{p,q}(X,E)$ and then composing with the corresponding projections onto $(p+1,q)$ and $(p,q+1)$ forms respectively and obey the Leibniz rule since d_B does.

Proposition 2.3.2. *For every Cauchy-Riemann operator on E, there exists a unique Hermitian connection $B \in \mathcal{A}(P)$ such that $D'' = \overline{\partial}_B$*

Proposition 2.3.3. *For every Hermitian connection $B \in \mathcal{A}(P)$, the associated Cauchy-Riemann operator satisfies*

$$\overline{\partial}_B \overline{\partial}_B s = F_B^{0,2} s - \frac{1}{4}(\partial_B s) \circ N_J$$

when X is a complex manifold, $N_J = 0$ implies that $\overline{\partial}_B \circ \overline{\partial}_B = F_B^{0,2}$

Definition 2.3.4. Let X be a complex manifold. A holomorphic vector bundle on X is a complex vector bundle on X such that there exists a trivialising cover for which the transition functions are holomorphic.

If $E \to X$ is a holomorphic vector bundle on a complex manifold X equipped with a Hermitian structure, then there is a canonical Cauchy-Riemann operator. This is given by passing to a local trivialisation and taking the connection locally to be $\overline{\partial}$. Since the transition functions are holomorphic, they satisfy $\overline{\partial} g_{\alpha\beta} = 0$ so indeed it behaves appropriately on overlaps and gives a well-defined Cauchy-Riemann operator that satisfies $\overline{\partial}^2 = 0$. Comparing Proposition 2.3.3 and Proposition 2.3.2, there is a unique Hermitian connection B which induces this canonical operator $\overline{\partial}$ that must satisfy $F_B^{0,2} = 0$, it is called the *Chern connection.* It follows that curvature of B is a $(1,1)$-form which we call the *Chern curvature.*

In fact the condition that $F_B^{0,2} = 0$ is the precise condition required for E to be a holomorphic vector bundle, as per the following theorem of Newlander and Nirenberg.

Theorem 2.3.5 (Newlander-Nirenberg)**.** *Let (X, J) be a complex manifold and $E \longrightarrow X$ a Hermitian vector bundle with connection B. Then the Cauchy-Riemann operator $\overline{\partial}_B : \Omega^0(X, E) \longrightarrow \Omega^{1,0}(X, E)$ determines a Holomorphic structure on E if and only if $F_B^{0,2} = 0$.*

Observe that if X is a complex manifold and $E \longrightarrow X$ is a holomorphic line bundle then $\overline{\partial}$ induces the twisted Dolbeault complex

$$\Omega^{0,0}(X, E) \xrightarrow{\overline{\partial}} \Omega^{0,1}(X, E) \to \cdots \to \Omega^{0,n}(X, E).$$

the cohomology groups of the complex are in fact complex vector spaces and denoted by $H^{p,q}_{\overline{\partial}}(X, E)$. These cohomology groups coincide with the sheaf cohomology of holomorphic p-forms valued in E, i.e. the following isomorphism

$$H^q(X, \Omega^{p,0}(X, E)) \cong H^{p,q}_{\overline{\partial}}(X, E).$$

Fixing metrics on X and E we may obtain an L^2 inner product, obtain a Hodge star operator and define a Laplacian valued in E. Analogous to Section 2.2, the space of E-valued harmonic forms is defined to be the kernel of Δ_{∂}, the space of E-valued harmonic (p, q)-forms is denoted $\mathcal{H}^{p,q}(E)$, it is finite dimensional and the Dolbeault lemma implies the following isomorphism

$$\mathcal{H}^{p,q}(E) \cong H^{p,q}_{\overline{\partial}}(X, E)$$

The Hodge star operator also gives the following isomorphism.

Theorem 2.3.6. *Let $E \to X$ be a holomorphic vector bundle over an n-dimensional complex manifold X, then there is an isomorphism*

$$H^q(X, \Omega^{p,0}(X, E)) \cong H^{n-q}(X, \Omega^{n-p,0}(E^*))^*$$

in particular for $p = 0$ we have Serre duality

$$H^q(X, E) \cong H^{n-q}(X, K_X \otimes E^*)^*$$

where $K_X = \Lambda^n T^* X$ is the canonical bundle of X.

Denote $\dim_{\mathbb{C}}(H^{0,i}(X, E)) = \dim_{\mathbb{C}}(H^{i,0}(X, E))$ by $h^i(E)$ and define the *holomorphic Euler characteristic* to be the alternating sum

$$\chi(X, E) := \sum_{i=0}^{n} (-1)^i h^i(E)$$

which is an invariant of the holomorphic vector bundle E. In particular define $\rho_g := \dim_{\mathbb{C}}(H^{2,0}(X))$ to be the *geometric genus* of X, we then have the following lemma on cohomological restrictions of line bundles on compact complex surfaces.

Lemma 2.3.7. *Let L be a holomorphic line bundle on a compact complex surface X*

- *if $h^0(L) > 0$, then $h^2(L) \leq \rho_g$,*

- *if $h^2(L) > 0$, then $h^0(L) \leq \rho_g$.*

Proof. Suppose that $h^0(L) > 0$, then there must exist a holomorphic section s of L which is not identically zero, then it must be non-zero on a dense open subset of X. We claim that multiplication by s defines an injective map $H^0(X, K \otimes L^*) \to H^0(X, K)$, this holds since if t is a section $H^0(X, K \otimes L^*)$ and $st = 0$ then $t = 0$ on the dense subset of X for which s is non-zero. By continuity $t \equiv 0$ is zero everywhere and the map is injective. By Serre duality, $H^2(X, L)^* \cong H^0(X, K \otimes L^*)$ so $h^2(L)$ is the dimension of $H^0(X, K \otimes L^*)$. Hence, by injectivity of the above map $h^2(L) \leq \dim(H^0(X, K)) = \rho_g$.

The second result then follows from the first by replacing L with $K \otimes L^*$ and applying Serre duality. □

Remark 2.3.8. The holomorphic Euler characteristic can be recovered when X has an almost complex structure even though there is no canonical $\overline{\partial}$ operator. Given a connection B on E, we can consider the operators $\overline{\partial}_B$:

$\Omega^{0,k}(X,E) \longrightarrow \omega^{0,k+1}(X,E)$, even though the composition $\overline{\partial}_B \circ \overline{\partial}_B$ will generally be non-zero, one can wrap this up into a two-term sequence

$$\overline{\partial}_B + \overline{\partial}_B^* : \Omega^{0,\mathrm{ev}}(X,E) \longrightarrow \Omega^{0,\mathrm{odd}}(X,E)$$

in the integrable case, the index of this operator recovers the holomorphic euler characteristic, leading to the following definition.

Definition 2.3.9. Let (X,J) be an almost complex manifold and $L \longrightarrow X$ a holomorphic line bundle with connection B, the *holomorphic Euler Characteristic* is then defined to be the Fredholm index of the operator $\overline{\partial}_B + \overline{\partial}_B^*$, that is

$$\chi(X,L) := \mathrm{ind}_{\mathbb{C}}(\overline{\partial}_B + \overline{\partial}_B^*)$$

Remark 2.3.10. $\overline{\partial}_B + \overline{\partial}_B^*$ is in fact an example of a Dirac operator, and the Hirzebruch-Riemann-Roch theorem stated later in this section is a special case of the Atiyah-Singer Index Theorem which computes the index of Dirac operators in terms of topological invariants.

Definition 2.3.11. Let $E \longrightarrow X$ be a complex vector bundle,

Todd class of E: to be the integral cohomology class defined by

$$\mathrm{td}(E) := \prod_{j=1}^{m} \frac{x_j}{1 - e^{-x_j}}$$

Chern character of E: to be the integral cohomology class defined by

$$\mathrm{ch}(E) := \sum_{j=1}^{m} e^{x_j}$$

where we have employed the splitting principle and the x_j are the Chern roots of E.

In particular for 4-manifolds we have the following formulae

$$\mathrm{td}(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1(E)^2 + c_2(E))$$

and

$$\mathrm{ch}(E) = m + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E))$$

where m is the rank of the vector bundle E.

Theorem 2.3.12 (Hirzebruch-Riemann-Roch)**.** *Let $E \longrightarrow X$ be a complex vector bundle over an almost complex manifold (X, J), then the holomorphic Euler characteristic is given by*

$$\chi(X, E) = \int_X ch(E) \wedge td(TX)$$

where TX is viewed as a complex vector bundle via the almost complex structure J.

With this one can obtain a general computation of the holomorphic Euler characteristic on complex vector bundles over almost complex compact manifolds of dimension 4. If X is a compact oriented connected 4-manifold, then the cup product on cohomology and Poincaré duality gives rise to a non-degenerate symmetric bilinear form $Q : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}$ called the *intersection form* of X. This may be extended to a bilinear form on $H^2(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ and diagonalised such that the only entries along the diagonal are 0 and ± 1. The number of $+1$ and -1 entries defines $b^+(X)$ and $b^-(X)$ respectively and the difference $\sigma(X) := b^+(X) - b^-(X)$ is called the *signature* of X. All three of $b^{\pm}(X)$ and $\sigma(X)$ are topological invariants of the 4-manifold X. It is also of note that $b^{\pm}(X)$ can be computed as the dimension of the space of self-dual and anti-self dual harmonic 2-forms on X respectively.

Corollary 2.3.13. *The holomorphic Euler characteristic of a compact connected almost complex manifold (X, J) is given by*

$$\chi(X, \mathcal{O}) = \frac{1}{4}\sigma(X) + \frac{1}{4}\chi(X) = \frac{1 - b_1 + b^+}{2}$$

after twisting by a line bundle L, we have

$$\chi(X, L) = \frac{1}{8}\left\langle c_1(K^* \otimes L^2)^2, [X]\right\rangle - \frac{1}{8}\sigma(X)$$

where $K_X = det(T^*X) = \Lambda^{2,0}T^*X$ is the canonical bundle

2.4 Kähler Manifolds

Definition 2.4.1. Let X be a Hermitian manifold, we say that X is a *Kähler manifold* if the associated 2-form ω is closed, i.e.

$$d\omega = 0.$$

More generally, any smooth manifold of dimension $2n$ equipped with a non-degenerate closed 2-form ω is called a *symplectic manifold*. Thus a Kähler manifold is a manifold with mutually compatible Riemannian, complex and symplectic structures. Of particular interest are *Kähler surfaces*, these being the Kähler manifolds of complex dimension 2.

Proposition 2.4.2. *Let X be a symplectic 4-manifold and $L \longrightarrow X$ be a Hermitian line bundle. Let $B \in \mathcal{A}(L)$ be a Hermitian connection, then*

$$\overline{\partial}_B^* \overline{\partial}_B = \frac{1}{2} d_B^* d_B \varphi_0 - i(F_B)_\omega \varphi_0$$

for $\varphi_0 \in \Omega^{0,0}(X, E)$

Proposition 2.4.3. *Let X be a Kähler surface and $L \longrightarrow X$ be a holomorphic line bundle with a non-zero section $s : X \longrightarrow L$. Then either L is the trivial line bundle or*

$$[\omega] \cdot c_1(L) > 0$$

Since Kähler manifolds have a canonical 2-form ω there is the following *Lefschetz operator*

$$L : \Lambda^k X \longrightarrow \Lambda^{k+2} X$$

defined by

$$\alpha \mapsto \alpha \wedge \omega$$

with adjoint

$$L^* a : \Lambda^k X \longrightarrow \Lambda^{k-2} X$$

defined in terms of the Lefschetz operator and the Hodge star operator by

$$L^* := \star^{-1} \circ L \circ \star$$

with which one can prove the following *Kähler identities*

Proposition 2.4.4. *Let X be a complex manifold with Kähler metric g. Then the following identities hold, they are called the Kähler identities.*

i.)

$$[\overline{\partial}, L] = [\partial, L] = 0$$
$$[\overline{\partial}^*, L^*] = [\partial^*, L^*] = 0$$

ii.)

$$[\overline{\partial}^*, L] = i\partial, [\partial^* L] = -i\partial$$

and

$$[L^*, \overline{\partial}] = -i\partial^*, [L^*, \partial] = i\overline{\partial}^*$$

iii.) $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta$ *and* Δ *commutes with* $\star, L, L^*a, \partial, \overline{\partial}, \partial^*, \overline{\partial}^*$

importantly this implies that when X is a compact Kähler manifold, a (p,q)-form α is d-harmonic if and only if it is ∂-harmonic if and only if it is $\overline{\partial}$-harmonic. Hence one obtains the following orthogonal Hodge decomposition of cohomology on Kähler manifolds.

Theorem 2.4.5. *Let X be a compact Kähler manifold, the following decomposition holds.*

$$H^k(X;\mathbb{C}) \cong \bigoplus_{k=p+q} H^{p,q}(X).$$

2.5 Divisors

In this section we recall the theory of divisors on a complex manifold. These are related to holomorphic line bundles on a complex manifold and will turn out to be of relevance when solving the Seiberg-Witten equations on a Kähler surface.

We begin by introducing the notion of analytic and irreducible analytic hypersurfaces which mimic the concepts of algebraic sets and algebraic varieties from algebraic geometry.

Definition 2.5.1. Let X be a complex manifold and $V \subseteq X$ a closed subset. We say that V is a *analytic hypersurface*, if for every point $x \in V$, there exists a neighbourhood U of x and a non-zero local holomorphic function $f : U \longrightarrow \mathbb{C}$ such that $V \cap U = f^{-1}(0)$

The non-zero holomorphic function f, is given a decomposition of f into irreducibles of $\mathcal{O}_x$, the set of germs of holomorphic functions at $x \in X$. That is, $f = p_1^{a_1} \dots p_k^{a_k}$, since $f_0 = p_1 \dots p_k$ has the same zero set, f can always be taken of this form and divides any $f \in \mathcal{O}_x$ which vanishes near x. Any such minimal function f is called a *local defining function* for V at x.

This determines a non-zero principal ideal $\mathcal{I}_x$ defined as the set of germs of holomorphic functions which vanish on V, any local defining function is a generator for $\mathcal{I}_x$. We may define a map $m_V : X \longrightarrow \mathbb{N}$ given by

$$m_V(x) := \inf_{f \in \mathcal{I}_x} m_x(f)$$

where $m_x(f)$ is simply the vanishing order of f. This quantity is simply the vanishing order of any local defining function at x and is independent of choice of generator.

Definition 2.5.2. Let X be a complex manifold and $V \subseteq X$ an analytic hypersurface, we say that V is *irreducible* if whenever $V_1, V_2 \subseteq X$ are analytic hypersurfaces with $V_1 \cup V_2 = V$, either $V_1 = V$ or $V_2 = V$.

Irreducible analytic hypersurfaces on a complex surface can be characterised by the following property.

Proposition 2.5.3. *Let X be a complex surface, then $V \subseteq X$ is an irreducible analytic hypersurface if and only if there is a compact connected Riemann surface Σ and a holomorphic map ι with $\iota(\Sigma) = V$.*

We may now state the definition of a Weil divisor. This is a conceptually easy definition to work with, although it is not always suitable, leading to the notion of a Cartier divisor. We will see that on a complex manifold, Weil divisors can be viewed as Cartier divisors under an equivalence relation.

Definition 2.5.4. Let X be a complex surface, the group of *Weil divisors*, denoted $\mathrm{Div}(X)$ is the free abelian group of irreducible analytic hypersurfaces in X. Thus a *Weil divisor* is simply a finite formal sum

$$V = \sum_i m_i V_i$$

where $m_i \in \mathbb{Z}$.

If the $m_i \geq 0$ we say that the divisor V is *effective*.

Any Weil divisor defines a map $m : X \longrightarrow \mathbb{Z}$ called the *order* of the divisor, given by

$$m(x) := \sum_i m_i m_{V_i}(x)$$

and it is a non-trivial fact that any such Weil divisor is determined by such a map which can be written as $m(x) = m_x(f) - m_x(g)$ for holomorphic functions f, g defined in a neighbourhood U of x. Consequently, this can be taken as an alternate definition of a Weil divisor.

We now give the definition of a *Cartier divisor*, this is distinct from the notion of a Weil divisor, although closely related. It will prove to be the most useful definition for characterising line bundles.

Definition 2.5.5. Let X be a complex manifold, a *Cartier divisor* consists of an open cover $\{U_\alpha\}_{\alpha \in I}$ of X and a collection of non-trivial meromorphic functions (i.e. f_α / g_α where $f_\alpha, g_\alpha : U_\alpha \longrightarrow \mathbb{C}$ are holomorphic which do not vanish on any U_α), such that there exist nowhere vanishing holomorphic functions $u_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}$ such that

$$\frac{f_\beta}{g_\beta} = \frac{u_{\beta\alpha} f_\alpha}{g_\alpha}$$

on $U_\alpha \cap U_\beta$.

Two Cartier divisors $\{U_\alpha, f_\alpha, g_\alpha\}$ and $\{V_\beta, f'_\beta, g'_\beta\}$ are called *equivalent* if their union is also a divisor. This leaves everything unchanged when U_α and V_β don't intersect, but when the intersection is nonempty this means there exists a nowhere zero holomorphic function $v_{\alpha\beta} : U_\alpha \cap V_\beta \longrightarrow \mathbb{C}$ with $f'_\beta / g'_\beta = v_{\beta\alpha} f_\alpha / g_\alpha$. There is then a group isomorphism between Cartier divisors modulo equivalence and Weil divisors. The notion of effective divisor carries over under this correspondence to be $g_\alpha \equiv 1$ for all α.

Since we have non-zero holomorphic functions $u_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$ and it is easy to see they satisfy the cocycle conditions

$$\begin{aligned} u_{\alpha\alpha} &= 1 \\ u_{\alpha\beta} &= u_{\beta\alpha}^{-1} \\ u_{\alpha\beta} u_{\beta\gamma} &= u_{\alpha\gamma} \end{aligned}$$

hence any Cartier divisor $E = \{U_\alpha, f_\alpha, g_\alpha\}$ necessarily defines a holomorphic line bundle $\mathcal{O}(E)$ (sometimes we simply refer to the line bundle as E for simplicity), where equivalent divisors determine isomorphic line bundles. The total space of this line bundle is given by equivalence classes

$$[x, z, \alpha] \equiv [x, u_{\beta\alpha}(x) z, \beta]$$

where $x \in U_{\alpha\beta}$ and $z \in \mathbb{C}$. There is then a natural group homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ into the Picard group, the group of isomorphism classes of holomorphic line bundles on X given by $E \mapsto \mathcal{O}(E)$. This induces a map $\mathrm{Div}(X) \to H^2(X; \mathbb{Z})$ given by

$$E \mapsto c_1(\mathcal{O}(E)).$$

If a divisor is written in terms of irreducible analytic hypersurfaces V_i as $E = \sum_i m_i V_i$, since the V_i can be written as the image of Riemann surfaces under holomorphic maps, they correspond to submanifolds of X, hence have associated fundamental cycles $[V_i]$. The first Chern class of E is obtained as

$$\sum_i m_i \mathrm{PD}([V_i])$$

where PD refers to Poincaré duality.

A holomorphic section $s : X \to E$ constitutes a collection of holomorphic maps $v_\alpha : U_\alpha \to \mathbb{C}$ satisfying $v_\beta = u_{\beta\alpha} v_\alpha$. Any such section of E determines a meromorphic function $v : X \to \mathbb{C}$ given by

$$v(x) = \frac{v_\alpha(x) g_\alpha(x)}{f_\alpha(x)}$$

for $x \in U_\alpha$, the right hand is well-defined on $U_{\alpha\beta}$ and is only defined when $f_\alpha(x) \neq 0$. The meromorphic function has a multiplicity function which satisfies

$$m(x) + m_v(x) \geq 0$$

for all $x \in X$. Conversely, any meromorphic function $v : X \to \mathbb{C}$ that satisfies the above defines a holomorphic section of E, hence there is a correspondence

$$H^0(X, E) \cong \{v : X \to \mathbb{C} : v \text{ is meromorphic}, \ m_v + m \geq 0\}.$$

Observe that a divisor E is effective if and only if it admits a meromorphic section v with $m_v(x) = 0$, in particular this implies that v is in fact a non-zero holomorphic section s, consequently we obtain the following equivalence

$$\mathrm{Div}^{\mathrm{eff}}(X) \cong \frac{E \to X \text{ is a hol. line bundle with hol. section } s \neq 0}{\text{isomorphism of line bundles}}$$

Note that the first Chern class gives a map $\mathrm{Pic}(X) \to H^2(X; \mathbb{Z})$, if we assume that $b_1(X) = 0$ then it is injective and its image is the set of classes in $H^2(X; \mathbb{Z})$ whose image in $H^2(X; \mathbb{R}))$ lies in $H^{1,1}(X)$. It follows from this fact and the above that for complex manifolds with $b_1(X) = 0$, that the set of effective divisors with specified first Chern class c can be described in terms of a fixed complex line bundle $E \to X$ as follows

$$\mathrm{Div}^{\mathrm{eff}}(X, c) \cong \frac{\{(\overline{\partial}, s) : \overline{\partial} \circ \overline{\partial} = 0, \overline{\partial} s = 0, s \neq 0\}}{(\overline{\partial}, s) \equiv u^*(\overline{\partial}, s) = (u^{-1} \circ \overline{\partial} \circ u, u^{-1} s)}$$

that is, the set of holomorphic structures on E with a non-zero holomorphic section s modulo gauge equivalence.

2.6 Blowup of a Manifold

In this section we describe the blowup of a complex manifold, this will provide an example of a family manifolds which is investigated later in Chapter 7.

We shall develop the basic theory of blowups and then proceed to overview some tools to compute the cohomology of holomorphic line bundles on the blowup of a complex manifold, these will be used later for the computation of the families Seiberg-Witten invariant.

Let Δ be a disc in $\mathbb{C}^2$ centred at the origin and define $\widetilde{\Delta} \subset \Delta \times \mathbb{CP}^1$ by

$$\widetilde{\Delta} := \{(z_1, z_2) \times [y_1, y_2] \in \Delta \times \mathbb{CP}^1 : z_1 y_2 = z_2 y_1\}$$

and $p : \widetilde{\Delta} \longrightarrow \Delta$ to be the projection onto the first factor $p(z, [y]) = z$. Note that $z_1 y_2 = z_2 y_1$ if and only if (z_1, z_2) lies in the span of (y_1, y_2), thus $p^{-1}(\{0\})$ can be identified with a copy of all lines through the origin, i.e. a copy of $\mathbb{CP}^1$, this is denoted $E = \{(0,0)\} \times \mathbb{CP}^1$ and called the *exceptional divisor*. Away from 0 there is a unique point in $\widetilde{\Delta}$ lying over (z_1, z_2), namely $(z_1, z_2) \times [z_1, z_2]$ hence $p|_{\widetilde{\Delta} \setminus E} : \widetilde{\Delta} \setminus E \longrightarrow \Delta \setminus \{0\}$ is a bijection and in fact a biholomorphism.

Proposition 2.6.1. *The exceptional divisor E is an effective Cartier divisor.*

Proof. Let $U_i := \{(z, [y]) \in \widetilde{\Delta} : y_i \neq 0\}$, these form an open cover of $\widetilde{\Delta}$. Define the holomorphic functions $f_i : U_i \longrightarrow \mathbb{C}$ by

$$f_i(z, [y]) := z_i$$

and let $g_i : U_i \longrightarrow \mathbb{C}$ be the constant function equal to 1 on U_i. The f_i vanish on $E \cap U_i$, which is not open and the g_i do not vanish anywhere, moreover on $U_i \cap U_j$ the condition $z_i y_j = z_j y_i$ implies that z_i and z_j are non-zero, so we have the existence of nowhere vanishing holomorphic functions $u_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}$ defined by

$$u_{ij} := \frac{z_i}{z_j}$$

satisfying

$$f_i = u_{ij} f_j$$

hence E is a Cartier divisor, since the g_i are all identically equal to 1, it is an effective divisor. □

Since E is a Cartier divisor, there is a corresponding line bundle on $\widetilde{\Delta}$ which is trivial away from E. However, restricted to E the open sets U_i and transition functions u_{ij} can be identified precisely with the standard trivialisation for the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^1$, hence

$$\mathcal{O}(E)|_E \cong \mathcal{O}(-1).$$

We can now transport this construction to a complex surface X. Suppose $x \in X$, $U \subseteq X$ is an open neighbourhood and $\varphi : U \longrightarrow \Delta$ is a biholomorphism sending x to 0, thus defining a holomorphic coordinate chart centred at x. Define $p' : \widetilde{U} \longrightarrow U$ to be the pullback of $p : \widetilde{\Delta} \longrightarrow \Delta$ under φ, that is, $\widetilde{U} = \varphi^* \widetilde{\Delta}$, and define the manifold

$$\widetilde{X}(\varphi) := \frac{X \setminus \{x\} \sqcup \widetilde{U}}{\sim}$$

where $X \setminus \{x\} \ni x' \sim u \in \widetilde{U}$ if $p'(u) = x'$, to be the blowup of X at x.

There is an obvious projection map $p : \widetilde{X} \longrightarrow X$ sending $x' \mapsto x'$ if $x' \in X \setminus \{x\}$ and $u \mapsto p'(x')$ if $x' \in \widetilde{U}$. As with the local case, the restriction $p|_{p^{-1}(X \setminus \{x\})} : \widetilde{X} \setminus E \longrightarrow X \setminus \{x\}$ is a biholomorphism and E is a divisor which we call the exceptional divisor. This all follows immediately from the local results.

Proposition 2.6.2. *Let $\varphi, \varphi' : U, U' \longrightarrow \Delta$ be two coordinate biholomorphisms centered at x, then $\widetilde{X}(\varphi, x)$ and $\widetilde{X}(\varphi', x)$ are biholomorphic.*

Proof. It suffices to show that $\widetilde{U}$ and $\widetilde{U}'$ are biholomorphic. Given φ we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\ h\ } & \widetilde{\Delta} \\ \Big\downarrow {\scriptstyle p'} & & \Big\downarrow {\scriptstyle p} \\ U & \xrightarrow{\ \varphi\ } & \Delta \end{array}$$

note that $\widetilde{U}$, by nature of being a pullback is defined set theoretically as

$$\widetilde{U} = \varphi^* \Delta = \{(x_0, y_0) \in U \times \widetilde{\Delta} : \varphi(x_0) = p(y_0)\}$$

and the map h simply sends (x_0, y_0) to y_0. This is clearly holomorphic. We claim that this is in fact a biholomorphism.

Define $g : \widetilde{\Delta} \longrightarrow \widetilde{U}$ by

$$g(y) := (\varphi^{-1}(p(y)), y)$$

since p and φ are holomorphic, this is as well. This indeed actually maps into $\widetilde{U}$ since $\varphi \circ \pi_1(g(y)) = \varphi \circ \varphi^{-1} \circ p(y) = p(y) = p \circ \pi_2(g(y))$.

Now observe that

$$\begin{aligned} g \circ h(x_0, y_0) &= g(y_0) \\ &= (\varphi^{-1}(p(y_0)), y_0) \\ &= (\varphi^{-1}(\varphi(x_0)), y_0) \\ &= (x_0, y_0) \\ &= \mathrm{id}_{\widetilde{U}}(x_0, y_0) \end{aligned}$$

and

$$h \circ g(y) = h(\varphi^{-1}(p(y)), y) = y = \mathrm{id}_{\widetilde{\Delta}}(y).$$

Hence, given φ and φ', we have the following commutative diagram

$$\begin{array}{ccccc} \widetilde{U} & \xrightarrow{h} & \widetilde{\Delta} & \xleftarrow{h'} & \widetilde{U}' \\ \Big\downarrow{\scriptstyle p'} & & \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p''} \\ U & \xrightarrow{\varphi} & \Delta & \xleftarrow{\varphi'} & U' \end{array}$$

since h and h' are biholomorphisms, we have $\widetilde{U} \cong \widetilde{U}'$ with $p'' \circ h'^{-1} \circ h = \varphi'^{-1} \circ \varphi \circ p$. □

The blowup at $x \in X$ can be understood purely topologically in terms of connected sums

Proposition 2.6.3. *Let X be a complex surface, then $Bl_x(X)$ diffeomorphic to $X\#\overline{\mathbb{CP}}^2$ via an oriented diffeomorphism.*

Proof. See [Huy05][p.102] □

This categorisation of the blowup then yields a simple computation of some topological properties of the blowup

Proposition 2.6.4. *Let X be a compact complex surface and $\widetilde{X} = Bl_x(X)$ with exceptional divisor E, then the following hold*

(1) $\pi_1(\widetilde{X}) \cong \pi_1(X)$,

(2) $b_1(\widetilde{X}) = b_1(X)$,

(3) $b_+^2(\widetilde{X}) = b_+^2(X)$,

(4) $b^2_-(\widetilde{X}) = b^2_-(X) + 1$,

(5) if X *is compact so that we have Poincaré duality and* $[E]$ *is the Poincaré dual of the exceptional divisor, then* $[E]^2 = -1$.

Proof. Since $\widetilde{X} \cong X \# \overline{\mathbb{CP}}^2$ the intersection of X and $\mathbb{CP}^2$ in $X \# \overline{\mathbb{CP}^2}$ is a disc and $\pi_1(\mathbb{CP}^2)$ is trivial, it follows from the Seifert van-Kampen theorem that $\pi_1(\widetilde{X}) \cong \pi_1(\widetilde{X})$, this immediately implies (2).

It also follows from Proposition 2.6.3 that the intersection form of $\widetilde{X}$ decomposes as a direct sum

$$Q_{\widetilde{X}} = Q_X \oplus Q_{\overline{\mathbb{CP}}^2}.$$

Since $b^+(\overline{\mathbb{CP}}^2) = 0$ and $b^-(\overline{\mathbb{CP}}^2) = 1$, both (3) and (4) then follow, and since E satisfies the same property in $\overline{\mathbb{CP}}^2$ (5) follows. □

Proposition 2.6.5. *Let* X *be a compact complex surface, then*

$$Div(\widetilde{X}) = \pi^*(Div(X)) \oplus \mathbb{Z}\{E\}$$

consequently we have isomorphisms

$$Pic(\widetilde{X}) \cong Pic(X) \oplus \mathbb{Z}$$

given by

$$Pic(X) \oplus \mathbb{Z} \ni (L, k) \mapsto p^*L \otimes \mathcal{O}(kE)$$

and

$$H^2(\widetilde{X}; \mathbb{Z}) \cong H^2(X; \mathbb{Z}) \oplus \mathbb{Z}$$

given by

$$H^2(X; \mathbb{Z}) \oplus \mathbb{Z} \ni (n, m) \mapsto p^*(n) + m[E]$$

where $[E]$ *is the Poincaré dual of* E.

Proof. The last two assertions follow from the first. The first is easy to see since if Σ is any irreducible analytic hypersurface on X not containing E, then it is simply the proper transform of its image $p(\Sigma)$. Hence the irreducible analytic hypersurfaces of $\widetilde{X}$ are just those of X as well as the exceptional divisor. Since the group of divisors is just the free abelian group on this, our isomorphism follows. □

From this result, we shall sometimes denote line bundles on the blowup $\widetilde{X}$ as $L + kE$ where L is a line bundle on X and $k \in \mathbb{Z}$.

A more general construction will also later be required, namely the blowup along a submanifold of codimension k. The construction is quite similar to the blowup around a point. As before let be a disc in $\mathbb{C}^n$ and V be defined by $z_{k+1} = \cdots = z_n = 0$ and define $\widetilde{\Delta} := \{(z, [y]) \in \Delta \times \mathbb{CP}^{n-1}\} : z_i y_j = z_j y_i, k+1 \leq i, j \leq n$ and the projection map $\pi : \widetilde{\Delta} \longrightarrow \Delta$ by $\pi(z, [y]) = z$. Now given a complex manifold X of complex dimension n and a submanifold Y of codimension k, then we may choose local coordinates (U_i, φ_i) with $\varphi : U_i \longrightarrow \Delta_i$ where the Δ_i are discs in $\mathbb{C}^n$ on X such that $U_i \cap Y$ is given by the zero set $(\varphi(x)_{k+1}, \ldots \varphi(x)_n)$. Consider the pullback bundles given by $\varphi^* \widetilde{\Delta}_i$, there are isomorphisms on overlaps, hence the union $\cup_i \varphi^* \widetilde{\Delta}_i / \sim$, with the equivalence provided by isomorphism on overlaps, defines a manifold. There is a natural projection onto X which is gives a biholomorphism $X \setminus \cup_i U_i$ and we may glue together $X \setminus Y$ and $\cup_i \widetilde{\Delta}_i$ to obtain the blowup of X along Y, denoted $\mathrm{Bl}_Y(X)$

In the interest of computing the families Seiberg-Witten invariant of a blowup family in 7 we aim to develop tools for the computation of the cohomology of holomorphic line bundles on the blowup of X, specifically an exact sequence. Let I_x be the ideal sheaf of $x \in X$, that is, the sheaf of holomorphic functions on X vanishing at x, note that for open sets not containing x this agrees with $\mathcal{O}(X)$ where $\mathcal{O}$ is the structure sheaf of X. Given $k \geq 0$, I_x^k is then the sheaf of holomorphic functions on X vanishing at x to order at least k, also let $\widetilde{\mathcal{O}}_x$ be the skyscraper sheaf of holomorphic functions on X at x. There is then the following lemma.

Lemma 2.6.6. *Let $m \geq 0$, then*

$$\pi_* \mathcal{O}_{\widetilde{X}}(-kE) \cong I_x^k$$

and

$$R^j \pi_* \mathcal{O}_{\widetilde{X}}(-kE) = 0$$

for $j > 0$

Proof. Clearly $\pi_* \mathcal{O}_{\widetilde{X}} \subset \mathcal{O}_X$ and since any holomorphic function on $X \setminus \{x\}$ correpsonds to one on $\widetilde{X} \setminus E$ which extends uniquely to all of X due to Levi's extension theorem [Bar+04, Theorem 8.7] The exceptional divisor E then gives an exact sequence ([GH94] p.139)

$$0 \to \mathcal{O}_{\widetilde{X}}(-E) \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_E \to 0.$$

Note that the higher direct image sheaves satisfy $R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$ [Bar+04, Theorem 9.1], the induced long exact sequence of higher direct image sheaves gives the following exact sequence

$$0 \to \pi_*\mathcal{O}_{\widetilde{X}}(-E) \to \mathcal{O}_X \xrightarrow{\mathrm{ev}_x} \widetilde{\mathcal{O}}_x \to R^1\pi_*\mathcal{O}_{\widetilde{X}}(-E) \to 0$$

and $R^j p_*\mathcal{O}_{\widetilde{X}}(-E) = 0$ for $j > 1$. Since the evaluation map $\mathrm{ev}_x : \mathcal{O}_X \to \widetilde{\mathcal{O}}_x$ is surjective, $R^1\pi_*\mathcal{O}_{\widetilde{X}}(-E) = 0$, this reduces the above exact sequence. There also a map $I_x \to \pi_*\mathcal{O}_{\widetilde{X}}(-E)$, this and the structure sequence of X gives the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_*\mathcal{O}_{\widetilde{X}}(-E) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \widetilde{\mathcal{O}}_x & \longrightarrow & 0 \\
 & & \uparrow & & \Vert & & \Vert & & \\
0 & \longrightarrow & I_x & \hookrightarrow & \mathcal{O}_X & \longrightarrow & \widetilde{\mathcal{O}}_x & \longrightarrow & 0
\end{array}$$

since the rightmost two arrows are isomorphisms, it follows that $\pi_*\mathcal{O}_{\widetilde{X}}(-E) \cong I_x$.

Now proceed inductively on k, again the exceptional divisor gives an exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-kE) \to \mathcal{O}_{\widetilde{X}}(-(k-1)E) \to \mathcal{O}_E(-(k-1)E) \to 0$$

the long exact sequence of higher direct image sheaves and the inductive hypothesis gives the following long exact sequence

$$0 \to \pi_*\mathcal{O}_{\widetilde{X}}(-kE) \to \pi_*\mathcal{O}_{\widetilde{X}}(-(k-1)E) \to \pi_*\mathcal{O}_E(-(k-1)E) \to R^1\pi_*\mathcal{O}_{\widetilde{X}}(-kE) \to 0.$$

and that

$$R^j\pi_*(-(k-1)E) \cong R^{j+1}\pi_*\mathcal{O}_{\widetilde{X}}(-kE)$$

for $j > 0$. Since $\mathcal{O}(E)|_E \cong \mathcal{O}(-1)$ over $\mathbb{CP}^1$, it follows that $\mathcal{O}(-(k-1)E) \cong \widetilde{\mathcal{O}}_x(S^k(T_x^*X))$ and $R^j\pi_*\mathcal{O}_E(-(k-1)E) = 0$ for $j > 0$.

