Autonomous Navigation of Multiple Unmanned Aerial Vehicles

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B.Sc., M.Eng.(EE)

Thesis Submitted to The University of Adelaide in fulfilment of the requirement for the degree of Doctor of Philosophy

Recommended for Acceptance
by the Faculty of
Sciences, Engineering and Technology
The University of Adelaide
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September 2022

Contents

Abstract

The focus of the present thesis is on the analysis and design of guidance, control and navigation algorithms for unmanned aerial vehicles, specifically in the context of drone warfare and aerospace battles.

In aerospace engagement scenarios involving autonomous agents, the synthesis of intelligent actions must consider the potential strategies by the adversary. When analysing the possible outcomes of an engagement, unpredictability of the adversary's decisions presents the main challenge, the design of our strategies must be robust to a very broad set of possible counter strategies employed by the adversary. Differential game theory provides the correct framework to analyse and design optimal strategies in these dynamic engagement scenarios. Here the goal is the find the state-feedback Nash equilibrium, the optimal outcome of an engagement scenario, in which all parties with knowledge of the strategies deployed, cannot increase their payoff by altering their decision making process.

The contents of this thesis uncovers significant new results in the area of pursuit-evasion differential games. The main contributions are, the discovery of a new geometric mechanism to verify the Hamilton-Jacobi-Bellman equations, and uncovering new symmetries in simple-motion pursuit-evasion games.

More specifically, the thesis primarily examines the differential game of active target defence, otherwise known as the Target-Attacker-Defender pursuit-evasion game. This simple-motion, two-team, zero-sum differential game emulates a common aerospace engagement scenario found in defence applications. Here an explosive carrying Attacker is tasked with neutralising a Target, and the Target in its defence launches an agent named the Defender, from another platform in an integrated/fused air defence.

The present thesis identifies and proves the value and optimal state-feedback strategies for both teams in this engagement scenario. This is done via the analysis of the discrete-time turn-based variant of the differential game, also known as the upper or lower value. Moreover, we unearth new symmetries in the differential game, named *Target Symmetry* and *Defender Symmetry*. A symmetry is a transformation of the state of the differential game that leaves the optimal strategies unchanged. Using the newly discovered symmetries

we develop unified optimality principles, culminating in the *Holographic Theorem* for the differential game of active target defence, and more generally, the *Holographic Principle* for simple-motion differential games.

Acknowledgements

I would like to thank Professor Cheng-Chew Lim for providing this wonderful opportunity to undertake a PhD in the University of Adelaide, in the exciting field of differential games; Cheng-Chew provided valuable feedback on my manuscripts, helping to improve the clarity of my work before submission. I also would like to thank Professor Peng Shi for introducing me to the Systems and Control group, and his feedback given in my presentations.

Of-course I give thanks to the University of Adelaide itself for awarding me the Faculty of Engineering Computer and Mathematical Sciences Divisional Scholarship. I give thanks to DefendTex, who have provided me a great opportunity to collaborate with the defence industry as part of the Counter Improved Threat Grand Challenge. Dr Kim Jijoong from DST also gets many thanks in this regard.

Many thanks goes to Dr Bernard Evans and Hamish Pratt with whom I have had a wonderful time collaborating with in the development of the Target and Track algorithms. Also I would like to thank Keren Reynolds and Matthew Gervasoni from Lockheed Martin Australia, with whom I undertook an internship during my candidature.

Finally I would like to thank my family for all their support and encouragement.

List of Publications

1. Mammadov, K., Lim, C., & Shi, P. (2020). State-feedback optimal strategies for the differential game of cooperative target defence: a geometric approach. *International Journal of Control*, *94*(10), 2615–2622.

2. Mammadov, K., Lim, C., & Shi, P. (2021). A state-feedback Nash equilibrium for the general Target-Attacker-Defender differential game of degree in arbitrary dimensions. *International Journal of Control*, *95*(1), 93–103.

3. Mammadov, K., Lim, C., & Shi, P. (2022). Generalising the capture the flag scenario to active target defence. *Australian and New Zealand Control Conference 2022*, accepted for publication.

4. Mammadov, K., Lim, C., & Shi, P. (2022). Unified optimality criteria for the Target–Attacker–Defender pursuit-evasion game. *European Journal of Control*, under review.

5. Mammadov, K., Lim, C., & Shi, P. (2022). The holographic principle for the differential game of active target defence. *International Journal of Control*, doi:10.1080/00207179.2022.2111369.

Declaration

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Chapter 1

Synopsis

The research presented in this thesis was jointly funded by the Faculty of Sciences, Engineering and Technology, the University of Adelaide; and by DefendTex as part of the Next Generation Technologies Fund (NGTF) into unmanned aerial systems (UAS).

Small, low-cost, unmanned aerial vehicles (UAV) are emerging and proliferating as a key decisive weapon in the modern battlefield. Threat UAVs can disrupt military operations as both a surveillance platform and by delivering explosive devices. Common counter measures for these low-cost threats are themselves expensive, often ineffective and have the potential for collateral damage.

In response to this emerging threat, the Defence Science and Technology Group (DSTG) initiated the Counter Improvised Threat Grand Challenge in NGTF; for which DefendTex, in collaboration with the University of Adelaide, were tasked with the development of a low-cost unmanned aerial vehicle, specifically designed to counter these small enemy UAV platforms.

Professor Cheng-Chew Lim and myself, with support from Dr Jijoong Kim from DSTG, were assigned to the T2 Target and Track team, responsible for the design of robust decision making, control and guidance algorithms for the drone. This unmanned aerial vehicle is expected to be deployed as an effector, with other kinetic and electromagnetic effectors, complemented with a suite of radar, thermal and visual sensors, in an integrated air defence system.

