Equivariant Bundle Gerbes and Simplicial Extensions

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Dedication

This thesis is dedicated to my family and friends for their support throughout the last two years.

Abstract

Bundle gerbes, introduced in Michael Murray's 1996 paper "Bundles Gerbes" are a way of geometrically representing degree three integral cohomology for a manifold in the same way that line bundles represent degree two integral cohomology. We want to explore the notion of a bundle gerbe on a simplicial manifold and relate this to simplicial cohomology. We define the simplicial extension of a bundle gerbe and show that appropriate equivalence classes of simplicial extensions are classified by degree two $U(1)$ cohomology and in some cases degree three integral cohomology. One important example we come across is the simplicial space generated by the action of a Lie group on a manifold. These simplicial techniques give us a method of classifying degree three integral equivariant cohomology using the idea of weak actions and strong actions of a Lie group on a bundle gerbe. We introduce the universal strongly equivariant bundle gerbe which is universal in the sense that every strongly equivariant bundle gerbe is a pullback of this bundle gerbe.

Chapter 1

Introduction

This thesis is mostly focused on the construction and definition of various types of cohomology on simplicial manifolds. In order to do so, we build up an understanding of simplicial objects and cohomology on topological spaces in chapter 2. We define a simplicial manifold to be a simplicial object in the category of manifolds, in particular this is a contravariant functor $\Delta \to \mathbf{Man}$. The category $[\Delta^{op}, \mathbf{Man}]$ will remain the focus of our interest throughout the thesis. We define the notions of Čech, sheaf, and singular cohomology for manifolds, each playing its own part throughout the thesis.

We also look at relations between co-chain complexes via the algebraic techniques of double complexes and spectral sequences. In the same way that double complexes relate the cohomology of single complexes we define the notions of triple-complexes in order to relate the cohomology of double complexes. This results in an understanding of triple complexes with reference to double complexes and analogues of theorems of which spectral sequences produce for double complexes.

In chapter 3 we start to talk more explicitly about simplicial manifolds. In particular we define the important example $EG(M)_\bullet$ which we will relate throughout the thesis to the equivariant cohomology of a manifold M. We consider the constructions of the fat and geometric realisations and then move on to define cohomology on a simplicial manifold $X_\bullet$. In order to define simplicial-Čech cohomology we first are required to define the notion of a sheaf on a simplicial manifold. There are two equivalent ways of constructing sheaves on simplicial manifolds, either locally or in terms of Grothendieck topos. We then discuss how the two are related in computing the cohomology of a simplicial manifold. Doing so allows us to define both sheaf and Čech cohomology of a simplicial manifold with respect to a sheaf $A \in \mathbf{AbSh}(\mathbf{X}_\bullet)$.

Our notion of Čech cohomology with respect to a covering then leads to defining coverings of a simplicial manifold $X_\bullet$ and in particular good coverings of a simplicial manifold. Following this definition we show that the category of simplicial-covers on a simplicial manifold is a directed category in which the subcategory of good simplicial covers are cofinal, thus allowing simplicial-Čech cohomology to be computed as the direct limit of simplicial-Čech cohomology with respect to good covers on $X_\bullet$.

We then show that simplicial-Čech cohomology with respect to a good cover is isomorphic to that of sheaf cohomology by using triple complexes. This is a major turning point in the thesis as this allows us to compute simplicial-Čech cohomology by computing the simplicial-Čech cohomology with respect to a good simplicial cover of $X_\bullet$, simplifying the classification of geometric objects on simplicial manifolds. We then attain similar results for simplicial-singular cohomology, in particular this isomorphism of integral simplicial-singular cohomology and integral simplicial-Čech cohomology allows us to prove an analogue of Theorem 5.15 of [1]

$$\check{H}^\bullet(\|X_\bullet\|, \mathbb{Z}) \simeq \check{H}^\bullet(\mathcal{U}^{(\bullet)}, \mathbb{Z}).$$

We will see that this isomorphism will relate geometric objects on the simplicial manifold $X_\bullet$ to that of geometric objects on the fat realisation.

We move on to more common differential geometry concepts. In particular we look at G-bundles and their classification in terms of simplicial manifolds. We define descent for G-bundles and look at the pre-G bundle which will become useful later when we look at simplicial extensions and the lifting gerbe. Equipped with this knowledge we are able to define the famous bundle gerbe due to Michael Murray [2]. We define the Dixmier-Douady class and show that the stable isomorphism classes of bundle gerbes are in bijection with degree three integral Čech cohomology. An important example of a bundle gerbe is the lifting gerbe of a central extension, this will be used when we wish to classify equivariant bundle gerbes by equivariant integral cohomology.

We now look for classification theorems of simplicial $U(1)$-bundles on a simplicial manifold. We get that isomorphism classes of $U(1)$-bundles on $X_\bullet$ are in bijection with $\check{H}^1(\mathcal{U}^{(\bullet)}, U(1))$. Furthermore we can look at the explicit example of $EG(M)_\bullet$ and we find that G-equivariant $U(1)$-valued principle bundles on M are in bijection with the equivariant cohomology $H^1_G(M, U(1))$. Furthermore if we look at G-equivariant $U(1)$-bundles we find that these objects are classified by the second degree equivariant integral cohomology on M. We look at the definition of pre-G and equivariant pre-A bundles, this leads into the definition of a simplicial extension of a bundle gerbe.

Given the information that we have so far we are at this point able to define the notion of a simplicial extension over $X_\bullet$ of a bundle gerbe on X_0 which is done in Definition 6.2.1. We then define notions of product, dual, and triviality for a simplicial extension which are derived from the simplicial-Dixmier-Douady class. We then show that the stable-isomorphism classes of simplicial extensions are in bijection with the $U(1)$-valued simplicial-Čech cohomology $\check{H}^2(X_\bullet, U(1))$. In some special cases we have that $\check{H}^2(X_\bullet, U(1)) \simeq \check{H}^3(X_\bullet, \mathbb{Z})$, in particular this is true for simplicial manifold $EG(M)_\bullet$. In these cases we then get a classification of simplicial extensions in terms of third degree integral simplicial-Čech cohomology, and so by our analogue of theorem 5.15 [1] we get that isomorphism classes of simplicial extensions on $X_\bullet$ are in direct bijection with stable isomorphism classes of bundle gerbes on $\|X_\bullet\|$. The notions of simplicial extensions can be extended to bundle p-gerbes and A-valued bundle gerbes on X_0 in a similar fashion. We then get the appropriate classification theorems relating these objects. We then define the 2-category of simplicial extensions and bundle gerbes on a simplicial manifold $X_\bullet$.

With the theory of simplicial extensions and simplicial cohomology we look at the main application of both of these objects. This allows us to define the notion of equivariant cohomology in terms of simplicial manifolds. If G is a Lie group which acts on a manifold M we define the equivariant cohomology of M by $H^n_G(M,\mathbb{Z}) := H^n(M \times EG/G, \mathbb{Z})$, as we have that EG is homotopic to $\|EG_\bullet\|$ and so we can compute the equivariant cohomology using simplicial methods, in particular

$$H^n_G(M,\mathbb{Z}) \simeq \check{H}^n(EG(M)_\bullet, \mathbb{Z}).$$

This allows us to relate the theory of simplicial extensions to that of the theory of equivariant cohomology. In particular the isomorphism above tells us that the isomorphism classes of simplicial extensions of bundle gerbes are in bijection with degree three equivariant integral cohomology for a manifold M. We then define the notions of a weakly-equivariant bundle gerbe (by a simplicial extension over $EG(M)_\bullet$) and a strongly equivariant bundle gerbe. We have a natural morphism from strongly equivariant bundle gerbes into weakly equivariant bundle gerbes. We define a notion of triviality for strongly equivariant bundle gerbes and this shows that the morphism described is well defined in terms of stable isomorphism classes. Finally, we define the object known as the *universal strongly equivariant bundle gerbe* which helps us describe the isomorphism between classes of strongly equivariant bundle gerbes and weakly equivariant bundle gerbes. In particular if we are given a bundle gerbe on $M \times_G EG$ we show that there exists a strongly equivariant bundle gerbe $\mathcal{G}$ on M such that $\mathcal{G} \times_G EG$ is stably isomorphic to the bundle gerbe that we started with. This construction gives a bijection between the stable isomorphism classes of strongly equivariant bundle gerbes and $H^3_G(M;\mathbb{Z})$.

Chapter 2

Background

2.1 The Simplex Category

We refer to Tom Leinster's book "Basic Category Theory" [3] for standard notions in category theory. We assume that the reader is familiar with the notions of categories, functors, natural transformations, and adjoint functors.

Definition 2.1.1 (Simplex category)**.** *We define the category Δ to have objects the finite ordered sets $[n] := \{0, \ldots, n\}$ for $n \geq 0$ with morphisms given by weakly order preserving maps, $f : [n] \to [m]$ such that if $i \leq j$ then $f(i) \leq f(j)$. We call Δ the simplex category.*

Proposition 2.1.2 (Generator for the morphisms of the simplex category)**.** *The morphisms in the simplex category can be generated by a sequence of co-face and co-degeneracy maps denoted by d^i and s^i respectively, this is a well known fact, for instance see [4]. We represent these maps by*

$$\begin{aligned} d^i &: [n] \to [n+1] \quad i = 0, \ldots, n+1 \\ s^i &: [n+1] \to [n] \quad i = 0, \ldots, n \end{aligned}$$

Where d^i is the injective map that does not attain the value $i \in [n+1]$ and s^i is the surjective map that sends $i, i+1 \mapsto i \in [n]$.

Figure 2.1: The underlying graph of the category Δ can be conveniently pictured in this way.

$$[0] \underset{s_i}{\overset{d_i}{\rightleftarrows}} [1] \underset{s_i}{\overset{d_i}{\rightleftarrows}} [2] \underset{s_i}{\overset{d_i}{\rightleftarrows}} [3] \cdots$$

There are functions $\mu^k : [0] \to [n]$ for $k = 0, \ldots, n$ such that $\mu^k : 0 \mapsto k$ in Δ. We can represent these functions in terms of co-face maps d^i in a consistent way.

Proposition 2.1.3. *We have for* $\mu^j : [0] \to [n]$

$$\mu^j = (d^0)^j \circ d^{n-j} \circ d^{n-j-1} \circ \cdots \circ d^1.$$

Proof. To prove this we simply compute this value at zero. If $i \geq j$ then $d^j(i) = i + 1$. This is due to the fact that omitting the number $j \leq i$ will 'push' i up one. Firstly let $j = n$, we have $(d^0)^n(0)$ pushes the number 0 up n times, and thus $(d^0)^n(0) = n = \mu^n(0)$. Now suppose that $0 < j < n$, we have that

$$\begin{aligned}(d^0)^j \circ d^{n-j} \circ d^{n-j-1} \circ \cdots \circ d^1(0) &= (d^0)^j(0)\\ &= j\\ &= \mu^j(0)\end{aligned}$$

due to the fact that $d^k(0) = 0$ for $k \neq 0$. Finally for $j = 0$ we have that

$$\begin{aligned}d^n \circ \cdots \circ d^1(0) &= 0\\ &= \mu^0(0)\end{aligned}$$

due to the same fact, and so we have expressed μ^j in some regular form. □

Proposition 2.1.4 (co-simplicial identities)**.** *We have that the co-face and co-degeneracy maps satisfy the following identities (as seen in [4], for example).*

$$\begin{aligned}
d^j \circ d^i &= d^i \circ d^{j-1} && \text{if } i < j\\
s^j \circ s^i &= s^{i-1} \circ s^j && \text{if } i > j\\
s^j \circ d^i &= \begin{cases} d^i \circ s^{j-1} & \text{if } i < j\\ id & \text{if } i = j \text{ or } i = j+1\\ d^{i-1} \circ s^j & \text{if } i > j+1\end{cases}
\end{aligned}$$

Definition 2.1.5 (The category of simplicial objects)**.** *A simplicial object in a category* $\mathscr{C}$ *is a contravariant functor from* Δ *to* $\mathscr{C}$*, i.e.* $X : \Delta^{op} \to \mathscr{C}$*. We can also talk about the category of simplicial objects in* $\mathscr{C}$ *denoted* $\mathbf{S}(\mathscr{C})$*. The morphisms of simplicial objects are the natural transformations between them.*

Given a simplicial object $X : \Delta^{op} \to \mathscr{C}$, we define the degeneracy maps $s_k = X(s^k)$ and the face maps $d_k = X(d^k)$.

Proposition 2.1.6 (Simplicial Identities). *Given a simplicial object X we have that the face and degeneracy maps satisfy the following identities*

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i && \text{if } i < j \\ s_i \circ s_j &= s_j \circ s_{i-1} && \text{if } i > j \\ d_i \circ s_j &= \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ id & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases} \end{aligned}$$

We can represent a simplicial object $X : \Delta^{op} \to \mathscr{C}$ by a sequence of objects $X_\bullet = X([0]), X([1]), \cdots \in \mathscr{C}$ which we will call $X_0, X_1, \cdots$ with face and degeneracy maps $d_k = X(d^k)$, $s_k = X(s^k)$ that satisfy the simplicial identities described in Proposition 2.1.6. We will often write $X_\bullet$ to refer to a simplicial object. This definition is often how we will define simplicial objects without reference to functors or the simplex category. This is particularly convenient for describing simplicial objects. In particular we will see simplicial topological spaces, simplicial sets, simplicial abelian groups, and simplicial manifolds.

Example 2.1.7. *Given a set S we can define the simplicial set $S^{\bullet+1}$ given by the sequence of sets $S, S^2, S^3, \ldots$ with face and degeneracy morphisms given by*

$$\begin{aligned} d_i : S^{n+1} &\to S^n \\ (x_1, \cdots, x_{n+1}) &\mapsto (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}) \\ s_i : S^{n+1} &\to S^{n+2} \\ (x_1, \cdots, x_{n+1}) &\mapsto (x_1, \cdots, x_i, x_i, \cdots, x_{n+1}) \end{aligned}$$

Definition 2.1.8 (Semi-simplicial object). *A semi-simplicial object $C_\bullet$ in $\mathscr{C}$ is a sequence of objects $C_0, C_1, \ldots \in \mathscr{C}$ similar to that of a simplicial object but we do not have the information of the degeneracy maps. In particular, we only have maps $d_i : C_{k+1} \to C_k$ satisfying the identities $d_i \circ d_j = d_{j-1} \circ d_i$ if $i < j$.*

Notice that every simplicial object has an underlying semi-simplicial object, we simply forget about the degeneracy morphisms. We will find the notion of semi-simplicial object useful when we work with simplicial covers of a simplicial manifold. Furthermore we can define the category Δ_{semi} Using this category we can also define a semi-simplicial object in $\mathscr{C}$ as a contravariant functor $\Delta_{semi}^{op} \to \mathscr{C}$.

Definition 2.1.9 (cosimplicial object). *A cosimplicial object is simply a functor $F : \Delta \to \mathscr{C}$. We can also similarly define a semi-cosimplicial object by forgetting about co-degeneracy maps.*

Example 2.1.10. *Although an uninteresting one, the identity functor id_Δ is a cosimplicial object.*

2.2 Cohomology

2.2.1 Simplicial Cohomology

We first wish to look at simplicial cohomology, we will find that other types of cohomology can be defined in terms of simplicial objects. Let $A^\bullet$ be a semi-cosimplicial abelian group, notice that we only require a semi-cosimplicial object to define simplicial cohomology. We can define a co-chain complex $0 \to A^0 \to A^1 \to \cdots$ by defining

$$\begin{aligned} \delta : A^k &\to A^{k+1} \\ a &\mapsto \sum_{i=0}^{k+1} (-1)^i d^i(a) \end{aligned}$$

Proposition 2.2.1. *$A^\bullet$ forms a co-chain complex as claimed.*

Proof. This is a standard construction, see [4] for a proof of this fact. □

In fact this construction extends to give a categorical equivalence between the category of cosimplicial abelian groups and the category of (non-negative, graded) co-chain complexes, see [4] for more details. This is known as the Dold-Kan correspondence.

2.3 Sheaf Cohomology

We refer to [5] for a summary of sheaf cohomology. Briefly, we have the following important definitions and results from [5].

Definition 2.3.1. *A* presheaf *$\mathcal{F}$ on a topological space X is a contravariant functor $\mathscr{O}(X) \to$* **Set** *where $\mathscr{O}(X)$ is the category of open sets in X with morphisms given by inclusion maps, in particular we have functions $\rho_{U,V} : F(U) \to F(V)$ if $U \subseteq V$. A* sheaf *is a presheaf such that for every open set $V \subseteq X$ with covering $\mathcal{U} = (U_i)_{i \in I}$ of V, if we are given sections $(s_i)_{i \in I}$ such that $s_i \in \mathcal{F}(U_i)$ satisfying the compatibility condition*

$$\rho_{U_i \cap U_j, U_i}(s_i) = \rho_{U_i \cap U_j, U_j}(s_j)$$

there exists a unique $s \in \mathcal{F}(V)$ such that $\rho_{V,U_i}(s) = s_i$.

If we are given a function $s : X \to Y$ and a subset $U \subseteq X$ we will often use the notation $s|_U : U \to Y$ to denote the restriction of s to the subset of U.

Example 2.3.2. *Given an abelian Lie group A we have the sheaf $\underline{A}$ which consists of smooth A valued functions, $\underline{A}(U) = C^\infty(U, A)$.*

Definition 2.3.3. *Given a sheaf F on X we denote $\Gamma(X, F) := F(X)$ the global sections of the sheaf F.*

Proposition 2.3.4. *Equality of subsheaves can be detected on stalks. Let $\mathcal{G}$ be a sheaf on X, and let $\mathcal{F}$ and $\mathcal{F}'$ be subsheaves of $\mathcal{G}$. The subsheaves $\mathcal{F} = \mathcal{F}'$ if and only if the stalks $\mathcal{F}_x = \mathcal{F}'_x$ for each $x \in X$.*

Proof. We see this proof in [5, Lemma 1.1.8]. □

Corollary 2.3.5. *[5] For a smooth manifold X there is an exact sequence of sheaves*

$$0 \to \mathbb{Z}(1) \to \underline{\mathbb{C}}_X \to \underline{\mathbb{C}}^*_X \to 0$$

where the morphism $\underline{\mathbb{C}}_X \to \underline{\mathbb{C}}^*_X$ is induced by the exponential map.

Definition 2.3.6. *A sheaf $\mathcal{F}$ of abelian groups on a topological space X is called* injective *if for every diagram of sheaves of abelian groups on X*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & B \\ \mathcal{F} & \end{tikzcd}$$

with $\ker(i) = 0$ *there exists a morphism $g : B \to \mathcal{F}$ such that $g \circ i = f$.*

Lemma 2.3.7. *[5] For any sheaf $\mathcal{F}$ of abelian groups on X, there exists an injective sheaf I of abelian groups and a monomorphism $\mathcal{F} \to I$.*

For a proof of the above fact see [5].

Definition 2.3.8. *A* resolution $K^\bullet$ *of a sheaf $A \in \mathbf{AB}(X)$ is a complex of sheaves $K^\bullet$ with a morphism $i : A \to K^0$ such that*

1. *i is a monomorphism with image equal to* $\ker(d^0)$.
2. *For $n \geq 1$* $\ker(d^n) = im(d^{n-1})$.

Proposition 2.3.9. *[5] For any abelian sheaf $A \in \mathbf{AB}(X)$ there exists a resolution $A \xrightarrow{i} I^\bullet$ of A in which each I^n is injective. This is an injective resolution of A.*

For a proof of the above fact see [5]. This then allows us to define sheaf cohomology for a sheaf $A \in \mathbf{AB}(X)$.

Definition 2.3.10. *Given a sheaf A with injective resolution $A \xrightarrow{i} I^\bullet$ we define the sheaf cohomology groups $H^n(X, A)$ by the cohomology of the complex*

$$\cdots \to \Gamma(X, I^n) \to \Gamma(X, I^{n+1}) \to \Gamma(X, I^{n-1}) \to \cdots$$

2.3.1 Čech Cohomology

We refer to [5] for a summary of Čech cohomology. Consider a presheaf of abelian groups A over some space X with an open cover $\mathcal{U} = \{U_i\}_{i\in I}$. For simplicity we will denote $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0\cdots i_p}$.

Definition 2.3.11. *[5] Let $p \geq 0$ and define $\check{C}^p(\mathcal{U}, A) := \prod_{i_0\cdots i_p\in I} A(U_{i_0\cdots i_p})$ to be the product over $p+1$ tuples of elements of I. An element $\underline{\alpha} \in \check{C}^p(\mathcal{U}, A)$ is a family of $\alpha_{i_0\cdots i_p} \in A(U_{i_0\cdots i_p})$ such that $i_0 \cdots i_p \in I$.*

Definition 2.3.12. *[5] We define a homomorphism $d : \check{C}^p(\mathcal{U}, A) \to \check{C}^{p+1}(\mathcal{U}, A)$ as follows*

$$d(\underline{\alpha})_{i_0\cdots i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j (\alpha_{i_0\cdots i_{j-1} i_{j+1}\cdots i_{p+1}})|_{U_{i_0\cdots i_{p+1}}}$$

Proposition 2.3.13. *The homomorphism d defines a complex of abelian groups*

$$\cdots \xrightarrow{d} \check{C}^p(\mathcal{U}, A) \xrightarrow{d} \check{C}^{p+1}(\mathcal{U}, A) \xrightarrow{d} \cdots \tag{2.1}$$

Proof. All that needs to be shown is that $dd = 0$. We have that the Čech complex can be defined as a semi-cosimplicial abelian group $\check{C}^\bullet(\mathcal{U}, \underline{A}) : \Delta_{\text{semi}} \to \mathbf{Ab}$, specifically $\check{C}^\bullet(\mathcal{U}, \underline{A}) = \underline{A}(N\mathcal{U}_\bullet)$ where $N\mathcal{U}_\bullet$ is the nerve of the covering $\mathcal{U}$ (see example 3.1.4). The map $d^j : \check{C}^p(\mathcal{U}, \underline{A}) \to \check{C}^{p+1}(\mathcal{U}, \underline{A})$ is defined by

$$d^j(\underline{\alpha})_{i_0\cdots i_{p+1}} = \underline{\alpha}_{i_0\cdots\widehat{i_j}\cdots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}$$

where $\widehat{i_j}$ indicates omission of that index. These coface maps form a semi-cosimplicial complex and thus from Proposition 2.2.1 we have that the morphism d forms a chain complex $\check{C}^\bullet(\mathcal{U}, \underline{A})$. It is clear that d as defined in Definition 2.3.12 is the same as that in Proposition 2.2.1. □

Definition 2.3.14. *[5, Definition 1.3.1] The Čech cohomology with respect to $\mathcal{U}$ is defined by the complex in 2.1 where the degree p cohomology group is denoted by $\check{H}^p(\mathcal{U}, A)$ and is called the degree p Čech cohomology of the covering $\mathcal{U}$ with coefficients in the presheaf A.*

Definition 2.3.15. *[5, Definition 1.3.1] An element $\underline{\alpha} \in \check{C}^p(\mathcal{U}, A)$ is called a Čech p-cochain of the covering $\mathcal{U}$ with coefficients in the presheaf A. If $\underline{\alpha} \in \ker d$ then $\underline{\alpha}$ is called a Čech p-cocycle of the cover $\mathcal{U}$ with coefficients in A. We say that $d(\underline{\alpha})$ is co-boundary of $\underline{\alpha}$. The image of the p-cocycle $\underline{\alpha}$ in $\check{H}^p(\mathcal{U}, A)$ is called the cohomology class of $\underline{\alpha}$. The complex in equation 2.1 is called the Čech complex of $\mathcal{U}$ with coefficients in A and is denoted $\check{C}^\bullet(\mathcal{U}, A)$.*

Proposition 2.3.16. *[5] If A is a sheaf then we have that $\check{H}^0(\mathcal{U}, A) = \Gamma(X, A)$.*

Definition 2.3.17 (Čech Cohomology)**.** *The Čech cohomology groups are defined by taking the directed limit on open covers* $\mathcal{U}$*,*

$$\check{C}^p(X, A) = \varinjlim_{\mathcal{U}} \check{C}^p(\mathcal{U}, A)$$

where $\mathcal{U}$ *is ordered by refinement, we say that* $\mathcal{U} < \mathcal{V}$ *if there exists a morphism of indexing sets* $f : I \to J$ *such that* $U_i \subseteq V_{f(i)}$*.*

Definition 2.3.18 (Good Cover)**.** *An open cover* $\mathcal{U} = (U_i)_{i \in I}$ *of a topological space* X *is called* good *if for every finite non-empty intersection of sets* $U_{i_0}, \cdots, U_{i_p} \in \mathcal{U}$*,* $U_{i_0 \cdots i_p} \subseteq X$ *is contractible.*

It can be seen that the Čech cohomology can be computed with respect to a good cover, this is shown in [5] by the fact that Čech cohomology with respect to a good cover is isomorphic to sheaf cohomology. Furthermore good covers are cofinal in the category of covers of a manifold X, so the direct limit on $\mathcal{U}$ can be computed with respect to good covers [6].

Lemma 2.3.19. *[5, Lemma 3.1.10] Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of sheaves in* $\mathbf{AB}(X)$*, there exists a long exact sequence of groups*

$$0 \to \check{H}^0(X, A) \to \check{H}^0(X, B) \to \check{H}^0(X, C) \to \check{H}^1(X, A) \to \cdots$$

Proof. See [5, Lemma 3.1.10] for a proof of this fact. □

2.4 Singular Cohomology

For this section we reference both [6] and [7]. We need some ideas on singular cohomology to compare cohomology of simplicial objects with topological spaces. In order to look at singular cohomology we must first define the sets of simplices.

Definition 2.4.1 (n-simplex)**.** *We define the standard* n*-simplex by*

$$\Delta^n := \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1, \quad t_i \geq 0\}$$

equipped with the subspace topology from $\mathbb{R}^{n+1}$*.*

Definition 2.4.2 (Singular n-simplex and singular n-chain)**.** *We define a singular* n*-simplex to be a continuous map* $s : \Delta^n \to X$*. A singular* n*-chain in* X *is an element of the free abelian group on the set of singular* n*-simplices. This abelian group will be referred to as* $S_n(X)$*.*

Definition 2.4.3 (Face map on the standard n-simplex)**.** *The k-th face map of the standard n-simplex is the function*

$$\begin{aligned} d_n^k : \Delta_{n-1} &\to \Delta_n \\ (t_0, \cdots, t_{n-1}) &\mapsto (t_0, \cdots, t_{k-1}, 0, t_k, \cdots, t_{n-1}) \end{aligned}$$

We can use this to define a differential operator on the groups of singular n-chains.

Definition 2.4.4 (Differential operator on the group of n-chains)**.** *Using the definition of the face map from definition 2.4.3 we define a differential operator on $S_n(X)$ by*

$$\begin{aligned} \delta : S_n(X) &\to S_{n-1}(X) \\ s &\mapsto \sum_{k=0}^{n} (-1)^k s \circ d_n^k \end{aligned}$$

In particular we have defined the group of n-chains as the free $\mathbb{Z}$ module on singular n-simplices but we can do this for general abelian groups A.

Proposition 2.4.5. *The differential operator δ satisfies the property that $\delta^2 = 0$.*

Proof. This follows from the proof that the simplicial differential operator is nilpotent. $\square$

Definition 2.4.6 (Singular n-cochain)**.** *A singular n-cochain is a linear functional on the $\mathbb{Z}$-module $S_n(X)$ of singular n-chains. The group of singular n-cochains is denoted $S^n(X)$ which is defined by $S^n(X) = Hom(S_n(X), \mathbb{Z})$ where $S_n(X)$ is the group of singular n-chains as before. Furthermore we can define a coboundary operator d defined by the pullback of δ:*

$$\begin{aligned} d : S^{n-1} &\to S^n \\ \varphi &\mapsto \varphi \circ \delta. \end{aligned}$$

Proposition 2.4.7. *The differential operator $d : S^{n-1} \to S^n$ makes the graded complex $\oplus_{n \in \mathbb{N}} S^n$ a differential complex.*

Proof. As d is the pullback of δ and $\delta^2 = 0$ then $d^2 = 0$. $\square$

Definition 2.4.8 (Singular cohomology)**.** *The singular cohomology of X with coefficients in an abelian group A is the homology of the complex of n-cochains. Explicitly the cohomology groups:*

$$H^n(X; A) := \ker(d)/Im(d)$$

Notice that the construction of $\Delta^n \to \mathbf{Top}$ extends to define a functor $\Delta \xrightarrow{\Delta} \mathbf{Top}$. If $X \in \mathbf{Top}$ then we obtain a simplicial set $Sing(X)$ defined by $\mathbf{Top}(\Delta, X) : \Delta^{op} \to \mathbf{Set}$, piecewise we have that $Sing(X)_n := Top(\Delta^n, X)$, an element of this set is a singular n-simplex. We have that $S_n(X)$ is then defined by the free $\mathbb{Z}$-module $\mathbb{Z}Sing(X)_n$. Notice that this then defines a simplicial abelian group $\mathbb{Z}Sing(X)$, thus using the Dold-Kan correspondence we know that this forms a graded chain-complex.

Proposition 2.4.9. *If X is a triangularisable space with a good cover $\mathcal{U}$ then, the Čech cohomology $\check{H}^n(\mathcal{U}, \mathbb{Z})$ and singular cohomology $H^n(X, \mathbb{Z})$ are isomorphic.*

Proof. See [6, Theorem 15.8] for a proof of this fact. □

2.4.1 Double and Triple Chain Complexes

A significant tool in working with cohomology of simplicial objects will be double and triple cochain complexes. In particular where cohomology is calculated from a single cochain complex associated to a manifold, we will associate double cochain complexes to a simplicial manifold. We also consider double complexes over these spaces to exhibit isomorphisms between different types of cohomology, this leads to the notion of spectral sequences [5, 6] which we will not go into detail about here.

Definition 2.4.10 (A Double Chain Complex)**.** *A double cochain complex is a collection of abelian groups $A^{p,q}$ such that $p, q \in \mathbb{Z}$, equipped with differentials $d_P : A^{p,q} \to A^{p+1,q}$ and $d_Q : A^{p,q} \to A^{p,q+1}$ such that $d_P^2 = 0$ and $d_Q^2 = 0$. Furthermore we have the condition that $d_P d_Q + d_Q d_P = 0$.*

Alternatively we can use the condition that $d_P d_Q = d_Q d_P$, we will show that these both produce the same cohomology, however we will be using the first condition to make algebra easier later on.

Definition 2.4.11 (Total Complex)**.** *Given a double cochain complex $A^{\bullet,\bullet}$ we can form the total complex*

$$C^k := \bigoplus_{p+q=k} A^{p,q}$$

with differential $D := d_P + d_Q$.

Notice that

$$\begin{aligned} D^2 &= (d_P + d_Q)^2 \\ &= d_P^2 + d_P d_Q + d_Q d_P + d_Q^2 \\ &= 0 \end{aligned}$$

and so forms a cochain complex. We say that the cohomology of the double complex $A^{\bullet,\bullet}$ is the cohomology of the total complex $C^\bullet$.

Remark. *Notice that if we have* $d_P d_Q + d_Q d_P = 0$ *then if we set* $d'_Q = (-1)^p d_Q$ *we get that*

$$\begin{aligned} d_P d_Q + d_Q d_P &= 0 \\ d_P d_Q &= -d_Q d_P \\ d_P (-1)^p d_Q &= (-1)^{p+1} d_Q d_P \\ d_P d'_Q &= d'_Q d_P \end{aligned}$$

and so from the first definition of a double complex we can get the second by changing the sign. In the second case we define the differential $D' = d_P + (-1)^p d'_Q$ *which is precisely* $D = d_P + d_Q$*. So the cohomology of both of these total complexes is precisely equal, this will make defining and working with triple cochain complexes easier.*

Definition 2.4.12. *We say that a single cochain* $A^\bullet$ *complex is bounded below if there exists some* $n \in \mathbb{Z}$ *such that* $A^k = \{0\}$ *for* $k < n$*.*

Notice that a bounded below cochain complex can be renumerated to start at zero, similarly one can extend this idea to a first quadrant double complex where $A^{p,q} = 0$ if either $p < n$ or $q < m$ for some $m, n \in \mathbb{Z}$, again this can be renumerated such that $A^{p,q}$ is zero if either p pr q is less than zero.