Consequently $R^j\pi_*\mathcal{O}_{\widetilde{X}}(-kE) = 0$ for $j > 1$ and the above exact sequence is

$$0 \to \pi_*\mathcal{O}_{\widetilde{X}}(-kE) \to \pi_*\mathcal{O}_{\widetilde{X}}(-(k-1)E) \to \widetilde{\mathcal{O}}_x(S^kT_x^*X) \to R^1\pi_*\mathcal{O}_{\widetilde{X}}(-kE) \to 0$$

Since an element of $p \in S^k(T_x^*X)$ can be lifted to a degree k polynomial $\widetilde{p}$ in local coordinates around x, we obtain a locally defined holomoprhic function which vanishes to order $k-1$ at x, defining a local section of $\pi_*\mathcal{O}_{\widetilde{X}}(-(k-1)E)$ which evaluates to p at x. Hence the map $\pi_*\mathcal{O}_{\widetilde{X}}(-(k-1)E) \to \widetilde{\mathcal{O}}_x(S^kT_x^*X)$

is surjective so $R^1\pi_*\mathcal{O}_{\widetilde{X}}(-kE) = 0$, hence we have proven the second claim. This also reduces the exact sequence to

$$0 \longrightarrow \pi_*\mathcal{O}_{\widetilde{X}}(-kE) \to \pi_*\mathcal{O}_{\widetilde{X}}(-(k-1)E) \to \widetilde{\mathcal{O}}_x(S^kT_x^*X) \to 0.$$

A section of I_x^k can be regarded as a map which vanishes at x to order at least k, there are natural maps $I_x^k \longrightarrow \pi_*\mathcal{O}_{\widetilde{X}}(-kE)$ making the following diagram commute

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_x^k & \longrightarrow & I_x^{k-1} & \longrightarrow & \widetilde{\mathcal{O}}_x(S^kT_x^*X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_*\mathcal{O}_{\widetilde{X}}(-kE) & \longrightarrow & \pi_*\mathcal{O}_{\widetilde{X}}(-(k-1)E) & \longrightarrow & \widetilde{\mathcal{O}}_x(S^kT_x^*X) & \longrightarrow & 0
\end{array}$$

where the rows are exact. The rightmost map is the identity and the middle is an isomorphism by the inductive hypothesis. Consequently, the leftmost vertical map is an isomorphism and

$$\pi_*\mathcal{O}_{\widetilde{X}}(-kE) \cong I_x^k$$

so we obtain the first claim by induction. □

Theorem 2.6.7. *Let L be a holomorphic line bundle on X. For each $k \geq 0$, the following holds.*

(1) There is a long exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(\widetilde{X}, L-kE) & \longrightarrow & H^0(X,L) & \xrightarrow{ev_x} & L_x \otimes (\mathcal{O}_X/I_x^k) & \\
 & H^1(\widetilde{X}, L-kE) & \longrightarrow & H^1(X,L) & \longrightarrow & 0 &
\end{array} \tag{2.1}$$

(2) There is an isomorphism $H^2(\widetilde{X}, L-kE) \cong H^2(X,L)$

(3) For $m \geq 1$, there is a long exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow \to & H^1(X,L) & \longrightarrow & H^1(\widetilde{X}, L+kE) & \longrightarrow & L_x \otimes (K_X^*)_x \otimes (\mathcal{O}_X/I_x^k)^* \\
 & H^2(X,L) & \longrightarrow & H^2(\widetilde{X}, L+kE) & \longrightarrow & 0
\end{array} \tag{2.2}$$

(4) and an isomorphism (for $m \geq 1$)

$$H^0(\widetilde{X}, L + kE) \cong H^0(X, L)$$

Proof. Since L is locally trivial, it corresponds to a locally free sheaf, hence via the projection formula (see [Dim04][Theorem 2.3.29]), it follows that

$$R^q\pi_*\mathcal{O}_{\widetilde{X}}(-kE) \otimes \mathcal{O}_X(L) \cong R^q\pi_*(\mathcal{O}_{\widetilde{X}}(-kE) \otimes \mathcal{O}_{\widetilde{X}}(L)) = R^q\pi_*\mathcal{O}_{\widetilde{X}}(L - kE)$$

Consequently, from Lemma 2.6.6 we have that

$$R^q\pi_*\mathcal{O}_{\widetilde{X}}(L - kE) = 0$$

for $q > 0$, and

$$\pi_*\mathcal{O}_{\widetilde{X}}(L - kE) = \mathcal{O}_X(L) \otimes_{\mathcal{O}_X} I_x^k$$

in the case when $q = 0$.

The E_2 page of the Leray spectral sequence for $\mathcal{O}_{\widetilde{X}}(L - kE)$ under the map $\pi : \widetilde{X} \longrightarrow X$ is given by $E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{O}_{\widetilde{X}}(L-kE))$, this converges to $H^{p+q}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(L - kE)) \equiv H^{p+q}(\widetilde{X}, L - kE)$.

However, for any p, $E_2^{p,q}$ is zero for $q > 0$ by the above computation, hence the spectral sequence degenerates at the E_2 level and so

$$H^j(\widetilde{X}, L - kE) \cong H^j(X, \pi_*\mathcal{O}_{\widetilde{X}}(L - kE)) \cong H^j(X, \mathcal{O}_X(L) \otimes_{\mathcal{O}_X} I_x^k).$$

Since $\mathcal{O}_X(L)$ is a locally free sheaf, by tensoring the exact sequence defining the quotient $\mathcal{O}_X/I_x^k$ the following sequence is exact.

$$0 \to \mathcal{O}_X(L) \otimes_{\mathcal{O}_X} I_x^k \to \mathcal{O}_X(L) \to L_x \otimes (\mathcal{O}_X/I_x^k) \to 0.$$

Taking the induced long exact sequence in cohomology and using the fact that $\mathcal{O}_X/I_x^k$ is a skyscraper sheaf, thus it has no cohomology in degrees bigger than zero, (1) and (2) immediately follow. If we then consider (1) and (2) for line bundles over $\widetilde{X}$ of the form $L - K_X - (k-1)E$ when $k \geq 1$, then (3.) and (4.) follow from applying Serre duality, the fact that $K_{\widetilde{X}} = \pi^*(K_X) + [E]$ and dualising the resulting exact sequence and isomorphism respectively. □

Chapter 3

Spin^c Structures and Dirac Operators

In contrast to Yang-Mills theory, the Seiberg-Witten equations have spinors as one of their unknowns. To define these objects on a manifold it is integral to discuss spin^c structures on a manifold, this naturally leads to a discussion on Dirac operators, these also appear in the Seiberg-Witten equations and are differential operators that act on spinors. Hence this chapter shall the collect necessary facts on spin^c structures and some of the analytic facts on Dirac operators for use in Seiberg-Witten theory. Although when discussing ordinary Seiberg-Witten theory one only needs to discuss spin^c structures on the tangent bundle, we shall discuss the theory for general vector bundles since it allows an easy transportation of results to the families case. The results here are all standard and discussions relevant to Seiberg-Witten theory can be found in texts such as [Nic00] and [Sal99]. A detailed account of the closely related theory of spin structures can be found in [LM89].

3.1 Spin^c Structures

3.1.1 Clifford Algebras and Spin$^c(n)$

Closely related to the notion of a spin^c structure are the groups Spin$^c(n)$. Of particular application to 4-manifolds is Spin$^c(4)$, due to an exceptional isomorphism this is simply

$$\mathrm{Spin}^c(4) \cong \frac{SU(2) \times SU(2) \times U(1)}{\mathbb{Z}_2}$$

where the equivalence relation is given by $(A, B, C) \sim (-A, -B, -C)$ for $(A, B) \in SU(2) \times SU(2), C \in U(1)$. However, in this section we shall take the approach of defining Spin$^c(n)$ as a subgroup of the complexified Clifford algebra, since the development of the machinery surrounding Clifford algebras is required for the definition of a spin^c structure and makes many of the relevant properties to spin^c structures apparent.

Definition 3.1.1. Let V be an n-dimensional real inner product space and $e_1, ..., e_n$ an orthonormal basis for V, the *Clifford algebra* C(V) is the associative algebra with unit 1 over R generated by $e_1, ..., e_n$ satisfying

$$vw + wv = -2 \langle v, w \rangle$$

for all $v, w \in C(V)$.

This can be viewed as a real 2^n dimensional vector space, a basis is given by elements of the form

$$e_0 = 1, \qquad e_I = e_{i_1}...e_{i_k}$$

where $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ with $i_1 < \cdots < i_k$ is any multi-index, viewing e_0 as corresponding to $I = \emptyset$ we may then write $x = \sum_I x_I e_I$ for any $x \in C(V)$. The algebra structure is then obtained from the multiplication rules

$$e_i^2 = -1, \qquad e_i e_j = -e_j e_i$$

and $C(V)$ becomes a graded algebra with $C_0(V) = \mathbb{R}$ containing elements $x_0 \in \mathbb{R}$, and $C_k(V)$ containing elements of the form $x_I e_I$ with $|I| = k$, producing a splitting of vector spaces into even and odd degree elements, denoted by $C(V) = C^{\text{ev}}(V) \oplus C^{\text{odd}}(V)$.

All of this machinery can be complexified, giving rise to the following.

Definition 3.1.2. Let V be an n-dimensional real inner product space, the *complexified Clifford algebra* is simply the tensor product of algebras

$$C^c(V) := C(V) \otimes_{\mathbb{R}} \mathbb{C}.$$

This has elements of the form $x = \sum_I x_I e_I$, $x_I \in \mathbb{C}$, an involution and Hermitian inner product are defined by

$$\tilde{x} = \sum_I (-1)^{|I|(|I|+1)/2} \overline{x}_I e_I$$

and

$$\langle x, y \rangle = \sum_I \widetilde{x_I} y_I$$

respectively, both satisfying

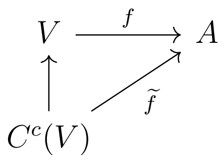
$$\widetilde{xy} = \widetilde{y}\widetilde{x}, \qquad ((\widetilde{x}y)_0 = \langle x, y \rangle \, .$$

where $(\widetilde{x}y)_0$ denotes the degree 0 part of $\widetilde{x}y$. There is also the following universal property satisfied by the complexified Clifford algebra.

Proposition 3.1.3. *Let V be an n-dimensional real inner product space and A a finite dimensional associative $\mathbb{C}$-algebra with unit $\mathbb{1}$ and involution $a \mapsto a^*$. If $f : V \longrightarrow A$ is an $\mathbb{R}$-linear map satisfying*

$$f(v)^* + f(v) = 0, \qquad f(v)^* f(v) = |v|^2 \mathbb{1} \tag{3.1}$$

then there is a unique $\mathbb{C}$-algebra homomorphism $\widetilde{f} : C^c(V) \longrightarrow A$ extending f as in the following commutative diagram



where $C^c(V) \to V$ is the projection $C(V) \to C_1(V) \cong V$.

We say such a map f as above satisfying 3.1 is a *Clifford map* and satisfies the *Clifford properties.* Furthermore, it is important to note that if n is even, the extension $\widetilde{f} : C(V) \longrightarrow A$ is injective.

3.1.2 Spin^c Structures on Vector Spaces

The spin^c structures we consider in the following section are simply vector bundle generalisations of spin^c structures on vector spaces. Therefore, it is useful to collect some of the facts relevant to the simpler cases vector spaces as done in this section.

Definition 3.1.4. Let V be a real inner product space of dimension $2n$, a (vector space) *spin^c structure* consists of a pair (W, Γ) where W is a 2^n-dimensional complex Hermitian vector space and $\Gamma : V \longrightarrow \mathrm{End}(W)$ is a linear map satisfying the *Clifford properties*

$$\Gamma(v)^* + \Gamma(v) = 0, \qquad \Gamma(v)^* \Gamma(v) = |v|^2 \mathbb{1}_W \tag{3.2}$$

We denote the space of spin^c isomorphisms by $\mathrm{Hom}^{\mathrm{spin}^c}(W_0, W_1)$

The Clifford condition implies that Γ extends uniquely to an algebra isomorphism $\Gamma : C^c(V) \longrightarrow \mathrm{End}(W)$ by Proposition 3.1.3 since V is even dimensional.

Consequently, we can embed the space of 2-forms on V in $\mathrm{End}(W)$ as follows. There is an algebra isomorphism $\Lambda^2 V^* \to C_2(V)$, where if $e_1, \ldots e_{2n}$ is an orthonormal basis of V, it is given by identifying basis elements $e_i^* \wedge e_j^*$ with $e_i e_j \in C_2(V)$, composing this with the extension of Γ to the Clifford algebra gives the map $\rho : \Lambda^2 V^* \longrightarrow \mathrm{End}(W)$

$$\rho\left(\sum_{i<j} \eta_{ij} e_i^* \wedge e_j^*\right) := \sum_{i<j} \eta_{ij} \Gamma(e_i)\Gamma(e_j) \tag{3.3}$$

the map ρ is independent of the choice of orthonormal basis and extends to complex valued 2-forms, giving a map $\rho : \Lambda^2 V^* \otimes \mathbb{C} \longrightarrow \mathrm{End}(W)$, it satisfies the following universal property

$$\begin{array}{ccc} \Lambda^2(V^*) \otimes \mathbb{C} & \xrightarrow{\ \rho\ } & \mathrm{End}(W) \\ \uparrow & \nearrow_{\Gamma} & \\ C_2(V^*) \otimes \mathbb{C} & & \end{array}$$

where the leftmost map is the identification between the Clifford algebra and the exterior algebra. The inverse of the map ρ is denoted $\sigma : \Lambda^2 V^* \otimes \mathbb{C} \to \mathrm{End}(W)$ and satisfies

$$\sigma(T)(v, w) = \frac{1}{2^n} \mathrm{trace}(\Gamma(v) T \Gamma(w)). \tag{3.4}$$

An orientation of V gives rise to splitting of W as follows.

Proposition 3.1.5. *Let V be a real $2n$-dimensional inner product space and fix an orientation on V. Suppose (W, Γ) is a spin^c structure on V, then W splits as*

$$W = W^+ \oplus W^-$$

where $W^\pm$ are 2^{n-1} dimensional complex vector spaces.

Proof. Choose a positively oriented orthonormal basis $e_1, \ldots, e_{2n}$ and define $\varepsilon := e_{2n} \ldots e_1$, given any other orthonormal basis $e'_1, \ldots e'_{2n}$, the two bases are related by $e_i = \sum_j A_{ij} e'_j$ for some matrix $A_{ij} \in SO(2n)$. It is easy to see that ε would change by a factor of $\det(A_{ij})$ which is 1 since $A_{ij} \in SO(2n)$, hence ε is independent of choice of orthonormal basis. It follows from the multiplication rules for the Clifford algebra that $\varepsilon^2 = (-1)^n$, via the algebra isomorphism $\Gamma : C^c(V) \longrightarrow \mathrm{End}(W)$ there is an endomorphism $\Gamma(\varepsilon) : W \longrightarrow W$ with $\Gamma(\varepsilon)^2 = (-1)^n$. Consequently $\Gamma(\varepsilon)$ has eigenvalues $\pm i^n$ and W splits into eigenspaces $W = W^+ \oplus W^-$ where

$$W^\pm := \{\psi \in W : \Gamma(\varepsilon)\psi = \pm i^n \psi\}.$$

□

This splitting satisfies $\Gamma(v)W^\pm \subset W^\mp$ for all $v \in V$ and is in fact the unique such splitting determined by this condition up to an interchange of signs. Note that interchanging the orientations of V interchanges the spaces $W^\pm$.

For even dimensional V, the spaces $W^\pm$ are invariant under the action of $\rho(\eta)$ for all $\eta \in \Lambda^2 V^*$, so define the restricted maps $\rho^\pm : \Lambda^2 V^* \to \mathrm{End}(W^\pm)$ by $\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$.

Of particular interest is when dim(V) = 4, there is a Hodge star operator $\star : \Lambda^2 V^* \to \Lambda^2 V^*$ with eigenvalues ± 1 yielding a splitting of vector spaces $\Lambda^2 V^* = \Lambda^{2,+} V^* \oplus \Lambda^{2,-} V^*$ into the self-dual and anti self-dual spaces of 2-forms

$$\Lambda^{2,\pm} V^* := \{\eta \in \Lambda^2 : \star\eta = \pm\eta\}.$$

Under the map ρ this map corresponds to the splitting $\Gamma(\mathrm{Spin(V)}) = SU(W^+) \times SU(W^-)$ and ρ gives an isomorphism $\rho^\pm : \Lambda^{2,\pm} \to \mathfrak{su}(W^\pm)$ where $\mathfrak{su}(W^\pm)$ consists of the traceless skew-hermitian endomorphisms of $W^\pm$. Consequently via the complexified map there are isomorphisms

$$\rho^\pm : \Lambda^{2,\pm} V^* \otimes \mathbb{C} \longrightarrow \mathrm{End}_0(W^\pm)$$

where $\mathrm{End}_0(W^\pm)$ are the traceless endomorphisms of $W^\pm$ with inverses $\sigma^\pm : \mathrm{End}_0(W^\pm) \to \Lambda^{2,\pm} \otimes \mathbb{C}$ satisfying the same property as in Equation (3.4).

For the Seiberg-Witten equations we are particularly interested in elements of $\mathrm{End}_0(W^+)$ of the form $(\Phi\Phi^*)_0$ where $\Phi \in W^+$ and the action on $\tau \in W^+$ by $\Phi\Phi^*$ is given by $\Phi\Phi^*\tau := \langle \Phi, \tau \rangle \Phi$ and $(\Phi\Phi^*)_0 = \Phi\Phi^* - \frac{1}{2}\mathrm{trace}(\Phi\Phi^*)\mathbb{1} \in \mathrm{End}_0(W^+_{\mathrm{can}})$ denotes the traceless part. The following computations involving the norms of ρ, σ and $(\Phi\Phi^*)_0$ are useful in later analysing the Seiberg-Witten equations

Lemma 3.1.6. *Let $\Gamma : V \longrightarrow End(W)$ be a spin^c structure on a 4-dimensional oriented real inner product space. Then if $\eta \in \Lambda^{2,+} V$, $T \in End_0(W^+)$ and*

$\Phi \in W^+$, then the following identities hold.

$$|\rho^+(\eta)|^2 = 2|\eta^+|^2 \tag{3.5}$$

$$|\sigma^+(T)|^2 = \frac{1}{2}|T|^2 \tag{3.6}$$

$$|(\Phi\Phi^*)_0|^2 = \frac{1}{4}|\Phi|^4 \tag{3.7}$$

$$\langle T, (\Phi\Phi^*)_0\rangle = \frac{1}{2}\langle T\Phi, \Phi\rangle \tag{3.8}$$

3.1.3 Spin^c Structures on Vector Bundles

Let X be a smooth manifold and $V \longrightarrow X$ a real oriented Riemannian vector bundle of dimension $2n$, a *spin^c structure* s consists of a pair (W, Γ), where $W \longrightarrow X$ is a complex Hermitian vector bundle of rank 2^n and $\Gamma : V \longrightarrow \mathrm{End}(W)$ is a vector bundle homomorphism satisfying the *Clifford properties*

$$\Gamma(v)^* + \Gamma(v) = 0 \tag{3.9}$$

$$\Gamma(v)^*\Gamma(v) = |v|^2 \mathbb{1}_{\mathrm{End}(W)} \tag{3.10}$$

for all $v \in V$.

We shall say that two spin^c structures $s_i = (W_i, \Gamma_i)$, $i = 1, 2$ are *isomorphic* if there is some unitary vector bundle isomorphism $\Phi : W_1 \longrightarrow W_2$ satisfying $\Phi\Gamma_1 = \Gamma_2\Phi$, that is the following diagram commutes for all $v \in V$.

$$\begin{array}{ccc} W_1 & \xrightarrow{\ \Phi\ } & W_2 \\ {\scriptstyle \Gamma_1(v)}\uparrow & & \uparrow{\scriptstyle \Gamma_2(v)} \\ W_1 & \xrightarrow[\ \Phi\ ]{} & W_2 \end{array}$$

The set of all isomorphism classes of spin^c structures on V is denoted $\mathcal{S}^c(V)$.

Remark 3.1.7. Spin^c structures can also be defined through the formalism of principal bundles, which is used widely throughout the literature. Given an oriented Riemannian vector bundle of rank k, a spin^c structure is a principal bundle $P \longrightarrow X$ with structure group $G = \mathrm{Spin}^c(k)$ such that there is an isomorphism of oriented Riemannian vector bundles $P \times_{\mathrm{ad}} \mathbb{R}^m \cong V$. The second statement is equivalent to P being a lift of the $SO(k)$ frame bundle of V.

Given a spin^c structure (W, Γ), the fibrewise splitting of W as in Proposition 3.1.5 implies that the W splits into two vector bundles of rank 2^{n-1}.

We shall call sections of W *spinors* or *spinor fields*, if it takes values entirely in W^+ or W^- then the spinor is said to have *positive* or *negative chirality* respectively.

It is natural to ask about the existence and uniqueness of spin^c structures, an answer to this is given in the following theorem. Generally, given an oriented Riemannian vector bundle $V \longrightarrow X$ of rank $2n$, a spin^c structure exists if and only if the second Steifel-Whitney class $w_2(V) \in H^2(X;\mathbb{Z}_2)$ has an integral lift $c \in H^2(X;\mathbb{Z})$ and isomorphism classes of spin^c structures are related via tensoring W by line bundle.

Theorem 3.1.8. *Let $V \to X$ be an oriented Riemannian vector bundle of rank $2n$.*

i. If $\Gamma : V \to End(W)$ is a spin^c structure on V, then the first Chern class of the associated line bundle $c_1(L_\Gamma) \in H^2(X;\mathbb{Z})$ is an integral lift of the second Steifel-Whitney class $w_2(V) \in H^2(X;\mathbb{Z}_2)$.

ii. For every integral lift $c \in H^2(X;\mathbb{Z})$ of $w_2(V) \in H^2(X;\mathbb{Z}_2)$ there exists a spin^c structure with $c_1(L_\Gamma) = c$.

iii. If $\Gamma : V \to End(W)$ is a spin^c structure on V and $E \to X$ is a Hermitian line bundle, then the line bundle of the twisted spin^c structure $\overline{\Gamma} : V \to End(W \otimes E)$ given by $\overline{\Gamma} = \Gamma \otimes \mathbb{1}$ has associated line bundle

$$L_{\overline{\Gamma}} = L_\Gamma \otimes E^{\otimes 2}.$$

iv. Suppose that $\Gamma_i : V \to End(W_i)$, $i = 1, 2$ are two spin^c structures on V, then there exists a Hermitian line bundle such that $W_2 \cong W_1 \otimes E$ and $\Gamma_2 \cong \Gamma_1 \otimes \mathbb{1}$, and the spin^c structures are isomorphic if and only if $c_1(E) = 0$.

Consequently we see that a Riemannian vector bundle V admits a spin^c structure if and only if the second Stiefel-Whitney class $w_2(V)$ has an integral lift. Of particular interest is when V is the tangent bundle of a manifold X, in which case we simply say that the smooth manifold X admits a spin^c structure. The vanishing of $w_2(V)$ is of particular interest and corresponds to the existence of a spin structure, any spin^c structure is locally determined by a spin structure.

There are a variety of conditions under which a spin^c structure can exist. If X admits a spin structure, then via the inclusion $\mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}^c(n)$ there is a canonical spin^c structure and its associated line bundle L_Γ is simply the trivial line bundle. In such a case there is a canonical origin and we may identify as groups $\mathcal{S}^c(X) \cong H^2(X;\mathbb{Z})$, it is also precisely in this case, when X admits a spin structure, that the associated line bundle admits a square root.

Since the second Stiefel-Whitney class $w_2(TX)$ of an oriented smooth manifold with dimension $n \leq 3$ vanishes, all such manifolds admit spin structures, hence spin^c structures as well. Orientable 4-manifolds provide the first examples of smooth manifolds which admit spin^c structures, yet not necessarily spin structures, one being the Enriques surface $X_4/\mathbb{Z}_2$ where X_4 is the vanishing set of the homogeneous polynomial $z_0^4 + z_1^4 + z_2^4 + z_3^4$ in $\mathbb{CP}^3$ and the group action is induced by complex conjugation. This has signature $\sigma = -8$, but by Rohlin's theorem [Roh52], every smooth compact manifold that admits a spin structure must have signature divisible by 16. However, all orientable 4-manifolds admit a spin^c structure. This was first proved by Hirzebruch and Hopf for compact oriented 4-manifolds in [HH58]

3.2 Dirac Operators and Connections

Since we have defined spinors, it is natural to ask how we may differentiate them in a way which is compatible with the spin^c structure. This leads us to the notion of spin^c connections and the Dirac operator.

3.2.1 Spin^c Connections

Definition 3.2.1. Let X be a Riemannian manifold, $\Gamma : TX \longrightarrow \mathrm{End}(W)$ be a spin^c structure and ∇ a Hermitian connection on W. We say that ∇ is a *spin^c connection* if there exists a connection on TX (also denoted by ∇) such that

$$\nabla_v(\Gamma(w)\Phi) = \Gamma(w)\nabla_v\Phi + \Gamma(\nabla_v w)\Phi \tag{3.11}$$

for all $\Phi \in C^\infty(X, W)$ and $v, w \in C^\infty(X, TX)$.

Proposition 3.2.2. *Let X be a Riemannian manifold, $\Gamma : TX \longrightarrow End(W)$ a spin^c structure and ∇ a spin^c connection on W, then the induced connection on TX is unique and Riemannian.*

Proof. Suppose ∇', ∇'' are the induced connections on TX, then it follows that for all $v, w \in C^\infty(X, TX)$ and $\Phi \in C^\infty(X, W)$

$$\Gamma(\nabla'_v w - \nabla''_v w)\Phi = 0$$

and thus $\nabla'_v w = \nabla''_v w$, so $\nabla' = \nabla''$ and the connection is unique.

From the Clifford properties and the polarisation identity, it follows that

$$\Gamma(u)\Gamma(v) + \Gamma(v)\Gamma(u) = -2g(u, v)\mathbb{1}$$

by using this to act on an arbitrary spinor $\Phi \in C^\infty(X, W)$, differentiating with the spin^c connection ∇ and applying 3.11 it follows that

$$\begin{aligned} -2dg(u, v)\Phi = &\left(\Gamma(\nabla u)\Gamma(v) + \Gamma(v)\Gamma(\nabla u)\right)\Phi \\ &+ \left(\Gamma(u)\Gamma(\nabla v) + \Gamma(\nabla v)\Gamma(u)\right)\Phi \end{aligned}$$

or equivalently

$$dg(u, v) = g(\nabla u, v) + g(u, \nabla v)$$

which is precisely the Riemannian condition. □

Although a spin^c connection induces a unique Riemannian connection on TX, the converse does not hold, given a Riemannian connection on TX, there are many possible spin^c connections on W. Given a spin^c connection the induced connection on TX is not necessarily torsion-free. However, when it is, it must be the Levi-Civita connection, it is also precisely the case of interest for the Seiberg-Witten equations.

Definition 3.2.3. Let ∇ be a spin^c connection on W. We say that ∇ is *compatible with the Levi-Civita connection* if the induced connection on TX is the Levi-Civita connection.

Given a spin^c connection ∇ that is compatible with the Levi-Civita connection, it induces a virtual connection on the virtual line bundle $L_\Gamma^{1/2}$ which does in fact uniquely determine ∇, we write the space of all such connections as $\mathcal{A}(\Gamma)$. Hence we denote ∇_A to refer to the spin^c connection on W which is compatible with the Levi-Civita connection and induces A. Note that the virtual connection A induces a genuine connection $2A$ on the characteristic line bundle L_Γ and many authors refer to a choice of this connection as opposed to our virtual one, at the cost of some factors of 2 in formulae involving the Seiberg-Witten equations.

Lemma 3.2.4. *Let ∇^1, ∇^2 be two spin^c connections on W. Then there exists a 1-form $\alpha \in \Omega^1(X, C_2(TX) \oplus i\mathbb{R})$ such that*

$$\nabla^1_v \Phi - \nabla^2_v \Phi = \Gamma((\alpha)(v))\Phi$$

for all $\Phi \in \Gamma(X, W)$ *and* $v \in \Gamma(X, TX)$.

Conversely, if ∇ *is a spin*^{c} *connection on* W *and* $\alpha \in \Omega(X, C_2(TX) \oplus i\mathbb{R})$, *then* $\nabla + \Gamma(\alpha)$ *is also a spin*^{c} *connection.*

Naturally a spin^c connection has an endomorphism valued 2-form, $F^\nabla \in \Omega^2(X, \mathrm{End}(W))$ which is the curvature, computed by by

$$F^\nabla(v, w)\Phi = \nabla_v \nabla_w \Phi - \nabla_w \nabla_v \Phi + \nabla_{[v,w]}\Phi$$

where $v, w \in C^\infty(X, TX)$ and $\Phi \in C^\infty(X, W)$.

If ∇ is a spin^c connection which is compatible with the Levi-Civita connection, then it follows that the traceless part of F^∇ is given by the Riemann curvature tensor. Hence

$$F^\nabla(v, w) - \frac{1}{2^n}\mathrm{trace}(F^\nabla(v, w)) = \rho(R(v, w))$$

where ρ is the homomorphism $\rho : \mathfrak{so}(TX) \longrightarrow \mathrm{End}(W)$ is defined by $\rho \circ \mathrm{Ad} = \Gamma : C_2(TX) \longrightarrow \mathrm{End}(W)$ where $\mathrm{Ad}(\zeta) \in \mathfrak{so}(TX)$ is defined by $\mathrm{Ad}(\zeta)v = [\zeta, v] = \zeta v - v\zeta$. Moreover the induced curvature on the line bundle L_Γ is $1/2^{n-1}\mathrm{trace}(F^\nabla(v, w))$

3.2.2 Dirac Operators

Definition 3.2.5. Let $A \in \mathcal{A}(\Gamma)$ with corresponding spin^c connection compatible with the Levi-Civita connection ∇_A, define the corresponding *Dirac operator*

$$\mathcal{D}_A : C^\infty(X, W) \longrightarrow C^\infty(X, W)$$

by

$$\mathcal{D} = \Gamma \circ \nabla_A$$

where $\nabla_A : C^\infty(X, W) \to C^\infty(X, T^*X \otimes W)$ is the spin^c connection induced by A compatible with the Levi-Civita connection and $\Gamma : C^\infty(X, T^*X \otimes W) \to C^\infty(X, W)$ is the Clifford multiplication map given by $v^* \otimes \Phi \mapsto \Gamma(v)\Phi$ where we identify v and v^* via the metric.

Given an orthonormal basis e_ν, $\nu \in \{1, ..., 2n\}$ of TX the Dirac operator can be written as

$$\mathcal{D}_A\Phi = \sum_\nu \Gamma(e_\nu)\nabla_{e_\nu}\Phi$$

If X is even dimensional, then $\mathcal{D}$ restricts to maps $\mathcal{D}_A^\pm : C^\infty(X, W^\pm) \longrightarrow C^\infty(X, W^\mp)$ these two maps can be viewed as the adjoints of each other with respect to the L^2 inner product and we often write $D_A := \mathcal{D}_A^+$.

The Dirac operator satisfies a *unique continuation theorem*

Theorem 3.2.6. *Let X be a connected Riemannian manifold of real dimension m equipped with a spin^c structure $\Gamma : TX \to End(W)$ and assume that the metric and spin^c structure are C^3. Suppose that $A \in \mathcal{A}^{1,p}(X)$ with $p > m$ and $\Phi \in L^p_2(X, W)$ satisfies*

$$D_A\Phi = 0.$$

If Φ vanishes on some open set of X then $\Phi \equiv 0$ is zero everywhere.

For a proof, see [Sal99][Appendix E]. This ultimately follows from a result of [AN67] and the same techniques were used to prove a unique continuation theorem for anti self-dual instantons (see [DK07]).

The initial motivation for Dirac to define the Dirac operator was to find a 'square-root' for the Laplacian on flat Minkowski space. The Dirac operator as defined in Definition 3.2.5 is simply a generalisation to curved manifolds with a Riemannian metric instead, and satisfies an analogous property with the presence of additional curvature terms. On flat Euclidean space, these curvature terms become zero and the initial property desired by Dirac is satisfied. This property is known as the *Weitzenböck formula* and is as follows

Theorem 3.2.7. *Let X be an oriented Riemannian manifold equipped with a spin^c structure Γ. Let $A \in \mathcal{A}(\Gamma)$, $\Phi \in C^\infty(X, W)$, then*

$$\mathcal{D}_A\mathcal{D}_A\Phi = \nabla_A^*\nabla_A\Phi + \frac{1}{4}s\Phi + \rho(F_A)\Phi$$

where ∇_A^* is the L^2 adjoint of ∇_A and $s : X \to \mathbb{R}$ is the scalar curvature of X.

Since the calculation of the Weitzenböck formula for spin^c structures is local, it can be proved using the corresponding formula for Dirac operators on spin structures. This is done by using the fact that locally, the complex spinor bundle W for a spin^c structure can be written as $W = S \otimes L_\Gamma^{1/2}$ for some local spin structure with spinor bundle S. This is the approach taken to prove the Weitzenböck formula in [LM89], although the formula can be proven via explicit calculation as in [Sal99][p.205].

The Weitzenböck formula immediately implies the following formula for D_A, the Dirac operator restricted to spinors of positive chirality.

Corollary 3.2.8. *Let X be an oriented Riemannian manifold equipped with a spin^c structure Γ, then the following formula holds*

$$D_A^*D_A\Phi = \nabla_A^*\nabla_A\Phi + \frac{1}{4}s\Phi + \rho^+(F_A)\Phi$$

for $A \in \mathcal{A}(\Gamma)$ *and* $\Phi \in C^\infty(X, W^+)$.

3.3 The Canonical Spin^c Structure on Symplectic and Kähler Manifolds

Let (X, ω) be a symplectic manifold with compatible almost complex structure J. Note that J is not necessarily integrable, and hence does not inherently determine a complex structure and thus X is not assumed to be Kähler. Of course, all of the following will apply in the Kähler case. In such a case, there is a canonical spin^c structure and Dirac operator.

Definition 3.3.1. Let (X, ω) be a symplectic manifold with compatible almost complex structure J, then there is a canonical spin^c structure on X. The spinor bundle is given by

$$W_{\text{can}} = \Lambda^{0,*}T^*X$$

and has a Hermitian structure induced by $\langle v, w\rangle = g(v, w) + i\omega(v, w)$ where $g(v, w) = \omega(v, Jw)$ and the spin^c representation is given by

$$\Gamma_{\text{can}} v = \frac{1}{\sqrt{2}} v'' \wedge \tau - \sqrt{2}\iota(v)\tau$$

there is a splitting of W_{can} into the $(0, p)$-forms where p is of even and odd degree as follows

$$W^+_{\text{can}} = \Lambda^{0,\text{ev}}T^*X, \; W^-_{\text{can}} = \Lambda^{0,\text{odd}}T^*X.$$

and the characteristic line bundle is the anticanonical bundle

$$L_{\Gamma_{\text{can}}} = K^* = \Lambda^{0,n}T^*X$$

Proposition 3.3.2. *Let* (X, ω) *be a symplectic manifold with compatible almost complex structure* J, *let* g *be the metric given by* $g(v, w) = \omega(v, Jw)$ *then there is a Hermitian connection called the Chern connection on* TX *which preserves the spaces* $\Omega^{p,q}(X)$ *given by*

$$\widetilde{\nabla}_v w = \nabla_v w - \frac{1}{2} J(\nabla_v J) w$$

which extends to $\Lambda TX^* \otimes \mathbb{C}$ *by*

$$\widetilde{\nabla}_v \tau = \nabla_v \tau + \frac{1}{2} \iota(J\nabla_v J)\tau$$

where ∇ *is the Levi-Civita connection.*

Lemma 3.3.3. *The connection $\widetilde{\nabla}$ is a Hermitian connection on W_{can}, moreover it is a spin^c connection which is compatible with the Hermitian connection $\widetilde{\nabla}$ on TX.*

Proof. [Sal99][p. 199] □

Given a Hermitian line bundle $L \to X$ and a connection B on L, there is a natural analogue of the Chern connection $\widetilde{\nabla}_B$ on $TX \otimes L$ defined by $\widetilde{\nabla}_B(\tau \otimes s) = \widetilde{\nabla}\tau \otimes s + \tau \otimes \nabla_B s$. This extends to $\Lambda T^*X \otimes L$ and is a connection preserving $\Omega^{p,q}(X, L)$, the space of (p, q) forms valued in L. The connection restricts to on $W_{\mathrm{can}} \otimes L$ which is is a spin^c connection compatible with $\widetilde{\nabla}$ on TX for the spin^c structure given by $\Gamma = \Gamma \otimes \mathbb{1}$.

Definition 3.3.4. We define the map $\mu : \mathfrak{so}(TX) \longrightarrow \mathrm{End}(W_{\mathrm{can}})$ to be the unique homomorphism such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{so}(TX) & \xrightarrow{\ \mu\ } & \mathrm{End}(W_{\mathrm{can}}) \\ {\scriptstyle \mathrm{ad}}\uparrow & \nearrow_{\Gamma} & \\ C_2(TX) & & \end{array}$$

That is, the homomorphism is characterised by the identity

$$[\mu(A), \Gamma(v)] = \Gamma(Av)$$

for all $A \in \mathfrak{so}(T_xX)$ and $v \in T_xX$.

Proposition 3.3.5. *There is a Hermitian connection ∇_{can} defined by*

$$\nabla_{can,v}\tau = \widetilde{\nabla}_v\tau + \frac{1}{2}\mu(J(\nabla_v J))\tau$$

which is a Hermitian connection on W_{can}, moreover it is a spin^c connection which is compatible with the Levi-Civita connection on TX.

Just as the Chern connection before, given a Hermitian line bundle L over X and Hermitian connection B on L, this extends to a connection on L-valued (p, q) forms and is a spin^c connection compatible with the Levi-Civita connection for the spin^c obtained by twisting the canonical spin^c structure by L.

