The Closing Lemma for Riemann Surfaces

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Contents

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Abstract

The closing lemma is a result in dynamical system theory originating from the study of orbits of celestial bodies. In general, it refers to the problem of perturbing a dynamical system so as to obtain an arbitrarily close system for which there is a periodic orbit passing through a given point with a recurrence property. The problem often takes a variety of forms depending on the constraints one imposes and the setting of the given dynamical system, with many closing lemmas still unproven today.

In this thesis, we prove closing lemmas in the setting of Riemann surfaces with dynamical systems determined by holomorphic endomorphisms, and with points given the non-wandering property. We aim to provide elementary proofs of these results using the techniques and powerful machinery available to us from Riemann surface theory and the theory of holomorphic dynamics in one complex variable, amongst other areas. Detailed proofs that the closing lemma holds for endomorphisms of the plane $\mathbb{C}$, punctured plane $\mathbb{C}^*$, complex tori, and all Riemann surfaces of hyperbolic type will be presented, with the former two cases forming the main body of the thesis. For the case of the Riemann sphere $\mathbb{P}$, we furnish a proof that the closing lemma holds provided that the given endomorphism admits no Siegel discs and Herman rings.

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"Like what you do, and then you will do your best."

– Katherine Johnson

Chapter 1

Introduction

1.1 Context

The problem known as the *closing lemma* is well known amongst dynamical system theorists, with origins from Poincaré's paper concerning orbits of celestial bodies [14]. It refers to the problem of perturbing a dynamical system so as to obtain an arbitrarily close system (with respect to the topology on the space of considered dynamical systems) with periodic orbit passing through a given point. The given point is normally endowed with a recurrence property. Solutions to this problem depend on the type of recurrence property, the topology on the given space of dynamical systems, and constraints that one may have on obtaining the required perturbations. Consequently, closing lemma problems can vary drastically in both level of difficulty and available solution methods. For example, the closing lemma for $\mathcal{C}^1$ diffeomorphisms on compact smooth manifolds with a given recurrent point was proved by Pugh in 1967 [15], but generalisations to $\mathcal{C}^r$ diffeomorphisms for $r \geq 2$ remain elusive in the present day. In fact, the closing lemma for smooth dynamics is Smale's 10th problem, a list of unresolved mathematical problems to be tackled by mathematicians in the 21st century [19]. For more information on the many varieties of the closing lemma, we point to the comprehensive survey article written by Anosov and Zhuzhoma [1].

The closing lemma problem has now recently been asked in the setting of holomorphic dynamics. That is, for a complex manifold X and a holomorphic endomorphism f of X, if $p \in X$ is non-wandering, then is it true that every open neighbourhood (with respect to the compact-open topology) of f in the space of endomorphisms $\operatorname{End} X$ contains an endomorphism g for which p is periodic? Here, we say that p is periodic under g if there exists $n \in \mathbb{N}$ for which the n-th composition g^n maps p to itself. As far as we know, variations of the closing lemma in the setting of holomorphic dynamics in $\mathbb{C}^n$ first appear in Fornæss and Sibony's landmark papers [6] and [7]. The results in these papers were then generalised and adapted to Oka-Stein manifolds in Arosio and Lárusson's paper [3]. In fact, they give an affirmative answer to the question we posed above in the case that X is an Oka-Stein manifold. However, the techniques and machinery they apply are rooted in the theory of several complex variables, as is to be expected by the generality of the problem, and hence tend to be complicated and subtle.

The focus of our research is to give an affirmative answer to the closing lemma problem in the case that X is a Riemann surface, a connected 2-dimensional topological manifold modelled on the plane $\mathbb{C}$. To achieve this, we will use some of the ideas in [3] as inspiration. Our contribution will not only pertain to elaborating some of these complicated and subtle arguments, but to also furnish elementary and accessible arguments in our proofs wherever possible. For example, a key theorem that will see repeated use in our proofs is the *Runge approximation theorem* (Theorem 2.1.6), which has no analogous result as elegant in the theory of several complex variables.

1.2 Structure of the thesis and our results

With the above context, we look to prove that the closing lemma holds for a variety of Riemann surfaces. Indeed, the significant contribution of this thesis is in providing a proof to this theorem for *almost* all kinds of Riemann surfaces. In our journey, we will begin by providing a detailed overview of the many background results that will be present in our research. The areas that we draw upon are varied, with the backbone of our work constituted by Riemann surface theory and holomorphic dynamics. These areas are themselves quite general and underpinned by complex analysis seen in standard undergraduate courses. Hence, we will assume a base understanding of the ideas in complex analysis throughout the thesis. We will, however, endeavour to make this thesis as self-contained as possible, and so we provide as many relevant results as needed. This includes detailed treatments of *Weil's lemma* (Theorem 2.4.4) and *Šura-Bura's theorem* (Theorem 2.2.4), both of which are results that are perhaps rarer to encounter in the literature. The collection of background material is collated in Chapter 2.

To clarify what we mean by "almost" all Riemann surfaces, our strategy will be to utilise the power of the *uniformisation theorem* (Theorem 2.1.7) which classifies all the possible types of Riemann surfaces that can be encountered (up to isomorphism). Namely, these are the elliptic type, the parabolic type, and the hyperbolic type. There is only one Riemann surface of elliptic type, the Riemann sphere $\mathbb{P}$, and the surfaces of parabolic type are the plane $\mathbb{C}$, the punctured plane $\mathbb{C}^*$, and complex tori $\mathbb{T}$. All other surfaces are of hyperbolic type, in particular, the unit disc $\mathbb{D}$. We prove the closing lemma for the plane, punctured plane, complex tori, and all hyperbolic surfaces. The case of the Riemann sphere is not quite complete, but we will produce a proof of a version of the closing lemma with an additional assumption.

The crown jewels of our work will be proving the closing lemma for the two non-compact parabolic Riemann surfaces, $\mathbb{C}$ and $\mathbb{C}^*$. These results are highly non-trivial and so each have dedicated chapters, namely Chapters 3 and 4 respectively. In Chapter 3, we come across our first main result.

Main Theorem 1 (Closing lemma for $\mathbb{C}$)**.** *Let $p \in \mathbb{C}$ be a non-wandering point of an endomorphism $f : \mathbb{C} \to \mathbb{C}$. Then every open neighbourhood of f in* $\operatorname{End}\mathbb{C}$ *contains an endomorphism of which p is a periodic point.*

The chapter consists of two sections, each detail the proof of the closing lemma for $\mathbb{C}$ under conflicting assumptions. More precisely, we introduce a definition concerning our holomorphic endomorphism f and non-wandering point p, first seen in [2], called the *robustly non-expelling property*. Ignoring the details of the definition, the idea of the property is that it allows us to separate the proof of the closing lemma for $\mathbb{C}$ to when we can apply Montel's theorem on the family of iterates of f, and when we cannot. This is according to whether f has the robustly non-expelling property at p or not, respectively. Interestingly, the case that f does not exhibit the robustly non-expelling property at p is easier for us to deal with in comparison to when f does enjoy the property. In fact, we will see that the bulk of the background material are ingredients to the proof for the case when f has the robustly non-expelling property at p.

The general argument for the case that f does not enjoy the robustly non-expelling property at p, as seen in [3], utilises results in Oka theory and differential geometry. We however, furnish a proof that only relies on simple arguments involving normed vector spaces, polynomial interpolation, and Runge's approximation theorem. For the case that f enjoys the robustly non-expelling property at p, we provide a significantly streamlined and simplified proof using some of the ideas in [3]. Here, we exploit the unique topological properties of $\mathbb{C}$ in comparison with a general complex manifold. In particular, we use the fact that connected Runge sets in $\mathbb{C}$ are necessarily simply connected.

Chapter 4 is our treatment of the closing lemma for $\mathbb{C}^*$ and thus where we present our second main result.

Main Theorem 2 (Closing lemma for $\mathbb{C}^*$)**.** *Let $p \in \mathbb{C}^*$ be a non-wandering point of an endomorphism $f : \mathbb{C}^* \to \mathbb{C}^*$. Then every open neighbourhood of f in* $\operatorname{End}\mathbb{C}^*$ *contains an endomorphism of which p is a periodic point.*

It should be noted that our proof of the closing lemma for $\mathbb{C}^*$ follows the same lines as our proof of the closing lemma for $\mathbb{C}$. The chapter is again split into two sections, when our endomorphism f of $\mathbb{C}^*$ has the robustly non-expelling property at p, and when it does not. Once again, we will see that the case that f does not enjoy the robustly non-expelling property at p is easier to deal with; it closely resembles our corresponding proof for $\mathbb{C}$ with the appropriate adjustments. On the other hand, we will see many divergences from our corresponding arguments in the case of $\mathbb{C}$ for when f is robustly non-expelling at p. These diverging arguments originate from the different topological properties carried by $\mathbb{C}^*$ compared to $\mathbb{C}$. The greatest difference is that connected Runge sets in $\mathbb{C}^*$ are no longer necessarily only simply connected, but may also be doubly connected.

Finally, Chapter 5 concludes this thesis and contains proofs of the closing lemma for the remaining Riemann surfaces: hyperbolic Riemann surfaces, complex tori $\mathbb{T} = \mathbb{C}/\Gamma$ (where Γ denotes a lattice), and the Riemann sphere $\mathbb{P}$. This gives our final three main results.

Main Theorem 3 (Closing lemma for hyperbolic surfaces)**.** *Let $p \in X$ be a non-wandering point of an endomorphism f of a hyperbolic Riemann surface X. Then every open neighbourhood of f in* $\operatorname{End} X$ *contains an endomorphism of which p is a periodic point.*

Main Theorem 4 (Closing lemma for $\mathbb{T}$)**.** *Let $p + \Gamma \in \mathbb{T}$ and let f be an endomorphism of $\mathbb{T}$. Then every open neighbourhood of f in* $\operatorname{End}\mathbb{T}$ *contains an endomorphism of which*

$p + \Gamma$ is a periodic point.

Main Theorem 5 (Closing lemma for $\mathbb{P}$)**.** *Let $p \in \mathbb{P}$ be a non-wandering point of a rational map f on $\mathbb{P}$. Suppose f does not admit Siegel discs and Herman rings. Then every open neighbourhood of f in* $\operatorname{End}\mathbb{P}$ *contains a rational map of which p is a periodic point.*

This chapter is heavily reliant on the ideas seen in holomorphic dynamics, where we will take advantage of the power of several classification theorems. In particular, we use the dichotomy of the Fatou and Julia sets and its consequences, the classification of dynamics on hyperbolic Riemann surfaces (Theorem 2.3.4), and the classification of Fatou components on the Riemann sphere (Theorem 2.3.5). We also observe that our statement of the closing lemma for $\mathbb{T}$ does not include the hypothesis that p has some recurrence property. This implies that the closing lemma on $\mathbb{T}$ is independent of the behaviour of the given point. Further note however, that we do not quite manage to affirm the closing lemma for the Riemann sphere. We instead adjust our problem to omit the case that our endomorphism of $\mathbb{P}$ admits Siegel discs and Herman rings. Following the proof of our adjusted closing lemma for $\mathbb{P}$, we provide detailed justification on the reason behind these omitted scenarios. As mentioned above, this is the only case where we have not been able to verify the closing lemma for Riemann surfaces.

1.3 Further research

Summarising our work, we prove that the closing lemma holds for almost all Riemann surfaces. The only unresolved case is when we have a rational map on the Riemann sphere that exhibits cycles of either Siegel discs or Herman rings. More precisely, it seems especially non-trivial to find a perturbed rational map for which the given non-wandering point p lies on either a Siegel disc or Herman ring exhibited by the original map. As such, the natural direction in which one can take to continue the work presented in this thesis is to prove or disprove the closing lemma for $\mathbb{P}$ for this specific case.

Another possible direction is to explore variations of the closing lemma. As the nature of the problem is dependent on our specification of the type of recurrence of our point p in our Riemann surface, we may generalise our non-wandering hypothesis to, say, chain recurrent. Other directions include obtaining results on controlling our new-found periodic orbit. It may be of interest to produce perturbed maps with periodic point p whose orbit stays close to its orbit under the original map. Our results and proofs seemingly do not suggest any sort of control since we were only interested in producing perturbed maps with specified periodic point. It should be noted that progress on this matter can be found in [7].

Chapter 2

Background material

This chapter will introduce the main ideas and the common ingredients we will see in the ensuing proofs presented in later chapters. We will see that the areas of mathematics involved in our proofs are varied; from Riemann surface theory to holomorphic dynamics, then from general topology to topological and Lie group theory. Although the majority of the results that we will call upon will be cited, we will provide proofs of the results that are more difficult to obtain. In particular, we dedicate entire sections to detailed expositions of Šura-Bura's theorem and Weil's lemma, both are major contributors to our proof of the closing lemma for $\mathbb{C}$ and $\mathbb{C}^*$.

2.1 Riemann surfaces

We begin by providing a glossary of the definitions and theorems on Riemann surfaces that underpin the majority of the complex geometric and analytic theory we use. Here, we will assume some basic manifold and covering space theory, and will use Forster's *Lectures on Riemann Surfaces* as our primary reference (unless explicitly stated otherwise) [8].

For a $2n$-dimensional manifold X, we say that $\mathfrak{U} = \{\psi_i : U_i \to V_i : U_i \subset X, V_i \subset \mathbb{C}^n \text{ for all } i \in I\}$ is a *complex atlas* on X if $\mathfrak{U}$ is a system of holomorphically compatible charts (that is, the maps $\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ are biholomorphic for all $i, j \in I$) and the collection of open sets $(U_i)_{i \in I}$ cover X. We also say that two complex atlases $\mathfrak{U}$ and $\mathfrak{U}'$ are *analytically equivalent* if every chart of $\mathfrak{U}$ is holomorphically compatible with every chart of $\mathfrak{U}'$. Moreover, we give the remark that as the composition of biholomorphic maps is also biholomorphic, analytic equivalence of complex atlases is an equivalence relation.

We now specialise to the case when X is a 2-dimensional manifold.

Definition 2.1.1. A *Riemann surface* is a pair (X, Σ), where X is a connected two-dimensional manifold and Σ is a complex structure on X. That is, X is a Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^2$ and is given an equivalence class of analytically equivalent complex atlases on X.

Observe that every Riemann surface is metrisable. For by Radó's theorem [8, Theorem 23.3], every Riemann surface is second-countable and hence metrisable by Urysohn's metrisation theorem. Thus, we may employ the arguments seen in geometry and analysis on metric spaces to deduce results on Riemann surfaces.

When the context is clear, we will omit the complex structure Σ and will refer to a Riemann surface by only its underlying two-dimensional manifold X. To further illustrate these objects, we provide some standard examples of Riemann surfaces that will also arise in the ensuing chapters.

Examples.

1. The complex plane $\mathbb{C}$ is a Riemann surface, where its complex structure is defined by the atlas whose only chart is the identity map id on $\mathbb{C}$.

2. Any domain $Y \subset X$, that is, connected open subset, of a Riemann surface X is

itself a Riemann surface. Indeed, Y inherits the complex structure on X by taking the atlas of those complex charts $U \to V$ on X such that $U \subset Y$. In particular, the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ are domains of $\mathbb{C}$.

3. The Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ is a Riemann surface, where the symbol ∞ is not contained in $\mathbb{C}$. Here, the topology on $\mathbb{P}$ is given by the usual open sets on $\mathbb{C}$ together with sets of the form $V \cup \{\infty\}$, where $V = \mathbb{C} \setminus K$ for compact $K \subset \mathbb{C}$. Now set $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C}^* \cup \{\infty\}$, and define the maps $\psi_1 : U_1 \to \mathbb{C}$ and $\psi_2 : U_2 \to \mathbb{C}$ by $\psi_1(z) = z$ and
$$\psi_2(z) = \begin{cases} 1/z & \text{for } z \in \mathbb{C}^*, \\ 0 & \text{for } z = \infty. \end{cases}$$
One can easily verify that ψ_1 and ψ_2 are holomorphically compatible. We therefore obtain a complex structure on $\mathbb{P}$, given by the atlas consisting of the charts ψ_1 and ψ_2. We further note that with this construction, $\mathbb{P}$ is a compact, connected Hausdorff space that is homeomorphic to the 2-sphere S^2.

4. Finally, the following construction of a quotient space is a Riemann surface. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Define $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ to be the lattice spanned by ω_1 and ω_2. We define an equivalence relation on $\mathbb{C}$ by decreeing that $z, z' \in \mathbb{C}$ are equivalent mod Γ if $z - z' \in \Gamma$. Denote $\mathbb{C}/\Gamma$ to be the set of all equivalence classes defined by this relation and let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ be the canonical projection. Then by endowing $\mathbb{C}/\Gamma$ with the quotient topology, $\mathbb{C}/\Gamma$ is a connected, compact Hausdorff space, and π is continuous and an open map.

   We can then give a complex structure on $\mathbb{C}/\Gamma$ in the following way. Let $V \subset \mathbb{C}$ be an open set for which no two points in V are equivalent under Γ. Then $U = \pi(V)$ is open and the restriction $\pi|_V : V \to U$ is a homeomorphism. The inverse of $\pi|_V$

is then a complex chart on $\mathbb{C}/\Gamma$ and we let $\mathfrak{U}$ be the set of all charts defined in this way. After noting that Γ is discrete, it is straightforward to check the charts in $\mathfrak{U}$ are holomorphically compatible and hence that $\mathfrak{U}$ defines a complex structure on $\mathbb{C}/\Gamma$. With this construction, we call such space $\mathbb{T} = \mathbb{C}/\Gamma$ a complex torus.

We now define what we mean by a holomorphic map between Riemann surfaces.

Definition 2.1.2. Let X and Y be Riemann surfaces. A continuous map $f : X \to Y$ is called *holomorphic* if for every pair of charts $\psi_1 : U_1 \to V_1$ on X and $\psi_2 : U_2 \to V_2$ on Y with $f(U_1) \subset U_2$, the map

$$\psi_2 \circ f \circ \psi_1^{-1} : V_1 \to V_2$$

is holomorphic in the usual sense. A map $f : X \to Y$ is called *biholomorphic* if it is bijective and both $f : X \to Y$ and its inverse $f^{-1} : Y \to X$ are holomorphic. We say that X and Y are *isomorphic* if there exists a biholomorphic map $f : X \to Y$. For the special case of $Y = X$, we say that a general holomorphic map $f : X \to X$ is a *holomorphic endomorphism* of X, and a *holomorphic automorphism* whenever f is also biholomorphic.

Throughout the following chapters, we will omit the word "holomorphic" when dealing with either holomorphic endomorphisms or automorphisms if the context is clear.

Example. As it happens, the holomorphic endomorphisms of $\mathbb{P}$ have a very special form. First, let X be a general Riemann surface and let $Y \subset X$ be open. Then by a *meromorphic* map on Y, we mean a holomorphic map $f : Y' \to \mathbb{C}$ on an open subset $Y' \subset Y$ such that $Y \setminus Y'$ contains only isolated points (called poles) and $\lim_{z\to p} |f(z)| = \infty$ for every $p \in Y \setminus Y'$. With this definition, we can identify meromorphic maps on X as the holomorphic maps $X \to \mathbb{P}$ by defining $p \mapsto \infty$ for each pole p (except for the constant map with value ∞, which is not considered a meromorphic map). It is then a standard fact that every meromorphic map $\mathbb{P} \to \mathbb{C}$ is a rational map, that is, can be written as the quotient of two polynomials. Hence, under the above identification, the endomorphisms of $\mathbb{P}$ are precisely the rational maps (see [8] for more details). In particular, the automorphisms of $\mathbb{P}$ are precisely the Möbius transformations: $z \mapsto \dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$.

Example. For a general complex torus $\mathbb{T} = \mathbb{C}/\Gamma$ with lattice $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, it is a standard fact that $\mathbb{T}$ is isomorphic to a torus of the form $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ where τ has positive imaginary part. Hence, for our purposes, we may assume that our complex torus $\mathbb{T}$ is of the form $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Furthermore, for any two lattices Γ and Γ', $f : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ is a non-constant holomorphic map if and only if there exists $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma \subset \Gamma'$ and $\beta \in \mathbb{C}$ for which $f(z + \Gamma) = \alpha z + \beta + \Gamma'$. In particular, f is biholomorphic if and only if $\alpha\Gamma = \Gamma$. Thus, the only non-constant holomorphic maps between complex tori are precisely the affine linear maps. We point to [8] and [12] for further information.

With these definitions, we can introduce the essential results in Riemann surface theory that will see repeated usage in later chapters.

Theorem 2.1.3 (Identity theorem)**.** *Suppose X and Y are Riemann surfaces and $f, g : X \to Y$ are two holomorphic maps which coincide on a set in X with an accumulation point. Then f and g are identically equal.*

Proof. See [8, Theorem 1.11]. □

Theorem 2.1.4 (Maximum modulus principle)**.** *Suppose X is a Riemann surface and $f : X \to \mathbb{C}$ is a non-constant holomorphic map. Then the modulus of f, $|f|$, does not attain its maximum.*

Proof. See [8, Corollary 2.6] □

Remark. We observe that $\mathbb{P}$, $\mathbb{C}$, and $\mathbb{D}$ are not pairwise isomorphic. Indeed, any holomorphic map $\mathbb{P} \to \mathbb{C}$, $\mathbb{P} \to \mathbb{D}$, and $\mathbb{C} \to \mathbb{D}$ must be constant by the maximum modulus principle and Liouville's theorem.

Next, we introduce a topological hull operator that will be essential in many of our results. In fact, this operator is key in the proof of the celebrated Runge's approximation theorem.

Definition 2.1.5. Let X be a non-compact Riemann surface. For any subset $Y \subset X$, let $\widehat{Y}$ denote the union of Y with all the relatively compact connected components of $X \setminus Y$. If Y is open, then we say that Y is *Runge* if $Y = \widehat{Y}$, that is, none of the connected components of $X \setminus Y$ are compact.

Remark. Observe that this hull operator is dependent on the Riemann surface X. For example, take the annulus $\mathbb{A}_r = \{z \in \mathbb{C} : 1 < |z| < r\}$ for $r > 1$. Then treated as a subset of $\mathbb{C}$, $\widehat{\mathbb{A}_r} = D(0,r) = \{z \in \mathbb{C} : |z| < r\}$. But if treated as a subset of $\mathbb{C}^*$, we see that $\widehat{\mathbb{A}_r} = \mathbb{A}_r$.

This hull operator also satisfies the following properties (see [8, Theorem 23.5] for details):

(i) $\widehat{\widehat{Y}} = \widehat{Y}$ for any subset $Y \subset X$,

(ii) $\widehat{Y_1} \subset \widehat{Y_2}$ whenever $Y_1 \subset Y_2$,

(iii) $\widehat{Y}$ is closed whenever Y is closed, and

(iv) $\widehat{Y}$ is compact whenever Y is compact.

Moreover, if Y is a Runge subset of a Riemann surface X, then every connected component of Y is also Runge (see [8, Theorem 23.8]).

Note that we have denoted this hull operator with the "wide hat" symbol in reference to the standard notation for the holomorphically convex hull of a set in an n-dimensional complex manifold X. More precisely, for a subset K of a domain $D \subset X$, its *holomorphi-*

cally convex hull $\widehat{K}_{\mathscr{O}(D)}$ in D is defined by

$$\widehat{K}_{\mathscr{O}(D)} = \left\{ z \in D : |f(z)| \leq \sup_{w \in K} |f(w)| \text{ for all } f \in \mathscr{O}(D) \right\},$$

where $\mathscr{O}(D)$ denotes the set of holomorphic maps from D to $\mathbb{C}$. In particular for Riemann surfaces, the definition of the holomorphically convex hull coincides with our more topological definition of the hull operator (one can see this via the maximum modulus principle).

As mentioned above, Runge sets are key in the Runge approximation theorem.

Theorem 2.1.6 (Runge approximation theorem)**.** *Let X be a non-compact Riemann surface and Y a Runge subset of X. Then every holomorphic map on Y can be approximated uniformly on every compact subset of Y by holomorphic maps on X.*

Proof. See [8, Theorem 25.5]. □

Finally, we reference the generalisation of the classic Riemann mapping theorem for simply connected domains on $\mathbb{C}$ to arbitrary Riemann surfaces: the uniformisation theorem.

Theorem 2.1.7 (Uniformisation theorem)**.** *Suppose X is a simply connected Riemann surface. Then X is isomorphic to either the Riemann sphere $\mathbb{P}$, the complex plane $\mathbb{C}$, or the unit disc $\mathbb{D}$.*

Proof. See [8, Theorem 27.9]. □

An important application of the uniformisation theorem is the following. From covering space theory, if X and Y are connected manifolds and $p : X \to Y$ is a covering map, then p is the universal covering of Y if and only if X is simply connected. In particular, any Riemann surface Y has universal covering $p : X \to Y$ where X is also a Riemann surface and p is holomorphic. Hence, we may apply the uniformisation theorem on X.

Let $\operatorname{Deck}(p)$ denote the group of covering transformations of the universal covering $p : X \to Y$ of a Riemann surface Y. Then any $\sigma \in \operatorname{Deck}(p)$ is a holomorphic automorphism of X and $\operatorname{Deck}(p)$ acts freely and discretely on X, that is, the only automorphism in $\operatorname{Deck}(p)$ with fixed points is the identity map itself and every orbit under $\operatorname{Deck}(p)$ is a discrete subset of X. It can then be shown that any Riemann surface Y with universal covering map $p : X \to Y$ is isomorphic to the quotient $X/\operatorname{Deck}(p)$. Here, the equivalence relation is defined by identifying points of X that can be transformed to each other by elements of $\operatorname{Deck}(p)$. Thus, by the uniformisation theorem, every Riemann surface Y is isomorphic to either $\mathbb{P}/\operatorname{Deck}(p)$, $\mathbb{C}/\operatorname{Deck}(p)$, or $\mathbb{D}/\operatorname{Deck}(p)$ according to whether Y has universal covering $p : \mathbb{P} \to Y$, $p : \mathbb{C} \to Y$, or $p : \mathbb{D} \to Y$ respectively.

Now, as every automorphism of $\mathbb{P}$ has a fixed point, $\operatorname{Deck}(p)$ is trivial for the universal covering $p : \mathbb{P} \to Y$. Thus Y is isomorphic to $\mathbb{P}$ itself. On the other hand, the group of automorphisms of $\mathbb{C}$ which act freely and discretely on $\mathbb{C}$ is either trivial, consists of all translations of the form $z \mapsto z + n\gamma$ for $\gamma \in \mathbb{C}^*$ and $n \in \mathbb{Z}$, or consists of all translations of the form $z \mapsto z + m\gamma_1 + n\gamma_2$ where $\gamma_1, \gamma_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$ and $m, n \in \mathbb{Z}$. Hence, Y is isomorphic to either $\mathbb{C}$, $\mathbb{C}^*$, or torus $\mathbb{C}/\Gamma$ according to whether $\operatorname{Deck}(p)$ consists of the aforementioned types of automorphisms, respectively. Every other Riemann surface is isomorphic to $\mathbb{D}/\operatorname{Deck}(p)$ where $\operatorname{Deck}(p)$ is the group of automorphisms of $\mathbb{D}$ acting freely and discretely. We point to [8] for further details.

Remark. With the above classification, we say that $\mathbb{P}$ is *elliptic*, $\mathbb{C}$, $\mathbb{C}^*$, and any complex tori $\mathbb{C}/\Gamma$ are *parabolic*, and that every other Riemann surface is *hyperbolic*.