Definition 2.4.13. *We say that a sequence of abelian groups* $A \xrightarrow{f} B \xrightarrow{g} C$ *is exact if* $\ker g = \operatorname{Im} f$*. Given a complex* $A^\bullet$ *with differential* d *we define exactness in the same way, in other words we require* $\ker(d_n : A_n \to A_{n+1}) = im(d_{n-1} : A_{n-1} \to A_n)$ *for all* $n \in \mathbb{Z}$*.*

Proposition 2.4.14. *If* $A^{p,q}$ *is a first quadrant double complex with differential* d_Q *exact, then the total cohomology of* $A^{\bullet,\bullet}$ *is isomorphic to the cohomology of the complex* $\ker(d_Q : A^{\bullet,0} \to A^{\bullet,1})$ *with differential induced by* d_P*.*

Proof. This proof is given in [6] in terms of spectral sequences of double complexes. □

Definition 2.4.15 (Triple Cochain Complex)**.** *A triple cochain complex is a collection of abelian groups* $A^{p,q,r}$ *indexed by* $\mathbb{Z}^3$ *with morphisms* $d_P : A^{p,q,r} \to A^{p+1,q,r}$*,* $d_Q : A^{p,q,r} \to A^{p,q+1,r}$*, and* $d_R : A^{p,q,r} \to A^{p,q,r+1}$ *which satisfy* $d_P^2 = d_Q^2 = d_R^2 = 0$*. Furthermore we must have that*

$$\begin{aligned} d_P d_Q + d_Q d_P &= 0 \\ d_P d_R + d_R d_P &= 0 \\ d_Q d_R + d_R d_Q &= 0. \end{aligned}$$

We define the total complex of $A^{\bullet,\bullet,\bullet}$ *by*

$$Tot(A)^k := \bigoplus_{p+q+r=k} A^{p,q,r}$$

with differential $D = d_P + d_Q + d_R$*.*

Notice that

$$\begin{aligned} D^2 &= (d_P + d_Q + d_R)^2 \\ &= d_P^2 + d_P d_Q + d_P d_R + d_Q^2 + d_Q d_P + d_Q d_R + d_R^2 + d_R d_P + d_R d_Q \\ &= 0 \end{aligned}$$

and so forms a cochain complex. We then define the cohomology of the triple complex $A^{\bullet,\bullet,\bullet}$ to be the cohomology of the total complex $C^\bullet$.

Proposition 2.4.16. *Given a triple complex $A^{\bullet,\bullet,\bullet}$ one can form a double complex in three different ways by summing up any two of the indices*

$$\begin{aligned} Tot_{\mathrm{I}}(A)^{i,r} &= \bigoplus_{p+q=i} A^{p,q,r} \\ Tot_{\mathrm{II}}(A)^{j,q} &= \bigoplus_{p+r=j} A^{p,q,r} \\ Tot_{\mathrm{III}}(A)^{p,k} &= \bigoplus_{q+r=k} A^{p,q,r} \end{aligned}$$

with corresponding differentials given by

$$\begin{aligned} d_I &:= d_P + d_Q : A_{\mathrm{I}}^{i,r} \to A_{\mathrm{I}}^{i+1,r} \\ d_J &:= d_P + d_R : A_{\mathrm{II}}^{j,q} \to A_{\mathrm{II}}^{j+1,q} \\ d_K &:= d_Q + d_R : A_{\mathrm{III}}^{p,k} \to A_{\mathrm{III}}^{p,k+1} \end{aligned}$$

Forming the total complex of any of these double complexes will give the total complex of a triple complex as in definition 2.4.15, in particular this can be neatly stated

$$Tot(Tot_{\mathrm{I}}(A)) = Tot(A).$$

Proof. We will prove this statement for the first complex $Tot_{\mathrm{I}}(A)^{\bullet,\bullet}$ as the argument is identical otherwise. We have differentials $d_I = d_P + d_Q$ and d_R. Notice that

$$\begin{aligned} d_I d_R + d_R d_I &= (d_P + d_Q) d_R + d_R (d_P + d_Q) \\ &= d_P d_R + d_Q d_R + d_R d_P + d_R d_Q \\ &= 0 \end{aligned}$$

and so $Tot_{\mathrm{I}}(A)$ forms a double complex. Thus taking the total complex of this double complex with differential $D = d_I + d_R = d_P + d_Q + d_R$ we can see that

$$\begin{aligned}\bigoplus_{i+r=n} Tot_{\mathrm{I}}(A)^{i,r} &= \bigoplus_{i+r=n}\bigoplus_{p+q=i} A^{p,q,r} \\ &= \bigoplus_{p+q+r=n} A^{p,q,r}\end{aligned}$$

with identical differential, thus forming the same complex as the total complex of the triple complex. □

Lemma 2.4.17. *Given a non-negative, graded triple complex $A^{\bullet,\bullet,\bullet}$ such that for fixed p, q the differential $d_R : A^{(p,q,r)} \to A^{(p,q,r+1)}$ is exact, we have that the cohomology of the total complex and the cohomology of* $\ker d_R$ *are isomorphic, in particular*

$$H^n(\ker d_R : Tot_{\mathrm{I}}(A)^{\bullet,0} \to Tot_{\mathrm{I}}(A)^{\bullet,1}) \simeq H^n(A^{\bullet,\bullet,\bullet})$$

where the cohomology on the right is the total cohomology and $Tot_{\mathrm{I}}(A)^{(\bullet,r)}$ is defined as in Proposition 2.4.16.

Proof. We understand that the total cohomology of $A^{\bullet,\bullet,\bullet}$ is equal to the total cohomology of $Tot_{\mathrm{I}}(A)^{\bullet,\bullet}$ through Proposition 2.4.16, we then want to use Proposition 2.4.14 to prove that $H^n(\ker d_R : Tot_{\mathrm{I}}(A)^{\bullet,0} \to Tot_{\mathrm{I}}(A)^{\bullet,1}) \simeq H^n(Tot_{\mathrm{I}}(A)^{\bullet,\bullet})$, so we need to show that the differential d_R is exact in $Tot_{\mathrm{I}}(A)^{\bullet,\bullet}$. Suppose that $(\alpha) \in Tot_{\mathrm{I}}(A)^{k,r}$ with $d_R(\alpha) = 0$. Thus (α) is a tuple $(\alpha) = (\alpha_{0,k,r}, \alpha_{1,k_1,r}, \cdots, \alpha_{k,0,r})$ with $d_R(\alpha_{p,q,r}) = 0$ for all p, q. Since d_R is exact on $A^{p,q,\bullet}$, then there exists $\beta_{p,q,r-1}$ such that $d_R(\beta_{p,q,r-1}) = \alpha_{p,q,r}$ for all p, q. Thus $(\beta) = (\beta_{0,k,r-1}, \beta_{1,k-1,r-1}, \cdots, \beta_{k,0,r-1}) \in Tot_{\mathrm{I}}(A)^{k,r-1}$ satisfies the condition that $d_R(\beta) = \alpha$. It follows that $d_R : Tot_{\mathrm{I}}(A)^{k,r} \to Tot_{\mathrm{I}}(A)^{k,r+1}$ is exact. Now that we know that d_R in $Tot_{\mathrm{I}}(A)^{\bullet,\bullet}$ is exact and we are done.

□

These results lay the groundwork to be able to prove results regarding cohomology on simplicial spaces, which we will come to in the next chapter. In particular we will be able to produce analogues of the fact that singular, Čech, and sheaf cohomology are all isomorphic under sufficient conditions.

Chapter 3

Simplicial Manifolds

3.1 Definitions

Simplicial manifolds will be the main object of study for the majority of this thesis. We wish to understand how to compute the cohomology of a simplicial manifold so that we can classify objects on simplicial manifolds. We look at [8] for some of the following definitions.

Definition 3.1.1 (Simplicial Manifold)**.** *A simplicial manifold is a simplicial object in the category of smooth manifolds and smooth maps,* **Man***.*

An explicit description of a simplicial manifold can be given using the fact that the simplex category Δ has generating morphisms given by d^i and s^i. So a contravariant functor $X_\bullet : \Delta^{op} \to \mathbf{Man}$ can be described by a sequence of manifolds $X_0, X_1, X_2, \cdots$ together with smooth face and degeneracy maps d_i and s_i satisfying the simplicial identities (see Proposition 2.1.6).

Example 3.1.2 (Constant simplicial manifold)**.** *Let X be a manifold, then the sequence of manifolds $X^{(\bullet)} = X, X, \cdots$ where each map between manifolds is the identity map forms a simplicial manifold.*

Example 3.1.3. *Let X be a manifold, the sequence of manifolds $X^{\bullet+1} = X, X^2, X^3, \ldots$ with face and degeneracy maps given by*

$$d_i(x_1, \cdots, x_n) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$
$$s_i(x_1, \cdots, x_n) = (x_1, \cdots, x_i, x_i, \cdots, x_n).$$

is a simplicial manifold.

Example 3.1.4. *Given a surjective submersion $f : Y \to X$ we can form the Čech nerve $\check{C}(f)$ with $\check{C}(f)_n = Y^{[n]}$, where $Y^{[n]}$ denotes the iterated fiber product, and with face and degeneracy maps defined by*

$$\begin{aligned} d_i(y_1, \cdots, y_n) &= (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n) \\ s_i(y_1, \cdots, y_n) &= (y_1, \cdots, y_i, y_i, \cdots, y_n) \end{aligned}$$

.

Example 3.1.5. *Let G be a Lie group. We define a simplicial manifold BG which plays an important role in algebraic topology as follows. We take the sequence of manifolds $BG_\bullet = \{1\}, G, G^2, \cdots$ with maps:*

$$\begin{aligned} d_i(g_0, \cdots g_n) &= \begin{cases} (g_1, \cdots, g_n) & i = 0 \\ (g_0, \cdots, g_{i-1}g_i, \cdots, g_n) & 1 \leq i \leq n-1 \\ (g_0, \cdots, g_{n-1}) & i = n \end{cases} \\ s_i(g_0, \cdots, g_n) &= (g_0, \cdots, g_{i-1}, 1, g_i, \cdots, g_n) \end{aligned}$$

Example 3.1.6. *Let G be a Lie group and M be a manifold with a smooth right action of G. We define a simplicial manifold $EG(M)_\bullet$ where $EG(M)_n = M \times G^n$ with face and degeneracy maps defined by*

$$\begin{aligned} d_i(m, g_0, \cdots g_n) &= \begin{cases} (mg_0, g_1, \cdots, g_n) & i = 0 \\ (m, g_0, \cdots, g_{i-1}g_i, \cdots, g_n) & 1 \leq i \leq n-1 \\ (m, g_0, \cdots, g_{n-1}) & i = n \end{cases} \\ s_i(m, g_0, \cdots, g_n) &= (m, g_0, \cdots, g_{i-1}, 1, g_i, \cdots, g_n) \end{aligned}$$

This object will be of particular interest in Chapter 7.

Definition 3.1.7 (Morphisms of simplicial manifolds)**.** *A morphism of a simplicial manifolds $\phi : Y_\bullet \to X_\bullet$ consists of a sequence of smooth maps $\phi_k : Y_k \to X_k$ that commute with the face and degeneracy maps, this is to say for any $\alpha : [m] \to [n]$ the diagram*

$$\begin{tikzcd} X_n \arrow[r, "X_\bullet(\alpha)"] & X_m \\ Y_n \arrow[u, "\phi_n"] \arrow[r, "Y_\bullet(\alpha)"] & Y_m \arrow[u, "\phi_m"'] \end{tikzcd}$$

commutes. In particular this is a natural transformation of functors $Y_\bullet$ and $X_\bullet$.

3.2 Realisation

Here we define some constructions of topological spaces given by simplicial manifolds. We reference [1] on notions of simplicial spaces.

Definition 3.2.1 (Fat Realisation)**.** *The fat realisation of a simplicial manifold $X_\bullet$ is given by the topological space*

$$\|X_\bullet\| = \coprod_{n\geq 0} \Delta^n \times X_n / \sim$$

where

$$(d^i(t), x) \sim (t, d_i(x))$$

for $x \in X_n$, $t \in \Delta^{n-1}$, $i = 0, \ldots, n$, $n = 1, 2, \ldots$.

Definition 3.2.2 (Geometric Realisation)**.** *The geometric realisation of a simplicial manifold $X_\bullet$ is defined in the same way as the fat realisation*

$$|X_\bullet| = \coprod_{n\geq 0} \Delta^n \times X_n / \sim$$

with the added condition that

$$(s^i(t), x) \sim (t, s_i(t))$$

where $x \in X_n$, $t \in \Delta^{n+1}$, $i = 0, \cdots, n$, $n = 0, 1, 2, \cdots$.

We have that both $\|-\|$ and $|-|$ are functors $[\Delta^{op}, \mathbf{Man}] \to \mathbf{Top}$, the geometric realisation preserves finite limits however the fat realisation may not [9]. Notice that there is a canonical morphism $\|X_\bullet\| \to |X_\bullet|$ for any simplicial manifold $X_\bullet$. This morphism is a homotopy equivalence; when $X_\bullet$ is a simplicial manifold each degeneracy map is a closed cofibration - in other words $X_\bullet$ is "good" in the sense of [10]. This condition on degeneracy maps is a sufficient condition to ensure that the canonical map $\|X_\bullet\| \to |X_\bullet|$ is a homotopy equivalence. If $X_\bullet$ is a simplicial manifold both the geometric realisation $|X_\bullet|$ and the fat realisation $\|X_\bullet\|$ will form a topological space with the homotopy type of a CW-complex when computed in the subcategory of compactly generated topological spaces, see [11].

3.3 Sheaves on Simplicial Manifolds

We will follow the format laid out by [5] in defining sheaf cohomology for simplicial manifolds. We need to define terms such as a sheaf on a simplicial manifold and then talk about injective resolutions of such sheaves as well as existence and uniqueness thereof. We start this journey with the definition of a sheaf on a simplicial manifold.

Definition 3.3.1. *A sheaf $A_\bullet$ on a simplicial manifold $X_\bullet$ is a sequence of sheaves $A_0, A_1, A_2, \ldots,$ on $X_0, X_1, X_2, \ldots,$ respectively with morphisms of sheaves for each $\alpha : [m] \to [n]$ in Δ*

$$A_\bullet(\alpha) : A_m \to (X_\bullet(\alpha))_*(A_n)$$

such that

$$A_\bullet(\alpha \circ \beta) = A_\bullet(\alpha) \circ A_\bullet(\beta).$$

Note. *As we will be talking about various open subsets U in reference to a simplicial manifold $X_\bullet$, for ease of notation we will write $U^{(m)}$ where the superscript m indicates that $U^{(m)} \subseteq X_m$ is an open subset of X_m.*

Example 3.3.2. *An important example is given by a Lie group A, we define the sheaf of smooth A valued functions on $X_\bullet$ denote by $\underline{A}_\bullet := \underline{A}_{X_\bullet}$ with morphisms generated by $\alpha : [m] \to [n]$ given by the pullback of a function*

$$\begin{aligned} A_\bullet(\alpha) : A_m(U^{(m)}) &\to (X_\bullet(\alpha)_*(A_n))(U^{(m)}) \\ (f : U^{(m)} \to A) &\mapsto (f \circ X_\bullet(\alpha) : X_\bullet(\alpha)^{-1}(U^{(m)}) \to A). \end{aligned}$$

Example 3.3.3. *Let X be a manifold. Consider an abelian sheaf $A_\bullet$ over the constant simplicial manifold $X^{(\bullet)}$. This simply amounts to a complex of sheaves $A_0, A_1, A_2, \cdots$ over X. Due to the simplicial identities we can construct a morphism of sheaves $\delta : A_n \to A_{n+1}$ such that $\delta^2 = 0$. Explicitly this morphism is*

$$\delta = \sum_{i=0}^{n+1} (-1)^i A_\bullet(d_i)$$

and $\delta^2 = 0$ due to the simplicial identities per usual.

3.4 Grothendieck Topologies

We will need a more general definition of a sheaf, in order to prove that injective resolutions exist for abelian sheaves on simplicial manifolds. We will briefly introduce Grothendieck topologies and sheaves on a category $\mathcal{C}$ and apply this to the context of simplicial manifolds.

Definition 3.4.1 (Grothendieck Topology [12])**.** *A Grothendieck Topology on a category $\mathcal{C}$ is a set T of families of maps in $\mathcal{C}$, $\{\phi_i : U_i \to U\}_{i \in I}$ called coverings such that*

1. *for any isomorphism ϕ in $\mathcal{C}$, $\{\phi\} \in T$;*

2. *if* $\{U_i \to U\} \in T$ *and* $\{V_{ij} \to U_i\} \in T$ *for each* i*, then* $\{V_{ij} \to U\} \in T$*;*

3. *if* $\{U_i \to U\} \in T$ *and* $V \to U$ *is a morphism, then the fibre products* $U_i \times_U V$ *exist and* $\{U_i \times_U V \to V\} \in T$*.*

Definition 3.4.2 (Grothendieck Pretopology [12])**.** *A Grothendieck pretopology on a category* $\mathcal{C}$ *with pullbacks is defined as, for each* $U \in \mathcal{C}$ *we have a set* $P(U)$ *of families of morphisms of the form* $\{U_i \xrightarrow{\alpha_i} U | i \in I\}$*, called covering families of the pretopology, such that*

1. *For any object* U*, the family* $U \xrightarrow{id} U$ *is in* $P(U)$*,*

2. *If* $V \to U$ *is a morphism of* $\mathcal{C}$ *and* $\{U_i \to U | i \in I\}$ *is in* $P(U)$ *then* $\{V \times_U U_i \xrightarrow{\pi_1} V | i \in I\}$ *is in* $P(V)$*,*

3. *If* $\{U_i \xrightarrow{\alpha_i} U | i \in I\} \in P(U)$ *and* $\{V_{i,j} \xrightarrow{\beta_{ij}} U_i | j \in J_i\} \in P(U_i)$ *for each* i*, then* $\{V_{ij} \xrightarrow{\alpha_i \beta_{ij}} U | i \in I, j \in J\} \in P(U)$*.*

Definition 3.4.3. *Following Friedlander [13], we define the site of a simplicial manifold* $X_\bullet$ *to be the category* $\mathscr{O}(X_\bullet)$ *with objects* $U^{(n)} \subseteq X_n$ *equipped with their inclusion mapping and morphisms to be commuting squares*

$$\begin{tikzcd} U^{(n)} \arrow[r, "X_\bullet(\alpha)|_U"] \arrow[d, hook] & V^{(m)} \arrow[d, hook] \\ X_n \arrow[r, "X_\bullet(\alpha)"] & X_m \end{tikzcd}$$

Proposition 3.4.4. *The category* $\mathscr{O}(X_\bullet)$ *admits a Grothendieck pretopology.*

Proof. For each object $U \hookrightarrow X_n$ we take all families of open covers of U by sets $U_i \hookrightarrow X_n$. In particular we immediately have the first condition $U \overset{id}{\hookrightarrow} U$ is contained in the set of covering families as U covers itself.

Now let $V \xrightarrow{X_\bullet(\alpha)|_V} U$ be a morphism in $\mathcal{O}(X_\bullet)$ and let $\{U_j\}_{j \in J}$ be a covering family of U. We need to show that

$$\{V \times_U U_j \xrightarrow{\pi_1} V | j \in J\}$$

is a covering family of V. Firstly we must unravel the definition of $V \times_U U_j$, this object is the pullback of the diagram

$$\begin{tikzcd} & V \arrow[d, "X_\bullet(\alpha)|_U"] \\ U_j \arrow[r, hook] & U \end{tikzcd}$$

which is $V \cap X_\bullet(\alpha)^{-1}(U_j)$, where the pullback maps the are inclusion, $V \cap X_\bullet(\alpha)^{-1}(U_j) \hookrightarrow V$ and $X_\bullet(\alpha) : V \cap X_\bullet(\alpha)^{-1}(U_j) \to U_j$. So the family

$$\{V \times_U U_j \xrightarrow{\pi_1} V | j \in J\} = \{V \cap X_\bullet(\alpha)^{-1}(U_j) | j \in J\}$$

is a covering of V because $V \subseteq X_\bullet(\alpha)^{-1}(U)$ and $\{U_j\}_{j \in J}$ covers U. So our selection of covering families contains pullbacks. The final condition required to admit a pretopology follows immediately from the fact that $\{U_i \cap V_{ij}\}_{i \in I, j \in J_i}$ is a cover for U and thus we are done. □

Proposition 3.4.4 means that we can define the category of sheaves over $X_\bullet$ given this Grothendieck topology. We now wish to compare our two definitions of sheaves for a simplicial manifold.

Proposition 3.4.5. *There is a categorical isomorphism between* $\mathbf{Sh}(\mathscr{O}(X_\bullet))$ *and* $\mathbf{Sh}(X_\bullet)$*.*

Proof. We will construct functors $F : \mathbf{Sh}(\mathcal{O}(X_\bullet)) \to \mathbf{Sh}(X_\bullet)$ and $G : \mathbf{Sh}(X_\bullet) \to \mathbf{Sh}(\mathcal{O}(X_\bullet))$ and then show that F and G are inverse functors of each other. Let A be a sheaf on $\mathcal{O}(X_\bullet)$. Define $(F(A))_n := F(A|_{X_n})$, where $A|_{X_n}$ is the sheaf A restricted to the subcategory $\mathcal{O}(X_n)$ of $\mathcal{O}(X_\bullet)$. Another way we can view this is on an open set $U \subseteq X_n$ we have that $F(A)_n(U) := A(U \hookrightarrow X_n)$. Given a morphism $\alpha : [m] \to [n]$ we define a morphism of sheaves

$$F(A)_m \to X_\bullet(\alpha)_* F(A)_n.$$

If $U \subseteq X_M$ is an open set then

$$F(A)_m(U) \to X_\bullet(\alpha)_* F(A)_n(U) = F(A)_n(X_\bullet(\alpha)^{-1}(U))$$

is defined by the morphism

$$A(\alpha) : A(U \hookrightarrow X_m) \to A(X_\bullet(\alpha)^{-1}(U) \hookrightarrow X_n).$$

We need to show that these maps $F(A)_\bullet(\alpha)$ satisfy the functorial properties of a sheaf, in particular, compatibility with composition. Let $\alpha : [m] \to [n]$, $\beta : [k] \to [m]$. We have that

$$\begin{aligned}(X_\bullet(\beta)_*(F(A)_\bullet)(\alpha))\,(U) \circ F(A)_\bullet(\beta)(U) &= (A(\alpha))(X_\bullet(\beta)^{-1}(U)) \circ (A(\beta))(U) \\ &= A(\alpha) \circ A(\beta)(U) \\ &= A(\alpha \circ \beta)(U) \\ &= F(A)_\bullet(\alpha \circ \beta)(U)\end{aligned}$$

due to the fact that A is a functor $\mathcal{O}(X_\bullet)^{op} \to \mathbf{Set}$. So we have that the maps $F(A)_\bullet(\alpha)$ satisfy the required compatibility condition with face and degeneracy morphisms and furthermore is compatible with the restriction morphism. Let $\varphi : A \to B$ be a morphism of sheaves in $\mathbf{Sh}(\mathcal{O}(X_\bullet))$. We define $F(\varphi)_\bullet : F(A)_\bullet \to F(B)_\bullet$ on objects $U \subseteq X_n$ by

$$\begin{aligned} &F(\varphi)_n(U) : F(A)_n(U) \to F(B)_n(U) \\ = &\varphi(U \hookrightarrow X_n) : A(U \hookrightarrow X_n) \to B(U \hookrightarrow X_n) \end{aligned}$$

we can see that $F(f)$ commutes with the simplicial maps $F(A)_\bullet(\alpha)$ and $F(B)_\bullet(\alpha)$ by the fact that it does so in $\mathcal{O}(X_\bullet)$ as that is how these maps are defined, so $F(f)$ defines a morphism in $\mathbf{Sh}(X_\bullet)$. Suppose $\varphi : A \to B$ and $\psi : B \to C$. We have that

$$\begin{aligned} F(\psi)_n(U) \circ F(\varphi)_n(U) &= \psi(U \hookrightarrow X_n) \circ \varphi(U \hookrightarrow X_n) \\ &= \psi \circ \varphi(U \hookrightarrow X_n) \\ &= F(\psi \circ \varphi)_n(U) \end{aligned}$$

and so F defines a functor.

Now we will define the functor $G : \mathbf{Sh}(X_\bullet) \to \mathbf{Sh}(\mathcal{O}(X_\bullet))$. Let $A \in \mathbf{Sh}(X_\bullet)$. Define G on objects A by

$$G(A_\bullet)(U \hookrightarrow X_n) = A_n(U).$$

Given a morphism $U \xrightarrow{X_\bullet(\alpha)} V$ we define the morphism

$$G(A_\bullet)(U \xrightarrow{X_\bullet(\alpha)} V) := A_m(V) \xrightarrow{A_\bullet(\alpha)} A_n(X_\bullet(\alpha)^{-1}(V)) \xrightarrow{\rho_{X_\bullet(\alpha)^{-1}(V),U}} A_n(U)$$

where ρ represents the respective restriction mapping. Notice that $U \subseteq X_\bullet(\alpha)^{-1}(V)$ and this allows us to define this morphism. Now suppose that we have a composable pair of morphisms $U \xrightarrow{X_\bullet(\alpha)} V \xrightarrow{X_\bullet(\beta)} W$, we wish to show that G respects composition. As $A_\bullet(\alpha)$ is a morphism of sheaves then the following diagram commutes

$$\begin{array}{ccc} A_m(X_\bullet(\beta)^{-1})(W) & \xrightarrow{A_\bullet(\alpha)} & X_\bullet(\alpha)_*(A_m)(X_\bullet(\beta)^{-1}(W)) \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ A_m(V) & \xrightarrow{\quad A_\bullet(\alpha) \quad} & X_\bullet(\alpha)_*(A_m)(V), \end{array}$$

where ρ is the respective restriction mapping. Therefore we have that

$$\begin{aligned} &G(A_\bullet)(X_\bullet(\alpha) : U \to V) \circ G(A_\bullet)(X_\bullet(\beta) : V \to W) \\ &= \rho_{X_\bullet(\alpha)^{-1}(V),U} \circ A_\bullet(\alpha) \circ \rho_{X_\bullet(\beta)^{-1}(W),V} \circ A_\bullet(\beta) \\ &= \rho_{X_\bullet(\alpha)^{-1}(V),U} \circ \rho_{X_\bullet(\alpha)^{-1}X_\bullet(\beta)^{-1}(W),X_\bullet(\alpha)^{-1}(V)} \circ A_\bullet(\alpha) \circ A_\bullet(\beta) \\ &= \rho_{X_\bullet(\alpha\circ\beta)^{-1}(W),U} \circ A_\bullet(\alpha) \circ A_\bullet(\beta) \\ &= \rho_{X_\bullet(\alpha\circ\beta)^{-1}(W),U} \circ A_\bullet(\alpha \circ \beta) \\ &= G(A_\bullet)(X_\bullet(\alpha) \circ X_\bullet(\beta)). \end{aligned}$$

Finally we will show that $F \circ G$ and $G \circ F$ are the identity. Firstly consider $F \circ G$. On an object $A_\bullet$ we have that

$$\begin{aligned}(F \circ G)(A_\bullet)(U) &= G(A_\bullet)(U \hookrightarrow X_n)\\ &= A_n(U)\end{aligned}$$

and on morphisms $A_\bullet(\alpha)$ we have that

$$\begin{aligned}(F \circ G)(A_\bullet)(\alpha) &= A_m(U) \xrightarrow{G(A_\bullet)(\alpha)} X_\bullet(\alpha)_*(A_n)(U)\\ &= A_m(U) \xrightarrow{G(A_\bullet)(\alpha)} (A_n)(X_\bullet(\alpha)^{-1}(U))\\ &= G(A_\bullet)(U) \xrightarrow{G(A_\bullet)(\alpha)} G(A_\bullet)(X_\bullet(\alpha)^{-1}(U))\\ &= G(A_\bullet)(U) \xrightarrow{G(A_\bullet)(\alpha)} X_\bullet(\alpha)_* G(A_\bullet)(U)\\ &= A_m(U) \xrightarrow{A_\bullet(\alpha)} X_\bullet(\alpha)_*(A_n)(U)\end{aligned}$$

and so $F \circ G$ is the identity functor.

Now consider $G \circ F$. Given an open set $U \subseteq X_n$, on an object $A_\bullet$ we have

$$\begin{aligned}(G \circ F)(A)(U \hookrightarrow X_n) &= F(A)_n(U)\\ &= A(U \hookrightarrow X_n)\end{aligned}$$

and on morphisms $U \xrightarrow{X_\bullet(\alpha)} V$

$$\begin{aligned}(G \circ F)(A)(U \to V) &= \rho_{X_\bullet(\alpha)^{-1}(V),U} \circ F(A)(\alpha)\\ &= A(\rho) \circ A(X_\bullet(\alpha)^{-1}(V) \to V)\\ &= A(U \to V).\end{aligned}$$

and so F and G define an isomorphism of categories. □

3.5 Simplicial-Čech Cohomology

Now we will define simplicial-Čech cohomology using simplicial sheaves and Čech cohomology as in [5].

Definition 3.5.1 (Simplicial Cover). *We define a simplicial cover $\mathcal{U}^{(\bullet)}$ of a simplicial manifold $X_\bullet$ to be a sequence of open covers $\mathcal{U}^{(n)}$ for X_n with indexing set I_n. We also require that for each $n \in \mathbb{N}$ there exist a collection of maps $d_k : I_n \to I_{n-1}$ for $k = 0, \cdots, n$, satisfying the simplicial identities for face maps $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$, and satisfying the following condition on open sets*

$$U_i^{(n)} \subseteq d_k^{-1}(U_{d_k(i)}^{(n-1)}).$$

The condition above that is required of a simplicial cover ensures that we are able to pull back a map from $U^{(n-1)}_{d_k(i)}$ onto $U^{(n)}_i$, furthermore we can pull back sheaves in a similar fashion. We will see that this is necessary to be able to define simplicial-Čech cohomology.

Definition 3.5.2. *Given a simplicial manifold $X_\bullet$ with simplicial cover $\mathcal{U}^{(\bullet)}$ and an abelian sheaf A on $X_\bullet$, define the group*

$$\check{C}^p(\mathcal{U}^{(q)}, A) := \check{C}^p(\mathcal{U}^{(q)}, r_q(A))$$

where r_q is the restriction functor $\mathbf{Sh}(\mathcal{O}(X_\bullet)) \to \mathbf{Sh}(\mathcal{O}(X_q))$. The group on the right hand side is just the group of Čech cochains for X_q with respect to the cover $\mathcal{U}^{(q)}$ with values in $r_q(A)$.

Given a morphism $f : Y \to X$, an abelian sheaf $\underline{A}$ on X, and an open cover $\mathcal{U}$ of X indexed by I, we have the pullback morphism

$$f^* : \check{C}^p(\mathcal{U}, \underline{A}) \to \check{C}^p(f^{-1}(\mathcal{U}), f^{-1}(\underline{A})).$$

Given an $\alpha_{i_0\cdots i_p} \in \underline{A}(U_{i_0\cdots i_p})$ we have that $f^*(\alpha)_{i_0\cdots i_p} \in f^{-1}\underline{A}(f^{-1}(U_{i_0\cdots i_p}))$, the group $f^{-1}\underline{A}(f^{-1}(U_{i_0\cdots i_p}))$ is naturally isomorphic to that of $A(U_{i_0\cdots i_p})$ and so we have $f^*(\alpha)_{i_0\cdots i_p} := \alpha_{i_0\cdots i_p}$. Similarly, given a morphism of simplicial spaces $f_\bullet : Y_\bullet \to X_\bullet$ one can form the pull back morphism

$$f_\bullet^* : \check{C}^p(\mathcal{U}^\bullet, \underline{A}) \to \check{C}^p(f_\bullet^{-1}(\mathcal{U}^\bullet), f_\bullet^{-1}(\underline{A}))$$

which is defined piecewise in terms of each map f_n.

Lemma 3.5.3. *Given a simplicial sheaf A and a simplicial cover $\mathcal{U}^{(\bullet)}$ on $X_\bullet$ indexed by the simplicial set $I_\bullet$, the groups $\check{C}^\bullet(\mathcal{U}^{(\bullet)}, A)$ form a double complex.*

Proof. Let $\underline{\alpha} \in \check{C}^p(\mathcal{U}^{(n-1)}, r_{n-1}(A))$ and let $i_0, \ldots, i_p \in I^{n-1}$. We will define the differential $\delta : \check{C}^p(\mathcal{U}^{(n-1)}, r_{n-1}(A)) \to \check{C}^p(\mathcal{U}^{(n)}, r_n(A))$ by the map

$$\delta(\underline{\alpha})_{i_0\cdots i_p} := \sum_{k=0}^{n} (-1)^k d_k^*(\underline{\alpha})_{i_0\cdots i_p}$$

Firstly we will define the morphisms $d_k^* : \check{C}^p(\mathcal{U}^{(n-1)}, r_{n-1}(A)) \to \check{C}^p(\mathcal{U}^{(n)}, r_n(A))$. Firstly we have the pullback morphism

$$d_k^* : \check{C}^p(\mathcal{U}^{(n-1)}, r_{n-1}(A)) \to \check{C}^p(d_k^{-1}(\mathcal{U}^{(n-1)}), d_k^{-1}(r_{n-1}(A)))$$

we then compose this morphism with the function

$$\check{C}^p(d_k^{-1}(\mathcal{U}^{(n-1)}), d_k^{-1} r_{n-1}(A)) \xrightarrow{f} \check{C}^p(\mathcal{U}^{(n)}, d_k^{-1} r_{n-1}(A))$$

which is defined by

$$f(\underline{\alpha})_{i_0\cdots i_p} = \alpha_{d_k(i_0)\cdots d_k(i_p)}.$$

Finally we compose these functions with the morphism of sheaves $A(d^k) : d_k^{-1}(r_{n-1}(A)) \to r_n(A)$. Thus we define the map d_k^* on co-chains by

$$\begin{aligned} d_k^* : \check{C}^p(\mathcal{U}^{(n-1)}, r_{n-1}(A)) &\to \check{C}^p(d_k^{-1}(\mathcal{U}^{(n-1)}), d_k^{-1}(r_{n-1}(A))) \\ d_k^*(\alpha)_{i_0\cdots i_p} &= A(d^k)(\alpha)_{d_k(i_0)\cdots d_k(i_p)}. \end{aligned}$$

We will show that the differential δ forms a double complex as in Figure 3.1.