Proposition 3.3.6. *If X is Kähler, then $\nabla_{can} = \widetilde{\nabla} = \nabla$ is simply the standard Levi-Civita connection on forms. Moreover, the induced connection on the characteristic line bundle $L_{\Gamma_{can}}$ is the Levi-Civita connection on $\Lambda^{0,n}T^*X$*

Proof. Since X is Kähler, $\nabla J = 0$ and so all the extra terms are zero in the above definitions. Consequently the Chern connection and the connection in coincide with the Levi-Civita Connection ☐

Recall that all other spin^c structures on X are obtained by tensoring with a Hermitian line bundle. Consequently, given a Hermitian line bundle $E \longrightarrow X$, we denote the *twisted spin^c structure by*

$$W_E := \Lambda^{0,*}T^*X \otimes E$$

this has a natural splitting

$$W_E^{\pm} = W_{\text{can}}^{\pm} \otimes E$$

to determine a spin^c connection on W_E one requires a Hermitian connection on E. We denote this by B and its associated covariant derivative operator by d_B, and the resulting spin^c connection on W_E by $\nabla_A = \nabla_{A_{\text{can}}+B}$, this connection is defined by

$$\nabla_A(\tau \otimes s) := (\nabla_{\text{can}}\tau) \otimes s + \tau \otimes d_B s$$

where $\tau \in W_{\text{can}}$ and $s \in C^\infty(X, E)$.

Now we turn to the Dirac operator, note that because of the canonical spin^c structure, sections of $C^\infty(X, W_E^{\pm})$ are identified with $\Omega^{0,\text{ev}}(X, E)$ and $\Omega^{0,\text{odd}}(X, E)$ respectively. The Dirac operator corresponding to $A_{\text{can}} + B$ is then written as follows

Theorem 3.3.7. *Let (X, ω) be a symplectic manifold with compatible almost complex structure J and $E \longrightarrow X$ a Hermitian line bundle and B a Hermitian connection on E so that $\nabla_A := \nabla_{A_{can}+B}$ determines a spin^c connection on W_E, then the (positive chirality part) of the Dirac operator is given by the following formula*

$$\frac{1}{\sqrt{2}} D_{A_{can}+B} = \overline{\partial}_B + \overline{\partial}_B^*$$

It is also useful to examine the representation maps $\rho_{\text{can}}^{\pm} : \Lambda^{2,\pm}T^*X \to \text{End}_0(W^{\pm})$ and their inverses for the canonical spin^c structure. In particular we look at the case of 4-dimensional manifolds where the following lemma will inevitably lead to a simplification of the Seiberg-Witten equations in Chapter 5. The symplectic form ω gives rise to a natural isomorphism $\Lambda^{2,+}T^*X \cong \mathbb{R}\omega \oplus \Lambda^{0,2}T^*X$ given by $\eta \mapsto (\eta^{1,1}, \eta^{0,2})$. One can then use this and some algebra to prove the following lemma.

Lemma 3.3.8. *Let X be a symplectic manifold with compatible almost complex structure J, $\eta \in \Lambda^{2,+}T^*X \otimes i\mathbb{R}$ and $\tau = (\tau_0, \tau_2) \in W_{can} = \mathbb{C} \oplus \Lambda^{0,2}T^*X$, then*

$$\rho^+_{can}(\eta) : \begin{pmatrix} \tau_0 \\ \tau_2 \end{pmatrix} \mapsto 2 \begin{pmatrix} \eta_0\tau_0 + \langle \eta_2, \tau_2 \rangle \\ \tau_0\eta_2 - \eta_0\tau_2 \end{pmatrix}$$

where $\eta^{1,1} = i\eta_0\omega$ and $\eta^{0,2} = \eta_2$.

For the Seiberg-Witten equations, we are particularly interested in the case when $\rho^+(\eta)$ is $(\Phi\Phi^*)_0$ for some positive spinor Φ. The above lemma then gives rise to the following.

Lemma 3.3.9. *Let X be a symplectic manifold with compatible almost complex structure J, $\eta \in i\Omega^{2,+}(X)$ and $\Phi \in C^\infty(X, W^+_{can})$, then*

$$\rho^+_{can}(\eta) = (\Phi\Phi^*)_0$$

if and only if

$$2\eta^{0,2} = \overline{\varphi}_0\varphi_2, \qquad 2i\eta \wedge \omega = \frac{|\varphi_0|^2 - |\varphi_2|^2}{2}\omega \wedge \omega.$$

Proof. The natural isomorphism $\Omega^{2,+}(X) = \mathbb{R}\omega \oplus \Omega^{0,2}(X)$ implies that $\eta = \sigma^+_{\text{can}}((\Phi\Phi^*)_0)$ decomposes as $\eta = \eta_2 + i\eta_0\omega - \overline{\eta}_2$ where $\eta_0 \in \mathbb{R}$, $\eta^{0,2} = \eta$ and $\eta^{2,0} = -\overline{\eta}_2$. Given $\tau = (\tau_0, \tau_2) \in W^+_{\text{can}}$ and writing $\Phi = (\varphi_0, \varphi_2)$, we have that

$$\Phi\Phi^*\tau = (\overline{\varphi}_0\tau_0 + \langle \varphi_2, \tau_2 \rangle)\varphi$$

and

$$\text{trace}(\Phi\Phi^*) = |\varphi_0|^2 + |\varphi_2|^2.$$

Comparing the definition of $(\Phi\Phi^*)_0$ and the expression for $\rho^+_{\text{can}}(\eta)$ in Lemma 3.3.8 and the identity $\langle \varphi_2, \tau_2 \rangle \varphi_2 = |\varphi_2|^2\tau_2$ gives the following two equations

$$\begin{aligned} 2\eta_0 + \tau_0 + 2\langle \eta_2, \tau_2 \rangle &= \frac{|\varphi_0|^2 - |\varphi_2|^2}{2}\tau_0 + \langle \overline{\varphi}_0\varphi_2, \tau_2 \rangle \\ 2\tau_0\eta_2 - 2\eta_0\tau_2 &= \frac{|\varphi_0|^2 - |\varphi_2|^2}{2}\tau_2 + \tau_0\overline{\varphi}_0\varphi_2. \end{aligned}$$

which holds for all τ if and only if

$$2\eta_2 = \overline{\varphi}_0\varphi_2, \qquad 2\eta_0 = \frac{|\varphi_0|^2 - |\varphi_2|^2}{2}.$$

Since ω is a $(1,1)$ form, $\eta^{0,2} \wedge \omega = 0$ and so combining $i\eta \wedge \omega = i\eta_0\omega \wedge \omega$ and the second equation above shows that it is equivalent to

$$2i\eta \wedge \omega = \frac{|\varphi_0|^2 - |\varphi_2|^2}{2}\omega \wedge \omega.$$

□

Remark 3.3.10. If $L \to X$ is a Hermitian line bundle, then Lemma 3.3.9 extends to the spin^c structure given by tensoring by L.

3.4 The Index of the Dirac Operator

The Dirac operator can be naturally extended to appropriate Sobolev completions of the space of smooth sections, in particular, by viewing it as a map of Hilbert spaces $\mathcal{D}_A : W^{1,p}(X, W) \longrightarrow L^p(X, W)$ one can see that it is a Fredholm operator. Its index is a topological invariant of X and such is the content of the famed Atiyah-Singer index theorem. Although the original result was in the context of spin structures, a corresponding result exists for spin^c Dirac operators. We state the index D_A in the case when X is a compact 4-manifold.

Theorem 3.4.1 (Atiyah-Singer)**.** *Let X be a compact smooth 4-manifold and $\Gamma : TX \longrightarrow End(W)$ a spin^c structure with associated line bundle L_Γ. The real Fredholm index of D_A is given by*

$$ind(D_A) = \frac{\langle c_1(L_\Gamma)^2, [X]\rangle - \sigma(X)}{4}$$

where $\sigma(X) = b^+ - b^-$ is the signature of X.

3.5 Technical Properties of the Dirac Operator

In order to rigorously justify many results on the Seiberg-Witten equations and the moduli space, we require the following technical results on the Dirac operator and its extension to suitable Sobolev completions. Proofs of these facts can be found in [Sal99].

Lemma 3.5.1. *Fix integers j, k and $p, q \in \mathbb{R}$ with $p, q \geq 1$ such that*

$$0 \leq j \leq k, \qquad j - \frac{4}{p} \leq k - \frac{4}{q}, \qquad (k+1)q > 4$$

then for any two connections $A_0, A_1 \in \mathcal{A}^{k,q}(\Gamma)$, the linear operator

$$D_{A_1} - D_{A_0} : W^{j+1,p}(X, W^+) \longrightarrow W^{j,p}(X, W^-)$$

is a compact operator.

Lemma 3.5.2. *Let $A \in \mathcal{A}^{j,p}$ and $\Psi \in W^{j,p}(X, W^-)$ for some constant $p \geq 1$ and integer $j \geq 1$ with $(j+1)p > 4$. Suppose that $\Phi \in L^q(X, W^+)$ with $\frac{1}{p} + \frac{1}{q} = 1$ satisfies*

$$\int_X \langle D_A^* \psi, \Phi \rangle \, vol = \int_X \langle \psi, \Psi \rangle \, vol$$

for all $\psi \in C^\infty(X, W^-)$. Then $\Phi \in W^{j+1,p}(X, W^+)$ and $D_A \Phi = \Psi$.

Lemma 3.5.3. *If $kq > 4$, for every smooth reference connection $A_0 \in \mathcal{A}(\Gamma)$, there exists a constant $c = c(A_0, j, k, p, q) > 0$ such that*

$$\|\Phi\|_{W^{j+p,1}} \leq c \left(\|D_A \Phi\|_{j,p} + (1 + \|A - A_0\|_{k,q}) \|\Phi\|_{j,p} \right)$$

for every $A \in \mathcal{A}^{k,q}(\Gamma)$ and every $\Phi W^{j+1,p}(X, W^+)$

These two results alongside Theorem 3.4.1 then imply that the Dirac operator is Fredholm with the usual index from the Atiyah-Singer index theorem, provided we have sufficient regularity assumptions on the spaces.

Proposition 3.5.4. *Let $A \in \mathcal{A}^{k,q}(\Gamma)$ for some constant $q > 1$ and some integer $k \geq 0$. Let $j \in \mathbb{Z}$ and $p > 1$ such that*

$$0 \leq j \leq k, \qquad j - \frac{4}{p} \leq k - \frac{4}{q}, \qquad (k+1)q > 4.$$

Then the Dirac operator $D_A : W^{j+1,p}(X, W^+) \longrightarrow W^{j,p}(X, W^-)$ is Fredholm with index

$$ind(D_A) = \frac{c_1(L_\Gamma) \cdot c_1(L_\Gamma) - \sigma(X)}{4}$$

3.6 The Gauge Group

As is a general theme in gauge theory, one often studies equations on principal bundles or their associated bundles and their behaviour under certain automorphism groups of said bundles. In studying the Seiberg-Witten equations on a 4-manifold with spin^c structure, automorphisms of the spin^c structure are essentially determined by automorphisms of the line bundle L_Γ, hence we shall consider the following group.

Definition 3.6.1. Let

$$\mathcal{G} := \text{Maps}(X, S^1) = \{u : X \to S^1 : u \text{ is smooth}\}.$$

This shall often be referred to as the *gauge group* or the *gauge group of the Seiberg-Witten equations.*

Definition 3.6.2. Let $u \in \mathcal{G}$, we say that u is *harmonic* if

$$d^*(u^{-1}du) = 0.$$

Since $d(u^{-1}du) = 0$, u is harmonic if and only if the 1-form $u^{-1}du \in \Omega^1(X, i\mathbb{R})$ is harmonic in the usual sense.

The space of harmonic maps into the circle are denoted $\mathcal{G}_{\mathcal{H}}$ and forms a subgroup of $\mathcal{G}$.

We have the following short exact sequence

$$0 \to S^1 \to \mathcal{G}_{\mathcal{H}} \to H^1(X;\mathbb{Z}) \to 0$$

and the following theorem.

Theorem 3.6.3. *Let X be a compact connected manifold*

(i.) *Every component of $\mathcal{G} = Map(X, S^1)$ contains a harmonic representative which is unique up to multiplication by a constant.*

(ii.) *The map $Map(X, S^1) \longrightarrow Hom(\pi_1(X), \mathbb{Z})$ denoted $u \mapsto \rho_u$ which is given by its action on a homotopy class of loops (with representative) γ*

$$\rho_u(\gamma) := deg(u \circ \gamma) = \frac{1}{2\pi i}\int_\gamma u^{-1}du$$

induces an isomorphism $\pi_0(Map(X, S^1)) \longrightarrow Hom(\pi_1(X), \mathbb{Z})$

(iii.) *A map $u : X \longrightarrow S^1$ is even if and only if $\rho_u \in Hom(\pi_1(X), 2\mathbb{Z})$*

Part II

Seiberg-Witten Theory

Chapter 4

Standard Seiberg-Witten Theory

In this chapter we shall give an overview of ordinary Seiberg-Witten theory on 4-manifolds. This will be instructive since much of Seiberg-Witten theory in the families setting proceeds using the same or similar techniques. First we shall introduce the Seiberg-Witten equations and outline the construction of the Seiberg-Witten moduli space. This moduli space depends on a choice of perturation of the Seiberg-Witten equations, Riemannian metric and spin^c structure. We shall argue for a generic choice of perturbation and metric the moduli space is a smooth compact oriented manifold. In the case when the manifold has $b^+ > 1$, given a generic choice of metric and perturbation these moduli spaces can be joined by a smooth cobordism, while for $b^+ = 1$ a cobordism can be obtained between two choices of generic metrics and perturbations, provided they are both in one of two connected components of the total parameter space called a chamber. Because of this cobordism, the Seiberg-Witten invariant can be defined, being the integral over the moduli space of a particular cohomology class. For a more detailed exposition, we refer to [Nic00] and [Sal99].

4.1 The Seiberg-Witten Equations

Throughout, let X be a compact, connected, oriented, smooth 4-manifold. Fix a metric g and spin^c structure $\Gamma : TX \longrightarrow \mathrm{End}(W)$, this gives a natural splitting of the spinor bundle W into positive and negative chirality spinors

$$W \simeq W^+ \oplus W^-$$

and since the manifold is 4-dimensional, the characteristic line bundle of the spin^c structure is given by

$$L_\Gamma \simeq \det(W^+) \simeq \det(W^-).$$

Let $\mathcal{A}(\Gamma)$ be the space of virtual connections on the virtual line bundle $L_\Gamma^{1/2}$, this space is in correspondence with spin^c connections on W compatible with the Levi-Civita connection. Given $A \in \mathcal{A}(\Gamma)$, denote the corresponding spin^c connection by $\nabla_A : C^\infty(X, W) \longrightarrow \Omega^1(X, W)$ and the corresponding Dirac operator acting on the positive chirality spinors by $D_A : C^\infty(X, W^+) \longrightarrow C^\infty(X, W^-)$. Given these choices, the Seiberg-Witten equations are defined as follows.

Definition 4.1.1. Let X be a compact, connected and oriented smooth Riemannian manifold of real dimension 4 and $\Gamma : TX \longrightarrow \mathrm{End}(W)$ a spin^c structure on X, the *Seiberg-Witten equations* for (X, Γ, g) are the following

$$D_A\Phi = 0 \tag{4.1}$$
$$\rho^+(F_A) = (\Phi\Phi^*)_0 \tag{4.2}$$

where $\Phi \in C^\infty(X, W)$ is a smooth section of the spinor bundle, $A \in \mathcal{A}(\Gamma)$, $F_A = \frac{1}{4}\mathrm{trace}_c(F^{\nabla_A}) \in \Omega^2(X, i\mathbb{R})$ is the curvature of A and F^{∇_A} is the curvature of the corresponding spin^c connection and $\Phi\Phi^* \in C^\infty(X, \mathrm{End}(W))$ is defined by

$$\Phi\Phi^*\tau = \Phi\left\langle \Phi, \tau \right\rangle$$

for $\tau \in C^\infty(X, W^+)$ and $(\Phi\Phi^*)_0$ is its traceless part given by

$$(\Phi\Phi^*)_0\tau = \Phi\left\langle \Phi, \tau \right\rangle - \frac{1}{2}|\Phi|^2\tau.$$

Remark 4.1.2. Note that the map ρ^+ can be inverted so that eq. (4.2) of Seiberg-Witten equations is also written in the following form on a 4-manifold

$$F_A^+ = \sigma^+((\Phi\Phi^*)_0) \tag{4.3}$$

Note that this necessitates that X is a 4-manifold, since the Hodge star operator does not map 2-forms to 2-forms in dimensions other than four, there is not an obvious notion of self-duality for 2-forms on such manifolds and (4.3) would not make any sense.

These equations are the minimum of the following action

$$S(A, \Phi) = E(A, \Phi) := \int_X \left(|\nabla_A\Phi|^2 + \frac{s}{2}|\Phi|^2 + \frac{1}{4}|\Phi|^4 + |F_A|^2 \right) \mathrm{vol}$$

which is easily seen from the Weitzenböck formula.

Remark 4.1.3. These equations are generally definable on any even dimensional manifold with the above assumptions. However, interesting consequences only arise in $d = 4$. If $n = 2k > 4$ then the equations as in (4.1) and (4.2) are overdetermined. Moreover, the notion of a spin^c structure requires $\dim(TX) \geq 3$, consequently the Seiberg-Witten equations as is, cannot be defined on surfaces.

4.2 Construction of the Moduli Space

We shall now outline the construction of the moduli space and argue that it is a smooth, compact, oriented manifold (up to a generic choice of metric and perturbation for the Seiberg-Witten equations). The general theme is as follows, we consider the space of solutions to the Seiberg-Witten equations with an equivalence relation induced by the gauge group of smooth maps into the circle $\mathcal{G} = \mathrm{Maps}(X, S^1)$.

The gauge group acts on pairs of solutions (A, Φ) by

$$u \bullet (A, \Phi) = (u^* A, u^{-1}\Phi) := (u^{-1}du + A, u^{-1}\Phi).$$

Let $\mathcal{C}(\Gamma) := \mathcal{A}(\Gamma) \times C^\infty(X, W^+)$ be the configuration space of the Seiberg-Witten equations amd $\mathcal{Z}(\Gamma) \subset \mathcal{C}(\Gamma)$ be the solution set to the Seiberg-Witten equations. The *Seiberg-Witten moduli space* is then the space $\mathcal{M} = \mathcal{M}(g, \Gamma) = \mathcal{Z}(\Gamma)/\mathcal{G}$.

The stabiliser of the group action is trivial unless $\Phi = 0$, in which case it is the constant maps of the circle. Because of this it is convenient to say that a pair of solutions to the Seiberg-Witten equations is *reducible* if $\Phi = 0$, otherwise it is *irreducible*.

Consequently the gauge group acts freely on the space $\mathcal{C}^*(\Gamma) := \mathcal{A}(\Gamma) \times C^\infty(X, W^+)^*$, but not on the reducible solutions. This leads us to define the *irreducible moduli space* $\mathcal{M}^* := \mathcal{Z}^*/\mathcal{G}$ where $\mathcal{Z}$ is the set of irreducible solutions to the Seiberg-Witten equations, one can shown that for a generic choice of metric g, this is a smooth manifold (see [Fri00][Appendix A]). However, it is more convenient to introduce a perturbation to the Seiberg-Witten equations and study the resulting moduli space, this lends itself to simpler proofs of the transversality of the moduli space, moreover the notion of a perturbation will be crucial for studying the moduli space on Kähler surfaces in Chapter 5. Given any imaginary-valued self-dual 2-form $\eta \in \Omega^{2,+}(X, i\mathbb{R})$, the *perturbed Seiberg-Witten equations* are the following

$$D_A\Phi = 0 \tag{4.4}$$
$$F_A^+ + \eta = \sigma^+((\Phi\Phi^*)_0) \tag{4.5}$$

Note that when (A, Φ) is reducible, the Seiberg-Witten equations become

$$\Phi = 0 \tag{4.6}$$
$$F_A^+ + \eta = 0 \tag{4.7}$$

for generic η there are no solutions to this equation, in such a case we have $\mathcal{M} = \mathcal{M}^*$ and it can be shown that the moduli space of solutions to the perturbed Seiberg-Witten equations is a smooth oriented manifold, allowing us to define the Seiberg-Witten invariant. Moreover, the invariant is independent of a choice of metric and perturbation when $b^+(X) > 1$, for $b^+(X) = 1$ this is almost the case, with the space of suitable metrics and perturbations split into two connected components and the invariant dependent on a choice of such a component, these components are called *chambers*.

The following sections shall briefly outline arguments for the transversality, compactness and orientability of the moduli space. It is of note that the arguments for the same properties in Chapter 6 when discussing the moduli space for the families Seiberg-Witten equations are almost identical and specialise to the unparametrised case when the parameter space for the family is a point.

The following theorem of Uhlenbeck is quite integral to formally proving the compactness of the Seiberg-Witten moduli space and also ensures when desired we can make a smart choice of gauge. This will allow us to eliminate all but an S^1 component of the gauge group action when studying the moduli space. A more general version of the theorem exists for any principal G bundle over a compact manifold [Uhl82] and involves difficult analysis in its proof, although when specialising the case of $G = S^1$ for the sake of Seiberg-Witten theory it becomes a simple consequence of Hodge theory.

Fix some smooth reference connection A_0 and $p > 2$, define the L_1^p space of connections

$$\mathcal{A}^{1,p}(\Gamma) := \{A_0 + \alpha : \alpha \in L_1^p(X, T^*X \otimes i\mathbb{R})\}$$

and the gauge group

$$\mathcal{G}^{2,p} = L_2^p(X, S^1) = \{u : X \to S^1 : u^{-1}du \in L_1^p\}$$

Theorem 4.2.1. *Fix a connection $A_0 \in \mathcal{A}(\Gamma)$ and a constant $p > \frac{1}{2}dimX$. Then there is some constant $c > 0$ such that for every $A \in \mathcal{A}^{1,p}(\Gamma)$, there exists $u \in L_2^p(X, S^1)$ such that*

$$d^*(u^*A - A_0)$$

and

$$\|u^*A - A_0\|_{L_2^p} \leq c(1 + \|F_A\|_{L^p})$$

Proof. Let $\mathcal{H}^1(X, i\mathbb{R})$ be the space of imaginary valued harmonic 1-forms and $\Lambda = \mathcal{H}^1(X, 2\pi i\mathbb{Z})$ be the lattice of harmonic 1-forms $\alpha \in \mathcal{H}^1(X, i\mathbb{R})$ whose integral over every loop is $2\pi i$. Recall that by Theorem 3.6.3, that Λ is equivalently the space of $\alpha = u^{-1}du$ where $u : X \longrightarrow S^1$ is smooth and $d^*(u^{-1}du) = 0$. Choose a bounded fundamental domain in $\mathcal{H}^1(X, i\mathbb{R})$ with respect to the lattice Γ

We may write $A = A_0 + \alpha$ for some $\alpha \in L_1^p(X, T^*X \otimes i\mathbb{R})$ and by Hodge theory this decomposes as

$$\alpha = \alpha_0 + d\zeta + \star d\eta$$

Where α has harmonic part α_0.

Since Γ is a lattice, there exists some function $u_0 : X \longrightarrow S^1$ such that $u_0^{-1}du_0 \in \Gamma$ is harmonic and $\alpha_0 + u_0^{-1}du_0$ lies in the fundamental domain. Boundedness of the domain implies that there is some constant $c_0 > 0$ such that

$$\left\|\alpha_0 + u_0^{-1}du_0\right\|_{L^p} \leq c_0.$$

Define

$$u(x) := e^{-\zeta(x)}u_0(x)$$

then $u \in L_2^p(X, S^1)$ and satisfies

$$\alpha + u^{-1}du = \alpha_0 + u_0^{-1}du_0 + \star d\eta.$$

Since α_0 and u_0 are harmonic and $\star d\eta$ is in the image of d^* for some 3-form that

$$d^*(\alpha + u^{-1}du) = 0.$$

This proves the first part of the theorem since $u^*A = A_0 + \alpha + u^{-1}du$.

Consider the operator $L : \alpha \mapsto (\alpha_0, d\alpha, d^*\alpha)$ from $L_1^p(X, T^*X \otimes i\mathbb{R})$ to $L^p(X, T^*X \otimes i\mathbb{R}) \oplus L^p(X, \Lambda^2 T^*X \otimes i\mathbb{R}) \oplus L^p(X, \Lambda^2 T^*X \otimes i\mathbb{R})$. This is easily verified to be an injective linear map and satisfies the following inequality

$$\|\alpha\|_{L_1^p} \leq c(\|L\alpha\|_{L^p} + \|\alpha\|_{L^p}).$$

By Rellich's theorem, the inclusion of α into L^p space is compact so by Lemma 1.1.2 the map L has closed range.

Consequently, the image of L is a Banach space, by injectivity there is an inverse from $\mathrm{im}(L) \longrightarrow L^p_1(X, T^*X \otimes i\mathbb{R})$, since this is surjective, the open mapping theorem implies that the inverse is bounded, hence there is a constant c independent of α such that

$$\|\alpha\|_{L^p_1} \leq c(\|\alpha_0\|_{L^p} + \|d\alpha\|_{L^p} + \|d^*\alpha\|_{L^p}).$$

Since $u^{-1}du$ is closed $d(\alpha + u^{-1}du) = d\alpha$ so the above inequality implies

$$\left\|\alpha + u^{-1}du\right\|_{L^p_1} \leq c(c_0 + \|d\alpha\|_{L^p})$$

which is the desired inequality. □

With Theorem 4.2.1, one can then show the following theorem that ensures any solution to the Seiberg-Witten equations on appropriate Sobolev completions, is gauge equivalent to a smooth solution.

Theorem 4.2.2. *Let $A \in \mathcal{A}^{1,p}$ and $\Phi \in L^p_1(X, W^+)$ with $p > 2$ and suppose (A, Φ) is a solution to the Seiberg-Witten equations. Then there exists a gauge transformation $g \in \mathcal{G}^{2,p}$ such that the pair $(u^*A, u^{-1}\Phi)$ is smooth.*

Proof. Choose a smooth reference connection A_0, without loss of generality we may further assume it is Yang-Mills, i.e. it satisfies $d^*F_{A_0} = 0$. By Theorem 4.2.1 we may choose gauge transformation $u \in \mathcal{G}^{2,p}$ such that $d^*(u^*A - A_0) = 0$, we wish to show that the pair (A, Φ) is smooth, and will do so by showing $\alpha = A - A_0$ and Φ are smooth.

It is simple to show that for every $\beta \in \Omega^1(X, i\mathbb{R})$, the above conditions imply the following

$$\langle \beta, d^*d\alpha + dd^*\alpha \rangle_{L^2} = \langle \beta, 2d^*F^+_A \rangle_{L^2} \tag{4.8}$$

and since (A, Φ) satisfies the Seiberg-Witten equations, it follows from Lemma 3.5.2 that $\Phi \in L^p_2(X, W^+)$ and the invariance of products in Sobolev spaces for $2p > 4$ gives $F^+_A \in L^p_2$. Equation (4.8) above implies that $\alpha \in L^p_3$. Furthermore, 4.8 is a weak form of the strong equation

$$d^*d\alpha + dd^*\alpha = 2d^*F^+_A$$

which is indeed defined since we have shown that α is of class at least L^p_2.

We also obtain that $A \in \mathcal{A}^{3,p}$ and $\Phi \in L^p_4$ and with this setup via an elliptic bootstrapping procedure we can show that the pair (A, Φ) is smooth. By the same argument if $\Phi \in L^p_k$ with $k \geq 2$, then this implies that $F^+_A \in L^p_k$, consequently $\alpha \in L^p_{k+1}$, hence $\Phi \in L^p_{k+2}$, proceeding inductively shows this holds for all k and thus α and Φ are smooth.

□

Corollary 4.2.3. *The moduli space $\mathcal{M}(X, \Gamma, g, \eta)$ can be naturally identified with the space*

$$\mathcal{M}(X, \Gamma, g, \eta) \cong \frac{\{(A, \Phi) \in \mathcal{A}^{1,p}(\Gamma) \times L^p_1(X, W^+) : 4.4, 4.5\}}{\mathcal{G}^{2,p}}.$$

Compactness of the Moduli Space

To obtain the compactness of the moduli space we require the following key estimate for the Seiberg-Witten equations which can be shown with the Weitzenböck formula and Lemma 3.1.6.

Lemma 4.2.4. *Let X be a compact, oriented 4-manifold equipped with a Riemannian metric and spin^c structure and (A, Φ) be a smooth solution to the Seiberg-Witten equations for a perturbation η, then either $\Phi \equiv 0$ or*

$$\sup_X |\Phi| \leq \frac{1}{2} \sup_X \left(4\sqrt{2}|\eta| - s\right)$$

where $s : X \to \mathbb{R}$ is the scalar curvature of X.

It is interesting to contrast the situation to the study of anti-self dual Yang-Mills instantons. It is precisely this universal bound that forces the moduli space to be compact, and hence we avoid the difficult compactification procedures that are required.

Theorem 4.2.5. *Let X be a compact, oriented 4-manifold equipped with a Riemannian metric and spin^c structure, then the Seiberg-Witten moduli space $\mathcal{M}$ is compact.*

Proof. Fix a constant $p > 4$ and a smooth reference connection A_0, by Theorem 4.2.1, every solution (A, Φ) of the perturbed Seiberg-Witten equations is gauge equivalent to one which satisfies

$$d^*\alpha = 0, \qquad \|\alpha\|_{W^{1,p}} \leq c(1 + \|d\alpha\|_{L^p})$$

where $\alpha = A - A_0$ and the proof of Theorem 4.2.2 implies that this solution is smooth.

Since $F_A^+ + \eta = \sigma^+((\Phi\Phi^*)_0)$, Lemma 3.1.6 implies the pointwise inequality

$$|F_A^+|^2 \leq |\eta|^2 + \frac{1}{8}\left(|\Phi|^2\right)^2$$

taking the supremum of both sides over X and applying Lemma 4.2.4, compactness of X gives

$$\sup_X |F_A^+| \leq c_0$$

for some c_0 which is independent of (A, Φ).

Equation (4.8) can then be used to give the following inequality with $1/p + 1/q = 1$

$$\|d\alpha\|_{L^p} \leq c \sup_\beta \frac{2\left\langle d\beta, F_A^+ - F_{A_0}^+\right\rangle}{\|\beta\|_{L_1^q}}$$

the Cauchy-Schwarz inequality and the uniform bound on the curvature above implies that $d\alpha$ is bounded in the L^p norm, in conjunction with the fact that $\|\alpha\|_{W^{1,p}} \leq c(1 + \|d\alpha\|_{L^p})$ this gives

$$\|\alpha\|_{L_1^p} \leq c_1.$$

Since $p > 4$, Theorem 1.2.2 implies that α is continuous and uniformly bounded in the sup-norm.

Lemma 3.5.2 implies that $\Phi \in L_2^p$, thus we may apply Lemma 3.5.3 and obtain

$$\|\Phi\|_{L_1^p} \leq c(1 + \|\alpha\|_{L_1^p})\,\|\Phi\|_{L^4} \leq c_1'$$

and so Φ is also uniformly bounded in L_1^p.

Equation (4.8) and Lemma 3.5.3 as above imply that for each integer $k \geq 1$, there is a constant $c > 0$ such that

$$\begin{aligned} \|\alpha\|_{L_{k+1}^p} &\leq c(1 + \|\Phi\Phi^*\|_{L_k^p}) \\ \|\Phi\|_{L_{k+1}^p} &\leq c(1 + \|\alpha\|_{L_k^p})\,\|\Phi\|_{L_k^p} \end{aligned}$$

via inductive application of the above inequalities one obtains

$$\|\alpha\|_{L_k^p} + \|\Phi\|_{L_k^p} \leq c_k.$$

Now suppose that (A_ν, Φ_ν) is a sequence of L_1^p solutions to the Seiberg-Witten equations, we may take these to satisfy $d^*\alpha = 0$ and $\|\alpha\|_{W^{1,p}} \leq c(1 + \|d\alpha\|_{L^p})$. By the above string of inequalities, any such sequence is uniformly bounded in L^p_k for all $k > 0$, since the inclusion $L^p_{k+1} \subset L^p_k$ is compact via Rellich's theorem, the sequence has a convergent subsequence in L^p_k for all $k > 0$. Theorem 1.2.2 asserts that $L^p_k \subset C^\ell$ for $kp > \ell p + 4$, consequently this subsequence converges in the C^∞ topology. When quotienting by the remainder of the group action to obtain a smooth manifold, we obtain that any sequence in the moduli space has a convergent subsequence, hence the moduli space is compact. □

Transversality of the Moduli Space

The content of this section is to outline an argument for the transversality of the moduli space for sufficiently generic perturbations.

The first observation is that we may gauge fix the Seiberg-Witten equations away from reducible solutions and for a fixed smooth connection A_0, we introduce the following spaces

$$\begin{aligned} \widetilde{\mathcal{M}} &= \{(A, \Phi) \in \mathcal{Z} : d^*(A - A_0)\} \\ \widetilde{\mathcal{M}}^* &= \{(A, \Phi) \in \widetilde{\mathcal{M}} : \Phi \neq 0\} \end{aligned}$$

the harmonic gauge group $\mathcal{G}_{\mathcal{H}}$ acts freely and properly on $\tilde{\mathcal{M}}$, hence provided a perturbation away from the wall is chosen, if it can be shown that the moduli space of gauge fixed solutions fixed to Coulomb gauge constitutes a smooth manifold, it immediately follows that the moduli space obtained from acting on with the full gauge group action is a smooth manifold.

We first aim to prove that the map between appropriate Sobolev spaces

$$f(A, \Phi) = (D_A\Phi, \sigma(\Phi) - F_A^+, d^*(A - A_0))$$

is Fredholm and compute its index. However, by fixing a smooth reference connection A_0 and setting $\alpha = A - A_0$ we may rewrite this as

$$f(\alpha, \Phi) = (D_{A_0+\alpha}\Phi, d^+\alpha + \sigma((\Phi\Phi^*)_0) - F_{A_0}, d^*\alpha).$$

The spaces involved in the map $f : V \to W$ are

$$V = V_{\mathbb{C}} \oplus V_{\mathbb{R}}, \qquad W = W_{\mathbb{C}} \oplus W_{\mathbb{R}}$$

with $V_{\mathbb{C}}, W_{\mathbb{C}}$ and $V_{\mathbb{R}}, W_{\mathbb{R}}$ given by

$$\begin{aligned} V_{\mathbb{C}} &:= L^2_k(X, W^+), & V_{\mathbb{R}} &:= iL^2_k(X, \Lambda^1 T^*X) \\ W_{\mathbb{C}} &:= L^2_{k-1}(X, W^-), & W_{\mathbb{R}} &:= iL^2_{k-1}(\Lambda^{2,+}T^*X) \oplus L^2_{k-1}(X, i\mathbb{R})_0 \end{aligned}$$

where $L^2_q(X,\mathbb{R})_0$ is the subspace satisfying $\int_X f\text{vol} = 0$.

The map f can be further decomposed as $f = l + c$ where

$$l(\alpha,\Phi) = (D_{A_0}\Phi, d^+\alpha, d^*\alpha), \qquad c(\alpha,\Phi) = (\Gamma(\alpha)\Phi, \sigma((\Phi\Phi^*)_0) - F_{A_0}, 0).$$

Now consider the linearisation of f at (α,Φ), note that since all the spaces involved are vector spaces, they are their own tangent spaces. Observe that

$$d_{(\alpha,\Phi)}l(a,\psi) = (D_{A_0}\psi, d^+a, d^*a)$$

and

$$d_{(\alpha,\Phi)}c(a,\psi) = (\Gamma(a)\Psi, \sigma^+((\Phi\psi^* + \psi\Phi^*)_0))$$

Since the difference of Dirac operators is compact (Lemma 3.5.1) and $\sigma^+((\Phi\psi^*+\psi\Phi^*)_0)$ is a map involving addition and multiplication on a Sobolev space which is continuous for our choice of p composed with the inclusion $L^p_k \hookrightarrow L^p_{k-1}$ which is compact, hence $d_{(\alpha,\Phi)}c(a,\psi)$ is a compact operator.

It remains to show that the operator $d_{(\alpha,\Phi)}l(a,\psi)$ is Fredholm. The operator dl can be written as $D_{A_0} \oplus D^+$, where the operator D^+ is simply $d^+ \oplus d^*$. The Dirac operator is Fredholm with index given by

$$\frac{\langle c(L_\Gamma)^2, [X]\rangle - \sigma(X)}{4}$$

by Proposition 3.5.4, and the operator D^+ is also known to be Fredholm with index

$$b_1(X) - 1 - b^+(X) = -\frac{\chi(X) - \sigma(X)}{2} - 1$$

(see [Sal99], the -1 term comes from the fact that we have restricted the rightmost factor of the codomain of D^+ to be to the smooth functions with mean value zero).

Consequently, f is Fredholm with index

$$\frac{\langle c(L_\Gamma)^2, [X]\rangle - 2\chi(X) - 3\sigma(X)}{4} + 1$$

Hence for any regular value η, the implicit function theorem, i.e. Theorem 1.1.3 implies that $\widetilde{\mathcal{M}} = f^{-1}(0,\eta,0)$ is a smooth manifold of dimension $\text{ind}(df)$, factoring by the remaining group action gives the moduli space and has dimension

$$d(X,\Gamma) = \frac{\langle c(L_\Gamma)^2, [X]\rangle - 2\chi(X) - 3\sigma(X)}{4}$$

The fact that the space of regular perturbations $\Omega^{2,+}_{\text{reg}}(X,g)$ is generic, i.e. it forms a set which is second category in the sense of Baire in $\Omega^{2,+}(X,g)$ follows from the Sard-Smale theorem. Strictly speaking the above map does not give this result since we are only interested in perturbations of the form $(0,\eta,0)$. However, one may prove with similar methods to the above, that the space of pairs (A,Φ)

$$\mathcal{N}^{k,p} \subset \mathcal{A}^{k,p}(\Gamma) \times L^p_k(X,W^+)$$

satisfying $D_A\Phi = 0, d^*(A-A_0) = 0$ and $\Phi \neq 0$ is a smooth paracompact Banach manifold (for $k \geq 1, p > 4$). The moduli space $\widetilde{\mathcal{M}}$ can then equivalently be obtained as the preimage of a point η in the codomain of the map

$$f' : \mathcal{N}^{k,p} \to L^p_{k-1}(X, \Lambda^{2,+}T*X \otimes i\mathbb{R})$$

given by

$$f'(A,\Phi) = F^+_A - \sigma^+((\Phi\Phi^*)_0)$$

This is a smooth Fredholm map and its index agrees with the map f above, by the Sard-Smale theorem the space of regular perturbations is then indeed generic in the L^p_k topology, one may approximate such perturbations by smooth perturbations and obtain that the space of regular perturbations is in fact generic in the C^∞ topology.