2.2 Šura-Bura's theorem and its consequences

In the previous section, we saw what it means for an open subset of a Riemann surface X to be Runge and the celebrated Runge approximation theorem. This theorem will be essential in many of our arguments in the ensuing chapters, so it is key for us to determine when an open set $U \subset X$ yields compact components in $X \setminus U$. As such, this section is dedicated to providing another tool for us to apply to this problem.

We will see that $X \setminus U$ has compact components if and only if $X \setminus U$ has non-empty open compact sets, and that these arguments will be purely topological in nature. The driving force behind this equivalence is Šura-Bura's theorem, a somewhat obscure topological result that we will prove in full detail. We acknowledge Remmert's *Classical Topics in Complex Function Theory* as our reference material for this section, particularly [16, p. 289–307].

Let X be a compact Hausdorff topological space. For any compact $K \subset X$, denote by $\mathcal{F}$ the family of all open compact sets in X that contain K. Note that $\mathcal{F}$ is non-empty, for $X \supset K$. Moreover, as X is Hausdorff, every member in $\mathcal{F}$ is closed and hence the intersection $\bigcap \mathcal{F}$ of all the sets in $\mathcal{F}$ is compact and contains K. With this notation, we prove the following lemma.

Lemma 2.2.1. *Every open set $U \subset X$ containing $\bigcap \mathcal{F}$ contains an element of $\mathcal{F}$.*

Proof. Let U be an open set in X containing $\bigcap \mathcal{F}$. Then $(X \setminus U) \cap \bigcap \mathcal{F} = \varnothing$. We claim that there exist finitely many sets $F_1, \ldots, F_r \in \mathcal{F}$ such that

$$(X \setminus U) \cap \bigcap_{i=1}^{r} F_i = \varnothing.$$

To see this, consider the family $\mathcal{G} = \mathcal{F} \cup \{X \setminus U\}$. As $\mathcal{F}$ is a family of (open) compact sets in a Hausdorff space, every member of $\mathcal{G}$ is closed in X. Note that $\bigcap \mathcal{G} = \varnothing$. Further observe that $(X \setminus G)_{G \in \mathcal{G}}$ is an open cover of $X \setminus U$. For if there exists $x \in X \setminus U$ that is not contained in $X \setminus G$ for any $G \in \mathcal{G}$, then $x \in \bigcap \mathcal{G}$, a contradiction. Now, as $X \setminus U$ is a closed subset of X, it is compact. So there exists a finite subcover of $X \setminus U$ in $(X \setminus G)_{G \in \mathcal{G}}$, say $(X \setminus G_i)_{i=1,\dots,m}$. Without loss of generality, we may assume that $G_i \neq X \setminus U$ for all $i = 1, \dots, m$ (for if $G_i = X \setminus U$ for some i, then $X \setminus G_i = U$). Thus, after relabelling, $G_i \in \mathcal{F}$ for all $i = 1, \dots, r$. It follows that $(X \setminus U) \cap \bigcap_{i=1}^{r} G_i = \varnothing$. This verifies the claim. But $\bigcap_{i=1}^{r} G_i \in \mathcal{F}$, so as $(X \setminus U) \cap \bigcap_{i=1}^{r} G_i = \varnothing$, $\bigcap_{i=1}^{r} G_i \subset U$, as required. □

We now introduce the topological objects called quasicomponents.

Definition 2.2.2. For a topological space X and $x \in X$, the *quasicomponent* of x in X is the intersection of all the sets that are both open and closed in X containing x.

Hence, for an arbitrary subset $U \subset X$, the quasicomponent of U is the intersection of all the sets that are both open and closed in X and contain U. With this definition, we prove the following theorem.

Theorem 2.2.3. *Every compact connected component K of a compact Hausdorff space X coincides with the quasicomponent containing K.*

Proof. Let K be a compact connected component of X and let $\mathcal{F}$ be defined as above. If we prove that $\bigcap \mathcal{F}$ is connected, then it will follow from the fact that $K \subset \bigcap \mathcal{F}$ that $K = \bigcap \mathcal{F}$. In other words, it suffices to prove that if $\bigcap \mathcal{F} = I_1 \cup I_2$, where I_1 and I_2 are disjoint closed subsets of X, then either I_1 or I_2 is empty. As $K = (K \cap I_1) \cup (K \cap I_2)$ and since K is a connected component, either $K = K \cap I_1$ or $K = K \cap I_2$.

Without loss of generality, suppose $K \subset I_1$. Then as I_1 and I_2 are closed subsets of the Hausdorff space X, they are compact. So as I_1 and I_2 are disjoint, we can find disjoint open sets U_1 and U_2 in X such that $I_1 \subset U_1$ and $I_2 \subset U_2$ (recall that compact Hausdorff spaces are normal). Now, because $\bigcap \mathcal{F} \subset U_1 \cup U_2$, there exists $F \in \mathcal{F}$ for which

$\bigcap \mathcal{F} \subset F \subset U_1 \cup U_2$ by Lemma 2.2.1. But then

$$F \cap (X \setminus U_2) = F \cap U_1.$$

Since F and U_1 are open (by definition of $\mathcal{F}$), $F \cap U_1$ is open. Moreover, as U_2 is open, $X \setminus U_2$ is closed in X and therefore compact. Hence, as F is compact, $F \cap U_1$ is compact. Since we have the inclusions

$$K \subset \bigcap \mathcal{F} \subset F \quad \text{and} \quad K \subset I_1 \subset U_1,$$

it follows that $F \cap U_1$ is an open compact subset of X satisfying $K \subset F \cap U_1$. Thus, $F \cap U_1 \in \mathcal{F}$ and $\bigcap \mathcal{F} \subset F \cap U_1 \subset U_1$ and consequently $U_2 \cap \bigcap \mathcal{F} = \varnothing$. Hence, $\bigcap \mathcal{F}$ is connected, for $I_2 = \varnothing$, and therefore $K = \bigcap \mathcal{F}$, as required. □

Remark. Since we have shown that in a compact Hausdorff space X, any compact connected component K coincides with the quasicomponent containing it, Lemma 2.2.1 implies that for any open set U containing K, there is an open compact set in U that contains K. Hence, we deduce that in a compact Hausdorff space, every compact connected component has a neighbourhood basis of open compact subsets. In fact, this remains true if we weaken X to be locally compact: this is known as Šura-Bura's theorem.

Theorem 2.2.4 (Šura-Bura's theorem)**.** *Every compact connected component K of a locally compact Hausdorff space X has a neighbourhood basis in X consisting of open compact subsets of X.*

Proof. As we know that the theorem is true in the case that X is compact, we may assume this result when proving the case when X is locally compact. Let U be an open neighbourhood of K in X. Since X is locally compact, there exists an open neighbourhood V of K in X whose closure $\overline{V}$ is a compact subset of U. Note that any connected subspace of $\overline{V}$ is connected as a subspace of X. Thus, we deduce that K is a connected component of the space $\overline{V}$.

By hypothesis, K has a neighbourhood basis in $\overline{V}$ consisting of open compact subsets of $\overline{V}$. Hence, there exists an open compact subset C of $\overline{V}$ such that $K \subset C \subset V$, as V is an open neighbourhood of K in X. But then C is also open in V and hence in X. This implies that C is an open compact subset of X for which $K \subset C \subset U$. As U was an arbitrary open neighbourhood of K in X, we conclude that K has a neighbourhood basis in X consisting of open compact subsets of X. □

Remark. Observe that every subspace Y of a Hausdorff space X that is both open and closed is a union of connected components of X. Indeed, suppose $Y \subset X$ is both open and closed. Consider the set $\bigcup_{y \in Y} X_y \subset X$, where X_y denotes the connected component of X containing $y \in Y$. Clearly $Y \subset \bigcup_{y \in Y} X_y$, so it suffices to show the reverse inclusion. In other words, it suffices to show that for any $y \in Y$, $X_y \subset Y$. Let $U = X_y \cap Y$ and $V = X_y \cap (X \setminus Y)$. We have $X_y = U \cup V$, where U and V are disjoint. Moreover, by definition of the subspace topology on X_y and the fact that every connected component of a topological space is closed, U and V are closed in X_y. Since X_y is connected, it follows that either $U = \varnothing$ or $V = \varnothing$. But $y \in X_y \cap Y = U$, so we must have $V = \varnothing$ and hence that $X_y \subset Y$.

With this observation, we have the following corollary of Šura-Bura's theorem.

Corollary 2.2.5. *A locally compact Hausdorff space X has compact connected components if and only if there exist non-empty open compact sets in X. The union of all the compact connected components of X coincides with the union of all the open compact subsets of X, and in particular, is open in X.*

Proof. ($\Longrightarrow$) Suppose K is a compact connected component of X. Then by Šura-Bura's theorem, K has a neighbourhood basis in X consisting of open compact subsets of X. Since K is non-empty (for it is a connected component), we conclude that there exist non-empty open compact subsets of X.

($\Longleftarrow$) Suppose there exists a non-empty open compact subset Y in X. Then as X is Hausdorff, Y is closed. Thus, by the previous remark, Y must be a union of connected components of X. Since Y is compact and non-empty, these components must also be compact.

Next, we prove the latter statement. Let $\mathcal{K}$ denote the family of all compact connected components of X and let $\mathcal{F}$ denote the family of all non-empty open compact subsets of X. We will show that $\bigcup \mathcal{K} = \bigcup \mathcal{F}$. Let $x \in \bigcup \mathcal{K}$. Then there exists a compact connected component $K \subset X$ containing x. By Šura-Bura's theorem, there exists an open compact set $F \supset K$ in X (as X is an open set containing K). So $x \in F$ and hence $x \in \bigcup \mathcal{F}$.

On the other hand, let $x \in \bigcup \mathcal{F}$. Then there exists an open compact subset $F \subset X$ containing x. As X is Hausdorff, F is an open and closed subspace of X and therefore is a union of connected components in X. Once again, as F is compact, these connected components are also compact and hence x is contained in a compact connected component of X. Thus, $x \in \bigcup \mathcal{K}$. It follows that $\bigcup \mathcal{K} = \bigcup \mathcal{F}$, as was to be proved. $\square$

Corollary 2.2.5 is the first of two topological results in this section that will see direct application in proofs presented in later chapters, for any Riemann surface is a locally compact Hausdorff topological space. Moreover, any open or closed subspace of a Riemann surface is locally compact and Hausdorff.

To be more precise, recall that an open set U in a Riemann surface X is Runge if and only if $X \setminus U$ has no compact connected components. Since $X \setminus U$ is a closed subspace of X, by Corollary 2.2.5, U is Runge if and only if the only open compact subset of $X \setminus U$ is empty. So when proving when certain sets are Runge, it can be more practical to work with open compact subsets instead of compact connected components.

To complete this section, we provide the second topological result that will be implemented in conjunction with Corollary 2.2.5. An analogous result can be found in [16, Chapter 13, Section 2], whose proof applied the various machinery afforded by working on $\mathbb{C}$, such as the Heine-Borel theorem. However, in the spirit of the preceding text within this section, we prove a more general result that only relies on the locally compact Hausdorff topology on a set X. Using this theorem in the context of Riemann surfaces, it allows us to find a relatively compact, open neighbourhood of any given open compact set for which we can apply the maximum modulus principle in a clever manner. We will see such usage in Chapters 3 and 4.

Theorem 2.2.6. *Let $U \subset X$ be an open subset of a locally compact Hausdorff space X. Then for every open compact subset K of $X \setminus U$, there exists a relatively compact, open neighbourhood V of K in X with $\partial V \subset U$, where ∂V denotes the boundary of V in X.*

Proof. Since K is open in $X \setminus U$, we can write $X \setminus U = K \cup C$, where $C = (X \setminus U) \setminus K$ is closed in $X \setminus U$. Because $X \setminus U$ is closed in X, C must also be closed in X. Note that K is a compact subset of X. As X is locally compact and Hausdorff, it is regular. Hence, we can find disjoint open sets $W, W' \subset X$ such that $K \subset W$ and $C \subset W'$.

We may assume that $K \neq \varnothing$, for otherwise the required open neighbourhood V in X is the empty set itself. So fix $x \in K$. Then as X is locally compact, there exists a compact neighbourhood N_x of x in X. Given the subspace topology, N_x is a compact Hausdorff space and hence normal. Note that the set $W \cap N_x$ is open in N_x and so $N_x \setminus (W \cap N_x)$ is closed in N_x. Hence, as $\{x\}$ is closed in N_x and disjoint to $N_x \setminus (W \cap N_x)$, we can find disjoint open sets V_x and V'_x in N_x such that $\{x\} \subset V_x$ and $N_x \setminus (W \cap N_x) \subset V'_x$. By definition of subspace topology, we can find open sets Y_x and Y'_x in X such that $V_x = N_x \cap Y_x$ and $V'_x = N_x \cap Y'_x$. Take the interior V_x° of V_x in X. Then $V_x^\circ = N_x^\circ \cap Y_x$ and hence is an open neighbourhood of x in X. Furthermore, $V_x^\circ \subset V_x \subset N_x$ and so taking closures in X, we obtain

$$\overline{V_x^\circ} \subset \overline{N_x} = N_x,$$

for N_x is compact and thus closed in X. Additionally,

$$V_x^\circ \cap Y_x' \subset V_x \cap Y_x' = (N_x \cap Y_x) \cap Y_x' = (N_x \cap Y_x) \cap (N_x \cap Y_x') = V_x \cap V_x' = \varnothing.$$

Thus, V_x° and Y_x' are disjoint open subsets of X, and consequently $\overline{V_x^\circ}$ and Y_x' are also disjoint subsets of X. Therefore, $\overline{V_x^\circ} \subset X \setminus Y_x'$. As $\overline{V_x^\circ} \subset N_x$ and $N_x \setminus (W \cap N_x) \subset V_x' \subset Y_x'$, we obtain

$$\begin{aligned}
\overline{V_x^\circ} &\subset N_x \cap (X \setminus Y_x') \\
&\subset N_x \cap (X \setminus (N_x \setminus (W \cap N_x))) \\
&= N_x \cap [(X \cap (W \cap N_x)) \cup (X \setminus N_x)] \\
&= N_x \cap (X \cap (W \cap N_x)) \\
&= N_x \cap W \\
&\subset W.
\end{aligned}$$

Hence, as $x \in K$ was arbitrary, this shows that for any $x \in K$, we can find a relatively compact, open neighbourhood V_x of x in X such that $V_x \subset \overline{V_x} \subset W$. Taking such open neighbourhoods, we see that $K \subset \bigcup\limits_{x \in K} V_x \subset W$. Thus, $(V_x)_{x \in K}$ is an open cover of K and so as K is compact, there exists finitely many $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup\limits_{i=1}^{n} V_{x_i}$. Choose $V = \bigcup\limits_{i=1}^{n} V_{x_i}$ to be the candidate open neighbourhood of K in X. Taking closures in X, we obtain

$$K \subset \overline{V} = \overline{\bigcup_{i=1}^{n} V_{x_i}} = \bigcup_{i=1}^{n} \overline{V_{x_i}} \subset W.$$

Now, as V is a finite union of relatively compact subsets of X, it is itself relatively compact in X. Moreover, since W and W' are disjoint, $\overline{V} \subset X \setminus W' \subset X \setminus C$ and so $\partial V \cap C = \varnothing$.

But V is open and contains K, so $\partial V \cap K = \varnothing$. Consequently, as $X \setminus U = K \cup C$, $\partial V \subset U$, completing the proof. $\square$

2.3 Holomorphic dynamics

Next, we establish some of the fundamentals of holomorphic dynamics in one complex variable. That is, the dynamics of holomorphic endomorphisms on Riemann surfaces. Many of the results that we cite in this section will be key in the ensuing chapters. Here, we appeal to Milnor's *Dynamics in One Complex Variable* as our reference material [12].

Throughout this thesis, we will say that a sequence of holomorphic maps $(f_n)_{n\in\mathbb{N}}$ on an open subset U of a Riemann surface *converges locally uniformly* to mean that $(f_n)_{n\in\mathbb{N}}$ uniformly converges on compact subsets of U. (The usual, more familiar, notion of local uniform convergence seen in analysis coincides with uniform convergence on compact subsets since Riemann surfaces are locally compact.)

Definition 2.3.1. For an endomorphism f of an arbitrary Riemann surface X, the *Fatou set* $\mathcal{F}(f)$ of f is the union of all open sets $U \subset X$ such that every sequence of iterates $(f^{n_j}|_U)$ either

(i) contains a subsequence $(f^{n_{j_k}}|_U)$ that converges locally uniformly, or

(ii) contains a subsequence $(f^{n_{j_k}}|_U)$ which diverges locally uniformly from X. That is, for every compact subsets $K, K' \subset X$, $f^{n_{j_k}}(K) \cap K' = \varnothing$ for all k sufficiently large.

We say that the complement of the Fatou set $\mathcal{J}(f) = X \setminus \mathcal{F}(f)$ is the *Julia set* of f.

Remark. The definition of Fatou set is motivated by the more general concept of normal families of holomorphic maps. We say that a family $\mathcal{H}$ in the space of holomorphic maps $\mathscr{O}(X, Y)$ between Riemann surfaces X and Y is *normal* if every sequence of maps in $\mathcal{H}$ contains either a subsequence which converges locally uniformly, or a subsequence that converges locally uniformly to the point at infinity in Y. (We say that a sequence $(f_n)_{n\in\mathbb{N}}$ of maps $X \to Y$ *converges locally uniformly to the point at infinity* in Y if for every compact sets $K \subset X$ and $K' \subset Y$, $f_n(K) \cap K' = \varnothing$ for all sufficiently large n.)

For the case of non-constant holomorphic maps between compact Riemann surfaces X and Y, there is a unique integer that can be associated to each of these maps. This integer will play a pivotal role in many of our arguments in Chapter 5. Let $x \in X$ and $f : X \to Y$ be a non-constant holomorphic map. Set $b = f(x)$. Then by [8, Theorem 2.1], there exists an integer $k \geq 1$ such that f is locally of the form z^k near x (up to chart). Denote $\mathrm{mult}(f, x) = k$, which we call the *multiplicity* of f at x. Imprecisely, for each number $y \neq b$ near b, we can think of $\mathrm{mult}(f, x)$ as the number of times for which f takes the value y on an open neighbourhood of x.

Definition 2.3.2. Let $f : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then for any $y \in Y$, the number

$$\deg(f) = \sum_{x\in f^{-1}(y)} \mathrm{mult}(f, x)$$

is a well-defined positive integer called the *degree* of f [8, Theorem 4.24].

Remark. There is an analogous characterisation of $\deg(f)$ using de Rham cohomology. Let $f : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then there is an integer d such that

$$\iint\limits_X f^*\omega = d \iint\limits_Y \omega$$

for every differentiable 2-form ω on Y. Moreover, d is necessarily equal to $\deg(f)$. (The above is a special case of applying de Rham cohomologies on smooth manifolds, see [11] and [13] for further details.)

We now look at some of the types of possible behaviour of points under iteration of an endomorphism f of a Riemann surface X. We say that $p \in X$ is *periodic* under f if there exists $n \in \mathbb{N}$ such that $f^n(p) = p$. For the special case that $n = 1$, we say that p is a *fixed point* of f. More generally, we say that p is *preperiodic* under f if there exists $m, n \in \mathbb{N}$ such that $f^m(p) = f^{m+n}(p)$.

Definition 2.3.3. Let X be a Riemann surface and let $z_0 \in X$. Consider a *periodic orbit* or *cycle*

$$z_0,\ f(z_0) = z_1,\ \ldots,\ f^{m-1}(z_0) = z_{m-1},\ f^m(z_0) = z_m = z_0$$

for an endomorphism f of X. If the points $z_1, \ldots, z_m$ are all distinct, then the integer $m \geq 1$ is called the *period*. Furthermore, the first derivative of the m-fold iterate f^m at a point of the cycle is a well-defined (that is, independent of chart) complex number λ called the *multiplier* of the cycle [12, Definition 4.5].

There are two generalisations of periodic points that we will frequently use. We say that $p \in X$ is *recurrent* for an endomorphism f of X if there exists a subsequence of $(f^n(p))_{n\in\mathbb{N}}$ that converges to p. We also say that $p \in X$ is *non-wandering* for an endomorphism f of X if for every open neighbourhood U of p in X, there exists $k \in \mathbb{N}$ such that $U \cap f^k(U) \neq \varnothing$. It is easy to see that every periodic point is recurrent, and every recurrent point is non-wandering. We also note that the set of non-wandering points of f form a closed subset of X, or equivalently, the set of wandering points of f is open in X. Indeed, if $p \in X$ is wandering, then there exists an open neighbourhood U of p in X such that $U \cap f^k(U) = \varnothing$ for all $k \in \mathbb{N}$. But then every point in U must also be wandering, and so U is an open neighbourhood of p contained in the set of wandering points of f.

We say that a periodic orbit is either *attracting* or *repelling* if its multiplier λ satisfies either $|\lambda| < 1$ or $|\lambda| > 1$, respectively. The periodic orbit will be called *superattracting* if $\lambda = 0$. Also, we say that a periodic orbit is *indifferent* if $|\lambda| = 1$, and specifically *parabolic* or *irrationally indifferent* if λ is a root of unity or not a root of unity, respectively.

Next, if $\mathcal{O}$ is an attracting periodic orbit of period m, we define the *basin of attraction* to be the open set $\mathcal{A} \subset X$ consisting of all points $z \in X$ for which the successive iterates $f^m(z), f^{2m}(z), \ldots$ converge towards some point in $\mathcal{O}$. For a given attracting periodic point p in $\mathcal{O}$, we say that the connected component in $\mathcal{A}$ containing p is the *immediate basin of attraction* of p. There also exist basins of attraction for parabolic periodic orbits, but we will cite [12] for the details.

Finally, in the interests of brevity as the following objects have highly technical definitions, we cite [12] for precise treatments of *parabolic petals*, *Siegel discs*, and *Herman rings*. Informally however, a parabolic petal is a connected open set for which a parabolic periodic point lies on its boundary, whereas Siegel discs and Herman rings correspond to irrational rotations on discs and annuli, respectively.

Remark. One of the most useful tools for studying dynamical systems is to be able to *conjugate* the map defining the dynamical system to a map whose dynamics is simpler to study. Informally, the dynamical properties exhibited by f will carry over to the map $\varphi \circ f \circ \varphi^{-1}$, where φ is a bijection, since we are essentially looking at the dynamics of f in a different coordinate system. In particular, conjugation by a homeomorphism preserves all topological dynamics exhibited by the original map. One can see this via the following defining equation: if $g = \varphi \circ f \circ \varphi^{-1}$, then for all $n \in \mathbb{Z}$,

$$g^n = \varphi \circ f^n \circ \varphi^{-1}$$

(whenever f^n is well-defined for $n < 0$). Hence, the orbits of f are in a one-to-one correspondence with the orbits of g.

With these definitions, we are prepared to list the following results that describe the behaviour of orbits of points lying in $\mathcal{F}(f)$ under an endomorphism $f : X \to X$. The ensuing theorem is underpinned by hyperbolic geometry. For those interested, we cite [12] for the details of the machinery behind this theorem.

Theorem 2.3.4 (Classification of dynamics on hyperbolic surfaces)**.** *For any endomorphism f of a hyperbolic Riemann surface X, exactly one of the following four possibilities holds:*

- ***Attracting case.*** *If f has an attracting fixed point z_0, then all orbits under f converge towards z_0. In fact, (f^n) converges locally uniformly to the constant map with value equal to z_0.*
- ***Escape case.*** *If some orbit under f has no accumulation point in X, then no orbit has an accumulation point. In fact, for any compact set $K \subset X$ there exists an integer n_K so that $K \cap f^n(K) = \varnothing$ for $n \geq n_K$.*
- ***Finite order case.*** *If f has two distinct periodic points, then some iterate f^n is the identity map and every point of X is periodic.*
- ***Irrational rotation case.*** *In all other cases, (X, f) is a rotation domain. That is, X is isomorphic either to the unit disc $\mathbb{D}$, the punctured disc $\mathbb{D} \setminus \{0\}$, or to the annulus $\mathbb{A}_r = \{z : 1 < |z| < r\}$ where $r > 1$, and f is conjugate to an irrational rotation $z \mapsto e^{2\pi i \alpha} z$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

Proof. See [12, Theorem 5.2]. □

This fundamental result is key in the proof of the celebrated classification theorem of Fatou components.

Theorem 2.3.5 (Classification of Fatou components)**.** *Let $f : \mathbb{P} \to \mathbb{P}$ be a rational map of degree at least 2. If f maps the Fatou component $U \subset \mathbb{P}$ onto itself, then there are just four possibilities, as follows:*

- *U is the immediate basin of attraction for an attracting fixed point,*
- *U is the immediate basin of attraction for a parabolic fixed point,*

- *U is a Siegel disc,*
- *U is a Herman ring.*

Proof. See [12, Theorem 16.1]. □

Remark. In fact, we can generalise Theorem 2.3.5 to Fatou components which cycle periodically under f, for U would be mapped onto itself by some iterate of f. Hence, the Fatou component U is either the immediate attractive basin for an attracting periodic point, the immediate basin for some petal of a parabolic periodic point, or it is either a member of a cycle of Siegel discs or a cycle of Herman rings.

The final major result in holomorphic dynamics on Riemann surfaces that we will cite is Sullivan's non-wandering theorem.

Theorem 2.3.6 (Sullivan's non-wandering theorem)**.** *Every Fatou component U for a rational map $f : \mathbb{P} \to \mathbb{P}$ of degree at least 2 is preperiodic. That is, there exists integers $n \geq 0$ and $m \geq 1$ so that the n-th forward image $f^n(U)$ is mapped onto itself by f^m.*

Proof. An outline of the proof can be found in [12], but one can also find the complete proof in [5]. □

Interestingly, it follows from Sullivan's non-wandering theorem that every Fatou component is either a branched covering or a biholomorphic copy of some periodic Fatou component. These components must then belong to one of the four types described by Theorem 2.3.5. As such, the dynamics of the Fatou set for a rational map of degree at least 2 have been completely determined.

We end this section by referencing an interesting equivalent definition of the Julia set $\mathcal{J}(f)$ of a rational map of degree at least 2.

Theorem 2.3.7. *The Julia set for any rational map of degree at least* 2 *is equal to the closure of its set of repelling periodic points.*

Proof. See [12, Theorem 14.1]. □

2.4 Weil's lemma

In this section, we give a brief overview of topological groups and provide a proof of Weil's lemma, whose statement and proof first appears in Weil's 1940 monograph *L'intégration dans les groupes topologiques et ses applications* [20]. Weil's lemma will be a useful tool in the proof of the closing lemma for $\mathbb{C}$ and $\mathbb{C}^*$, and the proof we provide here has been carefully translated and expanded upon from the original text.

Definition 2.4.1. A *topological group* G is a topological space that is also a group, such that the maps

$$\cdot \ : G \times G \to G, \quad (x, y) \mapsto x \cdot y, \qquad {}^{-1} : G \to G, \quad x \mapsto x^{-1},$$

are continuous, where $G \times G$ is given the product topology.

Remark. Note that with this definition, the translation map $x \mapsto g \cdot x$ for any $g \in G$ are homeomorphisms, since its inverse $x \mapsto g^{-1} \cdot x$ is continuous by definition. Furthermore, many authors require that the topology on G is Hausdorff. We opt to make no such assumption, to keep our definition as general as possible.

Henceforth, we will omit the product symbol denoting the group operation and simply juxtapose if the operation is clear.

Proceeding to Weil's lemma, we start by introducing two lemmas. The first of which is a well-known result, but we have provided a proof for the convenience of the reader.