Figure 3.1: Simplicial Čech complex

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r, "\delta"] & C^{p+1}(\mathcal{U}^{(q)}, r_q(A)) \arrow[r, "\delta"] \arrow[u, "d"] & C^{p+1}(\mathcal{U}^{(q+1)}, r_{q+1}(A)) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
\cdots \arrow[r, "\delta"] & C^{p}(\mathcal{U}^{(q)}, r_q(A)) \arrow[r, "\delta"] \arrow[u, "d"] & C^{p}(\mathcal{U}^{(q+1)}, r_{q+1}(A)) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
 & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] &
\end{tikzcd}$$

In order to show that this forms a double complex one must show that $\delta d = d\delta$, $\delta^2 = 0$, and $d^2 = 0$. We already have that $d^2 = 0$ from proposition 2.3.13. To see that $\delta^2 = 0$, we will show that $d_k^* \circ d_j^* = d_j^* \circ d_{k-1}^*$ for $j < k$,

$$\begin{aligned} d_k^*\left(d_j^*(\alpha)\right)_{i_0\cdots i_p} &= A(d^k)\big(d_j^*(\alpha)_{d_k(i_0)\cdots d_k(i_p)}\big) \\ &= A(d^k)A(d^j)\big(\alpha_{d_j d_k(i_0)\cdots d_j d_k(i_p)}\big) \\ &= A(d^k d^j)\big(\alpha_{d_j d_k(i_0)\cdots d_j d_k(i_p)}\big) \\ &= A(d^k d^{k-1})\big(\alpha_{d_{k-1} d_j(i_0)\cdots d_{k-1} d_j(i_p)}\big) \\ &= A(d^k)A(d^{k-1})\big(\alpha_{d_{k-1} d_j(i_0)\cdots d_{k-1} d_j(i_p)}\big) \\ &= d_j^*\left(d_{k-1}^*(\alpha)\right)_{i_0\cdots i_p} \end{aligned}$$

Thus forming a simplicial complex and so $\delta^2 = 0$. Now we can prove commutation of differentials, this boils down to showing that the index omission maps p_k which make up the Čech differential and the face pullback maps d_j^* commute. We have

$$\begin{aligned} d_j^* \circ p_k(\alpha)_{i_0 \cdots i_{p+1}} &= d_j^*(\alpha)_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}} \\ &= A(d^j)(\alpha_{d_j(i_0) \cdots \widehat{i_k} \cdots d_j(i_{p+1})}) \\ &= A(d^j)(\alpha_{d_j(i_0) \cdots \widehat{d_j(i_k)} \cdots d_j(i_{p+1})}) \\ &= p_k\left(A(d^j)(\alpha_{d_j(i_0) \cdots d_j(i_{p+1})})\right) \\ &= p_k \circ d_k^*(\alpha)_{i_0 \cdots i_{p+1}} \end{aligned}$$

Thus the complex described in figure 3.1 is a double complex. □

Definition 3.5.4 (Simplicial-Čech complex). *We define the simplicial-Čech complex to be the double chain complex $\check{C}^\bullet(\mathcal{U}^{(\bullet)}, A)$. We then define the simplicial-Čech cohomology $\check{H}^*(\mathcal{U}^{(\bullet)}, A)$ of $X_\bullet$ with respect to $\mathcal{U}^{(\bullet)}$ to be the total cohomology of the double complex.*

3.6 Good Simplicial Covers

Similar to that of Čech cohomology for manifolds we wish to show that having a good cover allows us to compute the Čech cohomology for a simplicial manifold. In particular we will have to relate this to simplicial-sheaf cohomology. Firstly we will look at the properties of good simplicial covers and decide if they exist. Many of the following propositions rely on the fact that for a manifold good covers always exist and are cofinal in the category of open covers over a manifold [6].

Definition 3.6.1 (Good Simplicial Cover). *A good simplicial cover of a simplicial manifold $X_\bullet$ is a simplicial cover in which every covering $\mathcal{U}^{(n)}$ of X_n is good.*

Lemma 3.6.2. *A simplicial cover always exists for any simplicial manifold $X_\bullet$.*

Proof. One can always take the constant open cover $\mathcal{U}^{(n)} = \{X_n\}$ where the face maps on indexing sets is the constant map. Notice that the condition $X_n \subseteq d_k^{-1}(X_{n-1})$ is always satisfied and so this is a simplicial cover, although an uninteresting one. □

Definition 3.6.3 (Simplicial Cover Refinement). *We say that a simplicial cover $\mathcal{V}^{(\bullet)}$ indexed by $J_\bullet$ is a refinement of $\mathcal{U}^{(\bullet)}$ indexed by $I_\bullet$ if there exists a semi-simplicial map $f_\bullet : J_\bullet \to I_\bullet$ such that if $f_n(j) = i$ then $V_j^{(n)} \subseteq U_i^{(n)}$.*

Lemma 3.6.4. *Given an open cover $\mathcal{U}$ of a manifold X indexed by I, one can find a good refinement $\mathcal{V}$ of X indexed by J such that the refinement map $f : J \to I$ is surjective, this is stated in [6], we will call such a refinement a surjective refinement.*

Proof. Firstly we endow X with a Riemannian metric in order to generate geodesically convex neighborhoods. Now for each $U_i \in \mathcal{U}$ take a good open cover of U_i made up of geodesically convex neighborhoods, see [6]. Call this cover $\mathcal{V}_i$ indexed by J_i. Define a new cover $\mathcal{V} := \cup_{i\in I}\mathcal{V}_i$ indexed by $J = \cup_{i\in I} J_i$. Define a map $f : J \to I$ given by $J_i \ni j \mapsto i$. Thus as we created a $\mathcal{V}_i$ which is non-empty for every i this map is surjective. Furthermore as $V_j \in \mathcal{V}_i$ is a cover for U_i then $V_j \subseteq U_{f(j)}$, so this is a surjective refinement of open covers. Finally as each set V_j is geodesically convex with respect to the Riemannian metric we picked then so are intersections of these sets. So this cover is a good cover. □

Lemma 3.6.5. *Given a simplicial open cover $\mathcal{U}^{(\bullet)}$ of $X_\bullet$ and a surjective refinement $\mathcal{V}^{(m)}$ of $\mathcal{U}^{(m)}$ for some $m \in \mathbb{N}$, there exists a simplicial refinement $\mathcal{V}^{(\bullet)}$ of $\mathcal{U}^{(\bullet)}$ such that $\mathcal{V}^{(n)} = \mathcal{U}^{(n)}$ for $n < m$, $\mathcal{V}^{(n)} = \mathcal{V}^{(m)}$ for $n = m$.*

Proof. We will prove that such a refinement exists by induction on $n > m$. First we will prove the base case $n = m + 1$. Let J_m be the indexing set of $\mathcal{V}^{(m)}$ and $I_\bullet$ be the semi-simplicial indexing set of $\mathcal{U}^{(\bullet)}$. Let $f^{(m)} : J_m \to I_m$ be the morphism of indexing sets describing the surjective refinement of $\mathcal{U}^{(m)}$ to $\mathcal{V}^{(m)}$. Define the collection of open subsets

$$\mathcal{V}^{(m+1)} := \left\{ \bigcap_{k=0}^{m+1} d_k^{-1}(V^{(m)})_{j_k} \cap U_i^{(m+1)} : f(j_k) = d_k(i), j_k \in J_m, i \in I_{m+1} \right\}$$

of X_{m+1}. This collection of subsets is indexed by $J_{m+1} \subseteq J_m^{m+2} \times I_m$ where the condition $d_k(i) = f(j_k)$ for all $k = 0, \cdots, m+1$ is satisfied. We need to prove (i) there exists face maps between indexing sets $J_{m+1} \to J_m$, (ii) this cover is a refinement of $\mathcal{U}^{(m+1)}$, and (iii) $\mathcal{V}^{(m+1)}$ is in-fact a cover of X_{m+1}.

Notice that each index for $\mathcal{V}^{(m+1)}$ contains an element of I_{m+1}, so we define the map $f^{(m+1)} : J_{m+1} \to I_{m+1}$ by the projection map onto I_{m+1}. It is clear that the map $f^{(m+1)}$ defines a refinement as $V^{(m+1)}_{j_0\cdots j_{m+1} i} \subseteq U_i^{(m+1)}$ by definition. Define the face map $d_l : J_{m+1} \to J_m$ for $0 \le l \le m+1$ by $d_l : (j_0, \cdots, j_{m+1}, i) \mapsto j_l$. We will show that if $0 \le k \le l \le m+1$ then $d_k \circ d_l = d_{l-1} \circ d_k$

$$I_{m-1} \quad \begin{matrix} \longleftarrow \\ \vdots \\ \longleftarrow \end{matrix} \quad J_m \quad \begin{matrix} \longleftarrow \\ \vdots \\ \longleftarrow \end{matrix} \quad J_{m+1}$$

where $d_l : J_m \to I_{m-1}$ is defined by $d_l \circ f^{(m)}$. Let $k < l$ we have that

$$\begin{aligned} d_k \circ d_l(j_0, \cdots, j_{m+1}, i) &= d_k(j_l) \\ &= d_k \circ f^{(m)}(j_l) \\ &= d_k(d_l(i)) \\ &= d_{l-1}(d_k(i)) \\ &= d_{l-1} \circ f^{(m)}(j_k) \\ &= d_{l-1} \circ d_k(j_0, \cdots, j_{m+1}, i) \end{aligned}$$

thus satisfying the simplicial identities. Furthermore we have that

$$V^{(m+1)}_{j_0\cdots j_{m+1}i} \subseteq d_k^{-1}\left(V^{(m)}_{j_k}\right) = d_k^{-1}\left(V^{(m)}_{d_k(j_0,\cdots,j_{m+1},i)}\right)$$

by definition of the set $V^{(m+1)}_{j_0\cdots j_{m+1}i}$ and so $\mathcal{V}^{(m+1)}$ satisfies the conditions for being a simplicial cover.

Finally, we must show that it is actually a cover of X_{m+1}. Let $x \in X_{m+1}$. As $\mathcal{U}^{(m+1)}$ is a cover of X_{m+1} then $x \in U_i^{(m+1)}$ for some $i \in I_{m+1}$. Furthermore as $\mathcal{U}^{(\bullet)}$ is a simplicial cover then $d_k(x) \in U^{(m)}_{d_k(i)}$. Assuming that $\mathcal{V}^{(m)}$ is produced from $\mathcal{U}^{(m)}$ as in Lemma 3.6.4 then there exists $j_0 \cdots j_{m+1} \in J_m$ such that $d_k(x) \in V_{j_k}$ and $f^{(m)}(j_k) = d_k(i)$. Thus we have that $x \in d_k^{-1}(V_{j_k})$ and $x \in U_i^{(m+1)}$ and so $x \in V^{(m+1)}_{j_0\cdots j_k i}$ and therefore $\mathcal{V}^{(m+1)}$ covers X_{m+1}. Furthermore, notice that this is a surjective refinement in the sense that $f^{(m+1)}$ is surjective and given any $x \in U_i^{(m+1)}$ there exists $V_j^{(m+1)}$ such that $f(j) = i$ for $j \in J_{m+1}$. This completes the proof for the base case $n = m + 1$.

Now we can assume that $\mathcal{V}^{(m)}, \cdots, \mathcal{V}^{(n)}$ indexed by $J_m, \cdots, J_n$ satisfy the appropriate conditions where $n > m$. We will prove that the open cover $\mathcal{V}^{(n+1)}$ defined by

$$\mathcal{V}^{(n+1)} := \left\{ \bigcap_{k_{n+1}\cdots k_{m+1}} d^{-1}_{k_{n+1}} \cdots d^{-1}_{k_{m+1}}(V_{j_{k_{n+1}\cdots k_{m+1}}}) \cap U_i^{(n+1)} \; : \; \begin{array}{l} f^{(m)}(j_{k_{n+1}\cdots k_{m+1}}) = d_{m+1}\cdots d_{n+1}(i) \\ j_{l_1 l_2 k_{n-1}\cdots k_{m+1}} = j_{(l_2-1)l_1 k_{n-1}\cdots k_{m+1}} \\ \text{if } l_1 < l_2 \end{array} \right\}$$

makes $\mathcal{U}^{(0)}, \cdots, \mathcal{V}^{(m)}, \cdots, \mathcal{V}^{(n+1)}$ a (truncated) simplicial refinement of $\mathcal{U}^{(0)}, \cdots, \mathcal{U}^{(n+1)}$. The indexing set is given inductively by $J_{n+1} \subseteq J_n^{n+2} \times I_{n+2}$ with the compatibility condition described above between indices. Let $(\mathbf{j}, i) \in J_{n+1}$ where $\mathbf{j} = (j_{k_{n+1}\cdots k_{m+1}})_{k_{n+1}\cdots k_{m+1}}$, we define $d_l : J_{n+1} \to J_n$ by fixing the $n+1$-th index in $\mathbf{j}$ to l. In particular we send each $j_{k_{n+1}k_n\cdots k_{m+1}}$ to $j_{lk_n\cdots k_{m+1}}$ and $i \mapsto d_l(i)$. Firstly we will show that this is a cover of X_{n+1}. Let $x \in U_i^{(n+1)} \subseteq X_{n+1}$ for some $i \in I_{n+1}$. Then choose $j_{k_{n+1}\cdots k_{m+1}} \in J_m$ such hat

$$d_{k_{m+1}} \cdots d_{k_{n+1}}(x) \in V^{(m)}_{j_{k_{n+1}\cdots k_{m+1}}} \subseteq X_m.$$

This is possible as $\mathcal{V}^{(m)}$ is a good cover for X_m. We also choose our indices such that $j_{l_1 l_2 k_{n-1}\cdots k_{m+1}} = j_{(l_2-1)l_1 k_{n-1}\cdots k_{m+1}}$ if $l_1 < l_2$, this is again possible as $d_{l_1} \circ d_{l_2} = d_{l_2-1} \circ d_{l_1}$ if $l_1 < l_2$. From these conditions, we immediately have that

$$x \in \bigcap_{k_{n+1}\cdots k_{m+1}} d^{-1}_{k_{n+1}} \cdots d^{-1}_{k_{m+1}}(V_{j_{k_{n+1}\cdots k_{m+1}}}) \cap U_i^{(n+1)}$$

and thus $\mathcal{V}^{(n+1)}$ is a cover for X_{n+1} with refinement map $f^{(n+1)}(\mathbf{j}, i) = i \in I_{n+1}$. This refinement map is surjective in the sense of Lemma 3.6.4, we have that

$$\begin{aligned} d_l \circ f^{(n+1)}(j,i) &= d_l(i) \\ &= f^{(n)}(d_l(i), d_l(j)) \\ &= f^{(n)} \circ d_l(j,i) \end{aligned}$$

and so $f^{(0)} \cdots f^{(n+1)}$ form a truncated simplicial map. Thus we have a truncated simplicial refinement. Finally we must verify that this cover satisfies the appropriate simplicial identities. Notice that due to the fact that $j_{l_1 l_2 k_{n-1} \cdots k_{m+1}} = j_{(l_2-1) l_1 k_{n-1} \cdots k_{m+1}}$ if $l_1 < l_2$ we automatically have that J_{n+1} satisfies the simplicial identities with J_n and J_{n-1}. Finally notice that

$$\begin{aligned} d_l\left(V_{j,i}^{(n+1)}\right) &\subseteq d_l\left(\bigcap_{k_n \cdots k_{m+1}} d_l^{-1} d_n^{-1} \cdots d_{m+1}^{-1}\left(V^{(m)}_{j_l j_n \cdots j_{m+1}}\right) \cap U_i^{(n+1)}\right) \\ &\subseteq \bigcap_{k_n \cdots k_{m+1}} d_n^{-1} \cdots d_{m+1}^{-1}\left(V^{(m)}_{j_l j_n \cdots j_{m+1}}\right) \cap d_l(U_i^{(n+1)}) \\ &\subseteq \bigcap_{k_n \cdots k_{m+1}} d_n^{-1} \cdots d_{m+1}^{-1}\left(V^{(m)}_{j_l j_n \cdots j_{m+1}}\right) \cap (U^{(n)}_{d_l(i)}) \\ &= V^{(n)}_{d_l(j,i)} \end{aligned}$$

and thus forms a simplicial refinement as described. □

Proposition 3.6.6. *Every simplicial cover $\mathcal{U}^{(\bullet)}$ of $X_\bullet$ refines to a good simplicial cover $\mathcal{V}^{(\bullet)}$.*

Proof. This proof is given by iterating Lemma 3.6.5, firstly we refine the cover $\mathcal{U}^{(0)}$ to a good cover $\mathcal{V}_0^{(0)}$ and then perform the steps given in Lemma 3.6.5 to create a refinement $\mathcal{V}_0^{(\bullet)} \to \mathcal{U}^{(\bullet)}$ in which the cover $\mathcal{V}_0^{(0)}$ is good. We then repeat this process with a good refinement $\mathcal{V}_1^{(1)}$ of $\mathcal{V}_0^{(1)}$ giving a refinement $\mathcal{V}_1^{(\bullet)} \to \mathcal{V}_0^{(\bullet)}$. Repeating this process inductively gives us a good simplicial cover $\mathcal{V}^{(\bullet)}$ of $X_\bullet$. □

Theorem 3.6.7. *Good simplicial covers are cofinal in the set of simplicial covers for a simplicial manifold $X_\bullet$.*

Proof. This follows from Lemma 3.6.5 and proposition 3.6.6. We then define the category of open covers of $X_\bullet$. We have morphisms defined by simplicial refinements $f^{(\bullet)} : \mathcal{U}^{(\bullet)} \to \mathcal{V}^{(\bullet)}$ and we say that $\mathcal{U}^{(\bullet)} < \mathcal{V}^{(\bullet)}$ if there exists a refinement $\mathcal{U} \to \mathcal{V}$, this relation forms a directed set. As every simplicial cover refines to a good simplicial cover (Proposition 3.6.6) then the category of good simplicial covers is cofinal in the category of simplicial covers. □

Lemma 3.6.8. *Let $X_\bullet$ be a simplicial manifold and $\mathcal{U}^{(\bullet)}, \mathcal{V}^{(\bullet)}$ be open simplicial covers of $X_\bullet$. If $\mathcal{U}^{(\bullet)} < \mathcal{V}^{(\bullet)}$ then the induced map $\check{C}^*(\mathcal{V}^{(\bullet)}, A) \to \check{C}^*(\mathcal{U}^{(\bullet)}, A)$ doesn't depend on the choice of refinement $f : \mathcal{U}^{(\bullet)} \to \mathcal{V}^{(\bullet)}$.*

Proof. Suppose we have two simplicial refinements $f_\bullet, f'_\bullet : \mathcal{V}^{(\bullet)} \to \mathcal{U}^{(\bullet)}$. Let $I_\bullet, J_\bullet$ be the simplicial indexing sets of $\mathcal{U}^{(\bullet)}$ and $\mathcal{V}^{(\bullet)}$ respectively, thus $f_\bullet, f'_\bullet$ are simplicial maps $J_\bullet \to I_\bullet$. We wish to design a homotopy between the induced maps $(f_\bullet)_*, (f'_\bullet)_* : \check{H}^*(\mathcal{V}^{(q)}, A_q) \to \check{H}^*(\mathcal{U}^{(q)}, A_q)$. Consider the map $H_q : \check{C}^p(\mathcal{U}^q, A_q) \to \check{C}^{p-1}(\mathcal{V}^q, A_q)$ defined by

$$(H_q \underline{\alpha})_{j_0 \cdots j_{p-1}} = \sum_{k=1}^{p-1} (-1)^k (\underline{\alpha}_{f(j_0) \cdots f(j_k) f'(j_k) \cdots f'(j_{p-1})})|_{V_{j_0 \cdots j_{p-1}}}.$$

Notice that we are following the proof of [5, Lemma 1.3.8] and so we already know that $dH_q = H_q d$, now we need to show that $\delta H_q = H_{q+1}\delta$, we will do so by showing that the diagram

$$\begin{array}{ccc} \check{C}^p(\mathcal{U}^{(q)}, \underline{A}_q) & \xrightarrow{d_i^*} & \check{C}^p(\mathcal{U}^{(q+1)}, \underline{A}_{q+1}) \\ \downarrow{\scriptstyle H_q} & & \downarrow{\scriptstyle H_{q+1}} \\ \check{C}^{p-1}(\mathcal{V}^{(q)}, \underline{A}_q) & \xrightarrow{d_i^*} & \check{C}^{p-1}(\mathcal{V}^{(q+1)}, \underline{A}_{q+1}) \end{array}$$

commutes. Due to the fact that f and f' are simplicial maps we get the fact that $d_i \circ f = f \circ d_i$, and similarly for f', from this we get that

$$\begin{aligned}
d_i^*(H_q \underline{\alpha})_{j_0 \cdots j_{p-1}} &= A(d^i)(H_q(\alpha))_{d_i(j_0) \cdots d_i(j_{p-1})} \\
&= A(d^i) \left(\sum_{k=1}^{p-1} (-1)^k (\underline{\alpha}_{f(d_i j_0) \cdots f(d_i j_k) f'(d_i j_k) \cdots f'(d_i j_{p-1})})|_{V_{d_i j_0 \cdots d_i j_{p-1}}} \right) \\
&= \sum_{k=1}^{p-1} (-1)^k A(d^i) (\underline{\alpha}_{f(d_i j_0) \cdots f(d_i j_k) f'(d_i j_k) \cdots f'(d_i j_{p-1})})|_{V_{d_i j_0 \cdots d_i j_{p-1}}} \\
&= \sum_{k=1}^{p-1} (-1)^k A(d^i) (\underline{\alpha}_{d_i(f j_0) \cdots d_i(f j_k) d_i(f' j_k) \cdots d_i(f' j_{p-1})})|_{V_{d_i j_0 \cdots d_i j_{p-1}}} \\
&= \sum_{k=1}^{p-1} (-1)^k d_i^*(\alpha)_{f(j_0) \cdots f(j_k) f'(j_k) \cdots f'(j_{p-1})}|_{V_{d_i j_0 \cdots d_i j_{p-1}}} \\
&= H_q(d_i^*(\alpha))_{j_0 \cdots j_{p-1}}
\end{aligned}$$

and thus $\delta H_q = H_{q+1}\delta$. This means that the maps H_q piece together to form a simplicial homotopy $H_\bullet$ such that $DH_\bullet + H_\bullet D = (f_\bullet)_* - (f'_\bullet)_*$, thus a homotopy is formed between the maps $(f_\bullet)_*$, and $(f'_\bullet)_*$ and therefore the morphism $\check{H}^*(\mathcal{V}^{(q)}, A_q) \to \check{H}^*(\mathcal{U}^{(q)}, A_q)$ is independent upon choice of refinement. $\square$

Definition 3.6.9 (Simplicial Čech Cohomology). *Let $X_\bullet$ be a simplicial manifold and let $A \in \mathbf{Sh}(X_\bullet)$, we define the Čech cohomology $\check{H}^*(X_\bullet, A)$ by*

$$\check{H}^n(X_\bullet, A) \coloneqq \lim_{\mathcal{U}^{(\bullet)} \to} \check{H}^n(\mathcal{U}^{(\bullet)}, A).$$

From the above discussion we can see that good simplicial covers of $X_\bullet$ are cofinal in the directed set of simplicial covers, thus the simplicial Čech cohomology can be computed using good covers. What we really wish to show here is that the simplicial Čech cohomology is isomorphic for any two good simplicial covers. We will see that this will follow from results of sheaf cohomology. After attaining this result we will see that the simplicial Čech cohomology is isomorphic to that of simplicial Čech cohomology with respect to a good simplicial cover of $X_\bullet$.

3.7 Sheaf Cohomology of $X_\bullet$

Now that we understand that sheaves (and abelian sheaves) over the site $\mathcal{O}(X_\bullet)$ can be understood locally as a sequence of sheaves $\mathcal{F}_n$ over X_n we wish to understand how the sheaf cohomology of $\mathcal{F} \in \mathbf{AbSh}(\mathcal{O}(X_\bullet))$ relates to understanding sheaves locally.

In [13] it is shown that the restriction functor of sheaves $r_n : \mathbf{Sh}\mathcal{O}(X_\bullet) \to \mathbf{Sh}\mathcal{O}(X_n)$ has both a right and left adjoint, R^n and L^n respectively. This fact can also be recovered from the fact that the restriction functor is produced by the inclusion functor $\iota_n : \mathcal{O}(X_n) \to \mathcal{O}(X_\bullet)$.

Definition 3.7.1 (Sheaf Cohomology). *We define the sheaf cohomology of a sheaf $\mathcal{F}$ as the i-th right derived functor of the functor $\mathcal{F} \mapsto \mathbf{AbSh}(X_\bullet)(\underline{\mathbb{Z}}, \mathcal{F})$, the global sections functor.*

This process amounts to taking an injective resolution $I^\bullet$ of $\mathcal{F}$ and computing the cohomology of the complex

$$\cdots \to \mathbf{AbSh}(X_\bullet)(\underline{\mathbb{Z}}, I^q) \to \mathbf{AbSh}(X_\bullet)(\underline{\mathbb{Z}}, I^{q+1}) \to \cdots$$

In [13] the cohomology of this complex is related to the total cohomology of the double complex $\mathbf{AbSh}(X_\bullet)(\underline{\mathbb{Z}}_{X_*}, I^\bullet)$ by the fact that $\underline{\mathbb{Z}}_{X_*}$ forms an augmentation of the sheaf $\underline{\mathbb{Z}}$, this complex of sheaves is defined by the left adjoint to the restriction functor

$$\cdots \to L^2(\underline{\mathbb{Z}}_{X_2}) \to L^1(\underline{\mathbb{Z}}_{X_1}) \to L^0(\underline{\mathbb{Z}}_{X_0}) \to \underline{\mathbb{Z}}$$

Because of the way that these sheaves are defined we have that the sheaf cohomology can now be defined by the total cohomology of the global sections functors

$$\begin{aligned}\mathbf{AbSh}(X_\bullet)(\underline{\mathbb{Z}}_{X_p}, I^q) &= \mathbf{AbSh}(X_\bullet)(L^p(\underline{\mathbb{Z}}), I^q)\\ &= \mathbf{AbSh}(X_\bullet)(\underline{\mathbb{Z}}, r_p(I^q))\\ &= \Gamma(X_p, I_p^q)\end{aligned}$$

and so the sheaf cohomology is equal to the total cohomology of the double complex $\Gamma(X_p, I_p^q)$, which is similar to our definition of Čech cohomology. We now wish to show that sheaf cohomology on $X_\bullet$ and Čech cohomology with respect to a good simplicial cover $\mathcal{U}^{(\bullet)}$ are isomorphic. We will use the techniques of spectral sequences in order to prove this fact.

Theorem 3.7.2. *Let $X_\bullet$ be a simplicial manifold and A be an abelian Lie group. Let the sheaf $\underline{A} \in Sh(X_\bullet)$ be the sheaf defined by functions on open sets into A. The simplicial-Čech cohomology $\check{H}^p(\mathcal{U}^{(\bullet)}, \underline{A})$ is isomorphic to that of sheaf cohomology $H^p(X_\bullet, \underline{A})$ when $\mathcal{U}^{(\bullet)}$ is a good simplicial covering of $X_\bullet$.*

Proof. We will be considering the triple complex $\check{C}^p(\mathcal{U}^{(q)}, I_q^r)$ with differentials d_P, d_Q, and d_R which are Čech, simplicial, and sheaf differentials respectively. Let $I^\bullet$ be an injective resolution of the sheaf $\underline{A}$. Thus by the fact that the restriction functor $r_q(-)$ has both left and right adjoints [13] we have that each I_q^r is an injective sheaf. Furthermore from [14, Section 09WB] we have that $I_q^\bullet$ forms a injective resolution of A_q. So as $\cdots \to I_q^{r-1} \xrightarrow{d_R} I_q^r \xrightarrow{d_R} I_q^{r+1} \to \cdots$ is a long exact sequence of sheaves we get a long exact sequence of groups [15]

$$\cdots \xrightarrow{d_R} \check{C}^p(\mathcal{U}^{(q)}, I_q^{r-1}) \xrightarrow{d_R} \check{C}^p(\mathcal{U}^{(q)}, I_q^r) \xrightarrow{d_R} \check{C}^p(\mathcal{U}^{(q)}, I_q^{r+1}) \xrightarrow{d_R} \cdots$$

Therefore, the differential d_R is exact. Now we wish to show that the differential d_P is exact. We know from [5] that the sheaf cohomology $H^p(X, I)$ of an injective sheaf I is zero for $p > 0$. We also understand that if $\mathcal{U}^{(q)}$ is a good cover for X_q then we have that the Čech cohomology of I_q^r computes the sheaf cohomology of I_q^r with respect to $\mathcal{U}^{(q)}$, as this is zero then we know that the Čech cohomology is zero, and thus d_P is exact for $p > 0$. We have that Proposition 2.4.14 tells us the cohomology of $\ker(d_P)$ is isomorphic to the cohomology of $\ker(d_R)$. Now we need to compute each of these kernels, we have that

$$\begin{aligned}
\ker(d_R) &= \ker\left(d_R : \bigoplus_{p+q=n} \check{C}^p(\mathcal{U}^{(q)}, I_q^0) \to \bigoplus_{p+q=n} \check{C}^p(\mathcal{U}^{(q)}, I_q^1)\right) \\
&= \bigoplus_{p+q=n} \ker\left(d_R : \check{C}^p(\mathcal{U}^{(q)}, I_q^0) \to \check{C}^p(\mathcal{U}^{(q)}, I_q^1)\right) \\
&= \bigoplus_{p+q=n} \check{C}^p(\mathcal{U}^{(q)}, A_q)
\end{aligned}$$

thus the cohomology of $\ker(d_R)$ is the total cohomology of the Čech-simplicial complex. Now we will inspect the kernel of d_P

$$\begin{aligned}
\ker(d_P) &= \ker\left(d_R : \bigoplus_{q+r=n} \check{C}^0(\mathcal{U}^{(q)}, I_q^r) \to \bigoplus_{q+r=n} \check{C}^1(\mathcal{U}^{(q)}, I_q^r)\right) \\
&= \bigoplus_{q+r=n} \ker\left(d_R : \check{C}^0(\mathcal{U}^{(q)}, I_q^r) \to \check{C}^1(\mathcal{U}^{(q)}, I_q^r)\right) \\
&= \bigoplus_{q+r=n} \Gamma(X_q, I_q^r)
\end{aligned}$$

so the cohomology of $\ker(d_P)$ is the total cohomology of the double complex $\Gamma(X_q, I_q^r)$ which we understand computes the sheaf cohomology $H^\bullet(X_\bullet, A)$, and so from Proposition 2.4.14 we understand that the Čech-simplicial cohomology with respect to a good cover $\check{H}^n(\mathcal{U}^{(\bullet)}, A_\bullet)$ is isomorphic to that of the sheaf cohomology $H^\bullet(X_\bullet, A)$. □

Corollary 3.7.3. *Simplicial-Čech cohomology is isomorphic to simplicial-Čech cohomology with respect to a good simplicial cover.*

Proof. This follows from Theorem 3.7.2 as we have that the simplicial-Čech cohomology can be computed by the direct limit of good simplicial covers. As the simplicial-Čech cohomology with respect to a good cover is always isomorphic to the sheaf cohomology then we have that we are taking the limit of a diagram of constant abelian groups and thus the simplicial-Čech cohomology can be computed by taking a simplicial good cover of $X_\bullet$ and computing the respective simplicial-Čech cohomology. □

3.8 An Isomorphism Theorem

If we consider a simplicial manifold $X_\bullet$ and it's corresponding Čech cohomology $\check{H}^\bullet(X, A)$ we wish to know how this relates to the cohomology of the fat realisation $\|X_\bullet\|$. In particular we can compare these two cohomologies when looking at the sheaf $\mathbb{Z}$ in which we use [1, Proposition 5.15].

Theorem 3.8.1. *If $X_\bullet$ is a simplicial manifold then for any good simplicial cover $\mathcal{U}^{(\bullet)}$ of $X_\bullet$ then we have that*

$$H^\bullet(\mathcal{U}^{(\bullet)}, \mathbb{Z}) \simeq H^\bullet(\|X_\bullet\|, \mathbb{Z})$$

We will show that this is true by using singular cohomology which is defined for a simplicial manifold in [1] and then using [1, Proposition 5.15] we know this is isomorphic to singular cohomology on the fat realisation $\|X_\bullet\|$, as $\|X_\bullet\|$ has the homotopy type of a CW-complex we have that singular and Čech cohomology is isomorphic on $\|X_\bullet\|$.

We now need to consider the Čech-singular-simplicial triple complex and the related differentials.