Orientation of the Moduli Space

The aim of this section is to discuss the orientability of the Seiberg-Witten moduli space, this in fact depends on a choice of orientation of the real vector spaces $\mathcal{H}^1(X,i\mathbb{R})$ and $\mathcal{H}^{2,+}(X,i\mathbb{R})$ which we fix from now on.

Since the moduli space $\widetilde{\mathcal{M}}^*$ is obtained from the preimage of regular values of the map f as in Section 4.2, its tangent space at (A,Φ) is given by the kernel of $df_{(A,\Phi)}$, hence an orientation can be obtained via a trivialisation of the determinant line bundle $Det \to \widetilde{\mathcal{M}}^*$, this is a line bundle with fibres $\det(df_{A,\Phi}) = \Lambda^{\max}\ker(df_{A,\Phi}) \otimes \Lambda^{\max}\text{coker}(df_{A,\Phi})$, which for regular values is simply $\Lambda^{\max}\ker(df_{A,\Phi})$.

We can obtain an orientation from the family of operators $df_{A,t\Phi}$, $t \in [0,1]$ since the determinant line bundle trivialises over the path and hence we can transport an orientation of $\det(\mathcal{D}_{A,0})$. Since $\mathcal{D}_{A,0} = D_A \oplus D^+$ we have $\det(\mathcal{D}_{A,0}) = \det(D_A) \oplus \det(D^+)$, moreover D_A is a complex linear operator between complex vector spaces, thus its kernel and cokernel have natural orientations. We also have $\ker(D^+) = \mathcal{H}^1(X;i\mathbb{R})$ and $\mathcal{H}^0(X;i\mathbb{R}) \oplus \mathcal{H}^{2,+}(X;i\mathbb{R})$, since $\mathcal{H}^0(X; i\mathbb{R}) \cong \mathbb{R}$ it carries a natural orientation, then our fixed choice of orientations on $\mathcal{H}^1(X; i\mathbb{R})$ and $\mathcal{H}^{2,+}(X; i\mathbb{R})$ fixes an orientation on $\det(\mathcal{D}_{A,0}$, consequently inducing an orientation on $\det(df_{A,\Phi})$. Any path from $(A_0, 0)$ to (A, Φ) gives rise to an orientation and since the determinant line bundle Det extends to locally trivial bundle $Det \to \mathcal{A}(\Gamma) \times C^\infty(X, W^+)$, and base space is contractible, it is trivial, thus the orientation on $\det(df_{A,\Phi})$ is independent of the choice of path and basepoint $(A_0, 0)$.

Furthermore we claim that the harmonic gauge group $\mathcal{G}_{\mathcal{H}}$ acts by orientation preserving diffeomorphisms, inducing an orientation on $\mathcal{M}^*$. Let (A_0, Φ_0) be a solution to the Seiberg-Witten equations and $u \in \mathcal{G}_{\mathcal{H}}$, set $(A_1, \Phi_1) = (u^* A_0, u^{-1}\Phi_0)$ and choose paths between these two $t \mapsto A_t$ and $t \mapsto \Phi_t$ with $t \in [0, 1]$. Trivialising the determinant line bundle over the path $t \mapsto \det(df_{A_t,\Phi_t}))$ gives an isomorphism $\det(df_{A_0,\Phi_0}) \to \det(df_{A_1,\Phi_1})$, the linearisation of the gauge group action gives isomorphisms

$$\begin{aligned} \ker(df_{A_0,\Phi_0}) \to \ker(df_{A_1,\Phi_1}) &: (\alpha, \varphi) \mapsto (\alpha, u^{-1}\varphi) \\ \operatorname{coker}(df_{A_0,\Phi_0}) \to \operatorname{coker}(df_{A_1,\Phi_1}) &: (\beta, \tau, \psi) \mapsto (\beta, \tau, u^{-1}\psi) \end{aligned}$$

thus also inducing an isomorphism $\det(df_{A_0,\Phi_0}) \to \det(df_{A_1,\Phi_1})$, we now show the two isomorphisms are in fact the same. Consider the family of operators $df_{A_t,s\Phi_t}$, $s \in [0, 1]$, this provides a homotopy between the family of operators df_{A_t,Φ_t} and $df_{A_t,0}$, we may trivialise the determinant line bundle along this allowing us to transport orientations, so without loss of generality we may take $\Phi_t = 0$ for all t.

Recall that $\det(df_{A_t,0}) = \det(D_{A_t}) \otimes \det(D^+)$, this determines a line bundle over $[0, 1]$, since the orientation of D^+ is independent of t it suffices to look at the operators D_{A_t}. Observe they are complex linear and hence provides a trivialisation of this bundle by identifying the complex structures, in particular producing an isomorphism $\det(D_{A_0}) \to \det(D_{A_1})$ by identifying the orientations induced by the complex structure. However, since $\Phi_t = 0$ for all t the isomorphisms induced by the linearisation of the gauge group acts trivially on the D^+ component and the induced isomorphisms

$$\begin{aligned} \ker(D_{A_0}) \to \ker(df_{A_1}) &: \varphi \mapsto u^{-1}\varphi \\ \operatorname{coker}(D_{A_0}) \to \operatorname{coker}(D_{A_1}) &: \psi \mapsto u^{-1}\psi \end{aligned}$$

are complex linear and so induce the same map $\det(D_{A_0}) \to \det(D_{A_1})$ as the trivialisation of the determinant line bundle.

The Chamber Structure of the Parameter Space

To analyse the chamber structure of the invariants when $b^+(X) = 1$, define the *parameter space* Π, consisting of pairs of metrics g and imaginary-valued, g-self dual perturbations η, and define $\mathcal{W}$ to be the subset of Π for which the pair (g, η) admits reducible solutions, this is called *the wall.* Then set $\Pi^* = \Pi \setminus \mathcal{W}$ to be the space of parameters for which the Seiberg-Witten equations admit no reducible solutions.

For a fixed metric g on X, let $\Omega^{2,+}_\Gamma(X, i\mathbb{R}) := \{\eta \in \Omega^{2,+}(X, i\mathbb{R}) : \text{there exists } A \in \mathcal{A}(\Gamma) \text{ such that } F^+_A + \eta = 0\}$. The following proposition states that this is an affine subspace of codimension $b^+(X)$.

Proposition 4.2.6. *The set $\Omega^{2,+}_\Gamma(X, i\mathbb{R})$ is an affine subspace of codimension b^+ whose parallel vector space is the image of the operator $d^+ : \Omega^1(X, i\mathbb{R}) \longrightarrow \Omega^{2,+}(X, i\mathbb{R})$*

Proof. First we prove that this is indeed an affine subspace with the desired parallel vector space. Fix $\eta_0 \in \Omega^{2,+}(X, i\mathbb{R})$ and $A_0 \in \mathcal{A}(\Gamma)$ such that $F^+_{A_0} + \eta_0 = 0$. Now suppose that $\eta \in \Omega^{2,+}_\Gamma(X, i\mathbb{R})$, hence there exists some $A \in \mathcal{A}(\Gamma)$ such that $F^+_A + \eta = 0$. We may write $\eta - \eta_0 = F^+_{A_0} - F^+_A = F^+_{A-A_0}$ but recall that the curvature is obtained locally by $F_A = dA$. Hence $\eta - \eta_0 = d^+(A_0 - A)$ and so $\Omega^{2,+}_\Gamma(X, i\mathbb{R}) \subseteq \eta_0 + \mathrm{im}(d^+)$.

To see the reverse inclusion, suppose that $\eta = \eta_0 + d^+\alpha$, then see that $F^+_{A_0-\alpha} + \eta = d^+A_0 + \eta_0 = 0$ hence $\eta + \mathrm{im}(d^+) \subseteq \eta \in \Omega^{2,+}_\Gamma(X, i\mathbb{R})$ so indeed, $\Omega^{2,+}_\Gamma = \eta_0 + \mathrm{im}(d^+)$.

Therefore $\Omega^{2,+}_\Gamma(X, i\mathbb{R})$ an affine subspace whose parallel vector space is the image of d^+. It remains to show it has the desired codimension $\Omega^{2,+}(X, i\mathbb{R})$. To do so, we simply prove the following direct sum decomposition.

$$\Omega^{2,+}(X, i\mathbb{R}) = \mathcal{H}^{2,+}(X, i\mathbb{R}) \oplus \mathrm{im}(d^+)$$

where $\mathcal{H}^{2,+}(X, i\mathbb{R})$ denotes the space of self-dual imaginary-valued harmonic 2-forms.

Suppose that $\tau \in \Omega^{2,+}(X, i\mathbb{R})$, by the Hodge decomposition

$$\tau = \tau_0 + d\alpha + d^*\gamma$$

where τ_0 is harmonic, $\alpha \in \Omega^1(X, i\mathbb{R})$ and $\gamma \in \Omega^3(X, i\mathbb{R})$. Define $\beta := - \star \gamma$, then

$$\tau = \tau_0 + d\alpha + \star d\beta$$

note that this decomposition is unique with respect to β, α. Since τ is self-dual and $\star^2 = 1$ on 2-forms

$$\tau_0 + d\alpha + \star d\beta = \tau = \star\tau = \star\tau_0 + \star d\alpha + d\beta.$$

since this decomposition is a direct sum decomposition, it follows that

$$\tau_0 = \star\tau_0, \qquad d\alpha = d\beta$$

therefore

$$\tau = \tau_0 + d\alpha + \star d\alpha = \tau_0 + 2d^+\alpha$$

where $\tau_0 \in \mathcal{H}^{2,+}(X, i\mathbb{R})$ and $\alpha \in \Omega^1(X, i\mathbb{R})$ and moreover every such choice of τ_0, α gives us an element $\tau \in \Omega^{2,+}(X, i\mathbb{R})$.

Since every self-dual harmonic 2-form is orthogonal to the image of d^+ it follows that we have the direct sum decomposition

$$\Omega^{2,+}(X, i\mathbb{R}) = \mathcal{H}^{2,+}(X, i\mathbb{R}) \oplus \mathrm{im}(d^+).$$

and the result on the codimension immediately follows. □

It follows that $\Omega^{2,+}(X, i\mathbb{R}) \setminus \Omega^{2,+}_{\Gamma}(X, i\mathbb{R})$ is connected for $b^+(X) > 1$ and split into two connected components for $b^+(X) = 1$. When $b^+(X) = 0$, every perturbation admits a reducible solution, since the moduli space will never be a smooth manifold, we avoid attempting to define a Seiberg-Witten invariant in this case. Consequently the space Π^* exhibits the same properties, namely that it is connected for $b^+(X) > 1$ and split into two connected components for $b^+(X) = 1$.

Recall from Hodge theory, given a metric g on X and an orientation on $\mathcal{H}^{2,+}(X, i\mathbb{R})$, there exists a unique self-dual harmonic 2-form $\omega_g \in H^{2,+}(X, i\mathbb{R})$ with $\|\omega_g\|_{L^2} = 1$ and represents the orientation chosen on $\mathcal{H}^{2,+}(X, i\mathbb{R})$. Define

$$\varepsilon_\Gamma(g, \eta) := -\int_X \langle i\eta, \omega_g \rangle \,\mathrm{vol} - \pi[\omega_g] \cdot c_1(L_\Gamma) \tag{4.9}$$

since $(i/\pi)F_A$ is a representative for $c_1(L_\Gamma)$ in deRham cohomology, it is easy to show that

$$(g, \eta) \in \mathcal{W} \iff \varepsilon_\Gamma(g, \eta) = 0.$$

4.3 The Seiberg-Witten Invariant

We now proceed to define the Seiberg-Witten invariant on a compact oriented 4-manifold X, this is dependent on the choice of spin^c structure and is a diffeomorphism invariant of the 4-manifold when $b^+(X) > 1$.

The idea of the construction is as follows. Choose a sufficiently generic metric and perturbation η such that the moduli space $\mathcal{M}$ is a smooth compact oriented 4-manifold of dimension d. Being a compact oriented submanifold of $\mathcal{B}(\Gamma) := \mathcal{C}(\Gamma)^*/\mathcal{G}$, the moduli space has a fundamental class, thus we may pair it with cohomology classes in $\mathcal{B}(\Gamma)$ to obtain an integer, at the level of deRham cohomology this simply corresponds to integration over $\mathcal{M}$. The Seiberg-Witten invariant is then obtained by considering a natural cohomology class $\tau \in H^2(\mathcal{C}(\Gamma)^*/\mathcal{G};\mathbb{Z})$, if d is even the Seiberg-Witten invariant will be the integral of $\tau^{d/2}$ over $\mathcal{M}$, otherwise it is defined to be zero. If $b_1(X) = 0$, then the the space $\mathcal{B}(\Gamma)$ has the homotopy type of $\mathbb{CP}^\infty$, thus its cohomology ring is $\mathbb{Z}[x]$ and τ corresponds to its generator.

We shall detail two methods to construct this cohomology class τ, the first being a simple example of a more general gauge-theoretic construction of Donaldson, namely the μ-map. This involves a choice of basepoint $x_0 \in X$, although the cohomology class obtained in this way is independent of such a choice. However, this construction is not particularly amenable to obtaining an invariant in families Seiberg-Witten theory, requiring the existence of a non-canonical section of the family, hence we also outline a simpler construction of τ in the special case when $b_1(X) = 0$ which generalises easily in the families case as seen in Chapter 6.

For the first construction we introduce the *based gauge group* $\mathcal{G}(x_0) = \{u \in \mathcal{G} : u(x_0) = 1\}$ and consider the space

$$\mathcal{B}(\Gamma, x_0) = \frac{\mathcal{A}(\Gamma) \times C^\infty(X, W^+)^*}{\mathcal{G}(x_0)}.$$

Note that $\mathcal{G}/\mathcal{G}_{x_0} \cong S^1$ so $\mathcal{B}(\Gamma, x_0)$ forms a principal circle bundle over $\mathcal{B}(\Gamma)$ with circle action given by $[A,\Phi] \mapsto [A, e^{i\theta}\Phi]$. If x_1 is another point in X and X is connected then by choosing a smooth path $\gamma : [0,1] \to X$, fix a smooth reference connection A_0 and define $\rho_\gamma : \mathcal{A}(\Gamma) \to S^1$ by

$$\rho_\gamma(A_0 + \alpha) = \exp\left(\int_0^1 \alpha_{\gamma(t)}(\gamma'(t))dt\right)$$

this gives rise to a bundle isomorphism $\mathcal{C}(\Gamma, x_0)/\mathcal{G} \to \mathcal{C}(\Gamma, x_1)/\mathcal{G}$ given by

$$[A,\Phi]_0 \mapsto [A, \rho_\gamma(A)\Phi].$$

Consequently, the isomorphism class of $\mathcal{B}(\Gamma, x_0)$ is independent of the base-point x_0, we then let $\tau \in H^2(\mathcal{B}(\Gamma); \mathbb{Z})$ be the Euler class of the circle bundle $\mathcal{B}(\Gamma, x_0) \to \mathcal{B}(\Gamma)$. Alternatively one can observe that by nature of being a principal circle bundle, there is an associated unitary line bundle $\mathcal{L} \to \mathcal{B}(\Gamma)$ and take τ to be the Chern class, this gives the same cohomology class but lends itself to an easier computation of the invariant.

For the second construction, assume that $b_1(X) = 0$ and define the *reduced gauge group*

$$\mathcal{G}_0 := \left\{ g \in \mathcal{G} : g = e^{if} \text{ for some } f : X \to \mathbb{R} \text{ such that } \int_X f\text{vol} = 0 \right\}.$$

Since $b_1(X) = 0$, every such g can be written as e^{if} for some smooth function $f : X \to \mathbb{R}$, hence we obtain the following exact sequence.

$$0 \to \mathcal{G}_0 \to \mathcal{G} \to S^1 \to 0$$

where the map $\mathcal{G}_0 \to \mathcal{G}$ is inclusion and the map $\mathcal{G} \to S^1$ is given by $e^{if} \mapsto e^{\int_X f\text{vol}}$. Therefore $\mathcal{G}/\mathcal{G}_0 \cong S^1$ and identically to the previous construction we may define the space

$$\mathcal{B}(\Gamma)_0 = \frac{\mathcal{A}(\Gamma) \times C^\infty(X, W^+)^*}{\mathcal{G}_0}$$

which forms a principal circle bundle over $\mathcal{B}(\Gamma)$ and we may again take the τ to be the Chern class of the associated line bundle, this also represents the generator of $\mathbb{CP}^\infty$ and so indeed both methods produce the same cohomology class.

Definition 4.3.1. Let X be a compact oriented connected 4-manifold with $b^+(X) > 0$, choose a metric g, a spin^c structure on X and η a self-dual imaginary valued 2-form. If $d(X, \Gamma)$ is even, the *Seiberg-Witten invariant* is defined as

$$\text{SW}(X, \Gamma; g, \eta) = \int_{\mathcal{M}(X,\Gamma,g,\eta)} \tau^{\frac{d}{2}}$$

and zero otherwise.

Theorem 4.3.2. *Assume $b^+(X) > 1$, then the Seiberg-Witten invariant $SW(X, \Gamma; g, \eta)$ is independent of the choice of g and η and only depends on the isomorphism class of Γ.*

Proof. See Theorem 6.2.2 in the following chapter, the proof in the case of $B = \{\text{pt}\}$ gives the result. □

4.4 Important Results

There is a natural involution on spin^c structures $\Gamma \mapsto \overline{\Gamma}$ obtained by reversing the complex structure on W. There is a bijection between the moduli spaces $\mathcal{M}(X, \Gamma, g, \eta)$ and $\mathcal{M}(X, \overline{\Gamma}, g, -\eta)$ which is a diffeomorphism when η is a regular perturbation for the moduli space corresponding to Γ (equivalently $-\eta$ is a regular perturbation for the moduli space corresponding to $\overline{\Gamma}$). However, this diffeomorphism is not orientation preserving in general, changing the sign via the complex index of the Dirac operator, it also reverses the sign of the first Chern class of the line bundle $\mathcal{L} \to \mathcal{M}$ used to define the Seiberg-Witten invariant. It follows that the Seiberg-Witten invariant under this diffeomorphism changes by a sign of

$$(-1)^{\mathrm{ind}_{\mathbb{C}}(D_A) - \dim(\mathcal{M})/2} = (-1)^{(\sigma(X)+\chi(X))/4}.$$

Furthermore, when $b^+(X) = 1$ we have $\varepsilon_{\overline{\Gamma}}(g, -\eta) = -\varepsilon_{\Gamma}(g, \eta)$, since this switches the sign of ε, the chamber of perturbation for the Seiberg-Witten invariant is switched under the involution $\Gamma \mapsto \overline{\Gamma}$, giving the following result.

Proposition 4.4.1. *Let X be a compact oriented smooth 4-manifold and $\Gamma : TX \to End(W)$ be a spin^c structure. Then if $b^+(X) > 1$*

$$SW(X, \overline{\Gamma}) = (-1)^{\frac{\sigma(X)+\chi(X)}{4}} SW(X, \Gamma)$$

and if $b^+(X) = 1$

$$SW^+(X, \overline{\Gamma}) = (-1)^{\frac{\sigma(X)+\chi(X)}{4}} SW^-(X, \Gamma).$$

From our earlier discussion when $b^+(X) = 1$ the Seiberg-Witten invariant depends on a choice of chamber, nonetheless there is still a relation between the invariants on both sides of the wall, a so called *wall-crossing formula.* The first instance of a wall-crossing formula for the Seiberg-Witten invariants was given by Kronheimer and Mrowka in [KM94] when $b_1(X) = 0$ for zero-dimensional moduli spaces, and a general relation was later obtained by Li and Liu in [LL95] and independently by Ohta and Ono in [OO96]. The following theorem is a special case of the general wall crossing formula when $b_1(X) = 0$ and the moduli space is non-negative

Theorem 4.4.2. *Let X be a compact, oriented and connected smooth 4-manifold with $b^+(X) = 1$ and $b_1(X) = 0$. Let Γ be a spin^c structure on X and further suppose that the dimension of the moduli space is non-negative, that is $c_1(L_\Gamma)^2 - 2\chi(X) - 3\sigma(X) \geq 0$. Then*

$$SW^+(X, \Gamma) - SW^-(X, \Gamma) = 1.$$

Chapter 5

Kähler Seiberg-Witten Theory

This chapter concerns the simplifications that occur in Seiberg-Witten theory when the manifold of discussion is Kähler, which inevitably allows for a computation of the Seiberg-Witten invariant. In particular we cover simplifications to the Seiberg-Witten equations that will, given a particular choice of perturbation, allow for an easy description of the moduli space. This in conjunction with a particular exact sequence allows for a computation of the Seiberg-Witten invariant for Kähler surfaces with $b_1(X) = 0$. We then conclude the chapter by overviewing some constraints that are obtained from this computation on the cohomology of line bundles required for a non-vanishing invariant. Much of the content covered in this chapter can be found in [Nic00],as well as [Sal99], which in particular details the computation of the invariant on Kähler surfaces.

5.1 Simplifications in the Kähler Case

Recall that a Kähler surface is a smooth 4-manifold equipped with three mutually compatible objects, an integrable complex structure $J \in \mathrm{End}(TX)$ making it a complex surface, a Riemannian metric g and a symplectic structure, i.e. a closed, non-degenerate 2-form $\omega \in \Omega^2(X)$. Symplectic and almost-complex structures give rise to a canonical spin^c structure as in Definition 3.3.1, this causes the Seiberg-Witten equations to simplify greatly.

Recall from Proposition 3.3.6 that since X is Kähler, the Levi-Civita connection induces a canonical spin^c connection ∇_{can} on W_{can}. Any other spin^c structure via twisting by a Hermitian line bundle $L \to X$, given a Hermitian connection $B \in \mathcal{A}(L)$ there is a spin^c connection compatible with the Levi-Civita connection for the twisted spin^c structure $\Gamma_L = \Gamma_{\text{can}} \otimes L$ and the induced virtual connection is denoted $A_{\text{can}+B} \in \mathcal{A}(\Gamma_L)$. Because of this we often write the Seiberg-Witten invariant of X corresponding to the spin^c structure Γ_L as $\mathrm{SW}(X, L)$ when X is Kähler. The Seiberg-Witten equations in the Kähler case then simplify to the following form.

Proposition 5.1.1. *Let X be a Kähler surface and $\eta \in i\Omega^{2,+}(X)$, the perturbed Seiberg-Witten equations then acts on pairs (A_{can+B}, Φ) and are given by*

$$\overline{\partial}_B \varphi_0 + \overline{\partial}_B^* \varphi_2 = 0 \tag{5.1}$$

$$2(F_B + \eta)^{0,2} = \overline{\varphi_0}\varphi_2 \tag{5.2}$$

$$4i(F_{A_{can}} + F_B + \eta)_\omega = |\varphi_2|^2 - |\varphi_0|^2 \tag{5.3}$$

where $\Phi = (\varphi_0, \varphi_2) \in \Omega^{0,0}(X, E) \times \Omega^{0,2}(X, E)$, η is a self-dual imaginary valued 2-form and for a 2-form $\tau \in \Omega^2(X, \mathbb{C})$, $\tau_\omega : X \longrightarrow \mathbb{C}$ is defined by

$$\omega \wedge \tau := \tau_\omega \omega \wedge \omega.$$

Proof. The content of Theorem 3.3.7 is precisely 5.1, Lemma 3.3.9 gives 5.3 and after applying that $F_{A_{\text{can}}}$ is a $(1,1)$ form when X is Kähler, Equation (5.2). □

Remark 5.1.2. Note that φ_0 is valued in the line bundle L and there is no naturally defined complex conjugation. Hence $\overline{\varphi}$ should be interpreted as a section of the bundle $\overline{L} = L^*$ with the reversed complex structure and the product $\overline{\varphi}_0\varphi_2$ should be interpreted as the tensor product.

Proposition 5.1.3. *Suppose that X is connected, let $B \in \mathcal{A}(E), \varphi_0 \in \Omega^{0,0}(X, E)$ and $\varphi_2 \in \Omega^{0,2}(X, E)$ which satisfy the Seiberg-Witten equations for a Kähler surface 5.1, 5.2, 5.3, with $\eta \in \Omega^{1,1} \cap \Omega^{2,+}$. Then either $\varphi_0 = 0$ or $\varphi_2 = 0$.*

Proof. Apply $\overline{\partial}_B$ to both sides of 5.1, since $\overline{\partial}_B^2 = 0$ it follows that

$$0 = \overline{\partial}_B \overline{\partial}_B^* \varphi_2 + \overline{\partial}_B \overline{\partial}_B \varphi_0$$

hence

$$\overline{\partial}_B \overline{\partial}_B^* \varphi_2 = -\overline{\partial}_B \overline{\partial}_B \varphi_0 = -F_B^{0,2} \varphi_0$$

since by assumption η is a $(1,1)$-form, its $(0,2)$ component is zero, so the above and 5.2 gives

$$\overline{\partial}_B \overline{\partial}_B^* \varphi_2 = -\frac{1}{2}|\varphi_0|^2 \varphi_2$$

taking the L^2 inner product of both sides of this with φ_2 gives

$$\int_X \left(|\overline{\partial}^*{}_B \varphi_2|^2 + \frac{1}{2}|\varphi_0|^2|\varphi_2|^2 \right) \mathrm{dvol} = 0$$

and hence $\overline{\partial}^* \varphi_2 = 0$, $\overline{\partial}_B \varphi_0 = 0$ and $\overline{\varphi_0}\varphi_2 = 0$.

Suppose that φ_2 does not vanish everywhere, then φ_0 must vanish on some open set by the third equation above, since the pair $(\varphi_0, 0)$ is in the kernel of the Dirac operator by being a solution to the Seiberg-Witten equations, by Theorem 3.2.6 φ_0 vanishes everywhere. If this does not hold, then φ_2 vanishes everywhere. Consequently, one of φ_0 and φ_2 must vanish. □

Note that similar methods to the proof above can be used to show that the Seiberg-Witten invariant is zero if the line bundle $L \to X$ is not holomorphic.

Given a perturbation η eq. (5.3) gives a means to determine which of φ_0 or φ_2 vanishes, integrating both sides gives

$$\frac{\|\varphi_2\|^2_{L^2} - \|\varphi_0\|^2_{L^2}}{2} = \pi(2c_1(L) - c_1(K^*)) \cdot [\omega] + \int_X i\eta \wedge \omega = \varepsilon_{\Gamma_L}(g, \eta)$$

where ε_{Γ_L} is as defined in Section 4.2. It is clear that if $\varepsilon_{\Gamma_L}(g, \eta) < 0$ then φ_2 vanishes.

When X is a Kähler surface, if $L \to X$ is a line bundle determining a spin^c structure $\Gamma_L : TX \to \mathrm{End}(W_L)$ and $\overline{\Gamma}_L : TX \to \mathrm{End}(\overline{W_L})$ is its dual, then there is a natural isomorphism $\overline{W}_L \cong W_{K\otimes L^*}$ which respects the Hermitian structure. If $k = 0, 1, 2$ and $\varphi_k \in \Omega^{0,k}(X, L)$ then there is a corresponding element $\widetilde{\varphi}_k \in \Omega^{k,0}(X, L^*)$, and the isomorphism is the following composition of maps $\Lambda^{k,0} \to \Lambda^{2,2-k} \to \Lambda^{0,2-k} \otimes K$ induced by

$$\Omega^{k,0}(X, L^*) \to \Omega^{2,2-k}(X, L^*) \to \Omega^{0,2-k}(X, K \otimes L^*)$$

given by

$$\widetilde{\varphi}_k \mapsto \widetilde{\varphi}_k \wedge \frac{(i\omega)^{2-k}}{(2-k)!} \mapsto \widetilde{\varphi}_k$$

consequently the dual spin^c structure on a Kähler surface is simply obtained by interchanging the line bundle L with $K \otimes L^*$. Via the previous duality formula in Proposition 4.4.1, it follows that

Proposition 5.1.4. *There is a natural bijection*

$$\mathcal{M}(X, \Gamma_L, g, \eta) \to \mathcal{M}(X, \Gamma_{K\otimes L^*}, g, \eta)$$

given by $(B, \varphi_0, \varphi_2) \mapsto (-B - 2A_{can}, \widetilde{\varphi}_0, \widetilde{\varphi}_2)$ *and* η *is regular for* Γ_L *if and only if* $-\eta$ *is regular for* $\Gamma_{K\otimes L^*}$ *and the Seiberg-Witten invariants are related by*

$$SW(X, K \otimes L^*) = (-1)^{\frac{\sigma(X)+\chi(X)}{4}} SW(X, L)$$

for $b^+(X) > 1$ *and*

$$SW^+(X, K \otimes L^*) = (-1)^{\frac{\sigma(X)+\chi(X)}{4}} SW^-(X, L)$$

for $b^+(X) = 1$.

It is natural to look at the linearisation of the Seiberg-Witten equations, we shall specifically consider the case when $\varphi_2 = 0$, for a solution $(B, \varphi_0, 0)$ the linearised equations in the Kähler case are as follows

$$-2i(d\alpha)_\omega - \mathrm{Re}(\overline{\varphi}_0 \tau_0) = 0 \tag{5.4}$$

$$\overline{\partial}\tau_0 + \overline{\partial}^* \tau_2 + \alpha^{0,1}\varphi_0 = 0 \tag{5.5}$$

$$2(d\alpha)^{0,2} - \overline{\varphi}_0 \tau_2 = 0 \tag{5.6}$$

acting on a triple $(\alpha, \tau_0, \tau_2) \in i\Omega^1(X, L) \oplus \Omega^{0,0}(X, L) \oplus \Omega^{0,2}(X, L)$.

It is convenient to introduce the gauge fixing condition

$$d^*\alpha - i\langle i\varphi_0, \tau_0\rangle \tag{5.7}$$

this asserts that (α, τ_0, τ_2) is orthogonal to the orbit of $(B, \varphi_0, 0)$ under the gauge action, completely eliminating the gauge group action. Equations 5.10 and 5.7 can equivalently be stated as

$$2\overline{\partial}\alpha_1 - \overline{\varphi}_0 \tau_0 = 0 \tag{5.8}$$

where $\alpha_1 = \alpha^{0,1}$.

This leads us to define the linearised operator

$$\tilde{\mathcal{D}}_{B,\varphi} \begin{pmatrix} \tau_0 \\ \alpha_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} \overline{\partial}^* \alpha_1 - \overline{\varphi_0}\tau_2/2 \\ \overline{\partial}_B \tau_0 + \overline{\partial}^*_B \tau_0 + \alpha_1 \varphi_0 \\ \overline{\partial}\alpha_1 - \overline{\varphi_0}\tau_2/2 \end{pmatrix} \tag{5.9}$$

which fits into a useful long exact sequence

Lemma 5.1.5. *(Mrowka) Let* $B \in \mathcal{A}(L)$ *and* $\varphi_0 \in C^\infty(X, L)$ *be nonzero with* $F_B^{0,2}$ *and* $\overline{\partial}_B \varphi_0 = 0$. *Then there is an exact sequence as follows.*

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(X,\mathcal{O}) & \longrightarrow & H^0(X,\mathcal{E}_B) & \longrightarrow & ker(\widetilde{\mathcal{D}}_{B,\varphi}) \\
 & & H^1(X,\mathcal{O}) & \longrightarrow & H^1(X,\mathcal{E}_B) & \longrightarrow & coker(\widetilde{\mathcal{D}}_{B,\varphi}) \\
 & & H^2(X,\mathcal{O}) & \longrightarrow & H^2(X,\mathcal{E}_B) & \longrightarrow & 0
\end{array}$$

where $\mathcal{O}$ is the structure sheaf of holomorphic functions on X, $\mathcal{E}_B$ is the sheaf of holomorphic sections of L with holomorphic structure given by $\overline{\partial}_B$.

Proof. [Sal99][Lemma 12.6] □

Theorem 5.1.6. *Let X be a Kähler surface with $b^+ > 1$, then X has Seiberg-Witten invariants*

$$SW(X, \Gamma_{can}) = 1, \qquad SW(X, \Gamma_{K^*}) = (-1)^{\frac{\sigma+\chi}{4}}$$

moreover, if $SW(X, \Gamma_L) \neq 0$, then $c_1(L)$ can be represented by a harmonic 2-form of type $(1,1)$ and we have

$$0 \leq c_1(L) \cdot [\omega] \leq c_1(K^*) \cdot [\omega]$$

equality can occur only if $L = \mathbb{C}$ or $L = K^*$.

Proof. By duality it suffices to prove that the Seiberg-Witten invariant for the canonical spin^c structure is 1.

Hence, take $L = \mathbb{C}$ to be the trivial bundle and consider the perturbation

$$\eta = -F^+_{A_{\text{can}}} + i\pi\lambda\omega, \qquad \lambda > 0$$

Then equation Equation (5.3) becomes

$$4i(F_B + i\pi\lambda\omega)_\omega = |\varphi_2|^2 - |\varphi_0|^2$$

which is equivalently

$$4i(dB)_\omega = 4\pi\lambda + |\varphi_2|^2 - |\varphi_0|^2$$

since L is a line bundle.

Since vol $= \frac{1}{2}\omega \wedge \omega$, integrating both sides gives

$$\begin{aligned}\int_X 4\lambda\pi + |\varphi_2|^2 - |\varphi_0|^2 \text{vol} &= 2i\int_X dB \wedge \omega \\ &= -\frac{4}{\pi} c_1(L) \cdot [\omega]\end{aligned}$$

since B is a connection on the line bundle L and hence the curvature $dB = F_B$ defines a representative for the first Chern class by $c_1(L) := [\frac{i}{2\pi} F_B]$. Since $L = \mathbb{C}$ is the trivial line bundle in this case, $c_1 = 0$, hence we have

$$\|\varphi_2\|^2 - \|\varphi\|^2 = -\lambda \int_X 4\pi \text{vol} < 0.$$

Since this particular choice of η is a self-dual $(1,1)-$form, one of φ_0, φ_2 must be zero and the other non-zero. But to have the above inequality, it must be the case that $\varphi_2 = 0$ everywhere and φ_0 is the non-zero component.

Consequently, the Kähler Seiberg-Witten equations for the trivial bundle reduce to

$$\overline{\partial}_B \varphi_0 = 0 \tag{5.10}$$

$$(dB)^{0,2} = 0 \tag{5.11}$$

$$4i(dB)_\omega = 4\pi\lambda - |\varphi_0|^2 \tag{5.12}$$

This has an obvious solution $B = 0, \varphi_0 = \sqrt{4\pi\lambda}$, this is in fact the only up to gauge equivalence.

Observe that if (B, φ_0) satisfies 5.10, then applying $2\overline{\partial}_B^*$ and by Proposition 2.4.2

$$\begin{aligned}0 &= 2\overline{\partial}_B^* \overline{\partial}_B \varphi_0 \\ &= d_B^* d_B \varphi_0 - 2i(dB)_\omega \varphi_0\end{aligned}$$

Then, taking the inner product of both sides and rearranging, see that

$$\|d_B\varphi_0\|_{L^2}^2 = \int_X 2i(dB)_\omega |\varphi_0|^2 \text{vol}$$

and since a trivial bundle has vanishing first Chern class, the integral of $(dB)_\omega$ over X is zero. We may then insert a term so that

$$\begin{aligned}\|d_B\varphi_0\|_{L^2}^2 &= \int_X 2i(dB)_\omega \left(|\varphi_0|^2 - 4\pi\lambda\right) \text{vol} \\ &= -\frac{1}{2}\int_X \left(|\varphi_0|^2 - 4\pi\lambda\right)^2 \text{vol}\end{aligned}$$

where the last result comes from 5.12, hence

$$\|d_B\varphi_0\|_{L^2}^2 + \frac{1}{2}\int_X \left(|\varphi_0|^2 - 4\pi\lambda\right)^2 \mathrm{vol} = 0$$

since both terms are non-negative, $d_B\varphi_0 = 0$ and $|\varphi_0|^2 \equiv 4\pi\lambda$

The second result implies that there is some smooth function $u' : X \longrightarrow S^1$ such that $\varphi_0(x) = u'(x)\sqrt{4\pi\lambda}$, note that $u(x) := u'(x)^{-1}$ is also a smooth map into the circle, so equivalently we have the existence of a smooth map $u(x) : X \longrightarrow S^1$ such that $\varphi_0 = u(x)^{-1}\sqrt{4\pi\lambda}$.

Now examine the first equation $d_B\varphi_0 = 0$, this gives

$$\begin{aligned} 0 &= d_B\varphi_0 \\ &= (d+B)(u^{-1}\sqrt{4\pi\lambda}) \\ &= d(u^{-1}\sqrt{4\pi\lambda}) + u^{-1}\sqrt{4\pi\lambda}B \\ &= -u^{-2}du\sqrt{4\pi\lambda} + u^{-1}B\sqrt{4\pi\lambda} \\ &= \left(u^{-1}B - u^{-2}du\right)\sqrt{4\pi\lambda} \end{aligned}$$

since $\lambda > 0$, we must have $u^{-1}B - u^{-2}du = 0$ and so $B = u^{-1}du$.