Lemma 2.4.2. *Let G be a Hausdorff topological group and let $H \subset G$ be a locally compact subgroup. Then H is closed in G.*

Proof. We claim that H is open in its closure $\overline{H}$ in G. Let $x \in H$. As H is locally compact, we can find an open neighbourhood U of x in H whose closure $\overline{U}$ in H is compact. By definition of the subspace topology, we can write $U = V \cap H$ for some open set $V \subset \overline{H}$. Since $\overline{U}$ is compact and $\overline{H}$ is Hausdorff, $\overline{U}$ is closed in $\overline{H}$. Thus, $V \setminus \overline{U} = (\overline{H} \setminus \overline{U}) \cap V$ is open in $\overline{H}$. Now, as $V \cap H = U \subset \overline{U}$, then $(V \setminus \overline{U}) \cap H = \varnothing$. But H is dense in $\overline{H}$, so we must have $V \setminus \overline{U} = \varnothing$. Consequently, $V \subset \overline{U} \subset H$ and is thus an open subset of $\overline{H}$ containing x. Since $x \in H$ was arbitrary, we conclude that H is open in $\overline{H}$. But an open subgroup of a Hausdorff topological group is also closed for it can be written as the complement of a union of cosets. Hence, $H = \overline{H}$, and so H is closed in G. □

Lemma 2.4.3. *Let H denote either of the additive groups $\mathbb{Z}$ or $\mathbb{R}$ and $f : H \to G$ be a homomorphism from H into a locally compact Hausdorff topological group G. Suppose there is an open neighbourhood V of the identity $e \in G$ and a number $M > 0$ such that every $t \in H$ satisfying $f(t) \in V$ also satisfies $|t| \leq M$. Then f is an isomorphism onto a closed subgroup of G.*

Proof. We begin by noting that $\ker f = \{0\}$ and hence f is injective. For if there exists $t_0 \neq 0$ in H such that $f(t_0) = e$, then $nt_0 \in \ker f$ for any $n \in \mathbb{Z}$ since f is a homomorphism. Thus, $\ker f$ is unbounded, a contradiction as $|nt_0| > M$ for sufficiently large $|n|$. It follows that f is bijective onto its image, the subgroup $f(H)$ of G given the subspace topology.

Moreover, the set $H' = \{t \in H : |t| \leq M\}$ is clearly compact. As G is Hausdorff, then so too is $f(H')$ with respect to the subspace topology. It follows that f is a continuous and closed bijective map from H' to $f(H')$ and hence that H' is homeomorphic to $f(H')$. Thus, $f^{-1} : f(H) \to H$ is a continuous map when restricted to $V \cap f(H)$, since $V \cap f(H) \subset f(H')$

is open and non-empty. This implies that f^{-1} is continuous at e and hence must be continuous everywhere on $f(H)$, as the translation maps $x \mapsto gx$ are homeomorphisms for all $g \in G$. This shows that $f : H \to f(H)$ is an isomorphism.

Finally, we are left to show that $f(H)$ is closed in G. By Lemma 2.4.2 it suffices to show that $f(H)$ is locally compact. But we saw that $f : H \to f(H)$ is an isomorphism, so $f(H)$ is indeed locally compact since H is locally compact. $\square$

Theorem 2.4.4 (Weil's lemma). *Let H denote either of the additive groups $\mathbb{Z}$ or $\mathbb{R}$ and $f : H \to G$ be a homomorphism from H into a locally compact Hausdorff topological group G. Then either f is an isomorphism onto a closed subgroup of G or the closure of $f(H)$ in G is a compact abelian subgroup. In the second case, we can, for any open neighbourhood V of the identity in G, find a corresponding $T > 0$ such that any interval in $\mathbb{R}$ of length T contains an element of H whose image under f lies in V.*

Proof. Suppose that f is not an isomorphism onto a closed subgroup of G. Then by Lemma 2.4.3, we may assume that for any open neighbourhood V of the identity e in G and $M > 0$, there is an element $t \in H$ such that $f(t) \in V$ and $|t| > M$. Without loss of generality we may also assume that $G = \overline{f(H)}$. We claim that for any non-empty open set U in G, there is $t > 0$ such that $f(t) \in U$. Indeed, the image of H under f being dense in G, there is $z \in H$ such that $f(z) \in U$. Since U is open, we may find a symmetric open neighbourhood $V = V^{-1}$ of e in G such that $f(z) \cdot V = \{f(z)v : v \in V\} \subset U$. Then by assumption, there is $u \in H$ such that $f(u) \in V$ and $|u| > |z|$, and hence $f(z + |u|) \in U$ where $z + |u| > 0$.

Next, let $V = V^{-1}$ be a symmetric relatively compact neighbourhood of e in G. Then for any $x \in G$, there is some $t > 0$ such that $f(t) \in x \cdot V$, that is, $x \in f(t) \cdot V$ by symmetry. In particular, any $x \in \overline{V}$ belongs to a set $f(t) \cdot V$ with $t > 0$. Thus, the collection of open sets $(f(t) \cdot V)_{t>0}$ is an open cover of $\overline{V}$. As $\overline{V}$ is compact, there are finitely many $t_i > 0$ such that $\overline{V} \subset \bigcup_i f(t_i) \cdot V$. Let T be the largest of these t_i. For any $x \in G$, let $\tau \geq 0$ be the smallest element of H such that $f(\tau)x^{-1} \in \overline{V}$. Such τ exists by continuity of the group operation on G and by denseness of $f(H)$ in G. Then $f(\tau)x^{-1}$ belongs to some open set $f(t_i) \cdot V$ and thus $f(\tau - t_i)x^{-1} \in V$. By the definition of τ, we must have $\tau - t_i < 0$ and therefore $0 \leq \tau < T$. Since then we have $x \in \overline{V} \cdot f(\tau)$, it follows that if I is the image of the set of elements of H such that $0 \leq \tau \leq T$ under f, we have $G \subset \overline{V} \cdot I$. As $\overline{V}$ and I are compact, the same is true for G.

Moreover, as $f(H)$ is dense in G, if we take $x = f(-t)$, then we see that there exists $\tau \in H$ such that $0 \leq \tau < T$ and $f(t + \tau) \in \overline{V}$. The second claim of Weil's lemma then follows after noting that locally compact Hausdorff topological spaces are regular. Indeed, for any open neighbourhood U of e in G, we can find a symmetric relatively compact neighbourhood V of e such that $\overline{V} \subset U$. Finally, G is abelian since it contains a dense abelian subgroup. This completes the proof. $\square$

2.5 Lie groups and Lie algebras

Extending our notion of groups endowed with a topology, we now consider groups endowed with a notion of differentiability. We will assume familiarity with Lie groups and Lie algebras. Our first theorem in this section shows that compact abelian Lie groups are all have a certain form.

Theorem 2.5.1 (Structure of compact abelian Lie groups)**.** *Let G be a compact abelian Lie group. Then G is isomorphic to the direct product $(S^1)^n \times A$ for some $n \geq 0$, where S^1 is the unit circle in the complex plane and A is a finite abelian group.*

Proof. See [18, Theorem 5.2]. (The whole proof is quite involved, requiring multiple non-trivial but elementary lemmas.) □

Continuing the theme of compact Lie groups, we are interested in the "largest" possible compact Lie subgroup of a given Lie group. To make this precise, we give the following definition.

Definition 2.5.2. Let G be a topological group. A compact subgroup K in G is *maximal compact* if whenever K' is a compact subgroup of G such that $K \subset K'$, then $K = K'$.

Applied to Lie groups, there is a non-trivial result pertaining to maximal compact subgroups: for a Lie group with finitely many connected components, every compact subgroup is contained in a maximal compact subgroup, and any two maximal compact subgroups are conjugate. (To find a proof of this result, we point to [4, Chapter VII, Theorem 1.2].) This means that for Lie groups with finitely many connected components, maximal compact subgroups always exist and that they are unique up to conjugation. We now introduce some relevant Lie algebra theory. The following background is taken from Knapp's *Lie Groups: Beyond an Introduction* [9].

Definition 2.5.3. Let $\mathfrak{g}$ be a semisimple Lie algebra and B denote the Killing form on $\mathfrak{g}$. A *Cartan involution* θ of $\mathfrak{g}$ is an automorphism of $\mathfrak{g}$ satisfying $\theta^2(X) = X$ for all $X \in \mathfrak{g}$ and such that the symmetric bilinear form

$$B_\theta(X, Y) = -B(X, \theta(Y))$$

is positive definite.

Note that every semisimple matrix Lie algebra has a faithful representation in which the map $\theta(X) = -X^* = -\overline{X}^\top$ is a Cartan involution.

Next, we observe that as θ^2 is equal to the identity mapping, then it only has eigenvalues ± 1. Thus, a Cartan involution θ yields an eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ correspond to the $+1$ and -1 eigenspaces respectively. Furthermore, as θ is a Lie algebra automorphism, these eigenspaces must bracket to the rules

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}. \tag{2.1}$$

Using these rules, we observe that for $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, the adjoint map $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ carries $\mathfrak{k}$ to $\mathfrak{p}$ and $\mathfrak{p}$ to $\mathfrak{k}$. It follows that the trace of $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ is equal to zero (since the only eigenvalue of $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ is zero) and thus that the Killing form on $\mathfrak{g}$ satisfies $B(X,Y) = 0$ for all $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$. Moreover, as $\theta(Y) = -Y$, then $B_\theta(X,Y) = B(X,Y) = 0$ for all $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$. Hence, $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal under B and B_θ. In fact, as B_θ is positive definite, the eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ have the additional property that the Killing form

$$B \text{ is } \begin{cases} \text{negative definite on } \mathfrak{k} \\ \text{postive definite on } \mathfrak{p}. \end{cases} \tag{2.2}$$

Using these facts, we say that a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into a direct sum of eigenspaces that satisfies (2.1) and (2.2) is a *Cartan decomposition* of $\mathfrak{g}$. This construction yields the following.

Theorem 2.5.4. *Let G be a semisimple Lie group and θ be a Cartan involution of its Lie algebra $\mathfrak{g}$. Let $\mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition and K the analytic subgroup of G with the Lie algebra $\mathfrak{k}$. Then*

(i) *there exists a Lie group automorphism Θ of G with differential θ and $\Theta^2 = 1$,*

(ii) *the subgroup of G fixed by Θ is K,*

(iii) *the mapping $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k\exp(X)$ is a surjective diffeomorphism,*

(iv) *K is closed,*

(v) *K contains the center Z of G,*

(vi) *K is compact if and only if Z is finite, and*

(vii) *when Z is finite, K is a maximal compact subgroup of G.*

Proof. See [9, Theorem 6.31]. □

To see this theorem in action, we apply it to the Lie group of automorphisms of the unit disc $\mathbb{D}$: $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$. Moreover, the following proposition will be a very useful tool in the proof of the closing lemma for $\mathbb{C}$ and $\mathbb{C}^*$.

Proposition 2.5.5. *The 3-dimensional, connected, real Lie group* $\mathrm{PSL}(2,\mathbb{R})$ *contains a 1-dimensional maximal compact Lie subgroup.*

Proof. Let K be a maximal compact Lie subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and denote its dimension as m. Note that m is at most equal to 3. In fact, m is at most equal to 2 for if $m = 3$, then K must be isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. This is impossible since $\mathrm{PSL}(2,\mathbb{R})$ is not compact. Now we rule out the case that $m = 2$ as follows.

Observe that if K is a maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$, then $K/\{\pm I\}$ is a maximal compact subgroup of $\mathrm{PSL}(2,\mathbb{R})$. So it suffices to determine a maximal compact subgroup of the semisimple Lie group $\mathrm{SL}(2,\mathbb{R})$. Take $\theta(X) = -X^*$ to be the Cartan involution to the corresponding Lie algebra of $\mathrm{SL}(2,\mathbb{R})$: the algebra of traceless 2×2 matrices with real entries, $\mathfrak{sl}(2,\mathbb{R})$. Then we may write

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \{X \in \mathfrak{sl}(2,\mathbb{R}) : X = -X^\top\}$ and $\mathfrak{p} = \{X \in \mathfrak{sl}(2,\mathbb{R}) : X = X^\top\}$. Computing the matrix exponential on $\mathfrak{k}$, we see that the corresponding analytic subgroup of $\mathrm{SL}(2,\mathbb{R})$ with Lie algebra $\mathfrak{k}$ is

$$K = \{A \in \mathrm{SL}(2,\mathbb{R}) : AA^\top = I\} = \mathrm{SO}(2).$$

As the center of $\mathrm{SL}(2,\mathbb{R})$ is precisely $Z = \{\pm I\}$ and as $\mathrm{SO}(2)$ is closed, we conclude that $\mathrm{SO}(2)$ is a maximal compact Lie subgroup of $\mathrm{SL}(2,\mathbb{R})$ by Theorem 2.5.4. Thus, $\mathrm{SO}(2)/\{\pm I\}$ is a maximal compact Lie subgroup of $\mathrm{PSL}(2,\mathbb{R})$. In particular, $\mathrm{SO}(2)/\{\pm I\}$ is a one-dimensional real Lie group. $\square$

Chapter 3

The closing lemma for $\mathbb{C}$

In this chapter, we will furnish a complete, detailed proof of the closing lemma for $\mathbb{C}$. We use the methods devised in [2] and [3] as a guide. Let X be a complex manifold and let $\operatorname{End} X$ denote the monoid of holomorphic endomorphisms on X endowed with the compact-open topology. In particular, we note that $\operatorname{End} \mathbb{C}$ is a Fréchet space over $\mathbb{C}$, for the compact-open topology on $\operatorname{End} \mathbb{C}$ is induced by a family of seminorms.

Theorem 3.0.1 (Closing lemma for $\mathbb{C}$)**.** *Let $p \in \mathbb{C}$ be a non-wandering point of an endomorphism $f : \mathbb{C} \to \mathbb{C}$. Then every open neighbourhood of f in* $\operatorname{End} \mathbb{C}$ *contains an endomorphism of which p is a periodic point.*

Before we begin the proof of the closing lemma for $\mathbb{C}$, we first introduce a key definition.

Definition 3.0.2. Let X be a complex manifold and let $p \in X$. We say that $f \in \operatorname{End} X$ is *robustly non-expelling at p* if there is an open neighbourhood W of f in $\operatorname{End} X$, an open neighbourhood V of p in X, and a compact subset $K \subset X$ such that $g^j(V) \subset K$ for all $g \in W$ and $j \geq 0$.

Remark. A set that we will frequently see for the case that f is robustly non-expelling at

$p \in X$ is defined as follows. Since f is assumed to be robustly non-expelling at $p \in X$, there exists an open neighbourhood W of f in $\operatorname{End} X$, an open neighbourhood V of p in X, and a compact set $K \subset X$ such that the iterated images $g^j(V)$ are contained in K for all $g \in W$ and $j \geq 0$. Let U be the interior of the closed set

$$T = \{(x, g) \in X \times \operatorname{End} X : g^j(x) \in K \text{ for all } j \geq 0\}$$

and let U_f be the slice $\{x \in X : (x, f) \in U\}$. Then U_f is a non-empty open subset of X that is forward invariant under f. Moreover, U_f is relatively compact.

With this definition, the proof of the closing lemma for $\mathbb{C}$ will be completed in two steps. That is, we will see that the closing lemma for $\mathbb{C}$ holds both in the case that our endomorphism f is robustly non-expelling at our non-wandering point $p \in \mathbb{C}$, and when it is not. We will also see that many ingredients in the proofs presented in this chapter are transferable to the closing lemma for $\mathbb{C}^*$, which we prove in Chapter 4.

Interestingly, the latter case is easier to prove and the assumption that p is non-wandering is unnecessary. Hence, we will tackle the case that f is not robustly non-expelling at p first.

3.1 The case that f is not robustly non-expelling at p

Define the set

$$\mathcal{P}_m = \{P : \mathbb{C} \to \mathbb{C} : P \text{ is a polynomial of degree} \leq m - 1\}$$

and fix a compact set $K \subset \mathbb{C}$ containing at least m distinct points. Note that $\mathcal{P}_m$ is an m-dimensional vector space over $\mathbb{C}$, which we can equip with a norm with respect to K by giving it the sup norm $\|P\|_K = \sup_{z \in K} |P(z)|$. Now fix a finite set $M = \{z_1, \ldots, z_m\}$ in K, where $z_1, \ldots, z_m$ are distinct. Then the function $\|\cdot\|_M : \mathcal{P}_m \to \mathbb{R}$ defined by

$$\|P\|_M = \max_{z \in M} |P(z)|$$

is also a norm on $\mathcal{P}_m$. Using the fact that any two norms on a finite-dimensional vector space are equivalent and hence induce the same topology on the vector space, we can find constants $0 < C_1 \leq C_2$ such that

$$C_1 \|P\|_M \leq \|P\|_K \leq C_2 \|P\|_M$$

for all $P \in \mathcal{P}_m$. Knowing this, we prove the subsequent theorem.

Theorem 3.1.1. *Let $p \in \mathbb{C}$ and $f \in \operatorname{End}\mathbb{C}$. Suppose that f is not robustly non-expelling at p. Then every open neighbourhood of f in $\operatorname{End}\mathbb{C}$ contains an endomorphism of which p is a periodic point.*

Remark. Since $\mathbb{C}$ is a metric space, the compact-open topology on $\operatorname{End}\mathbb{C}$ coincides with the topology of compact convergence. As the latter topology is generated by basis elements of the form $B_K(h, \epsilon) = \left\{ g \in \operatorname{End}\mathbb{C} : \sup_{z \in K} |h(z) - g(z)| < \epsilon \right\}$, where $h \in \operatorname{End}\mathbb{C}$, $K \subset \mathbb{C}$ is compact, and $\epsilon > 0$, it suffices to prove the theorem on arbitrary basis elements containing f.

Proof. Let $B_K(f, \epsilon)$ be a basis element containing f in $\operatorname{End}\mathbb{C}$, where $K \subset \mathbb{C}$ is compact and $\epsilon > 0$. Pick a relatively compact, open neighbourhood V of p in $\mathbb{C}$. Observe that for any $g \in B_K(f, \epsilon)$, $g(K)$ is contained in the ϵ-neighbourhood of $f(K)$. Since $f(K)$ is compact, for all sufficiently large $r > 0$, the disc $D(0, r)$ contains the ϵ-neighbourhood of $f(K)$. Choose $r > 0$ such that $D(0, r)$ contains V and the ϵ-neighbourhood of $f(K)$. Then by hypothesis, there exists $g \in B_K(f, \epsilon)$ and a point $q \in V$ whose g-orbit is not contained in $\overline{D}(0, r)$. Assume that $g^k(q) \in \overline{D}(0, r)$ for $0 \leq k \leq m-1$ and $g^m(q) \notin \overline{D}(0, r)$. Note that this implies that the set $\{q, g(q), \ldots, g^m(q)\}$ consists of $m+1$ distinct elements.

Since $g^m(q) \in \mathbb{C} \setminus \overline{D}(0, r)$, there exists $r' > 0$ such that $\overline{D}(0, r) \cap D(g^m(q), r') = \varnothing$. After possibly shrinking $r' > 0$, we can find disjoint open discs U_1 and U_2 containing

$\overline{D}(0,r)$ and $\overline{D}(g^m(q),r')$, respectively. Define a holomorphic map $\phi : U_1 \cup U_2 \to \mathbb{C}$ by

$$\phi(z) = \begin{cases} z & \text{if } z \in U_1, \\ q & \text{if } z \in U_2. \end{cases}$$

Next, let $M = \{g(q), g^2(q), \ldots, g^m(q)\}$. Choose a constant $C > 0$ such that $\|P\|_{K'} \leq C\|P\|_M$ for all $P \in \mathcal{P}_m$ with $K' = \overline{D}(0,r) \cup \overline{D}(g^m(q),r')$. We claim that we can find a polynomial $P : \mathbb{C} \to \mathbb{C}$ arbitrarily close to ϕ on $\overline{D}(0,r) \cup \overline{D}(g^m(q),r')$ satisfying $P(g^k(q)) = g^k(q)$ for $1 \leq k \leq m-1$ and $P(g^m(q)) = q$.

Let $\varepsilon > 0$. As $U_1 \cup U_2$ is Runge in $\mathbb{C}$, the Runge approximation theorem gives a polynomial $Q : \mathbb{C} \to \mathbb{C}$ satisfying

$$\|Q - \phi\|_{K'} < \frac{\varepsilon}{C+1}$$

on $\overline{D}(0,r) \cup \overline{D}(g^m(q),r')$. Set $a_k = Q(g^k(q)) - \phi(g^k(q))$ for $1 \leq k \leq m$ and note that $|a_k| < \varepsilon/(C+1)$ for all such k. Via polynomial interpolation, take the unique polynomial $R : \mathbb{C} \to \mathbb{C}$ of degree at most $m-1$ satisfying $R(g^k(q)) = a_k$ for $1 \leq k \leq m$. Then the polynomial

$$P(z) = Q(z) - R(z)$$

satisfies $P(g^k(q)) = g^k(q)$ for $1 \leq k \leq m-1$ and $P(g^m(q)) = q$ by definition of ϕ. Since $R \in \mathcal{P}_m$, we see that

$$\|R\|_{K'} \leq C\|R\|_M = C \max_{z \in M} |R(z)| = C \max_{1 \leq k \leq m} |a_k| < \frac{C\varepsilon}{C+1}.$$

Hence, by the triangle inequality,

$$\|P - \phi\|_{K'} \leq \|Q - \phi\|_{K'} + \|R\|_{K'} < \frac{\varepsilon}{C+1} + \frac{C\varepsilon}{C+1} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

By taking such a polynomial $P : \mathbb{C} \to \mathbb{C}$, we obtain an endomorphism $P \circ g : \mathbb{C} \to \mathbb{C}$ that is arbitrarily close to g on $g^{-1}\left(\overline{D}(0,r)\right)$ with a periodic point q. In particular, as

$g(K) \subset \overline{D}(0,r)$, we can find a polynomial $P : \mathbb{C} \to \mathbb{C}$ with these properties such that $P \circ g \in B_K(f, \epsilon)$. Finally, let $T(z) = z - p + q$ be the translation map sending p to q. By shrinking V if necessary, we can ensure that $\|T - \mathrm{id}\,\|_K$ is as small as desired. Hence, the map

$$T^{-1} \circ P \circ g \circ T : \mathbb{C} \to \mathbb{C}$$

is an endomorphism with periodic point p and lies in $B_K(f, \epsilon)$ after choosing T sufficiently close to the identity map. Since $B_K(f, \epsilon)$ was an arbitrary basis element containing f in $\mathrm{End}\,\mathbb{C}$, the proof is complete. □

3.2 The case that f is robustly non-expelling at p

To complete the proof of the closing lemma for $\mathbb{C}$, we are left to tackle the case that our endomorphism f is robustly non-expelling at our non-wandering point $p \in \mathbb{C}$. Contrasting the proof in the previous section, this step is more difficult. The idea of the proof is to show that our non-wandering point p, with the added assumption that f is robustly non-expelling at it, is already a periodic point of f.

First, the assumption that f is robustly non-expelling at p allows us to conclude the following.

Lemma 3.2.1. *Let X be a non-compact Riemann surface and let $f \in \mathrm{End}\,X$. Let $p \in X$ be a non-wandering point of f. If f is robustly non-expelling at p, then p is a recurrent point of f.*

Remark. The proof of Lemma 3.2.1 uses a result pertaining to normal families of holomorphic maps. In particular, we use the fact that if X and Y are hyperbolic Riemann surfaces, then every family in the space $\mathscr{O}(X,Y)$ of holomorphic maps from X to Y is normal [12, Corollary 3.3]. This is a generalisation of the classic Montel's theorem for uniformly bounded families of holomorphic maps on open subsets of $\mathbb{C}$.

Proof. With the notation and setting described in the beginning of this section, we consider the open set U_f. Let U_0 denote the connected component of U_f containing p. Since p is non-wandering, there exists a smallest integer $\ell \geq 1$ so that $f^\ell(U_0) \cap U_0 \neq \varnothing$. As U_0 is a connected component and as U_f is forward invariant under f, f must map U_0 into a connected component of U_f. Thus, $f^\ell(U_0) \subset U_0$ and so p is non-wandering for the map $g = f^\ell$. We claim that p is in fact recurrent for g (and hence for f). To prove this, let $(V_k)_{k\in\mathbb{N}}$ be a decreasing neighbourhood basis of p, that is, $V_k \supset V_{k+1}$ for all $k \geq 1$. As p is non-wandering, for each k there is an integer j_k such that $g^{j_k}(V_k) \cap V_k \neq \varnothing$. Now, either the sequence $(j_k)_{k\in\mathbb{N}}$ has a strictly increasing subsequence or a constant subsequence. In the latter case, as (V_k) is a decreasing neighbourhood basis of p, we immediately conclude that p is a periodic point of g, and hence of f.

So after passing to a subsequence, let us assume that (j_k) is strictly increasing. We claim that every sequence in the family $\{g^{j_k} : U_f \to U_f : k \in \mathbb{N}\}$ has a subsequence that converges locally uniformly. Utilising the uniformisation theorem, we will argue according to whether X is hyperbolic or parabolic (X cannot be elliptic for it is non-compact).

If X is hyperbolic, then every family of maps in $\operatorname{End} X$ is normal. In particular, $\{f^n : X \to X : n \in \mathbb{N}\}$ is a normal family. Restricting f to U_f and using the fact that f is robustly non-expelling at p, we see that there does not exist subsequences of $(g^{j_k})_{k\in\mathbb{N}}$ on U_f that converge locally uniformly to the point at infinity in X. Hence, every sequence in the family $\{g^{j_k} : U_f \to U_f : k \in \mathbb{N}\}$ has a subsequence that converges locally uniformly to a holomorphic limit $h : U_f \to X$. On the other hand, suppose X is parabolic. Since X is non-compact, it is either $\mathbb{C}$ or $\mathbb{C}^*$. But as U_f is relatively compact in X and forward invariant under f, the family $\{f^n : X \to X : n \in \mathbb{N}\}$ is uniformly bounded on U_f. By Montel's theorem, it follows that every sequence in the family $\{g^{j_k} : U_f \to U_f : k \in \mathbb{N}\}$

has a subsequence that converges locally uniformly to a holomorphic limit $h : U_f \to X$. This proves the claim.

We are now ready to prove that p is a recurrent point of f. After possibly passing to a subsequence, $(g^{j_k})_{k\in\mathbb{N}}$ converges locally uniformly on U_f to a holomorphic limit $h : U_f \to X$. If $h(p) \neq p$, then $h(V_k) \cap V_k = \varnothing$ for large enough k, in which case $g^{j_k}(V_k) \cap V_k = \varnothing$ for large enough k. But this contradicts the fact that p is a non-wandering point of g, and so we must have $h(p) = p$. It follows that p must be a recurrent point of g and therefore of f, completing the proof. $\square$

The last result we will need is a perturbation lemma.

Lemma 3.2.2. *Let $z_1, \ldots, z_n$ be distinct points in $\mathbb{C}$. For every open neighbourhood U of the identity map* id *in* $\operatorname{End}\mathbb{C}$*, there is an open neighbourhood V of 1 in $\mathbb{C}$ such that for all $\lambda \in V$, there exists a polynomial $h \in U$ so that:*

(i) $h(z_j) = z_j$ *for all* $j = 1, \ldots, n$,

(ii) $h'(z_j) = 1$ *for all* $j = 2, \ldots, n$,

(iii) $h'(z_1) = \lambda$.