Definition 3.8.2. *The Čech-singular-simplicial complex is the triple complex $\check{C}^\bullet(\mathcal{U}^{(\bullet)}, S^\bullet)$ where $\check{C}^n(\mathcal{U}^{(m)}, S^k)$ is defined by Čech n-cochain with respect to the cover $\mathcal{U}^{(m)}$ of the manifold X_m the values in singular k-cochains. Equivalently*

$$\check{C}^n(\mathcal{U}^{(m)}, S^k) := \coprod_{i_0, \cdots, i_n} Hom\left(S_k(U^{(m)}_{i_0 \cdots i_n}), \mathbb{Z}\right)$$

which will be our working definition. We will use the shorthand $C^{(n,m,k)}$ for this complex with differentials

$$\begin{aligned} d^{(1)} &: \check{C}^n(\mathcal{U}^{(m)}, S^k) \to \check{C}^{n+1}(\mathcal{U}^{(m)}, S^k) \\ d^{(2)} &: \check{C}^n(\mathcal{U}^{(m)}, S^k) \to \check{C}^n(\mathcal{U}^{(m+1)}, S^k) \\ d^{(3)} &: \check{C}^n(\mathcal{U}^{(m)}, S^k) \to \check{C}^n(\mathcal{U}^{(m)}, S^{k+1}). \end{aligned}$$

Definition 3.8.3. *Given a cover $\mathcal{U} = (U_i)_{i \in I}$ of a manifold M we can define the following free abelian group*

$$S^{\mathcal{U}}_q(M) := \mathbb{Z}\{\sigma : \Delta^q \to M | \exists i \in I \text{ such that } \sigma(\Delta^q) \subseteq U_i\}.$$

This is called the group of $\mathcal{U}$-small singular q-chains.

Lemma 3.8.4. *The sequence*

$$0 \leftarrow S^{\mathcal{U}}_q(M) \leftarrow \bigoplus_{i \in I} S_q(U_i) \leftarrow \bigoplus_{i,j \in I} S_q(U_{ij}) \leftarrow \cdots$$

is exact.

Proof. This fact is shown in [6, Proposition 15.2] □

Lemma 3.8.5. *The differentials $d^{(1)}$ and $d^{(3)}$ in the triple complex $C^{(n,m,k)}$ are exact, given that $\mathcal{U}^{(\bullet)}$ is a good simplicial cover of $X_\bullet$.*

Proof. Notice that $d^{(1)} : \check{C}^n(\mathcal{U}, S^k(M)) \to \check{C}^{n+1}(\mathcal{U}, S^k(M))$ is the differential in Lemma 3.8.4 but after applying the functor $Hom(-,\mathbb{Z})$. As the sequence in Lemma 3.8.4 is a sequence of free abelian groups then the functor $Hom(-,\mathbb{Z})$ preserves exactness, giving us the fact that $d^{(1)}$ is exact for $n > 0$, the kernel of $d^{(1)}$ is $S^k_{\mathcal{U}^{(m)}}(X_m) \coloneqq Hom(S_k^{\mathcal{U}^{(m)}}(X_m), \mathbb{Z})$.

We now wish to show that $d^{(3)}$ is exact. Notice that

$$\check{C}^n(\mathcal{U}^{(m)}, S^k(X_m)) = \bigoplus_{i_0\cdots i_n \in I_m} S^k(U^{(m)}_{i_0\cdots i_n})$$

which is an arbitrary sum of groups $S^k(U^{(m)}_{i_0\cdots i_n})$, notice that the differential $d^{(3)}$ sends elements in $S^k(U^{(m)}_{i_0\cdots i_n})$ to elements in $S^{k+1}(U^{(m)}_{i_0\cdots i_n})$. As $\mathcal{U}^{(\bullet)}$ is a good simplicial cover then the set $U^{(m)}_{i_0\cdots i_n}$ is contractible, therefore the differential $d^{(3)}$ restricted to the complex $S^\bullet(U^{(m)}_{i_0\cdots i_n})$ is exact. The differential

$$d^{(3)} : \bigoplus_{i_0\cdots i_n \in I_m} S^k(U^{(m)}_{i_0\cdots i_n}) \to \bigoplus_{i_0\cdots i_n \in I_m} S^{k+1}(U^{(m)}_{i_0\cdots i_n})$$

is simply the sum of these exact differentials, and thus must be exact itself. □

Lemma 3.8.6. *The kernel of $d^{(1)}$ describes an element of the $\mathcal{U}$-small singular-simplicial complex.*

Proof. As in the proof of Lemma 3.8.5 we have that the kernel of $d^{(1)} : C^{0,m,k} \to C^{1,m,k}$ is the group of $\mathcal{U}$-small singular cochains $S^k_{\mathcal{U}^{(m)}}(X_m)$, thus forming

$$\begin{aligned} \ker\left(d^{(1)} : \bigoplus_{m+k=N} C^{0,m,k} \to \bigoplus_{m+k=N} C^{1,m,k}\right) &= \bigoplus_{m+k=N} \ker\left(d^{(1)} : C^{0,m,k} \to C^{1,m,k}\right) \\ &= \bigoplus_{m+k=N} S^k_{\mathcal{U}^{(m)}}(X_m) \end{aligned}$$

the total complex of the double complex $S^\bullet_{\mathcal{U}^{(\bullet)}}(X_\bullet)$.

□

Lemma 3.8.7. *The total cohomology of the $\mathcal{U}$-small singular-simplicial complex and the singular-simplicial complex are isomorphic.*

Proof. There is a canonical inclusion of double complexes $f^{\bullet,\bullet} : S^\bullet_{\mathcal{U}^{(\bullet)}}(X_\bullet) \hookrightarrow S^\bullet(X_\bullet)$, moreover for each $m \geq 0$ the map $f^{m,\bullet} : S^\bullet_{\mathcal{U}^{(m)}}(X_m) \hookrightarrow S^\bullet(X_m)$ induces an isomorphism on cohomology [6]. We can use the comparison theorem [4, Theorem 5.2.12] applied to the spectral sequences associated to the canonical filtrations of the double complexes $S^\bullet_{\mathcal{U}^{(\bullet)}}(X_\bullet)$ and $S^\bullet(X_\bullet)$ to deduce that the canonical map $S^\bullet_{\mathcal{U}^{(\bullet)}}(X_\bullet) \to S^\bullet(X_\bullet)$ induces an isomorphism on total cohomology. □

Lemma 3.8.8. *The kernel of $d^{(3)}$ describes an element of the total Čech-simplicial complex.*

$$\ker\left(d^{(3)} : \bigoplus_{n+m=N} C^{(n,m,0)} \to \bigoplus_{n+m=N} C^{(n,m,1)}\right) \simeq \bigoplus_{n+m=N} \check{C}^n(\mathcal{U}^{(m)}; \mathbb{Z})$$

Proof. We will prove this directly;

$$\begin{aligned}
&\ker\left(d^{(3)} : \bigoplus_{n+m=N} C^{(n,m,0)} \to \bigoplus_{n+m=N} C^{(n,m,1)}\right)\\
&= \bigoplus_{n+m=N} \ker\left(d^{(3)} : \coprod_{i_0\cdots i_n} S^0(U^{(m)}_{i_0\cdots i_n}) \to \coprod_{i_0\cdots i_n} S^1(U^{(m)}_{i_0\cdots i_n})\right)\\
&= \bigoplus_{n+m=N} \coprod_{i_0\cdots i_n} \ker\left(d^{(3)} : S^0(U^{(m)}_{i_0\cdots i_n}) \to S^1(U^{(m)}_{i_0\cdots i_n})\right)
\end{aligned}$$

as each $\mathcal{U}^{(m)}$ is a good cover of X_m we have that the kernel of $d^{(3)}$ in each separate instance is precisely $\mathbb{Z}$ as this is the singular complex on a contractible space.

$$\begin{aligned}
&= \bigoplus_{n+m=N} \coprod_{i_0\cdots i_n} \mathbb{Z}\\
&= \bigoplus_{n+m=N} \coprod_{i_0\cdots i_n} \underline{\mathbb{Z}}(U^{(m)}_{i_0\cdots i_n})\\
&= \bigoplus_{n+m=N} \coprod_{i_0\cdots i_n} \underline{\mathbb{Z}}(U^{(m)}_{i_0\cdots i_n})\\
&= \bigoplus_{n+m=N} \check{C}^n(\mathcal{U}^{(m)}; \mathbb{Z}).
\end{aligned}$$

Furthermore an isomorphism of cohomology is immediate from this equality. □

Finally we can complete the proof for 3.8.1.

Proof. As $\|X_\bullet\|$ has the homotopy type of a CW-complex we have that $H^n(\|X_\bullet\|;\mathbb{Z}) \simeq \check{H}^n(\|X_\bullet\|;\mathbb{Z})$, the left being singular cohomology and the right being Čech. Through proposition 5.15 of [1] we have that $H^n(\|X_\bullet\|;\mathbb{Z}) \simeq H^n(X_\bullet;\mathbb{Z})$. From the lemmas above we have that $H^n(X_\bullet;\mathbb{Z}) \simeq H^n\left(\oplus_{i+j+k=n} C^{(i,j,k)}\right) \simeq \check{H}^n(\mathcal{U}^{(\bullet)};\mathbb{Z})$ and so we have that

$$\check{H}^n(\|X_\bullet\|;\mathbb{Z}) \simeq \check{H}^n(\mathcal{U}^{(\bullet)};\mathbb{Z})$$

and we are done. □

3.9 $\mathbb{R}$-Valued Simplicial Cohomology

Here we will focus on computing the $\mathbb{R}$ valued Čech cohomology of the simplicial manifold $EG(M)_\bullet$ discussed in Example 3.1.5, precisely this is the group $H^n(EG(M_\bullet), \mathbb{R})$ where $\mathbb{R}$ is the sheaf of $\mathbb{R}$ valued functions in this context. Firstly we will note that the $\mathbb{R}$ valued sheaf cohomology groups of M and G are all zero as M and G are both smooth manifolds [5]. This gives rise to the usual isomorphism between $U(1)$ and $\mathbb{Z}$ valued Čech cohomology. So from Proposition 2.4.14 we understand that the cohomology $H^n(EG(M_\bullet), \mathbb{R})$ can be computed by the cohomology group $H^n(\ker(d : \check{C}^0(M \times G^p; \mathbb{R}) \to \check{C}^1(M \times G^p; \mathbb{R})))$. This is the complex of $\mathbb{R}$-valued functions from $M \times G^p$ with differential defined by the alternating pullback of face maps δ.

Proposition 3.9.1. *The cohomology of the complex $C^\infty(M \times G^p; \mathbb{R})$ with differential δ is zero for $p > 0$.*

Proof. Let $f : M \times G^p \to \mathbb{R}$ such that $\delta(f) = 0$, explicitly we have

$$\begin{aligned}\delta(f)(m, g_1, \cdots, g_{p+1}) = f(mg_1, g_2, \cdots, g_{p+1}) - f(m, g_1 g_2, \cdots, g_{p+1}) + \cdots \\ + (-1)^{p+1} f(m, g_1, \cdots, g_p)\end{aligned}$$

rearranging we get that

$$\begin{aligned}(-1)^{p+1} f(m, g_1, \cdots, g_p) = f(mg_1, g_2, \cdots, g_{p+1}) - f(m, g_1 g_2, \cdots, g_{p+1}) + \cdots \\ + (-1)^p f(m, g_1, \cdots, g_p g_{p+1})\end{aligned}$$

and notice that the left hand side is independent of g_{p+1} and so in this calculation we can choose g_{p+1} to be any $g \in G$. Let

$$h(m, g_1, \cdots, g_{p-1}) = \int_{g \in G} f(m, g_1, \cdots, g_{p-1}, g) dg$$

we have that

$$\begin{aligned}\delta(h)(m, g_1, \cdots, g_p) = &\int_{g \in G} f(mg_1, g_2, \cdots, g_p, g) dg \\ &- \int_{g \in G} f(m, g_1 g_2, \cdots, g_p, g) dg + \cdots \\ &+ (-1)^p \int_{g \in G} f(m, g_1, \cdots, g_{p-1}, g) dg\end{aligned}$$

notice that in the last line we can replace g by $g_p g$ as we are summing over all values in G giving

$$\begin{aligned}\delta(h)(m, g_1, \cdots, g_p) &= \int_{g\in G} f(mg_1, g_2, \cdots, g_p, g)dg \\ &- \int_{g\in G} f(m, g_1g_2, \cdots, g_p, g)dg + \cdots \\ &+ (-1)^p \int_{g\in G} f(m, g_1, \cdots, g_{p-1}, g_pg)dg \\ &= \int_{g\in G} f(mg_1, g_2, \cdots, g_p, g) - f(m, g_1g_2, \cdots, g_p, g)dg + \cdots \\ &+ (-1)^p f(m, g_1, \cdots, g_{p-1}, g_pg)\end{aligned}$$

notice from the fact $\delta(f) = 0$ that equation gives us the fact that

$$\begin{aligned}\delta(h)(m, g_1, \cdots, g_p) &= \int_{g\in G} (-1)^{p+1} f(m, g_1, \cdots, g_p)dg \\ &= (-1)^{p+1} f(m, g_1, \cdots, g_p)\end{aligned}$$

and so changing h by a factor of $(-1)^{p+1}$ we get that $\delta(h) = f$ and thus δ is exact. □

We have an immediate corollary that $H^n(EG(M)_\bullet; \mathbb{R}) = 0$, furthermore we have that $H^n(EG(M)_\bullet; U(1)) \simeq H^{n+1}(EG(M)\bullet; \mathbb{Z})$ via the long exact sequence in cohomology.

Chapter 4

Principal G-Bundles

4.1 Definitions

We refer to the text [16] for the theory and definitions regarding principal G bundles.

Definition 4.1.1 (Principal G-Bundle)**.** *Let M be a manifold and G a Lie group. A smooth principal fibre bundle over M with structure group G consists of a manifold P, a smooth map $\pi : P \to M$, and a smooth right action of G on P such that*

1. *G acts freely on P and $\pi(p \cdot g) = \pi(p)$ for $g \in G$, $p \in P$.*
2. *P is locally trivial, meaning that given a point $x \in M$ there exists an open set $x \in U$ such that $\pi^{-1}(U)$ is isomorphic to $U \times G$ through an equivariant isomorphism that commutes through the projection map π.*

Example 4.1.2. *Given any manifold M and any Lie group G we have the example $P = M \times G$ with the canonical projection onto M. This is the trivial principal G bundle, called the trivial bundle.*

Example 4.1.3. *We can consider the principal $\mathbb{Z}$ bundle $\mathbb{R} \to U(1)$ defined by the map $t \mapsto e^{2\pi i t}$. The $\mathbb{Z}$ action on $\mathbb{R}$ is given by addition.*

Definition 4.1.4 (Morphisms of G bundles)**.** *A morphism between two G bundles $Q \to N$ and $P \to M$ is a map $f : Q \to P$ that is smooth and is equivariant with respect to the action of G. This map f covers a morphism between base spaces $N \to M$.*

Definition 4.1.5 (Triviality)**.** *We say that a principal G bundle $P \to M$ is trivialisable if there exists an isomorphism of G bundles $P \to M \times G$.*

This notion of triviality is equivalent to saying that there exists a global section $s : M \to P$ of the bundle P.

Proposition 4.1.6. *If X is a contractible space then any principal G bundle on X is trivial.*

We wish to understand the classification and construction of these principal bundles in terms of transition functions. Take a good cover of our manifold M and let P be a principal G bundle over M. As each open set $U_i \subseteq M$ is contractible then there exists a local section $s_i : U_i \to P$ of the G bundle, this fact is a result of Proposition 4.1.6. Furthermore as G acts freely on P over a non-empty intersection $U_{ij} := U_i \cap U_j$ we have a unique function $\varphi_{ij} : U_{ij} \to G$ which is defined by the equation

$$s_i(x) = s_j(x)\varphi_{ij}(x).$$

These functions satisfy the relation $\varphi_{jk}(x)\varphi_{ik}(x) = \varphi_{ik}(x)$ since we have

$$\begin{aligned} s_i(x) &= s_j(x)\varphi_{ij}(x) \\ &= s_k(x)\varphi_{jk}(x)\varphi_{ij}(x) \\ &= s_i(x)\varphi_{ki}(x)\varphi_{jk}(x)\varphi_{ij}(x) \\ &= s_i(x)(\varphi_{ik}(x))^{-1}\varphi_{jk}(x)\varphi_{ij}(x) \end{aligned}$$

and so

$$s_i(x)\left(\varphi_{jk}(x)\varphi_{ij}(x)\right)^{-1} = s_i(x)(\varphi_{ik}(x))^{-1}$$

which by uniqueness gives the fact that

$$\varphi_{jk}(x)\varphi_{ij}(x) = \varphi_{ik}(x).$$

This is the called the cocycle condition which we will see when talking about representing these objects in terms of Čech cohomology. Notice that given a cover $\mathcal{U}$ and a set of functions $\varphi_{ij} : U_{ij} \to G$ satisfying the cocycle condition we can construct a principal G bundle $P \to M$. Let

$$P := \left(\coprod_{i \in I} U_i \times G\right) / \sim$$

where we say that $(x, i, g) \sim (y, j, h)$ if $x = y$ and $\varphi_{ij}(x) = hg^{-1}$. We find that the transition functions of P are simply $\varphi_{ij} : U_i \cap U_j \to G$.

4.2 Constructions

For this section we will be working exclusively with abelian Lie groups in order for these constructions to make sense.

Definition 4.2.1 (Product Bundle)**.** *Given two principal G bundles P and Q on a manifold X we can form a new principal G bundle $P \otimes Q$ given by*

$$P \otimes Q := P \times Q / \sim$$

where $p \cdot g \otimes q \sim p \otimes q \cdot g$ for $g \in G$.

Definition 4.2.2 (Dual Bundle)**.** *Given a principal G bundle P we can form the dual bundle P^* which is given by $P = P^*$ but the G action on P^* is defined by $(p^*) \cdot g \mapsto (pg^{-1})^*$.*

Definition 4.2.3 (Pullback Bundle)**.** *Given a G bundle P over M and a smooth map $f : N \to M$ we have a G bundle $f^*(P) \to N$ defined by*

$$f^*(P) := \{(n, p) : f(n) = \pi(p)\}$$

with G action defined by $(n, p)g \mapsto (n, pg)$.

Proposition 4.2.4. *Let P and Q be principal G bundles over M with transition functions φ_{ij} and ψ_{ij} respectively. Let $f : N \to M$ be a smooth map between manifolds. The transition functions for products, duals, and pullbacks respectively are given by $\varphi_{ij}\psi_{ij}$, φ_{ij}^{-1}, and $f^*(\varphi_{ij})$.*

Proof. Let s_i be local sections of P and r_i be local sections of Q. We have that $s_i \otimes r_i$ defines local sections of $P \otimes Q$. Similarly we have that

$$\begin{aligned} s_i(x) \otimes r_i(x) &= (s_j(x)\varphi_{ij})(x) \otimes (r_j(x)\psi_{ij}(x)) \\ &= (s_j(x) \otimes r_j(x))\varphi_{ij}(x)\psi_{ij}(x) \end{aligned}$$

and so defining the transition functions for $P \otimes Q$. For the dual bundle P^* we have that $s_i(x)$ continues to define local sections and so we have that

$$\begin{aligned} (s_i(x))^* &= (s_j(x)\varphi_{ij}(x))^* \\ &= (s_j(x))^*(\varphi_{ij}(x))^{-1}. \end{aligned}$$

Finally given the pullback bundle $f^*(P)$ we also form the pullback cover $f^*(\mathcal{U})$ of N where $V_i := f^{-1}(U_i)$, similarly we have local sections of $f^*(P)$ defined by $s_i \circ f(x) : V_i \to f^*(P)$. So we have

$$s_i \circ f(x) = s_j \circ f(x)\varphi_{ij} \circ f(x)$$

and we are done. $\square$

4.3 Classification Theorem

Proposition 4.3.1. *Let A be an abelian Lie group. The isomorphism classes of principal A bundles on a manifold M are classified by the Čech cohomology group $H^1(M;\underline{A})$.*

Proof. We have seen that there is certainly a morphism of A-bundles on M into the set of cocycles on M. We have also seen that this morphism is surjective to the set of cocycles. We need to show that this is a bijection of isomorphism classes. Let $[P]$ denote the Čech 1-cocycle associated to a principal A bundle P. From proposition 4.2.4 we have that $[P \otimes Q] = [P] + [Q]$ and $[P^*] = -[P]$. Furthermore the map $P \mapsto [P]$ is surjective due to the clutching construction. We need to show that this map is injective which boils down to showing that the class of a bundle P is trivial if and only if the bundle P is trivial.

Suppose that P is trivial with transition functions given by $s_i : U_i \to P$, there exists a global section $s : M \to P$. Define a Čech 0-cochain by the equation $s_i(x)\alpha_i(x) = s(x)$. We have that

$$\begin{aligned} s_i(x) &= s_j(x)\varphi_{ij}(x) \\ s(x)(\alpha_i(x))^{-1} &= s(x)(\alpha_j(x))^{-1}\varphi_{ij}(x) \\ (\alpha_i(x))^{-1} &= (\alpha_j(x))^{-1}\varphi_{ij}(x) \\ (\alpha_j(x))(\alpha_i(x))^{-1} &= \varphi_{ij}(x) \\ d(\alpha)_{ij} &= \varphi_{ij} \end{aligned}$$

and so φ is in the image of the Čech derivative operator.

Suppose that $\varphi_{ij}(x) = \alpha_j(x)(\alpha_i(x))^{-1}$. Define a section $s : M \to P$ by $s(x)|_{U_i} := s_i(x)\alpha_i(x)$. Notice that this is well defined by the fact that $s_i(x)\alpha_i(x) = s_j(x)\alpha_j(x)$ over U_{ij} and thus forms a global function $s(x)$ by the stitching lemma, therefore P is trivial and we are done.

We have that the morphism $P \mapsto [P]$ is well defined on isomorphism classes and is a bijection. Thus we have classified principal A bundles by A-valued Čech cohomology. □

4.4 Descent

In [5] Brylinkski describes the descent of sheaves on a space M. In the preamble to this discussion he considers the simpler case of descent of G bundles. Given a surjective submersion $Y \to M$ and a G bundle $P \to Y$ we can decide when P is isomorphic to the pullback of a G bundle from M. The information required here is the descent data which consists of a section s of the bundle $p_1^*(P) \otimes p_2^*(P)^* \to Y^{[2]}$ such that $p_1^*(s) \otimes (p_2^*(s))^* \otimes (p_3^*(s)) = 1$ with respect to the canonical trivialisation of $d^2(P)$.

Definition 4.4.1. *Given a surjective submersion $Y \to M$ and a principal G-bundle $P \to Y^{[k]}$ we define a principal G bundle $d(P) \to Y^{[k+1]}$*

$$d(P) := p_1^*(P) \otimes p_2^*(P)^* \otimes p_3^*(P) \otimes \cdots \otimes p_k^*(P)^{*^{k+1}}$$

Similarly one can extend this definition to functions, given a function $f : Y^{[k]} \to A$ where A is some abelian Lie group we can define $d(f)$ by the alternating product of projection morphisms. Notice that this emulates the Čech differential when Y is given by an open cover of our space.

Proposition 4.4.2. *A principal G-bundle $P \to Y$ descends through a surjective submersion $Y \xrightarrow{\pi} M$ if and only if there exists descent data.*

Proof. This proof is detailed in [5, Section 5.1]. □

In particular [5] states that there exists a categorical equivalence between the category of G bundles and G bundles over surjective submersions with descent data. In particular this becomes useful when talking about trivialisations of bundle gerbes where we will see this situation arise frequently. Furthermore, this appears in simplicial extensions as a part of the definition.

4.5 Pre-G Bundles

If we consider the category of G bundles $P \to Y$ with descent data we can look at the subcategory consisting of trivial G bundles $G \times Y \to Y$ with descent data. This amounts to a function $\varphi : Y^{[2]} \to G$ such that $d(\varphi) : Y^{[3]} \to G$ is precisely the trivial map. We call this object a pre-G bundle and we can forget about the G bundle data over Y and only consider the map φ.

Definition 4.5.1 (Pre-G Bundle)**.** *A pre-G bundle consists of a surjective submersion $Y \to M$ and a smooth map $\varphi : Y^{[2]} \to G$ such that $d(\varphi) = 1$.*

Definition 4.5.2 (Morphisms of pre-G bundles)**.** *A morphism between two pre-G bundles (Y, s) and (Z, r) is a smooth map $\varphi : Y \times_M Z \to G$ such that $d(\varphi)(y_1, y_2, z_1, z_2) = s(y_1, y_2)(r(z_1, z_2))^{-1}$.*

Proposition 4.5.3. *To every G bundle $P \to M$ there exists an associated pre-G bundle.*

Proof. Notice that $P \to M$ is a surjective submersion, furthermore there exists a function $P^{[2]} \xrightarrow{f} G$ which is defined by $p_1 = p_2 f(p_1, p_2)$. Notice that $d(f) = 1$ due to the fact that the action of G on P is associative. □

Proposition 4.5.4. *To every pre-G bundle there exists a G bundle $P \to M$.*

Proof. Notice that this follows directly from Proposition 4.4.2. □

Remark. *We expect that G bundles and pre-G bundles forms an equivalence of categories in this way.*

Chapter 5

Bundle Gerbes

5.1 Background

Just as hermitian line bundles and principal $U(1)$ bundles are classified by $H^2(M, \mathbb{Z})$, bundle gerbes are an analogue to this in the sense that isomorphism classes of bundle gerbes are classified by degree three integral cohomology. Bundle gerbes were invented by Michael Murray in his paper, appropriately titled "Bundle Gerbes" [2]. The pioneering paper was based on the advertising material for the book "Loop Spaces, Characteristic Classes, and Geometric Quantization" [5] where gerbes were considered as sheaves of groupoids, and gave the completely geometric theory of bundle gerbes without reference to sheaves.

Definition 5.1.1 (Surjective Submersion)**.** *If X, Y are manifolds a surjective submersion $Y \xrightarrow{\pi} X$ is a map which is surjective and where the differential is also surjective.*

More generally, we look at *locally split maps* $Y \to X$ which have the same desired properties of surjective submersions but don't require the structure of a smooth manifold. In particular given a point $x \in X$ there exists an open neighborhood $x \in U$ such that there exists a section $s : U \to Y$ of the locally split map. Notice that this implies that the morphism $Y \to X$ is surjective. From this definition we have that the projection in principal G-bundles $P \to X$ is a locally split map.

Example 5.1.2. *We can think of surjective submersions as generalisations of open covers, which is how in a sense we get Čech cohomology representatives of these geometric objects. In particular given an open cover $\mathcal{U}$ of X one can construct the disjoint union $Y_{\mathcal{U}} := \{(x, i) | x \in U_i\}$ with a map $\pi : (x, i) \mapsto x$, this is a surjective submersion.*

Definition 5.1.3 (Fibre Products)**.** *Given surjective submersions $Y \xrightarrow{\pi_Y} X$, $Z \xrightarrow{\pi_Z} X$ we can form the fibre product of Y and Z over X given by*

$$Y \times_X Z := \{(y, z) | \pi_Y(y) = \pi_Z(z)\} \subseteq Y \times Z$$

notice that this is also a surjective submersion on X by the obvious mapping. Often we are concerned with the fibre product of a manifold with itself. This is denoted $Y^{[2]} = Y \times_X Y$, this of course generalises to $Y^{[p]}$ in the obvious manner.

Example 5.1.4. *Given an open cover $\mathcal{U}$ of a manifold M and taking the surjective submersion $Y_{\mathcal{U}} \to M$ we can form the space $Y_{\mathcal{U}}^{[p]}$ of p-fold ordered intersections of open sets. This aids in the discussion of Čech cohomology as a function $Y_{\mathcal{U}}^{[p]} \to U(1)$ is the same as a $U(1)$-valued Čech $p-1$ cochain.*

Example 5.1.5. *The projection map $\pi : P \to X$ in a smooth principal G-bundle is a surjective submersion. Fibre products of P are given by the diffeomorphism $P^{[k]} \to P \times G^{k-1}$.*

Example 5.1.6. *Given a smooth map between manifolds $f : X' \to X$ and a surjective submersion $Y \xrightarrow{\pi} X$ we can form the pullback surjective submersion $f^*(Y) \to X'$ defined by*

$$f^*(Y) := \{(x', y) | f(x') = \pi(y)\}$$

with projection defined by $\pi((x', y)) = x'$.

Proposition 5.1.7. *If $P \to Y^{[p]}$ is an abelian principal G bundle and $Y \to X$ is a surjective submersion then we can define a principal G bundle on $Y^{[p+1]}$ by*

$$\delta(P) := \pi_1^*(P) \otimes \pi_2^*(P)^* \otimes \pi_3^*(P) \otimes \cdots .$$

We can see that $\delta^2(P)$ is canonically trivial. This construction should be somewhat reminiscent of the Čech boundary operator, one can see this when comparing to a surjective submersion $Y_{\mathcal{U}}$.

Proof. Notice that the projection maps $Y^{[p+1]} \xrightarrow{\pi_k} Y^{[p]}$ for $k = 0, \ldots, k+1$ satisfy the simplicial identities, and $\delta(P)$ is defined by the alternating pullback of face maps in the same way we define the simplicial boundary operator. Due to the simplicial identities for each $\pi_i^* \pi_j^*(P)$ term in $\delta^2(P)$ there exists a $\pi_i^* \pi_j^*(P)^*$ term. Notice that

$$\pi_i^* \pi_j^*(P) \otimes \pi_i^* \pi_j^*(P)^*$$

is canonically trivial by the section $y \mapsto p \otimes p^*$ for any $p \in \pi^{-1}(\{y\})$. Thus defining a global section of the principal G bundle $\delta^2(P)$. □

5.2 Definition

Definition 5.2.1 (Bundle Gerbe)**.** *A bundle gerbe [2] over X is a pair (P, Y) where $Y \to X$ is a surjective submersion and $P \to Y^{[2]}$ is a principal $U(1)$ bundle together with* *a bundle gerbe multiplication* $m : \pi_3^*(P) \otimes \pi_1^*(P) \to \pi_2^*(P)$ *which is a smooth isomorphism of* $U(1)$ *bundles over* $Y^{[3]}$. *Furthermore we require that this multiplication* m *is associative, i.e. the diagram*

$$\begin{array}{ccc} P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes P_{(y_3,y_4)} & \longrightarrow & P_{(y_1,y_3)} \otimes P_{(y_3,y_4)} \\ \downarrow & & \downarrow \\ P_{(y_1,y_2)} \otimes P_{(y_2,y_4)} & \longrightarrow & P_{(y_1,y_4)} \end{array}$$

commutes where $(y_1, y_2, y_3, y_4) \in Y^{[4]}$.

Alternatively one can define the bundle gerbe multiplication m as a section $s : Y^{[3]} \to \delta(P)$ such that $\delta(s) = 1$ with respect to the canonical trivialisation of $\delta^2(P)$. We can see that $s : Y^{[3]} \to \pi_1^*(P) \otimes \pi_2^*(P)^* \otimes \pi_3^*(P)$ at a point (y_1, y_2, y_3) is the same thing as an isomorphism $\pi_3^*(P) \otimes \pi_1^*(P) \to \pi_2^*(P)$. Similarly the associativity condition of bundle gerbe multiplication comes from the fact that $\delta(s) = 1$.

5.3 Constructions

If $f : X' \to X$ is a smooth map between manifolds and $\mathcal{G} = (P, Y)$ is a bundle gerbe on X we can construct the pullback bundle gerbe $f^*(\mathcal{G})$ over X'. We can pullback the surjective submersion to $f^*(Y) \to X'$ with a map $\hat{f} : f^*(Y) \to Y$ covering f (a morphism of surjective submersions). There is an induced map $\hat{f}^{[2]} : f^*(Y)^{[2]} \to Y^{[2]}$ and thus we can pullback the $U(1)$ bundle $P \to Y^{[2]}$ to a $U(1)$ bundle $(\hat{f}^{[2]})^*(P) \to f^*(Y)^{[2]}$. Thus giving the bundle gerbe $f^*(\mathcal{G}) = ((\hat{f}^{[2]})^*(P), f^*(Y))$. Similarly we define the bundle gerbe multiplication by the original bundle gerbe multiplication as every fibre of the pullback bundle is that of the original $U(1)$ bundle P. If we view the bundle gerbe multiplication as a section of $\delta(P)$ then we can pullback the section $s : Y^{[2]} \to \delta(P)$ to a section $(\hat{f}^{[3]})^*(s) : (\hat{f}^*(Y))^{[3]} \to \delta((\hat{f}^{[2]})^*(P))$.

Given bundle gerbes $\mathcal{G} = (P, Y)$, $\mathcal{H} = (Q, Z)$ we can form the product bundle gerbe $\mathcal{G} \otimes \mathcal{H} := (P \otimes Q, Y \times_X Z)$ where the multiplication is defined by the product section $s \otimes t$.

Again given a bundle gerbe $\mathcal{G} = (P, Y)$ over X we can define the dual $\mathcal{G}^* = (P^*, Y)$, this is also a bundle gerbe over X where our bundle gerbe multiplication is defined by the dual multiplication s^*.