Hence there is some smooth function $u : X \longrightarrow S^1$ such that $B = u^{-1}du$ and $\varphi_0(x) = u(x)^{-1}\sqrt{4\pi\lambda}$. However, the entire gauge group is $\mathrm{Maps}(X, S^1)$ which acts on the obvious solution in precisely this way, hence there is always a gauge transformation to the obvious solution. Therefore, it is the only solution up to gauge equivalence.

Now recall the exact sequence of Lemma 5.1.5. All the maps $H^i(X,\mathcal{O}) \longrightarrow H^i(X,\mathcal{E}_B)$ are multiplication by φ_0, since L is trivial, the maps are all isomorphisms. Hence the kernel and cokernel of $\tilde{\mathcal{D}}_{B,\varphi}$ is zero. Hence the perturbation η is regular and the moduli space is zero-dimensional. Since there is only one solution up to gauge equivalence, it must be the case that

$$\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = \pm 1.$$

To determine the actual value of the invariant, the orientation of the moduli space must be considered. The sign can be obtained by trivialising the determinant line bundle over the following path of operators

$$t \mapsto \widetilde{\mathcal{D}}_{B,t\varphi}.$$

Note that for $t > 0$, that $t\varphi \neq 0$ and the pair $(B, t\varphi_0)$ satisfy the conditions for Lemma 5.1.5 and so by the same exact sequence, the kernels and cokernels are 0-dimensional for $t > 0$ as well, i.e. the operators are bijective. Therefore it remains to examine $t = 0$ where there is a crossing along the path. Notice that the operator for all t is given by

$$\widetilde{\mathcal{D}}_{B,t\varphi}\begin{pmatrix}\tau_0\\ \alpha_1\\ \tau_2\end{pmatrix} = \begin{pmatrix}\overline{\partial}^*\alpha_1\\ \overline{\partial}_B\tau_0 + \overline{\partial}^*\tau_2\\ \overline{\partial}\alpha_1\end{pmatrix} + t\sqrt{4\pi\lambda}\begin{pmatrix}-\tau_0/2\\ \alpha_1\\ -\tau_2/2\end{pmatrix}$$

and so $\widetilde{\mathcal{D}}_{B,0}$ is simply given by

$$\widetilde{\mathcal{D}}_{B,0} = \begin{pmatrix}\overline{\partial}^*\alpha_1\\ \overline{\partial}_B\tau_0 + \overline{\partial}^*_B\tau_2\\ \overline{\partial}\alpha_1\end{pmatrix}$$

with

$$\begin{aligned}\ker(\widetilde{\mathcal{D}}_{B,0}) &= \mathcal{H}^{0,0}(X) \oplus \mathcal{H}^{0,1}(X) \oplus \mathcal{H}^{0,2}(X)\\ \operatorname{coker}(\widetilde{\mathcal{D}}_{B,0}) &= \mathcal{H}^{0,0}(X) \oplus \mathcal{H}^{0,1}(X) \oplus \mathcal{H}^{0,2}(X)\end{aligned}$$

by making identifications between harmonic forms and cohomology groups, we know that the kernel and cokernel are even dimensional dimensional. Hence the contribution of the $t = 0$ crossing is $+1$. A trivialisation of the determinant line bundle over the path gives an isomorphism $\det(\widetilde{\mathcal{D}}_{B,0}) \longrightarrow \det(\widetilde{\mathcal{D}}_{B,\varphi})$ which sends $\sigma(\dot{D}_0) \longrightarrow \nu(\{D_t\})\sigma(\dot{D}_1)$ where $\sigma(D)$ is defined by $x_1 \wedge \cdots \wedge x_k \otimes y_1 \wedge \ldots y_k \in \det(D)$ where are bases of $x_1, ..., x_k \in \ker(D)$ and $y_1, ..., y_k \in \ker(D)^*$ such that $\langle y_i, \dot{D}x_j\rangle = \delta_{ij}$ and $\nu(\{D_t\})$ is $(-1)^{\dim(\ker(D_0))}$ and we write $\dot{D}_{t_0}$ for the operator defined by $d/dt(D_t x)|_{t_0}$ for a path of operators D_t.

Note that the sign of $\sigma(\dot{\mathcal{D}}_{B,0})$ is determined by whether the isomorphism $\ker(\mathcal{D}_{B,0}) \longrightarrow \operatorname{coker}(\mathcal{D}_{B,0})$, given by the restriction of $\mathcal{D}_{B,0}$ to the kernel followed by the projection onto the cokernel, is orientation preserving. If the above mentioned map is orientation preserving then $\sigma(\dot{D}_1)$ will induce the standard orientation and the opposite orientation otherwise.

Now observe that $\dot{\mathcal{D}}_{B,0}$ is complex linear, hence the mentioned isomorphism between the kernel and cokernel is orientation preserving. It follows that $\sigma(\dot{\mathcal{D}}_{B,0})$ is simply the standard form induced by the complex orientation, so the induced orientation from $\sigma(\dot{\mathcal{D}}_{B,\varphi})$ is the standard orientation. Consequently $\nu(B, \varphi_0, 0) = 1$ and so $\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1$.

Now to show the second part of the theorem. If $\mathrm{SW}(X, \Gamma_L) \neq 0$, then the moduli space $\mathcal{M}(X, \Gamma, g, i\lambda\omega)$ must be nonempty for any $\lambda \in \mathbb{R}$, consequently there exists a solution $(B, \varphi_0, \varphi_2)$ to the Seiberg-Witten equations for a Kähler surface, (5.1),(5.2) and (5.3) with $\eta = i\lambda\omega$. Take $\lambda > 0$ to be sufficiently large such that $\varepsilon_\Gamma < 0$ and so $\varphi_2 = 0$.

Then 5.10,5.11 become

$$\overline{\partial}_B \varphi_0 = 0 \tag{5.13}$$

$$F_B^{0,2} = 0 \tag{5.14}$$

which is equivalent to saying that L is a holomorphic line bundle with holomorphic structure $\overline{\partial}_B$, which admits a non-zero holomorphic section φ_0. From Proposition 2.4.3, it follows that

$$c_1(L) \cdot [\omega] \geq 0$$

with equality if and only if L is the trivial line bundle.

By duality, it is also the case that $\mathrm{SW}(X, \Gamma_{K^* \otimes L^*}) \neq 0$, by identical reasoning

$$c_1(K^* \otimes L^*) \cdot [\omega] \geq 0$$

with equality if and only if $K^* \otimes L^* = \mathbb{C}$ is the trivial line bundle, equivalently $L^* = K$, i.e. $L = K^*$.

Moreover, since $c_1(K^* \otimes L^*) = c_1(K^*) + c_1(L^*) = c_1(K^*) - c_1(L)$, then the above condition can equivalently be written as $c_1(K^*) \cdot [\omega] \geq c_1(L) \cdot [\omega]$.

Combining the two inequalities, then gives

$$0 \leq c_1(L) \cdot [\omega] \leq c_1(K^*) \cdot [\omega]$$

as required. □

A very important property of Kähler surfaces is that, provided $b^+(X) > 1$, they are of *simple type*, that is, only zero dimensional moduli spaces may yield a non-zero Seiberg-Witten invariant. This is precisely stated in the following proposition.

Proposition 5.1.7. *Let X be a compact Kähler surface and Γ a spin^c structure on X such that $SW(X, \Gamma) \neq 0$. Then the expected dimension of the moduli space is zero, i.e*

$$\langle c(L_\Gamma)^2, X \rangle = 2\chi(X) + 3\sigma(X).$$

In fact this result extends to the symplectic category as proven by Taubes in [Tau96]. Hence Kähler surfaces and more generally symplectic 4-manifolds provide a wealth of examples of manifolds that are of simple-type. To date, all known examples of 4-manifolds with $b^+(X) > 1$ are either of simple-type or it is unknown whether they are.

We now present a theorem, first proven by Bradlow in [Bra90] when studying the moduli space of *vortex pairs*, which is key to obtaining a more general computation of the Seiberg-Witten invariant for Kähler surfaces.

Theorem 5.1.8 (Bradlow)**.** *Let (X, ω, J, g) be a Kähler surface and $L \longrightarrow X$ a Hermitian line bundle, if*

$$0 \leq c_1(E) \cdot [\omega] < \frac{c_1(K^*) \cdot [\omega]}{2} + \lambda\, Vol(X),$$

then there is a natural bijection

$$\mathcal{M}(X, \Gamma_L, g, i\pi\lambda\omega) \cong Div^{eff}(X, c_1(L))$$

and if

$$\frac{c_1(K^*) \cdot [\omega]}{2} + \lambda\, Vol(X) < c_1(L) \cdot [\omega] \leq c_1(K^*) \cdot [\omega],$$

then there is a natural bijection

$$\mathcal{M}(X, \Gamma_L, g, i\pi\lambda\omega) \cong Div^{eff}(X, c_1(K^*) - c_1(L)).$$

Proof. Suppose that $(B, \varphi_0, \varphi_2)$ is a solution to the Kähler Seiberg-Witten equations 5.1,5.2,5.3 for the perturbation $\eta = i\lambda\omega$. Since $\eta \in \Omega^{1,1}(X)$, from Proposition 5.1.3, one of φ_0 or φ_2 must vanish. If the first condition above holds, this corresponds to $\varphi_2 = 0$, the other $\varphi_0 = 0$. The second assertion of the theorem follows from the first by applying duality, so without loss of generality assume that $\lambda > 0$ is sufficiently large so that φ_2 vanishes.

The Seiberg-Witten equations then become

$$F_B^{0,2} = 0 \tag{5.15}$$
$$\overline{\partial}\varphi_0 = 0 \tag{5.16}$$
$$4i(F_{A_{can}} + F_B)_\omega = 4\pi\lambda - |\varphi_0|^2 \tag{5.17}$$

Given a pair (B, φ_0) satisfying 5.15 and 5.16, up to unitary gauge equivalence there is in fact exactly one such pair in every complex gauge equivalence class.

A real gauge transformation of the form e^θ with $\theta : X \to \mathbb{R}$ acts on (B, φ_0) by

$$u^*B - B = \overline{\partial}\theta - \partial\theta, \qquad u^*\varphi_0 = e^{-\theta}\varphi_0$$

and the gauge transformed pair also satisfies 5.15 and 5.16. This gauge transformed pair satisfies 5.17 if and only if

$$4i(F_{u^*B})_\omega + |u^*\varphi_0|^2 = 4\pi\lambda - 4i(F_{A_{can}})_\omega$$

which can equivalently be written as the Kazdan-Warner equation

$$\Delta_g(-2\theta) + e^{-2\theta}|\varphi_0|^2 = 4\pi\lambda - 4i(F_B + F_{A_{can}})_\omega$$

since the integral of the term on the right hand side is positive by our assumptions on λ, this has a unique solution $\theta : X \to \mathbb{R}$.

Recall from the discussion in Section 2.3 that a connection B corresponds to a unique Cauchy-Riemann operator $\overline{\partial} = \overline{\partial}_B$ and that the condition 5.15 is equivalent to $\overline{\partial} \circ \overline{\partial} = 0$ and defining a holomorphic structure on the line bundle L. The condition of 5.16 then asserts the existence of a non-zero holomorphic section φ_0 of L. We have then shown that

$$\mathcal{M}(X, \Gamma_E, g, i\pi\lambda\omega) \cong \frac{\{(\overline{\partial}, s) : \overline{\partial} \circ \overline{\partial} = 0, \overline{\partial}s = 0, s \neq 0\}}{(\overline{\partial}, s) \equiv u^*(\overline{\partial}, s) = (u^{-1} \circ \overline{\partial} \circ u, u^{-1}s)}$$

but from Section 2.5 this can be identified with $\mathrm{Div}^{\mathrm{eff}}(X, c_1(L))$, giving the result. □

This observation allows one to obtain a computation of the Seiberg-Witten invariant on Kähler surfaces with $b_1(X) = 0$. A proof can be found in [Sal99], although the main result of Chapter 7, namely Theorem 7.1.1 being the computation of the families Seiberg-Witten invariant implies the following result for the unparametrised case, consequently we do not provide proof.

It follows from the above theorem and its proof that by choosing a perturbation $\eta = i\pi\lambda\omega$ with $\lambda > 0$ sufficiently large, the moduli space is empty unless L is a holomorphic line bundle. If L is indeed a holomorphic line bundle when $b_1(X) = 0$ the holomorphic structure is unique, hence the points of the moduli space can then be identified with the non-zero holomorphic sections of L up to gauge equivalence of line bundles.

$$\mathcal{M}(X, \Gamma_L, g, i\pi\lambda\omega) \cong \frac{H^0(X, L)) \setminus \{0\}}{\mathbb{C}^*} = \mathbb{P}(H^0(X, L)).$$

One may then obtain the following computation of the Seiberg-Witten invariant on Kähler surfaces with $b_1(X) = 0$.

Theorem 5.1.9. *Let X be a Kähler surface with $b_1(X) = 0$ and $L \longrightarrow X$ a holomorphic line bundle, if $p_g > 0$ and $h^0(L) > 0$ then*

$$SW(X, L) = \binom{h^1(L) - h^2(L)}{h^1(L) - h^2(L) + \rho_g}$$

If $\rho_g = 0$, and $\chi(X, L) \geq 1$ then

$$SW^+(X, L) = 1$$

and is zero otherwise.

Note that when $\rho_g > 0$ and $\chi(X, L) = \rho_g + 1$, then the formula for the Seiberg-Witten invariant in Theorem 5.1.9 can be rewritten as

$$\mathrm{SW}(X, L) = (-1)^{h^0-1}\binom{p_g - 1}{h^0 - 1}, \qquad \text{if } h^1 - h^2 < 0 < h^0$$

furthermore the Seiberg-Witten invariant is only non-zero when $h^1 - h^2 < 0 < h^0$.

5.2 Cohomological Restrictions from the Seiberg-Witten Invariants

We now present a few cohomological restrictions on Kähler surfaces with $b_1(X) = 0$ obtained from the Seiberg-Witten invaraints.

Proposition 5.2.1. *Let X be a Kähler surface with $b_1(X) = 0$ and $\rho_g > 0$. Let L be a holomorphic line bundle on X with $h^0(L) > 0$ and $\chi(L) \neq 1 + \rho_g$. Then $h^1(L) - h^2(L) \geq 0$*

Proof. Since $h^0(L) > 0$ implies that $h^1(L) - h^2(L) - \rho_g \geq 0$, it also implies via the computation of the Seiberg-Witten invariant in Theorem 5.1.6 that $\mathrm{SW}(X, L) = \binom{h^1(L)-h^2(L)}{h^1(L)-h^2(L)-\rho_g}$. However, since $\chi(L) \neq 1 + \rho_g$, the dimension of the moduli space is non-zero, since X is Kähler it is of simple type (Proposition 5.1.7) so $\mathrm{SW}(X, L) = 0$. The only way for this to occur when $h^1(L) - h^2(L) - \rho_g \geq 0$ with $\rho_g > 0$ is when $h^1(L) - h^2(L) \geq 0$. □

Proposition 5.2.2. *Let X be a Kähler surface with $b_1(X) = 0$ and $\rho_g > 0$. Let L be a holomorphic line bundle on X with $\chi(L) = 1 + \rho_g$ and assume that $h^0(L), h^2(L) > 0$. Then $SW(X, L) \neq 0$, $h^1(L)$ is even and either $h^0(L) = h^2(L)$ or $h^1(L) = 0$. If ρ_g is even then $h^1(L)$ is necessarily zero.*

Proof. Since $h^2(L) > 0$ by Serre duality there is a non-zero holomorphic section s of $K \otimes L^*$, multiplication by s then defines an injection $H^0(X, L) \to H^0(X, K)$, since ρ_g is the dimension of $H^0(X, K)$, $h^0(L) \leq \rho_g$.

Recall that from Theorem 5.1.9 the Seiberg-Witten invariant is given by

$$\mathrm{SW}(X, L) = (-1)^{h^0(L)-1} \binom{\rho_g - 1}{h^0(L) - 1}$$

since $0 < h^0(L) \leq \rho_g$ this is non-zero. Applying charge conjugation duality and Serre duality gives

$$\mathrm{SW}(X, L) = (-1)^{h^2(L)+\rho_g} \binom{\rho_g - 1}{h^2(L - 1)}.$$

Firstly, by comparing signs it is clear that $h^1(L) + h^2(L) = 1 + \rho_g (\mathrm{mod} 2)$, this alongside the condition $\chi(L) = 1 + \rho_g$ implies that $h^1(L)$ is even.

We now compare binomial coefficients, either $h^0(L) = h^2(L)$ in which case the condition that $\chi(L) = 1+\rho_g$ forces ρ_g to be odd, or $h^0(L)+h^2(L) = 1+\rho_g$, comparing this with $\chi(L) = 1 + \rho_g$ implies that $h^1(L) = 0$. □

Proposition 5.2.3. *Let X be a Kähler surface with $b_1(X) = 0$ and $\rho_g > 0$. Let L be a holomorphic line bundle on X with $h^0(L), h^2(L) > 0$. Then $d(X, L) \leq 0$*

Proof. Suppose that $d(X, L) \neq 0$, by 5.2.1 it follows that $h^2(L) - h^1(L) \leq 0$ and 2.3.7 gives $h^0(L) \leq \rho_g$, hence

$$\begin{aligned} \frac{d(X, L)}{2} &= \chi(X) - 1 - \rho_g \\ &\leq h^0(L) - 1 - \rho_g \\ &\leq -1 \\ &< 0 \end{aligned}$$

thus $d(X, L) < 0$. Consequently, $d(X, L) \leq 0$. □

Part III

Families Seiberg-Witten Theory on Kähler Manifolds

Chapter 6

Families Seiberg-Witten Theory

First we shall introduce families of 4-manifolds, these are smooth fibre bundles with a base B and fibres diffeomorphic to a fixed 4-manifold X. Then we shall define the various constructions on families which pertain to Seiberg-Witten theory, this allows an exposition on the moduli space of solutions to the Seiberg-Witten equations in a family. We shall provide proofs for the key properties of the moduli space, namely transversality and compactness. The proofs of these results are simple adaptations of standard techniques to prove the corresponding results in the case of the ordinary Seiberg-Witten moduli space in the unparametrised case

We then proceed to give an overview of Seiberg-Witten theory for families of 4-manifolds. As mentioned in the introductory chapter, these techniques were first used in results such as [Nis02], [Rub98] and [Rub02] with the first general approach outlined in [LL01]. However, we opt for a slightly different generalisation to the Seiberg-Witten invariant, namely a series of Seiberg-Witten invariants for each integer $n \geq 0$. This construction is discussed in detail and more generality in [BK21], with the moduli space and resulting Seiberg-Witten invariants being a special case of the construction of the moduli spaces and invariants of families monopole maps. These families Seiberg-Witten invariants are valued in the cohomology ring of the base of the family with the n-th invariant being an element of the $2n-d$-th degree cohomology of the base where d is the expected dimension of the ordinary Seiberg-Witten moduli space. When $2n - d = 0$ this recovers the ordinary Seiberg-Witten invariant of X.

In particular, we define the families Seiberg-Witten invariants in the case when $b_1(X) = 0$, this can be done more generally for $b_1(X) > 0$ if the family $E \to B$ admits a section, although the resulting invariant is dependent on such a choice. Analogous to the ordinary Seiberg-Witten invariant, these require a choice of a smooth family of metrics and perturbations of the Seiberg-Witten equations, although the resulting invariants are independent of such choices given some minor restrictions. However, when these conditions are not met the chamber structure is much richer than the unparametrised case, and we shall briefly outline some of the differences with the unparametrised theory.

In the final section of the chapter, we shall discuss the case of Kähler families, leading up to the computation of the invariant on Kähler families in the following chapter.

6.1 Families of Manifolds and Seiberg-Witten Theory

Definition 6.1.1. Let X be a compact, oriented, smooth 4-manifold with $b^+ > 0$ and B a compact smooth manifold. We say that a *smooth family of 4-manifolds* with fibres X and base B is a smooth locally trivial fibre bundle $\pi : E \longrightarrow B$ which is fibrewise oriented and for each b there exists an orientation preserving diffeomorphism $\pi^{-1}(b) \cong X$ for all $b \in B$. Since the fibres are diffeomorphic to X, the fibre over $b \in B$ shall be denoted X_b.

We often call such a (smooth) family of X's over B. It is obvious that this definition could be generalised in many ways, for example removing compactness of the fibre or the base or the restriction on b^+. However, these assumptions are crucial to be able to define the Seiberg-Witten invariants and to obtain compactness and transversality of the moduli space, similar to the unparametrised case.

We now wish to define the required constructions for the families Seiberg-Witten invariants. Recall to define the Seiberg-Witten invariant we required a choice of spin^c structure, metric and self-dual 2-form as a perturbation of the Seiberg-Witten equations. The idea to define the families Seiberg-Witten invariants, is to choose a smoothly varying family of spin^c structures, metrics and perturbations which induce corresponding constructions on each fibre. Then the Seiberg-Witten equations can be considered in a family for each b, from which we can obtain a family of moduli spaces, yielding a total families moduli space.

Note that for any construction the induced object on the fibres will be denoted by subscript and the family may sometimes be referred to by the collection of objects on the fibres. For example, given a families metric g, the induced metric on the fibre X_b is denoted g_b and the family of metrics may be referred to as $g = \{g_b\}_{b \in B}$.

Definition 6.1.2. Let $\pi : E \longrightarrow B$ be a smooth fibre bundle, the differential $\pi_* : TE \to TB$ induces a map $d\pi : TE \to \pi^*(TB)$. The *vertical tangent bundle* is defined as $T(E/B) := \ker(d\pi)$ and is a vector bundle $T(E/B) \longrightarrow E$.

Since the kernel of a surjective submersion can be identified with the tangent spaces of submanifolds of the domain defined by the preimages of points, it follows that $T(E/B)|_{X_b} \cong TX_b$. As a consequence, the vertical tangent bundle can be viewed as the smooth collection of the tangent bundles of the fibres X_b. This gives rise to the following definitions of families metrics, perturbations and spin^c structures.

Definition 6.1.3. Let $\pi : E \longrightarrow B$ be a smooth fibre bundle, then a *smoothly varying family of metrics* is a metric on the vertical tangent bundle. That is, a smooth section g of $S^2T^*(E/B)$ for which the induced map $g|_{(X_b)_p} : T_pX_b \times T_pX_b \longrightarrow \mathbb{R}$ is a positive definite inner product for all $b \in B$ and $p \in X_b$.

Definition 6.1.4. Let $\pi : E \longrightarrow B$ be a smooth fibre bundle which is fibrewise oriented and g a smoothly varying family of metrics on the family, a *families perturbation* η with respect to g, is a smooth section of $\Lambda^2(T^*(E/B)) \otimes i\mathbb{R}$ where the induced 2-forms $\eta_b \in \Omega^2(X, i\mathbb{R})$ are self-dual with respect to the induced metrics g_b on X_b.

As in the unparametrised case, the families moduli space depends on a choice of perturbation and metric. The *parameter space* of metrics and perturbations for the Seiberg-Witten equations defines a subset $\Pi_E \subset \mathrm{Met}(T(E/B)) \times i\Omega^2(T(E/B))$

$$\Pi_E := \{(g, \eta) : \star_b \eta_b = \eta_b\}.$$

The subset of Π for which the families Seiberg-Witten equations corresponding to the pair (g, η) admit no reducibles is denoted Π_E^* and its complement $\Pi_E \setminus \Pi_E^*$ called the wall is denoted $\mathcal{W}$.

Definition 6.1.5. Let $\pi : E \longrightarrow B$ be a smooth family of 4-manifolds, we say that a *spin^c structure for the family* is a spin^c structure on the vertical tangent bundle $T(E/B) \longrightarrow E$. This amounts to a pair (W, Γ) where W is a complex hermitian vector bundle of rank 4 over E and a vector bundle homomorphism $\Gamma : T(E/B) \longrightarrow \mathrm{End}(W)$ satisfying

$$\Gamma(v)^* + \Gamma(v) = 0, \qquad \Gamma(v)^*\Gamma(v) = |v|^2 \mathrm{id}_{\mathrm{End}(W)}.$$

It immediately follows from the above definition that given a spin^c structure for a smooth family $E \longrightarrow B$, any other spin^c structure is given by tensoring by a complex line bundle on E. Consequently, the spin^c structures on a family is a torsor for the group $H^2(E;\mathbb{Z})$. It is then of interest to understand the cohomology of the total space of the family, the following computation can be made when the fibre X and the base space B are both simply connected.

Proposition 6.1.6. *Let $X \hookrightarrow E \longrightarrow B$ be a family of 4-manifolds with X and B both simply-connected and $H^3(X;\mathbb{Z}) = 0$ then*

$$H^2(X;\mathbb{Z}) \cong H^2(B;\mathbb{Z}) \oplus H^2(X;\mathbb{Z})$$

Proof. Since X is simply-connected, by the Hurcewicz theorem, $H_1(X;\mathbb{Z}) = 0$, then by Poincaré duality $H^3(X;\mathbb{Z}) = 0$. Since X is compact its homology groups are finitely generated so by the universal coefficient theorem, it follows that $H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z}) \oplus T_0$ where T_0 is the torsion subgroup of $H_0(X;\mathbb{Z})$, but X is connected so $H_0(X;\mathbb{Z}) = \mathbb{Z}$ which has no torsion. Hence $H^1(X;\mathbb{Z}) = 0$.

Since B is simply-connected, the action of $\pi_1(B)$ on the fibres is trivial and the E^2 page of the Leray-Serre spectral sequence consists of $E^2_{p,q} = H^p(B; H^q(X;\mathbb{Z}))$ and abuts to $H^{p+q}(E;\mathbb{Z})$, this then reads as

4	$H^0(B;\mathbb{Z})$	$H^1(B;\mathbb{Z})$	$H^2(B;\mathbb{Z})$	$\cdots$
3	0	0	0	$\cdots$
2	$H^0(B;H^2(X;\mathbb{Z}))$	$H^1(B;H^2(X;\mathbb{Z}))$	$H^2(B;H^2(X;\mathbb{Z}))$	$\cdots$
1	0	0	0	$\cdots$
0	$H^0(B;\mathbb{Z})$	$H^1(B;\mathbb{Z})$	$H^2(B;\mathbb{Z})$	$\cdots$
	0	1	2	3

the differentials all map to or out of 0 so the entries of the E^3 page are identical to the E^2 page and the differentials of the E^r page for $r \geq 4$ are clearly all zero, so in particular there is an exact sequence

$$H^2(B;\mathbb{Z}) \to H^2(E;\mathbb{Z}) \to H^0(B; H^2(X;\mathbb{Z})) \to H^3(B;Z)$$

but $H^0(B; H^2(X;\mathbb{Z}))$ is just $H^2(X;\mathbb{Z})$ since B is connected and $H^3(B;Z)$ is assumed to be zero. The exact sequence then reduces to

$$H^2(B;\mathbb{Z}) \to H^2(E;\mathbb{Z}) \to H^0(B; H^2(X;\mathbb{Z})) \to 0.$$

Again by the universal coefficient theorem we have $H^2(X;\mathbb{Z}) \cong \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) \oplus T_1$ where T_1 is the torsion subgroup of $H_1(X;\mathbb{Z})$, but $T_1 = 0$ since $\pi_1(X) = 0$. Since $\mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$ is free, it follows that $H^2(X;\mathbb{Z})$ is free and thus the above exact sequence splits and so

$$H^2(X;\mathbb{Z}) \cong H^2(B;\mathbb{Z}) \oplus H^2(X;\mathbb{Z}).$$

□

Remark 6.1.7. Note that the existence of a spin^c structure on the vertical tangent bundle is a stronger condition than necessary to define the Seiberg-Witten moduli space for the family. In fact, only a monodromy invariant spin^c structure on X is required [Bar19]. However, this condition is insufficient for defining a families Seiberg-Witten invariant valued in cohomology classes $H^{2n-d}(B;\mathbb{Z})$ with $n > 0$, consequently we shall only consider such spin^c structures which extend to the vertical tangent bundle, leading to the preceding definition.

It is also of note that distinct spin^c structures on the vertical tangent bundle can induce the same family of spin^c structures on the fibres, suppose S is a spin^c structure on the vertical tangent bundle of a family $E \longrightarrow B$ and that L is a line bundle over B with non-zero first Chern class. We then obtain another spin^c structure on the vertical tangent bundle by

$$S' = S \otimes \pi^*(L)$$

distinct from S. However, this induces the same family of spin^c structures on the family by local triviality of B.

With the setup now complete, the Seiberg-Witten equations can be defined for a family. Choose a smooth family of metrics $\{g_b\}$, smooth family of perturbation by self-dual 2-forms $\{\eta_b\}$ with respect to the choice of metric and spin^c structure for the family $\Gamma : T(E/B) \longrightarrow W$. The smooth family of metrics $\{g_b\}$ defines a unique canonical torsion-free, metric-compatible Levi-Civita connection on $T(E/B)$ constructed identically to the case on Riemannian manifolds which restricts to the Levi-Civita connection on each fibre.

Choose a spin^c connection on W compatible with the families Levi-Civita connection with associated virtual connection $\{A_b\} \in \mathcal{A}(\Gamma)$ on $\mathcal{L} = L_\Gamma^{1/2}$ where $L_\Gamma = \det(W)$ (recall such a connection and $\mathcal{L}$ may only exist locally but is notationally more convenient to work with compared to the globally induced connection $2A$ on L_Γ). There is then an induced Dirac operator $\mathcal{D}_A : C^\infty(E, W) \longrightarrow C^\infty(E, W)$ as in Definition 3.2.5, the splitting of W into $W^\pm$ then gives the restricted Dirac operator $D_A : C^\infty(E, W^+) \longrightarrow C^\infty(E, W^-)$ which induces a family of Dirac operators $D_{A_b} : C^\infty(X_b, W_b^+) \to C^\infty(X_b, W_b^-)$, the families Seiberg-Witten equations are then the following.

Definition 6.1.8. Let $X \hookrightarrow E \longrightarrow B$ be a smooth family of 4-manifolds equipped with a smooth family of metrics, self-dual perturbations with respect to the metric and a families spin^c structure. The *(perturbed) families Seiberg-Witten equations* are precisely the analogue of the ordinary Seiberg-Witten equations on the vertical tangent bundle, that is

$$D_A\Phi = 0 \tag{6.1}$$
$$\rho^+(F_A + \eta) = (\Phi\Phi^*)_0 \tag{6.2}$$

where $(\Phi, A) \in C^\infty(E, W^+) \times \mathcal{A}(\Gamma)$ and $\rho^+, (\Phi\Phi^*)_0$ are defined as in Chapter 3.

By restricting all the objects involved to fibres X_b, this determines a family of equations for each $b \in B$

$$D_{A_b}\Phi_b = 0 \tag{6.3}$$
$$\rho^{+_b}(F_{A_b} + \eta_b) = (\Phi_b\Phi_b^*)_0 \tag{6.4}$$

These are equations in $A_b \in \mathcal{A}(\Gamma_b)$ and $\Phi_b \in C^\infty(X, W_b^+)$

For each $b \in B$, the Seiberg-Witten equations on X_b are invariant under the action of the group $\mathcal{G}_b := \mathrm{Maps}(X_b, S^1)$, the disjoint union of these gauge groups determines the total space for a bundle of groups $\mathcal{G} \to B$ with fibres $\mathcal{G}_b$. Although $\mathcal{G}$ is not a group, it is a groupoid and it induces an action on the families Seiberg-Witten equations by acting fibrewise.

The *families moduli space* is then defined to be the solutions to the families Seiberg-Witten equations equations modulo the groupoid action of $\mathcal{G}$. The total moduli space is then the disjoint union of the ordinary moduli spaces

$$\mathcal{M}(E, S, g, \eta) := \bigsqcup_{b \in B} \mathcal{M}_b.$$

with obvious projection map $\mathcal{M} \longrightarrow B$ obtained from the disjoint union.

As in the unparametrised theory, the families Seiberg-Witten invariant will only depend on a choice of connected component of $\Pi \setminus \mathcal{W}$ called a *chamber*, the set of chambers denoted $\mathcal{CH}$. The chamber structure for the unparametrised theory was relatively simple, with one connected component when $b^+(X) > 1$ and two in the case of $b^+(X) = 1$. However, it is significantly more complicated in the parameterised theory. It was shown in [LL01], that the set of chambers is identified with fibrewise homotopy classes of sections into the bundle $\mathcal{P} \to B$, where $\mathcal{P}$ is called the *period bundle* $\mathcal{P} \to B$ and is a subbundle of $H^2 \to B$ where H^2 has fibres $H^2(X_b; \mathbb{R})$ over $b \in B$. We denote such homotopy classes of sections by $[B, \mathcal{P}]_f$. The period bundle $\mathcal{P}$ is homotopic to an S^p bundle over B, and the the space of fibrewise homotopy classes into such bundles is very complicated. When the family E is a trivial product bundle $E = X \times B$ or if the base B is simply connected, the set $[B, \mathcal{P}]_f$ can be identified with $[B, S^{b^+(X)-1}]$, the $b^+(X) - 1$-th *cohomotopy set of* B. This is the set of homotopy classes of maps $B \to S^{b^+(X)-1}$, given restrictions on B and $b^+(X)$ this can simplify the chamber structure somewhat, as done in Chapter 8 when we analyse the invariant for Kähler families over $B = S^2$.

Definition 6.1.9. We say that a solution $\{(A_b, \Phi_b)\}$ to the families Seiberg-Witten equations is *reducible* if (A_b, Φ_b) is reducible for some $b \in B$, that is, $\Phi_b \equiv 0$ for some $b \in B$.

Since we may represent $c_1(L_\gamma)$ by $(i/\pi)F_A$ in deRham cohomology, the families Seiberg-Witten equations admits reducible solutions if and only if

$$[\eta_b] = i\pi c_1(L_{\Gamma_b})^{+_{g_b}} =: w_b$$

for some b, where $\eta_b \mapsto [\eta_b]$ is the projection onto $H^2_+(X_b, g_b)$. If this condition *does not* occur then the families equations admit no reducible solutions.

Just as in the families case, we can impose restrictions on $b^+(X)$ to guarantee no reducibles.

Proposition 6.1.10. *Let $X \hookrightarrow E \to B$ be a smooth family of 4-manifolds with a families spin^c structure and assume that $b^+(X) > dim(B)$, then for a generic perturbation there are no reducible solutions to the families Seiberg-Witten equations.*

Proof. Let $H^{2,+}$ denote the disjoint union $\bigsqcup_{b\in B} H^2_+(X_b, g_b)$, since the spaces $H^2_+(X_b, g_b)$ all have dimension $b^+(X)$, they are all of constant dimension and so this defines a vector bundle $H^+ \longrightarrow B$.

Choose a generic section $[\eta_b]$ we may take this to be transverse to w_b. Since both $[\eta_b]$ and w_b are sections of this vector bundle, their image constitutes $\dim(B)$-dimensional submanifolds of the total space H^+, it follows that they both have codimension $b^+(X)$ in H^+. Since codimension is additive over transverse intersection, it follows that

$$U := \{(b, [\eta_b]) : [\eta_b] = w_b\}$$

has codimension $2b^+(X)$. However, it follows from the definition of codimension that $\dim(U) = b^+(X) + \dim(B) - \operatorname{codim}(U)$ so

$$\dim(U) = \dim(B) - b^+(X)$$

since $b^+(X) > \dim(B)$ this dimension is negative, hence U must be empty. □

We note that the following proofs of transversality, compactness and orientability proceed identically to the corresponding proofs for moduli spaces of families monopole maps as in [BK21][Section 2.1] and are also quite similar to the proofs in the unparametrised case as seen in Section 4.2.

Via a families version of the map and Theorem 1.1.6 as in Section 4.2 one can prove that the set of regular families perturbations is generic. For such perturbations the moduli space is a smooth manifold.

Theorem 6.1.11. *Let $X \hookrightarrow E \to B$ be a smooth family of 4-manifolds $(g, \eta) \in \Pi^*_{reg}$ and families spin^c structure Γ, then the families Seiberg-Witten moduli space is a smooth manifold of dimension dim$(B) + d(X, \Gamma)$ where $d(X, \Gamma)$ is the expected dimension of the unparametrised moduli space*

$$d(X, \Gamma) = \frac{\langle c(L_\Gamma)^2, [X]\rangle - 2\chi(X) - 3\sigma(X)}{4}$$

Proof. Fix a smooth reference connection $A_0 \in \mathcal{A}(\Gamma)$, it suffices to prove that the moduli space $\widetilde{\mathcal{M}}$ obtained by adding the gauge fixing condition $d^*(A - A_0) = 0$ is a smooth manifold. Since $b_1(X) = 0$, the only remaining non-trivial part of the action of $\mathcal{G}$ is an S^1 action which acts freely away from reducibles, consequently the moduli space $\mathcal{M}$ is a smooth manifold.

It suffices to consider the map $f : V \to W$ between Hilbert bundles given by

$$f(\alpha, \Phi) = (D_{A_0+\alpha}\Phi, d^+\alpha + \sigma((\Phi\Phi^*)_0) - F_{A_0}, d^*\alpha)$$

acting fibrewise, where

$$V = V_{\mathbb{C}} \oplus V_{\mathbb{R}}, \qquad W = W_{\mathbb{C}} \oplus W_{\mathbb{R}}$$

and $V_{\mathbb{C}}, W_{\mathbb{C}}, V_{\mathbb{R}}, W_{\mathbb{R}}$ have fibres

$$\begin{aligned} V_{\mathbb{C}} &:= L^2_k(X_b, W^+), & V_{\mathbb{R}} &:= iL^2_k(X_b, \Lambda^1 T^* X_b) \\ W_{\mathbb{C}} &:= L^2_{k-1}(X_b, W^-), & W_{\mathbb{R}} &:= iL^2_{k-1}(\Lambda^{2,+} T^* X_b) \oplus L^2_{k-1}(X_b, i\mathbb{R})_0 \end{aligned}$$

where $L^2_q(X_b, \mathbb{R})_0$ is the subspace satisfying $\int_{X_b} f \mathrm{vol}_b = 0$.