Proof. Let U be an open neighbourhood of id in $\operatorname{End}\mathbb{C}$. We will show that a polynomial h of degree $2n-1$ will satisfy every condition as stated. Let

$$A = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{2n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{2n-1} \\ 0 & 1 & 2z_1 & \cdots & (2n-1)z_1^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2z_n & \cdots & (2n-1)z_n^{2n-2} \end{bmatrix}.$$

Then to prove the existence of a polynomial h satisfying conditions (i) – (iii) is equivalent to solving the equation $A\mathbf{v} = (z_1, \ldots, z_n, \lambda, 1, \ldots, 1)^\top$ for some $\mathbf{v} \in \mathbb{C}^{2n}$.

To do this, we will show that A is invertible and hence the map $\mathbf{x} \mapsto A\mathbf{x}$ is a homeomorphism of $\mathbb{C}^{2n}$ endowed with the Euclidean topology to itself. Let $\mathbf{u} \in \ker A$ be a non-zero vector, say $\mathbf{u} = (u_1, \ldots, u_{2n})^\top$. Pick the largest $j \in \{1, \ldots, 2n\}$ so that $u_j \neq 0$. Then as $A\mathbf{u} = \mathbf{0}$, we see that

$$\begin{aligned} u_1 + u_2 z_1 + u_3 z_2^2 + \cdots + u_j z_1^{j-1} &= 0, \\ &\vdots \\ u_1 + u_2 z_n + u_3 z_n^2 + \cdots + u_j z_n^{j-1} &= 0, \\ u_2 + 2u_3 z_1 + \cdots + (j-1)u_j z_1^{j-2} &= 0, \\ &\vdots \\ u_2 + 2u_3 z_n + \cdots + (j-1)u_j z_n^{j-2} &= 0. \end{aligned}$$

This implies that $z_1, \ldots, z_n$ are distinct roots of the polynomial $f(z) = u_1 + u_2 z + u_3 z^2 + \cdots + u_j z^{j-1}$ and its derivative $f'(z)$. It follows that $z_1, \ldots, z_n$ must each have multiplicity at least 2 and hence that f has at least $2n$ roots, counted with multiplicity. But f has degree at most $2n-1$ and so must be the zero polynomial by the fundamental theorem of algebra, a contradiction to the fact that $u_j \neq 0$. Thus, $\ker A$ is trivial and so A is indeed invertible. Since the map $\mathbf{x} \mapsto A\mathbf{x}$ is continuous with respect to the Euclidean topology on $\mathbb{C}^{2n}$, we conclude that it is also a homeomorphism for its inverse $\mathbf{x} \mapsto A^{-1}\mathbf{x}$ is continuous.

Now, the finite-dimensional subspace $\mathcal{P}_m \subset \operatorname{End} \mathbb{C}$ of polynomials with degree bounded by $m-1$ inherits the compact-open topology on $\operatorname{End} \mathbb{C}$. But since finite-dimensional Hausdorff topological vector spaces of the same dimension are topologically isomorphic, it suffices to work with the usual Euclidean topology on $\mathbb{C}^m$. In particular, we can consider U as an open neighbourhood of the vector $(0, 1, 0, \ldots, 0)^\top$ in $\mathbb{C}^{2n}$.

Let $p_{n+1} : \mathbb{C}^{2n} \to \mathbb{C}$ denote the projection map on the $(n+1)$-th component. Then p_{n+1} is continuous and open. Since the map $\mathbf{x} \mapsto A\mathbf{x}$ is a homeomorphism, $A(U)$ is an open neighbourhood of $(z_1, \ldots, z_n, 1, \ldots, 1)^\top$ in $\mathbb{C}^{2n}$. Take $V = p_{n+1}(A(U))$ as an open neighbourhood of 1 in $\mathbb{C}$. Then for any $\lambda \in V$, U is an open neighbourhood of $A^{-1}(z_1, \ldots, z_n, \lambda, 1, \ldots, 1)^\top = (a_1, \ldots, a_{2n})^\top$ in $\mathbb{C}^{2n}$. It clearly follows that the polynomial

$$h(z) = a_1 + a_2 z + \cdots + a_{2n} z^{2n-1}$$

is a map satisfying the conditions in the lemma. $\square$

Theorem 3.2.3. *Let $p \in \mathbb{C}$ and $f \in \operatorname{End}\mathbb{C}$. Suppose that p is a non-wandering point of f and that f is robustly non-expelling at p. Then p is a periodic point of f.*

The strategy of our proof is as follows.

1. Use our hypotheses to construct a certain relatively compact Runge set $U_f \subset \mathbb{C}$ containing p that is forward invariant under f.

2. Examining the connected component U_0 of U_f containing p, we deduce that there exists a smallest positive integer ℓ for which U_0 is forward invariant under $g = f^\ell$.

3. Use Lemma 3.2.1 to deduce that p is a recurrent point of f and g.

4. Show that a subsequence of $(g^n)_{n\in\mathbb{N}}$ converges locally uniformly to a map ρ on U_0 that fixes p.

5. By the identity theorem, the set of fixed points of ρ is either discrete or U_0. If the former holds, it will follow that p is a periodic point of f. Hence, the proof will be complete once we show that the case that ρ pointwise fixes U_0 is impossible. We will do this via contradiction.

6. So supposing that the set of fixed points of ρ is U_0, we show that $g|_{U_0}$ is necessarily an automorphism of U_0. It then follows from Weil's lemma (Theorem 2.4.4) that

the group $G = \overline{\{(g|_{U_0})^n : n \in \mathbb{Z}\}}$ is isomorphic to either the discrete group $\mathbb{Z}$ or a compact abelian subgroup of $\operatorname{Aut} U_0$.

7. If G is isomorphic to the discrete group $\mathbb{Z}$, then it will follow that G is finite since some iterate of $g|_{U_0}$ must be the identity map. But this is absurd, so we are left to deal with the case that G is a compact abelian subgroup of $\operatorname{Aut} U_0$.

8. For this case, we will derive a contradiction using the fact that U_f is Runge in $\mathbb{C}$, and Lemma 3.2.2 together with the Cauchy estimates. This will show that ρ cannot pointwise fix U_0, thus completing the proof.

Proof. We remind readers of the following construction. By assumption, there exists an open neighbourhood W of f in $\operatorname{End}\mathbb{C}$, an open neighbourhood V of p in $\mathbb{C}$, and a number $r > 0$ for which the iterated images $g^j(V)$ are contained in $\overline{D}(0, r)$ for all $g \in W$ and $j \geq 0$. Let U be the interior of the closed set

$$T = \big\{(z, g) \in \mathbb{C} \times \operatorname{End}\mathbb{C} : g^j(z) \in \overline{D}(0, r) \text{ for all } j \geq 0\big\}$$

and let U_f be the slice $\{z \in \mathbb{C} : (z, f) \in U\}$. Recall that U_f is a non-empty, relatively compact, open subset of $\mathbb{C}$ such that $f(U_f) \subset U_f$.

Claim 1. *U_f is Runge in $\mathbb{C}$.* To see this, we will show that $\mathbb{C} \setminus U_f$ has no compact components. Our strategy will be to utilise Corollary 2.2.5 and Theorem 2.2.6. Suppose there exists an open compact subset $K \subset \mathbb{C} \setminus U_f$. Then by Theorem 2.2.6, we can find a relatively compact, open set $V' \subset \mathbb{C}$ for which $K \subset V'$ and $\partial V' \subset U_f$. We will argue that $V' \subset U_f$. As $\partial V' \subset U_f$, $\partial V' \times \{f\} \subset U$. Observe that $\partial V'$ is compact in $\mathbb{C}$, for it is closed and V' is relatively compact. So there exist open sets $U' \subset \mathbb{C}$ and $W' \subset \operatorname{End}\mathbb{C}$ such that $\partial V' \times \{f\} \subset U' \times W' \subset U$. Hence, $\partial V' \times W' \subset U$. Let $g \in W'$. Then as $\partial V' \times \{g\} \subset U \subset T$, $g^j(\partial V') \subset \overline{D}(0, r)$ for all $j \geq 0$. Thus, for any $z \in \partial V'$, we have $|g^j(z)| \leq r$ for all $j \geq 0$. Now, by the maximum modulus principle, for each $j \geq 0$, the modulus of $g^j|_{\overline{V'}}$ attains a maximum at some point on $\partial V'$. Therefore, for any $z \in V'$ and $j \geq 0$,

$$|g^j(z)| < \max_{x \in \overline{V'}} |g^j(x)| = \max_{x \in \partial V'} |g^j(x)| \leq r.$$

As $z \in V'$ was arbitrary, $g^j(V') \subset \overline{D}(0,r)$ for all $j \geq 0$ and thus $V' \times \{g\} \subset T$. But as $g \in W'$ was arbitrary, we deduce that $V' \times W' \subset T$. Since $V' \times W'$ is open, it follows that $V' \times W' \subset U$. Consequently, $V' \times \{f\} \subset U$ and so $V' \subset U_f$, as required. But since $V' \supset K$ and $K \subset \mathbb{C} \setminus U_f$, the only open compact subset of $\mathbb{C} \setminus U_f$ must be the empty set. By Corollary 2.2.5, this shows that $\mathbb{C} \setminus U_f$ has no compact connected components, proving Claim 1.

It follows that each connected component of U_f is also Runge. Let U_0 denote the connected component of U_f containing p. As p is non-wandering, there exists a smallest integer $\ell \geq 1$ for which $f^\ell(U_0) \cap U_0 \neq \varnothing$. In fact, as U_f is forward invariant under f and since U_0 is a connected component of U_f, we deduce that $f^\ell(U_0) \subset U_0$. Set $g = f^\ell$. Note that by Lemma 3.2.1, p is a recurrent point for g (and hence of f). Further observe from the proof of Lemma 3.2.1 that we can extract a subsequence $(g^{j_k})_{k \in \mathbb{N}}$ that converges locally uniformly on U_f to a holomorphic limit $h : U_f \to \mathbb{C}$ that fixes p.

Since (g^{j_k}) converges locally uniformly on U_f to $h : U_f \to \mathbb{C}$, (g^{j_k}) also converges locally uniformly on U_0 to $\rho = h|_{U_0}$. Let $M \subset U_0$ be the set of fixed points of ρ and let M_0 be the connected component of M containing p. Observe that by the identity theorem, either M is discrete and hence $M_0 = \{p\}$, or $M = M_0 = U_0$ (the case that ρ is the identity map on U_0).

Since ρ is the uniform limit of a subsequence of (g^j) on U_0 and as $g(U_0) \subset U_0$, g and ρ commute on U_0 and it follows that $g(M) \subset M$. As p is a recurrent point of g, there exists a sequence of points $g^{j_k}(p) \to p$ as $k \to \infty$. In particular, if M is discrete, then as $g^{j_k}(p) \in M$ for all $k \in \mathbb{N}$, we have a convergent sequence of points in a discrete set. Such a sequence is eventually constant. That is, there exists $N \in \mathbb{N}$ such that $g^{j_k}(p) = p$ for all $k \geq N$. Hence, p is a periodic point of g and therefore of f, and we have completed the proof.

Now we specialise to the case that $M = U_0$ and ρ is the identity map on U_0. We will show that this case can never arise by obtaining a contradiction. To begin, we prove that under this hypothesis, the following holds.

Claim 2. *The map* $g|_{U_0} : U_0 \to U_0$ *is an automorphism of* U_0. It suffices to show that $g|_{U_0} : U_0 \to U_0$ is bijective. Recall that $((g|_{U_0})^j)_{j\in\mathbb{N}}$ has a subsequence $((g|_{U_0})^{j_k})_{k\in\mathbb{N}}$ that converges locally uniformly to the identity map ρ on U_0. First, we show that $g|_{U_0}$ is injective. Suppose that $g|_{U_0}(z_1) = g|_{U_0}(z_2)$ for some $z_1, z_2 \in U_0$. Then as $k \to \infty$, we have $(g|_{U_0})^{j_k}(z_1) \to \rho(z_1) = z_1$ and $(g|_{U_0})^{j_k}(z_2) \to z_2$ as $k \to \infty$. But since $(g|_{U_0})^{j_k}(z_1) = (g|_{U_0})^{j_k}(z_2)$ for all $k \in \mathbb{N}$, $z_1 = z_2$ by uniqueness of limits.

Next, we show that an iterate of $g|_{U_0}$ is surjective. Let $a \in U_0$ and pick $r > 0$ such that $\overline{D}(a, r) \subset U_0$. Then the sequence $((g|_{U_0})^{j_k}(z) - a)_{k\in\mathbb{N}}$ uniformly converges to $\rho(z) - a$ on $\overline{D}(a, r)$. Note that $\rho(z) - a \neq 0$ on the boundary $\partial\overline{D}(a, r)$. Thus, as $\rho(z) - a$ is continuous and the boundary $\partial\overline{D}(a, r)$ is compact, there is $m > 0$ so that $|\rho(z) - a| \geq m$ for all $z \in \partial\overline{D}(a, r)$. By uniform convergence, we can choose $N \in \mathbb{N}$ such that for all $k > N$ and all $z \in \partial\overline{D}(a, r)$ we have

$$\left|\left((g|_{U_0})^{j_k}(z) - a\right) - (\rho(z) - a)\right| < \frac{m}{2} < m \leq |\rho(z) - a|.$$

It follows from Rouché's theorem that $\rho(z) - a$ and $(g|_{U_0})^{j_k}(z) - a$ have the same number of zeros in $D(a, r)$ for all sufficiently large k, and hence that $(g|_{U_0})^{j_k}(z) - a$ has a single zero in $D(a, r)$ for all sufficiently large k. Thus, for all such k, $a \in (g|_{U_0})^{j_k}(U_0) = g|_{U_0}\left((g|_{U_0})^{j_k - 1}(U_0)\right) \subset g|_{U_0}(U_0)$. Hence, as $a \in U_0$ was arbitrary, $U_0 \subset g|_{U_0}(U_0)$ and it follows that $g|_{U_0}$ is surjective. We conclude that $g|_{U_0}$ must be an automorphism of U_0, thus completing Claim 2.

Now, since U_f is Runge in $\mathbb{C}$ and relatively compact, we see that the connected component U_0 of U_f is isomorphic to the unit disc $\mathbb{D}$ by the Riemann mapping theorem. Hence, the group of automorphisms $\operatorname{Aut} U_0$ is isomorphic to $\operatorname{Aut} \mathbb{D} \cong \mathrm{PSL}(2, \mathbb{R})$. It follows that $\operatorname{Aut} U_0$ is a 3-dimensional real Lie group. As the group generated by $g|_{U_0}$ is an abelian subgroup of $\operatorname{Aut} U_0$, the closure $G = \overline{\{(g|_{U_0})^n : n \in \mathbb{Z}\}}$ of this subgroup in $\operatorname{Aut} U_0$ is also an abelian subgroup of $\operatorname{Aut} U_0$. Consequently, G is either isomorphic to the discrete group $\mathbb{Z}$ or is compact by Weil's lemma. If G is isomorphic to the discrete group $\mathbb{Z}$, then some iterate of $g|_{U_0}$ must be the identity map, for a sequence of iterates of $g|_{U_0}$ converges locally uniformly to the identity map on U_0. But then G must be a finite group, which is clearly absurd if it is isomorphic to $\mathbb{Z}$. So we are left to deal with the case that G is a compact abelian subgroup of $\operatorname{Aut} U_0$. We will show that this is impossible by deriving a contradiction as follows.

By Theorem 2.5.1, we see that G is isomorphic to $(S^1)^n \times A$, where $n \geq 0$ and A is a finite abelian group. But by Proposition 2.5.5, we deduce that $n \leq 1$, for the maximal compact subgroups of $\mathrm{PSL}(2, \mathbb{R})$ are 1-dimensional. Take the connected component containing the identity map G_0 in G, which is a compact subgroup of G. Then $G_0 \cong (S^1)^n$ where $n \leq 1$. Hence, the orbit $G_0 z$ for any $z \in U_0$ is either the singleton $\{z\}$, or a 1-dimensional, connected, compact submanifold of U_0. For the latter, note that $G_0 z$ is necessarily diffeomorphic to the unit circle.

Let us suppose, for the moment, that $G_0 q = \{q\}$ for some $q \in U_0$. Then we can obtain our sought-after contradiction as follows. Recall that U is the interior of the closed set $T = \{(z, g) \in \mathbb{C} \times \operatorname{End} \mathbb{C} : g^j(z) \in \overline{D}(0, r) \text{ for all } j \geq 0\}$. Note that U is an open neighbourhood of (q, f) in $\mathbb{C} \times \operatorname{End} \mathbb{C}$. Since every connected component of G is diffeomorphic to G_0, the orbit Gq is finite. Consequently, q is a periodic point of $g|_{U_0}$ and therefore of g, say with period τ. Since $g = f^\ell$ and ℓ was the smallest positive integer such that $f^\ell(U_0) \subset U_0$, we deduce that q is a periodic point of f with period $\tau\ell$. Thus, for all $j = 0, \ldots, \tau\ell - 1$, $f^j(q)$ are distinct points in U_f. Moreover, as a subsequence of (f^n) converges locally uniformly on U_0 to the identity map, $\left|\left(f^{\tau\ell}\right)'(q)\right| = 1$. (Note that q is a parabolic or irrationally indifferent periodic point of f.)

Now, since U is an open neighbourhood of (q, f), for all $\alpha \in \operatorname{End}\mathbb{C}$ sufficiently close to the identity map, $(q, \alpha \circ f)$ lies in U. In other words, there exists an open neighbourhood U' of the identity map in $\operatorname{End}\mathbb{C}$ such that $(q, \alpha \circ f) \in U$ for all $\alpha \in U'$. By Lemma 3.2.2, there is a corresponding open neighbourhood V' of 1 in $\mathbb{C}$ such that for any $\lambda \in V'$, there is α in U' fixing $f^j(q)$ for all $j = 0, \ldots, \tau\ell - 1$ and whose derivative satisfies

$$\alpha'(q) = \lambda, \quad \alpha'(f(q)) = 1, \quad \ldots, \quad \alpha'\left(f^{\tau\ell-1}(q)\right) = 1.$$

In particular, take $|\lambda| > 1$. Picking such α and setting $f_0 = \alpha \circ f$, we observe that $f_0^j(q) = f^j(q)$ for any $j \geq 1$. Thus, q is a periodic point of f_0 with period $\tau\ell$. Calculating the multiplier of $f_0^{\tau\ell}$ at q, we obtain

$$\left(f_0^{\tau\ell}\right)'(q) = \prod_{j=1}^{\tau\ell} f_0'\left(f_0^{j-1}(q)\right) = \prod_{j=1}^{\tau\ell} (\alpha \circ f)'\left(f^{j-1}(q)\right) = \prod_{j=1}^{\tau\ell} \alpha'\left(f^j(q)\right) \prod_{j=1}^{\tau\ell} f'\left(f^{j-1}(q)\right).$$

But since $\prod_{j=1}^{\tau\ell} f'\left(f^{j-1}(q)\right) = \left(f^{\tau\ell}\right)'(q)$,

$$\left|\left(f_0^{\tau\ell}\right)'(q)\right| = |\alpha'(q)| \left(\prod_{j=1}^{\tau\ell-1} \left|\alpha'\left(f^j(q)\right)\right|\right) \left|\left(f^{\tau\ell}\right)'(q)\right| = |\lambda| > 1$$

by definition of α.

Finally, we note that the slice $U_{f_0} = \{z \in \mathbb{C} : (z, f_0) \in U\}$ is open and forward invariant under f_0, and hence is forward invariant under $f_0^{\tau\ell}$. Since U_{f_0} is an open neighbourhood of q, we can choose $r_0 > 0$ so that $\overline{D}(q, r_0) \subset U_{f_0}$. By definition of U_{f_0},

$$\max_{|z|=r_0} \left|f_0^{\tau\ell j}(z)\right| \leq r$$

for all $j \geq 1$. The Cauchy estimates at q then yield

$$|\lambda|^j = \left| \left(f_0^{\tau \ell j} \right)' (q) \right| \leq \frac{1}{r_0} \max_{|z|=r_0} \left| f_0^{\tau \ell j}(z) \right| \leq \frac{r}{r_0}$$

for all $j \geq 1$, where r and r_0 are clearly independent of j. But $|\lambda|^j \to \infty$ as $j \to \infty$, the required contradiction.

Thus, we have completed the proof if we can find $q \in U_0$ that is fixed by G_0. To see that such $q \in U_0$ exists, recall the hull operator as introduced in Chapter 2, Section 2.1. Observe that since U_0 is Runge, the hull (with respect to $\mathbb{C}$) $\widehat{G_0 z} \subset U_0$ for any $z \in U_0$. We also note that $z \in U_0$ is fixed by G_0 if and only if $G_0 z = \widehat{G_0 z}$, since $\widehat{G_0 z}$ is the union of the connected set $G_0 z$ and the unique bounded connected component of $\mathbb{C} \setminus G_0 z$.

Claim 3. *There exists $q \in U_0$ such that $\widehat{G_0 q} = G_0 q$.* We first assert that if $w, z \in U_0$ such that $z \in \widehat{G_0 w} \setminus G_0 w$, then $\widehat{G_0 z} \subset \widehat{G_0 w} \setminus G_0 w$. Indeed, if $G_0 w = \{w\} = \widehat{G_0 w}$, then this assertion is vacuously true. So suppose that $G_0 w$ is diffeomorphic to the unit circle. If $G_0 z = \{z\} = \widehat{G_0 z}$, then clearly $\widehat{G_0 z} \subset \widehat{G_0 w} \setminus G_0 w$ by assumption. Otherwise, $G_0 z$ is diffeomorphic to the unit circle, and we have $G_0 z \subset \widehat{G_0 w} \setminus G_0 w$ since $G_0 w \cap G_0 z \neq \varnothing$ would imply $G_0 w = G_0 z$. Hence, $\widehat{G_0 z} \subset \widehat{G_0 w}$. Thus, as $\widehat{G_0 z}$ is the union of $G_0 z$ and the unique bounded connected component of $\mathbb{C} \setminus G_0 z$, $\widehat{G_0 z} \cap G_0 w = \varnothing$ for $G_0 w$ lies in the unbounded connected component of $\mathbb{C} \setminus G_0 z$. It follows that $\widehat{G_0 z} \subset \widehat{G_0 w} \setminus G_0 w$.

Knowing this, we define a partial order on the set $\left\{ \widehat{G_0 z} : z \in U_0 \right\}$ by reverse inclusion, that is, $\widehat{G_0 x} \leq \widehat{G_0 y}$ if and only if $\widehat{G_0 y} \subset \widehat{G_0 x}$. Hence, if $\widehat{G_0 q}$ is maximal with respect to this partial order, then $\widehat{G_0 q} = \widehat{G_0 y}$ holds whenever $\widehat{G_0 y} \subset \widehat{G_0 q}$. Moreover, we see that $\widehat{G_0 q} = G_0 q$, for if $x \in \widehat{G_0 q} \setminus G_0 q$, then $\widehat{G_0 x}$ is a proper subset of $\widehat{G_0 q}$ by the assertion above, a contradiction to maximality. So to verify Claim 3, it suffices to exhibit a maximal element of $\left\{ \widehat{G_0 z} : z \in U_0 \right\}$.

Let $\mathscr{C}$ be a totally ordered subset of $\left\{ \widehat{G_0 z} : z \in U_0 \right\}$. Then as $\mathscr{C}$ is a chain (with respect to the given partial order) of non-empty compact subsets of U_0, the intersection

$\bigcap \mathscr{C}$ is not empty. Pick $q \in \bigcap \mathscr{C}$. Then $\widehat{G_0 q}$ is an upper bound for $\mathscr{C}$. Indeed, as $q \in \bigcap \mathscr{C}$, $q \in \widehat{G_0 z}$ for all $\widehat{G_0 z} \in \mathscr{C}$. But then for all $\widehat{G_0 z} \in \mathscr{C}$, q is either contained in $G_0 z$, or it is contained in $\widehat{G_0 z} \setminus G_0 z$. If $q \in G_0 z$, then $G_0 q = G_0 z$ and so $\widehat{G_0 q} = \widehat{G_0 z}$. Otherwise, the assertion above gives $\widehat{G_0 q} \subset \widehat{G_0 z} \setminus G_0 z \subset \widehat{G_0 z}$. Thus, $\widehat{G_0 q} \subset \widehat{G_0 z}$ and so $\widehat{G_0 z} \leq \widehat{G_0 q}$ for all $\widehat{G_0 z} \in \mathscr{C}$. Since $\mathscr{C}$ was an arbitrary totally ordered subset of $\left\{\widehat{G_0 z} : z \in U_0\right\}$, we conclude from Zorn's lemma that the set $\left\{\widehat{G_0 z} : z \in U_0\right\}$ has a maximal element. This completes Claim 3 and hence the proof of the theorem. $\square$

The closing lemma for $\mathbb{C}$ then follows from Theorems 3.1.1 and 3.2.3.

Chapter 4

The closing lemma for $\mathbb{C}^*$

Following from the closing lemma for $\mathbb{C}$, we ask whether the closing lemma also holds on $\mathbb{C}^*$. As it will turn out, this is an affirmative and we will adapt the reasoning provided in Chapter 3 in our proofs.

Theorem 4.0.1 (Closing Lemma for $\mathbb{C}^*$)**.** *Let $p \in \mathbb{C}^*$ be a non-wandering point of an endomorphism $f : \mathbb{C}^* \to \mathbb{C}^*$. Then every open neighbourhood of f in $\operatorname{End}\mathbb{C}^*$ contains an endomorphism of which p is a periodic point.*

An important result that underpins many of the ensuing arguments presented in this chapter is the structure theorem for endomorphisms of $\mathbb{C}^*$.

Theorem 4.0.2. *Every holomorphic endomorphism f of $\mathbb{C}^*$ is of the form*

$$f(z) = z^n \exp(F(z)),$$

where $n \in \mathbb{Z}$ and $F : \mathbb{C}^* \to \mathbb{C}$ is holomorphic.

Remark. To prove this result, we take advantage of the fact that a closed differential 1-form (in particular, a holomorphic 1-form) ω on a Riemann surface X has a primitive

f on X, that is $df = \omega$, if and only if $\int_\gamma \omega = 0$ for any homotopy class γ in X (see [8, Theorem 10.15]). Here, we compute $\int_\gamma \omega$ by choosing a loop representing γ and integrating along that loop. Note that the integer n in the theorem is in fact the winding number of the given endomorphism f. Further details can be found in [8].