5.4 The Dixmier-Douady Class

Perhaps the most important piece of information that results from a bundle gerbe is the Dixmier-Douady class which gives a bijection between isomorphism classes of bundle gerbes on a manifold X and $H^3(M;\mathbb{Z})$. Let $\mathcal{G} = (P, Y)$ be a bundle gerbe over X. To define the Dixmier-Douady class one chooses a good cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of X such that we have local sections $s_\alpha : U_\alpha \to Y$. We have local sections $s_{\alpha\beta} = (s_\alpha, s_\beta) : U_\alpha \cap U_\beta \to Y^{[2]}$. We now choose sections $\sigma_{\alpha\beta}$ of $P_{\alpha\beta} = s^*_{\alpha\beta}(P)$, precisely these are functions $\sigma_{\alpha\beta} : U_{\alpha\beta} \to P$ such that $\sigma_{\alpha\beta}(x) \in P_{(s_\alpha(x), s_\beta(x))}$. Thus over triple overlaps $U_{\alpha\beta\gamma}$ we define a Čech co-chain $g_{\alpha\beta\gamma}$ by

$$m(\sigma_{\alpha\beta}(x), \sigma_{\beta\gamma}(x)) = g_{\alpha\beta\gamma}(x)\sigma_{\alpha\gamma}(x).$$

The fact that $g_{\alpha\beta\gamma}$ is in fact a Čech 2-cocycle follows from the associativity of the bundle gerbe multiplication. It can be shown that the class associated to $g_{\alpha\beta\gamma}$ is independent of all choices. We denote this class by

$$DD(\mathcal{G}) = [g_{\alpha\beta\gamma}] \in H^2(X, U(1)) \simeq H^3(X, \mathbb{Z}).$$

This is called the Dixmier-Douady class of $\mathcal{G}$.

Proposition 5.4.1. *Due to [17] we have that*

1. $DD(f^*(\mathcal{G})) = f^*(DD(\mathcal{G}))$.
2. $DD(\mathcal{G} \otimes \mathcal{H}) = DD(\mathcal{G}) + DD(\mathcal{H})$.
3. $DD(\mathcal{G}^*) = -DD(\mathcal{G})$.

5.5 Triviality

Given a surjective submersion $Y \to X$ and a principal $U(1)$ bundle $T \to Y$ we have that the pair $(\delta(T), Y)$ forms a bundle gerbe with bundle gerbe multiplication $Y^{[3]} \to \delta^2(T)$ given by the canonical trivialisation of $\delta^2(T)$. The bundle gerbe $(\delta(T), Y)$ is a trivial bundle gerbe.

Definition 5.5.1. *A bundle gerbe $\mathcal{G} = (P, Y)$ is* trivialisable *if there is a $U(1)$ bundle $T \to Y$ such that there is an isomorphism $\varphi : P \simeq \delta(T)$ compatible with the bundle gerbe multiplication $\delta(\varphi) \circ m = 1$. The morphism φ defines a bundle gerbe* isomorphism.

Definition 5.5.2 (Strongly Trivial Bundle Gerbe)**.** *We say a bundle gerbe is strongly trivial if there exists a section $s : Y^{[2]} \to P$ such that $\delta(s) = m$. This is the same as a trivial bundle gerbe in the usual sense with $T = U(1) \times Y$.*

Proposition 5.5.3. *A bundle gerbe $\mathcal{G}$ has zero Dixmier-Douady class if and only if it is trivialisable.*

Proof. We see an explicit description of this proof later, see lemma 6.4.4. □

Definition 5.5.4 (Stable isomorphism)**.** *Two bundle gerbes* $\mathcal{G} = (P, Y)$ *and* $\mathcal{H} = (Q, Z)$ *on* X *are called stably isomorphic if* $\mathcal{G}^* \otimes \mathcal{H}$ *is trivialisable. A choice of trivialisation for* $\mathcal{G}^* \otimes \mathcal{H}$ *is called a stable isomorphism.*

Proposition 5.5.5. *Two bundle gerbes are stably isomorphic if and only if they have the same Dixmier-Douady class [17].*

Proof. This follows directly from proposition 5.4.1 and proposition 5.5.3. Suppose that $\mathcal{G}^* \otimes \mathcal{H}$ is trivial. Then

$$\begin{aligned} 0 =& DD(\mathcal{G}^* \otimes \mathcal{H}) \\ =& - DD(\mathcal{G}) + DD(H) \\ \Longrightarrow& DD(\mathcal{G}) = DD(\mathcal{H}). \end{aligned}$$

Alternatively suppose that $DD(\mathcal{G}) = DD(\mathcal{H})$. Then we have that $DD(\mathcal{G}^* \otimes \mathcal{H}) = 0$. Therefore $\mathcal{G}^* \otimes \mathcal{H}$ is trivial. □

Proposition 5.5.6. *Stable isomorphism classes of bundle gerbes are in bijection with* $H^3(X, \mathbb{Z})$*.*

Proof. See [17, Section 4.3] on the classification of bundle gerbes. □

Chapter 6

Simplicial Extensions

6.1 Background

We will develop some of the knowledge and constructions required to be able to define a simplicial extension. The theory of simplicial extensions combines ideas from simplicial manifolds, bundle gerbes, and simplicial-Čech cohomology. We want to define the simplicial Dixmier-Douady class for a simplicial extension on a simplicial manifold $X_\bullet$ in $H^2(X_\bullet; U(1))$ and show this gives a bijection with equivalence classes of simplicial extensions. We will look at various constructions with simplicial extensions analogous to those of bundle gerbes and line bundles and how they affect the simplicial Dixmier-Douady class. We discuss some limitations in defining degree three integral simplicial cohomology.

Definition 6.1.1 (Simplicial Surjective Submersion)**.** *From [8] we define a simplicial surjective submersion to be a morphism of simplicial manifolds $Y_\bullet \xrightarrow{\pi_\bullet} X_\bullet$ such that each $\pi_i : Y_i \to X_i$ is a surjective submersion.*

Following [8] we can define maps $\mu_k : X_n \to X_0$ which are induced by the morphisms $\mu^k : [0] \to [n]$, this allows us to define a morphism $X_k \to X_0^{k+1}$ by $\mu = (\mu_0, \cdots, \mu_k)$. This morphism becomes important when talking about simplicial extensions later on. We have

Lemma 6.1.2. *[8] If $Y_\bullet \to X_\bullet$ is a simplicial surjective submersion there is a morphism of simplicial surjective submersions*

$$\mu_\bullet : (Y_\bullet \to X_\bullet) \to (\mu^{-1}(Y_0^{\bullet+1}) \to X_\bullet).$$

As with surjective submersions we have analogous constructions for simplicial surjective submersions. For example given simplicial surjective submersions $Y_\bullet \to X_\bullet$ and $Z_\bullet \to X_\bullet$ we can form the simplicial surjective submersion $Y_\bullet \times_{X_\bullet} Z_\bullet \to X_\bullet$. Similarly we can also form the p-fold fibre product $Y_\bullet^{[p]} \to X_\bullet$ naturally with face and degeneracy maps defined by repeating the face and degeneracy maps p times.

Definition 6.1.3. *Given a simplicial manifold $X_\bullet$ and a line bundle (or $U(1)$ bundle) $Q \to X_k$ we can define a line bundle $\delta(Q) \to X_{k+1}$ defined by*

$$\delta(Q) = d_0^{-1}(Q) \otimes d_1^{-1}(Q)^* \otimes d_2^{-1}(Q) \otimes \cdots$$

Using this we have the following fact.

Proposition 6.1.4. *Given a line bundle ($U(1)$ bundle) over X_k, the line bundle $\delta^2(Q) \to X_{k+2}$ is canonically trivial.*

Proof. This follows directly from the simplicial identities. For simplicity of notation we will write Q^* as $-Q$. We have that

$$\begin{aligned}
\delta^2(Q) &= \delta\left(\bigotimes_{p=0}^{k}(-1)^p d_p^{-1}(Q)\right) \\
&\simeq \bigotimes_{p=0}^{k}(-1)^p \delta\left(d_p^{-1}(Q)\right) \\
&\simeq \bigotimes_{p=0}^{k}\bigotimes_{q=0}^{k+1}(-1)^{p+q}(d_p d_q)^{-1}(Q)
\end{aligned}$$

where each $\simeq$ is a canonical isomorphism. Using the fact that $d_p d_q = d_{q-1} d_p$ for $p < q$ we can split the tensor product into two cases

$$\begin{aligned}
\delta^2(Q) &\simeq \left(\bigotimes_{p<q}(-1)^{p+q}(d_p d_q)^{-1}(Q)\right) \otimes \left(\bigotimes_{p\geq q}(-1)^{p+q}(d_p d_q)^{-1}(Q)\right) \\
&\simeq \left(\bigotimes_{p<q}(-1)^{p+q}(d_{q-1} d_p)^{-1}(Q)\right) \otimes \left(\bigotimes_{p\geq q}(-1)^{p+q}(d_p d_q)^{-1}(Q)\right)
\end{aligned}$$

and we can see that for the same reason the square simplicial boundary operator is zero we get that for each $(d_p d_q)^{-1}(Q)$ there is a $(d_p d_q)^{-1}(Q)^*$ and so as $(d_p d_q)^{-1}(Q \otimes Q^*)$ is canonically trivial we have that $\delta^2(Q)$ is canonically trivial. □

We will also be talking about bundle gerbes over a manifold X_k in our simplicial manifold. If we have a simplicial surjective submersion $Y_\bullet \to X_\bullet$ and a bundle gerbe $\mathcal{G} = (P, Y_k)$ over X_k we define $\delta(\mathcal{G}) = (\delta(P), Y_{k+1})$. The multiplication in $\delta(\mathcal{G})$ is defined by the alternating pullback of the bundle gerbe multiplication $m : Y^{[3]} \to d(P)$.

Proposition 6.1.5. *Let $Y_\bullet \to X_\bullet$ be a simplicial surjective submersion and $\mathcal{G} = (P, Y_k)$ be a bundle gerbe over X_k. If $\delta(\mathcal{G}) = (\delta(P), Y_{k+1})$ has a trivialisation $T \to Y_{k+1}$ then $\delta(T) \to Y_{k+2}$ descends to a line bundle A_T on X_{k+2}.*

Proof. Notice that the bundle gerbe $\delta^2(\mathcal{G})$ is canonically trivialised by $U(1) \times Y_{k+2} \to Y_{k+2}$ and so we have that there are two trivialisations for $\delta^2(\mathcal{G})$. Thus we have descent information for $\delta(T) \otimes (U(1) \times Y_{k+2})^*$ which is canonically isomorphic to the $U(1)$ bundle $\delta(T)$ and thus $\delta(T)$ descends. □

6.2 Definition of a Simplicial Extension

With the results from the previous section in hand we are now in a position to define the simplicial extension.

Definition 6.2.1 (Simplicial Extension [8]). *Let $X_\bullet$ be a simplicial manifold and $\mathcal{G} = (P, Y)$ be a bundle gerbe on X_0. A simplicial extension of $\mathcal{G}$ over $X_\bullet$ is a triple $(Y_\bullet, T, t)$ consisting of*

1. *$Y_\bullet \to X_\bullet$ a simplicial surjective submersion with $Y_0 = Y$ as in the bundle gerbe $\mathcal{G}$;*
2. *a trivialisation $T \to Y_1$ of $\delta(\mathcal{G}) = (\delta(P), Y_1)$ over X_1; and*
3. *a section $t : X_2 \to A_T$ of the descended line bundle $\delta(T)$, satisfying $\delta(t) = 1$ with respect to the canonical trivialisation of $\delta(A_T)$.*

Definition 6.2.2 (Simplicial bundle gerbe). *This is simply an amalgamation of all of the above information into one object. We call a simplicial bundle gerbe $\mathcal{G}_\bullet$ on $X_\bullet$ the quadruple $(P, Y_\bullet, T, t)$, where $(P, Y_0) = \mathcal{G}_0$ is a bundle gerbe on X_0 and the remaining information is a simplicial extension thereof.*

6.3 Constructions of Simplicial extensions

Given a principal bundle P on Y_0 for some simplicial surjective submersion $Y_\bullet \to X_\bullet$ there are two ways we can define $\delta(P)$, either $\delta(P) \to Y_0^{[2]}$ or $\delta(P) \to Y_1$. In future use we will refer to the former by $d(P) \to Y_0^{[2]}$ see Definition 4.4.1, and the latter by $\delta(P) \to Y_1$ see Definition 6.1.3. This is meant to represent the relation between Čech and simplicial differentials.

Proposition 6.3.1 (Dual). *Given a simplicial extension $(Y_\bullet, T, t)$ of $\mathcal{G}$ over $X_\bullet$ we can define a dual extension $(Y_\bullet, T^*, t^*)$ for $\mathcal{G}^*$ over $X_\bullet$.*

Proof. Notice that $\mathcal{G}^* = (P^*, Y_0)$, we firstly need to show that T^* is a trivialisation for $\delta(\mathcal{G}^*)$. Note that $\delta(P^*)$ is canonically isomorphic to $\delta(P)^*$. We have that $d(T) \simeq \delta(P)$, similarly we have $d(T)^* \simeq \delta(P)^*$ and so $d(T^*) \simeq \delta(P^*)$, thus T^* is a trivialisation of $\delta(\mathcal{G}^*)$. We have that $\delta(T^*) \simeq \delta(T)^*$ and so descends canonically to A_T^*, we naturally have the section $t^* : X_2 \to A_T^*$, and so we have defined a dual extension $\mathcal{G}_\bullet^*$. □

Proposition 6.3.2 (Product). *Given simplicial bundle gerbes $\mathcal{G}_\bullet = (P, Y_\bullet, T, t)$ and $\mathcal{H}_\bullet = (Q, Z_\bullet, R, r)$ we can define a product simplicial extension, denoted $\mathcal{G} \otimes \mathcal{H}$, and hence a product simplicial bundle gerbe $\mathcal{G}_\bullet \otimes \mathcal{H}_\bullet = (P \otimes Q, Y_\bullet \times_{X_\bullet} Z_\bullet, T \otimes R, t \otimes r)$.*

Proof. Firstly note that $\delta(P \otimes Q)$ is canonically isomorphic to $\delta(P) \otimes \delta(Q)$. We have that $d(T) \simeq \delta(P)$ and $d(R) \simeq \delta(Q)$ and so $d(T \otimes R) \simeq \delta(P \otimes Q)$ over $Y_1^{[2]} \times_{X_1} Z_1^{[2]}$. Similarly we have that $\delta(T)$ descends to A_T and $\delta(R)$ descends to A_R on X_2. Note that $\delta(T \otimes R)$ is canonically isomorphic to $\delta(T) \otimes \delta(R)$ and so $\pi^*(A_T \otimes A_R) \simeq \pi^*(A_T) \otimes \pi^*(A_R) \simeq \delta(T) \simeq \delta(R) \simeq \delta(T \otimes R)$. Therefore we have that $A_{T \otimes R}$ is canonically isomorphic to $A_T \otimes A_R$ and so there exists a section $t \otimes r : X_2 \to A_{T \otimes R}$. $\square$

Proposition 6.3.3 (Pullback). *Given a simplicial bundle gerbe $\mathcal{G}_\bullet$ over $X_\bullet$ and a morphism of simplicial surjective submersions $f_\bullet : Z_\bullet \to Y_\bullet$ we can define the pullback simplicial bundle gerbe $f^{-1}(\mathcal{G}_\bullet) = \Big((f_0^{[2]})^{-1}(P), Z_\bullet, f_1^{-1}(T), t\Big)$ over $X_\bullet$.*

Proof. As per the last two proofs we have that the pullback, dual, and tensor product canonically commute and so $d(f_1^{-1}(T)) = f_1^{-1}(d(T)) \simeq f_1^{-1}(\delta(P)) \simeq \delta(f_0^{-1}(P))$. The second important note here is that $\delta(T)$ and $\delta(f^{-1}(T))$ descend to the same line bundle A_T on X_2. $\square$

Notice that this can be extended to the case where we have a simplicial smooth map $X_\bullet \to Y_\bullet$ and pull back a simplicial extension of $\mathcal{G}$ on Y_0 to a simplicial extension of $f_0^*(\mathcal{G})$ on X_0.

Similarly to the case of bundle gerbes we will see that these operations will give rise to the appropriate operations in the cohomology setting once the simplicial Dixmier-Douady class is defined. Similarly once defining a notion of triviality we will be able to define 'simplicial stable isomorphism' which will give a bijection to the second degree $U(1)$ simplicial-Čech cohomology.

6.4 The Simplicial Dixmier-Douady Class

We wish to show results on cocycles related to manipulating bundle gerbes and the objects to which they belong. This will aid in defining and computing the Simplicial Dixmier-Douady class.

Lemma 6.4.1. *Given a simplicial manifold $X_\bullet$ and a line bundle $P \to X_k$, the transition functions of the line bundle $\delta(P) \to X_{k+1}$ are precisely the alternating pullback by face maps of the transition functions of P to X_{k+1}.*

Proof. Let $\mathcal{U}^{(\bullet)}$ be a good simplicial cover of $X_\bullet$. Therefore we can find local sections $s_\alpha^{(k)} : U_\alpha^{(k)} \to P$ and transition functions $s_\alpha^{(k)} = s_\beta^{(k)} g_{\alpha\beta}$. We can define local sections

$\delta(s)_\alpha^{(k+1)} : X_{k+1} \to \delta(P)$ by

$$\delta(s)_\alpha^{(k+1)} = d_0^*(s_{d_0(\alpha)}^{(k)}) \otimes d_1^*(s_{d_1(\alpha)}^{(k)})^* \otimes \cdots \otimes d_0^*(s_{d_{k+1}(\alpha)}^{(k)})^{*+k}$$

notice that this defines a section $\delta(s)_\alpha^{(k+1)} : U_\alpha^{(k+1)} \to \delta(P)$ due to the simplicial condition on our good cover

$$U_\alpha^{(k+1)} \subseteq d_i^{-1}\left(U_{d_i(\alpha)}^{(k+1)}\right).$$

So now we have that

$$\begin{aligned}
\delta(s)_\beta^{(k+1)} h_{\alpha\beta} &= \delta(s)_\alpha^{(k+1)} \\
&= d_0^*(s_{d_0(\alpha)}^{(k)}) \otimes d_1^*(s_{d_1(\alpha)}^{(k)})^* \otimes \cdots \otimes d_0^*(s_{d_{k+1}(\alpha)}^{(k)})^* \\
&= d_0^*(s_{d_0(\beta)}^{(k)} g_{d_0(\alpha)d_0(\beta)}) \otimes d_1^*(s_{d_1(\beta)}^{(k)} g_{d_1(\alpha)d_1(\beta)})^* \otimes \cdots \otimes d_{k+1}^*(s_{d_1(\beta)}^{(k)} g_{d_{k+1}(\alpha)d_{k+1}(\beta)})^* \\
&= \delta(s)_\beta^{(k)} g_{d_0(\alpha)d_0(\beta)} g_{d_1(\alpha)d_1(\beta)}^{-1} \cdots g_{d_{k+1}(\alpha)d_{k+1}(\beta)}^{(-1)^{k+1}} \\
&= \delta(s)_\beta^{(k)} \delta(g)_{\alpha\beta}
\end{aligned}$$

and so the transition functions $h_{\alpha\beta} = \delta(g)_{\alpha\beta}$. □

Lemma 6.4.2. *Given a bundle gerbe $\mathcal{G} = (P, Y_k)$ on X_k, the Dixmier-Douady class of $\delta(\mathcal{G}) = (\delta(P), Y_{k+1})$ is given by $\delta(DD(\mathcal{G})) = DD(\delta(\mathcal{G}))$. Where δ in the left hand side is the simplicial boundary operator and on the right hand side is the alternating pullback by face maps of $\mathcal{G}$.*

Proof. Consider the bundle gerbe $d_i^*(\mathcal{G}) = ((\widehat{d_i}^{[2]})^*(P), d_i^*(Y_k))$. Notice that there exists a unique map $Y_{k+1} \to d_i^*(Y_k)$ as they both satisfy the same pullback diagram

$$\begin{tikzcd}
 & & Y_{k+1} \arrow[dll, bend right, "d_i"'] \arrow[dl, dashed] \arrow[ddl, bend left] \\
Y_k \arrow[d] & d_i^*(Y_k) \arrow[l] \arrow[d] & \\
X_k & X_{k+1} \arrow[l, "d_i"] &
\end{tikzcd}$$

therefore we have that $(\widehat{d_i}^{[2]})^*(P) \to d_i^*(Y_k)^{[2]}$ pulled back by this unique map is precisely equal to the $U(1)$ bundle $d_i^*(P) \to Y_{k+1}^{[2]}$ pulled back by the map $d_i : Y_{k+1} \to Y_k$. Therefore we have that the bundle gerbe $d_i^*(\mathcal{G})$ is equal to the pullback over the identity morphism on X_{k+1} with covering map defined by this unique map

$$\begin{tikzcd} d_i^*(Y_k) \arrow[d] & Y_{k+1} \arrow[l, dashed] \arrow[d] \\ X_{k+1} & X_{k+1} \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$

so we have that the bundle gerbe $(d_i^*(P), Y_{k+1})$ has Dixmier-Douady class described by $DD(\mathrm{id}^*(d_i^*(\mathcal{G}))) = DD(d_i^*(\mathcal{G}))$. This is convenient as this means that the cocycles which describe $(d_i^*(P), Y_{k+1})$ are precisely the pullback of the cocycles of $\mathcal{G}$ by d_i. From this we can perform the following computation

$$\begin{aligned} \delta(DD(\mathcal{G})) &= d_0^{-1}(DD(\mathcal{G})) - d_1^{-1}(DD(\mathcal{G})) + \cdots \\ &= DD(d_0^{-1}(\mathcal{G})) + DD(d_1^{-1}(\mathcal{G}^*)) + \cdots \\ &= DD(\delta(\mathcal{G})) \end{aligned}$$

and we are done. □

Lemma 6.4.3. *We can assign a Čech 0 co-chain to a trivialisation of a $U(1)$ bundle such that the Čech derivative of the 0 cochain gives precisely the transition functions of P.*

Proof. Given a $U(1)$ bundle $P \to X$ take a good open cover of X and define transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to U(1)$ for P from local sections $\sigma_\alpha : U_\alpha \to P$. Now given a trivialisation of P, some section $s : X \to P$ define the following 0 cochain $s_\alpha : U_\alpha \to U(1)$ via

$$s|_{U_\alpha} s_\alpha = \sigma_\alpha.$$

We have that

$$\begin{aligned} \sigma_\alpha &= \sigma_\beta g_{\alpha\beta} \\ s|_{U_{\alpha\beta}} s_\alpha &= s|_{U_{\alpha\beta}} s_\beta g_{\alpha\beta} \\ s_\alpha s_\beta^{-1} &= g_{\alpha\beta} \end{aligned}$$

and so $g_{\alpha\beta}$ is in the image of the 0 cochain $(s_\alpha)_{\alpha\in I}$. □

The above lemma serves the purpose of a warm-up for the same process but working with bundle gerbes as below. Both are required to be able to define the simplicial Dixmier-Douady class. There will be some difficulty in showing the uniqueness of such sections.

Lemma 6.4.4. *We can assign a Čech 1 co-chain to a trivialisation of a bundle gerbe such that the Čech derivative of this cochain is precisely the representative of the Dixmier-Douady class of $\mathcal{G}$.*

Proof. Let $\mathcal{G} = (P, Y)$ be a bundle gerbe with trivialisation (T, φ) such that

$$\varphi \circ 1 = m : Y^{[3]} \to U(1)$$

where m is the bundle gerbe multiplication of $\mathcal{G}$ and 1 is the canonical trivialisation of $d^2(T)$. Let $\{U_\alpha\}_{\alpha \in I}$ be a good cover of the base space M. Choose local sections $s_\alpha : U_\alpha \to Y$. Notice that this gives us local sections

$$(s_\alpha, s_\beta) : U_{\alpha\beta} \to Y^{[2]}$$

and we extend this to local sections $\sigma_{\alpha\beta} : U_{\alpha\beta} \to P$. We can then compare these local sections to the bundle gerbe multiplication in order to define a cocycle $g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)$

$$\sigma_{\beta\gamma} \otimes \sigma^*_{\alpha\gamma} \otimes \sigma_{\alpha\beta} = m \circ (s_\alpha, s_\beta, s_\gamma) g_{\alpha\beta\gamma}.$$

We also have local sections of T

$$h_\alpha : U_\alpha \to T$$

through the sections s_α. Notice that we get sections $h_\beta \otimes h^*_\alpha : U_{\alpha\beta} \to d(T)$. Now through the isomorphism φ we get that

$$\varphi(h_\beta \otimes h^*_\alpha)\zeta_{\alpha\beta} = \sigma_{\alpha\beta}$$

and so notice that

$$\begin{aligned}
\sigma_{\beta\gamma} \otimes \sigma^*_{\alpha\gamma} \otimes \sigma_{\alpha\beta} &= \varphi(h_\gamma \otimes h^*_\beta)\zeta_{\beta\gamma} \otimes \varphi(h_\gamma \otimes h^*_\beta)^* \zeta^{-1}_{\alpha\gamma} \otimes \varphi(h_\beta \otimes h^*_\alpha)\zeta_{\alpha\beta} \\
&= (d(\varphi) \circ 1) d(\zeta)_{\alpha\beta\gamma} \\
&= m d(\zeta)_{\alpha\beta\gamma} \\
&= m g_{\alpha\beta\gamma}
\end{aligned}$$

and so by uniqueness we have that $d(\zeta)_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}$ and we are done. □

Proposition 6.4.5. *Given a simplicial bundle gerbe $\mathcal{G}_\bullet = (P, Y_\bullet, T, t)$ we can define a class $DD_\bullet(\mathcal{G}_\bullet) := [\omega_0 \oplus \omega_1 \oplus \omega_2] \in H^2(X_\bullet, U(1))$ which is unique up to choices.*

Proof. Firstly we define ω_0 to be the Dixmier-Douady class of the bundle gerbe $\mathcal{G} = (P, Y_0)$ which is unique up to 1-cochains on X_0. Secondly we define ω_1 to be the cochain given by the trivialisation T of $\delta(\mathcal{G})$ as described in Lemma 6.4.4. We have that $d(\omega_1) = \delta(\omega_0)$ by the fact that $\delta(\omega_0)$ is the representative of the Dixmier-Douady class for $\delta(\mathcal{G})$ which is trivialised by $d(\omega_1)$ as in the discussion of Lemma 6.4.4. Similarly we have that $\delta(T)$ descends to a line bundle A_T on X_2 and has trivialisation t, as in the discussion of Lemma 6.4.3 we have that ω_2 on X_2 is a 0 cochain given by the trivialisation of A_T and that $d(\omega_2) = \delta(\omega_1)$ as it trivialises the representative for $\delta(\omega_1)$. Finally we have that

$\delta(\omega_2) = 0$ as $\delta(t) = 1$ on X_3. Therefore we have that $(d+\delta)(\omega_0 \oplus \omega_1 \oplus \omega_2) = 0$ and thus we have a class $[\omega_0 \oplus \omega_1 \oplus \omega_2] \in H^2(X_\bullet, U(1))$.

Finally we wish to show that this class is unique of all the choices that have been made up to the image of simplicial 1 cochains $(\rho_0 \oplus \rho_1)$. We will leave this until later once we have notions of triviality and isomorphism and show that these are unique. Specifically this comes up in Theorem 6.5.4. □

Proposition 6.4.6. *Given a class $[\omega_0 \oplus \omega_1 \oplus \omega_2] \in H^2(X_\bullet, U(1))$ we can construct a simplicial extension of the bundle gerbe given by $[\omega_0] \in H^2(X_0, U(1))$ such that calculating the class as in Proposition 6.4.5 gives us the class we started with.*

Proof. Take a good simplicial cover $\mathcal{U}^{(\bullet)}$ of $X_\bullet$ and construct the simplicial surjective submersion $\mu^{-1}(Y^{\bullet+1}_{\mathcal{U}^{(0)}})$. There exists a line bundle P over $Y^{[2]}_{\mathcal{U}^{(0)}}$ which is given by the Čech 2-cochain ω_0 and defines a bundle gerbe $(P, Y_{\mathcal{U}^{(0)}})$. Notice that the alternating pullback of the bundle gerbe $(P, Y_{\mathcal{U}^{(0)}})$ onto X_1 is precisely the bundle gerbe $(\delta(P), \mu^{-1}(Y^2_{\mathcal{U}^{(0)}}))$, thus the Čech 2-cochain representing this bundle gerbe is precisely $\delta(\omega_0)$. Explicitly this cochain is constructed by taking the local sections of $\mu^{-1}(Y^2_{\mathcal{U}^{(0)}})) \to X_1$ defined by $s^{(1)}_i := (s_{\mu_0(i)} \circ \mu_0, s_{\mu_1(i)} \circ \mu_1)$ which defines a section $U^{(1)}_i \to \mu^{-1}(Y^2_{\mathcal{U}^{(0)}}))$ and defines the class $\delta(\omega_0)$ explicitly.

Now that we have the bundle gerbe $(\delta(P), \mu^{-1}(Y^2_{\mathcal{U}^{(0)}}))$ with class $\delta(\omega_0)$, we know that $d(\omega_1) = \delta(\omega_0)$, in particular the bundle gerbe $\delta(\mathcal{G})$ is trivial. Using the cochain ω_1 we construct a trivialisation of $(\delta(P), \mu^{-1}(Y^2_{\mathcal{U}^{(0)}}))$ called T. We have that the cochain representing T is precisely ω_1 as discussed in Lemma 6.4.4. We do the same for the trivialisation of $\delta(T)$ as discussed in Lemma 6.4.3. This results in a simplicial extension of the bundle gerbe $\mathcal{G}$ with class $[\omega_0 \oplus \omega_1 \oplus \omega_2]$. □

6.5 Triviality and Isomorphism

Now we wish to define a notion of trivialisation for a simplicial extension $\mathcal{G}_\bullet$ and its respective simplicial Dixmier-Douady class. As there is already a bundle gerbe $\mathcal{G}$ on X_0 the first part of a trivial simplicial extension is a trivialisation of $\mathcal{G}$.

Definition 6.5.1 (Trivial simplicial extension)**.** *Given a bundle gerbe $\mathcal{G}$ on X_0 and a simplicial extension $\mathcal{G}_\bullet = (Y_\bullet, T, t)$ over $X_\bullet$ we say that $\mathcal{G}_\bullet$ is trivial if there exists a trivialisation R of $\mathcal{G}$ on X_0 such that $T \otimes \delta(R)^*$ descends to a bundle $T \oslash \delta(R)$ that is trivialised by some section $r : X_1 \to T \oslash \delta(R)$ and $\delta(r) = s \otimes 1$ where 1 is the canonical section of $\delta^2(R) \to X_2$ which is the canonical descent of $\delta^2(R) \to Y_2$. The pair (R, r) is a called a trivialisation and we say that the simplicial extension $\mathcal{G}_\bullet$ is trivialisable.*

Proposition 6.5.2. *A simplicial extension $\mathcal{G}_\bullet$ is trivial if and only if its simplicial Dixmier-Douady class is trivial.*

Proof. Suppose that $\mathcal{G}_\bullet$ is trivial. Then we have a trivialisation R of $\mathcal{G}$ on X_0. We know that from R we get a Čech 1-cochain ρ_0 such that $d(\rho_0) = \omega_0$. The section r also gives us a Čech 0-cochain ρ_1 such that $\delta(\rho_1) = \omega_2$, this is because $\delta(r) \equiv s \otimes 1$. We just need to show that $\delta(\rho_0) + d(\rho_1) = \omega_1$. We have local sections $s_\alpha^{(1)} : U_\alpha^{(1)} \to Y_1$ and local sections $s'_\alpha : U_\alpha^{(1)} \to T$ such that $s'_\alpha = s'_\beta h_{\alpha\beta}$ for $h_{\alpha\beta} : U_{\alpha\beta}^{(1)} \to U(1)$, this is the cochain assigned to our trivialisation of $\delta(\mathcal{G})$. We naturally have that the line bundle $\delta(R)^* \otimes T$ can be assigned the cochain $(\delta(g))_{\alpha\beta}^{-1} h_{\alpha\beta}$ where g is the cochain assigned to the trivialisation R. Furthermore we have that $d(\delta(g)^{-1}h) = 1$ thus defining a line bundle, in particular this line bundle is $T \oslash \delta(R)$ that has cocycle $\delta(g)_{\alpha\beta}^{-1} h_{\alpha\beta}$, we have that r trivialises this bundle and we have that the cochain k_α assigned to r, when taking the Čech derivative we get that $d(k)_{\alpha\beta} = h_{\alpha\beta}\delta(g)_{\alpha\beta}^{-1}$ and so $d(k) + \delta(g) = h$ in particular this says that $\delta(\rho_0) + d(\rho_1) = \omega_1$ so every trivial simplicial extension has trivial class.