The map f can be further decomposed as $f = l + c$ where

$$l(\alpha, \Phi) = (D_{A_0}\Phi, d^+\alpha, d^*\alpha), \qquad c(\alpha, \Phi) = (\Gamma(\alpha)\Phi, \sigma((\Phi\Phi^*)_0) - F_{A_0}, 0).$$

The linearisation of f at (α, Φ) in the fibre directions is computed as

$$d_{(\alpha,\Phi)} l(a, \psi) = (D_{A_0}\psi, d^+ a, d^* a)$$

and

$$d_{(\alpha,\Phi)} c(a, \psi) = (\Gamma(a)\Psi, \sigma^+((\Phi\psi^* + \psi\Phi^*)_0)).$$

This is a perturbation of a Fredholm operator of index $d(X, \Gamma) + 1$ by a compact operator as in Section 4.2, consequently the total map is Fredholm with same index. Since the family of perturbations η is a section and can be viewed as a $\dim(B)$ dimensional submanifold of the codomain, it follows from Theorem 1.1.6 that $\tilde{\mathcal{M}} = f^{-1}(\eta)$ is a smooth submanifold of dimension

$$d(E, \Gamma) = d(X, \Gamma) + \dim(B) = \frac{\langle c(L_\Gamma)^2, [X] \rangle - 2\chi(X) - 3\sigma(X)}{4} + \dim(B).$$

□

Theorem 6.1.12. *Let $X \hookrightarrow E \to B$ be a smooth family of 4-manifolds with X and B both compact equipped with a smooth family of metrics and a families spin^c structure. Then the families Seiberg-Witten moduli space is compact.*

Proof. Assume $p > 4$, since B is compact, there is a finite trivalising cover U_α of the family $X \hookrightarrow E \to B$ such that $\pi^{-1}(U_\alpha) \cong U_\alpha \times X$. Applying the proof as in Theorem 4.2.5, we may gauge fix into Coulomb gauge and this gives a uniform bound on families of solutions over U_α, since the cover is finite, there is a uniform bound on solutions to the perturbed families Seiberg-Witten equations in L^p_k for all $k > 0$.

As before, it follows from Rellich's theorem and Theorem 1.2.2 that given any sequence of solutions to the families Seiberg-Witten equations which we may take to be gauge fixed into Coulomb gauge, there is a subsequence that converges in the C^∞ topology. Hence the families moduli space is compact.

□

In the families setting the moduli space is not necessarily oriented, although to obtain an invariant it suffices to require a more general notion of orientation.

Definition 6.1.13. Let $\pi : E \to B$ be a smooth fibre bundle, we say that it has a *relative orientation* if

$$TE \oplus \pi^*(TB)$$

is equipped with an orientation.

We can obtain a relative orientation similar to the unparametrised as in the following result

Theorem 6.1.14. *There is a natural isomorphism*

$$det(T\mathcal{M} \oplus \pi^*(TB)) \cong \pi^*(det(ind(l_{\mathbb{R}})))$$

where $ind(l_{\mathbb{R}})$ is the K-theory class of the family of operators $l_{\mathbb{R}}$ as in Theorem 6.1.11 and $\pi : \mathcal{M} \to B$ is the obvious projection.

Proof. The S^1 action acts via orientation preserving diffeomorphisms, so we must show there is an S^1 equivariant isomorphism

$$\det(T\widetilde{\mathcal{M}} \oplus \pi'^*(TB)) \cong \pi'^*(\det(\mathrm{ind}(l_{\mathbb{R}})))$$

where $\pi' : \widetilde{\mathcal{M}} \to B$ is the obvious projection.

As in the unparametrised case, there is an S^1-equivariant homotopy between the families of maps $df_{v_{v\in\tilde{\mathcal{M}}}}$ and $\{l\}_{v\in\tilde{\mathcal{M}}}$, consequently we obtain an S^1 equivariant isomorphism

$$\det(T\widetilde{\mathcal{M}}) \cong \pi'^*\det(\mathrm{ind}(l) \oplus \det(TB))$$

since l decomposes as $l_{\mathbb{C}}+l_{\mathbb{R}}$ and $l_{\mathbb{C}}$ is a complex Fredholm operator, $\det(\mathrm{ind}(l_{\mathbb{C}}))$ is trivial and we have

$$\det(\mathrm{ind}(l)) = \det(\mathrm{ind}(l_{\mathbb{C}})) + \det(\mathrm{ind}(l_{\mathbb{R}})) = \det(\mathrm{ind}(l_{\mathbb{R}}))$$

giving the desired result. □

Recall from the unparametrised case that over each fibre the kernel of l is given by $H^1(X_b; i\mathbb{R})$ and the cokernel by $H^0(X_b; i\mathbb{R}) \oplus H^{2,+}(X_b; i\mathbb{R})$, since we assume that $b_1(X) = 0$ and becuase $H^0(X_b; i\mathbb{R})$ already carries a natural orientation. Assume that $H^{2,+}(X_b; i\mathbb{R})$ has constant dimension over B so that it defines a vector bundle, one may then transport a relative orientation onto the families moduli space by assuming that the vector bundle H^+ with fibres $H^{2,+}(X_b; i\mathbb{R})$ is orientable, we shall make these assumptions from now on.

6.2 Defining the Families Seiberg-Witten Invariants

Recall in Section 4.3 the Seiberg-Witten invariant could defined by constructing a line bundle $\mathcal{L} \to \mathcal{B}$ via a subgroup of based gauge transformations and integrating its first Chern class over the moduli space $\mathcal{M}$. This involved a choice of basepoint x_0 but in the families case, this leads to a non-canonical choice of section x_0. Such a section does not necessarily exist for any smooth family and if one does exist for a given family, it is not known whether the families Seiberg-Witten invariant is independent of such a choice. Alternatively we described another method when $b_1(X) = 0$ which is independent of a choice of basepoint and we shall assume this throughout.

Consider the subgroup of the reduced gauge group $\mathcal{G}_b$

$$\mathcal{G}_{0,b} := \{g \in \mathcal{G}_b : g = e^{if}, f : X_b \longrightarrow \mathbb{R}, \int_{X_b} f \mathrm{vol}_{X_b} = 0\}$$

and recall for $b_1(X) = 0$ there is the following exact sequence

$$0 \to \mathcal{G}_0 \to \mathcal{G} \to S^1 \to 0$$

Define $\widetilde{\mathcal{M}}$ to be the moduli space factoring out by $\mathcal{G}_0$ instead of the full gauge group fibrewise. This then gives a principal circle bundle $\widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ since $\mathcal{G}/\mathcal{G}_0 \cong S^1$, denote the associated complex line bundle by $\mathcal{L} \longrightarrow \mathcal{M}$ and its first Chern class by $x \in H^2(\mathcal{M}; \mathbb{Z})$.

Since the moduli space $\mathcal{M}$ has a relative orientation there is a natural 'wrong way' map corresponding to integration over the fibres in deRham cohomology, if we further assume that B is oriented then $\mathcal{M}$ is oriented and it can be described as follows.

Since both $\mathcal{M}$ and B are both compact and oriented, via Poincaré duality there is a map $\pi_! : H^k(\mathcal{M}; \mathbb{Z}) \longrightarrow H^{k-d}(B; \mathbb{Z})$ where d is the expected dimension of the unparametrised moduli space. This map is defined via the following commutative diagram

$$\begin{array}{ccc} H^k(\mathcal{M}; \mathbb{Z}) & \xrightarrow{\pi_!} & H^{k-d}(B; \mathbb{Z}) \\ \Big\downarrow {\scriptstyle \text{Poincaré Duality}} & & \Big\uparrow {\scriptstyle \text{Poincaré Duality}} \\ H_{\dim(\mathcal{M})-k}(\mathcal{M}; \mathbb{Z}) & \xrightarrow[\pi_*]{} & H_{\dim(\mathcal{M})-k}(B; \mathbb{Z}) \end{array} .$$

This map is often denoted as

$$\pi_!(x) = \int_{E/B} x$$

due to its correspondence with integration over fibres.

Definition 6.2.1. For any $n \geq 0$ given a choice of families spin^c structure and chamber ϕ the *families Seiberg-Witten invariants* are

$$\mathrm{FSW}_n(E, s, \phi) \in H^{2n-d}(B; \mathbb{Z})$$

defined by

$$\mathrm{FSW}_n(E, s, \phi) = \pi_!(y^n)$$

where y is the first Chern class of the line bundle $\mathcal{L}$ defined previously and $\pi_!$ is the wrong way map.

Just as in the unparametrised case, provided $b^+(X) > \dim(B) + 1$ these invariants are independent of metric and a generic choice of perturbation. In the case when $1 \leq b^+(X) \leq \dim(B) + 1$ there is only a dependence of the invariants on a choice of chamber of the perturbation space.

Theorem 6.2.2. *Let $X \hookrightarrow E \longrightarrow B$ be a smooth family of 4-manifolds with $b^+(X) > dim(B) + 1$, then the Families Seiberg-Witten invariant is independent of choice of a smooth family of metrics and perturbations*

Proof. Suppose $g_{0,b}, g_{1,b}$ are both smooth families of metrics on $E \to B$ and $\eta_{0,b}$ and $\eta_{1,b}$ are choices of perturbations which are self-dual to $g_{0,b}$ and $g_{1,b}$ respectively. Since $b^+(X) > \dim(B) + 1$, the parameter space Π^* is path-connected, there is a generic path between $(g_{0,b}, \eta_{0,b})$ and $(g_{1,b}, \eta_{1,b})$ such that the perturbations intersect the map f as in Theorem 6.1.11 transversely and defines a $\dim(B) + 1$ dimensional submanifold of the target space.

It follows that the preimage of the path $\widetilde{\mathcal{M}}$ is a smooth compact manifold of dimension $d(X, \Gamma) + \dim(B) + 2$ with boundary $\partial\widetilde{\mathcal{M}} = \partial\widetilde{\mathcal{M}}_0 - \partial\widetilde{\mathcal{M}}_1$. The S^1 action acts freely giving the parametrised families moduli space $\mathcal{M}$ with the families moduli spaces corresponding to (g_0, η_0) and (g_1, η_1) on the boundary. There is a projection map $\pi : \mathcal{M} \to B \times [0, 1]$ that restricts to $\pi_1 : \mathcal{M}_1 \to B$ and $\pi_2 : \mathcal{M}_2 \to B$. The projection $\widetilde{\mathcal{M}} \to \mathcal{M}$ obtained from the S^1 action defines a line bundle $\mathcal{L}$ which restricts to the line bundles $\mathcal{L}_0 \to \mathcal{M}_0$ and $\mathcal{L}_1 \to \mathcal{M}_1$, this gives

$$FSW_n(E, \Gamma, g_0, \eta_0) = \pi_{0!} c_1(\mathcal{L})^n = \pi_{1!} c_1(\mathcal{L})^n = FSW_n(E, \Gamma, g_1, \eta_1).$$

□

The argument in the proof above also shows that when $1 \leq b^+(X) \leq \dim(B)+1$ the families Seiberg-Witten invariant does not depend on a choice of perturbation within a given chamber. Therefore the families Seiberg-Witten invariants only depend on a choice of families spin^c structure and chamber.

Remark 6.2.3. If $2n-d = \dim(B)$, then the invariant lies in the top degree cohomology of B and we can integrate it over B to obtain an integer that is

$$\int_B \int_{E/B} y^n = \int_{\mathcal{M}} y^n$$

which coincides with the traditional definition of the Seiberg-Witten invariant in the unparametrised case.

6.3 Charge Conjugation and the Families Wall-Crossing Formula

The families Seiberg-Witten invariant also exhibits a charge conjugation formula and wall-crossing formula. The proof of the charge conjugation formula is identical to the unparametrised case, the only distinction being the presence of a richer chamber structure since the index of the family of Dirac operators is constant along B, only depending on topological properties of X. For a choice of chamber c we denote the dual chamber by $-c$, note that when B is simply connected, the space of chambers is identified with homotopy classes of maps $B \to S^{b^+(X)-1}$ and the dual chamber obtained from charge conjugation is given by composing c with the antipodal map on the sphere. We then have the following result.

Theorem 6.3.1. *Let $X \hookrightarrow E \to B$ be a smooth family of 4-manifolds and $\Gamma : T(E/B) \to End(W)$ be a families spin^c structure. Then if $b^+(X) > dim(B) + 1$*

$$FSW(E, \overline{\Gamma}) = (-1)^{ind_{\mathbb{C}}(D_A)+b^+ +1+n} SW(E, \Gamma)$$

and if $b^+(X) \leq dim(B) + 1$

$$FSW(E, \overline{\Gamma}, c) = (-1)^{ind_{\mathbb{C}}(D_A)+b^+ +1+n} SW(E, \Gamma, -c)$$

where $ind_{\mathbb{C}}(D_A) = \frac{c(L_\Gamma)^2 - \sigma(X)}{2}$.

The families wall-crossing formula was first proven in [LL01], although a purely cohomological proof of the formula was obtained by Baraglia and

Konno in [BK21]. We state their result in the special case that B is simply-connected

Theorem 6.3.2. *Let $X \hookrightarrow E \to B$ be a smooth family of 4-manifolds with $b_1(X) = 0$ and $b^+(X) > 1$ equipped with a families spin^c structure Γ. Let D be the virtual index bundle of the family of Dirac operators obtained from the families spin^c structure, set $d = rank_{\mathbb{C}}(D)$ and let c_1, c_2 be two chambers, viewed as homotopy classes of maps $B \to S^{b^+(X)-1}$ where $S^{b^+(X)-1}$ is the unit sphere in $H^+(X)$. Since H^+ is oriented it induces and orientation on $S^{b^+(X)-1}$ and there is an associated volume form $\nu \in H^{b^+(X)-1}(S^{b^+(X)-1})$ consistent with the orientation, set deg$(c) := c^*(\nu)$. It then follows that*

$$FSW_n(E, \Gamma, c_1) - FSW_n(E, \Gamma, c_2) = 0$$

if $n < d - 1$ and

$$FSW_n(E, \Gamma, c_1) - FSW_n(E, \Gamma, c_2) = (-1)^n (deg(c_1) - deg(c_2)) s_{(n-d+1)}(ind(D))$$

if $n \geq d - 1$ where $s_j(ind(D))$ is the j-th Segre class of the virtual index bundle of D.

More generally, the $\deg(c_1) - \deg(c_2)$ term above is an instance of the primary difference class, see [Ste99][Chapter 36]. Using such one can also obtain a wall-crossing formula when $b^+(X) = 1$ similar to the above expression.

6.4 Kähler Families

In this section we shall discuss particular classes of smooth families called Kähler families. These are smooth families of manifolds for which there is a smoothly varying Kähler structure. Provided the relevant assumptions are made, the required properties for the computation of the Seiberg-Witten invariant of Kähler surfaces carry over in the families setting, allowing us to compute the invariant as is done later in Chapter 7.

Definition 6.4.1. Let $\pi : E \longrightarrow B$ be a smooth family, we say that the family is *Kähler* or *has a smoothly varying Kähler structure*, if there is a Kähler structure on the tangent bundle $T(E/B)$. That is, there exist the following

- a metric g on $T(E/B)$
- an almost-complex structure on $T(E/B)$ whose restriction to any fibre is integrable.

and the induced non-degenerate $(1,1)$-form $\omega \in C^\infty(E, \Lambda^2 T^*(E/B))$ defined by

$$\omega(v,w) = g(Jv,w)$$

restricts to a closed form on each fibre.

Remark 6.4.2. Note that a smooth family of Kähler structures restricts to a Kähler structure on each fibre, hence a Kähler family is indeed a smooth family of Kähler manifolds. Note that a smooth family of Kähler manifolds in the sense of Definition 6.1.1, i.e. a smooth family for which the fibres happen to be Kähler, does not necessarily determine a Kähler family since the Kähler structures may not vary smoothly.

Analogously to the unparametrised case, smooth Kähler family has a canonical families spin^c structure obtained from the complex structure as follows

Definition 6.4.3. Let $E \longrightarrow B$ be a family of Kähler surfaces with smoothly varying Kähler structure

The *canonical families spin^c structure* is defined by

$$W_{\text{can}} = \Lambda^{0,*} T^*(E/B)$$

where $\Gamma_{\text{can}} : T(E/B) \longrightarrow \text{End}(W_{\text{can}})$ is given by

$$\Gamma_{\text{can}}(v)\tau := \frac{1}{\sqrt{2}} v'' \wedge \tau - \sqrt{2}\iota(v)\tau$$

where $v \in T(E/B)$, $\tau \in W_{\text{can}}$ and $v'' := v^* - i(Jv)^* = g_{\mathbb{C}}(\cdot, v)$, where $*$ is the real dual.

Analogously to the canonical spin^c structure on a Kähler surface there is a characteristic line bundle

$$L_{\Gamma_{\text{can}}} = K^* = \Lambda^{0,2} T^*(E/B).$$

The families Levi-Civita connection on the vertical tangent bundle naturally extends to differential forms and provides a canonical spin^c connection on W_{can} which restricts to the canonical connection on the fibres W_{can_b}, as in the unparametrised case. Via the results of Chapter 5 we immediately obtain the following form of the families Seiberg-Witten equations for Kähler families.

Proposition 6.4.4. *Let $X \hookrightarrow E \to B$ be a smooth family of Kähler surfaces with smoothly varying Kähler structure, $L \longrightarrow E$ a Hermitian line bundle* *and $B \in \mathcal{A}(L)$ a Hermitian connection, then the families Seiberg-Witten equations are*

$$\overline{\partial}_{B,b}\varphi_{0,b} + \overline{\partial}^*_{B,b}\varphi_{2,b} = 0 \tag{6.5}$$

$$2(F_{B,b} + \eta_b)^{0,2} = \overline{\varphi_{0,b}}\varphi_{2,b} \tag{6.6}$$

$$4i(F_{A_{can}} + F_{B,b} + \eta_b)_{\omega_b} = |\varphi_{2,b}|^2 - |\varphi_{0,b}|^2 \tag{6.7}$$

where $\overline{\partial}_B$ is defined via the complex structure J on the family. On each fibre these are simply the equations 5.1,5.2 and 5.3 as in Proposition 5.1.1 with respect to the restriction of the line bundle L to X_b, denoted L_b and the restricted connection B_b on L_b.

Chapter 7

Computation of the Families Seiberg-Witten Invariants on Families of Kähler Surfaces

The goal of this chapter is to demonstrate the computation of the families Seiberg-Witten invariant on Kähler families with smooth Kähler structure when $b_1(X) = 0$, we shall require an addition assumption that certain cohomology groups have constant dimension over the base of the family B. Since we are strictly interested in Kähler families, the associated 2-form ω allows us to choose a particular perturbation and hence a chamber. As in the discussion preceding Theorem 5.1.9, this allows a computation of the invariant in the families case. As will be seen, this choice of perturbation is not necessarily generic, nonetheless a computation of the invariant can still be made by modifying the computation with a factor of the Euler class of the obstruction bundle of the moduli space.

After performing the computation in question and simplifying it into a more manageable expression, we further apply it to 3 specific classes of families. Namely a family of $\mathbb{CP}^2$'s, $\mathbb{CP}^1 \times \mathbb{CP}^1$'s and finally a family with fibres being the blowup of a Kähler surface X.

7.1 A Computation for a General Class of Kähler Families

Consider a smooth family of Kähler surfaces with smooth Kähler structure $E \longrightarrow B$ where B is compact and the fibres X_b which are diffeomorphic to some compact Kähler surface X with $b_1(X) = 0$. Choose a family of metrics g_b and the family of self-dual 2-forms $i\lambda\omega_b$ with $\lambda > 0$ sufficiently large so that $i\lambda\omega_b$ does not lie on the wall for any $b \in B$ as a family of perturbations, where the ω_b are the Kähler forms obtained from the Kähler structure, note that the existence of such a λ follows from the compactness of B. This determines a chamber for the families Seiberg-Witten invariant which we call the *Kähler chamber*.

There is a canonical spin^c structure on the vertical tangent bundle since the family has a smooth Kähler structure, thus we may obtain any other families spin^c structure s_L via tensoring by a hermitian line bundle L over E. This restricts to a line bundle over each X_b which we denote by L_b, note that a choice of a family of connections for the families Seiberg-Witten equations amounts to a choice of a smooth family of connections A_b on L_b. Denote the cohomology groups for the sheaf of holomorphic sections of L_b with respect to the connection A_b by $H^i(X_b, L_b, A)$, since we assume $b_1(X) = 0$ there is a unique holomorphic structure on each L_b, precisely the one given by A_b so we further drop the A_b in our notation where necessary. We shall make the following assumptions relating to the choice of line bundle L.

1. For each $b \in B$, the line bundle L_b has first Chern class $c_1(L_b)$ which is represented by a $(1,1)$ form, hence the line bundles L_b are all holomorphic.

2. The dimensions of $H^i(X_b, L_b)$ for $i = 0, 1, 2$ are independent of b.

The first assumption ensures the line bundles L_b are holomorphic which is simply a necessary condition for a non-zero Seiberg-Witten invariant, the second ensures that the higher direct image sheaves $R^i\pi_*\mathcal{O}(L)$ are locally free [BS76][Lemma 1.5]. These have stalks $H^i(X_b, L_b)$ and so the families of vector spaces $H^i(X_b, L_b)$ form locally trivial vector bundles over B

$$V^i \longrightarrow B$$

where the fibres are $H^i(X_b, L_b)$ and are of rank $h^i(L)$ although for simplicity we shall often denote the rank by h^i when L is understood.

The families Seiberg-Witten moduli space is then $\mathcal{M} = \bigsqcup_{b\in B} \mathcal{M}_b$ where each $\mathcal{M}_b$ is the ordinary Seiberg-Witten moduli space obtained from solutions to the Seiberg-Witten equations with the induced spin^c structure on each X_b with respect to the connection A_b and the bundle of gauge groups $\mathcal{G}_b = \mathrm{Maps}(X_b, S^1)$ acts fibrewise. This families moduli space has a natural projection map $\pi : \mathcal{M} \longrightarrow B$.

Since B is compact, by taking a finite trivialising cover of the family and applying Theorem 5.1.8, it follows that we can further choose λ sufficiently large such that such that on each fibre X_b, solutions to the Seiberg-Witten equations up to gauge equivalence will correspond to, up to gauge equivalence of complex line bundles, a holomorphic structure on the line bundle L_b with non-zero holomorphic section. It follows as in Chapter 5 the moduli space $\mathcal{M}_b$ is then given by

$$\mathcal{M}_b = \frac{H^0(X_b, L_b) \setminus \{0\}}{\mathbb{C}^*} = \mathbb{P}(H^0(X_b, L_b))$$

i.e. the projectivisation of the vector space $H^0(X, L_b)$. Therefore the families moduli space is the vector bundle

$$\mathcal{M} = \mathbb{P}(V^0).$$

Let $\widetilde{\mathcal{M}}$ be the families moduli space obtained factoring by the family of reduced gauge groups $\{\mathcal{G}_{0,b}\}_{b\in B}$ instead of the full gauge group. Since $\mathcal{G}_b \cong \mathcal{G}_{0,b} \times S^1$, there is a principal circle bundle

$$\widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$$

for which there is an associated line bundle over the total moduli space, denoted $\mathcal{O}_{V^0}(-1)$, this restricts to the tautological bundle $\mathcal{O}(-1)$ on each fibre. Define $y = c_1(\mathcal{O}_{V^0}(-1))$, and $x = c_1(\mathcal{O}_{V^0}(1))$ where $\mathcal{O}_{V^0}(1)$ is the dual of $\mathcal{O}_{V^0}(-1)$, then $y = -x$.

We now wish to compute the families Seiberg-Witten invariant. However, the particular choice of perturbation is not necessarily generic, to compute the invariant we must then insert a factor of the Euler class of the obstruction bundle $e(\mathrm{Obs})$ where $\mathrm{Obs} = \mathrm{coker}(\overline{f}) : V \longrightarrow W$, f is defined as in Theorem 6.1.11 and $\overline{f}$ is the induced map under the remaining S^1 action (see the proof of [FM99][Theorem 3.1]).

Note that the families moduli space is given by $\overline{f}^{-1}(\eta)$ and the obstruction bundle will be obtainable from the linearisation of the families Seiberg-Witten equations $\widetilde{\mathcal{D}}$ as seen in Equation (5.9), it will also be useful to define the linearisation of the families Seiberg-Witten equations with the Coulomb gauge fixing condition applied instead, we denote the corresponding operator by $\mathcal{D}$, unlike the operator $\widetilde{\mathcal{D}}$ this does not eliminate the full gauge group, but leaves the S^1 action remaining. Given some integer $n \geq 0$, the families Seiberg-Witten invariants for the family E and spin^c structure induced by a choice of line bundle L can be computed as

$$\mathrm{FSW}_n(E, L) = \int_{\mathbb{P}(V^0)/B} y^n e(\mathrm{Obs}).$$

Note that we have the following isomorphisms

$$\ker(df(A,\Phi)) \cong \ker(\mathcal{D}_{A,\Phi}), \qquad \operatorname{coker}(df) \cong \operatorname{coker}(\mathcal{D}_{A,\Phi}).$$

Consequently

$$\begin{aligned} \ker(d\overline{f}[A,\Phi]) &\cong \ker(\widetilde{\mathcal{D}}_{A,\Phi}) \cong \ker(\mathcal{D}_{A,\Phi})/\mathbb{R}(0,i\Phi) \\ \operatorname{coker}(d\overline{f}[A,\Phi]) &\cong \operatorname{coker}(\widetilde{\mathcal{D}}_{A,\Phi}) \cong \operatorname{coker}(\mathcal{D}_{A,\Phi}). \end{aligned}$$

7.1.1 The Obstruction Bundle in the Families Case

Since we are in the obstructed case, we must determine $e(\operatorname{coker}(\mathcal{D}))$ where $\mathcal{D}$ is the linearisation of the family Seiberg-Witten equations.

Recall for each fibre X_b, Lemma 5.1.5 gives the following exact sequence of vector spaces.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X_b,\mathcal{O}) & \longrightarrow & H^0(X_b,L_b,A_b) & \longrightarrow & \ker(\tilde{\mathcal{D}}_{A,\varphi,b}) \\ & & H^1(X_b,\mathcal{O}) & \longrightarrow & H^1(X_b,L_b,A_b) & \longrightarrow & \operatorname{coker}(\tilde{\mathcal{D}}_{A,\varphi,b}) \\ & & H^2(X_b,\mathcal{O}) & \longrightarrow & H^2(X_b,L_b,A_b) & \longrightarrow & 0 \end{array}$$

and $\mathcal{O}$ is the usual structure sheaf of X_b.

Since $b_1(X) = 0$, it follows that $H^1(X_b,\mathcal{O}) = 0$, hence there is a reduction of the exact sequence to

$$0 \to H^1(X_b,L_b) \to \operatorname{coker}(\mathcal{D}_{A,\varphi,b}) \to H^2(X_b,\mathcal{O}) \to H^2(X_b,L_b) \to 0$$

Since the dimensions of $H^i(X_b,L_b)$ are assumed to be constant over b, define vector bundles V^i with fibres $H^i(X_b,L_b)$ over B as above. We also define the vector bundle $H^{2,0}$ to be the vector bundle with fibres $H^2(X_b,\mathcal{O})$. The previous sequence then implies that the following sequence of smooth vector bundles over B is exact.

$$0 \to V^1 \to \operatorname{coker}(\mathcal{D}) \to H^{2,0} \to V^2 \to 0$$

This induces an exact sequence of vector bundles over $\widetilde{\mathcal{M}}$ by taking the relevant pullback of all the vector bundles involved. The remaining S^1 action acts fibrewise, identifying cokernels via

$$\mathrm{coker}(\mathcal{D}_{A,\Phi}) \cong \mathrm{coker}(\mathcal{D}_{A,e^{i\theta}\Phi})$$

and induces an action on L_b via rotation of complex numbers of unit length, further inducing an action on $H^i(X_b, L_b)$ by multiplication of the same complex number, thus applying the S^1 action will result in tensoring by a factor of the hyperplane line bundle $\mathcal{O}(1)$ for these terms. The action on $H^{2,0}$ is trivial. All the relevant maps in the exact sequence are also S^1 equivariant, hence applying the S^1 action gives rise to a well-defined exact sequence of vector bundles over $\mathcal{M}$.

$$0 \to \pi^* V^1 \otimes \mathcal{O}_{V^0}(1) \to \mathrm{Obs} \to \pi^* H^{2,0} \to \pi^* V^2 \otimes \mathcal{O}_{V^0}(1) \to 0$$

where π is the projection map $\pi : \mathcal{M} \longrightarrow B$.

It follows that the obstruction bundle as a complex vector bundle has rank as a complex vector bundle $\mathrm{rank}(\mathrm{Obs}) = h^1 - h^2 + \rho_g$, the total Chern class of the obstruction bundle is also given by

$$c(\mathrm{Obs}) = c(\pi^* H^{2,0}) c(\pi^* V^1 \otimes \mathcal{O}_{V^0}(1)) c(\pi^* V^2 \otimes \mathcal{O}_{V^0}(1))^{-1}$$

and we aim to extract the top degree term from this.

If E is a vector bundle of rank r and L is a line bundle, then the ith Chern class is obtained by

$$c_i(E \otimes L) = \sum_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(E) c_1(L)^j. \tag{7.1}$$

Meanwhile $\pi^* c(V^2 \otimes \mathcal{O}_{V^0}(1))^{-1}$ is the total Segre class, $s(\pi^* V^2 \otimes \mathcal{O}_{V^0}(1))$ where the i-th graded piece in cohomology of even degrees is the *i-th Segre class.* There is a similar formula to the Chern classes

$$s_i(E \otimes L) = \sum_{j=0}^{i} (-1)^{i-j} \binom{r+i-1}{r-1+j} s_j(E) c_1(L)^{i-j} \tag{7.2}$$

where the s_i can inductively be computed by

$$\begin{aligned} c_1(E) &= -s_1(E) \\ c_1(E) &= s_1(E)^2 - s^2(E) \\ &\vdots \\ c_r(E) &= -s_1(E) c_{n-1}(E) - s_2(E) c_{n-2}(E) - \cdots - s_n(E) \end{aligned}$$

it follows that

$$c(\pi^* V^1 \otimes \mathcal{O}_{V^0}(1)) = \sum_{\ell=0}^{h^1} \sum_{j=0}^{\ell} \binom{h^1 - \ell + j}{j} c_{\ell-j}(\pi^* V^1) x^j$$

where $x = c_1(\mathcal{O}_{V^0}(1))$

and similarly

$$c(\pi^* V^2 \otimes \mathcal{O}_{V^0}(1))^{-1} = \sum_{p=0}^{\infty} \sum_{i=0}^{p} (-1)^{p-i} \binom{h^2 + p - 1}{h^2 + i - 1} s_i(\pi^* V^2) x^{p-i}$$

the top degree term, which coincides with the Euler class of the obstruction bundle, is given by the sum of all products which are cohomology classes of degree $\mathrm{rank}_{\mathbb{R}}(\mathrm{Obs}) = 2(h^1 - h^2 + \rho_g)$, i.e.

$$e(\mathrm{Obs}) = \sum_{m=0}^{h^1 - h^2 + \rho_g} c_{h^1 - h^2 + \rho_g - m}(\pi^* H^{2,0}) \phi_m \tag{7.3}$$

where ϕ_m is the $2m$-th degree term of the product $c(\pi^* V^1 \otimes \mathcal{O}_{V^0}(1)) c(\pi^* V^2 \otimes \mathcal{O}_{V^0}(1))^{-1}$, by expanding this product explicitly, one obtains that

$$\begin{aligned} \phi_m = \sum_{p=0}^{m} \sum_{j=0}^{p} \sum_{i'=0}^{m-p} \Bigg[(-1)^{p-j} \binom{h^2 + p - 1}{h^2 + j - 1} \binom{h^1 - m + p + i'}{i'} \\ (\pi^* V^2) c_{m-p-i'}(\pi^* V^1) x^{p+i'-i} \Bigg]. \end{aligned} \tag{7.4}$$

7.1.2 Integration Over the Fibre

The other required term is the fibre integral of y^k where $y = c_1(\mathcal{O}_{V^0}(-1)) = -x$. Because of this relation, it suffices to understand the fibre integrals of powers of x.

Since the fibres of $\mathbb{P}(V^0) \longrightarrow B$ are $h^0 - 1$ complex dimensional, for $0 \leq k < h^0 - 1$ we have

$$\int_{\mathbb{P}(V^0)/B} x^k = 0$$

since in representing such via a differential form and working on a local trivialisation, the fiber part has to be of degree less than that of the dimension of the fiber.

To obtain x^{h^0-1}, consider the Euler sequence for the projectivisation of a vector bundle, this gives an exact sequence of vector bundles over $\mathcal{M} = \mathbb{P}(V^0)$.

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_{V^0}(1) \otimes \pi^* V^0 \longrightarrow T(\mathbb{P}(V^0)/B) \longrightarrow 0$$

due to the triviality of $\mathbb{C}$ as a vector bundle, we immediately have the equality of the following two total Chern classes

$$c(T(\mathbb{P}(V^0)/B)) = c(\mathcal{O}_{V^0}(1) \otimes \pi^* V^0)$$

from which it follows that the individual Chern classes of each degree are equal. Consequently

$$c_{h^0-1}(T(\mathbb{P}(V^0)/B)) = c_{h^0-1}(\mathcal{O}_{V^0}(1) \otimes \pi^* V^0).$$

Since $\mathcal{O}_{V^0}(1) \otimes V^0$ is a rank h^0 vector bundle, using the formula for the Chern class of the tensor product of a vector bundle with a line bundle we have

$$c_{h^0-1}(T(\mathbb{P}(V^0)/B)) = \sum_{j=0}^{h^0-1} \binom{j+1}{j} c_{h^0-1-j}(\pi^* V^0) x^j \tag{7.5}$$

in integrating both sides over the fibre, since these are forms with same degree as the dimension of the fibres, the fibre integrals are simply 0-forms. Observe that the fibre integral of $c_{h^0-1}(T(\mathbb{P}(V^0)/B))$ is simply the Euler class of vertical tangent bundle, since it fibrewise restricts to the Euler class of the tangent bundle of the fibres, integrating along the fibre gives the Euler characteristic of $\mathbb{P}(V^0)_b$. Since it is fibrewise isomorphic to $\mathbb{CP}^{h^0-1}$, the left hand side is h^0. By the projection formula it then follows that

$$h^0 = \sum_{j=0}^{h^0-1} \binom{j+1}{j} c_{h^0-1-j}(V^0) \int_{\mathbb{P}(V^0)/B} x^j$$

and only the $j = h^0 - 1$ term survives from the above argument, hence

$$h^0 = \binom{h^0}{h^0-1} \int_{\mathbb{P}(V^0)/B} x^{h^0-1} = h^0 \int_{\mathbb{P}(V^0)/B} x^{h^0-1}$$

and so

$$\int_{\mathbb{P}(V^0)/B} x^{h^0-1} = 1.$$

To compute higher powers of x, we shall obtain a relation which allows us to reduce any power of x larger than $h^0 - 1$ to a polynomial in x of degree at most $h^0 - 1$.

Since the total Chern classes of $T(\mathbb{P}(V^0)/B)$ and $\mathcal{O}_{V^0}(1) \otimes \pi^* V^0$ are equal, in particular their h^0-th Chern classes are equal. Since the vertical tangent bundle is a rank $h^0 - 1$ vector bundle and $\mathcal{O}_V(1) \otimes V$ is a rank h^0 vector bundle, it follows that

$$c_{h^0}(\mathcal{O}_V(1) \otimes V) = 0.$$

However, expanding out the Chern class of the tensor product via the formula we have

$$0 = x^{h^0} + c_1(\pi^* V^0) x^{h^0-1} + ... + c_{h^0}(\pi^* V^0)$$

which is the desired relation.

With this the fibre integral of x^{h^0-1+j} can be computed for $j > 0$, we shall show a general recursion relation. Let

$$\tau_j := \int_{\mathbb{P}(V)/B} x^{j+h^0-1}$$

with the initial conditions that $\tau_0 = 1$ and $\tau_j = 0$ for $j < 0$.