Proof. Clearly every map on $\mathbb{C}^*$ of the form $z \mapsto z^n \exp(F(z))$, where $n \in \mathbb{Z}$ and $F : \mathbb{C}^* \to \mathbb{C}$ is holomorphic, is an endomorphism of $\mathbb{C}^*$. So it suffices to show that for a given endomorphism f of $\mathbb{C}^*$, there exists $n \in \mathbb{Z}$ and $F : \mathbb{C}^* \to \mathbb{C}$ holomorphic such that $f(z) = z^n \exp(F(z))$. Let $z = x + iy$ be the usual coordinate on $\mathbb{C}^*$ and let $\omega = dz/z$ be a holomorphic 1-form on $\mathbb{C}^*$. Note that the fundamental group on $\mathbb{C}^*$ is isomorphic to $\mathbb{Z}$ and a generator of this fundamental group is represented by the loop $\gamma : [0,1] \to \mathbb{C}^*$, $\gamma(t) = e^{2\pi i t}$. The integral of the pullback $f^*\omega$ along γ is hence

$$\int_\gamma f^*\omega = \int_{f\circ\gamma} \omega = \int_{f\circ\gamma} \frac{dz}{z} = 2\pi i n,$$

where $n \in \mathbb{Z}$. Define the holomorphic endomorphism $g : \mathbb{C}^* \to \mathbb{C}^*$ by

$$g(z) = z^{-n} f(z).$$

Then the pullback $g^*\omega$ is

$$g^*\omega = \frac{dg}{g} = \frac{-nz^{-n-1}f + z^{-n}f'}{z^{-n}f} dz = -\frac{n}{z}dz + \frac{f'}{f}dz = -n\omega + f^*\omega.$$

The integral of $g^*\omega$ along γ is therefore

$$\int_\gamma g^*\omega = -n\int_\gamma \omega + \int_\gamma f^*\omega = 0.$$

Hence, by the remark above, there exists a holomorphic map $F : \mathbb{C}^* \to \mathbb{C}$ such that $dF = g^*\omega$. This implies that $F' = g'/g$ on $\mathbb{C}^*$. But then the non-vanishing holomorphic map $g(z)\exp(-F(z))$ defined on $\mathbb{C}^*$ has derivative $g'(z)\exp(-F(z)) - g(z)F'(z)\exp(-F(z)) = 0$. Thus, $g(z)\exp(-F(z)) = \exp(c)$ for some constant $c \in \mathbb{C}$ and so it follows that $f(z) = z^n \exp(F(z) + c)$. Absorbing c into F then completes the proof. $\square$

For a more extensive study of endomorphisms of $\mathbb{C}^*$, see [17].

With this in mind, we will prove the closing lemma for $\mathbb{C}^*$ in two stages: when f is robustly non-expelling at p, and when f is not. We again tackle the latter scenario first.

4.1 The case that f is not robustly non-expelling at p

Fix integers $m, n \geq 0$. Define the set

$$\mathcal{L}_{m,n} = \left\{ L : \mathbb{C}^* \to \mathbb{C} : L(z) = \sum_{j=-m}^{n} a_j z^j,\ a_j \in \mathbb{C} \text{ for all } j = -m, \ldots, 0, \ldots, n \right\}.$$

That is, $\mathcal{L}_{m,n}$ is the set of Laurent polynomials $\mathbb{C}^* \to \mathbb{C}$ of the form $P(z)/z^m$, where $P(z)$ is a polynomial of degree at most $m+n$. Now fix a compact set $K \subset \mathbb{C}^*$ containing at least $m+n+1$ distinct points. Observe that $\mathcal{L}_{m,n}$ is a $(m+n+1)$-dimensional vector space over $\mathbb{C}$, which we can equip with a norm with respect to K by giving it the sup norm $\|L\|_K = \sup_{z \in K} |L(z)|$. Furthermore, note that for integers $0 \leq m' \leq m$ and $0 \leq n' \leq n$, $\mathcal{L}_{m',n'}$ is a subspace of $\mathcal{L}_{m,n}$.

Next, let $M = \{z_1, \ldots, z_{m+n+1}\}$ be a set of $m+n+1$ distinct points in K. Then the function $\|\cdot\| : \mathcal{L}_{m,n} \to \mathbb{R}$ defined by

$$\|L\|_M = \max_{z \in M} |L(z)|.$$

is also a norm on $\mathcal{L}_{m,n}$. Once again using the fact that any two norms on a finite-dimensional vector space are equivalent and hence induce the same topology on the vector space, we can obtain constants $0 < C_1 \leq C_2$ such that

$$C_1 \|L\|_M \leq \|L\|_K \leq C_2 \|L\|_M$$

for all $L \in \mathcal{L}_{m,n}$. We are now fully equipped to prove the following theorem.

Theorem 4.1.1. *Let $p \in \mathbb{C}^*$ and $f \in \operatorname{End}\mathbb{C}^*$. Suppose that f is not robustly non-expelling at p. Then every open neighbourhood of f in $\operatorname{End}\mathbb{C}^*$ contains an endomorphism of which p is a periodic point.*

Proof. Beginning in a similar way to Theorem 3.1.1, $\mathbb{C}^*$ is a metric space and so the compact-open topology coincides with the topology of compact convergence on $\operatorname{End}\mathbb{C}^*$. Hence, it suffices to prove the theorem for arbitrary basis elements

$$B_K(f,\epsilon) = \left\{ g \in \operatorname{End}\mathbb{C}^* : \sup_{z\in K} |f(z) - g(z)| < \epsilon \right\}$$

containing f, where $K \subset \mathbb{C}^*$ is compact and $\epsilon > 0$.

So let $B_K(f,\epsilon)$ be a basis element containing f. Without loss of generality, we may assume that $\epsilon < \dfrac{1}{2} \inf_{z\in K} |f(z)|$, for K is compact and f has no zeros. Pick a relatively compact, open neighbourhood V of p in $\mathbb{C}^*$. Observe that for all $g \in B_K(f,\epsilon)$, $g(K)$ is contained in the ϵ-neighbourhood of $f(K)$ in $\mathbb{C}^*$. As $f(K)$ is compact and by our assumption on $\epsilon > 0$, the annulus $\mathbb{A}_{r,R} = \{z \in \mathbb{C}^* : r < |z| < R\}$ contains the ϵ-neighbourhood of $f(K)$ for all sufficiently small $r > 0$ and sufficiently large $R > r$. Choose $0 < r < R$ such that $\mathbb{A}_{r,R}$ contains the ϵ-neighbourhood of $f(K)$ and V. Then by assumption, there exist $g \in B_K(f,\epsilon)$ and $q \in V$ whose g-orbit is not contained in the closed annulus $\overline{\mathbb{A}_{r,R}} = \{z \in \mathbb{C}^* : r \le |z| \le R\}$. Choose the smallest $n \in \mathbb{N}$ such that $g^n(q) \notin \overline{\mathbb{A}_{r,R}}$. Note that this implies that the set $\{q, g(q), \ldots, g^n(q)\}$ consists of $n+1$ distinct elements.

We will construct a holomorphic endomorphism $h : \mathbb{C}^* \to \mathbb{C}^*$ arbitrarily close to the identity map on $\overline{\mathbb{A}_{r,R}}$ such that $h(g^k(q)) = g^k(q)$ for all $1 \le k < n$, and $h(g^n(q)) = q$. Since the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is surjective, we can pick $x \in \exp^{-1}(q/g^n(q))$. Let $d > 0$ denote the distance between $\overline{\mathbb{A}_{r,R}}$ and $g^n(q)$. Let $\overline{\mathbb{A}}$ denote a closed annulus centred at zero whose interior $\mathbb{A}$ contains $\overline{\mathbb{A}_{r,R}}$ and such that the distance between $\overline{\mathbb{A}}$ and

$g^n(q)$ is $d/2$. Then $\mathbb{A}$ is a Runge set in $\mathbb{C}^*$ containing $\overline{\mathbb{A}_{r,R}}$. Take $r' > 0$ such that the open disc $D(g^n(q), r')$ is contained in $\mathbb{C}^* \setminus \overline{\mathbb{A}}$. After possibly shrinking $r' > 0$, we can find an open disc U in $\mathbb{C}^*$ containing $\overline{D}(g^n(q), r')$ such that $\mathbb{A} \cap U = \varnothing$. Define a holomorphic function $\varphi : \mathbb{A} \cup U \to \mathbb{C}$ by

$$\varphi(z) = \begin{cases} 0 & \text{if } z \in \mathbb{A}, \\ x & \text{if } z \in U. \end{cases}$$

We claim that we can find a Laurent polynomial $L : \mathbb{C}^* \to \mathbb{C}$ arbitrarily close to φ on the compact set $K' = \overline{\mathbb{A}_{r,R}} \cup \overline{D}(g^n(q), r')$ satisfying $L(g^k(q)) = 0$ for all $1 \le k < n$ and $L(g^n(q)) = x$.

To see this, take $M = \{g(q), \dots, g^n(q)\}$ and let $\varepsilon > 0$. As discussed above, choose a constant $C > 0$ such that $\|L\|_{K'} \le C\|L\|_M$ for all $L \in \mathcal{L}_{0,n-1}$. As $\mathbb{A} \cup U$ is Runge in $\mathbb{C}^*$, we can employ Runge's approximation theorem to find a Laurent polynomial $Q : \mathbb{C}^* \to \mathbb{C}$ such that

$$\|Q - \varphi\|_{K'} < \frac{\varepsilon}{C+1}$$

on K'. Set $a_k = Q(g^k(q)) - \varphi(g^k(q))$ for $1 \le k \le n$ and note that $|a_k| < \varepsilon/(C+1)$ for all such k. Via polynomial interpolation, take the unique polynomial $S : \mathbb{C} \to \mathbb{C}$ of degree at most $n-1$ satisfying $S(g^k(q)) = a_k$ for $1 \le k \le n$. Then the restriction $S|_{\mathbb{C}^*}$ is a Laurent polynomial $\mathbb{C}^* \to \mathbb{C}$ of degree at most $n-1$ satisfying $S|_{\mathbb{C}^*}(g^k(q)) = a_k$ for all $1 \le k \le n$.

Hence, the Laurent polynomial

$$L(z) = Q(z) - S|_{\mathbb{C}^*}(z)$$

satisfies $L(g^k(q)) = 0$ for all $1 \le k < n$ and $L(g^n(q)) = x$ by definition of φ. Since $S|_{\mathbb{C}^*} \in \mathcal{L}_{0,n-1}$, we see that

$$\|S|_{\mathbb{C}^*}\|_{K'} \le C\|S|_{\mathbb{C}^*}\|_M = C \max_{z \in M} |S(z)| = C \max_{1 \le k \le m} |a_k| < \frac{C\varepsilon}{C+1}.$$

Consequently, by the triangle inequality,

$$\|L - \varphi\|_{K'} \leq \|Q - \varphi\|_{K'} + \|S|_{\mathbb{C}^*}\|_{K'} < \frac{\varepsilon}{C+1} + \frac{C\varepsilon}{C+1} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

Now, by taking such a Laurent polynomial $L : \mathbb{C}^* \to \mathbb{C}$, we can construct the required endomorphism $h : \mathbb{C}^* \to \mathbb{C}^*$ by defining

$$h(z) = z \exp(L(z)).$$

Indeed, the post-composition map $\exp_*$ is continuous on the space of holomorphic functions $\mathbb{C}^* \to \mathbb{C}$ (with the compact-open topology). Hence, $\exp(L(z))$ can be taken arbitrarily close to the constant map $z \mapsto 1$ on $\overline{\mathbb{A}_{r,R}}$, for L is arbitrarily close to the constant map $z \mapsto 0$ on $\overline{\mathbb{A}_{r,R}}$. Thus, h can be made arbitrarily close to the identity map on $\overline{\mathbb{A}_{r,R}}$ satisfying $h(g^k(q)) = g^k(q)$ for all $1 \leq k < n$ and $h(g^n(q)) = q$. It follows that the endomorphism $h \circ g : \mathbb{C}^* \to \mathbb{C}^*$ is arbitrarily close to g on $g^{-1}\left(\overline{\mathbb{A}_{r,R}}\right)$ with periodic point q. In particular, as $g(K) \subset \overline{\mathbb{A}_{r,R}}$, we can obtain such $h : \mathbb{C}^* \to \mathbb{C}^*$ so that $h \circ g \in B_K(f, \epsilon)$.

Finally, let $F : \mathbb{C}^* \to \mathbb{C}^*$ be the automorphism

$$F(z) = \frac{q}{p} z.$$

By shrinking V if necessary, we can ensure that $|1 - q/p|$ is as small as desired and hence that $\|F - \mathrm{id}\|_K$ is as small as desired. Thus, the map

$$F^{-1} \circ h \circ g \circ F : \mathbb{C}^* \to \mathbb{C}^*$$

is an endomorphism with periodic point p and lies in $B_K(f, \epsilon)$ after choosing F sufficiently close to the identity map. Since $B_K(f, \epsilon)$ was an arbitrary basis element containing f in $\operatorname{End} \mathbb{C}^*$, this finishes the proof. $\square$

4.2 The case that f is robustly non-expelling at p

As in the proof of the closing lemma for $\mathbb{C}$, we will require a perturbation lemma in our proof of the closing lemma for $\mathbb{C}^*$. Obtaining an endomorphism of $\mathbb{C}^*$ that satisfies the conditions in Lemma 3.2.2 is somewhat trickier than obtaining such an endomorphism of $\mathbb{C}$. However, we will see that much of our arguments will be of a similar spirit.

Lemma 4.2.1. *Let $z_1, \ldots, z_n$ be distinct points in $\mathbb{C}^*$. For every open neighbourhood U of the zero map 0 in $\operatorname{End}\mathbb{C}$, there is an open neighbourhood V of 1 in $\mathbb{C}$ such that for all $\lambda \in V$, there exists a polynomial $h \in U$ so that:*

(i) $h(z_j) = 0$ *for all* $j = 1, \ldots, n$,

(ii) $h'(z_j) = 0$ *for all* $j = 2, \ldots, n$,

(iii) $h'(z_1) = \dfrac{\lambda - 1}{z_1}$.

Proof. Let U be an open neighbourhood of the zero map 0 in $\operatorname{End}\mathbb{C}$. As we saw in the proof of the closing lemma for $\mathbb{C}$, we will show that a polynomial h of degree $2n-1$ will satisfy every condition as stated. Recall that the matrix

$$A = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{2n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{2n-1} \\ 0 & 1 & 2z_1 & \cdots & (2n-1)z_1^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2z_n & \cdots & (2n-1)z_n^{2n-2} \end{bmatrix}$$

is invertible and hence defines a homeomorphism $\mathbf{x} \mapsto A\mathbf{x}$ of $\mathbb{C}^{2n}$ endowed with the Euclidean topology to itself (see Lemma 3.2.2).

As finite-dimensional Hausdorff topological vector spaces of the same dimension are topologically isomorphic and as the subspace $\mathcal{P}_m \subset \operatorname{End}\mathbb{C}$ of polynomials with degree bounded by $m-1$ inherits the compact-open topology, it suffices to work with the usual Euclidean topology on $\mathbb{C}^m$. Hence, we can consider U to be an open neighbourhood of the zero vector $\mathbf{0}$ in $\mathbb{C}^{2n}$.

Let $p_{n+1} : \mathbb{C}^{2n} \to \mathbb{C}$ denote the projection map on the $(n+1)$-th component. Then p_{n+1} is continuous and open by definition. Since the map $\mathbf{x} \mapsto A\mathbf{x}$ is a homeomorphism, $A(U)$ is an open neighbourhood of $\mathbf{0}$ in $\mathbb{C}^{2n}$. Let $T : \mathbb{C} \to \mathbb{C}$ be the automorphism $T(z) = z_1 z + 1$. The composition $T \circ p_{n+1} : \mathbb{C}^{2n} \to \mathbb{C}$ is hence continuous and open. Take $V = (T \circ p_{n+1})(A(U))$ as the open neighbourhood of 1 in $\mathbb{C}$. Then for any $\lambda \in V$, $A(U)$ is an open neighbourhood of $(0, \ldots, 0, (\lambda-1)/z_1, 0, \ldots, 0)^\top$. Hence, U is an open neighbourhood of $A^{-1}(0, \ldots, 0, (\lambda-1)/z_1, 0, \ldots, 0)^\top = (a_1, \ldots, a_{2n})^\top$. But then it follows that the polynomial

$$h(z) = a_1 + a_2 z + \cdots + a_{2n} z^{2n-1}$$

is a map satisfying the conditions in the lemma. □

With this lemma, we obtain the following corollary and hence the perturbation result we require for the proof of the closing lemma for $\mathbb{C}^*$.

Corollary 4.2.2. *Let $z_1, \ldots, z_n$ be distinct points in $\mathbb{C}^*$. For every open neighbourhood U of* id *in* $\operatorname{End}\mathbb{C}^*$, *there is an open neighbourhood V of* 1 *in* $\mathbb{C}$ *such that for all $\lambda \in V$, there exists $H \in U$ so that:*

(i) $H(z_j) = z_j$ *for all* $j = 1, \ldots, n$,

(ii) $H'(z_j) = 1$ *for all* $j = 2, \ldots, n$,

(iii) $H'(z_1) = \lambda$.

Proof. Let U be an open neighbourhood of id in $\operatorname{End}\mathbb{C}^*$. Define an endomorphism H of

$\mathbb{C}^*$ by

$$H(z) = z\exp(h(z)),$$

where $h : \mathbb{C}^* \to \mathbb{C}$ is holomorphic. We will show that we may choose h so that every condition in the corollary is satisfied.

For Riemann surfaces X and Y, denote by $\mathscr{O}(X,Y)$ the set of all holomorphic maps $X \to Y$ endowed with the compact-open topology. For an open subset $Z \subset X$, observe that the restriction map $r : \mathscr{O}(X,Y) \to \mathscr{O}(Z,Y)$, $f \mapsto f|_Z$, is continuous. Also, for a given Riemann surface Z, we see that for any $g \in \mathscr{O}(Y,Z)$, the inclusion $\iota_g^1 : \mathscr{O}(X,Y) \hookrightarrow \mathscr{O}(Y,Z) \times \mathscr{O}(X,Y)$ defined by $f \mapsto (g,f)$ is continuous.

Next, we claim that the map $\phi : \mathscr{O}(X,Y) \times \mathscr{O}(X,Y) \to \mathscr{O}(X, Y \times Y)$ defined by

$$(f,g) \mapsto f \times g$$

where $(f \times g)(x) = (f(x), g(x))$ is continuous. Let $K \subset X$ be compact and let $W, W' \subset Y$ be open. Let $V(K, W \times W')$ be a subbasis element of the compact-open topology on $\mathscr{O}(X, Y \times Y)$. Then

$$\begin{aligned}
\phi^{-1}(V(K, W \times W')) &= \{(f,g) \in \mathscr{O}(X,Y) \times \mathscr{O}(X,Y) : f(K) \times g(K) \subset W \times W'\} \\
&= \{(f,g) \in \mathscr{O}(X,Y) \times \mathscr{O}(X,Y) : f(K) \subset W \text{ and } g(K) \subset W'\} \\
&= V(K,W) \times V(K,W').
\end{aligned}$$

But by definition, $V(K,W) \times V(K,W')$ is open in $\mathscr{O}(X,Y) \times \mathscr{O}(X,Y)$. Since $V(K, W \times W')$ was an arbitrary subbasis element, it follows that ϕ is continuous.

We now specialise to $\mathbb{C}$ and $\mathbb{C}^*$. Observe that the multiplication map $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$, $(z,w) \mapsto zw$, is holomorphic. Since the composition map $(f,g) \mapsto f \circ g$ is continuous whenever defined, we have a sequence of continuous maps as follows:

$$\begin{aligned}\operatorname{End}\mathbb{C} &\xrightarrow{r} \mathscr{O}(\mathbb{C}^*,\mathbb{C}) \xrightarrow{\iota^1_{\exp}} \mathscr{O}(\mathbb{C},\mathbb{C}^*)\times\mathscr{O}(\mathbb{C}^*,\mathbb{C}) \xrightarrow{\circ} \operatorname{End}\mathbb{C}^*\\ &\xrightarrow{\iota^1_{\mathrm{id}}} \operatorname{End}\mathbb{C}^*\times\operatorname{End}\mathbb{C}^* \xrightarrow{\phi} \mathscr{O}(\mathbb{C}^*,\mathbb{C}^*\times\mathbb{C}^*)\\ &\xrightarrow{\iota^1_m} \mathscr{O}(\mathbb{C}^*\times\mathbb{C}^*,\mathbb{C}^*)\times\mathscr{O}(\mathbb{C}^*,\mathbb{C}^*\times\mathbb{C}^*) \xrightarrow{\circ} \operatorname{End}\mathbb{C}^*,\\ &f(z)\mapsto \mathrm{id}(z)\exp(f|_{\mathbb{C}^*}(z)).\end{aligned}$$

Hence, as the composition of continuous maps is continuous, the above map is continuous. Call this map F. As U is an open neighbourhood of id in $\operatorname{End}\mathbb{C}^*$, $F^{-1}(U)$ is an open neighbourhood of the zero map 0 in $\operatorname{End}\mathbb{C}$. By Lemma 4.2.1, there is an open neighbourhood V of 1 in $\mathbb{C}$ such that for all $\lambda \in V$, there exists $h \in F^{-1}(U)$ so that $h(z_j) = 0$ for all $j = 1, \ldots, n$, $h'(z_j) = 0$ for all $j = 2, \ldots, n$, and $h'(z_1) = (\lambda - 1)/z_1$. Choose such $h \in F^{-1}(U)$ and such open neighbourhood $V \subset \mathbb{C}$. Then we have $H(z) = z\exp(h|_{\mathbb{C}^*}(z)) \in U$ with derivative $H'(z) = \exp(h|_{\mathbb{C}^*}(z))\,(1 + zh|'_{\mathbb{C}^*}(z))$. But by definition of h, $H(z_j) = z_j$ for all $j = 1, \ldots, n$, $H'(z_j) = 1$ for all $j = 2, \ldots, n$, and $H'(z_1) = \lambda$ for any $\lambda \in V$. This completes the proof. $\square$

The topology on $\mathbb{C}^*$ will also play a pivotal role in our proof. It is well known that the Riemann mapping theorem completely classifies any non-empty simply connected open subset of $\mathbb{C}$; every such subset is either $\mathbb{C}$ itself, or it is isomorphic to the unit disc $\mathbb{D}$. By Liouville's theorem, $\mathbb{C}$ and $\mathbb{D}$ are not isomorphic, and so we have a mutually disjoint classification.

For domains that are not simply connected, we do not have such a strong result. However, there is a lesser known theorem that addresses domains in $\mathbb{C}$ with finitely many connected components in its complement. But first, we introduce some terminology. We say that a domain $U \subset \mathbb{C} \subset \mathbb{P}$ is *doubly connected* if $\mathbb{P}\setminus U$ has two connected components, and in general, we say that U is *n-connected* if $\mathbb{P} \setminus U$ has $n \geq 1$ connected components. Note that $U \subset \mathbb{C}$ is simply connected if and only if $\mathbb{P} \setminus U$ is connected.

Theorem 4.2.3. *Let $U \subset \mathbb{C}$ be a domain that is n-connected, with $n \geq 2$. If none of the connected components of $\mathbb{P} \setminus U$ are singletons, then U is isomorphic to an annulus $\mathbb{A}_r = \{z \in \mathbb{C} : 1 < |z| < r\}$, where $r > 1$, minus $n-2$ mutually disjoint concentric closed arcs lying on circles in $\mathbb{A}_r$ centred at zero.*

Proof. See [10, Theorem 4.2.3]. □

In particular, if U is a doubly connected domain in $\mathbb{C}$ for which both of the connected components of $\mathbb{P}\setminus U$ are not singletons, then U is isomorphic to $\mathbb{A}_r$ for some $r > 0$. We note that this theorem is not as strong as the Riemann mapping theorem since annuli $\mathbb{A}_r$ and $\mathbb{A}_{r'}$ are isomorphic if and only if $r = r'$. As such, our classification of finitely connected domains is only up to the number of connected components of their complement in $\mathbb{P}$. For $n \geq 2$, two n-connected domains U and U' need not be isomorphic to each other, but they must be isomorphic to some annuli minus $n-2$ concentric arcs.

Next, we prove a classification theorem regarding Runge sets in $\mathbb{C}^*$.

Lemma 4.2.4. *Suppose $U \subset \mathbb{C}^*$ is relatively compact, connected, and Runge in $\mathbb{C}^*$. Then U is either simply connected or doubly connected. Moreover, if U is doubly connected, then $\mathbb{P} \setminus U$ is the union of two disjoint compact neighbourhoods W_0 and W_∞ of 0 and ∞ in $\mathbb{P}$ respectively.*

Proof. Since U is relatively compact in $\mathbb{C}^*$, there exists numbers $0 < s < t$ such that $U \subset \overline{\mathbb{A}_{s,t}} = \{z \in \mathbb{C} : s \leq |z| \leq t\}$. Hence, $\mathbb{P} \setminus \overline{\mathbb{A}_{s,t}} \subset \mathbb{P} \setminus U$. Observe that $\mathbb{P} \setminus \overline{\mathbb{A}_{s,t}}$ has precisely two connected components, one containing 0 and one containing ∞. Call these connected components V_0 and V_∞ respectively. As the inclusion $\mathbb{P} \setminus \overline{\mathbb{A}_{s,t}} \hookrightarrow \mathbb{P} \setminus U$ is continuous, V_0 and V_∞ must be contained in connected components W_0 and W_∞ in $\mathbb{P}\setminus U$, respectively. Note that either $W_0 \cap W_\infty = \varnothing$ or $W_0 = W_\infty$.

The lemma will be proven once we show that any connected component of $\mathbb{P}\setminus U$ must contain either 0 or ∞, for then W_0 and W_∞ are the only connected components of $\mathbb{P}\setminus U$. Suppose on the contrary that $C \subset \mathbb{P}\setminus U$ is a connected component that does not contain either 0 and ∞. We will show that C is a relatively compact component of $\mathbb{C}^* \setminus U$. As $C \cap U = \varnothing$,

$$C \subset \mathbb{C}^* \setminus U \subset \mathbb{P} \setminus U.$$

Thus, C is a connected subset of $\mathbb{C}^* \setminus U$.

Now, since C does not contain 0 and ∞ and as it is a connected component of $\mathbb{P}\setminus U$, we have $C \cap V_0 = \varnothing$ and $C \cap V_\infty = \varnothing$. Hence, $(\mathbb{P}\setminus \overline{\mathbb{A}_{s,t}}) \cap C = \varnothing$ and so $C \subset \overline{\mathbb{A}_{s,t}}$. Taking closures with respect to $\mathbb{C}^*$, we have $\overline{C} \subset \overline{\mathbb{A}_{s,t}}$. It follows that C is relatively compact in $\mathbb{C}^*$.

Next, assume that $C' \supset C$, where C' is connected in $\mathbb{C}^*\setminus U$. As the inclusion $\mathbb{C}\setminus U \hookrightarrow \mathbb{P}\setminus U$, is continuous, C' is connected in $\mathbb{P}\setminus U$. But C was assumed to be a connected component of $\mathbb{P}\setminus U$, and so we must have $C = C'$. Consequently, C must be a maximal connected set in $\mathbb{C}^*\setminus U$ and thus a connected component of $\mathbb{C}^*\setminus U$. We conclude that C is a relatively compact, connected component of $\mathbb{C}^*\setminus U$, contradicting the fact that U is Runge in $\mathbb{C}^*$. Thus, any connected component of $\mathbb{P}\setminus U$ contains either 0 or ∞, whence U is either simply connected or doubly connected.