Now suppose that $\omega_0 \oplus \omega_1 \oplus \omega_2$ is trivial. In particular we have a $\rho_0 \oplus \rho_1$ such that $(d+\delta)(\rho_0 \oplus \rho_1) = \omega_0 \oplus \omega_1 \oplus \omega_2$. As $d(\rho_0) = \omega_0$ we have a trivialisation of $\mathcal{G}$ with transition functions given by ρ_0 as discussed in Lemma 6.4.4. Furthermore as $\delta(\rho_0) + d(\rho_1) = \omega_1$ we have that ρ_1 defines a trivialisation of the line bundle $\omega_1 - \delta(\rho_0)$ which is defined by $T \oslash \delta(R)$, thus defining a section $r : X_1 \to T \oslash \delta(R)$ such that the 0-cochain as in *Lemma* 6.4.3 assigned to r is precisely ρ_1, by definition we have that $\delta(\rho_1) = \omega_2$ and so $\delta(r) = s$ and we are done. □

Proposition 6.5.3. *We have the following results for classes of simplicial extensions;*

1. $DD_\bullet(\mathcal{G}_\bullet \otimes \mathcal{H}_\bullet) = DD_\bullet(\mathcal{G}_\bullet) + DD_\bullet(\mathcal{H}_\bullet)$,
2. $DD_\bullet(\mathcal{G}_\bullet^*) = -DD_\bullet(\mathcal{G}_\bullet)$,
3. $DD_\bullet(f^*(\mathcal{G}_\bullet)) = f^*(DD_\bullet(G_\bullet))$.

Proof. (1): Notice that $T \otimes S$ trivialises $\delta(P) \otimes \delta(Q)$. A cochain representing T and a cochain representing S multiplied together will result in a cochain given by $T \otimes S$. Furthermore as the Čech derivative of either of these cochains gives rise to a representative of the DD class of $\delta(P)$ and $\delta(Q)$ their product gives rise to the representative of the DD class of $\delta(P) \otimes \delta(Q)$. This technique is repeated for the remainder of the simplicial DD class.

(2): Again we notice that T^* trivialises $\delta(P^*) \simeq \delta(P)^*$. So the cochain we obtain is given by the formula as in Lemma 6.4.4

$$\begin{aligned}(\varphi(h_\beta \otimes h_\alpha^*)\zeta_{\alpha\beta})^* &= (\sigma_{\alpha\beta})^* \\ \varphi((h_\beta \otimes h_\alpha^*)^*)\zeta_{\alpha\beta}^{-1} &= (\sigma_{\alpha\beta})^*\end{aligned}$$

thus describing the cochain which trivialises the representative for $\delta(P)^*$.

(3): Notice that the pullback f^* and the Čech derivative d commute and so we have that $df^*(\zeta) = f^*(d(\zeta)) = f^*(g)$, the rest of this proof follows in a similar fashion. □

Theorem 6.5.4. *The module $H^2(X_\bullet, U(1))$ is in bijection with the set of isomorphism classes of bundle gerbes $\mathcal{G}$ with simplicial extensions $\mathcal{G}_\bullet$.*

Proof. This follows directly from Propositions 6.4.5, 6.4.6, 6.5.2 and 6.5.3.

Propositions 6.4.5 and 6.4.6 tells us that for each class in $H^2(X_\bullet, U(1))$ there exists a corresponding bundle gerbe and simplicial extension thereof. Propositions 6.5.2 and 6.5.3 then tells us that this correspondence defines a bijection between stable isomorphism classes of simplicial extensions and $H^2(X_\bullet, U(1))$. □

6.6 Simplicial Extensions of A Bundle Gerbes

Given some abelian Lie group A we can classify A bundle gerbes by A-valued Čech cohomology $H^2(M, A)$. Similarly we should be able to classify extensions of A valued simplicial extensions with A-valued simplicial cohomology.

Proposition 6.6.1. *If we have an A valued function $f : X_0 \to A$ on a simplicial manifold such that $\delta(f) : X_1 \to A$ is precisely the zero map, we can classify such objects up to homotopy equivalence in f by $H^0(X_\bullet, A)$.*

Proof. Take a good simplicial cover of $X_\bullet$, $\mathcal{U}^{(\bullet)}$ and let $\alpha_i = f|_{U_i^{(0)}}$. As f is a function globally defined on X_0 we have that $d(\alpha_i) = 0$, and as $\delta(f) = 0$ we have that $\delta(\alpha) = \delta(f)_{U_i^{(1)}}\ \delta(\alpha) = 0$ and so $[\alpha]$ defines a class in $H^0(X_\bullet, A)$.

Let $\alpha \in H^0(X_\bullet, A)$, as $d(\alpha) = 0$ the function $f|_{U_i^{(0)}} := \alpha_i$ is well defined. Furthermore as $\delta(\alpha)_i = \delta(f)|_{U_i^{(1)}}$ f describes a function as mentioned in the hypothesis. □

Example 6.6.2. *Proposition 6.6.1 says that an element of degree* 0 *Čech equivariant cohomology (which we will see in Chapter 7) is a function $f : M \to A$ such that $f(mg)^{-1}f(m) = 1$, in particular $f(mg) = f(m)$. This is to say that f is invariant under the action of G on M.*

Proposition 6.6.3. *There is a bijection between classes $[\alpha^{(0)} \oplus \alpha^{(1)}] \in H^1(X_\bullet, A)$ and isomorphism classes of principal A bundles $P \to X_0$ and a trivialisation $s : X_1 \to \delta(P)$ such that $\delta(s) = 1$ with respect to the canonical trivialisation of $\delta^2(P) \to X_2$.*

Proof. Let (P, s) be as described, we will assign a 1-cocycle to (P, s) which is an element of $H^1(X_\bullet, A)$. Let $\mathcal{U}^{(\bullet)}$ be a good simplicial cover of $X_\bullet$. Take local sections $s_i : U_i^{(0)} \to P$ and thus define $\alpha_{ij}^{(0)} : U_{ij}^{(0)} \to A$ uniquely by the equation

$$s_i(x) = s_j(x)\alpha_{ij}^{(0)}(x)$$

furthermore we have that

$$\begin{aligned} s_i(x) &= s_j(x)\alpha^{(0)}_{ij}(x) \\ &= s_k(x)\alpha^{(0)}_{jk}\alpha^{(0)}_{ij}(x) \\ &= s_i(x)\alpha^{(0)}_{ki}(x)\alpha^{(0)}_{jk}(x)\alpha^{(0)}_{ij}(x) \end{aligned}$$

over $U^{(0)}_{ijk}$, thus satisfying the cocycle condition. Furthermore we take sections $s^{(1)}_i : X_1 \to \delta(P)$ defined by $s^{(1)}_i = (s^{(0)}_{d_0(i)} \circ d_0)^* \otimes (s^{(0)}_{d_1(i)} \circ d_1) : U^{(1)}_i \to \delta(P)$. We then compare our local sections $s^{(1)}_i$ to our global section $s : X_1 \to \delta(P)$ to generate $\alpha^{(1)}$ as so

$$s(x) = s^{(1)}_i(x)\alpha^{(1)}_i.$$

Notice that

$$s^{(1)}_i(x) = s^{(1)}_j(x)(\alpha^{(1)}_i)^{-1}\alpha^{(1)}_j$$

and so $d(\alpha^{(1)}) = \delta(\alpha^{(0)})$. Furthermore as $\delta(s) = 1$ we have that $\delta(\alpha^{(1)})_i = 1$, thus defining a cocycle as required.

Given a cocycle $[\alpha^{(0)} \oplus \alpha^{(1)}]$ we can form a principal A bundle $P \to X_0$ via the clutching construction. Similarly as $d(\alpha^{(1)}) = \delta(\alpha^{(0)})$ we produce a global section of $\delta(P)$ such that $\delta(s) = 1$. □

Similar to above we can continue this process for A bundle gerbes and thus classify $H^2(X_\bullet, A)$ by following an analogous procedure to that of Proposition 6.4.5. Furthermore one could consider extensions of bundle p-gerbes over a simplicial manifold $X_\bullet$, and then classify the extension of a bundle p-gerbe using $H^{p+1}(X_\bullet, U(1))$.

6.7 On the 2-Category of Simplicial Extensions

We can consider the category of bundle gerbes with simplicial extensions over a simplicial manifold $X_\bullet$, we will denote this category by $\mathbf{BG}_\bullet(X_\bullet)$. The morphisms in $\mathbf{BG}_\bullet(X_\bullet)$, $\mathcal{G}_\bullet \to \mathcal{H}_\bullet$ are defined by trivialisations of $\mathcal{G}^*_\bullet \otimes \mathcal{H}_\bullet$. Furthermore we can define morphisms between trivialisations $(R, r) \Rightarrow (S, s)$ which will be discussed here.

Proposition 6.7.1. $\mathbf{BG}_\bullet(X_\bullet)$ *forms a category.*

Proof. We already have our notion of objects and morphisms. Furthermore we have the identity morphism $\mathcal{G}_\bullet \to \mathcal{G}_\bullet$ is the canonical trivialisation of $\mathcal{G}^*_\bullet \otimes \mathcal{G}_\bullet$. Notice that $\mathcal{G}^*_\bullet \otimes \mathcal{G}_\bullet \otimes \mathcal{G}^*_\bullet \otimes \mathcal{H}_\bullet$ is canonically identified with $\mathcal{G}^*_\bullet \otimes \mathcal{H}_\bullet$ in this way, this shows that the canonical trivialisation is the identity morphism. This result also follows directly from the fact that $P \otimes U(1) \cong P$. Finally we need to show that composition of morphisms (tensor product) is associative. Suppose that $(R, r) : \mathcal{G}_\bullet \to \mathcal{H}_\bullet$, $(S, s) : \mathcal{H}_\bullet \to \mathcal{K}_\bullet$, and $(T, t) : \mathcal{K}_\bullet \to \mathcal{L}_\bullet$. Due to the tensor product $\otimes$ being associative we have that

$$((T \otimes S) \otimes R, (t \otimes s) \otimes r) = (T \otimes (S \otimes R), t \otimes (s \otimes r))$$

and so composition is associative and thus $\mathbf{BG}_\bullet(X_\bullet)$ forms a category. $\square$

Knowing that $\mathbf{BG}_\bullet(X_\bullet)$ forms a category allows us to then understand the 2-category of simplicial extensions and bundle gerbes over $X_\bullet$.

Proposition 6.7.2. $\mathbf{BG}_\bullet(X_\bullet)$ *forms a* 2*-category.*

Proof. We define a morphism between morphisms $\varphi : (R, r) \to (T, t)$ to be a trivialisation of the descended line bundle $(T \oslash R)$. Naturally we have the identity morphism $1 : T \oslash T \cong X_1 \times U(1)$ and vertical composition defined by the tensor product of two trivialisations. Our horizontal composition is also defined by the tensor product of trivialisations. $\square$

Chapter 7

Equivariant Bundle Gerbes

7.1 Background

Given a smooth lie group action of G on a manifold X we wish to consider the equivariant cohomology $H^n_G(X;\mathbb{Z})$ of X. We define $H^n_G(X;\mathbb{Z}) \coloneqq H^n(X \times_G EG;\mathbb{Z})$ and notice that $H^n(X \times_G EG;\mathbb{Z}) \simeq H^n(EG(X)_\bullet;\mathbb{Z})$, thus we are able to relate equivariant cohomology to simplicial cohomology. We will then relate $H^2_G(X;\mathbb{Z})$ and $H^3_G(X;\mathbb{Z})$ to associated equivariant line bundles and equivariant bundle gerbes respectively.

7.1.1 Equivariant Line Bundles

Let M be a manifold equipped with a smooth action of a Lie group G. We will consider the simplicial manifold $EG(M)_\bullet$ with face and degeneracy maps given in Example 3.1.6. Firstly we will look at the lower dimensional case of equivariant line bundles in order to compare and apply techniques to their more complex counterpart, the equivariant bundle gerbe. Firstly we will define strong and weak actions of a Lie group G on a line bundle L.

Definition 7.1.1 (Strongly Equivariant Line Bundle). *A strong action of G on a line bundle $L \to M$ is a G action on L for which the projection map is equivariant. We also require that the $\mathbb{C}^\times$ action on L and the G action on L commute.*

Definition 7.1.2 (Weakly Equivariant Line Bundle). *A weak action on a line bundle $L \to M$ is a non-vanishing section $s : M \times G \to d_0^*(L) \otimes d_1^*(L)^*$ such that $\delta(s) = 1$. Notice that this mimics the definition of a simplicial extension for a line bundle over the simplicial manifold $EG(M)_\bullet$.*

Note that the definition of a weak action alludes to something similar to a simplicial extension of a line bundle over $EG(M)_\bullet$, mirroring the definition presented in [8] but in a lower dimension. To relate the two we have the following

Proposition 7.1.3. *A strong action of $L \to M$ induces a weak action on $L \to M$.*

Proof. We can describe the section $s : M \times G \to \delta(L)$ by $s(m,g) = pg \otimes p^*$ for some non-zero $p \in L_m$. All we need to show is that this is well defined. Pick another $q \in L_m$ we will show that $pg \otimes p^* = qg \otimes q^*$, since $p, q \in L_m$ there exists a $z \in \mathbb{C}$ such that $p = qz$ and so we have that

$$\begin{aligned}(pg \otimes p^*) &= (pg \otimes p^*)zz^{-1} \\ &= (pgz \otimes p^*)\, z^{-1} \\ &= pzg \otimes p^* z^{-1} \\ &= pzg \otimes (pz)^* \\ &= qg \otimes q^*\end{aligned}$$

and so the section is well defined. Now we can check the condition $\delta(s) = 1$

$$\begin{aligned}\delta(s)(m,g,h) &= s(mg,h) \otimes s(m,gh)^* \otimes s(m,g) \\ &= pgh \otimes (pg)^* \otimes (pgh \otimes p^*)^* \otimes pg \otimes p^* \\ &= pgh \otimes (pg)^* \otimes (pgh)^* \otimes p \otimes pg \otimes p^* \\ &= 1\end{aligned}$$

with respect to the canonical trivialisation. So we have that a strong action of G on L induces a weak action on L. This proof is the same for line bundles and $U(1)$ bundles. □

7.1.2 Triviality and Isomorphism

Definition 7.1.4 (Weak Triviality). *We say that a weakly equivariant line bundle (L, s) is trivial if there exists a non-vanishing section $t : M \to L$ such that $\delta(t) = s$.*

Definition 7.1.5 (Strong G Triviality). *We say that a strongly equivariant line bundle $L \to M$ is trivial if there exists an equivariant non-vanishing section $t : M \to L$.*

Proposition 7.1.6. *A strong G action is trivial if and only if it is trivial as a weak G action.*

Proof. Suppose that $L \to M$ is strongly G trivial. We already have a section t of $L \to M$ and so we just need to prove that $\delta(t)(m,g) = s(m,g)$ as defined in the proof for proposition 7.1.3. We have that

$$\begin{aligned}\delta(t)(m,g) &= t(mg) \otimes t(m)^* \\ &= t(m)g \otimes t(m)^* \\ &= pg \otimes p^* \\ &= s(m,g).\end{aligned}$$

Now suppose that $L \to M$ is weakly trivial as a weak action on L. We wish to show that the trivialisation t is equivariant. Note that $\delta(t)(m,g) = pg \otimes p^*$. In particular if we choose $p = t(m)$ we get

$$\begin{aligned}
\delta(t)(m,g) &= pg \otimes p^* \\
\delta(t)(m,g) &= t(m)g \otimes t(m)^* \\
t(mg) \otimes t(m)^* &= t(m)g \otimes t(m)^* \\
t(mg) &= t(m)g
\end{aligned}$$

and so we have that t is equivariant and thus $L \to M$ is strongly G trivial. □

7.1.3 An Equivalence of Actions

Proposition 7.1.7. *If there is a weak action of G on $L \to M$ then there is an induced strong action of G on $L \to M$.*

Proof. We have a section $s : M \times G \to \delta(L)$. We define an action of G on L through our section s in the following way. Let $p \in L$ and let $g \in G$, then there is a unique element $q \in L_{mg}$ such that $s(m,g) = q \otimes p^*$, define $pg := q$. Now if we pick another element in the fibre $pz \in L_m$ we have also defined a G action implicitly on pz. We have

$$\begin{aligned}
q \otimes p^* &= (q \otimes p^*)\, zz^{-1} \\
&= qz \otimes (pz)^*
\end{aligned}$$

and we define $(pz)g = qz$ consistent with the definition above. This also gives us the fact that the G and $\mathbb{C}$ actions commute as we have

$$\begin{aligned}
(pz)g &= qz \\
&= (pg)z.
\end{aligned}$$

Finally we must show that this action is associative. This will primarily use the fact that $\delta(s) = 1$. We can see that

$$\begin{aligned}
1 &= \delta(s)(m,g,h) \\
&= s(mg,h) \otimes s(m,gh)^* \otimes s(m,g) \\
&= (pg)h \otimes (pg)^* \otimes (p(gh) \otimes p^*)^* \otimes pg \otimes p^* \\
p(gh) \otimes p^* &= (pg)h \otimes (pg)^* \otimes pg \otimes p^* \\
p(gh) &= (pg)h.
\end{aligned}$$

Using the fact that the $\mathbb{C}$ action and G action commutes it can then be shown for any $p \in L_m$ suppose that $qz = p$ then

$$\begin{aligned} p(gh) &= (pg)h \\ (qz)(gh) &= ((qz)g)h \\ q(gh)z &= ((qg)z)h \\ q(gh)z &= ((qg)h)z \\ q(gh) &= (qg)h \end{aligned}$$

thus every weak action on a line bundle gives rise to a strong action on a line bundle. This proof as again applicable to G-equivariant $U(1)$ bundles with the same definitions. $\square$

Definition 7.1.8. *We say that two strongly equivariant line bundles P and Q are strongly isomorphic if there exists a line bundle isomorphism $\varphi : P \to Q$ which is also equivariant. We say that two weakly equivariant line bundles (P, t), (Q, s) are weakly isomorphic if there exists an isomorphism $\varphi : P \to Q$ such that $\delta(\varphi) \circ t = s$. These two notions are equivalent in the sense that if P and Q are strongly isomorphic then they are weakly isomorphic and vice versa.*

Proposition 7.1.9. *Let P and Q be strongly equivariant line bundles on M. If P and Q are strongly isomorphic then they are weakly isomorphic. Furthermore, if P and Q are weakly isomorphic then they are also strongly isomorphic.*

Proof. Let $\varphi : P \to Q$ be an isomorphism of strongly equivariant line bundles. We have that this induces an isomorphism between (P, t) and (Q, s) as weakly equivariant line bundles through $\varphi : P \to Q$, we just need to show that $\varphi \circ t = s$. Notice that $t(m, g) = pg \otimes p^*$. We have that

$$\begin{aligned} \delta(\varphi) \circ t(m, g) &= \varphi(pg) \otimes \varphi(p)^* \\ &= \varphi(p)g \otimes \varphi(p)^* \\ &= s(m, g) \end{aligned}$$

and so induces an isomorphism of weakly equivariant line bundles.

Furthermore given an isomorphism $\varphi : P \to Q$ of weakly equivariant line bundles we can show that φ induces an isomorphism of strongly equivariant line bundles with smooth G actions defined by the sections t and s. Notice that pg is defined by the section $t(m, g) = q \otimes p^*$ where $pg := q$. So we have that

$$\begin{aligned} \varphi(pg) &= \varphi(q) \\ &= \varphi(p)g \end{aligned}$$

as $s(m, g) = \varphi(q) \otimes \varphi(p)^*$ and $\varphi(q) := \varphi(p)g$ thus φ is an isomorphism of strongly equivariant line bundles. Notice that we do not edit any of the information in φ so if we were to turn φ into a weakly equivariant isomorphism and then back into a strongly equivariant isomorphism we would get φ back again. This indicates that the functor turning strongly equivariant line bundles in the weakly equivariant line bundles is a categorical isomorphism. $\square$

Proposition 7.1.10. *There is a categorical equivalence between the category of strongly and weakly equivariant line bundles.*

Proof. Let $L \to M$ be a strongly equivariant line bundle. We want to show that the weak G action defined by the section $s(m, g) = pg \otimes p^*$ of $\delta(L) \to M \times G$ will give the original G action back. In particular we defined our section $s(m, g) = pg \otimes p^*$. If we note the proof of proposition 7.1.3 we define the action of G on p to be the left hand side of the tensor in the section $s(m, g)$ which is exactly our original G action pg.

Let (L, s) be a weakly equivariant line bundle. We define pg through $s(m, g) = q \otimes p^*$ to be $q = pg$. So when we reconstruct the section $s(m, g) := pg \otimes p^*$ this is precisely $q \otimes p^*$ as we started with and so it is exactly the same weakly equivariant line bundle.

Notice that the functor sending strongly equivariant line bundles to weakly equivariant line bundles and vice-versa does not affect morphisms and thus this is also an isomorphism on morphisms. $\square$

7.2 G-Equivariant Pre-H Bundles

If $P \to M$ is a principal H bundle we can form a pre-H bundle (P, f) given by the construction in Proposition 4.5.3. In particular the map $f : P^{[2]} \to H$ is the map given by comparing two elements $p, q \in P$, we have that $\delta(f) : P^{[3]} \to H$ is the trivial map. Similarly we can produce an H bundle out of a pre-H bundle (Y, f) given by the clutching construction. We wish to have an analogue of this theory for G-equivariant principal H-bundles.

Definition 7.2.1. *Given a Lie group action of G on M, a G-equivariant principal H bundle is a H bundle $P \to M$ with a Lie group action of G on P such that the map $P \to M$ is equivariant and the actions of G and H on P commute.*

Definition 7.2.2. *Given a Lie group action of G on M, a G-equivariant pre-H bundle is a pre-H bundle (Y, f) such that $Y \to M$ and $f : Y^{[2]} \to H$ are both G-equivariant.*

Proposition 7.2.3. *Given a G-equivariant principal H-bundle $P \to M$ where $(pg)h = (ph)g$ we can form a pair (Y, f) where $Y \to M$ is a surjective submersion and $f : Y^{[2]} \to H$ is a G invariant map with $\delta(f) = 1$.*

Proof. As in the construction given in Proposition 4.5.3 for a regular pre-H bundle we have the function $f : P^{[2]} \to H$ is given by the comparison map. We need to show that this map is G-invariant. Let $p, q \in P^{[2]}$ and $g \in G$

$$\begin{aligned} p &= qf(p,q) \\ pg &= qf(p,q)g \\ pg &= qgf(p,q) \\ pg &= qgf(pg,qg) \end{aligned}$$

and so $f(p,q) = f(pg,qg)$. As $\delta(f) = 1$ per normal we have a G-invariant pre-H bundle. □

Proposition 7.2.4. *Given a G-equivariant pre-H bundle (Y, f), performing the clutching construction yields a G-equivariant H-bundle with commuting G and H actions.*

Proof. Define $P = (Y \times H)/\sim$ where $(y_1, h_1) \sim (y_2, h_2)$ if $\pi(y_1) = \pi(y_2)$ and $f(y_1, y_2) = h_2 h_1^{-1}$. This is the standard clutching construction thus forming a principal H bundle on M. The H action on P is defined by $[y, h_1]h_2 = [y, h_1 h_2]$, we define a G action on P by $[y, h]g = [yg, h]$. We can see that $(pg)h = (ph)g$ based on this definition. We need to verify that this action is well defined and is equivariant through π. Suppose that $(y_1, h_1) \sim (y_2, h_2)$, we wish to show that $(y_1 g, h_1) \sim (y_2 g, h_2)$. Notice we immediately have that $\pi(y_1 g) = \pi(y_1)g = \pi(y_2)g = \pi(y_2 g)$ and because f is invariant we have that $f(y_1 g, y_2 g) = h_2 h_1^{-1}$. So the G action on P is well defined and the projection mapping is automatically equivariant because the map $Y \to M$ is equivariant. □

7.3 Definitions

Definition 7.3.1 (Strongly Equivariant Bundle Gerbe). *Given a smooth Lie group action of G on M, a strongly equivariant bundle gerbe with respect to this action is a bundle gerbe $\mathcal{G} = (P, Y)$, with Lie group actions on P and Y such that the morphisms in the bundle gerbe are all equivariant.*

In particular the Lie group action of G on Y induces a Lie group action on $Y^{[k]}$ in which the projection maps are equivariant. Furthermore the compatibility condition with morphisms in the bundle gerbe requires the fact that the projections $P \to Y^{[2]}$ and $Y \to M$ are both equivariant morphisms. We also require that the section $s : Y^{[3]} \to \delta(P)$ defining the bundle gerbe multiplication is equivariant.

Definition 7.3.2 (Weakly Equivariant Bundle Gerbe). *If G has a smooth action on M and $\mathcal{G}$ is a bundle gerbe on M, then a weakly equivariant bundle gerbe on M is a simplicial extension of $\mathcal{G}$ over the space $EG(M)_\bullet$.*

Proposition 7.3.3. *A strong action on $\mathcal{G}$ induces a weak action on $\mathcal{G}$. We will call the induced weak action on $\mathcal{G}$ the 'weak variant' of the strongly equivariant bundle gerbe $\mathcal{G}$.*

Proof. We get that $\delta(\mathcal{G})$ is trivialisable by the bundle $(Y \times G) \times U(1)$ with canonical section $X_2 \to A_T$, the details of this proof are covered in [8]. □

7.4 Triviality and Isomorphism

We already have a notion of triviality and isomorphism for weakly equivariant gerbes and through this we would recover a definition for triviality of a strongly equivariant gerbe. Firstly we start with a definition from chapter 5.

Definition 7.4.1. *A strongly equivariant bundle gerbe $\mathcal{G}$ on M is trivial if there exists a $U(1)$ bundle $T \to Y$ such that $d(T) \simeq P$ (recall this notation from Definition 4.4.1). A choice of T and isomorphism $d(T)$ is called a trivialisation.*

However we will soon find that this definition is not sufficient for defining a trivialisation in the equivariant sense. So we make the following definition for strongly equivariant gerbes

Definition 7.4.2 (G-Trivial)**.** *A strongly equivariant bundle gerbe is called G-trivial if there exists a $U(1)$ bundle $T \to Y$ with a G action which covers all projection maps such that $d(T) \simeq P$. We require that the isomorphism $d(T) \simeq P$ is equivariant.*

Proposition 7.4.3. *If a strongly equivariant bundle gerbe $\mathcal{G}$ is G-trivial then it is trivial as a simplicial extension over $EG(M)_\bullet$.*

Proof. As we already have a trivialisation of the bundle gerbe $\mathcal{G}$ from the G-trivialisation, to generate the rest of the trivialisation for the simplicial extension we require a trivialising section t of $\delta(T) \oslash U(1) \simeq \delta(T)$ such that $\delta(t) = 1$ (see Definition 6.5.1 for the definition of $P \oslash Q$). We will first take a section $s : Y \times G \to \delta(T)$ and then show this section descends to a section on $\delta(T) \oslash U(1)$.

We have that $\delta(T) \simeq d_0(T) \otimes d_1(T)^*$ which on fibres will look like $\delta(T)_{(y,g)} \simeq T_y \otimes T^*_{yg}$. As G acts smoothly on T we can define an isomorphism $1 \otimes (g^{-1})^* : T_y \otimes T^*_{yg} \to T_y \otimes Y^*_y$ which is naturally isomorphic to $U(1)$. So we simply have that $\delta(T) \simeq U(1)$ and therefore we can choose the canonical section 1 for the $U(1)$ bundle $U(1) \oslash \delta(T) \to M \times G$, naturally we have that $\delta(1) = 1$ and so we have that the G-trivial strongly equivariant bundle gerbe is trivial in the weakly equivariant sense. □

The main part of the above proof that is not possible with regular triviality is the fact that we can get a trivialisation of $\delta(T)$ that descends to $M \times G$, this is only possible with the smooth G-action on T that covers projections. We also wish to ask the converse of proposition 7.4.3.

Proposition 7.4.4. *If $\mathcal{G}$, a strongly equivariant gerbe, is trivial as a weakly equivariant gerbe then it is G-trivial.*

Proof. Suppose (S, s) is our trivialisation for the weak variant of the strongly equivariant bundle gerbe $(EG(M)_\bullet, U(1), 1)$. We have that $\delta(s) = 1$ and so we have that (S, s) is a weakly equivariant line bundle on $EG(Y)_\bullet$ and so we have a G action on S which trivialises $\mathcal{G}$ and so $\mathcal{G}$ is G-trivial. □

Now that we have notions of triviality we can define stable isomorphism in the obvious manner.

Definition 7.4.5 (G-Stable isomorphism)**.** *We say that two strongly equivariant bundle gerbes $\mathcal{G} = (P, Y)$ and $\mathcal{H} = (Q, Z)$ are G-stably isomorphic if the strongly equivariant gerbe $\mathcal{G}^* \otimes \mathcal{H}$ is G-trivial. A choice of trivialisation is then called a G-stable isomorphism.*

Remark. *Similarly we also have that two weakly equivariant gerbes are isomorphic if they are isomorphic in the simplicial sense.*

7.5 The Universal Strongly Equivariant Bundle Gerbe

We wish to now classify the isomorphism classes of strongly equivariant bundle gerbes over a manifold M equipped with a smooth Lie group action of G on the right. We will call the set of G-stable isomorphism classes of strongly equivariant bundle gerbes $\mathrm{BGrb}_G(M)$. We will take an element of $H^3_G(M;\mathbb{Z})$ and then assign it a G-stable isomorphism class of equivariant bundle gerbes. In what follows we will often shorten $M \times_G EG$ to M_G.

7.5.1 Injectivity

Proposition 7.5.1. *Given a G-equivariant $U(1)$ principal bundle P on M we can form a $U(1)$ principal bundle $P \times_G EG$ on $M \times_G EG$. This operation is well defined on tensor products and duals. Furthermore this assignment extends to define a functor $Pic^\infty_G(M) \to Pic^\infty(M \times_G EG)$.*

Proof. Let $P \to M$ and $Q \to M$ be equivariant $U(1)$ bundles on M. Notice that $(P^*)_G \simeq (P_G)^*$ through the fact that as manifolds $P^* = P$ and the Borel construction does not affect the group action of $U(1)$ on P. Now we wish to describe an isomorphism $\varphi : (P \otimes Q)_G \to P_G \otimes Q_G$. Let $p \in P$, $q \in Q$ and $x \in EG$, we have that $[p \otimes q, x] \in (P \otimes Q)_G$. We define $\varphi([p \otimes q, x]) := ([p, x] \otimes [q, x])$, this morphism respects the $U(1)$ action and is well defined. We need to verify that it is an isomorphism. We have an inverse function $\psi([p, x] \otimes [q, y]) = [p \otimes qg, x]$ where $g \in G$ is given uniquely by $x = yg$, the element $g \in G$ is given by the fact that

$$[\pi_P(p), x] = [\pi_Q(q), y]$$

and so there exists some unique $g \in G$ such that $(\pi_P(p), x) = (\pi_Q(q)g, yg)$ by the fact that the action of G on EG is free. This gives us the fact that $\pi(p) = \pi(qg)$. We have

$$\begin{aligned}
\varphi \circ \psi([p, x] \otimes [q, y]) &= \varphi([p \otimes qg, x]) \\
&= [p, x] \otimes [qg, x] \\
&= [p, x] \otimes [q, xg^{-1}] \\
&= [p, x] \otimes [q, y] \\
\psi \circ \varphi([p \otimes q, x]) &= \psi([p, x] \otimes [q, x]) \\
&= [p \otimes q, x]
\end{aligned}$$

and so we can see that they are inverse functions of each other thus defining an isomorphism. One can then see that this is a functor through the fact that a morphism $\varphi : P \to Q$ gives rise to a morphism $\varphi_G : P_G \to Q_G$ in the appropriate manner. $\square$

Proposition 7.5.2. *Given a strongly equivariant bundle gerbe $\mathcal{G} = (P, Y)$ on M we can form a bundle gerbe $\mathcal{G}_G = (Y_G, P_G)$ on M_G.*

Proof. The equivariant multiplication $m : Y^{[3]} \to \delta(P)$ gives rise to a multiplication $m \times_G EG : Y^{[3]} \times_G EG \to \delta(P) \times_G EG$. There is an isomorphism of $U(1)$-principal bundles $\delta(P) \times_G EG \cong \delta(P \times_G EG)$, let $(y_1, y_2, y_3) \in Y^{[3]}$, $p_i \in P_{(y_j, y_k)}$ where $i \neq j \neq k$, and let $x \in EG$, we define the isomorphism by

$$\begin{aligned}
&[[p_1, y_2, y_3] \otimes [p_2, y_1, y_3]^* \otimes [p_3, y_1, y_2], x] \\
&\mapsto [p_1, y_2, y_3, x] \otimes [p_2, y_1, y_3, x]^* \otimes [p_3, y_1, y_2, x].
\end{aligned}$$

We have that the construction $P \mapsto P \times_G EG$ preserves the structure of a $U(1)$ bundle and so $\mathcal{G} \times_G EG$ forms a bundle gerbe as required. $\square$

Remark. *Notice that the assignment $\mathcal{G} \mapsto \mathcal{G} \times_G EG$ extends to define a functor*

$$BGrb_G(M_G) \to BGrb(M).$$

We have that a G-trivialisation T of $\mathcal{G}$ will give a trivialisation of $\mathcal{G} \times_G EG$.