Since $x^{h^0} + c_1(\pi^* V^0) x^{h^0-1} + ... + c_{h^0}(\pi^* V^0)) = 0$, it follows that

$$\begin{aligned} 0 &= \int_{\mathbb{P}(V^0)/B} x^{j-1}(x^{h^0} + c_1(\pi^* V) x^{h^0-1} + ... + c_{h^0}(\pi^* V^0)) \\ &= \tau_j + c_1(V^0)\tau_{j-1} + \dots c_{h^0}(V^0)\tau_{j-h^0} \end{aligned}$$

thus

$$\tau_j = -(c_1(V^0)\tau_{j-1} + \dots c_{h^0}(V^0)\tau_{j-h^0})$$

this is precisely the same recursion relation as the Segre classes $s_j(V^0)$ defined to be the the $2j$-th graded component of $c(V^0)^{-1}$ where $c(V^0)$ is the total Chern class of V^0, hence

$$\int_{\mathbb{P}(V^0)/B} x^{j+h^0-1} = s_j(V^0). \tag{7.6}$$

7.1.3 Computing and Simpifying the Invariant

For each $n \geq 0$, the families Seiberg-Witten invariants can be computed as

$$\mathrm{FSW}_n(E, L) := \int_{\mathbb{P}(V^0)/B} y^n e(\mathrm{Obs})$$

It follows from the projection formula and Equation (7.3) that

$$\mathrm{FSW}_n(E, L) = \sum_{m=0}^{h^1-h^2+\rho_g} c_{h^1-h^2+\rho_g-m}(H^{2,0})\Gamma_{m,n}$$

where

$$\Gamma_{m,n} := \int_{\mathbb{P}(V^0)/B} y^n \phi_m.$$

Since y satisfies $y^{h^0} - c_1(V^0)y^{h^0-1} + c_2(V^0)y^{h^0-2} + \cdots = 0$ there is a recursion relation

$$\Gamma_{m,n+h^0} - c_1(V^0)\Gamma_{m,n+h^0-1} + c_2(V^0)\Gamma_{m,n+h^0-2} + \cdots = 0.$$

To further simplify the expression for the families Seiberg-Witten invariant, we wish to simplify the expression for $\Gamma_{m,n}$, from the expressions for ϕ_m and the fibre integrals of powers x (eq. (7.4) and eq. (7.6) respectively) and the fact that $y = -x$, we obtain the following expression for $\Gamma_{m,n}$

$$\Gamma_{m,n} = \sum_{p=0}^{m}\sum_{j=0}^{p}\sum_{i'=0}^{m-p}\Bigg[(-1)^{n+p-j}\binom{h^2+p-1}{p-j}\binom{h^1-m+p+i'}{i'} \\ s_j(V^2)c_{m-p-i'}(V^1)s_{p+i'+n-j-h^0+1}(V^0)\Bigg]. \tag{7.7}$$

We introduce the quantity $\delta := m + n - h^0 + 1$ which keeps track of the degree of the families Seiberg-Witten invariant.

We may adjust the indices in Equation (7.7) so that i' begins at $m-p-\delta$ and j ends at δ. To see this, observe that $0 \leq i' \leq m-p$, implies that $p+i'+n-j \leq m+n-i$, but $m+n = h^0-1+\delta$ and so $p+i'+n-j \leq h^0-1+\delta-j$. Since the fibre integral of x^k is non-zero only if $k \geq h^0-1$, the only non-zero terms which survive in the sum occur when $m-p-\delta \leq i' \leq m-p$ and $0 \leq j \leq \delta$. Moreover, since $\binom{a}{b} = 0$ for $a > 0, b < 0$, even if $m-p-\delta < 0$, we can start the i' index at $m-p-\delta$. Similarly, the j index may end at δ. Hence $\Gamma_{m,n}$ can be written as

$$\Gamma_{m,n} = (-1)^n \sum_{p=0}^{m} \sum_{i'=m-p-\delta}^{m-p} \sum_{j=0}^{\delta} \Bigg[(-1)^{p-j} \binom{h^2+p-1}{p-j} \binom{h^1-m-p+i'}{i'} s_j(V^2) c_{m-p-i'}(V^1) s_{p+i'+n-j-h^0+1}(V^0) \Bigg]$$

set $i = i' - m + p + \delta$ then the sum over i' can be rewritten as a sum over i, since the range of values for the j and i indices is independent of p, we may freely change the position of the sum over p, this in conjunction with the identity that $\binom{a}{b} = (-1)^b \binom{b-a-1}{b}$ applied to the first binomial coefficient yields the following expression

$$\Gamma_{m,n} = (-1)^n \sum_{i=0}^{\delta} \sum_{j=0}^{\delta} \sum_{p=0}^{m} \Bigg[\binom{-h^2-j}{p-j} \binom{h^1-\delta+i}{m-\delta+i-p} s_j(V^2) c_{\delta-i}(V^1) s_{i-j}(V^0) \Bigg]$$

the terms from the sum over p are zero unless $0 \le p \le m-\delta+i$ and also $p \ge j$ and so we can rewrite the above as

$$\Gamma_{m,n} = (-1)^n \sum_{i=0}^{\delta} \sum_{j=0}^{\delta} \Bigg[s_j(V^2) c_{\delta-i}(V^1) s_{i-j}(V^0) \sum_{p=j}^{m-\delta+i} \binom{-h^2-j}{p-j} \binom{h^1-\delta+i}{m-\delta+i-p} \Bigg]$$

reindexing the sum over p as $p' = p - j$ we have

$$\Gamma_{m,n} = (-1)^n \sum_{i=0}^{\delta} \sum_{j=0}^{\delta} \Bigg[s_j(V^2) c_{\delta-i}(V^1) s_{i-j}(V^0) \sum_{p'=0}^{m-\delta+i-j} \binom{-h^2-j}{p'} \binom{h^1-\delta+i}{m-\delta+i-j-p'} \Bigg]$$

so applying the Vandermonde-Chu identity gives.

$$\Gamma_{m,n} = (-1)^n \sum_{i=0}^{\delta} \sum_{j=0}^{\delta} s_j(V^2) c_{\delta-i}(V^1) s_{i-j}(V^0) \binom{h^1-h^2-\delta+i-j}{m-\delta+i-j}$$

Since $s_{i-j} = 0$ for $i \leq j$, we finally have the following expression for $\Gamma_{m,n}$

$$\Gamma_{m,n} = (-1)^n \sum_{i=0}^{\delta} \sum_{j=0}^{i} s_j(V^2) c_{\delta-i}(V^1) s_{i-j}(V^0) \binom{h^1 - h^2 - \delta + i - j}{m - \delta + i - j}.$$

The above computation is summarised in the following theorem

Theorem 7.1.1. *Let $X \hookrightarrow E \to X$ is a smooth family of compact Kähler surfaces with a smoothly varying Kähler structure and $b_1(X) = 0$ such that the following assumptions hold*

1. *For all $b \in B$, the line bundle L_b has first Chern class $c_1(L_b)$ which is represented by a $(1,1)$ form.*

2. *The dimensions of $H^i(X_b, L_b)$ for $i = 0, 1, 2$ are independent of b.*

then the families Seiberg-Witten invariant in the Kähler chamber is given by

$$FSW_n(E, L) = \sum_{m=0}^{h^1-h^2+\rho_g} c_{h^1-h^2+\rho_g-m}(H^{2,0}) \Gamma_{m,n} \tag{7.8}$$

where $\Gamma_{m,n} \in H^{2\delta}(B; \mathbb{Z})$ is given by

$$\Gamma_{m,n} = (-1)^n \sum_{i=0}^{\delta} \sum_{j=0}^{i} s_j(V^2) c_{\delta-i}(V^1) s_{i-j}(V^0) \binom{h^1 - h^2 - \delta + i - j}{m - \delta + i - j} \tag{7.9}$$

where $\delta = m + n - h^0 - 1$ and the $\Gamma_{m,n}$ satisfy the following recursion relation

$$\Gamma_{m,n+h^0} - c_1(V^0)\Gamma_{m,n+h^0-1} + c_2(V^0)\Gamma_{m,n+h^0-2} + \cdots = 0$$

note that terms where the denominator of the binomial coefficients are negative are to be disregarded.

Remark 7.1.2. The computation of Theorem 7.1.1 indeed reduces to the result of Theorem 5.1.9 when B is a point. In such a case the vector bundles H^2, V^i are all trivial so the only possibly non-vanishing invariant occurs when $m = h^1 - h^2 + \rho_g$, $i = j = \delta = 0$ and $n = \rho_g + 1 - \chi(L)$, i.e. the ordinary Seiberg-Witten invariant is given by $\Gamma_{h^1-h^2+\rho_g, \rho_g+1-\chi(L)}$. It is clear from the formula in Theorem 7.1.1 that this recovers the result in the unparametrised case.

It is useful to see the computations of $\Gamma_{m,n}$ when δ is small.

$\delta = 0$

From the above formula we have

$$\Gamma_{m,n} = (-1)^n \binom{h^1 - h^2}{m}$$

$\delta = 1$

$$\begin{aligned} \Gamma_{m,n} = & (-1)^n c_1(V^1) \binom{h^1 - h^2 - 1}{m - 1} \\ & + (-1)^n s_1(V^0) \binom{h^1 - h^2}{m} \\ & + (-1)^n s_1(V^2) \binom{h^1 - h^2 - 1}{m - 1} \end{aligned} \tag{7.10}$$

$\delta = 2$

$$\begin{aligned} \Gamma_{m,n} = & (-1)^n c_2(V^1) \binom{h^1 - h^2 - 2}{m - 2} \\ & + (-1)^n c_1(V^1) s_1(V^0) \binom{h^1 - h^2 - 1}{m - 1} \\ & + (-1)^n s_1(V^2) c_1(V^1) \binom{h^1 - h^2 - 2}{m - 2} \\ & + (-1)^n s_2(V^0) \binom{h^1 - h^2}{m} \\ & + (-1)^n s_1(V^2) s_1(V^0) \binom{h^1 - h^2 - 1}{m - 1} \\ & + (-1) s_2(V^2) \binom{h^1 - h^2 - 2}{m - 2} \end{aligned} \tag{7.11}$$

With a general computation for the families Seiberg-Witten invariants we shall now apply it to example cases of Kähler families.

7.2 The Families Seiberg-Witten Invariants for the Projectivisation Family

The first Kähler family we shall consider will be the projectivisation of a rank 3 complex vector bundle.

Let $\widetilde{\pi} : V \longrightarrow B$ be a complex rank 3 vector bundle over some compact base space B. We may define its projectivisation $\pi : \mathbb{P}(V) \longrightarrow B$ by taking the fibres to be the projectivisation of the fibres of V, that is $\mathbb{P}(V)_b := \mathbb{P}(V_b)$.

Proposition 7.2.1. *Let $V \longrightarrow B$ be a complex rank 3 vector bundle where B is compact. Then the projectivisation family $\pi : \mathbb{P}(V) \longrightarrow B$ is a smooth Kähler family.*

Proof. Since each V_b is a 3 dimensional vector space, $V_b \cong \mathbb{C}^3$ and hence the fibres of $\mathbb{P}(V)$ are diffeomorphic to $\mathbb{CP}^2$, hence this is a family of $\mathbb{CP}^2$'s. This diffeomorphism also preserves the natural orientations defined on the fibres and $\mathbb{CP}^2$, consequently, this is a smooth family. The cohomology ring of $\mathbb{CP}^2$ is well known with $b^+ = 1$ and $b_1(\mathbb{CP}^2) = 0$ so it remains to show this family has a Kähler structure.

Choose a Hermitian metric on V, there is then a naturally induced metric on the tautological line bundle $\mathcal{O}_V(1) \longrightarrow \mathbb{P}(V)$. Consider the Chern curvature with respect to this metric

Since the restriction of $\mathcal{O}_V(1)$ to the fibres of $\mathbb{P}(V)$ is isomorphic to the tautological line bundle over $\mathbb{CP}^2$ and the Chern curvature of any metric on the tautological line bundle over $\mathbb{CP}^2$ is the closed positive $(1,1)$ form induced by the Fubini-Study metric. Consequently, the Chern curvature of the metric on $\mathcal{O}_V(1)$ must be a closed $(1,1)$ form which is positive along the vertical tangent bundle, hence the projectivisation family is a smooth Kähler family. □

It is a well known fact that $b^+(\mathbb{CP}^2) = 1$, consequently $\rho_g(\mathbb{CP}^2) = 0$.

We have from the Leray-Hirsch theorem

$$H^2(E;\mathbb{Z}) \cong H^2(B;\mathbb{Z}) \oplus \mathbb{Z}$$

where the isomorphism is given by

$$H^2(B;\mathbb{Z}) \oplus \mathbb{Z} \ni (L, m) \mapsto \mathcal{O}_V(m) \otimes \pi^* L$$

consequently, any line bundle over $\mathbb{P}(V)$ is of the form:

$$\mathcal{O}_V(m) \otimes \pi^* L$$

for some line bundle $L \longrightarrow B$. Hence we are interested in the higher direct image sheaves $R^j\pi_*(\mathcal{O}_V(m) \otimes \pi^*L)$ and write the corresponding cohomology bundles as

$$V^j = H^j(\mathbb{P}(V), \mathcal{O}_V(k) \otimes \pi^*L)$$

where π is the projection map of the projectivisation family $\mathbb{P}(V) \to B$ induced by $\widetilde{\pi}$.

When $k < 0$ the line bundles $\mathcal{O}(k)$ over $\mathbb{CP}^2$ do not admit global sections, it follows that

$$H^0(\mathbb{CP}^2, \mathcal{O}(k)) = 0$$

this fiberwise calculation implies that

$$H^0(\mathbb{P}(V), \mathcal{O}_V(k)) = 0$$

and so $h^0 = 0$ when $k < 0$. Since the dimension of the fiberwise moduli spaces is $h^0 - 1$, they are empty for $k < 0$, hence the families Seiberg-Witten invariant is zero. It is then sufficient to consider the cohomology groups for $k \geq 0$.

By the Leray spectral sequence, we have for $k \geq 0$

$$H^j(\mathbb{P}(V), \mathcal{O}_V(k)) \cong \begin{cases} S^k(V^*) & j = 0 \\ 0 & j > 0 \end{cases}$$

so it follows from the projection formula that after twisting by the pullback of a line bundle L over B

$$V^j = H^j(\mathbb{P}(V), \mathcal{O}_V(k) \otimes \pi^*L) \cong \begin{cases} S^k(V^*) \otimes L & j = 0 \\ 0 & j > 0 \end{cases}.$$

It immediately follows that $h^1 = h^2 = 0$. Moreover h^0 is just the rank of $S^k(V^*)$. Since V is a 3 dimensional complex vector bundle

$$\operatorname{rank}(S^k(V^*)) = \binom{2+k}{k}$$

We now proceed to obtain a more explicit expression for the families Seiberg-Witten invariants for the projectivisation family. Recall from Theorem 7.1.1 that

$$\mathrm{FSW}_n(E, \mathcal{O}_V(k) \otimes \pi^*L) = \sum_{m=0}^{h^1-h^2+\rho_g} c_{h^1-h^2+\rho_g-m}(H^{2,0})\Gamma_{m,n}$$

where H^2 is the bundle with fibres $H^2(X_b, \mathcal{O})$. Since $h^1 - h^2 + \rho_g = 0$, m is forced to be zero, hence

$$\mathrm{FSW}_n(E, \mathcal{O}_V(k) \otimes \pi^* L) = \Gamma_{0,n}$$

and $\delta = n - h^0 + 1$. Applying the expression for $\Gamma_{m,n}$ in Theorem 7.1.1 gives

$$\mathrm{FSW}_n(E, \mathcal{O}_V(k) \otimes \pi^* L) = (-1)^n \sum_{i=0}^{n-h^0+1} \sum_{j=0}^{i} s_j(V^2) c_{n-i}(V^1) s_{i-j}(V^0).$$

Recall that V^1 and V^2 are zero bundles, so the only surviving terms occur when $j = 0, i = n - h^0 + 1$. This results in the following computation of the families Seiberg-Witten invariants for the projectivisation family

$$\mathrm{FSW}_n(E, \mathcal{O}_V(k) \otimes \pi^* L) = (-1)^n s_{n-h^0+1}(V^0) \tag{7.12}$$

This can be made further explicit, by expanding $s_{n-h^0+1}(V^0) = s_{n-h^0+1}(S^k(V^*) \otimes L)$ via 7.2 we obtain an explicit expression as displayed in the following theorem

Theorem 7.2.2. *Let $\pi : V \to B$ be a complex vector bundle of rank 3 and $\mathbb{P}(V) \to B$ be the corresponding projectivisation family, then the families Seiberg-Witten invariants in the Kähler chamber for the spin^c obtained by twisting the canonical structure by $\mathcal{O}_V(k) \otimes \pi^* L$ for $k \geq 0$ are given by*

$$FSW_n(E, \mathcal{O}_V(k) \otimes \pi^* L) = \sum_{i=0}^{n-h^0+1} (-1)^{h^0-1+i} \binom{n}{h^0 - 1 + i} s_i(S^k(V^*)) c_1(L)^{n-h^0+1-i} \tag{7.13}$$

where $h^0 = \binom{2+k}{k} = (k+1)(k+2)/2$ and are zero otherwise.

In principle, the expression obtained in the above theorem can be further simplified by computing the Segre classes of $S^k(V^*)$, although this becomes increasingly difficult and quite cumbersome for larger and and larger k. However, when $k = 0$ we have $S^k(V^*) = \mathbb{C}$ and $h^0 = 1$ and so an explicit expression can be obtained. Observe that the only surviving Segre class is the one corresponding to $i = 0$. Thus

$$FSW_n(E, \mathcal{O}_V(0) \otimes \pi^* L) = c_1(L)^n.$$

7.3 The Families Seiberg-Witten Invariants for a Family with Fibres $\mathbb{CP}^1 \times \mathbb{CP}^1$

We now investigate another family. Suppose that $\pi_1 : V_1 \longrightarrow B$ and $\pi_2 : V_2 \longrightarrow B$ are rank 2 complex vector bundles over B and consider their fibre product $\Pi : \mathbb{P}(V^1) \times_B \mathbb{P}(V^2) \longrightarrow B$. This also has a Kähler structure, since as in Proposition 7.2.1 we may endow $\mathbb{P}(V^1)$ and $\mathbb{P}(V^2)$ with Kähler structures which induces one on the product and consequently the fibre product. The Serre spectral sequence gives

$$H^2(E;\mathbb{Z}) \cong H^2(B;\mathbb{Z}) \times \mathbb{Z} \times \mathbb{Z}$$

where the map sends $H^2(B;\mathbb{Z}) \times \mathbb{Z} \times \mathbb{Z} \ni (L, m, n) \mapsto \mathcal{O}_{V_1}(k) \otimes \mathcal{O}_{V_2}(\ell) \otimes \Pi^* L$.

As in Section 7.2 unless $k, \ell \geq 0$ we have $h^0 = 0$ yielding a zero Seiberg-Witten invariant due to a lack of sections. In the case when $k, \ell \geq 0$, one may use the computation of the cohomology of line bundles as in the previous section on projective bundles and a Künneth formula for sheaf cohomology to obtain that the cohomology bundles are given by

$$H^j(E; \mathcal{O}_{V_1}(k) \otimes \mathcal{O}_{V_2}(\ell)) = \begin{cases} S^k(V_1^*) \otimes S^\ell(V_2^*) & j = 0 \\ 0 & j > 0 \end{cases}$$

and the general case when twisting by a line bundle involves in tensoring by the line bundle L via the projection formula as before. and thus $h^1 = h^2 = 0$ and $h^0 = (1+k)(1+\ell)$.

These conditions then imply as in Section 7.2 the following result

Theorem 7.3.1. *Let $\pi : V_1, V_2 \to B$ be complex vector bundles of rank 2 and $\mathbb{P}(V_1) \times_B \mathbb{P}(V_1) \to B$ be the family obtained from the fibre product of their projectivisations, then the families Seiberg-Witten invariants in the Kähler chamber for the spin^c structure obtained by twisting the canonical structure by $\mathcal{O}_{V_1}(k) \otimes \mathcal{O}_{V_2}(\ell) \otimes \Pi^*(L)$ are given by*

$$FSW_n(E, \mathcal{O}_{V_1}(k) \otimes \mathcal{O}_{V_2}(\ell) \otimes \Pi^* L = (-1)^n s_{n-h^0+1}(V^0) \qquad (7.14)$$

where $h^0 = (1+k)(1+\ell)$.

This can also be made more explicit via the formula for the Chern classes of products of vector bundles and expressions for the Chern classes of $S^k(V_1)$ and $S^\ell(V_2)$, but similar to Section 7.2 it becomes increasingly cumbersome as k, ℓ become larger. One can make an explicit calculation when $k = \ell = 0$, this proceeds identically to the $k = 0$ case as with the projectivisation family, hence

$$\mathrm{FSW}_n(E, \mathcal{O}_{V_1}(0) \otimes \mathcal{O}_{V_2}(0) \otimes \Pi^* L) = c_1(L)^n.$$

7.4 The Families Seiberg-Witten Invariants for Families of Blowups of Kähler Surfaces

As discussed in Section 2.6, given a Kähler manifold X and $x \in X$, one can construct the blowup of X at x. There are a variety of Kähler families we can obtain from this construction. Of primary interest shall be the *universal blowup family*. Given a Kähler surface X, this is a smooth family $Z \longrightarrow X$ with base X and fibre at $x \in X$ being $\mathrm{Bl}_x(X)$, the blowup of X at x and is a case of a more general construction of a sequence of spaces X_ℓ with fibres diffeomorphic to the blowup of X at ℓ possibly non-distinct points as in [Liu00].

Definition 7.4.1. Let X be a Kähler surface, let $\Delta : X \longrightarrow X \times X$ be the diagonal section $x \mapsto (x, x)$. The *universal blowup family* is then the smooth fibre bundle $\pi : Z \longrightarrow X$ where $Z := Bl_{\Delta(X)}(X \times X)$ is the blowup of $X \times X$ along the image of the diagonal section with projection $\Pi : Z \longrightarrow X \times X$ and $\pi = p_2 \circ \Pi$ where $p_2 : X \times X \longrightarrow X$ is the projection onto the right factor.

Recall for the blowup at a point $\widetilde{X}$,

$$\mathrm{Pic}(\widetilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$$

given by $(L, m) \mapsto p^*L \otimes \mathcal{O}(kE) \equiv L + kE$ where $k \in \mathbb{Z}$.

Therefore if X is simply connected, it follows that $H^3(X; \mathbb{Z}) = 0$ so by applying Proposition 6.1.6 one obtains for the universal blowup family $H^2(Z; \mathbb{Z}) \cong H^2(X; \mathbb{Z}) \oplus H^2(X; \mathbb{Z}) \oplus \mathbb{Z}$ with $(L_1, L_2, k) \mapsto \pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE)$ where $p : \widetilde{X} \longrightarrow X$ is the blowup projection for a fibre $Z_x = \widetilde{X}$ over some fixed point x.

The restriction $\pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE)$ to a fibre X_x is simply $p^*L_2 \otimes \mathcal{O}(kE)$ for the blowup at a point. To simplify notation we shall sometimes write $L = \pi^*L_1 \otimes p^*L_2$ and write the full line bundle $\pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE)$ as $L + kE$

Given $x \in X$, a fibre of the universal blowup family is $\pi^{-1}(x) = \mathrm{Bl}_x(X) \times \{x\} \cong \mathrm{Bl}_x(X)$, so it is indeed a family with fibres diffeomorphic to the blowup of X at x, it also satisfies the following universal property.

Proposition 7.4.2. *Suppose that $Z' \longrightarrow B$ is another family whose fibres are blowups of X where the point blown up by is specified by a smooth map*

$f : B \longrightarrow X$, then Z' is the pullback of $\pi : Z \longrightarrow X$ where under $f : B \longrightarrow X$.

If X is a compact Kähler surface, then the universal blowup family inherits a smoothly varying Kähler structure. There is also a more general construction one could consider, suppose that $E \longrightarrow B$ is a Kähler family with fibres diffeomorphic to X and suppose that $s : B \longrightarrow E$ is a section of this family. Then there is a new family $Z' = \mathrm{Bl}_s(E) \longrightarrow B$, this is the blowup of E along S, where the fibre of Z over $b \in B$ is the blowup of $\pi^{-1}(b)$ at $s(b)$ and also a Kähler family.

We now aim to compute the families Seiberg-Witten invariants for the universal blowup family. First recall that provided the dimensions of the cohomology groups $\{H^i(Z_b, (\pi^* L_1 \otimes p^* L_2 \otimes \mathcal{O}(kE)))\}_{b \in B}$ are constant over $b \in B$, they define vector bundles V^i and families Seiberg-Witten invariants are given by

$$\mathrm{FSW}_n(\pi^* L_1 \otimes p^* L_2 \otimes \mathcal{O}(kE)) = \sum_{m=0}^{h^1 - h^2 + \rho_g} c_{h^1 - h^2 + \rho_g - m}(H^{2,0}) \Gamma_{m,n}$$

where H^2 is the bundle with fibres $H^2(\mathrm{Bl}_x(X), \mathcal{O}_{\mathrm{Bl}_x(X)})$, the following proposition shows that H^2 is a trivial bundle.

Proposition 7.4.3. *The bundle $H^{2,0}$ with fibres $H^2(Bl_x(X), \mathcal{O}_{Bl_x(X)})$ is trivial for the universal blowup family.*

Proof. Let $\pi : Z \longrightarrow X$ be the universal blowup family and $\pi_x : \mathrm{Bl}_x(X) \longrightarrow X$ be the induced map from the fibre at x to X. This map corresponds to the projection map for the blowup of X at x and is a birational isomorphism, hence it induces an isomorphism between $H^0(X, \wedge^{2,0} TX)$ and $H^0(\mathrm{Bl}_x(X_x), \wedge^{2,0} T\mathrm{Bl}_x(X_x))$ [GH94, p. 494]. By Serre duality $H^0(\mathrm{Bl}_x(X_x), \wedge^{2,0} T\mathrm{Bl}_x(X_x)) \cong H^2(\mathrm{Bl}_x(X), \mathcal{O}_{\mathrm{Bl}_x(X)})$, hence π composed with Serre duality on the fibres induces an isomorphism between $H^{2,0}$ and the trivial bundle over X with fibres $H^0(X, \wedge^{2,0} TX)$, thus $H^{2,0}$ is trivial. □

The above proposition implies that the only surviving term in the expression for families Seiberg-Witten invariants occurs when $m = h^1 - h^2 + \rho_g$, so the invariants are computed as follows

$$\mathrm{FSW}_n(\pi^* L_1 \otimes p^* L_2 \otimes \mathcal{O}(kE)) = \Gamma_{h^1 - h^2 + \rho_g, n} \tag{7.15}$$

where $\Gamma_{h^1-h^2+\rho_g,n}$ is given by

$$\Gamma_{h^1-h^2+\rho_g,n} = (-1)^n \sum_{i=0}^{\delta}\sum_{j=0}^{i} s_j(V^2)c_{\delta-i}(V^1)s_{i-j}(V^0)\binom{h^1-h^2-\delta+i-j}{h^1-h^2+\rho_g-\delta+i-j}$$

where $h^i = \text{rank}(V^i) = \text{rank}(H^i(X, \pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE)))$, we also define $p^i = \dim(H^i(X, L_2))$.

Recall that the vector bundles V^i have fibres $\{H^i(Z_b, (\pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE)))\}_{b\in B}$ and correspond to the higher direct image sheaves $R^i\pi_*(\pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE))$. From the projection formula, one has

$$R^i\pi_*(\pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE)) = L_1 \otimes R^i\pi_*(p^*L_2 \otimes \mathcal{O}(kE)).$$

Thus the vector bundles V^i with fibres will be given by $V^i = L_1 \otimes W^i$ where $W^i = R^i\pi_*(p^*L_2 \otimes \mathcal{O}(kE))$.

Furthermore, since X is a Kähler surface, it only has cohomology up to degree 4, hence any terms of higher degree will not survive. These observations combined with standard formulae for the Chern class of the tensor product of a vector bundle with a line bundle and the computations made in Section 7.1.3 result in the following

Theorem 7.4.4. *Let X be a compact Kähler surface and $\pi : Z \to X$ be the universal blowup family and assume the the dimensions h^i of the cohomology groups $\{H^i(Z_b, (\pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE)))\}_{b\in B}$ are constant over $b \in B$, the families Seiberg-Witten invariants in the Kähler chamber for the spin^c structure obtained by twisting the canonical structure by the $\pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE)$ are given by the following formulae*

$$\Gamma_{h^1-h^2+\rho_g,n} = (-1)^n\binom{h^1-h^2}{h^1-h^2+\rho_g} \tag{7.16}$$

when $\delta = 0$

$$\begin{aligned}\Gamma_{h^1-h^2+\rho_g,n} &= (-1)^n(c_1(W^1) + h^1c_1(L_1))\binom{h^1-h^2-1}{h^1-h^2+\rho_g-1}\\ &+ (-1)^n(s_1(W^0) - h^0c_1(L_1))\binom{h^1-h^2}{h^1-h^2+\rho_g}\\ &+ (-1)^n(s_1(W^2) - h^2c_1(L_1))\binom{h^1-h^2-1}{h^1-h^2+\rho_g-1}\end{aligned} \tag{7.17}$$

when $\delta = 1$, *and*

$$\begin{aligned}\Gamma_{h^1-h^2+\rho_g,n} = {} & (-1)^n c_2(W^1\otimes L_1)\binom{h^1-h^2-2}{h^1-h^2+\rho_g-2} \\ & + (-1)^n c_1(W^1\otimes L_1)s_1(W^0\otimes L_1)\binom{h^1-h^2-1}{h^1-h^2+\rho_g-1} \\ & + (-1)^n s_1(W^2\otimes L_1)c_1(W^1\otimes L_1)\binom{h^1-h^2-2}{h^1-h^2+\rho_g-2} \\ & + (-1)^n s_2(W^0\otimes L_1)\binom{h^1-h^2}{h^1-h^2+\rho_g} \\ & + (-1)^n s_1(W^2\otimes L_1)s_1(W^0\otimes L_1)\binom{h^1-h^2-1}{h^1-h^2+\rho_g-1} \\ & + (-1)^n s_2(W^2\otimes L_1)\binom{h^1-h^2-2}{h^1-h^2+\rho_g-2}\end{aligned} \tag{7.18}$$

when $\delta = 2$, *where* $\delta = \rho_g + 1 - \chi(Z, L) + n$ *and* $\chi(Z, L) = h^0 - h^1 + h^2$. *The invariants are zero otherwise.*

We shall now use the results of Theorem 2.6.7 to compute the vector bundles W^i. Recall that depending on whether $k \geq 0$ or $k < 0$ there are two different sets of exact sequences and isomorphisms pertaining to the cohomology of line bundles over the blowup of X. Set $k \geq 0$ and write a line bundle on $\tilde{X} = \mathrm{Bl}_x(X)$ of the form $p^*L \otimes \mathcal{O}(\pm kE)$ as $L \pm kE$.

Line bundles of the form $L - kE$

From Theorem 2.6.7, we obtain a fibrewise isomorphism

$$H^2(Z_b, p^*L_2 \otimes \mathcal{O}(-kE)) \cong H^2(X, L_2)$$

hence it defines a trivial vector bundle $W^2 = H^2(Z, p^*L_2 \otimes \mathcal{O}(-kE))$ with fibres $H^2(X, L_2)$. We also have the following exact sequences of vector spaces on each fibre

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(Z_b, p^*L_2\otimes\mathcal{O}(-kE)) & \longrightarrow & H^0(X, L_2) & \xrightarrow{\mathrm{ev}_x} & L_{2,x}\otimes(\frac{\mathcal{O}_X}{I_x^k}) \\ & & H^1(Z_b, p^*L_2\otimes\mathcal{O}(-kE)) & \longrightarrow & H^1(X, L_2) & \longrightarrow & 0 \end{array} \tag{7.19}$$

There is an isomorphism $\mathcal{O}_X/I_x^k \cong \widetilde{\mathcal{O}}_x(S^{\leq(k-1)}(T_x^*X))$, since $S^k(V^*)$ can be identified with the degree k polynomials on V and we have an isomorphism given by a local chart containing p, $J_p^k \cong \mathbb{C}[z_1, z_2]/(z_1 - p_1, z_2 - p_2)^{k+1}$ between the jet bundle of X, it follows that $\mathcal{O}_X/I_x^k \cong J_x^{k-1}$ and so $L_{2,x} \otimes \mathcal{O}_X/I_x^k = J_x^{k-1}(L_2)$.

Assume the dimensions of the cohomology groups $\{H^i(Z_b, p^*L_2 \otimes \mathcal{O}(-kE))\}_{b\in B}$ are constant so that they define vector bundles W^i, a priori we have not established this, although we shall only be concerned with computations in cases where this can be established from the exact sequence of vector spaces. Proceeding with this assumption, there is an exact sequence of vector bundles over X

$$0 \longrightarrow W^0 \longrightarrow H^0(X, L_2) \longrightarrow J^{k-1}(L_2)$$
$$\longrightarrow W^1 \longrightarrow H^1(X, L_2) \longrightarrow 0 .$$

Note that the rank of $J^{k-1}(L_2)$ is easily shown to be $\dim(S^{\leq(k-1)}(T_x^*X)) = \sum_{j=0}^{k-1}(j+1) = k(k+1)/2$ and rank of W^i and V^i are identical. It then follows from the above exact sequence that

$$h^0 - p^0 + k(k+1)/2 - h^1 + p^1 = 0.$$

Since the fibrewise dimension of the families Seiberg-Witten moduli space is $h^0 - 1$, the families Seiberg-Witten invariants are necessarily zero unless

$$h^0 = p^0 + h^1 - p^1 - k(k+1)/2 > 0.$$

We now shall consider some special cases in further detail.

k=0

Assume that $k = 0$, then the exact sequence of vector spaces in 7.19 reduces to fibrewise isomorphisms $W^i \cong H^i(X, L_2)$ for $i = 0, 1$. It follows that the W^i are all trivial vector bundles with rank $h^i = p^i(L_2)$, thus $V^i \cong L_1^{\oplus p^i}$ and so the invariant is zero unless $p^0 > 0$, another immediate consequence is that $\delta = \rho_g + 1 - \chi(X, L_2) + n$. If $\delta = 0$ then the families invariants coincides with the ordinary Seiberg-Witten invariant of X. When $\delta = 1$, Equation (7.17) implies that the invariant is given by

$$\begin{aligned} \mathrm{FSW}(Z, \pi^*L_1 \otimes p^*L_2) = (-1)^n(p^1 - p^2)c_1(L_1)\binom{p^1 - p^2 - 1}{p^1 - p^2 + \rho_g - 1} \\ + (-1)^{n+1}p^0 c_1(L_1)\binom{p^1 - p^2}{p^1 - p^2 + \rho_g} \end{aligned} \quad (7.20)$$

For $\delta = 2$ the invariant is

$$\begin{aligned}\mathrm{FSW}(Z, \pi^* L_1 \otimes p^* L_2) = (-1)^n \frac{p^1(p^1-1)}{2} c_1(L_1)^2 \binom{p^1 - p^2 - 2}{p^1 - p^2 + \rho_g - 2} \\ + (-1)^{n+1}(p^1 + p^2) p_0 c_1(L)^2 \binom{p^1 - p^2 - 1}{p^1 - p^2 + \rho_g - 1} \\ + (-1)^{n+1} p^2 p^1 c_1(L)^2 \binom{p^1 - p^2 - 2}{p^1 - p^2 + \rho_g - 2} \\ + (-1)^n \frac{p^0(p^0+1)}{2} c_1(L)^2 \binom{p^1 - p^2}{p^1 - p^2 + \rho_g} \\ + (-1)^n \frac{p^2(p^2+1)}{2} c_1(L)^2 \binom{p^1 - p^2 - 2}{p^1 - p^2 + \rho_g - 2}\end{aligned} \tag{7.21}$$

k=1

We have $\mathcal{O}_X / I_x \cong \mathcal{O}_x(\mathbb{C})$ the exact sequence becomes

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(Z, p^* L_2 \otimes \mathcal{O}(-E)) & \longrightarrow & H^0(X, L_2) & \longrightarrow & L_2 & \\ & H^1(Z, p^* L_2 \otimes \mathcal{O}(-E)) & \longrightarrow & H^1(X, L_2) & \longrightarrow & 0 & \end{array}$$

We shall make a further assumption so that an explicit computation can be made, suppose that the line bundle L_2 is *basepoint-free*, that is, for each $x \in X$ there exists a non-zero holomporhic section of L_2 with $s(x) \neq 0$, then it necessarily follows that the map $H^0(X, L_2) \longrightarrow L_{2,x}$ is surjective for all x and dimensions of the cohomology groups are then constant along b. One then indeed has the above sequence of vector bundles. Furthermore, by exactness $H^1(Z, L_2 - E) \cong H^1(X, L_2)$ so W^1 is also trivial and the following sequence of vector bundles over X is exact

$$0 \to H^0(Z, \otimes p^* L_2 \otimes \mathcal{O}(-E)) \to H^0(X, L_2) \to L_2 \to 0$$

thus $h^0 = p^0 - 1$, hence the invariant is zero unless $p^0 > 1$ and since $H^0(X, L_2)$ is a trivial vector bundle, it follows that $c(L_2)c(W^0) = 1$. Hence $s(W^0) = c(L_2)$, it follows that $s_1(W^0 \otimes L_1) = c_1(L_2) + (1 - p^0) c_1(L_1)$ and $s_2(W^0 \otimes L_1) = c_1(L_1)^2 p^0(p^0 - 1)/2 - p^0 c_1(L_1) c_1(L_2)$. The expression of the invariant when

$\delta = 1$ is then following

$$
\begin{aligned}
\mathrm{FSW}(Z, \pi^* L_1 \otimes p^* L_2 \otimes \mathcal{O}(-E)) &= (-1)^n p^1 c_1(L_1) \binom{p^1 - p^2 - 1}{p^1 - p^2 + \rho_g - 1} \\
&+ (-1)^n (c_1(L_2) - (p^0 - 1) c_1(L_1)) \binom{p^1 - p^2}{p^1 - p^2 + \rho_g} \\
&+ (-1)^{n+1} p^2 c_1(L_1) \binom{p^1 - p^2 - 1}{p^1 - p^2 + \rho_g - 1}
\end{aligned}
\tag{7.22}
$$

and when $\delta = 2$

$$
\begin{aligned}
\mathrm{FSW}(Z, \pi^* L_1 \otimes p^* L_2 \otimes \mathcal{O}(-E)) &= (-1)^n \frac{p^1(p^1 - 1)}{2} c_1(L_1)^2 \binom{p^1 - p^2 - 2}{p^1 - p^2 + \rho_g - 2} \\
&+ (-1)^{n+1} p^1 (c_1(L_1) c_1(L_2) + (1 - p^0) c_1(L_1)^2) \binom{p^1 - p^2 - 1}{p^1 - p^2 + \rho_g - 1} \\
&+ (-1)^{n+1} p^2 p^1 c_1(L_1)^2 \binom{p^1 - p^2 - 2}{p^1 - p^2 + \rho_g - 2} \\
&+ (-1)^n \left(\frac{p^0(p^0 - 1)}{2} c_1(L_1)^2 - p^0 c_1(L_1) c_1(L_2) \right) \binom{p^1 - p^2}{p^1 - p^2 + \rho_g} \\
&+ (-1)^{n+1} p^2 (c_1(L_1) c_1(L_2) + (1 - p^0) c_1(L_1)^2) \binom{p^1 - p^2 - 1}{p^1 - p^2 + \rho_g - 1} \\
&+ (-1)^n \frac{p^2(p^2 - 1)}{2} c_1(L_1)^2 \binom{p^1 - p^2 - 2}{p^1 - p^2 + \rho_g - 2}.
\end{aligned}
\tag{7.23}
$$

General case for $k \geq 0$ with $L - kE$

Via similar assumptions to the case when $k = 1$, one can make computations for the families Seiberg-Witten invariants in principle. Recall the exact sequence of vector spaces

$$
\begin{array}{ccccccc}
0 \longrightarrow & H^0(\widetilde{X}, L - kE) & \longrightarrow & H^0(X, L_2) & \xrightarrow{\mathrm{ev}_x} & L_{2,x} \otimes (\frac{\mathcal{O}_X}{I_x^k}) & \\
& H^1(\widetilde{X}, L - kE) & \longrightarrow & H^1(X, L_2) & \longrightarrow & 0 &
\end{array}
$$

and the isomorphism $\mathcal{O}_X / I_x^k \cong \mathcal{O}_x(S^{\leq (k-1)}(T_x^* X))$.