Finally, if U is doubly connected, then $\mathbb{P}\setminus U = W_0 \cup W_\infty$, where $W_0 \cap W_\infty = \varnothing$. But as seen above, the connected components V_0 and V_∞ of $\mathbb{P}\setminus \overline{\mathbb{A}_{s,t}}$ are contained in W_0 and W_∞ respectively. Since V_0 and V_∞ are the only connected components of $\mathbb{P}\setminus\overline{\mathbb{A}_{s,t}}$, they are open in $\mathbb{P}\setminus\overline{\mathbb{A}_{s,t}}$. Hence, V_0 and V_∞ are open in $\mathbb{P}$ as $\mathbb{P}\setminus\overline{\mathbb{A}_{s,t}}$ is open in $\mathbb{P}$. Note that W_0 and W_∞ are closed in $\mathbb{P}\setminus U$ since they are connected components. As $\mathbb{P}\setminus U$ is closed in $\mathbb{P}$, W_0 and W_∞ are closed in $\mathbb{P}$ and hence compact. This shows that W_0 and W_∞ are disjoint compact neighbourhoods of 0 and ∞ respectively. $\square$

The final preparatory result that we will need concerns vector fields. Let X be a Riemann surface and let $z = x + iy$ denote a coordinate on an open set in X. Here, we adopt terminology presented in [3]. We say that a holomorphic vector field η on X is an $\operatorname{End} X$-*velocity* if there is a holomorphic map $\Psi : \mathbb{C} \times X \to X$ such that

- $\Psi_t = \Psi(t, \cdot) \in \operatorname{End} X$ for all $t \in \mathbb{C}$,
- $\Psi_0 = \mathrm{id}_X$, and
- $\left.\dfrac{\partial}{\partial t}\Psi(z)\right|_{t=0} \dfrac{d}{dz} = \eta.$

Remark. As in the proof of the closing lemma for $\mathbb{C}$, the following set will make a crucial appearance. Let $p \in \mathbb{C}^*$ be a non-wandering point of $f \in \operatorname{End}\mathbb{C}^*$ and suppose f is robustly non-expelling at p. Then we can take an open neighbourhood W of f in $\operatorname{End}\mathbb{C}^*$, an open neighbourhood V of p in $\mathbb{C}^*$, and numbers $0 < r < R$ such that $g^j(V) \subset \overline{\mathbb{A}_{r,R}} = \{z \in \mathbb{C}^* : r \leq |z| \leq R\}$ for all $g \in W$ and $j \geq 0$. Let U be the interior of the closed set

$$T = \left\{(z, g) \in \mathbb{C}^* \times \operatorname{End}\mathbb{C}^* : g^j(z) \in \overline{\mathbb{A}_{r,R}} \text{ for all } j \geq 0\right\}$$

and let U_f be the slice $\{z \in \mathbb{C}^* : (z, f) \in U\}$. Once again, observe that U_f is a non-empty, relatively compact, open subset of $\mathbb{C}^*$ such that $f(U_f) \subset U_f$.

The following lemma is a specific version of [3, Lemma 2], whose proof has been adapted and expounded upon for our purposes.

Lemma 4.2.5. *Let $f \in \operatorname{End}\mathbb{C}^*$ be robustly non-expelling at $p \in \mathbb{C}^*$. Let $C \subset U_f$ be a non-empty compact set that is forward invariant under f. Then there does not exist a continuous, zero-free, f-invariant vector field on C which is uniformly approximable by* $\operatorname{End}\mathbb{C}^*$*-velocities on C.*

Remark. Here, we say that a vector field ξ is *f-invariant* on C if $\xi(f(x)) = f'(x)\xi(x)$ for all $x \in C$.

Proof. Adopting the notation as defined in the previous remark, we proceed by contradiction. Let ξ be a continuous, zero-free, f-invariant vector field on C which is uniformly approximable on C by $\operatorname{End}\mathbb{C}^*$-velocities. Then there exists a continuous map $g : C \to \mathbb{C}^*$ such that $\xi = g\dfrac{d}{dz}$. We first claim that

$$M = \sup_{j\geq 0, x\in C} |(f^j)'(x)| \tag{4.1}$$

is finite. Indeed, for each $x \in C$, there exists $r_x > 0$ such that the closed disc $\overline{D}(x, r_x)$ is contained in U_f. The collection $(D(x, r_x/2))_{x\in C}$ is an open cover of C and so as C is compact, there exists a finite subcover $(D(x_m, r_{x_m}/2))_{m=1,\ldots,n}$ of C. Fix $j \geq 0$. For a given $m = 1, \ldots, n$, let $x \in C \cap D(x_m, r_{x_m}/2)$. By Cauchy's integral formula,

$$(f^j)'(x) = \frac{1}{2\pi i}\int_{\partial D(x_m, r_{x_m})} \frac{f^j(\zeta)}{(\zeta - x)^2}\, d\zeta.$$

Thus,

$$|(f^j)'(x)| \leq \frac{1}{2\pi}\int_{\partial D(x_m, r_{x_m})} \frac{|f^j(\zeta)|}{|\zeta - x|^2}\, |d\zeta| \leq r_{x_m} \sup_{\zeta\in\partial D(x_m, r_{x_m})} \frac{|f^j(\zeta)|}{|\zeta - x|^2}.$$

Now, as $f^j(w) \in \overline{\mathbb{A}_{r,R}}$ for all $j \geq 0$ and $w \in U_f$, $|f^j(\zeta)| \leq R$ for all $\zeta \in \partial D(x_m, r_{x_m})$. Since $x \in D(x_m, r_{x_m}/2)$ and $|\zeta - x_m| = r_{x_m}$ for any $\zeta \in \partial D(x_m, r_{x_m})$, the reverse triangle inequality yields

$$|\zeta - x| \geq |\zeta - x_m| - |x_m - x| = r_{x_m} - |x_m - x| > r_{x_m} - r_{x_m}/2 = r_{x_m}/2.$$

Hence,

$$\sup_{\zeta\in\partial D(x_m, r_{x_m})} \frac{|f^j(\zeta)|}{|\zeta - x|^2} \leq \frac{R}{(r_{x_m}/2)^2} = \frac{4R}{r_{x_m}^2}.$$

Combining everything, we get

$$|(f^j)'(x)| \leq \frac{4R}{r_{x_m}}$$

for any $x \in C \cap D(x_m, r_{x_m}/2)$. Since $m = 1, \dots, n$ was arbitrary, this inequality holds for any such m. Set $N = \max\{4R/r_{x_m} : m = 1, \dots, n\}$. Then for any $m = 1, \dots, n$, $|(f^j)'(x)| \leq N$ for all $x \in C \cap D(x_m, r_{x_m}/2)$. In fact, this inequality holds for any $x \in C \cap \bigcup_{m=1}^{n} D(x_m, r_{x_m}/2) = C$. Since $j \geq 0$ was arbitrary, we see that

$$|(f^j)'(x)| \leq N < \infty$$

for all $j \geq 0$ and $x \in C$. It follows that M is finite, as was claimed.

Next, since ξ is continuous and zero-free and as C is compact, $|g|$ attains a minimum value on C which cannot be zero. Call this value c. Since ξ is uniformly approximable on C by $\operatorname{End}\mathbb{C}^*$-velocities, there exists an $\operatorname{End}\mathbb{C}^*$-velocity η such that

$$\sup_{x \in C} |\xi(x) - \eta(x)| \leq \frac{c}{2M}. \tag{4.2}$$

Let $\Psi : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C}^*$ be the associated holomorphic map to η, as per the definition of an $\operatorname{End}\mathbb{C}^*$-velocity.

Because $C \times \{f\} \subset U$, where U is the interior of the set T as defined in the above remark, there is an open neighbourhood W' of f in $\operatorname{End}\mathbb{C}^*$ such that $H \times W' \subset U$. Since $\Phi_t \to \mathrm{id}_{\mathbb{C}^*}$ as $t \to 0$, by continuity, there exists $\delta > 0$ such that $\Psi_t \circ f \in W'$ when $|t| < \delta$. It follows from the robustly non-expelling property that

$$(\Psi_t \circ f)^j(x) \in \overline{\mathbb{A}_{r,R}}$$

when $|t| < \delta$, for any $x \in C$ and $j \geq 0$. We next claim that there exists a constant k such that for all $x \in C$ and $j \geq 0$,

$$\left| \frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\bigg|_{t=0} \right| \leq k. \tag{4.3}$$

Let $j \geq 0$ and fix $x \in C$. Then for any $t_0 \in D(0, \delta/4)$, Cauchy's integral formula gives

$$\frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\bigg|_{t=t_0} = \frac{1}{2\pi i} \int_{\partial D(0,\delta/2)} \frac{(\Psi_s \circ f)^j(x)}{(s - t_0)^2} ds$$

and hence we have

$$\left| \frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\Big|_{t=t_0} \right| \leq \delta \sup_{s\in\partial D(0,\delta/2)} \frac{|(\Psi_s \circ f)^j(x)|}{|s-t_0|^2}.$$

Since $(\Psi_s \circ f)^j(x) \in \overline{\mathbb{A}_{r,R}}$ when $|s| < \delta$, for any $x \in C$ and $j \geq 0$,

$$|(\Psi_s \circ f)^j(x)| \leq R$$

for all $s \in \partial D(0, \delta/2)$. As $|t_0| < \delta/4$ and $|s| = \delta/2$, the reverse triangle inequality yields

$$|s - t_0| \geq |s| - |t_0| = \delta/2 - |t_0| > \delta/2 - \delta/4 = \delta/4.$$

Hence, combining everything, we have

$$\left| \frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\Big|_{t=t_0} \right| \leq \delta \frac{R}{(\delta/4)^2} = \frac{16R}{\delta}$$

for all $t_0 \in D(0, \delta/4)$ and for all $j \geq 0$. Moreover, this inequality holds for any $x \in C$. In particular, taking $k = 16R/\delta$ and $t_0 = 0$ verifies the claim.

Now, by the chain rule,

$$\frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\Big|_{t=0} = \sum_{i=0}^{j} (f^i)'(f^{j-i}(x)) \, \frac{\partial}{\partial t}\Psi_t(f^{j-i}(x))\Big|_{t=0}$$

for any $x \in C$ and $j \geq 0$. As η is an $\operatorname{End}\mathbb{C}^*$-velocity, we see that

$$\frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\Big|_{t=0} \frac{d}{dz}\Big|_{z=x} = \sum_{i=0}^{j} (f^j)'(f^{j-i}(x))\eta(f^{j-i}(x)) \tag{4.4}$$

for any $x \in C$ and $j \geq 0$.

Fix $j \geq 1$. Then for $x \in C$ and $i = 0, \ldots, j-1$, by using the f-invariance of ξ and forward invariance of C under f, we get

$$
\begin{aligned}
&\left|(f^i)'(f^{j-i}(x))\eta(f^{j-i}(x)) - \xi(f^j(x))\right| \\
&\quad = \left|(f^i)'(f^{j-i}(x))\eta(f^{j-i}(x)) - \left(\prod_{a=1}^{i} f'(f^{j-a}(x))\right)\xi(f^{j-i}(x))\right| \\
&\quad = \left|(f^i)'(f^{j-i}(x))\eta(f^{j-i}(x)) - (f^i)'(f^{j-i}(x))\xi(f^{j-i}(x))\right| \\
&\quad = \left|(f^i)'(f^{j-i}(x))\right| \left|\eta(f^{j-i}(x)) - \xi(f^{j-i}(x))\right| \\
&\quad \leq M\frac{c}{2M} = \frac{c}{2},
\end{aligned}
$$

where we have used (4.1) and (4.2) to obtain the inequality. Hence, by using (4.4), for any $j \geq 0$ and $x \in H$ we obtain

$$
\left| \frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\bigg|_{t=0} \frac{d}{dz}\bigg|_{z=x} - j\xi(f^j(x)) \right| \leq \frac{cj}{2}.
$$

But by the reverse triangle inequality, for any $x \in C$ and $j \geq 0$,

$$
\begin{aligned}
&\left| \frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\bigg|_{t=0} \frac{d}{dz}\bigg|_{z=x} - j\xi(f^j(x)) \right| \\
&\qquad \geq \left| \frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\bigg|_{t=0} \frac{d}{dz}\bigg|_{z=x} \right| - j|\xi(f^j(x))| \\
&\qquad \geq cj - k,
\end{aligned}
$$

where we have used (4.3) to obtain the final inequality. It follows that for any $x \in C$ and $j \geq 0$,

$$
cj - k \leq \left| \frac{\partial}{\partial t}(\Psi_t \circ f)^j(x)\bigg|_{t=0} \frac{d}{dz}\bigg|_{z=x} - j\xi(f^j(x)) \right| \leq \frac{cj}{2}.
$$

In other words, $cj/2 \leq k$, which is impossible for all sufficiently large $j \geq 0$. □

We are now equipped to prove the closing lemma for $\mathbb{C}^*$ in the case that f is robustly non-expelling at p.

Theorem 4.2.6. *Let $p \in \mathbb{C}$ and $f \in \operatorname{End} \mathbb{C}^*$. Suppose that p is a non-wandering point of f and that f is robustly non-expelling at p. Then p is a periodic point of f.*

The strategy of this proof is comparatively more complicated than the analogous proof of Theorem 3.2.3.

1. We again use our hypotheses to construct the relatively compact Runge set $U_f \subset \mathbb{C}^*$ containing p that is forward invariant under f.

2. Use Lemma 4.2.4 to deduce that the connected component U_0 of U_f containing p is either simply connected or doubly connected. The proof will then bifurcate to dealing with the case when U_0 is simply connected, and the case when U_0 is doubly connected.

3. For the case that U_0 is simply connected, the proof proceeds similarly to the case of $\mathbb{C}$. As such, we provide an abridged version, with only the necessary details included.

4. The case that U_0 is doubly connected is harder. We begin by constructing a Runge set $\Omega \subset U_f$ out of certain connected components of U_f, with U_0 one such connected component. Then, we show that Ω is forward invariant under f.

5. Use Lemma 3.2.1 to deduce that p and some iterates $f^j(p)$ are recurrent points of f. We also show that a subsequence of $(f^n)_{n\in\mathbb{N}}$ converges locally uniformly to a map ρ on Ω that fix p and these iterates $f^j(p)$.

6. On each connected component of Ω, the identity theorem yields that the set of fixed points of ρ is either discrete or the connected component itself. If any of these sets are discrete, then it will follow that p is periodic.

7. So we are left to deal with the case that the set of fixed points of ρ is Ω itself. We will show that this case can never arise by obtaining a contradiction.

8. Assuming that ρ pointwise fixes Ω, we will prove that $f|_\Omega$ is an automorphism and hence that Ω must be connected. Consequently, $\Omega = U_0$.

9. We then use Weil's lemma to conclude that the group $G = \overline{\{(f|_{U_0})^n : n \in \mathbb{Z}\}}$ is either isomorphic to $\mathbb{Z}$ or it is compact in $\operatorname{Aut} U_0$. Similar to the case of $\mathbb{C}$, the former cannot arise, and so we reduce the proof to when G is compact.

10. Using Theorem 4.2.3, we deduce that $f|_{U_0}$ must correspond to an irrational rotation of an annulus. With this observation, we will extract a contradiction to Lemma 4.2.5 and thus conclude the proof.

Proof. Take an open neighbourhood W of f in $\operatorname{End}\mathbb{C}^*$, an open neighbourhood V of p in $\mathbb{C}^*$, and numbers $0 < r < R$ such that $g^j(V) \subset \overline{\mathbb{A}_{r,R}}$ for all $g \in W$ and $j \geq 0$. Let U be the interior of the closed set

$$T = \left\{(z, g) \in \mathbb{C}^* \times \operatorname{End}\mathbb{C}^* : g^j(z) \in \overline{\mathbb{A}_{r,R}} \text{ for all } j \geq 0\right\}$$

and let U_f be the slice $\{z \in \mathbb{C}^* : (z, f) \in U\}$. Note that U_f is a non-empty, relatively compact, open subset of $\mathbb{C}^*$ such that $f(U_f) \subset U_f$. With this setting, we will prove that p is a periodic point of f. As in the case for $\mathbb{C}$, we achieve this through a series of claims.

Claim 1. *U_f is Runge in $\mathbb{C}^*$, that is, $\mathbb{C}^* \setminus U_f$ has no compact connected components.* Our strategy will be to utilise Corollary 2.2.5 and Theorem 2.2.6. Suppose there exists an open compact subset $K \subset \mathbb{C}^* \setminus U_f$. Then by Theorem 2.2.6, we can find a relatively compact, open set $V' \subset \mathbb{C}^*$ for which $K \subset V'$ and $\partial V' \subset U_f$. We will argue that $V' \subset U_f$. As $\partial V' \subset U_f$, $\partial V' \times \{f\} \subset U$. Observe that $\partial V'$ is compact in $\mathbb{C}^*$, for it is closed and V' is relatively compact. So there exist open sets $U' \subset \mathbb{C}^*$ and $W' \subset \operatorname{End}\mathbb{C}^*$ such that $\partial V' \times \{f\} \subset U' \times W' \subset U$. Hence, $\partial V' \times W' \subset U$. Let $g \in W'$. Then as $\partial V' \times \{g\} \subset U \subset T$, $g^j(\partial V') \subset \overline{\mathbb{A}_{r,R}}$ for all $j \geq 0$. Thus, for any $z \in \partial V'$, we have $r \leq |g^j(z)| \leq R$ for all $j \geq 0$. Now, by the maximum modulus principle, for each $j \geq 0$, the modulus of $g^j|_{\overline{V'}}$ attains a maximum and minimum somewhere on $\partial V'$. Therefore, for any $z \in V'$ and $j \geq 0$,

$$r \leq \min_{x\in\partial V'} |g^j(x)| = \min_{x\in\overline{V'}} |g^j(x)| < |g^j(z)| < \max_{x\in\overline{V'}} |g^j(x)| = \max_{x\in\partial V'} |g^j(x)| \leq R.$$

As $z \in V'$ was arbitrary, $g^j(V') \subset \overline{\mathbb{A}_{r,R}}$ for all $j \geq 0$ and thus $V' \times \{g\} \subset T$. But as $g \in W'$ was arbitrary, we deduce that $V' \times W' \subset T$. Since $V' \times W'$ is open, it follows that $V' \times W' \subset U$. Consequently, $V' \times \{f\} \subset U$ and so $V' \subset U_f$, as required. But since $V' \supset K$ and $K \subset \mathbb{C}^* \setminus U_f$, the only open compact subset of $\mathbb{C}^* \setminus U_f$ must be the empty set. By Corollary 2.2.5, this shows that $\mathbb{C}^* \setminus U_f$ has no compact connected components, proving Claim 1.

It follows that each connected component of U_f is also Runge. Let U_0 denote the connected component of U_f containing p. Note that as U_f is relatively compact in $\mathbb{C}^*$, U_0 is relatively compact in $\mathbb{C}^*$. So by Lemma 4.2.4, U_0 is either simply connected or doubly connected. As p is non-wandering, there exists a smallest integer $\ell \geq 1$ for which $f^\ell(U_0) \cap U_0 \neq \varnothing$. Since U_f is forward invariant under f and as U_0 is a connected component of U_f, $f^\ell(U_0) \subset U_0$. Set $g = f^\ell$. Note that by Lemma 3.2.1, p is a recurrent point for g (and hence of f). Also recall that we can extract a subsequence $(g^{j_k})_{k\in\mathbb{N}}$ that converges locally uniformly on U_f to a holomorphic limit $h : U_f \to \mathbb{C}^*$ that fixes p.

Case 1: U_0 is simply connected. We proceed similarly to the proof of the closing lemma for $\mathbb{C}$. Since (g^{j_k}) converges locally uniformly on U_f to $h : U_f \to \mathbb{C}^*$, (g^{j_k}) also converges locally uniformly on U_0 to $\rho = h|_{U_0}$. Let $M \subset U_0$ be the set of fixed points of ρ and let M_0 be the connected component of M containing p. From the identity theorem, either M is discrete and so $M_0 = \{p\}$, or $M = M_0 = U_0$. The latter occurs when $\rho = \mathrm{id}$.

Because ρ is the uniform limit of a subsequence of (g^j) on U_0 and as $g(U_0) \subset U_0$, g and ρ commute on U_0 and it follows that $g(M) \subset M$. As p is a recurrent point of g, we have $g^{j_k}(p) \to p$ as $k \to \infty$. In particular, if M is discrete, then as $g^{j_k}(p) \in M$ for all $k \in \mathbb{N}$, we have a convergent sequence of points in a discrete set. Such a sequence is eventually constant, that is, $g^{j_k}(p) = p$ for all sufficiently large k. It follows that p is a periodic point of g and hence of f, thereby completing the proof.

On the other hand, if $M = U_0$ and ρ is the identity map on U_0, then we argue as follows. Our arguments will again be similar to the proof given for $\mathbb{C}$.

Claim 2. *The map $g|_{U_0} : U_0 \to U_0$ is an automorphism of U_0.* The proof of this claim proceeds verbatim as the one given in the proof for the closing lemma for $\mathbb{C}$.

Now, as U_0 is simply connected, it is isomorphic to the unit disc $\mathbb{D}$ by the Riemann mapping theorem. It follows that the group of automorphisms $\operatorname{Aut} U_0$ is isomorphic to $\operatorname{Aut} \mathbb{D} \cong \mathrm{PSL}(2, \mathbb{R})$. Subsequently, $\operatorname{Aut} U_0$ is a 3-dimensional real Lie group.

Next, the group generated by $g|_{U_0}$ is an abelian subgroup of $\operatorname{Aut} U_0$, and so the closure $G = \overline{\{(g|_{U_0})^n : n \in \mathbb{Z}\}}$ of this subgroup in $\operatorname{Aut} U_0$ is also an abelian subgroup. By Weil's lemma, G is isomorphic to the discrete group $\mathbb{Z}$ or it is compact. Analogously to the proof of the closing lemma for $\mathbb{C}$, if G is isomorphic to the discrete group $\mathbb{Z}$, then some iterate of $g|_{U_0}$ must be the identity map for a sequence of iterates converges locally uniformly to id on U_0. But then G is a finite group, which is clearly impossible. So we are again left to deal with the case that G is a compact abelian subgroup of $\operatorname{Aut} U_0$. We will show that this can never occur, and hence that p is a periodic point of f, by deriving a similar contradiction in the proof of Theorem 3.2.3.

By Theorem 2.5.1, we see that G is isomorphic to $(S^1)^n \times A$, where $n \geq 0$ and A is a finite abelian subgroup. But since the maximal compact subgroups of $\mathrm{PSL}(2, \mathbb{R})$ are 1-dimensional, we must have $n \leq 1$. Taking the connected component containing the identity map G_0 in G, which is a compact subgroup of G, we have $G_0 \cong (S^1)^n$ where $n \leq 1$. Hence, the orbit $G_0 z$ for any $z \in U_0$ is either the singleton $\{z\}$, or a 1-dimensional, connected, compact submanifold of U_0. As seen previously, we observe that

$G_0 z$ is diffeomorphic to the unit circle when we are in the latter situation.

Since U_0 is isomorphic to $\mathbb{D}$, we may apply the reasoning presented in the proof of the closing lemma for $\mathbb{C}$. Indeed, as U_0 is simply connected, it is Runge with respect to $\mathbb{C}$. Hence, taking the hull of a subset of U_0 with respect to $\mathbb{C}^*$ coincides with taking its hull with respect to $\mathbb{C}$. So considering U_0 as a Runge set in $\mathbb{C}$, we have $\widehat{G_0 z} \subset U_0$, where $\widehat{G_0 z}$ is the union of the connected set $G_0 z$ and the unique unbounded connected component of $\mathbb{C} \setminus G_0 z$ for any $z \in U_0$. It follows that $G_0 z = \widehat{G_0 z}$ if and only if $z \in U_0$ is fixed by G_0. Producing $q \in U_0$ that is fixed by G_0 as per Claim 3 in the proof of Theorem 3.2.3, we therefore obtain a periodic point of g and hence of f. We then proceed with our argument as seen towards the end of the proof of Theorem 3.2.3. If q has period τ under g, then q has period $\tau\ell$ under f. So for all $j = 0, \ldots, \tau\ell - 1$, $f^j(q)$ are distinct points in U_f. Furthermore, as a subsequence of (f^n) converges locally uniformly to the identity map on U_0, $\left|\left(f^{\tau\ell}\right)'(q)\right| = 1$.

Now, as U is the interior of the closed set

$$T = \left\{(z, g) \in \mathbb{C}^* \times \operatorname{End}\mathbb{C}^* : g^j(z) \in \overline{\mathbb{A}_{r,R}} \text{ for all } j \geq 0\right\},$$

it is an open neighbourhood of (q, f) in $\mathbb{C}^* \times \operatorname{End}\mathbb{C}^*$. Hence, there exists an open neighbourhood U' of the identity map in $\operatorname{End}\mathbb{C}^*$ such that $(q, \alpha \circ f) \in U$ for all $\alpha \in U'$. By Corollary 4.2.2, there is a corresponding open neighbourhood V' of 1 in $\mathbb{C}$ such that for all λ in V', there is α in U' fixing $f^j(q)$ for all $j = 0, \ldots, \tau\ell - 1$ and whose derivative satisfies

$$\alpha'(q) = \lambda, \quad \alpha'(f(q)) = 1, \quad \ldots, \quad \alpha'\left(f^{\tau\ell-1}(q)\right).$$

Take $|\lambda| > 1$. But then continuing with a completely verbatim argument with the Cauchy estimates as in the proof of Theorem 3.2.3 supplies the required contradiction. As such, G cannot be a compact abelian subgroup of $\operatorname{Aut} U_0$ and so we may dispose of the case when U_0 is isomorphic to $\mathbb{D}$.

Case 2: U_0 is doubly connected. By the reasoning presented at the beginning of the proof for Case 1, either U_0 is pointwise fixed by h, or p is a periodic point of g and hence of f. So suppose that U_0 is pointwise fixed by h. By Theorem 4.2.3 and Lemma 4.2.4, U_0 is isomorphic to an annulus $\mathbb{A}_{r'} = \{z \in \mathbb{C} : 1 < |z| < r'\}$, $r' > 1$. For $j = 0, \ldots, \ell - 1$, let U_j be the connected component of U_f containing $f^j(U_0)$. Note that $g(U_j) \subset U_j$ for all $j = 0, \ldots, \ell - 1$. Define $\Omega = \bigcup_{j=0}^{\ell-1} U_j$. Since Ω is a union of disjoint Runge sets, it is also Runge (observe that some of the U_j may coincide). Further note that Ω is relatively compact, for it is a subset of U_f. Observe that for each $j = 0, \ldots, \ell - 1$, we have $f(U_j) \subset U_{j+1}$, where $U_\ell = U_0$. Indeed, since U_j is the connected component of U_f containing $f^j(U_0)$,

$$f(U_j) \supset f(f^j(U_0)) = f^{j+1}(U_0)$$

and so $f(U_j) \cap U_{j+1} \neq \varnothing$. As the U_j are connected components of U_f and U_f is forward invariant under f, it follows that $f(U_j) \subset U_{j+1}$. Hence, $f(\Omega) = \bigcup_{j=0}^{\ell-1} f(U_j) \subset \bigcup_{j=0}^{\ell-1} U_j = \Omega$ and so Ω is forward invariant under f.

Since (g^{j_k}) converges locally uniformly on U_f to $h : U_f \to \mathbb{C}^*$, it converges locally uniformly on Ω to $\rho = h|_\Omega$. Let $M \subset \Omega$ be the set of fixed points of ρ in Ω. Note that $U_0 \subset M$ by assumption. Further observe that for $j = 0, \ldots, \ell - 1$, $g^{j_k}(f^j(p)) = f^j(g^{j_k}(p)) \to f^j(p)$ as $k \to \infty$, since p is recurrent for g. Hence, for $j = 0, \ldots, \ell - 1$, $f^j(p)$ is recurrent for g. It follows that $h(f^j(p)) = f^j(p)$ for all $j = 0, \ldots, \ell - 1$. Let M_j denote the connected component of M containing $f^j(p)$ and note that $M_0 = U_0$. Also note that $M_j \subset U_j$ for all $j = 0, \ldots, \ell - 1$.