Proposition 7.5.3. *The functor $\mathcal{G} \mapsto \mathcal{G} \times_G EG$ is well defined on isomorphism classes.*

Proof. Suppose that $\mathcal{G} = (P, Y)$ is G-trivial with equivariant trivialisation (T, φ). We have that $T \times_G EG \to Y \times_G EG$ is a $U(1)$ bundle and $\varphi \times_G EG : \delta(T \times_G EG) \to P \times_G EG$ is an isomorphism of $U(1)$ bundles. Now we need to show that $(\mathcal{G} \otimes \mathcal{H}) \times_G EG \simeq (\mathcal{G} \times_G EG) \otimes (\mathcal{H} \times_G EG)$. Let $\mathcal{H} = (Q, Z)$ and $\mathcal{G}$ be as before. We are comparing the bundle gerbes $((P \otimes Q) \times_G EG, (Y \times_M Z) \times_G EG)$ and $((P \times_G EG) \otimes (Q \times_G EG), (Y \times_G EG) \times_{M \times_G EG} (Z \times_G EG))$. Firstly notice that there exists $f : (Y \times_M Z) \times_G EG \to (Y \times_G EG) \times_{M \times_G EG} (Z \times_G EG)$ defined by

$$f\left([(y, z), x]\right) = ([y, x], [z, x])$$

Notice that

$$\pi[y, x] = [\pi(y), x] = [\pi(z), x] \in M \times_G EG$$

and

$$f([(yg, zg), xg]) = ([yg, xg], [zg, xg]) = ([y, x], [z, x])$$

so f is well defined. Furthermore f is a bijection, let $([y, x_1], [z, x_2]) \in (Y \times_G EG) \times_{M \times_G EG} (Z \times_G EG)$, we have that

$$\begin{aligned} \pi[y, x_1] &= \pi[z, x_2] \\ [\pi(y), x_1] &= [\pi(z), x_2] \\ (\pi(y), x_1)g &= (\pi(z), x_2) \end{aligned}$$

for some $g \in G$. Therefore we have that $x_1 g = x_2$ and $\pi(yg) = \pi(z)$ and so

$$\begin{aligned} f\left([(yg, z), x_1 g]\right) &= ([yg, x_1 g], [z, x_1 g]) \\ &= ([y, x_1], [z, x_2])\,. \end{aligned}$$

Alternatively suppose that

$$\begin{aligned} f([(y_1, z_1), x_1]) &= f([(y_2, z_2), x_2]) \\ ([y_1, x_1], [z_1, x_1]) &= ([y_2, x_2], [z_2, x_2]) \end{aligned}$$

as these two equivalence classes are equal then there exists $g \in G$ such that

$$(y_1 g, x_1 g), (z_1 g, x_1 g)) = ((y_2, x_2), (z_2, x_2))$$

and so we have that $(y_1, z_1, x_1)g = (y_2, z_2, x_2)$ and thus $[(y_1, z_1), x_1] = [(y_2, z_2), x_2]$ thus making f injective.

Similarly we can show that there is an isomorphism of $U(1)$ bundles

$$\begin{aligned} \varphi : (P \otimes Q) \times_G EG &\to (P \times_G EG) \otimes (Q \times_G EG) \\ [p \otimes q, x] &\mapsto [p, x] \otimes [q, x] \end{aligned}$$

this is both well defined and a homomorphism of $U(1)$ bundles. Finally we have that $P^* \times_G EG \cong (P \times_G EG)^*$ as this dual operation is unaffected by the functor $- \times_G EG$. This shows that $\mathcal{G} \mapsto \mathcal{G} \times_G EG$ is well defined. □

Lemma 7.5.4. *Let $f, f' : X \to Y$ be a pair of equivariant maps such that $f_G = f'_G : X_G \to Y_G$. We have that $f = f'$.*

Proof. Let $x \in X$ and $e \in EG$, notice that

$$\begin{aligned} f_G[x,e] &= [f(x), e] \\ &= f'_G[x,e] \\ &= [f'(x), e] \end{aligned}$$

and so

$$(f(x), e) = (f'(x), e)g$$

for $g \in G$, but notice that G acts on EG freely so $g = 1$ and thus $f(x) = f'(x)$. $\square$

Lemma 7.5.5. *Let $Y \to M$ be an equivariant surjective submersion and let $f : Y^{[n]} \to \mathbb{R}$ be an equivariant smooth function with respect to the trivial action of G on $\mathbb{R}$. If $\delta(f) = 0$ then there exists some equivariant $h : Y^{[n-1]} \to \mathbb{R}$ such that $\delta(h) = f$.*

Proof. We know that the complex $C^\infty(Y^{[\bullet]}, \mathbb{R})$ has no cohomology (see [2, § 8]). So we know that there exists some map $h : Y^{[n-1]} \to \mathbb{R}$ such that $\delta(h) = f$, we wish to now find an equivariant such map. Given $g \in G$ we define the map $h^g : Y^{[n-1]} \to \mathbb{R}$ by $h^g(y) = h(yg^{-1})$. Define a map $\alpha(h) : Y^{[n-1]} \to \mathbb{R}$ defined on $y \in Y^{[n-1]}$ by

$$\alpha(h)(y) = \int_G h^g(y) dg$$

where dg is the Haar measure of G. Recall, from [18] if G is a compact Lie group then there is a measure on G such that

1. $\int_G 1 dg = 1$

2. for any continuous function $f : G \to \mathbb{R}$, $\int_G f(gh)dg = \int_G f(hg)dg = \int_G f(g)dg$ for all $h \in G$.

Given a continuous function $f : X \to \mathbb{R}$ where there is a Lie G action on X, we can define a continuous function $f^-(x_0) : G \to \mathbb{R}$ which is defined on $g \in G$ by

$$f^g(x_0) = f(x_0 g^{-1})$$

for some fixed choice of $x_0 \in X$. Therefore, we can integrate this function with respect to the Haar measure. We can prove that α commutes with equivariant pullbacks, given an equivariant map $\psi : X \to Y^{n-1}$

$$\begin{aligned}\psi^*(\alpha(h))(y) &= \int_G h^g(\psi(y))dg \\ &= \int_G h(\psi(y)g^{-1})dg \\ &= \int_G h(\psi(yg^{-1}))dg \\ &= \int_G \psi^*(h)(yg^{-1})dg \\ &= \int_G \psi^*(h)^g(y)dg \\ &= \alpha(\psi^*(h))\end{aligned}$$

and so we have that $\delta(\alpha(h)) = \alpha(\delta(h))$. The morphism $\alpha(h)$ forms an equivariant morphism. We will go into more detail about α later, notice that if f is equivariant then α will not change f. We have that

$$\begin{aligned}f &= \alpha(f) \\ &= \alpha(\delta(h)) \\ &= \delta(\alpha(h))\end{aligned}$$

and so there exists an equivariant h such that $\delta(h) = f$. $\square$

Given a Lie group action of G on spaces X and Y we can define an associative G action on the mapping space $Map(X, Y)$ defined on a function $f \in Map(X, Y)$ point-wise by the formula

$$f^g(x) = f(xg^{-1})g.$$

Notice that this action is associative as

$$\begin{aligned}(f^g)^h(x) &= f^g(xh^{-1})h \\ &= f(xh^{-1}g^{-1})gh \\ &= f(x(gh)^{-1})gh \\ &= f^{gh}(x).\end{aligned}$$

Notation 7.5.6. *We will use the shorthand $f \approx f'$ to indicate that there is a homotopy between the functions f and f'. Where context is provided this will also indicate that two equivariant maps f and f' are homotopic through an equivariant homotopy.*

Definition 7.5.7. *Given a space M we define the function*

$$\begin{aligned}\varphi_M : Map(EG, M) \times EG &\to M \\ (f, x) &\mapsto f(x)\end{aligned}$$

which we will often refer to as the evaluation map. *Notice that if there is an action of G on M this map extends naturally to*

$$\begin{aligned}\varphi_M : Map(EG, M) \times_G EG &\to M \\ [f, x] &\mapsto f(x)\end{aligned}$$

which is well defined.

For spaces X and Y with a continuous group action of G on the right we define the set

$$C_G(X, Y) \coloneqq \{f : X \to Y : f(xg) = f(x)g\}.$$

Furthermore we have that $\pi_0 C_G(X, Y)$ is the set of *equivariant* homotopy classes.

Lemma 7.5.8. *Given the trivial action of G on $U(1)$ we have that $\pi_0 C_G(M, U(1)) \simeq \pi_0 C_G(M \times EG, U(1))$.*

Proof. Define the map $P : \pi_0 C_G(M, U(1)) \to \pi_0 C_G(M \times EG, U(1))$ by

$$[f] \mapsto [f \circ p_{EG}]$$

where $p_{EG} : M \times EG \to M$ is the projection map onto M. Firstly notice that this is well defined on equivariant homotopy classes as p_{EG} is equivariant. We will show that this is bijective on equivariant homotopy classes.

Let $f, f' : M \to U(1)$ be a pair of equivariant maps in $C_G(M, U(1))$. Suppose that $P(f) = P(f')$, this is to say that $f \circ p_{EG} \approx f' \circ p_{EG}$ through some equivariant map $h : M \times EG \times [0, 1] \to U(1)$. Notice that there is an adjunction

$$Map(- \times EG, U(1)) \cong Map(-, Map(EG, U(1)))$$

achieved by sending a map $f : M \times EG \to U(1)$ to the map $\bar{f} : M \to Map(EG, U(1))$ defined by

$$\bar{f}(m) = (x \mapsto f(m, x)).$$

Notice that if f is an equivariant map then we get the fact that

$$\begin{aligned}\bar{f}(mg) &= (x \mapsto f(mg, x)) \\ &= \big(x \mapsto f(m, xg^{-1})\big) \\ &= \bar{f}(m)^g\end{aligned}$$

and so $\bar{f}$ is an equivariant map. This way we construct an equivariant homotopy

$$\bar{h} : M \times [0,1] \to Map(EG, U(1))$$

and finally a homotopy

$$\phi \circ \bar{h} : M \times [0,1] \to U(1)$$

where ϕ is the function

$$\begin{aligned} \phi : Map(EG, U(1)) &\to U(1) \\ k &\mapsto \exp\left(\int_G \widehat{k}^g(x_0)dg\right) \end{aligned}$$

where $\widehat{k} : EG \to \mathbb{R}$ is a lift of $k \in Map(EG, U(1))$ into $\mathbb{R}$ and $x_0 \in EG$ is a chosen basepoint of EG. We will show that $\phi \circ \bar{h}$ is a homotopy of f and f' as required. Computing the value of $\phi \circ \bar{h}$ at $t = 0$ we get that

$$\begin{aligned} \phi \circ \bar{h}(m, 0) &= \phi\left(\overline{f \circ p_{EG}}(m)\right) \\ &= \phi\left(x \mapsto (f \circ p_{EG})(m, x)\right) \\ &= \phi\left(x \mapsto f(m)\right) \end{aligned}$$

we will write $c_{f(m)}$ for the constant function $x \mapsto f(m)$. Notice that $c_{f(m)}$ lifts to a map $\widehat{c_{f(m)}} : EG \to \mathbb{R}$ which is a constant map with some value $\widehat{f(m)}$ where $\exp(\widehat{f(m)}) = f(m)$. So we have that

$$\begin{aligned} \phi\left(x \mapsto f(m)\right) &= \exp\left(\int_G \widehat{c_{f(m)}}^g(x_0)dg\right) \\ &= \exp\left(\int_G \widehat{c_{f(m)}}(x_0 g^{-1})dg\right) \\ &= \exp\left(\int_G \widehat{f(m)}dg\right) \\ &= \exp\left(\widehat{f(m)}\right) \\ &= f(m) \end{aligned}$$

and so we have that $\phi \circ \bar{h}(m, 0) = f(m)$ and following the exact same process we get that $\phi \circ \bar{h}(m, 1) = f'(m)$ and so $\phi \circ \bar{h}$ defines an equivariant homotopy from f to f' and thus P is injective.

Let $f : M \times EG \to U(1)$, we will show that there exists some $f' : M \to U(1)$ such that $f' \circ p_{EG} \approx f$, where $p_{EG} : M \times EG \to M$ is the projection map. This is to say that the function P is surjective. Let $f' = \phi \circ \bar{f}$. Notice that

$$\begin{aligned} \phi \circ \bar{f} \circ p_{EG}(m, x) &= (\phi \circ p_{EG}) \circ (\bar{f} \circ id_{EG})(m, x) \\ &\approx \varphi_{U(1)} \circ (\bar{f} \circ id_{EG})(m, x) \\ &= \bar{f}(m)(x) \\ &= f(m, x) \end{aligned}$$

where the homotopy in the second line is given by Lemma 7.5.17 and so $f' \circ p_{EG} \approx f$ and thus P is bijective. □

Proposition 7.5.9. *The functor $\mathcal{G} \mapsto \mathcal{G} \times_G EG$ is injective.*

Proof. Let $\mathcal{G} = (P, Y)$ be a strongly equivariant bundle gerbe on M such that $\mathcal{G} \times_G EG = (P_G, Y_G)$ is trivial. There exists some $U(1)$ bundle $T \to Y_G$ with a bundle gerbe isomorphism

$$P_G \simeq \delta(T)$$

that trivialises $\mathcal{G} \times_G EG$. From the fact that $Pic^\infty_G(Y) \simeq Pic^\infty(Y_G)$ [19] we can choose an equivariant $U(1)$-bundle R with an isomorphism $R_G \simeq T$ that induces a bundle gerbe isomorphism

$$\psi : P_G \to \delta(R)_G$$

through the isomorphism $P_G \simeq \delta(T)$. Again using the fact that $Pic^\infty_G(Y^{[2]}) \simeq Pic^\infty(Y^{[2]}_G)$ we can choose an equivariant $U(1)$-bundle isomorphism

$$\phi : P \to \delta(R)$$

such that $\phi_G \approx \psi$. We define an equivariant morphism $h : Y^{[3]} \to U(1)$ which measures the failure of ϕ to be a bundle gerbe isomorphism through the formula

$$\begin{aligned} \delta(\phi)(m) \cdot h &= 1 \\ \delta(\phi)(m) &= 1 \cdot h^{-1} \end{aligned}$$

where m is the bundle gerbe multiplication and 1 is the canonical section of $\delta^2(R)$. We define a function $f : Y^{[2]}_G \to U(1)$ by the formula

$$\psi = \phi_G \cdot f$$

notice that $(-)_G$ acts as a functor on sections and bundle morphisms, so we have

$$\begin{aligned}
1 &= \delta(\psi)(m_G) \\
&= \delta(\phi_G \cdot f)(m_G) \\
&= \delta(\phi_G)(m_G) \cdot \delta(f) \\
&= \delta(\phi)(m)_G \cdot \delta(f) \\
&= (1 \cdot h^{-1})_G \cdot \delta(f) \\
&= 1_G \cdot h_G^{-1} \cdot \delta(f)
\end{aligned}$$

We have that the canonical section 1 of $\delta^2(R)$ gives rise to the section 1_G of $\delta^2(R)_G$, which is precisely the same as the canonical section 1 of $\delta^2(R_G)$. Therefore, we have that $h_G = \delta(f) : Y_G^{[3]} \to U(1)$. Due to the fact that ϕ_G and ψ are homotopic the morphism $\phi_G\psi^{-1} = f : Y_G^{[2]} \to U(1)$ is homotopic to the constant map, and so f must lift to a morphism $\widehat{f} : Y_G^{[2]} \to \mathbb{R}$. Let $\xi \in EG$, we define a map

$$\begin{aligned}
\widehat{h}_\xi(y_1, y_2, y_3) &= \widehat{f}([y_2, y_3, \xi]) - \widehat{f}([y_1, y_3, \xi]) + \widehat{f}([y_2, y_3, \xi]) \\
&= \delta(\widehat{f})([y_1, y_2, y_3, \xi])
\end{aligned}$$

and define a morphism $\widehat{k}_\xi : Y^{[2]} \to \mathbb{R}$ by

$$\widehat{k}_\xi(y_1, y_2) = \int_G \widehat{f}(y_1, y_2, \xi g) dg.$$

we have that

1. $\widehat{k}_\xi$ is an equivariant map.

2. $\delta(k_\xi) = h$, where $k_\xi = \exp(\widehat{k}_\xi)$.

which we show in Lemma 7.5.10. With this fact we have that the isomorphism

$$\phi \cdot k : P \to \delta(R)$$

gives rise to a bundle gerbe isomorphism as we have that

$$\begin{aligned}
\delta(\phi \cdot k)(m) &= \delta(\phi)(m) \cdot \delta(k) \\
&= \delta(\phi)(m) \cdot h \\
&= 1
\end{aligned}$$

and therefore R trivialises $\mathcal{G}$ and $\mathcal{G} \mapsto \mathcal{G} \times_G EG$ is injective. $\square$

Lemma 7.5.10. *For $\hat{k}_\xi$, k_ξ, and h as defined in the proof of Proposition 7.5.9 we have that*

1. *$\widehat{k}_\xi$ is an equivariant map.*

2. *$\delta(k_\xi) = h$, where $k_\xi = \exp(\widehat{k}_\xi)$.*

Proof. (1): The action of G on $\mathbb{R}$ is the trivial action, as such we wish to prove that $\widehat{k}_\xi(y_1, y_2) = \widehat{k}_\xi(y_1 h, y_2 h)$ for all $(y_1, y_2) \in Y^{[2]}$ and $h \in G$. We have that

$$\begin{aligned}
\widehat{k}_\xi(y_1 h, y_2 h) &= \int_G \widehat{f}([y_1 h, y_2 h, \xi g])dg \\
&= \int_G \widehat{f}([y_1, y_2, \xi g h^{-1}])dg \\
&= \int_G \widehat{f}([y_1, y_2, \xi g])dg \\
&= \widehat{k}_\xi(y_1, y_2)
\end{aligned}$$

and so $\widehat{k}_\xi$ is equivariant.

(2): Notice that $\delta(k_\xi) = \exp(\delta(\widehat{k}_\xi))$. We have that

$$\begin{aligned}
&\delta(\widehat{k})(y_1, y_2, y_3) \\
&= \int_G \Big(\widehat{f}(y_2, y_3, \xi g) - \widehat{f}(y_1, y_3, \xi g) + \widehat{f}(y_1, y_2, \xi g)\Big)\, dg \\
&= \int_G \widehat{h}_{\xi g}(y_1, y_2, y_3) dg
\end{aligned}$$

we fix some $\xi_0 \in EG$. For any $\xi \in EG$ we have

$$\exp \widehat{h}_{\xi_0}(y_1, y_2, y_3) = \exp \widehat{h}_\xi(y_1, y_2, y_3)$$

as

$$\begin{aligned}
\exp \widehat{h}_{\xi_0}(y_1, y_2, y_3) &= \exp(\delta(\widehat{f}))(y_1, y_2, y_3, \xi_0) \\
&= \delta(f)(y_1, y_2, y_3, \xi_0) \\
&= h_G(y_1, y_2, y_3, \xi_0) \\
&= h_G(y_1, y_2, y_3, \xi)
\end{aligned}$$

as the function h_G is invariant under the value on EG. Using this fact we get

$$\widehat{h}_{\xi_0}(y_1, y_2, y_3) = \widehat{h}_\xi(y_1, y_2, y_3) + 2\pi n_{\xi,\xi_0}(y_1, y_2, y_3)$$

for some $n_{\xi,\xi_0}(y_1, y_2, y_3) \in \mathbb{Z}$. Holding y_1, y_2, y_3 fixed we have that

$$\xi \mapsto n_{\xi,\xi_0}(y_1, y_2, y_3)$$

defines a continuous function $EG \to \mathbb{Z}$. As EG is connected we have that this function is constant. We can then define a function $n_{\xi_0} : Y^{[3]} \to \mathbb{Z}$

$$n_{\xi,\xi_0}(y_1, y_2, y_3) = n_{\xi_0}(y_1, y_2, y_3).$$

Therefore we have that

$$\widehat{h}_{\xi g}(y_1, y_2, y_3) = \widehat{h}_\xi(y_1, y_2, y_3) + n_\xi(y_1, y_2, y_3)$$

using our definition of $\widehat{k}$ we get that

$$\begin{aligned}
\delta(\widehat{k})(y_1, y_2, y_3) &= \int_G \widehat{h}_{\xi g}(y_1, y_2, y_3) dg \\
&= \int_G \widehat{h}_\xi(y_1, y_2, y_3) + n_\xi(y_1, y_2, y_3) dg \\
&= \widehat{h}_\xi(y_1, y_2, y_3) + n_\xi(y_1, y_2, y_3) \\
\delta(k)(y_1, y_2, y_3) &= \exp\left(\widehat{h}_\xi(y_1, y_2, y_3) + n_\xi(y_1, y_2, y_3)\right) \\
&= h(y_1, y_2, y_3)
\end{aligned}$$

and so $h = \delta(k)$. □

7.5.2 The Construction of the Strongly Equivariant Bundle Gerbe

Firstly we understand that $H^3_G(M; \mathbb{Z}) \simeq [M \times_G EG, B(PU(\mathcal{H}))]$. Furthermore we have that $[M \times_G EG, B(PU(\mathcal{H}))] \simeq [M, Map(EG, B(PU(\mathcal{H})))]^G$. If we are then able to build a strongly equivariant bundle gerbe on $Map(EG, B(PU(\mathcal{H})))$ then we are able to pull back such a bundle gerbe on to M, the assignment of a map in $[M \times_G EG, B(PU(\mathcal{H}))]$ to a strongly equivariant bundle gerbe will give us an equivalence of $H^3_G(M; \mathbb{Z})$ and isomorphism classes of strongly equivariant bundle gerbes. Thus we wish to build a bundle gerbe which is universal in some sense. We will be equipping the space $Map(EG, -)$ with a right G action given by $f^g(x) \mapsto f(xg^{-1})$.

Firstly we construct the equivariant lifting bundle gerbe with respect to the equivariant central extension

$$0 \to Map(EG, U(1)) \to Map(EG, U) \to Map(EG, PU) \to 0$$

notice that this will be a strongly equivariant $Map(EG, U(1))$ bundle gerbe, however we want a strongly equivariant $U(1)$ bundle gerbe. The described equivariant bundle gerbe is defined in the following way. Our base space is $Map(EG, BPU)$ with locally split map $Map(EG, EPU) \to Map(EG, BPU)$.

Definition 7.5.11. *A right G action on a simplicial manifold (space) $X_\bullet$ is a smooth (continuous) G action on each X_k such that each morphism $X_n \to X_m$ is G-equivariant. The notion of an equivariant simplicial map is then a simplicial map $f_\bullet$ such that each f_k is equivariant.*

Using this definition we have that fact that if $X_\bullet$ is a simplicial manifold with a right G action and $P \to X_k$ is a G-equivariant $U(1)$ bundle, the bundle $\delta(P) \to X_{k+1}$ is a G-equivariant $U(1)$ bundle.

Lemma 7.5.12. *There exists a morphism*

$$c_\bullet : Map(EG, EPU(\mathcal{H}))^{[\bullet+1]} \to EMap(EG, PU(\mathcal{H}))_\bullet$$

between simplicial spaces. The fibre product is produced through the locally split map $Map(EG, EPU) \to Map(EG, BPU)$. Furthermore, this morphism is an equivariant map between simplicial spaces with G action.

Proof. We define the morphism c_0 by the only possible morphism into $\{pt.\}$. We define the G action on $Map(EG, EPU(\mathcal{H}))^{[\bullet+1]}$ and $EMap(EG, PU(\mathcal{H}))_\bullet$ by pre-composition by g^{-1} in all parts. We claim that $Map(EG, EPU(\mathcal{H}))^{[\bullet+1]}$ and $EMap(EG, PU(\mathcal{H}))_\bullet$ are both G-simplicial spaces. $c_k : Map(EG, EPU(\mathcal{H}))^{[k+1]} \to Map(EG, PU(\mathcal{H}))^k$ is given by the comparison map of the i-th and $i+1$-th elements being sent to the i-th component of $Map(EG, PU(\mathcal{H}))^k$. We can prove that this is equivariant by showing that the comparison map is equivariant, the comparison map exists due to the transitivity of the $Map(EG, PU(\mathcal{H}))$ action on fibres.

$$\begin{aligned} \zeta_1 &= \zeta_2 c(\zeta_1, \zeta_2) \\ (\zeta_1)^g &= (\zeta_2 c(\zeta_1, \zeta_2))^g \\ (\zeta_1)^g &= (\zeta_2)^g c(\zeta_1, \zeta_2)^g \end{aligned}$$

and so $c(\zeta_1, \zeta_2)^g = c(\zeta_1^g, \zeta_2^g)$ as the action of $Map(EG, PU(\mathcal{H}))$ on fibres is free. Thus we have that $c_\bullet$ defines an equivariant map of spaces with G action. □

Over the space $Map(EG, PU)$ we have the $Map(EG, U(1))$ bundle $Map(EG, U) \to Map(EG, PU)$. However we wish to construct an equivariant simplicial line bundle over $EMap(EG, PU)_\bullet$, we will use the bundle $Map(EG, U) \to Map(EG, PU)$ and construct an associated equivariant $U(1)$ bundle to this bundle.

Lemma 7.5.13. *There exists an equivariant map $Map(EG, U(1)) \to U(1)$ where G acts on $U(1)$ by the trivial action. This is to say there exists a G invariant map $Map(EG, U(1)) \to U(1)$.*

Proof. Notice that $Map(EG, \mathbb{R}) \to Map(EG, U(1))$ is a $Map(EG, \mathbb{Z})$ bundle. As EG is path connected we have that $Map(EG, \mathbb{Z}) \simeq \mathbb{Z}$ and thus $Map(EG, \mathbb{R}) \to Map(EG, U(1))$ forms a $\mathbb{Z}$-covering map. In particular this allows us to lift maps $f : EG \to U(1)$ to maps $\hat{f} : EG \to \mathbb{R}$ because EG is contractible. We can then perform the averaging process, pick a basepoint $x_0 \in EG$ and define the map $\alpha : Map(EG, \mathbb{R}) \to \mathbb{R}$ by

$$\alpha(f) = \int_G f^g(x_0)dg$$

Notice that $\alpha(f_1 + f_2) = \alpha(f_1) + \alpha(f_2)$ and $\alpha(f^g) = \alpha(f)$. So the morphism $\alpha : Map(EG, \mathbb{R}) \to \mathbb{R}$ is a well defined G-equivariant homomorphism. This morphism induces a G-equivariant homomorphism $\phi : Map(EG, U(1)) \to U(1)$. Consider the following map ϕ. We define $\phi(f)$ by taking a lift $\hat{f}$ of f and then taking $\exp \alpha(f)$. This is to say

$$\phi = \exp \circ \alpha \circ \text{Lift}$$

Where $\text{Lift} : Map(EG, U(1)) \to Map(EG, \mathbb{R})$ is the process of lifting a $U(1)$ valued map into $\mathbb{R}$. We will show that ϕ defines a G-equivariant homomorphism.

$$\begin{aligned}
\phi(f_1 + f_2) - \phi(f_1) - \phi(f_2) &= \exp \circ \alpha \left(\text{Lift}(f_1 + f_2) - \text{Lift}(f_1) - \text{Lift}(f_2)\right) \\
&= \exp \circ \alpha \left(n : EG \to \mathbb{Z}\right) \\
&= \exp\left(\int_G n^g(c_0)dg\right) \\
&= \exp(n) \\
&= 0
\end{aligned}$$

and so ϕ defines a homomorphism. We have

$$\begin{aligned}
\phi(f^h) &= \exp \circ \alpha \circ \text{Lift}(f^h) \\
&= \exp \circ \alpha(\widehat{f^h}) \\
&= \exp\left(\int_G \widehat{f}^{hg}(x_0)dg\right) \\
&= \exp\left(\int_G \widehat{f}^{g}(x_0)dg\right) \\
&= \exp \circ \alpha(\widehat{f}) \\
&= \phi(f)
\end{aligned}$$

Therefore ϕ is an equivariant homomorphism. We need to show that this is well defined under choosing different lifts. Suppose $\hat{f}_1$ and $\hat{f}_2$ are two different lifts of f. Then we have that $\hat{f}_1 - \hat{f}_2 = 2k\pi$ for some $k \in \mathbb{Z}$, notice that this is an equivariant $\mathbb{R}$ valued map so $\alpha(\hat{f}_1 - \hat{f}_2) = 2k\pi$, thus $\exp \circ \widehat{\alpha(f_1)}(\exp \circ \alpha)(\hat{f}_2)^{-1} = 1$ and so ϕ is well defined under different lifts. $\square$

Proposition 7.5.14. *There exists an equivariant simplicial line bundle on the simplicial space $E(Map(EG, PU))_\bullet$.*

Proof. An equivariant simplicial line bundle is a simplicial line bundle in which the projection morphisms and sections are equivariant, this construction only works on equivariant simplicial spaces. We can take the $Map(EG, U(1))$ bundle $Map(EG, U) \to Map(EG, PU)$ and construct the associated $U(1)$ bundle through ϕ

$$\begin{aligned} Map(EG, U) \times_\phi U(1) &\coloneqq \{(f, z) | f : EG \to U, z \in U(1)\} / \sim \\ \text{where } (f, z) &\sim (f \cdot \chi, \phi(\chi)^{-1} z),\ \chi : EG \to U(1). \end{aligned}$$

where the projection map $Map(EG, U) \times_\phi U(1) \to Map(EG, PU)$ is defined by $[f, z] \mapsto \pi(f) : EG \to PU$. Notice that this is well defined on equivalence classes due to the fact that the projection $\pi : Map(EG, U) \to Map(EG, PU)$ will eliminate any terms of $U(1)$, thus multiplication by χ will give the same function $\pi(f\chi) = \pi(f)$. Furthermore the bundle $Map(EG, U) \times_\phi U(1)$ admits a right G action defined by

$$[f, z]g \coloneqq [f^g, z].$$

Notice that this is well defined as

$$\begin{aligned} [f\chi, \phi(\chi)^{-1}z]g &= [f^g \chi^g, \phi(\chi)^{-1} z] \\ &= [f^g, \phi((\chi^g)^{-1})^{-1} \phi(\chi)^{-1} z] \\ &= [f^g, \phi((\chi)^{-1})^{-1} \phi(\chi)^{-1} z] \\ &= [f^g, \phi((\chi)\phi(\chi)^{-1} z] \\ &= [f^g, z] \\ &= [f, z]g \end{aligned}$$

using both the fact that ϕ is a homomorphism of groups and is G-invariant. Therefore we have formed an equivariant $U(1)$ bundle $Map(EG, U) \times_\phi U(1)$. We now need to show that this is an equivariant simplicial $U(1)$ bundle.

Define a section $m : Map(EG, PU^2) \to \delta(Map(EG, U) \times_\phi U(1))$ in the following way, let $f_1, f_2 \in Map(EG, PU)$ and choose $[\hat{f}_1, z_1] \in (Map(EG, U) \times_\phi U(1))_{f_1}$, $[\hat{f}_2, z_2] \in (Map(EG, U) \times_\phi U(1))_{f_2}$. Notice that $\hat{f}_1 : EG \to U$ is a lift of $f_1 : EG \to PU$, and similarly for $\hat{f}_2$. Furthermore notice that $\hat{f}_1 \hat{f}_2$ is a lift for $f_1 f_2$ so we have that

$$m(f_1, f_2) = [\hat{f}_2, z_2] \otimes [\hat{f}_1 \hat{f}_2, z_1 z_2]^* \otimes [\hat{f}_1, z_1]$$

defines a section of $\delta(Map(EG,U)\times_\phi U(1))$. We can show this is well defined. Let $[\hat{f_1}', w_1]$ and $[\hat{f_2}', w_2]$ be two alternative points in the fibre of f_1 and f_2 respectively. Due to being in the same fibre we have that $\hat{f_1} = \hat{f_1}'\chi_1$ and $\hat{f_2} = \hat{f_2}'\chi_2$ where $\chi_1, \chi_2 : EG \to U(1)$, and thus

$$\begin{aligned} m(f_1, f_2) &= [\hat{f_2}', w_2] \otimes [\hat{f_1}'\hat{f_2}', w_1 w_2]^* \otimes [\hat{f_1}', w_1] \\ &= [\hat{f_2}'\chi_2, \phi(\chi_2)^{-1}w_2] \otimes [\hat{f_1}'\hat{f_2}'\chi_1\chi_2, \phi(\chi_1)^{-1}\phi(\chi_2)^{-1}w_1w_2]^* \otimes [\hat{f_1}'\chi_1, \phi(\chi_1)^{-1}w_1] \\ &= [\hat{f_2}'\chi_2, \phi(\chi_2)^{-1}w_2] \otimes [\hat{f_1}'\chi_1\hat{f_2}'\chi_2, \phi(\chi_1)^{-1}w_1\phi(\chi_2)^{-1}w_2]^* \otimes [\hat{f_1}'\chi_1, \phi(\chi_1)^{-1}w_1] \\ &= [\hat{f_2}, \phi(\chi_2)^{-1}w_2] \otimes [\hat{f_1}\hat{f_2}, \phi(\chi_1)^{-1}w_1\phi(\chi_2)^{-1}w_2]^* \otimes [\hat{f_1}, \phi(\chi_1)^{-1}w_1] \end{aligned}$$

we have that $z_i = \phi(\chi_i)^{-1}w_i c_i$ for some $c_i \in U(1)$. Thus

$$\begin{aligned} &= [\hat{f_2}, \phi(\chi_2)^{-1}w_2] \otimes [\hat{f_1}\hat{f_2}, \phi(\chi_1)^{-1}w_1\phi(\chi_2)^{-1}w_2]^* \otimes [\hat{f_1}, w_1\phi(\chi)^{-1}]c_1c_2c_1^{-1}c_2^{-1} \\ &= [\hat{f_2}, \phi(\chi_2)^{-1}w_2c_2] \otimes [\hat{f_1}\hat{f_2}, \phi(\chi_1)^{-1}w_1c_1\phi(\chi_2)^{-1}w_2c_2]^* \otimes [\hat{f_1}, w_1\phi(\chi)^{-1}c_1] \\ &= [\hat{f_2}, z_2] \otimes [\hat{f_1}\hat{f_2}, z_1z_2]^* \otimes [\hat{f_1}, z_1] \end{aligned}$$

and so the multiplication is well defined. Furthermore we have that m is equivariant as $\hat{f}^g$ is a lift of f^g. Finally the multiplication is required to be associative, i.e. $\delta(m) = 1$, this is true due to the fact that multiplication in $Map(EG, U)$ is associative. □

Proposition 7.5.15 (Universal Strongly Equivariant Lifting Gerbe)**.** *There exists a strongly equivariant bundle gerbe (fig. 7.1) over the space $Map(EG, BPU)$, we will call this bundle gerbe the Universal strongly equivariant lifting gerbe and represent it by $\mathcal{G}_1$.*

Proof. The morphism in lemma 7.5.12 allows us to pull back the equivariant simplicial line bundle on $E(Map(EG, PU))_\bullet$ to an equivariant simplicial line bundle on $Map(EG, EPU)^{[\bullet+1]}$. This is equivalent to a strongly equivariant bundle gerbe over the base space $Map(EG, BPU)$. □

Furthermore given the universal strongly equivariant lifting gerbe we get an identification of equivariant maps $f : M \to Map(EG, BPU(\mathcal{H}))$ with strongly equivariant bundle gerbes on M. This is achieved by taking the pullback of the universal strongly equivariant lifting gerbe by the map f. We understand that the set $[M, Map(EG, BPU(\mathcal{H}))]_G$ is in bijection with $H^3_G(M;\mathbb{Z})$. We wish to show that this construction gives us a bijection between strongly equivariant bundle gerbes and degree three integral equivariant cohomology.