If we assume as in the $k = 1$ case that the map in the long exact sequence induced by the evaluation map is surjective. Then by identical reasoning one obtains that W^1 is trivial and the following exact sequence of vector bundles

$$0 \to H^0(Z, p^*L_2 \otimes \mathcal{O}(-E)) \to H^0(X, L_2) \to J^{k-1}(L_2) \to 0$$

thus $s(W^0) = c(J^{k-1}(L))$. There is also an exact sequence

$$0 \to S^k(T^*X) \to J^k \to J^{k-1} \to 0$$

with the map on the right being the obvious projection map sending a k-jet to its corresponding $k-1$ jet and the map on the left is the inclusion of degree k-polynomials.

Tensoring by L_2 induces the following exact sequence

$$0 \to S^q(T^*X) \otimes L_2 \to J^q(L_2) \to J^{q-1}(L_2) \to 0$$

from which the total Chern class of the q-th jet bundle can be inductively computed as

$$c(J^q(L_2)) = c(S^q(T^*X) \otimes L_2)c(J^{q-1}(L_2))$$

consequently, one may compute the Segre classes $s_j(W^0 \otimes L_2)$ and use the formulae in Theorem 7.4.4 to obtain an expression for the invariants, although this results in more complicated expressions as k increases.

$L + kE$ with $k \geq 1$

There is an isomorphism $W^0 \cong H^0(X, L_2)$ and an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(X, L_2) & \longrightarrow & W^1 & \longrightarrow & J^{k-2}(L_2 \otimes K_X^*)^* \\ & & \swarrow & & & & \\ & & H^2(X, L_2) & \longrightarrow & W^2 & \longrightarrow & 0 \end{array}$$

hence W^0 is trivial and the invariants are zero unless $p^0 > 0$. In particular, one can also see if $k = 1$ then $W^1 \cong H^1(X, L_2)$ and $W^2 \cong H^2(X, L_2)$, thus the vector bundles W^i are all trivial and we obtain an identical computation for the families Seiberg-Witten invariants as in Equation (7.20) and Equation (7.21) when $\delta = 1$ and $\delta = 2$ respectively.

This exact sequence is obtained by applying Serre duality to 2.2 of Theorem 2.6.7 for a line bundle of the form $L_2^* \otimes K_X \otimes \mathcal{O}(-(k-1)E)$. Consequently, we may apply similar assumptions to those in the previous section to the line bundle $L_2 \otimes K_X^*$ to obtain a reduction of the exact sequence. If $L_2^* \otimes K_X$ is basepoint free it follows that $W^2 \cong H^2(X, L_2)$ and there is an exact sequence for $k = 2$

$$0 \to L_2 \otimes K_X^* \to W^2 \to H^2(X, L_2) \to 0$$

thus $c_i(L_2 \otimes K_X^*) = c_i(W^2)$ and one obtains formulae for the families Seiberg-Witten invariants similar to Equation (7.22) and Equation (7.23) with L_2 replaced by $L_2 \otimes K_X^*$ and the p^0, p^2 factors not inside the binomial coefficients interchanged. As seen previously, if we assume more generally that the map $H^0(X, L_2^* \otimes K_X) \to J^{k-2}(L_2^* \otimes K_X)$ is surjective then one obtains that W^1 is trivial and via the exact sequence above, it follows that the Chern classes of W^2 are those of $J^{k-2}(L_2^* \otimes K_X)$ and the invariants may be computed as discussed previously.

Chapter 8

Consequences of the Computation of the Families Seiberg-Witten Invariant

In this chapter we shall investigate relevant applications of the computations made of the Families Seiberg-Witten invariants in Chapter 7.

First we consider cohomological constraints for non-vanishing Families Seiberg-Witten invariants for Kähler families when B is simply-connected. Of particular interest is the case when $B = S^2$, also discussed are required constraints for when the families Seiberg-Witten invariants are genuine diffeomorphism invariants of the family alongside the non-vanishing conditions. We find that to obtain a non-vanishing families Seiberg-Witten invariant which is also a genuine diffeomorphism invariant of the family when $B = S^2$ is quite restrictive, and there are very few cases for when it is non-zero.

One application of the computing the families Seiberg-Witten invariants when $B = S^n$ is the detection of non-trivial homotopy classes of $\mathrm{Diff}_0(X)$. Via the clutching construction, any such family up to isomorphism is specified by a homotopy class of smooth map $f : S^{n-1} \longrightarrow \mathrm{Diff}_0(X)$. By definition, this is an element of $\pi_{n-1}(\mathrm{Diff}_0(X))$. Consequently, if we are in a situation where the families Seiberg-Witten invariant is a genuine diffeomorphism invariant of families, a non-zero Seiberg-Witten invariant of a family implies it is non-trivial which indicates the existence of a non-trivial homotopy class of $\mathrm{Diff}_0(X)$. Of particular interest is the case $B = S^2$ which can give nontrivial elements on $\pi_1(\mathrm{Diff}_0(X))$.

8.1 Cohomological Constraints for Non-Vanishing Invariants

Let $E \longrightarrow B$ be a smooth Kähler family with compact fibres diffeomorphic to a simply-connected 4-manifold X and compact base B. Furthermore, assume that B is simply-connected, then the set of chambers for the family E can be identified with homotopy classes of maps $B \longrightarrow S^{b^+(X)-1}$ for $b^+(X) > 1$ and the two connected components of S^0 for $b^+(X) = 1$. The Kähler chamber is denoted by $+$, the other chamber by $-$.

Let L be a line bundle on E and assume that $H^i(X_b, L)$ has constant dimension with respect to B so that it defines a vector bundle $V^i \longrightarrow B$ with fibres $H^i(X_b, L)$. Note that since the families moduli space has fibres which are $h^0 - 1$ dimensional, a necessary condition for the non-vanishing of the families Seiberg-Witten invariants is $h^0 > 0$.

We then have the following general results

Proposition 8.1.1. *Let $X \hookrightarrow E \to B$ be a Kähler family, suppose that $\rho_g(X) = 0$ and L is a holomorphic line bundle on E.*

If $h^0(L) > 0$, then

$$FSW_n(E, s_L, +) = (-1)^n s_{n-(\chi(L)-1)}(ind(L)), \qquad FSW_n(E, s_L, -) = 0,$$

and if $h^2(L) > 0$, then

$$FSW_n(E, s_L, +) = 0, \qquad FSW_n(E, s_L, -) = (-1)^{n+1} s_{n-(\chi(L)-1)}(ind(L)).$$

Proof. Since $\rho_g = 0$, it follows that $b^+ = 1$, it can be shown that the primary difference class of the $+$ and $-$ chambers 1, hence the wall-crossing formula gives the following identity

$$\mathrm{FSW}_n(E, s_L, +) - \mathrm{FSW}_n(E, s_L, -) = (-1)^{n+1} s_{n-(\chi(L)-1)}(\mathrm{ind}(L)).$$

First suppose that $h^0(L) > 0$, by Lemma 2.3.7 it follows that $h^2(L) = 0$. By Serre duality, $h^2(L) = h^0(K \otimes L^*)$ and thus $\mathrm{FSW}_n(E, s_{K\otimes L^*}, +) = 0$.

By charge conjugation of the Seiberg-Witten equations this gives the first half of the result for $h^0(L) > 0$ that $\mathrm{FSW}_n(E, s_L, -) = 0$, with the second half following from wall-crossing formula above.

Now suppose that $h^2(L) > 0$, again by Lemma 2.3.7 this implies that $h^0(L) = 0$ and so $\mathrm{FSW}_n(E, s_L, +) = 0$, the wall-crossing formula then gives

$$\mathrm{FSW}_n(E, s_L, -) = (-1)^{n+1} s_{n-(\chi(L)-1)}(\mathrm{ind}(L)).$$

□

There is also a computation for when $\rho_g > 0$ and $h^2(L) = 0$.

Proposition 8.1.2. *Let $X \hookrightarrow E \to B$ be a Kähler family with B simply-connected and dim$(B) = b^+(X) - 1$, suppose that $\rho_g(X) > 0$, $h^0(L) > 0$ and $h^2(L) = 0$ and L is a holomorphic line bundle on E. Then*

$$FSW_n(E, s_L, c^+) = (-1)^n s_{n-(\chi(L)-1)}(ind(L))2deg(c^+)$$

Proof. Since $h^2(L) = 0$, applying Serre duality gives $h^0(K \otimes L^*) = 0$ so $\mathrm{FSW}_n(E, s_{K\otimes L^*}, c^+) = 0$, charge conjugation then gives $\mathrm{FSW}_n(E, s_L, c^-) = 0$, this in conjunction with the wall-crossing formula then gives

$$\mathrm{FSW}_n(E, s_L, c^+) = (-1)^n s_{n-(\chi(L)-1)}(\mathrm{ind}(L))(\deg(c^+) - \deg(c^-)).$$

Since B is simply-connected, chambers are homotopy equivalence classes of maps $B \longrightarrow S^{b^+(X)-1}$ and charge conjugation corresponds to the antipodal map on chambers. Since $b^+(X)$ is odd the antipodal map is orientation reversing and so $\deg(c^+) = -\deg(c^-)$, giving the result. □

8.1.1 Constraints for Non-Vanishing Invariants when $B = S^2$

Since $\mathrm{FSW}_n(E, s_L, c^+) \in H^{2n-d}(B; \mathbb{Z})$ it is a necessary condition that $2n-d \in \{0, 2\}$ for a non-zero invariant, where $d = d(X, L)$ is the expected dimension of the unparametrised moduli space. However, if $2n - d = 0$ then FSW_n is the ordinary Seiberg-Witten invariant of X, for which the conditions under which it must vanish are already known. Hence, take $2n - d = 2$, this forces $n = 1 + d/2 = \chi(L) - \rho_g$, for simplicity denote the families Seiberg-Witten invariant for this n in the Kähler chamber by $\mathrm{FSW}(E, L)$. We shall now find non-vanishing criteria for the families Seiberg-Witten invariant, splitting the argument into cases depending on ρ_g.

Case 1: $B = S^2, \rho_g = 1$

Proposition 8.1.3. *Let $X \hookrightarrow E \to S^2$ be a smooth family of Kähler surfaces with smoothly varying Kähler structure, then $FSW(E, L) = 0$ except in the following cases:*

- *$h^2(L) = 0$ with*
$$FSW(E, L) = (-1)^{d/2} c_1(H^{2,0})$$

- $(h^0, h^1, h^2) = (1, 0, 1)$ *with*

$$FSW(E, L) = c_1(V^0)$$

- $(h^0, h^1, h^2) = (1, 1, 1)$ *with*

$$FSW(E, L) = -c_1(V^0) + c_1(V^1)$$

Proof. Recall that a necessary condition for the non-vanishing of the families Seiberg-Witten invariant is $h^0(L) > 0$, assume this holds. then it follows from Lemma 2.3.7 that $h^2(L) \leq \rho_g = 1$. Hence $h^2 \in \{0, 1\}$ and since $h^1 \geq 0$ we have $h^1 - h^2 \geq -1$. It then immediately follows from the computation of the families Seiberg-Witten invariant in Theorem 7.1.1

$$\mathrm{FSW}(E, L) = \begin{cases} (-1)^{d/2} c_1(H^{2,0}) & h^1 - h^2 > 0 \\ (-1)^{d/2}(c_1(H^{2,0}) - c_1(V^1) + c_1(V^2)) & h^1 - h^2 = 0 \\ (-1)^{d/2} c_1(V^0) & h^1 - h^2 = -1 \end{cases} \tag{8.1}$$

First assume that $h^2 = 0$, then $h^1 - h^2 = h^1 \geq 0$. If $h^1 = 0$ in this case, then since both h^1 and h^2 are zero, thus V^1 and V^2 are both zero bundles, hence their first Chern classes are zero. We then have from Equation (8.1) that $\mathrm{FSW}(E, L) = (-1)^{d/2} c_1(H^{2,0})$. If instead $h^2 > 0$ then by the above computation gives $\mathrm{FSW}(E, L) = (-1)^{d/2} c_1(H^{2,0})$ as well.

Now assume that $h^2 = 1$, then $h^0 = 1$ by Lemma 2.3.7 and the assumption that $h^0 > 0$. Hence for a non-zero invariant which lives in degree 2 cohomology, $n = \chi(L) - \rho_g = 1 - h^1(L)$, since this number is non-negative $h^1(L) = 0$ or 1.

If $h^1(L) = 0$ then $h^1 - h^2 = -1$ and the conditions on h^i and ρ_g imply that $d = 0$ so $\mathrm{FSW}(E, L) = c_1(V^0)$ by 8.1.

If $h^1(L) = 1$, instead $h^1 - h^2 = 0$ and $d/2 = 1$, 8.1 gives

$$\mathrm{FSW}(E, L) = -(c_1(H^{2,0}) - c_1(V^1) + c_1(V^2)).$$

Via the isomorphisms $H^{2,0} \cong H^0(X, K)$ and $(V^2)^* = (H^2(X, L))^* \cong H^0(X, K \otimes L^*)$ obtained via Serre duality, the natural map $H^0(X, L) \otimes H^0(X, K \otimes L^*) \longrightarrow H^0(X, K)$ induces a natural map $V^0 \otimes (V^2)^* \longrightarrow H^{2,0}$, moreover it is an isomorphism since if s, t are sections of L and $K \otimes L^*$ respectively which are not identically zero, then st is a section of K which is not identically zero, since $h^0 = h^2 = \rho_g = 1$ it follows that $V^0 \otimes (V^2)^* \longrightarrow H^{2,0}$ is a map of line bundles with trivial kernel, hence an isomorphism. It immediately follows that

$$c_1(V^0) - c_1(V^2) = c_2(H^{2,0})$$

and hence

$$\mathrm{FSW}(E, L) = c_1(V^1) - c(V^0).$$

□

Case 2: $B = S^2, \rho_g \geq 2$

In the case $\rho_g \geq 2$, one has $b^+(X) \geq 5$ so there is only one chamber since all maps $S^2 \longrightarrow S^{b^+(X)-1}$ are homotopic to a constant. Since there is only one chamber, duality gives

$$\mathrm{FSW}(E, K \otimes L^*) = (-1)^{\rho_g} \mathrm{FSW}(E, L)$$

hence the families Seiberg-Witten invariant is only non-zero if both $h^0(L)$ and $h^2(L)$ are non-zero by Serre duality.

Proposition 8.1.4. *If $d(X, L) = 0$, then*

$$FSW(E, L) = (-1)^{h^0 - 1}\Big[- c_1(H^{2,0}) \binom{\rho_g - 2}{h^0(L) - 2} + c_1(V^0) \binom{\rho_g - 1}{h^0(L) - 1} + (c_1(V^1) - c_1(V^2)) \binom{\rho_g - 1}{h^0(L) - 2} \Big]$$

If $d(X, L) \neq 0$ then $FSW(E, L)$ is non-zero only if $(h^0, h^1, h^2) = (\rho_g, \rho_g, \rho_g)$, in which case $FSW(E, L)$ is given by

$$FSW(E, L) = c_1(V^1) - c_1(V^0)$$

Proof. It follows from the computation of the families invariant in Theorem 7.1.1 and the fact that $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$ that

$$\begin{aligned} \mathrm{FSW}(E, L) = (-1)^{h^0(L)+1} \Big(& - c_1(H^{2,0}) \binom{\rho_g - 2}{h^1(L) - h^2(L) + \rho_g - 1} \\ & - c_1(V^0) \binom{\rho_g - 1}{h^1(L) - h^2(L) + \rho_g} \\ & + (c_1(V^1) - c_1(V^2)) \binom{\rho_g - 1}{h^1(L) - h^2(L) + \rho_g - 1} \Big) \end{aligned} \tag{8.2}$$

If $d(X, L) = 0$ then $\chi(L) = \rho_g + 1$, thus $h^1(L) - h^2(L) = h^0(L) - \rho_g - 1$, giving the desired result.

If $d(X, L) \neq 0$, it follows from Proposition 5.2.1 that $h^1(L) - h^2(L) \geq 0$, so by inspection of the binomial coefficients in 8.2, the only way for $\mathrm{FSW}(E, L)$ to be non-zero is to have $h^1(L) - h^2(L) = 0$, in which case the invariant is

$$\mathrm{FSW}(E, L) = (-1)^{h^0(L)+1}(c_1(V^1) - c_1(V^2)).$$

For the invariant to be non-zero we also require $n = \chi(L) - \rho_g = h^0(L) - \rho_g \geq 0$ and thus $h^0(L) \geq \rho_g$. This in conjunction with the requirement that $h^2(L) > 0$ and Lemma 2.3.7 implies that $h^0(L) = \rho_g$. An identical argument applied to $K \otimes L^*$ gives $h^2(L) = \rho_g$ via Serre duality, so $h^1(L) - h^2(L) = 0$ gives $h^1(L) = \rho_g$ as well. By replacing L with $K \otimes L^*$, applying charge conjugation and Serre duality then gives

$$\mathrm{FSW}(E, L) = (-1)^{\rho_g}\mathrm{FSW}(E, K \otimes L^*) = c_1(V^1) - c_1(V^0).$$

□

8.2 Further Constraints to Obtain a Diffeomorphism Invariant

Given a smooth Kähler family over S^2, there are two obstructions to the families Seiberg-Witten invariant being a diffeomorphism invariant of the family. First is the issue of chambers, since a diffeomorphism may not take a perturbation to another within the same chamber. Since there is only one chamber for $\rho_g \geq 2$, the chamber obstruction does not occur. In the case of $\rho_g = 0$ there are only two chambers with the Kähler chamber being the $'+'$ chamber. Hence by computing the families Seiberg-Witten invariant in the Kähler chamber, charge conjugation gives the invariant in the $'-'$ chamber, allowing two families to be compared. It is in the case of $\rho_g = 1$ where there is difficulty, the chambers being in correspondence with homotopy classes of maps $S^2 \longrightarrow S^2$, since $\pi_2(S^2) \cong \mathbb{Z}$ there is a chamber for each integer and the Kähler chamber cannot be easily identified as in the other cases.

The second obstruction comes from the families spin^c structure, since the lift of the spin^c structure on X to a families spin^c structure is non-canonical. Viewing S^2 as $\mathbb{CP}^1$, it follows from Proposition 6.1.6 that if s_L is a lift of a spin^c structure on X, then any other lift is of the form $s_L \otimes \pi^*(\mathcal{O}(m))$ for some integer m, then if $\nu = \mathrm{vol}_{S^2}$ is the generator of $H^2(S^2; \mathbb{Z})$ the first Chern class of the new line bundle $\mathcal{L}'$ is $x + m\pi^*(\nu)$ where π is the map $\mathcal{M} \longrightarrow B$. The families Seiberg-Witten invariant is then

$$\mathrm{FSW}_n(E, \pi^*\mathcal{O}(m) \otimes L) = \int_{T(E/B)} (x + m\pi^*(\nu))^n$$

applying the binomial theorem, the fact that only cohomology classes of degree 2 or less survive since the base is S^2 and the fact that the families Seiberg-Witten invariant when valued in degree 0 cohomology is the ordinary Seiberg-Witten invariant of X then gives the only possibly non-vanishing terms

$$\mathrm{FSW}_n(E, \pi^*\mathcal{O}(m) \otimes L) = \mathrm{FSW}_n(E, L) + mn\mathrm{FSW}_{n-1}(E, L)\nu$$

where $2n - d(X, L) = 0$ or 2. If $2n - d(X, L) = 0$ then this is simply the ordinary Seiberg-Witten invariant of X. Consequently, we take the other condition, that $n = 1 + d(X, L)/2$ and denote $\mathrm{FSW}_n(E, L)$ by FSW and $\mathrm{FSW}_{n-1}(E, L)$ so that to make this independent of m and thus a diffeomorphism invariant one could either take $d(X, L) = -2$ or $\mathrm{SW}(X, s_L) = 0$, but the first condition implies the second anyway, so it suffices to look for generally at when $\mathrm{SW}(X, L) = 0$.

If $d(X, L) \neq 0$ and $\rho_g > 0$, then $\mathrm{SW}(X, L) = 0$ since Kähler surfaces are simple-type. If in particular $\rho_g = 1$, since we want a non-zero invariant, we require $h^0(L) = 0$ which implies that $h^1(L) \geq h^2(L)$ by Proposition 5.2.1, but this means the condition $(h^0, h^1, h^2) = (1, 0, 1)$ in Proposition 8.1.3 cannot occur, leaving the other two cases as the only possibility for a non-zero diffeomorphism invariant when $d(X, L) \neq 0$ and $\rho_g > 0$, that is $h^2(L) = 0$ with $\mathrm{FSW}(E, L) = (-1)^{d/2} c_1(H^{2,0}(X))$ or $h^0 = h^1 = h^2 = 1$ with $\mathrm{FSW}(E, L) = -c_1(V^0) + c_1(V^1)$.

On the other hand if $d(X, L) = 0$ and $\rho_g > 0$ then by Proposition 5.2.2 a necessary condition for the Seiberg-Witten invariant to vanish is that h^0 or h^2 is zero. These are in fact sufficient, if $h^0 = 0$ then obviously the Seiberg-Witten invariant is zero and if $h^2(L) = 0$ then $h^0(K \otimes L^*) = 0$, thus $\mathrm{SW}(X, K \otimes L^*) = 0$ and so by charge conjugation $\mathrm{SW}(X, L) = 0$. However, if $\rho_g \geq 2$ we know this forces the families invariant to be zero as well, leaving the only interesting case when $\rho_g = 1$, in such a case, the requirement that $h^0(L) > 0$ and $h^2(L) = 0$ forces $\mathrm{FSW}(E, L) = (-1)^{d/2} c_1(H^{2,0}(X))$.

If $\rho_g \geq 2$ and $d(X, L) \neq 0$ then from Proposition 8.1.4 we know that $h^0 = h^1 = h^2 = \rho_g$ and $\mathrm{FSW}(E, L) = c_1(V^1) - c_1(V^0)$.

Finally if $\rho_g = 0$ and $d(X, L) \geq 0$ then $\mathrm{SW}^+(X, L) = 0$ requires that $h^0(L) = 0$, forcing the families invariant to be zero. However it is zero whenever $d(X, L) < 0$, in which case the fact that $\rho_g = 0$ gives $h^2(L) = 0$ and $H^{2,0} = 0$ so via the computation of the families invariant in the Kähler chamber is $\mathrm{FSW}(E, L) = -c_1(V^0) + c_1(V^1)$. Although this is in generally independent of the lift when $d(X, L) < 0$, due to the chamber structure it is not necessarily a diffeomorphism invariant of the family unless we take $n = 0$, i.e. $d(X, L) = -2$. We summarise these results in the following theorem.

Theorem 8.2.1. *Let $X \hookrightarrow E \to S^2$ be a smooth Kähler family over S^2, L be a holomorphic line bundle on E. Suppose that the cohomology groups of L have constant dimension over b and denote the resulting cohomology bundles by V^i. Then $FSW(E, L)$ is a non-zero diffeomorphism invariant of the family provided we are in one of the following cases*

- $\rho_g = 0, d(X, L) = -2, h^2(L) > 0$, *in which case*
$$FSW(E, L) = c_1(V^1) - c_1(V^0).$$
- $\rho_g = 1, h^0(L) > 0, h^2(L) = 0$, *in which case*
$$FSW(E, L) = (-1)^{d(X,L)/2} c_1(H^{2,0}).$$
- $\rho_g = 1, h^0(L) = h^1(L) = h^2(L) = 1$, *in which case*
$$FSW(E, L) = c_1(V^1) - c_1(V^0).$$
- $\rho_g \geq 2, h^0(L) = h^1(L) = h^2(L) = \rho_g$, *in which case*
$$FSW(E, L) = c_1(V^1) - c_1(V^0).$$

If $\rho_g > 0$ and $\alpha > 1$ is an integer then if we instead ask that the reduction of $\mathrm{FSW}(E, L)$ mod α is independent of the lift of spin^c structure, then it is a well-defined invariant of the family valued in $\mathbb{Z}/\alpha\mathbb{Z}$ when the $\mathbb{Z}$-valued invariant may not be. In this instance, one requires that $n\mathrm{SW}(E, L)$ is a multiple of α instead of being zero as done above to obtain invariance. If $d(X, L) \neq 0$, then $\mathrm{SW}(E, L) = 0$ and $\mathrm{FSW}(E, L)$ would be a $\mathbb{Z}$-valued invariant, so assume that $d(X, L) = 0$, hence we must assume $\mathrm{SW}(E, L)$ is a multiple of α and further must assume $h^0(L), h^2(L) > 0$ for it to be non-zero. Furthermore, since

$$\mathrm{SW}(E, L) = (-1)^{h^0(L)+1} \binom{\rho_g - 1}{h^0(L) - 1}$$

if we take $1 < h^0(L) < \rho_g$, which also forces $\rho_g \geq 3$, then $|\mathrm{SW}(E, L)| > 1$ and we may take $\alpha = |\mathrm{SW}(E, L)|$. In such a case, the computation for the families Seiberg-Witten invariant when reduced mod k gives

$$\mathrm{FSW}(E, L) = (-1)^{h^0+1}\left((c_1(V^1) - c^2(V^2))\binom{\rho_g - 1}{h^0 - 2} - c_1(H^{2,0})\binom{\rho_g - 2}{h^0 - 2}\right). \tag{8.3}$$

8.3 Analysing the Example Families when $B = S^2$

8.3.1 Remark on the $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ families

Since both $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ satisfy $\rho_g = 0$ we cannot seek a mod α invariant as done previously, moreover by Theorem 8.2.1 we obtain a non-zero diffeomorphism invariant only if $d(X, L) = -2$ and $h^0(L) > 0$. However, $d(X, L) = -2$ implies that $h^0(L) - h^1(L) + h^2(L) = 0$ since $\rho_g = 0$, but both families must satisfy $h^1(L) = h^2(L) = 0$, thus $h^0(L) = 0$, so there is no case in which we obtain a non-zero diffeomorphism invariant for both the projectivisation family and $\mathbb{CP}^1 \times \mathbb{CP}^1$ family when $B = S^2$.

8.3.2 The Blowup Family

Recall the universal blowup family $\pi : Z \longrightarrow X$ over a Kähler manifold X satisfies a universal property, namely that if $Z' \longrightarrow B$ is any other smooth family where the fibre is the blowup of X at a point where the specified blowup up point is defined by a map $f : B \longrightarrow X$, then $Z' \longrightarrow B$ is the pullback of $Z \longrightarrow X$ under f. Consequently smooth blowup families $Z' \longrightarrow S^2$ can be obtained from maps $f : S^2 \longrightarrow X$. For instance, if X is simply connected, there is an isomorphism between homotopy classes of such maps and $H_2(X; \mathbb{Z})$, hence such maps certainly exist for a simply connected Kähler surface can be known up to homotopy. Furthermore, this also means we only need to consider the computation for the universal blowup family, since the computation for any other blowup family can simply be obtained by taking the pullback under f.

Choose a spin^c structure with respect to the canonical spin^c structure via a choice of line bundle of the form $L = \pi^*L_1 \otimes p^*L_2 \otimes \mathcal{O}(kE)$ where $k \in \mathbb{Z}$. Since we are interested in blowup families which are the pullback of the universal family, the families Seiberg-Witten invariant for such families over S^2 can be computed via the formulae of Section 7.1 with the relevant vector bundles involved replaced by the pullback under f. Hence we may use the computations of Theorem 8.2.1 to find non-zero diffeomorphism invariants of the family.

Since the vector bundle $H^{2,0}(X)$ is trivial for the blowup family, we are only interested in the cases stated in Theorem 8.2.1 when the invariant is given by

$$\mathrm{FSW}(E, L) = c_1(f^*V^1) - c_1(f^*V^0).$$

Recall for the universal blowup family one has $V^i = W^i \otimes L_1$, hence

$$\mathrm{FSW}(E, L) = c_1(f^*W^1) - c_1(f^*W^0) + (h^1(L) - h^0(L))c_1(f^*L_1)$$

First assume that $h^0 > 0$, $\rho_g = 0$, $d(X, L) = -2$ and $k \leq 0$, then W^2 is trivial with fibres $H^2(X, L_2)$ and there is an exact sequence

$$0 \to W^0 \to H^0(X, L_2) \to J^{-1-k}(L_2) \to W^1 \to H^1(X, L_2) \to 0$$

from which it follows that $h^0 - p^0 - k(1-k)/2 - h^1 + p^1 = 0$ and $c_1(W^1) - c_1(W^0) = c_1(J^{-1-k}(L_2))$. Furthermore, since $h^0 > 0$ and $\rho_g = 0$, Lemma 2.3.7 implies that $h^2 = p^2 = 0$. Since $d(X, L) = -2$ we then have that $h^0 - h^1 + p^2 = 0$, thus $k^2 - k - 2\chi(X, L_2) = 0$ and $h^0 = h^1$. This has a unique non-positive integer solution for $k \leq 0$ if and only if $\Delta = 1 + 8\chi(X, L_2)$ is positive and an odd perfect square. Hence we may obtain non-zero proper diffeomorphism invariants provided the above conditions are satisfied and the families Seiberg-Witten invariant is given by

$$\mathrm{FSW}(E, L) = c_1(f^* J^{-1-k}(L_2))$$

Now suppose instead that $k > 0$, then W^0 is trivial with fibres $H^0(X, L_2)$ and there is an exact sequence

$$0 \to H^1(X, L_2) \to W^1 \to J^{k-2}(L_2 \otimes K_X^*)^* \to H^2(X, L_2) \to W^2 \to 0$$

which gives $p^1 - h^1 + k(k-1)/2 - p^2 + h^2 = 0$. The condition that $d(X, L) = -2$ implies that $p^0 - h^1 + h^2 = 0$ and applied to the previous expression, gives $k^2 - k - 2\chi(X, L_2) = 0$ which has a unique positive integer solution in k. Since we assume $h^0(L) = p^0(L_2) > 0$, Lemma 2.3.7 implies that W^2 and $H^2(X, L_2)$ are both zero bundles, hence $h^1 = p^0 = h^0$ and the exact sequence reduces to

$$0 \to H^1(X, L_2) \to W^1 \to J^{k-2}(L_2 \otimes K_X^*)^* \to 0$$

which gives $c_1(W^1) = c_1(J^{k-2}(L_2 \otimes K_X^*))^*$ and thus the families Seiberg-Witten invariant is given by

$$\mathrm{FSW}(E, L) = c_1(f^* J^{k-2}(L_2 \otimes K_X^*)^*)$$

and so a non-zero invariant of this form may be obtained provided we satisfy the above conditions and the above expression is non-zero.

We now seek to deal with the cases when $h^0 = h^1 = h^2 = \rho_g$, $\rho_g > 0$.

If $k \leq 0$ then as before, W^2 is trivial with fibres $H^2(X, L_2)$ and there is an exact sequence

$$0 \to W^0 \to H^0(X, L_2) \to J^{-1-k}(L_2) \to W^1 \to H^1(X, L_2) \to 0.$$

Since the map $W^0 \to H^0(X, L_2)$ is injective $p^0 \geq h^0 = \rho_g$ and $W^1 \to H^1(X, L_2)$ is surjective, $p^1 \leq h^1 = \rho_g$. Furthermore, since $p^2 = \rho_g > 0$, Lemma 2.3.7 implies that $p^0 \leq \rho_g$, therefore $p^0 = \rho_g$. It then follows from the above exact sequence that we must have $p^1 = k(1-k)/2 - \rho_g$ and via an identical computation of the Chern classes as done previously previously, we obtain that the families Seiberg-Witten invariant is given by

$$\mathrm{FSW}(E, L) = c_1(f^* J^{-1-k}(L_2))$$

Now assume that $k > 0$, then W^0 is trivial with fibres $H^0(X, L_2)$ and there is an exact sequence

$$0 \to H^1(X, L_2) \to W^1 \to J^{k-2}(L_2 \otimes K_X^*)^* \to H^2(X, L_2) \to W^2 \to 0.$$

from which we obtain the following identity in the first Chern classes

$$c_1(W^1) - c_1(W^2) = c_1(J^{k-2}(L_2 \otimes K_X^*))^* = c_1(J^{k-2}(L_2 \otimes K_X^*)^*)$$

and that $p^1 - k(1-k)/2 - p^2 = 0$. Since $H^1(X, L_2) \to W^1$ is injective, $p^1 \leq \rho_g$ and similarly since $H^2(X, L_2) \to W^2$ is surjective it follows that $\rho_g \leq p^2$. Applying Lemma 2.3.7 to L^2 gives $p^2 \leq p^0 = h^0 = \rho_g$, hence $p^2 = \rho_g$. Now when $\rho_g = 1$, recall that in Proposition 8.1.3 it was proven that $c_1(V^0) - c_1(V^2) = c_1(H^{2,0})$, since $H^{2,0}$ is trivial for the blowup family we have $c_1(V^1) = c_1(V^2)$ and it follows that the families Seiberg-Witten invariant is given by

$$\mathrm{FSW}(E, L) = c_1(f^* W^1) - c_1(f^* W^2) + (h^1 - h^0) c_1(f^* L_1)$$

and so, due to the above expression in the Chern classes and since $h^1 = h^0$

$$\mathrm{FSW}(E, L) = c_1(J^{k-2}(L_2 \otimes K_X^*)^*)$$

when $\rho_g = 1$. Furthermore when $\rho_g \geq 2$, it follows from the computations in Proposition 8.1.4 that

$$\mathrm{FSW} = (-1)^{\rho_g+1}(c_1(V^1) - c_1(V^2))$$

and so via the same computation used when in the case of $\rho_g = 1$, we have

$$\mathrm{FSW}(E, L) = (-1)^{\rho_g+1} c_1(J^{k-2}(L_2 \otimes K_X^*)^*).$$

If we instead seek a mod $\alpha = |\mathrm{SW}(E, L)| = \binom{\rho_g-1}{h^0-1}$ invariant as in the end of Section 8.2 then we must assume $d(X, L) = 0$ and $1 < h^0 < \rho_g$. If $k \leq 0$ then W^2 is trivial and there is an exact sequence

$$0 \to W^0 \to H^0(X, L_2) \to J^{-1-k}(L_2) \to W^1 \to H^1(X, L_2) \to 0$$

it then follows that $c_1(W^1) = c_1(J^{-1-k}(L_2)) + c_1(W^0)$ and combined with $d(X, L) = 0$, one can show that $k^2 - k - 2(\chi(X, L_2) - \rho_g - 1) = 0$. With this data and Equation (8.3), the mod α families Seiberg-Witten invariant for $k \leq 0$ is

$$\mathrm{FSW}(E, L)_\alpha = \left(c_1(f^* J^{-1-k}(L_2)) + c_1(f^* W^0) + h^1 c_1(f^* L_1)\right) \binom{\rho_g - 1}{h^0 - 2}$$

and this provides a non-zero invariant provided the above conditions hold.

If instead we have $k > 0$, then W^0 is trivial and there is an exact sequence

$$0 \to H^1(X, L_2) \to W^1 \to J^{k-2}(L_2 \otimes K_X^*)^* \to H^2(X, L_2) \to W^2 \to 0$$

from which again forces $k^2 - k - 2(\chi(X, L_2) - \rho_g - 1) = 0$ and we obtain $c_1(W^1) - c_1(W^2) = c_1(J^{k-2}(L_2 \otimes K_X^*)^*)$ and so using Equation (8.3), the mod α invariant is

$$FSW(E, L)_\alpha = \left(c_1(f^* J^{k-2}(L_2 \otimes K_X^*)^*) + (h^1 - h^2) c_1(f^* L_1)\right) \binom{\rho_g - 1}{h^0 - 2}.$$

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