Now, for each $j = 1, \ldots, \ell - 1$, either $M \cap U_j$ is discrete and so M_j is the singleton $\{p_j\}$, or $M_j = U_j$. Suppose that $M \cap U_j$ is discrete for some $j = 1, \ldots, \ell - 1$. Since ρ is the uniform limit of a subsequence of (g^n) on Ω, and therefore on U_j, and as $g(U_j) \subset U_j$, g and ρ commute on U_j. Hence, $g(M \cap U_j) \subset M \cap U_j$. Because $M \cap U_j$ is discrete and

$g^{j_k}(f^j(p)) \in M \cap U_j$ for all $k \in \mathbb{N}$, we must have $g^{j_k}(f^j(p)) = f^j(p)$ for all sufficiently large k. This implies that $f^j(p)$ is periodic under g and therefore under f. In other words, p is preperiodic under f. But since p is also recurrent under f, it follows that p must be periodic. So if $M \cap U_j$ is discrete for any $j = 1, \ldots, \ell - 1$, the proof will be complete.

Hence, we reduce the proof to the case that $M_j = U_j$ for all $j = 0, \ldots, \ell - 1$. This occurs when ρ is the identity map on each connected component U_j of Ω. Here, we will show that this case can never occur by deriving a contradiction and therefore conclude that some $M \cap U_j$ is discrete. By the reasoning presented in Claim 2, we see that $g|_{U_j} : U_j \to U_j$ is an automorphism of U_j for every $j = 0, \ldots, \ell - 1$. Hence, $g|_\Omega : \Omega \to \Omega$ is an automorphism of Ω. Since Ω is forward invariant under f, this implies that $f|_\Omega : \Omega \to \Omega$ is an automorphism of Ω. Moreover, f maps U_j biholomorphically onto U_{j+1} for all $j = 0, \ldots, \ell - 1$. Consequently, as U_0 is doubly connected, U_j is doubly connected for all $j = 0, \ldots, \ell - 1$. This fact will allow us to prove the following.

Claim 3. Ω *is connected.* Proceeding by contradiction, suppose U_j and U_k were distinct connected components of Ω. Intuitively, if we were to have two disjoint open annuli in $\mathbb{C}^*$ whose complement in $\mathbb{P}$ are compact neighbourhoods of 0 and ∞, the complement of the union of these annuli in $\mathbb{P}$ would consist of three compact connected components. One such compact connected component must lie in $\mathbb{C}^*$, and so if the union of these annuli was assumed to be Runge in $\mathbb{C}^*$, we would have an immediate contradiction. With this in mind, we will use the fact that for any connected topological space X and connected subset $Y \subset X$, if A and B form a separation of $X \setminus Y$ (that is, A and B are disjoint, non-empty open subsets of $X \setminus Y$ whose union is $X \setminus Y$), then $Y \cup A$ and $Y \cup B$ are connected. Since $U_j \subset \mathbb{C}^*$ is doubly connected, $\mathbb{P} \setminus U_j = W_{0,j} \cup W_{\infty,j}$, where $W_{0,j}$ and $W_{\infty,j}$ are connected, compact, disjoint neighbourhoods of 0 and ∞ respectively. Similarly, $\mathbb{P} \setminus U_k = W_{0,k} \cup W_{\infty,k}$. As U_k is connected and disjoint from U_j, either U_k is contained in

$W_{0,j}$ or contained in $W_{\infty,j}$. Note that if $U_k \subset W_{0,j}$, then

$$\mathbb{P} \setminus U_k = W_{0,k} \cup W_{\infty,k} \supset \mathbb{P} \setminus W_{0,j} = U_j \cup W_{\infty,j}.$$

Since $U_j \cup W_{\infty,j}$ is connected and contains ∞, it follows that $U_j \cup W_{\infty,j}$ is contained in $W_{\infty,k}$. Hence, U_j is contained in $W_{\infty,k}$.

So after relabelling, we may assume without loss of generality that $U_k \subset W_{\infty,j}$. Note that $\mathbb{P} \setminus U_k = W_{0,k} \cup W_{\infty,k} \supset \mathbb{P} \setminus W_{\infty,j} = U_j \cup W_{0,j}$. As $U_j \cup W_{0,j}$ is connected, $\mathbb{P} \setminus W_{\infty,j}$ is either contained in $W_{0,k}$ or contained in $W_{\infty,k}$. But $0 \in \mathbb{P} \setminus W_{\infty,j}$, so we must have $\mathbb{P} \setminus W_{\infty,j} \subset W_{0,k}$. Consequently, $W_{0,j} \subset W_{0,k}$. As $W_{0,k} \cap W_{\infty,k} = \varnothing$ and $\mathbb{P} \setminus W_{\infty,j} \subset W_{0,k}$, we also see that $W_{\infty,k} \subset W_{\infty,j}$.

Knowing this, we have

$$\begin{aligned}
\mathbb{P} \setminus (U_j \cup U_k) &= (\mathbb{P} \setminus U_j) \cap (\mathbb{P} \setminus U_k) \\
&= (W_{0,j} \cup W_{\infty,j}) \cap (W_{\infty,k} \cup W_{\infty,k}) \\
&= (W_{0,j} \cap W_{0,k}) \cup (W_{0,j} \cap W_{\infty,k}) \cup (W_{\infty,j} \cap W_{0,k}) \cup (W_{\infty,j} \cap W_{\infty,k}) \\
&= W_{0,j} \cup (W_{0,j} \cap W_{\infty,k}) \cup (W_{\infty,j} \cap W_{0,k}) \cup W_{\infty,k}.
\end{aligned}$$

Since $W_{0,j} \subset W_{0,k}$ and $W_{0,k} \cap W_{\infty,k} = \varnothing$, $W_{0,j} \cap W_{\infty,k} = \varnothing$. So $\mathbb{P} \setminus (U_j \cup U_k) = W_{0,j} \cup (W_{\infty,j} \cap W_{0,k}) \cup W_{\infty,k}$. Observe that $W_{\infty,j} \cap W_{0,k} \subset \mathbb{C}^* \setminus (U_j \cup U_k)$. Moreover, as $W_{\infty,j}$ and $W_{0,k}$ are compact subsets of $\mathbb{P}$, their intersection is a compact subset of $\mathbb{C}^* \setminus (U_j \cup U_k)$. Hence, to obtain the required contradiction, it suffices to prove that $W_{\infty,j} \cap W_{0,k}$ is non-empty, for connected components of a compact space are also compact.

Suppose that in fact $W_{\infty,j} \cap W_{0,k} = \varnothing$. Then $\mathbb{P} \setminus (U_j \cup U_k) = W_{0,j} \cup W_{\infty,k}$ and so $\mathbb{P} = (W_{0,j} \cup U_j) \cup (W_{\infty,k} \cup U_k)$. As $\mathbb{P} \setminus W_{\infty,j} = W_{0,j} \cup U_j$ and $\mathbb{P} \setminus W_{0,k} = W_{\infty,k} \cup U_k$, we see that $W_{0,j} \cup U_j$ and $W_{\infty,k} \cup U_k$ are open and non-empty in $\mathbb{P}$. As $W_{0,j}$ is contained in $W_{0,k}$ and as $W_{0,k}$ is disjoint from U_k, $W_{0,j} \cap (W_{\infty,k} \cup U_k) = \varnothing$. Furthermore, $U_j \cap (W_{\infty,k} \cup U_k) = \varnothing$, for U_j

and U_k are disjoint and $W_{\infty,k}$ is contained in $W_{\infty,j}$. Hence, $(W_{0,j} \cup U_j) \cap (W_{\infty,k} \cup U_k) = \varnothing$, and so $W_{0,j} \cup U_j$ and $W_{\infty,k} \cup U_k$ are disjoint. But this implies that we can write $\mathbb{P}$ as a union of disjoint, non-empty open subsets, a contradiction to the fact that $\mathbb{P}$ is connected. Hence, $W_{\infty,j} \cap W_{0,k}$ is the required non-empty compact subset of $\mathbb{C}^* \setminus (U_j \cup U_k)$.

Since the connected components of Ω are Runge and as U_j and U_k are distinct, $U_j \cup U_k$ is Runge in $\mathbb{C}^*$. But as argued above, $W_{\infty,j} \cap W_{0,k}$ is a non-empty compact subset of $\mathbb{C}^* \setminus (U_j \cup U_k)$, supplying the sought-after contradiction and completing Claim 3.

From Claim 3, we conclude that $\Omega = U_0$ and hence that $f|_{U_0}$ is an automorphism of U_0. Proceeding as in the proof for the case that U_0 is simply connected, the group generated by $f|_{U_0}$ is an abelian subgroup of $\operatorname{Aut} U_0$, and so its closure G in $\operatorname{Aut} U_0$ is also an abelian subgroup. Again by Weil's lemma, G is either isomorphic to the discrete group $\mathbb{Z}$ or it is compact. The case that G is isomorphic to $\mathbb{Z}$ is impossible, so we may assume that G is a compact abelian subgroup of $\operatorname{Aut} U_0$. Since $\operatorname{Aut} U_0$ is isomorphic to $S^1 \rtimes \mathbb{Z}_2$, the maximal compact subgroups of $\operatorname{Aut} U_0$ are 1-dimensional. Let G_0 be the connected component of G containing the identity map, which is itself a compact subgroup of G. Then $G_0 \cong (S^1)^n$ where $n \leq 1$. Note that if $n = 0$, then $G_0 = \{\mathrm{id}\}$ and we are again in the case that G is isomorphic to $\mathbb{Z}$. So we may assume that $G_0 \cong S^1$.

Observe that this is the case where $f|_{U_0}$ corresponds to an irrational rotation of $\mathbb{A}_{r'}$, and so has no periodic points in U_0. We will show that this is incompatible with the robustly non-expelling property and hence that G also cannot be compact. Subsequently, some $M \cap U_j$ is discrete and it follows that p is periodic under f, completing the proof. Let $\varphi : \mathbb{A}_{r'} \to U_0$ be a biholomorphic map and let $F : \mathbb{A}_{r'} \to \mathbb{A}_{r'}$ be the map defined by $F = \varphi^{-1} \circ f|_{U_0} \circ \varphi$. Then $F(z) = e^{2\pi i\alpha} z$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Observe that the set generated by $e^{2\pi i \alpha}$ is dense in S^1, and hence that $G_0 = G$.

For any $q \in U_0$, the orbit Gq is a 1-dimensional, compact connected submanifold of

U_0 and therefore is diffeomorphic to the unit circle. As $G \cong S^1$, we can write G as a 1-parameter group $(h_t)_{t\in\mathbb{R}}$ such that $h_1 = f|_{U_0}$. Note that for every $t \in \mathbb{R}$, h_t corresponds to the automorphism of $\mathbb{A}_{r'}$ defined by $H_t(z) = e^{2\pi i\alpha t}z$. With the usual local coordinate $z = x + iy$ on $\mathbb{C}^*$, consider the holomorphic vector field $\xi = \frac{\partial}{\partial t}h_t\Big|_{t=0}\frac{d}{dz}$ on U_0. By the chain rule, at each $q \in U_0$, we have

$$\begin{aligned}\frac{\partial}{\partial t}h_t(q)\Big|_{t=0} &= \frac{\partial}{\partial t}(\varphi \circ H_t \circ \varphi^{-1})(q)\Big|_{t=0} \\ &= 2\pi i\alpha\varphi'(H_0(\varphi^{-1}(q)))\varphi^{-1}(q)(\varphi^{-1})'(q) \\ &= 2\pi i\alpha\varphi^{-1}(q).\end{aligned}$$

But since φ is a biholomorphic map from $\mathbb{A}_{r'}$ to U_0, φ^{-1} is non-vanishing. Hence, $\frac{\partial}{\partial t}h_t\Big|_{t=0}$ has no zeros and so ξ is zero-free. Furthermore, for each $t \in \mathbb{R}$, as h_t is the uniform limit of a subsequence of (f^n) on compact subsets of U_0, h_t and $f|_{U_0}$ commute on U_0. Thus, for each $q \in U_0$,

$$\frac{\partial}{\partial t}(h_t \circ f)(q)\Big|_{t=0} = \frac{\partial}{\partial t}(f \circ h_t)(q)\Big|_{t=0} = f'(q)\,\frac{\partial}{\partial t}h_t(q)\Big|_{t=0}.$$

It follows that the vector field ξ is f-invariant.

We are now at the stage where we can derive a contradiction to Lemma 4.2.5 using our candidate holomorphic vector field ξ. Indeed, for any $q \in U_0$, the orbit $Gq \subset U_0$ is clearly a non-empty compact set that is forward invariant under f. So we obtain our contradiction once we show that ξ is uniformly approximable by $\operatorname{End}\mathbb{C}^*$-velocities on Gq. Since U_0 is Runge in $\mathbb{C}^*$, we can employ Runge's approximation theorem to find holomorphic maps on $\mathbb{C}^*$ that uniformly approximate $\frac{\partial}{\partial t}h_t\Big|_{t=0}$ on compact subsets of U_0. Let $\beta : \mathbb{C}^* \to \mathbb{C}$ be such a holomorphic map. Define the holomorphic vector field $\eta = \beta\frac{d}{dz}$ on $\mathbb{C}^*$. Then by construction, η uniformly approximates ξ on compact subsets of U_0. We claim that η

is an $\operatorname{End}\mathbb{C}^*$-velocity. Indeed, define the holomorphic map $\Psi : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C}^*$ by

$$\Psi_t(z) = z \exp\left(\frac{t\beta(z)}{z}\right),$$

for $t \in \mathbb{C}$ and $z \in \mathbb{C}^*$. Clearly, $\Psi_0 = \mathrm{id}_{\mathbb{C}^*}$, and for any $t \in \mathbb{C}$, $\Psi_t \in \operatorname{End}\mathbb{C}^*$ since $\beta : \mathbb{C}^* \to \mathbb{C}$ is holomorphic. Finally, we see that for any $q \in \mathbb{C}^*$,

$$\frac{\partial}{\partial t}\Psi_t(q)\bigg|_{t=0} \frac{d}{dz}\bigg|_{z=q} = \beta(q)\exp\left(\frac{t\beta(q)}{q}\right)\bigg|_{t=0} \frac{d}{dz}\bigg|_{z=q} = \eta(q).$$

So for $q \in U_0$, this shows that η is an $\operatorname{End}\mathbb{C}^*$-velocity that uniformly approximates ξ on Gq, contradicting Lemma 4.2.5. With this final contradiction, we conclude that some $M \cap U_j$, for $j = 0, \ldots, \ell - 1$, must be discrete. Hence, p is periodic under f. □

So as in the case for $\mathbb{C}$, the closing lemma for $\mathbb{C}^*$ follows from Theorems 4.1.1 and 4.2.6.

Remark. Clearly, our proof of the closing lemma for $\mathbb{C}^*$ is significantly more complicated than the corresponding proof for $\mathbb{C}$. This can be attributed to the fact that $\mathbb{C}^*$ is no longer simply connected and hence that a Runge set in $\mathbb{C}^*$ is not necessarily Runge in $\mathbb{C}$, and vice versa. Rather, in the proof of the closing lemma for $\mathbb{C}$ and Case 1 of the proof above, the Runge set U_0 is simply connected is crucial. It allows us to use a hull argument to extract a periodic point of f in U_0. Armed with this periodic point, we can use the theory of holomorphic dynamics in one complex variable to obtain a contradiction.

On the other hand, relatively compact Runge sets in $\mathbb{C}^*$ can also be doubly connected. In this case, before the conclusion of the preceding proof, f need not have any periodic points in U_0 a priori, preventing us from using our well acquainted theory of holomorphic dynamics. Hence, we require a completely separate argument to obtain our sought-after contradiction. This difference highlights the fact that although $\mathbb{C}^*$ is seemingly just $\mathbb{C}$ with the removal of a point, its different topological properties obstruct us from applying powerful theorems commonplace in the theory of Riemann surfaces and holomorphic dynamics.

Chapter 5

Hyperbolic surfaces, complex tori, and the Riemann sphere

For the final chapter, we will show that the closing lemma holds for any hyperbolic Riemann surface and any complex torus $\mathbb{T} = \mathbb{C}/\Gamma$ (where Γ is a lattice). We will also see that the closing lemma holds for the Riemann sphere, with the exception of two special cases. The proofs of the closing lemma for each of these Riemann surfaces are wildly different, utilising a variety of techniques from holomorphic dynamics in one complex variable.

As a precursor to the results presented in this chapter, we give the following useful remark. We will soon see that this observation will be essential in many of the ensuing proofs.

Remark. Let X and Y be Riemann surfaces and let $\mathscr{O}(X, Y)$ denote the space of holomorphic maps $X \to Y$ equipped with the compact-open topology. Note that $\mathscr{O}(X, X) = \operatorname{End} X$ for any Riemann surface X. Then since any Riemann surface is a locally compact Hausdorff topological space, the map $\mathscr{O}(Y, Z) \times \mathscr{O}(X, Y) \to \mathscr{O}(X, Z)$ defined by

$(g, f) \mapsto g \circ f$ is continuous for any Riemann surfaces X, Y and Z. Now, for any $g \in \mathscr{O}(Y, Z)$, the inclusion $\iota_g^1 : \mathscr{O}(X, Y) \hookrightarrow \mathscr{O}(Y, Z) \times \mathscr{O}(X, Y)$ defined by $\iota_g^1(f) = (g, f)$ is continuous. Analogously, the inclusion $\iota_h^2 : \mathscr{O}(Y, Z) \hookrightarrow \mathscr{O}(Y, Z) \times \mathscr{O}(X, Y)$ defined by $\iota_h^2(f) = (f, h)$ is continuous for any $h \in \mathscr{O}(X, Y)$. Hence, the composition

$$\mathscr{O}(Y, Z) \xrightarrow{\iota_h^2} \mathscr{O}(Y, Z) \times \mathscr{O}(X, Y) \xrightarrow{\circ} \mathscr{O}(X, Z)$$

$$\xrightarrow{\iota_g^1} \mathscr{O}(Z, W) \times \mathscr{O}(X, Z) \xrightarrow{\circ} \mathscr{O}(X, W),$$

$$f \mapsto g \circ f \circ h,$$

is continuous for all $g \in \mathscr{O}(Z, W)$ and $h \in \mathscr{O}(X, Y)$.

5.1 The closing lemma for hyperbolic surfaces

Theorem 5.1.1 (Closing lemma for hyperbolic surfaces)**.** *Let $p \in X$ be a non-wandering point of an endomorphism f of a hyperbolic Riemann surface X. Then every open neighbourhood of f in* End X *contains an endomorphism of which p is a periodic point.*

Remark. In the following proof, we will take for granted the fact that every hyperbolic Riemann surface X has a preferred Riemannian metric which induces a distance function dist_X. In fact, using the uniformisation theorem, this preferred metric comes from the Poincaré metric on the disc via the universal covering map. Hence, hyperbolic Riemann surfaces are metric spaces and we call dist_X the *Poincaré distance* on X. We cite [12] for further details.

Proof. We will prove this result in four cases by employing Theorem 2.3.4.

Finite order case. The closing lemma clearly holds since every point of X is a periodic point of f.

Irrational rotation case. For this case, we will not need to use the assumption that p is non-wandering. Indeed, suppose that X is isomorphic to either $Y = \mathbb{D}$, $\mathbb{D} \setminus \{0\}$, or $\mathbb{A}_r = \{z \in \mathbb{C} : 1 < |z| < r\}$, $r > 1$, and $f : X \to X$ is conjugate to an irrational rotation $z \mapsto e^{2\pi i\alpha} z$ on Y, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then we claim that for any open neighbourhood U of f in $\operatorname{End} X$, there exists $g \in U$ such that $g^\tau = \mathrm{id}$ for some $\tau \in \mathbb{N}$. In particular, we can find g arbitrarily close to f for which every $x \in X$ is a periodic point.

For $t \in \mathbb{R}$, let $f_t : Y \to Y$ denote the automorphism $f_t(z) = e^{2\pi i t} z$. Then there exists a biholomorphic map $\varphi : X \to Y$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $\varphi \circ f \circ \varphi^{-1} = f_\alpha$. Let U be an open neighbourhood of f in $\operatorname{End} X$. From the remark at the beginning of Chapter 5, the map $F : \operatorname{End} X \to \operatorname{End} Y$ defined by $h \mapsto \varphi \circ h \circ \varphi^{-1}$ is continuous with respect to the compact-open topology on $\operatorname{End} X$ and $\operatorname{End} Y$. Moreover, it has continuous inverse $\operatorname{End} Y \to \operatorname{End} X$ defined by $h \mapsto \varphi^{-1} \circ h \circ \varphi$, and so F defines a homeomorphism between $\operatorname{End} X$ and $\operatorname{End} Y$. Thus, $F(U) = \varphi U \varphi^{-1} = \{\varphi \circ h \circ \varphi^{-1} : h \in U\}$ is an open neighbourhood of f_α in $\operatorname{End} Y$.

Now, observe that the map $G : \mathbb{R} \to \operatorname{End} Y$ defined by $G(t) = f_t$ is continuous, where $\mathbb{R}$ has the usual topology. Indeed, this follows from the continuity of the map $\mathbb{R} \times Y \to Y$ given by $(t, z) \mapsto e^{2\pi i t} z$ (for this map is the product of the continuous maps $t \mapsto e^{2\pi i t}$ and $z \mapsto z$). Then for any open set V in $\operatorname{End} Y$, $G^{-1}(V)$ is open in $\mathbb{R}$. In particular, $G^{-1}(\varphi U \varphi^{-1})$ is open in $\mathbb{R}$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $\theta \in \mathbb{Q} \cap G^{-1}(\varphi U \varphi^{-1})$. Hence,

$$G(\theta) = f_\theta \in \varphi U \varphi^{-1},$$

where $f_\theta(z) = e^{2\pi i\theta} z$ is a rational rotation on Y. By definition of $\varphi U \varphi^{-1}$, there exists $g \in U$ such that $g = \varphi^{-1} \circ f_\theta \circ \varphi$. But then since f_θ is a rational rotation, there exists $\tau \in \mathbb{N}$ so that $g^\tau = \mathrm{id}$. As U was an arbitrary open neighbourhood of f in $\operatorname{End} X$, we conclude that we can find g arbitrarily close to f for which $g^\tau = \mathrm{id}$ for some $\tau \in \mathbb{N}$, and hence there is g arbitrarily close to f for which every $x \in X$ is a periodic point. The closing lemma then follows for the irrational rotation case.

Escape case. We will show that f has no non-wandering points and therefore the closing lemma holds vacuously. Let $x \in X$ and let $K = \overline{D}(x, r)$, $r > 0$, be a compact neighbourhood of x in X. Then there exists an integer n_K so that $f^n(K) \cap K = \varnothing$ for all $n \geq n_K$. Choose the smallest such n_K.

We claim that there exists $R_1 > 0$ such that $\overline{D}(x, r') \cap f\left(\overline{D}(x, r')\right) = \varnothing$ for all $0 < r' \leq R_1$. Suppose not. Then for all $R > 0$, there exists $0 < r' \leq R$ such that $\overline{D}(x, r') \cap f\left(\overline{D}(x, r')\right) \neq \varnothing$. In particular, for each $n \in \mathbb{N}$, there exists $0 < r'_n \leq 1/2^n$ such that $\overline{D}(x, r'_n) \cap f\left(\overline{D}(x, r'_n)\right) \neq \varnothing$. Thus, we have a sequence $(r'_n)_{n \in \mathbb{N}}$ of positive numbers converging to zero as $n \to \infty$, for which there exists $x_n \in \overline{D}(x, r'_n)$ satisfying $\mathrm{dist}_X(f(x_n), x) \leq r'_n$ for each $n \in \mathbb{N}$. But this implies that we have a sequence of points $x_n \to x$ satisfying $f(x_n) \to x$ as $n \to \infty$. Thus, by continuity of f, x is fixed by f. But $\{x\}$ is compact, so by assumption, there exists $N \in \mathbb{N}$ such that $f^n(\{x\}) \cap \{x\} = \varnothing$ for all $n \geq N$, which is not possible for a fixed point.

Observe that this claim also holds for the sets $f^2\left(\overline{D}(x, r')\right), \ldots, f^{n_K - 1}\left(\overline{D}(x, r')\right)$ in place of $f\left(\overline{D}(x, r')\right)$ (we would conclude that x is a periodic point of f instead of fixed). Hence, we can obtain corresponding positive numbers $R_1, R_2, \ldots, R_{n_K - 1}$ so that

$$\overline{D}(x, r'_j) \cap f^j\left(\overline{D}(x, r'_j)\right) = \varnothing$$

for all $0 < r'_j \leq R_j$, $j = 1, \ldots, n_K - 1$. Let $R = \min\{R_1, \ldots, R_{n_K - 1}, r\}$. Then for all $j \in \mathbb{N}$, the sets $f^j\left(\overline{D}(x, R)\right)$ are mutually disjoint to $\overline{D}(x, R)$. Moreover, $U = D(x, R)$ is an open neighbourhood of x such that $f^n(U) \cap U = \varnothing$ for all $n \in \mathbb{N}$, and thus x is a wandering point. Since $x \in X$ was arbitrary, f has no non-wandering points and therefore the closing lemma holds vacuously.

Attracting case. This case follows in a similar vein to the escape case. Here, we will show that the only non-wandering point of f in X is the attracting fixed point itself. It will hence follow that p is already a periodic point of f. Let x_0 be this attracting fixed point and let $x \in X \setminus \{x_0\}$. Choose $r > 0$ such that $x_0 \notin \overline{D}(x,r)$. Pick $\epsilon > 0$ such that $\overline{D}(x_0,\epsilon) \cap \overline{D}(x,r) = \varnothing$. Since $(f^n)_{n\in\mathbb{N}}$ converges locally uniformly to the constant map $x \mapsto x_0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $y \in \overline{D}(x,r)$,

$$\mathrm{dist}_X(f^n(y), x_0) < \frac{\epsilon}{2}.$$

Pick the smallest N for which this holds. Observe that

$$\mathrm{dist}_X\left(f^n\left(\overline{D}(x,r)\right), x_0\right) = \inf\left\{\mathrm{dist}_X(f^n(y), x_0) : y \in \overline{D}(x,r)\right\} \leq \frac{\epsilon}{2} < \epsilon$$

for all $n \geq N$. Therefore, $f^n\left(\overline{D}(x,r)\right) \cap \overline{D}(x,r) = \varnothing$ for all $n \geq N$.

Next, as in the escape case, we claim that there exists $R_1 > 0$ such that $\overline{D}(x,r') \cap f\left(\overline{D}(x,r')\right) = \varnothing$ for all $0 < r' \leq R_1$. Suppose not. Then for each $n \in \mathbb{N}$, there exists $0 < r'_n \leq 1/2^n$ such that $\overline{D}(x,r'_n) \cap f\left(\overline{D}(x,r'_n)\right) \neq \varnothing$. Thus, we have a sequence $(r'_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero as $n \to \infty$, for which there exists $x_n \in \overline{D}(x,r'_n)$ satisfying $\mathrm{dist}_X(f(x_n), x) \leq r'_n$ for each $n \in \mathbb{N}$. But this implies that we have a sequence of points $x_n \to x$ satisfying $f(x_n) \to x$ as $n \to \infty$. So as in the escape case, x is a fixed point of f. But since we are considering the attracting case, every orbit under f converges to the unique attracting fixed point x_0 and so we must have $x = x_0$. But this contradicts our choice of x.

Once again, we observe that this claim also holds for $f^2\left(\overline{D}(x,r')\right), \ldots, f^{N-1}\left(\overline{D}(x,r')\right)$ in place of $f\left(\overline{D}(x,r')\right)$. Here, we would conclude that x is a periodic point of f, and so as every orbit under f converges to x_0, every subsequence of $(f^n(x))_{n\in\mathbb{N}}$ converges to x_0. In particular, if τ was the period of x, the sequence $(f^{n\tau}(x))_{n\in\mathbb{N}}$ converges to x_0 and we obtain the same contradiction as above.