Figure 7.1: The universal strongly equivariant bundle gerbe

$$
\begin{tikzcd}
 & c_1^*(Map(EG,U)\times_\phi U(1)) \arrow[d] & Map(EG,U)\times_\phi U(1) \arrow[d] \\
 & Map(EG,EPU^{[2]}) \arrow[r, "c_1"] \arrow[dl, shift left] \arrow[dl, shift right] & Map(EG,PU) \arrow[dl, shift left] \arrow[dl, shift right] \\
Map(EG,EPU) \arrow[r, "c_0"] \arrow[d, "\pi"] & Map(EG,\{\mathbf{pt.}\}) & \\
Map(EG,BPU) & &
\end{tikzcd}
$$

7.5.3 Surjectivity

The notion of surjectivity here is equivalent to showing that the *universal* strongly equivariant bundle gerbe is universal as such. We want to know that this bundle gerbe is somewhat equivalent to looking at the universal lifting bundle gerbe formed by the central extension $U(1) \to U \to PU$. In order to relate these two constructions we need to look at the adjoint functors $- \times_G EG : \mathbf{Top}_G \to \mathbf{Top}$ and $Map(EG,-) : \mathbf{Top} \to \mathbf{Top}_G$.

Lemma 7.5.16. *The natural transformation $\varphi_M : Map(EG,M) \times_G EG \to M$ defined by evaluation describes the counit of the adjoint functors, $- \times_G EG \dashv Map(EG,-)$.*

Proof. We define $\varphi(f,x) = f(x)$. This is well defined as $\varphi((f,x)g) = \varphi(f^g, xg) = f(xg^{-1}g) = f(x)$. Let $\rho : X \to Y$. Notice that the diagram

$$
\begin{tikzcd}
Map(EG,X) \times_G EG \arrow[r, "\varphi_X"] \arrow[d, "Map(EG,\rho)\times_G EG"] & X \arrow[d, "\rho"] \\
Map(EG,Y) \times_G EG \arrow[r, "\varphi_Y"] & Y
\end{tikzcd}
$$

commutes due to

$$
\begin{aligned}
\rho \circ \varphi_X(f,x) &= \rho(f(x)) \\
\varphi_Y(\rho \circ f, x) &= \rho(f(x))
\end{aligned}
$$

and so is a natural transformation $Map(EG,-)\times_G EG \to 1_{\mathbf{Top}}$. One can see that this is the counit of the adjunction $-\times_G EG \dashv Map(EG,-)$. $\square$

The function $\varphi_{BPU} : Map(EG, BPU)\times_G EG \to BPU$ then allows us to pull back the universal lifting bundle gerbe on BPU to $Map(EG, BPU)\times_G EG$. We have the bundle gerbe $\mathcal{G}_1$ in proposition 7.5.15 which forms a bundle gerbe on $Map(EG, BPU)\times_G EG$ via the Borel construction $-\times_G EG$ due to proposition 7.5.2. We want to know if $\varphi^*_{BPU}(\mathcal{G}_0) \simeq \mathcal{G}_1 \times_G EG$. If this is true then we are able to use the fact that φ_{BPU} is the counit of the adjunction to say that $\varphi_{BPU}\circ(\bar{f}\times_G EG) = f : M\times_G EG \to BPU$ and so $[f^*(\mathcal{G}_0)] = [\bar{f}^*(\mathcal{G}_1)\times_G EG]$ which means that the diagram

$$\begin{tikzcd}
{[M, Map(EG,BPU)]_G} \arrow[r, "\bar{f}^*(\mathcal{G}_1)"] & \mathrm{BGrb}_G(M) \arrow[d, "-\times_G EG"] \\
H^3_G(M;\mathbb{Z}) \arrow[u, "f\mapsto\bar{f}"] \arrow[r, "f^*(\mathcal{G}_0)"] & \mathrm{BGrb}(M\times_G EG) \arrow[l, bend left, "{[\mathcal{G}]}"]
\end{tikzcd}$$

commutes. Thus as the map $f \mapsto f^*(\mathcal{G}_0)$ is an isomorphism on isomorphism classes of bundle gerbes we get that the map $\mathrm{BGrb}_G(M) \to \mathrm{BGrb}(M\times_G EG)$ is surjective and so the assignment of Dixmier-Douady classes through the functor $-\times_G EG$ is well defined.

Lemma 7.5.17. *There exists a G-equivariant homotopy $H : Map(EG, U(1)) \times EG \times [0,1] \to U(1)$ from $\varphi_{U(1)}$ to $\phi\circ p_1$.*

Proof. Let $h : EG\times[0,1]\to EG$ denote a contraction of EG with base point x_0. Through the functor $Map(-,\mathbb{R})$ we get a map

$$Map(EG,\mathbb{R}) \to Map(EG\times[0,1],\mathbb{R})$$

and by adjointness we get a map

$$Map(EG,\mathbb{R})\times EG\times[0,1]\to\mathbb{R}.$$

We then average over G to attain the map

$$\hat{H}(f,x,t) = \int_G f^g\left(h(xg,t)\right)dg$$

which is G equivariant by definition. We have the following

$$\begin{aligned}
\hat{H}(f, x, 0) &= \int_G f^g\left(h(xg, 0)\right) dg \\
&= \int_G f^g\left(xg\right) dg \\
&= \int_G f\left(xgg^{-1}\right) dg \\
&= \int_G f\left(x\right) dg \\
&= f(x) \\
&= \varphi_{\mathbb{R}}(f, x) \\
\hat{H}(f, x, 1) &= \int_G f^g\left(h(xg, 1)\right) dg \\
&= \int_G f^g\left(x_0\right) dg \\
&= \alpha(f) \\
&= \alpha \circ p_1(f, x)
\end{aligned}$$

Notice that the homotopy $\hat{H}$ is $\mathbb{Z}$-equivariant with respect to the natural $\mathbb{Z}$ actions on $Map(EG, \mathbb{R})$ and $\mathbb{R}$:

$$\begin{aligned}
\hat{H}(f + n, x, t) &= \int_G (f + n)^g(h(xg, t))dg \\
&= \int_G f^g(h(xg, t)) + ndg \\
&= \int_G f^g(h(xg, t))dg + \int_G ndg \\
&= \hat{H}(f, x, t) + n
\end{aligned}$$

Therefore, $\hat{H}$ defines an equivariant homotopy $H : Map(EG, U(1)) \times EG \times [0, 1] \to U(1)$ between $\varphi_{U(1)}$ and $\phi \circ p_1$. Similarly it also defines a homotopy $H : Map(EG, U(1)) \times_G EG \times [0, 1] \to U(1)$ between $\phi \circ p_{EG}$ and $\varphi_{U(1)}$. □

Proposition 7.5.18. *The square*

$$\begin{array}{ccc}
Map(EG, U(1)^2) \times EG \times [0, 1] & \xrightarrow{m \times id \times id} & Map(EG, U(1)) \times EG \times [0, 1] \\
\Big\downarrow{\scriptstyle (H,H)} & & \Big\downarrow{\scriptstyle H} \\
U(1)^2 & \xrightarrow{\quad m \quad} & U(1)
\end{array}$$

commutes, where "m" is the multiplication map.

Proof. To do this we need to show that H is made up of homomorphisms in each part. We will look at how H is defined in order to prove this. We have that H is the map defined by a covering map $\widehat{H} : Map(EG, \mathbb{R}) \times EG \times [0,1] \to \mathbb{R}$ which is defined in lemma 7.5.17. Notice that we have

$$\widehat{H}(f, x, t) \coloneqq \int_G f^g(h_t(xg))dg$$

and so

$$\begin{aligned} \widehat{H}(f_1 + f_2, x, t) &= \int_G (f_1 + f_2)^g(h_t(xg))dg \\ &= \int_G f_1^g(h_t(xg)) + f_2^g(h_t(xg))dg \\ &= \widehat{H}(f_1, x, t) + \widehat{H}(f_2, x, t). \end{aligned}$$

and projecting through the natural action of $\mathbb{Z}$ we have that

$$H(f_1 \cdot f_2, x, t) = H(f_1, x, t)H(f_2, x, t).$$

□

From proposition 7.5.18 we are able to construct a convenient simplicial line bundle over the space $EMap(EG, PU)_\bullet \times EG \times [0, 1]$. We naturally have the G-equivariant $Map(EG, U(1))$ bundle $Map(EG, U) \times EG \times [0, 1] \to Map(EG, PU) \times EG \times [0, 1]$, we then wish to produce an associated $U(1)$ bundle through the homotopy H. We define the following

$$Map(EG, U) \times EG \times [0, 1] \times U(1)/\sim$$

where

$$(f, x, t, z) \sim (f\chi, x, t, H(\chi, x, t)^{-1}z)$$

for $\chi : EG \to U(1)$. We will call this bundle $\hat{P}$ for convenience.

Proposition 7.5.19. *The $U(1)$ bundle $\hat{P}$ forms a simplicial line bundle over the simplicial space $EMap(EG, PU)_\bullet \times EG \times [0, 1]$.*

Proof. We need to show that $\delta(\hat{P})$ admits an equivariant section s such that $\delta(s) = 1$. Let $(\hat{f}_1, \hat{f}_2, x, t)$ be a point in $Map(EG, PU^2) \times EG \times [0,1]$. We wish to study the fibre of $\delta(\hat{P})$ over this point. We have that

$$\delta(\hat{P}) = \hat{P}_{(\hat{f}_2,x,t)} \otimes \hat{P}^*_{(\hat{f}_1\hat{f}_2,x,t)} \otimes \hat{P}_{(\hat{f}_1,x,t)}$$

now if we choose two points one in the fibre of $\hat{P}_{(f_2,x,t)}$ and the other in the fibre of $\hat{P}_{(\hat{f}_1,x,t)}$, let these be $[f_2, x, t, z_2]$ and $[f_1, x, t, z_1]$ respectively. We will show that the assignment

$$(\hat{f}_1, \hat{f}_2, x, t) \mapsto [f_2, x, t, z_2] \otimes [f_1 f_2, x, t, z_1 z_2]^* \otimes [f_1, x, t, z_1]$$

generates a well defined section of $\delta(\hat{P})$, call this assignment $s(f_1, f_2, x, t)$. Let $[g_2, x, t, w_2]$ and $[g_1, x, t, w_1]$ be another two points in the fibres, i.e. an alternative assignment of $s(f_1, f_2, x, t)$. As $\pi(f_1) = \pi(g_1)$ and $\pi(f_2) = \pi(g_2)$ we have that there exists morphisms $\chi_1, \chi_2 : EG \to U(1)$ such that $f_1 = g_1\chi_1$ and $f_2 = g_2\chi_2$. Therefore we have that

$$\begin{aligned}
&[g_2, x, t, w_2] \otimes [g_1 g_2, x, t, w_1 w_2]^* \otimes [g_1, x, t, w_1] \\
=&[g_2\chi_2, x, t, H_{(x,t)}(\chi_2)^{-1} w_2] \otimes [g_1 g_2 \chi_1 \chi_2, x, t, H_{(x,t)}(\chi_1\chi_2)^{-1} w_1 w_2]^* \\
\otimes&[g_1\chi_1, x, t, H_{(x,t)}(\chi_1)^{-1} w_1] \\
=&[g_2\chi_2, x, t, H_{(x,t)}(\chi_2)^{-1} w_2] \otimes [g_1\chi_1 g_2\chi_2, x, t, H_{(x,t)}(\chi_1)^{-1} H_{(x,t)}(\chi_2)^{-1} w_1 w_2]^* \\
\otimes&[g_1\chi_1, x, t, H_{(x,t)}(\chi_1)^{-1} w_1] \\
=&[f_2, x, t, H_{(x,t)}(\chi_2)^{-1} w_2] \otimes [f_1 f_2, x, t, H_{(x,t)}(\chi_1)^{-1} H_{(x,t)}(\chi_2)^{-1} w_1 w_2]^* \\
\otimes&[f_1, x, t, H_{(x,t)}(\chi_1)^{-1} w_1]
\end{aligned}$$

now also notice that there exists $c_1, c_2 \in U(1)$ such that

$$\begin{aligned}
z_1 &= H_{(x,t)}(\chi_1)^{-1} w_1 c_1 \\
z_2 &= H_{(x,t)}(\chi_2)^{-1} w_2 c_2
\end{aligned}$$

so we have that

$$\begin{aligned}
&[f_2, x, t, H_{(x,t)}(\chi_2)^{-1} w_2] \otimes [f_1 f_2, x, t, H_{(x,t)}(\chi_1)^{-1} H_{(x,t)}(\chi_2)^{-1} w_1 w_2]^* \otimes [f_1, x, t, H_{(x,t)}(\chi_1)^{-1} w_1] \\
&= \big([f_2, x, t, H_{(x,t)}(\chi_2)^{-1} w_2] \otimes [f_1 f_2, x, t, H_{(x,t)}(\chi_1)^{-1} H_{(x,t)}(\chi_2)^{-1} w_1 w_2]^* \\
&\otimes [f_1, x, t, H_{(x,t)}(\chi_1)^{-1} w_1]\big) c_1 c_2 c_1^{-1} c_2^{-1} \\
&= [f_2, x, t, H_{(x,t)}(\chi_2)^{-1} w_2 c_2] \otimes [f_1 f_2, x, t, H_{(x,t)}(\chi_1)^{-1} H_{(x,t)}(\chi_2)^{-1} w_1 w_2 c_1 c_2]^* \\
&\otimes [f_1, x, t, H_{(x,t)}(\chi_1)^{-1} w_1 c_1] \\
&= [f_2, x, t, z_2] \otimes [f_1 f_2, x, t, z_1 z_2]^* \otimes [f_1, x, t, z_1]
\end{aligned}$$

so the section $s(\hat{f}_1, \hat{f}_2, x, t)$ is well defined and is defined globally. We can show that s is equivariant we have that

$$\begin{aligned} s(f_1, f_2, x, t)g =&([f_2, x, t, z_2] \otimes [f_1 f_2, x, t, z_1 z_2]^* \otimes [f_1, x, t, z_1])g \\ =&[f_2, x, t, z_2]g \otimes [f_1 f_2, x, t, z_1 z_2]^* g \otimes [f_1, x, t, z_1]g \\ =&[f_2^g, xg, t, z_2] \otimes [f_1^g f_2^g, xg, t, z_1 z_2]^* \otimes [f_1^g, xg, t, z_1] \\ =&s((f_1, f_2, x, t)g). \end{aligned}$$

We have that $\delta(s) = 1$ with respect to the canonical trivialisation of $\delta^2(\hat{P})$ by the fact that group multiplication is associative in $U(1)$ and $Map(EG, U)$. □

Notice that this simplicial line bundle above gives us the description of associated $U(1)$ bundle to $Map(EG, U) \to Map(EG, PU)$ through ϕ. In particular if we fix $t = 1$ we get part of the defined universal bundle gerbe $\mathcal{G}_1$.

Lemma 7.5.20. *The equivariant homotopy H describes a bundle gerbe on $Map(EG, BPU) \times_G EG \times [0, 1]$ that is stably isomorphic to $\mathcal{G}_1 \times_G EG$ when restricted to $t = 1$. Call this bundle gerbe $\widehat{\mathcal{G}}$.*

Proof. We construct this bundle gerbe in precisely the same manner as in proposition 7.5.15 by using the simplicial morphism

$$c_\bullet \times \mathrm{id} \times \mathrm{id} : Map(EG, EPU^{[\bullet+1]}) \times_G EG \times [0, 1] \to EMap(EG, PU)_\bullet \times_G EG \times [0, 1]$$

to pull back the simplicial line bundle defined in proposition 7.5.19 over the fibre products of the principal fibre bundle $Map(EG, EPU) \times_G EG \times [0, 1] \to Map(EG, BPU) \times_G EG \times [0, 1]$ thus defining a bundle gerbe. Furthermore at $t = 1$ we have that $H(f, x, 1) = \phi(f)$ and thus the simplicial line bundle in proposition 7.5.19 restricted to 1 is precisely the simplicial line bundle proposition 7.5.14 and thus the bundle gerbe described restricted to $t = 1$ is precisely equal to $\mathcal{G}_1 \times_G EG$. □

Proposition 7.5.21. *There exists a stable isomorphism between the bundle gerbes $\widehat{\mathcal{G}}|_0$ and $\widehat{\mathcal{G}}|_1$.*

Proof. Notice that as $Map(EG, U) \times_G EG \times_H [0, 1] \to Map(EG, PU) \times_G EG \times [0, 1]$ is a $U(1)$ principal bundle when restricted to each $t \in [0, 1]$. There exists some abstract bundle isomorphism $\widehat{\Phi} : Map(EG, U) \times_G EG \times_H \{0\} \to Map(EG, U) \times_G EG \times_H \{1\}$. We can pull this isomorphism back through $c_\bullet$ and see that it defines a $U(1)$ bundle isomorphism

$$\begin{tikzcd} c_1^*(Map(EG, U) \times_G EG \times_{\varphi_{U(1)}} U(1)) \arrow[r, "\Phi"] \arrow[dr] & c_1^*(Map(EG, U) \times_G EG \times_{\phi p_1} U(1)) \arrow[d] \\ & Map(EG, EPU^{[2]}) \times_G EG \end{tikzcd}$$

we then want to edit the isomorphism Φ to form a strong bundle gerbe isomorphism. Let m_0, m_1 be the bundle gerbe multiplication of $\widehat{\mathcal{G}}|_0$ and $\widehat{\mathcal{G}}|_1$ respectively. We have that there exists some function $\chi : Map(EG, EPU^{[3]}) \times_G EG \to U(1)$ such that

$$(\delta(\Phi) \circ m_0) \cdot \chi = m_1.$$

Notice that

$$\begin{aligned}
\delta((\delta(\Phi) \circ m_0) \cdot \chi) &= \delta((\delta(\Phi) \circ m_0)) \cdot \delta(\chi) \\
&= (\delta^2(\Phi) \circ \delta(m_0)) \cdot \delta(\chi) \\
&= \delta(\chi) \\
&= \delta(m_1) \\
&= 1.
\end{aligned}$$

Furthermore we can lift the function $\chi : Map(EG, EPU^{[3]}) \times_G EG \to U(1)$ to a function $\widehat{\chi} : Map(EG, EPU^{[3]}) \times_G EG \to \mathbb{R}$ such that $\delta(\widehat{\chi}) = 2k\pi$ as $\exp \delta(\widehat{\chi}) = \delta(\chi) = 1$ and the space $Map(EG, EPU^{[4]}) \times_G EG$ is path connected. Notice that there are an odd number of face maps $Map(EG, EPU^{[5]}) \to Map(EG, EPU^{[4]})$ and so $\delta^2(\widehat{\chi}) = 2k\pi$ but must also be zero by definition of δ and so $k = 0$. This means that $\delta(\widehat{\chi}) = 0$ and as $C^\infty_G(Map(EG, EPU^{[\bullet+1]}), \mathbb{R})$ is a complex with no cohomology there exists some $\widehat{\zeta} : Map(EG, EPU^{[2]}) \times_G EG \to \mathbb{R}$ such that $\delta(\widehat{\zeta}) = \widehat{\chi}$. Thus taking $\zeta = \exp \circ \widehat{\zeta}$ we get that $\chi = \delta(\zeta)$ and furthermore the isomorphism $\Phi \cdot \zeta$ gives us a strong bundle gerbe isomorphism as

$$\begin{aligned}
\delta((\Phi \cdot \zeta)) \circ m_0 &= (\delta(\Phi) \cdot \delta(\zeta)) \circ m_0 \\
&= (\delta(\Phi) \circ m_0) \cdot \delta(\zeta) \\
&= (\delta(\Phi) \circ m_0) \cdot \chi \\
&= m_1.
\end{aligned}$$

□

Lemma 7.5.22. *The diagram of simplicial spaces*

$$\begin{tikzcd}
Map(EG, EPU^{[\bullet+1]}) \times_G EG \arrow[r, "\varphi_{EPU^{[\bullet+1]}}"] \arrow[d, "c_\bullet \times id"] & EPU^{[\bullet+1]} \arrow[d, "c_\bullet"] \\
EMap(EG, PU)_\bullet \times_G EG \arrow[r, "\varphi_{PU^\bullet}"] & E(PU_\bullet)
\end{tikzcd}$$

commutes.

Proof. If we look at the case $k = 1$ we can see that it extends for general $k \in \mathbb{N}$ quite easily as these morphisms are just products of the case of $k = 1$.

$$\begin{aligned}\varphi_{PU} \circ (c_1 \times id)\,((f_0, f_1), x) &= \varphi_{PU}(g, x) \\ &= g(x)\end{aligned}$$

where g is the unique function such that $f_0 \cdot g = f_1$.

$$\begin{aligned}c_1 \circ \varphi_{EPU^{[2]}}((f_0, f_1), x) &= c_1(f_0(x), f_1(x)) \\ &= w\end{aligned}$$

where w is the unique value in PU such that $f_0(x)w = f_1(x)$, notice that $g(x)$ also satisfies this condition and so $w = g(x)$. Therefore $\varphi_{PU} \circ (c_1 \times id) = c_1 \circ \varphi_{EPU^{[2]}}$, this argument extends naturally for any k. $\square$

Lemma 7.5.23. *The diagram of $U(1)$ bundles*

$$\begin{tikzcd} Map(EG, U) \times_G EG \times_{\varphi_{U(1)}} U(1) \arrow[r, "ev."] \arrow[d, "\pi \times id"] & U \arrow[d, "\pi"] \\ Map(EG, PU) \times_G EG \arrow[r, "\varphi_{PU}"] & PU \end{tikzcd}$$

describes a pullback diagram. Thus we have that $\varphi_{PU}^*(U) \simeq Map(EG, U) \times_G EG \times_{\varphi_{U(1)}} U(1)$.

Proof. The morphism ev. can be described by

$$\begin{aligned}\text{ev.} : Map(EG, U) \times_G EG \times_{\varphi_{U(1)}} U(1) &\to U \\ [f, x, z] &\mapsto f(x)z\end{aligned}$$

we need to show that this is well defined. Notice that

$$\begin{aligned}\text{ev.}(f\chi, x, \varphi_{U(1)}(\chi, x)^{-1}z) &= f(x)\chi(x)\varphi_{U(1)}(\chi, x)^{-1}z \\ &= f(x)\chi(x)\chi(x)^{-1}z \\ &= f(x)z \\ &= \text{ev.}(f, x, z)\end{aligned}$$

we also need to show that this is equivariant

$$\begin{aligned}\text{ev.}((f, x, z)g) &= \text{ev.}(f^g, xg, z) \\ &= f(xgg^{-1})z \\ &= f(x)z.\end{aligned}$$

Finally if we show that the diagram above commutes then we have that it is a pullback diagram of $U(1)$ bundles and therefore there is an isomorphism $\varphi_{PU}^*(U) \to Map(EG, U) \times_G EG \times_{\varphi_{U(1)}} U(1)$.

$$\begin{aligned}
\varphi_{PU} \circ (\pi \times id)([f, x, z]) &= \varphi_{PU}\left([\pi(f), x]\right) \\
&= \pi(f)(x) \\
&= \pi(f(x)) \\
&= \pi(f(x)z) \\
&= \pi \circ \text{ev.}([f, x, z]).
\end{aligned}$$

$\square$

Lemma 7.5.24. *The isomorphism described in lemma 7.5.23 extends to a strong isomorphism of simplicial line bundles.*

Proof. All we now need to prove is that the diagram

$$\begin{tikzcd}
\delta(Map(EG, U) \times_G EG \times_{\varphi_{U(1)}} U(1)) \arrow[r, "\delta(\text{ev.})"] \arrow[d, shift left, "\pi"] & \delta(U) \arrow[d, shift left, "\pi"] \\
Map(EG, PU^2) \times_G EG \arrow[r, "\varphi_{PU^2}"] \arrow[u, bend left, "m_0"] & PU^2 \arrow[u, bend left, "m"]
\end{tikzcd}$$

commutes and thus the pullback of the simplicial line bundle (U, m) will be isomorphic to the simplicial line bundle $(Map(EG, U) \times_G EG \times_{\varphi_{U(1)}}, m_0)$.

$$\begin{aligned}
\delta(\text{ev.}) \circ m_0(f_1, f_2, x) &= \delta(\text{ev.}) \left([\widehat{f_2}, x, z_2] \otimes [\widehat{f_1}\widehat{f_2}, x, z_1 z_2]^* \otimes [\widehat{f_1}, x, z_1]\right) \\
&= \widehat{f_2}(x) z_2 \otimes \left(\widehat{f_1}(x)\widehat{f_2}(x) z_1 z_2\right)^* \otimes \widehat{f_1}(x) z_1 \\
&= \widehat{f_2}(x) \otimes \left(\widehat{f_1}(x)\widehat{f_2}(x)\right)^* \otimes \widehat{f_1}(x) \\
m \circ \varphi_{PU}(f_1, f_2, x) &= m\left(f_1(x), f_2(x)\right) \\
&= w_2 \otimes w_1 w_2^* \otimes w_1
\end{aligned}$$

where $\pi(w_1) = f_1(x)$ and $\pi(w_2) = f_2(x)$. Notice that $\pi\left(\widehat{f_i}(x)\right) = f_i(x)$ and so

$$w_2 \otimes w_1 w_2^* \otimes w_1 = \widehat{f_2}(x) \otimes \left(\widehat{f_1}(x)\widehat{f_2}(x)\right)^* \otimes \widehat{f_1}(x)$$

thus the diagram commutes. Therefore ev. defines a strong isomorphism of simplicial line bundles. $\square$

Proposition 7.5.25. *The bundle gerbe $\widehat{\mathcal{G}}|_0$ is stably isomorphic to $\varphi^*_{BPU}(\mathcal{G}_0)$. Where $\mathcal{G}_0$ is the universal lifting gerbe on BPU.*

Proof. As the diagram in lemma 7.5.22 commutes and the two simplicial line bundles lemma 7.5.23 are isomorphic we have that

$$\begin{aligned}\varphi^*_{EPU^{[2]}} \circ c^*(U) &\simeq (c_1 \times \mathrm{id})^* \circ \varphi^*_{PU}(U) \\ &\simeq (c_1 \times \mathrm{id})^* \left(Map(EG, U) \times_G EG \times_{\varphi_{U(1)}} U(1) \right)\end{aligned}$$

furthermore this isomorphism is an isomorphism of simplicial line bundles over the space $Map(EG, EPU^{[\bullet+1]}) \times_G EG$ due to lemma 7.5.24 and thus forms a pair of strongly isomorphic bundle gerbes $\widehat{\mathcal{G}}|_0$ and $\varphi^*_{BPU}(\mathcal{G}_0)$. $\square$

Theorem 7.5.26. *The functor $- \times_G EG : BGrb_G(M) \to BGrb(M \times_G EG)$ is bijective and the universal strongly equivariant bundle gerbe is universal as claimed.*

Proof. We already know that the functor is injective due to proposition 7.5.9. Let $\mathcal{G}$ be a bundle gerbe on $M \times_G EG$. We have that $\mathcal{G}$ is stably isomorphic to the pullback of the lifting gerbe via some map $f : M \times_G EG \to BPU$. By adjointness we have that there exists some map $\bar{f} : M \to Map(EG, BPU)$ and we can form the strongly equivariant bundle gerbe $\bar{f}^*(\mathcal{G}_1)$. Now notice that through the counit φ_{BPU} and propositions 7.5.21 and 7.5.25 we have that

$$\begin{aligned}\bar{f}^*(\mathcal{G}_1) \times_G EG &\simeq (\bar{f} \times_G EG)^*(\mathcal{G}_1 \times_G EG) \\ &\simeq (\bar{f} \times_G EG)^*(\widehat{\mathcal{G}}|_1) \\ &\simeq (\bar{f} \times_G EG)^*(\widehat{\mathcal{G}}|_0) \\ &\simeq (\bar{f} \times_G EG)^* \circ \varphi_{BPU^*}(\mathcal{G}_0) \\ &\simeq f^*(\mathcal{G}_0)\end{aligned}$$

over $M \times_G EG$ and thus by injectivity of $- \times_G EG$ we have that the $\mathcal{G}$ we started with must be G-stably isomorphic to $\bar{f}^*(\mathcal{G}_1)$ over M and we are done. Furthermore as the choice of $\mathcal{G}$ was arbitrary we have shown the universal behavior of $\mathcal{G}_1$. $\square$

Bibliography

[1] J.L. Dupont. *Curvature and Characteristic Classes.* Lecture Notes in Mathematics, 640. Springer Berlin Heidelberg, Berlin, Heidelberg, 1st ed. 1978. edition.

[2] Michael Murray. Bundle gerbes. *Journal of the London Mathematical Society*, 54(2), 1996.

[3] Tom Leinster. *Basic category theory.* Cambridge studies in advanced mathematics ; 143. Cambridge University Press, Cambridge.

[4] Charles A Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, New York, 1994.

[5] J. L. (Jean Luc) Brylinski. *Loop spaces, characteristic classes, and geometric quantization.* Progress in mathematics ; v. 107. Birkhauser, Boston, c1993.

[6] R. Bott and L.W. Tu. *Differential Forms in Algebraic Topology.* Graduate Texts in Mathematics. Springer New York, 1995.

[7] Allen Hatcher. *Algebraic topology.* Cambridge Univ. Press, Cambridge, 2000.

[8] Michael Murray, David Roberts, Danny Stevenson, and Raymond Vozzo. Equivariant bundle gerbes. *Advances in Theoretical and Mathematical Physics*, 21, 06 2015.

[9] J.P May. *The Geometry of Iterated Loop Spaces*, volume 271 of *Lecture Notes in Mathematics.* Springer Berlin / Heidelberg, Berlin, Heidelberg, 2006.

[10] Graeme Segal. Categories and cohomology theories. *Topology*, 13(3):293–312, 1974.

[11] Tammo tom Dieck. On the homotopy type of classifying spaces. *manuscripta mathematica*, 11:41–49, 1974.

[12] P.T. Johnstone. *Topos Theory.* Dover Books on Mathematics. Dover Publications, 2014.

[13] ERIC M. FRIEDLANDER. *Etale Homotopy of Simplicial Schemes. (AM-104).* Princeton University Press, 1982.

[14] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2022.

[15] Jean-Luc Brylinski. Differentiable cohomology of gauge groups. 12 2000.

[16] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Volume 1*. A Wiley Publication in Applied Statistics. Wiley, 1996.

[17] Michael K Murray. An introduction to bundle gerbes. *arXiv preprint arXiv:0712.1651*, 2007.

[18] Roger W. Carter, Ian G. MacDonald, Graeme B. Segal, and M. Taylor. *Lectures on Lie Groups and Lie Algebras*. London Mathematical Society Student Texts. Cambridge University Press, 1995.

[19] Michael Atiyah and Graeme Segal. Twisted k-theory. *arXiv preprint math/0407054*, 2004.