Thus, we can find numbers $R_1, \ldots, R_{N-1} > 0$ such that

$$\overline{D}(x,r'_j) \cap f^j\left(\overline{D}(x,r'_j)\right) = \varnothing$$

for all $0 < r'_j \leq R_j$, and $j = 1, \ldots, N-1$. Take $R = \min\{R_1, \ldots, R_{N-1}, r\}$. Then by construction, the open neighbourhood $U = D(x, R)$ of x satisfies $f^n(U) \cap U = \varnothing$ for all $n \in \mathbb{N}$. Hence, x is a wandering point of f and so as $x \neq x_0$ was arbitrary in X, we conclude that the attracting fixed point x_0 is the only non-wandering point of f. Hence, $p = x_0$ and the closing lemma clearly follows for the attracting case. □

Remark. Interestingly, the case where our endomorphism f corresponds to an irrational rotation is the only case for which we need to perturb f. However, there are some parallels of the closing lemma for hyperbolic surfaces (ignoring the vacuous escape case) with the closing lemma for $\mathbb{C}$. In both proofs, we have seen that under certain assumptions, we can either make a small enough perturbation of our endomorphism f so that our non-wandering point p is periodic, or p was periodic under f from the very beginning.

As far as we know, our proof of the closing lemma for hyperbolic Riemann surfaces is original and the result is not available in the literature.

5.2 The closing lemma for complex tori

Recall from Section 2.1 that any complex torus $\mathbb{T}$ can be normalised to the form $\mathbb{C}/\Gamma$ where $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$ and τ has positive imaginary part. Hence, without loss of generality, we will assume that $\mathbb{T} = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ in the following results. Further recall that $f : \mathbb{T} \to \mathbb{T}$ is a non-constant endomorphism if and only if f is of the form $f(z + \Gamma) = \alpha z + \beta + \Gamma$ where $\alpha \in \mathbb{C}^*$ is a number such that $\alpha\Gamma \subset \Gamma$, and $\beta \in \mathbb{C}$.

Moreover, the degree of f is equal to $|\alpha|^2$, and so $|\alpha|^2$ is necessarily a positive integer. To see this, first suppose that $\alpha = 1$. Then f is a pure translation and hence has degree equal to 1. Now suppose that $\alpha \neq 1$. With respect to a chart, let $\omega = dz \wedge d\bar{z}$ be a 2-form on $\mathbb{T}$. Then the pullback of ω with respect to f is $f^*\omega = |\alpha|^2 dz \wedge d\bar{z}$. But then the number

$|\alpha|^2$ must necessarily be the degree of f (by the immediate remark after Definition 2.3.2), for it satisfies

$$\int_{\mathbb{T}} f^*\omega = |\alpha|^2 \int_{\mathbb{T}} dz \wedge d\bar{z} = |\alpha|^2 \int_{\mathbb{T}} \omega.$$

With this setting, we proceed to prove the closing lemma for $\mathbb{T}$. The following lemma is stated without proof in [12], but we will of course supply our own proof for the convenience of the reader.

Lemma 5.2.1. *Let $\alpha \notin \mathbb{Z}$ and suppose $f(z+\Gamma) = \alpha z + \beta + \Gamma$ is an endomorphism of $\mathbb{T}$. Then α satisfies the quadratic equation*

$$\alpha^2 + p\alpha + d = 0,$$

where d is the degree of f and p is an integer for which $p^2 \leq 4d$.

Proof. First observe that as we necessarily require $\alpha\Gamma \subset \Gamma$ for f to be an endomorphism of $\mathbb{T}$, we can write $\alpha = n_1 + \tau m_1$ and $\alpha\tau = n_2 + \tau m_2$ for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. Since τ has positive imaginary part, $m_1 \neq 0$ for otherwise $\alpha = n_1 \in \mathbb{Z}$. Moreover, we see that $\overline{\tau} \neq \tau$ and hence that $\overline{\alpha} \neq \alpha$. Next, we observe that as $\tau \neq 0$ and $\alpha \neq 0$,

$$\tau = \frac{\alpha\tau}{\alpha} = \frac{n_2 + \tau m_2}{n_1 + \tau m_1}.$$

Rearranging, we obtain the quadratic $m_1\tau^2 + (n_1 - m_2)\tau - n_2 = 0$. Since $\alpha = n_1 + \tau m_1$, we can rearrange for τ and substitute into this quadratic equation to obtain

$$\alpha^2 - (n_1 + m_2)\alpha + (n_1 m_2 - n_2 m_1) = 0. \tag{5.1}$$

We will now show that the degree of f is $|\alpha|^2 = n_1 m_2 - n_2 m_1$. Set $A = \begin{bmatrix} n_1 & m_1 \\ n_2 & m_2 \end{bmatrix}$. Note that $\det A = n_1 m_2 - n_2 m_1$ and $\operatorname{tr} A = n_1 + m_2$. By (5.1), α satisfies the characteristic equation $x^2 - (\operatorname{tr} A)x + \det A = 0$. But the solutions to this equation are precisely the eigenvalues of A. Since this characteristic equation has integer coefficients, $\overline{\alpha}$ is also an eigenvalue of A. It follows that $\det A = \alpha\overline{\alpha} = |\alpha|^2$, as required.

Now let $p = -\operatorname{tr} A$, which is an integer by definition. Then (5.1) is equivalent to $\alpha^2 + p\alpha + |\alpha|^2 = 0$ and $p^2 = (\operatorname{tr} A)^2 = (\alpha + \overline{\alpha})^2 = (2\,\mathrm{Real}(\alpha))^2 \leq 4|\alpha|^2$. This proves the lemma. $\square$

Theorem 5.2.2 (Closing lemma for $\mathbb{T}$). *Let $p + \Gamma \in \mathbb{T}$ and let f be an endomorphism of $\mathbb{T}$. Then every open neighbourhood of f in* $\operatorname{End}\mathbb{T}$ *contains an endomorphism of which $p + \Gamma$ is a periodic point.*

Proof. Let $f : \mathbb{T} \to \mathbb{T}$ be defined by $f(z + \Gamma) = \alpha z + \beta + \Gamma$ and let U be an open neighbourhood of f in $\operatorname{End}\mathbb{T}$. We will prove the closing lemma for complex tori via considering the possible cases of $\alpha \in \mathbb{C}^*$ satisfying $|\alpha| \geq 1$ (since $|\alpha|^2 \in \mathbb{N}$).

Case 1. $\alpha = 1$. Here, f is a pure translation. Observe that $z + \Gamma$ is a periodic point of f with period $n \in \mathbb{N}$ if and only if $n\beta \in \Gamma$ if and only if $\beta \in \frac{1}{n}\Gamma$. Hence, f has periodic points if and only if $\beta = \gamma/n$ for $\gamma \in \Gamma$ and $n \in \mathbb{N}$ (if fact, every point in $\mathbb{T}$ would be periodic).

Knowing this, we will show that the set $\mathcal{G} = \{\gamma/n \in \mathbb{C} : \gamma \in \Gamma \text{ and } n \in \mathbb{N}\}$ is dense in $\mathbb{C}$. Let $D(z_0, r)$ be an open disc centred at z_0 of radius $r > 0$. Since 1 and τ are linearly independent over $\mathbb{R}$, $\{1, \tau\}$ is a basis of the vector space $\mathbb{C}$ over $\mathbb{R}$. Hence, there exists $a, b \in \mathbb{R}$ such that $z_0 = a + b\tau$.

We claim that there exists $k, m \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that

$$\left|a - \frac{k}{N}\right| < \frac{r}{2} \quad \text{and} \quad \left|b - \frac{m}{N}\right| < \frac{r}{2|\tau|}.$$

To see this, choose $n_1, n_2 \in \mathbb{N}$ such that $1/n_1^2 < r/2$ and $1/n_2^2 < r/(2|\tau|)$. Set $N = \max\{n_1, n_2\}$. Then $1/N^2 \leq \min\{1/n_1^2, 1/n_2^2\}$. By Dirichlet's approximation theorem, there exists $k, m \in \mathbb{Z}$ such that

$$\left|a - \frac{k}{N}\right| < \frac{1}{N^2} \quad \text{and} \quad \left|b - \frac{m}{N}\right| < \frac{1}{N^2}.$$

This proves the claim, for $1/N^2 < r/2$ and $1/N^2 < r/(2|\tau|)$.

Now, pick $\gamma = k + m\tau$ and choose $N \in \mathbb{N}$ as in the claim above. Then

$$\left|z_0 - \frac{\gamma}{N}\right| = \left|\left(a - \frac{k}{N}\right) + \left(b - \frac{m}{N}\right)\tau\right| \leq \left|a - \frac{k}{N}\right| + \left|b - \frac{m}{N}\right||\tau| < \frac{r}{2} + \frac{r}{2} = r.$$

Hence, γ/N lies in $D(z_0, r)$ and so it follows that $\mathcal{G}$ is dense in $\mathbb{C}$.

After identifying the subspace $\{f \in \operatorname{End}\mathbb{T} : f(z+\Gamma) = z + \beta + \Gamma \text{ for } \beta \in \mathbb{C}\}$ of pure translations in $\operatorname{End}\mathbb{T}$ with $\mathbb{C}$, we conclude that the subset

$$\left\{f_{\gamma,n} \in \operatorname{End}\mathbb{T} : f_{\gamma,n}(z+\Gamma) = z + \frac{\gamma}{n} + \Gamma \text{ for } \gamma \in \Gamma, n \in \mathbb{N}\right\}$$

is dense in $\{f \in \operatorname{End}\mathbb{T} : f(z+\Gamma) = z+\beta+\Gamma \text{ for } \beta \in \mathbb{C}\}$ (with respect to the compact-open topology). Thus, there exists $f_{\gamma,n} \in U$, and so as U was an arbitrary open neighbourhood of f in $\operatorname{End}\mathbb{T}$, we can find an arbitrarily close endomorphism to f of which $p + \Gamma$ is a periodic point. Hence, the case that $\alpha = 1$ is complete.

Case 2. $|\alpha| = 1$. Here, f is the composition of a rotation fixing Γ and a translation. Suppose that $\alpha \in \mathbb{Z}$. Then $\alpha = 1$ or -1. The case that $\alpha = 1$ has already been covered, so we may assume that $\alpha = -1$. But then $f(z+\Gamma) = -z + \beta + \Gamma$ is an involution and it immediately follows that $p + \Gamma$ is already a periodic point of f.

Next, suppose that $\alpha \notin \mathbb{Z}$. Then by Lemma 5.2.1, α satisfies the quadratic equation $\alpha^2 + p\alpha + 1 = 0$ where p is an integer and $p^2 \leq 4$. Hence,

$$\alpha = \frac{-p \pm \sqrt{p^2 - 4}}{2}$$

and $p = 0, \pm 1$ or ± 2. If $p = \pm 2$, then $\alpha = \mp 1$ respectively. If $p = 1$, then $\alpha = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ and so either $\alpha = e^{2\pi i/3}$ or $e^{4\pi i/3}$. While if $p = -1$, then $\alpha = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ and so either

$\alpha = e^{2\pi i/6}$ or $e^{10\pi i/6}$. Finally, if $p = 0$, then $\alpha = \pm i$ and so either $\alpha = e^{2\pi i/4}$ or $e^{6\pi i/4}$. This shows that whenever $|\alpha| = 1$, α must be a root of unity.

Now, as $\alpha \neq 1$, f has $\frac{\beta}{1-\alpha} + \Gamma$ as a fixed point. Hence, by conjugating with the translation $\varphi(z + \Gamma) = z - \frac{\beta}{1-\alpha} + \Gamma$,

$$(\varphi \circ f \circ \varphi^{-1})(z + \Gamma) = \alpha z + \Gamma.$$

But as α is a root of unity, there exists some $n \in \mathbb{N}$ so that $\varphi \circ f^n \circ \varphi^{-1} = \mathrm{id}$. Thus, every point in $\mathbb{T}$ is periodic under f. In particular, $p + \Gamma$ is periodic under f. This completes the case when $|\alpha| = 1$.

Case 3. $|\alpha| > 1$. This case follows in a similar vein as the two preceding cases. Since $\alpha \neq 1$, then we may conjugate f by a translation φ to obtain the endomorphism

$$g = \varphi \circ f \circ \varphi^{-1}, \qquad z + \Gamma \mapsto \alpha z + \Gamma.$$

Observe that $z+\Gamma$ is a periodic point of g with period $n \in \mathbb{N}$ if and only if $\alpha^n z+\Gamma = z+\Gamma$ if and only if $z \in \frac{1}{\alpha^n - 1}\Gamma$.

Knowing this, we will show that the set $\mathcal{H} = \{\gamma/(\alpha^n - 1) : \gamma \in \Gamma \text{ and } n \in \mathbb{N}\}$ is dense in $\mathbb{C}$. First, we observe that as $|\alpha| > 1$, $|\alpha^n| > 1$ for any $n \in \mathbb{N}$. So by the reverse triangle inequality,

$$|\alpha^n - 1| \geq ||\alpha^n| - 1| > 0 \iff \frac{1}{|\alpha^n - 1|} \leq \frac{1}{||\alpha^n| - 1|}.$$

Hence, $1/|\alpha^n - 1| \to 0$ as $n \to \infty$. Write $b_n = 1/(\alpha^n - 1)$. We claim that the set

$$\mathcal{H} = \bigcup_{n \in \mathbb{N}} b_n \Gamma$$

is dense in $\mathbb{C}$. To see this, let $D(z_0, r)$ be an open disc centred at z_0 of radius $r > 0$. For each $n \in \mathbb{N}$, let R_n denote the longer diagonal in the fundamental parallelogram defined by the lattice $b_n\Gamma = b_n\mathbb{Z} + b_n\tau\mathbb{Z}$. Clearly $R_n \to 0$ as $n \to \infty$, for $|b_n| \to 0$ as $n \to \infty$.

Choose $n_0 \in \mathbb{N}$ for which $R_{n_0} < r/2$. Since the filled parallelograms determined by the lattices $b_n\Gamma$ tessellate $\mathbb{C}$, there exists a filled parallelogram determined by $b_{n_0}\Gamma$ that is completely contained in $D(z_0, r)$. Hence, $b_{n_0}\Gamma \cap D(z_0, r) \neq \varnothing$ and so we can find an element in $\mathcal{H}$ that lies in $D(z_0, r)$. It follows that $\mathcal{H}$ is dense in $\mathbb{C}$.

Now, note that the set $\varphi U \varphi^{-1} = \{\varphi \circ h \circ \varphi^{-1} : h \in U\}$ is an open neighbourhood of g in $\operatorname{End} \mathbb{T}$. As $\mathcal{H}$ is dense in $\mathbb{C}$, we can choose a point of the form $\gamma/(\alpha^n - 1) \in \mathcal{H}$ such that the translation $T(z + \Gamma) = z - \varphi(p + \Gamma) + \gamma/(\alpha^n - 1) + \Gamma$ sending $\varphi(p + \Gamma) + \Gamma$ to $\gamma/(\alpha^n - 1) + \Gamma$ is arbitrarily close to the identity map on $\mathbb{T}$. Thus, as $\varphi U \varphi^{-1}$ is open, we can find T so that $T^{-1} \circ g \circ T \in \varphi U \varphi^{-1}$. Consequently, $(T \circ \varphi)^{-1} \circ g \circ (T \circ \varphi) : \mathbb{T} \to \mathbb{T}$ is an endomorphism in U for which $p + \Gamma$ is a periodic point. Hence, the closing lemma holds whenever $|\alpha| > 1$.

Since U was an arbitrary open neighbourhood of f in $\operatorname{End} \mathbb{T}$, the closing lemma for $\mathbb{T}$ follows from Cases 1, 2, and 3. $\square$

Like the closing lemma for hyperbolic Riemann surfaces, we note that our proof of the closing lemma for $\mathbb{T}$ is original as far as we know, and that this result cannot be found in the literature.

5.3 The closing lemma for the Riemann sphere

For the Riemann sphere $\mathbb{P}$, we will see that the closing lemma holds with some reservations. However, this is by no means an assertion that the closing lemma does not hold for $\mathbb{P}$ at all. Rather, what we furnish in this section is a proof of the closing lemma for $\mathbb{P}$ provided that our rational map does not admit Siegel discs or Herman rings. We will later explain in this section why the presence of Siegel discs and Herman rings is such a difficult hurdle for us to surmount. Furthermore, in this section we will see many of the ideas behind holomorphic dynamics on the Riemann sphere behind our proofs. Here is where the results seen in Section 2.3 truly come to the fore, and we again refer to [12] as our main reference.

We first recall the following fact. Let f be an endomorphism of a Riemann surface X and let $\varphi : X \to Y$ be a biholomorphic map between Riemann surfaces. Then $p \in X$ is a non-wandering point of f if and only if $\varphi(p)$ is a non-wandering point of $\varphi \circ f \circ \varphi^{-1}$. That is, conjugation by biholomorphic maps preserve non-wandering points.

Theorem 5.3.1 (Closing lemma for $\mathbb{P}$)**.** *Let $p \in \mathbb{P}$ be a non-wandering point of a rational map f on $\mathbb{P}$. Suppose f does not admit Siegel discs and Herman rings. Then every open neighbourhood of f in* $\operatorname{End}\mathbb{P}$ *contains a rational map of which p is a periodic point.*

Proof. Let U be an open neighbourhood of f in $\operatorname{End}\mathbb{P}$. We will prove the closing lemma for $\mathbb{P}$ through the following cases.

Case 1. $\deg f = 1$. Here, f is a Möbius transformation. We may assume that f is not the identity map, for otherwise f fixes p and we are done. Therefore, since f is a Möbius transformation, f must either have a single fixed point or two distinct fixed points.

Suppose f has a single fixed point. After possibly conjugating f with a Möbius transformation, we may assume that this fixed point lies at infinity. Thus, $f(z) = z + c$ where $c \in \mathbb{C}^*$ is a constant. But then f is a pure translation on $\mathbb{P}$, so the only non-wandering point of f is the fixed point at infinity itself. Hence, $p = \infty$ is already a periodic point of f.

Now suppose f has two distinct fixed points. After again possibly conjugating f with a Möbius transformation, we may assume that f fixes 0 and the point at infinity. Then $f(z) = \lambda z$ where $\lambda \in \mathbb{C}^*$ is a constant. Thus, the behaviour of orbits under f is determined by whether $0 < |\lambda| < 1$, $|\lambda| > 1$, or $|\lambda| = 1$.

Assume that $0 < |\lambda| < 1$. Then 0 is an attracting fixed point and ∞ is a repelling fixed point. Observe that the sequence $(f^n)_{n\in\mathbb{N}}$ uniformly converges on $\mathbb{C}$ to the constant map $z \mapsto 0$. But as in the attracting case in the proof of Theorem 5.1.1, we conclude that the only non-wandering points of f are the fixed points 0 and ∞ themselves. Consequently, $p = 0$ or ∞ and is thus already periodic under f.

The case that $|\lambda| > 1$ is completely analogous to the case $0 < |\lambda| < 1$, for now we have that 0 is the repelling fixed point and ∞ the attracting fixed point of f. Thus, we are left to check the case of $|\lambda| = 1$, that is, $\lambda = e^{2\pi i\theta}$ for $\theta \in \mathbb{R}$. If $\theta \in \mathbb{Q}$, then λ is a root of unity and hence every point is periodic under f. In particular, p is a periodic point of f.

Finally, suppose $\theta \in \mathbb{R} \setminus \mathbb{Q}$. In other words, f is an irrational rotation on $\mathbb{P}$. But by similar reasoning as in the irrational rotation case in the proof of Theorem 5.1.1, we can find g in U so that g is a rational rotation of $\mathbb{P}$. Hence, there exists $\tau \in \mathbb{N}$ such that $g^\tau = \mathrm{id}$. It follows that there is g arbitrarily close to f for which every $z \in \mathbb{P}$ is a periodic point. In particular, p is a periodic point of g.

As there are no other cases to consider and as U was an arbitrary open neighbourhood of f, we conclude that the closing lemma holds whenever f has degree 1.

Case 2. $\deg f \geq 2$. We will prove this case in two sub-cases, namely whether p lies in the Julia set $\mathcal{J}(f)$ or the Fatou set $\mathcal{F}(f)$. Without loss of generality, we may assume that p is finite, for otherwise we simply conjugate f by the map $z \mapsto 1/z$.

Suppose $p \in \mathcal{J}(f)$. Recall that $\mathcal{J}(f)$ is equal to the closure of the set of repelling periodic points of f (Theorem 2.3.7). Define $T : \mathbb{P} \to \mathbb{P}$ to be the translation map $T(z) = z - p + q$ where $q \in \mathcal{J}(f)$ is a finite repelling periodic point of f. Since the set of repelling periodic points of f is dense in $\mathcal{J}(f)$, we can find such T arbitrarily close to the identity map on $\mathbb{P}$. Hence, as U is open in $\operatorname{End}\mathbb{P}$, we may choose T so that $T^{-1} \circ f \circ T \in U$. But then $T^{-1} \circ f \circ T$ is a rational map with periodic point p, and so the closing lemma holds whenever $p \in \mathcal{J}(f)$.

Now suppose $p \in \mathcal{F}(f)$. Let V be the connected component of $\mathcal{F}(f)$ containing p. Then V is either a wandering domain or preperiodic. That is, V is an open connected set such that either:

- the family $(f^n(V))_{n\in\mathbb{N}}$ is mutually disjoint, or
- there exist integers $n \geq 0$ and $m \geq 1$ so that $f^n(V) = f^{m+n}(V)$,

respectively. Note that if V is preperiodic and $n = 0$, then V is in fact periodic. Clearly the case that V is a wandering domain is impossible, for V itself is an open neighbourhood of the non-wandering point p. (In fact, wandering domains do not exist for rational maps by Sullivan's non-wandering theorem (Theorem 2.3.6).)

Thus, we may assume that V is a preperiodic Fatou component. We will show that V is necessarily periodic. Since V is an open set containing the non-wandering point p, there exists $j \in \mathbb{N}$ such that $f^j(V) \cap V \neq \varnothing$. Using the fact that V is connected and $\mathcal{F}(f)$ is completely invariant under the rational map f, that is, $f^{-1}(\mathcal{F}(f)) = \mathcal{F}(f) = f(\mathcal{F}(f))$, we conclude that $f^j(V) = V$. Hence, V must be a periodic Fatou component, as claimed.

Knowing this, we can use the classification theorem of periodic Fatou components (Theorem 2.3.5). We see that V is either:

- the immediate attractive basin of an attracting periodic point,
- the immediate basin of attraction for some petal of a parabolic periodic point,
- a member of a cycle of Siegel discs, or
- a member of a cycle of Herman rings.

By assumption, f does not admit any Siegel discs or Herman rings, so it suffices to prove the closing lemma when V is some immediate basin. Let $\tau \geq 1$ be the period of V under

f, that is, $\tau \geq 1$ is the smallest integer for which $f^\tau(V) = V$. Observe that p is also a non-wandering point of the iterate f^τ. Indeed, let $V' \subset V$ be an open neighbourhood of p. As p is a non-wandering point of f, there exists $n \in \mathbb{N}$ such that $f^n(V') \cap V' \neq \varnothing$. But as $f^n(V') \cap V' \subset f^n(V) \cap V$ and as V is a periodic Fatou component, n must be a positive multiple of τ, say $n = m\tau$. It follows that m is a positive integer for which $(f^\tau)^m(V') \cap V' \neq \varnothing$. Since V' was an arbitrary open neighbourhood of p, the observation follows.

Now, if V is an attracting basin, then p cannot be a non-wandering point of f^τ unless it is the attracting periodic point itself. Indeed, every orbit in V under f^τ must converge to a member, say $f^j(q)$ for $j \geq 0$, of this attracting periodic orbit. Then from similar reasoning to the attracting case in the proof of Theorem 5.1.1, $p = f^j(q)$. Consequently, p must be a periodic point of f and the closing lemma holds.

On the other hand, if V is an immediate basin of a petal of a parabolic periodic point, then every orbit in V under f^τ converges to a member, say $f^j(q)$ for $j \geq 0$, of this periodic orbit. But by definition of parabolic petal, $f^j(q)$ lies on the boundary of V. Thus, no orbit in V under f^τ can have an accumulation point in V. Hence, we can apply analogous reasoning as in the escape case in the proof of Theorem 5.1.1 to conclude that there are no non-wandering points of f^τ in V. Consequently, the case that V is an immediate basin of a petal of a parabolic periodic point is impossible, for it cannot contain p. As there are no other cases to consider, we conclude that the closing lemma holds for rational maps f of degree at least 2 under the assumption that f admits no Siegel discs or Herman rings.

Subsequently, the proof of the closing lemma for $\mathbb{P}$ is complete. $\square$

As promised, we will now explain why the presence of Siegel discs or Herman rings in the dynamics of rational maps gives us an obstacle which we cannot yet overcome. We first observe that Möbius transformations do not admit Siegel discs or Herman rings.

Indeed, the Julia set of such maps contain at most one point, while the entire topological boundary of Siegel discs and Herman rings (whenever they arise) is also contained in the Julia set. So in the setting of rational maps, Siegel discs and Herman rings only arise from maps of degree 2 or more.

Here, we will run into some issues. Let us assume, for simplicity, that our Siegel disc or Herman ring R is completely invariant under our rational map f of degree at least 2 (so a member of a cycle of length 1). Then since f is conjugate to an irrational rotation on the unit disc or an annulus, every point in R is recurrent under f and hence non-wandering. (In fact, since f is conjugate to an irrational rotation, no point in R is periodic except for the unique fixed point in the case that R is a Siegel disc.) Thus, there is nothing inherently special about the assumption that f has a non-wandering point p contained in R. This would imply, a priori, that we cannot garner further information from the existence of p.

Now let us turn to perturbing f. Hypothetically, suppose we can produce a small perturbation f_ϵ of f for which p is periodic. Then for a small enough perturbation, let us assume that the dynamics exhibited by f_ϵ closely resembles that of f. More precisely, if there exists a completely invariant Fatou component V of f, then there is a corresponding completely invariant Fatou component V' of f_ϵ such that $V \cap V'$ is an open neighbourhood of p. Hence, there is a Fatou component R' of f_ϵ containing p that corresponds to R, and so must be isomorphic to $\mathbb{D}$ or an annulus. By the classification theorem of periodic Fatou components, R' must be the immediate basin of an attracting fixed point or parabolic fixed point, for it contains a periodic point p. But as we have already seen above, p must be an attracting fixed point of f_ϵ and R' is necessarily the immediate attractive basin of p. Thus, for the closing lemma to hold for rational maps with Siegel discs or Herman rings, we would be asking the following question.

Question. Given a point p in a cycle of Siegel discs or Herman rings admitted by a rational map f, does there exist arbitrarily close rational maps to f for which p is an attracting periodic point?

We currently do not know the answer to this question, and it appears to be deeply non-trivial. Nevertheless, we can point to [7] for some variants of the closing lemma that are proved in the setting of holomorphic maps. In particular, Theorem 2.1 and Corollary 4.3 treat a version of the closing lemma for rational maps that admit Siegel discs and Herman rings respectively. As such, it may be fruitful to analyse these highly technical results to settle the above question. However, due to the difficulty of the proofs of these results exceeding those presented in this thesis, we will not provide a treatment of this paper and will leave the above question open.

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