During the PhD I also undertook an internship with STELaRLab, Lockheed Martin Australia. As such our research has benefited significantly from our collaboration with the defence industry, as well as interdisciplinary collaboration within the university.

Nevertheless, the thesis primarily focuses on our theoretical contributions to the field of pursuit-evasion differential games. As such, the remainder of the thesis is organised as follows. Chapter 2 and 3 introduces the groundwork, the fundamental principles underlying the theory of differential games; then it introduces a specific class of differential games, named simple-motion pursuit-evasion games; and delves specifically into the past literature studying the game of active target defence, which is the focus of the present thesis. We present our original research findings in a thesis by publication format. As such, the next five chapters are the five scholarly publications that resulted from the PhD. The final chapter summarises our accomplishments.

Chapter 2

Introduction and related work

2.1 Introduction to differential games

Driven by applications in defence, research in zero-sum differential games was first initiated by Rufus Isaacs in the late 1950s to 60s. The book Isaacs (1965) was the first book written in this field, followed closely by Blaquiere, Gerard, & Leitmann (1969) and Friedman (1971). These first works in differential games were primarily driven by applications in defence, and as such, focused on two-player zero-sum differential games.

Later, the theory of differential games was generalised to many player non zero-sum games in the seminal work of Sethi & Thompson (1981). Followed by the publication of books from Basar & Olsder (1982) and Dockner, Jorgensen, & Gerhard (2000) focusing on the application of differential games in management science and economics.

In the next section of this chapter, we provide the general definition of a continuous-time differential game.

2.2 System dynamics

In general, a differential game consists of the following components:

1. m players denoted by the set $M = \{1, \ldots, m\}$.

2. For each player $j \in M$, a state variable $x_j \in \mathbb{R}^n$, and a control input $u_j \in \mathbb{R}^{p_j}$.

3. Control constraints $u_j(t) \in U_j$ for each player $j \in M$.

4. A state equation

$$\dot{x}(t) = f(x(t), u(t), t), \qquad x(t_0) = x_0, \tag{2.1}$$

governing the state $x(t) = (x_1(t), \ldots, x_m(t))$ of the game given the initial state $x(t_0)$ and the control inputs $u(t) = (u_1(t), \ldots, u_m(t))$. Here $\dot{\mathbf{x}}(t)$ denotes the time derivative $\frac{d}{dt}\mathbf{x}(t)$.

5. Termination time t_f defined as the first time $t_f \geq t_0$ in which an equation of the form

$$\Theta(x(t), t) = 0 \tag{2.2}$$

holds.

6. A cumulative reward for each player $j \in M$ over the time horizon $[t_0, t_f]$ given by

$$J_j(u(\cdot), x_0, t_0) = \int_{t_0}^{t_f} g_j(x(t), u(t), t)dt + S_j(x(t_f), t_f).$$

Here S_j is the terminal reward and g_j is the instantaneous reward rate.

To complete the formulation of the differential game, we must specify the information available to each player. This information is used to make their choice for the control inputs u_j. The most common information structure used in differential games, are called the state-feedback information structure. Here, at every time t, each player has access to the entire state of the differential game $x(t)$, thus the controls u_j are functions of the state, hence the name 'state-feedback'.

2.3 Nash equilibrium

Each player $j \in M$ must select a state-feedback strategy γ_j according to

$$u_j(t) = \gamma_j(x(t), t), \qquad j \in M. \tag{2.3}$$

Let Γ_j denote the set of all permissible state-feedback strategies $\gamma_j(x(t), t)$ for each player $j \in M$. A state-feedback Nash equilibrium is an m tuple $\gamma^* = (\gamma_1^*, \ldots, \gamma_m^*)$ such that for any initial data $(x(t_0), t_0)$,

$$J_j(\gamma^*, x(t_0), t_0) \geq J_j([\gamma_j, \gamma_{-j}^*], x(t_0), t_0), \tag{2.4}$$

holds for all $\gamma_j \in \Gamma_j$, $j \in M$. Here $[\gamma_j, \gamma_{-j}^*] \coloneqq (\gamma_1^*, \ldots, \gamma_{j-1}^*, \gamma_j, \gamma_{j+1}^*, \ldots, \gamma_m^*)$.

A necessary condition of optimality is given by Pontryagin's maximum principle. Let

$$H_j(x, u, \lambda_j, t) = g_j(x, u, t) + \lambda_j(t) f(x, u, t), \tag{2.5}$$

be the Hamiltonian for each player $j \in M$. If $\gamma^*(x^*(t), t)$ is the state-feedback Nash equilibrium then

$$\gamma_j^*(x^*(t), t) = \arg\max_{u_j \in U_j} H_j(x^*(t), [u_j, \gamma_{-j}^*], \lambda_j, t), \tag{2.6}$$

$$\dot{\lambda}_j(t) = -\left(\frac{\partial}{\partial x} H_j + \sum_{i \in M \setminus j} \frac{\partial}{\partial u_i} H_j \frac{\partial}{\partial x} \gamma_i^*\right)|_{x^*(t), u^*(t), t} \tag{2.7}$$

$$\lambda_j(t_f) = \frac{\partial}{\partial x} S_j|_{x^*(t_f), t_f} \tag{2.8}$$

holds for every player $j \in M$. Nevertheless, the state-feedback Nash equilibrium (SFNE) in most cases cannot be uniquely determined on these principles alone. On the other-hand, one can use the necessary and sufficient condition given by the Hamilton-Jacobi-Bellman equation (also known as the Hamilton-Jacobi-Isaacs equation), to directly obtain the SFNE. These equations are the mathematical expression of Bellman's principle of optimality.

An m-tuple $\gamma^*(x(t), t) = (\gamma_1^*, \ldots, \gamma_m^*)$ is a SFNE at every initial point $(x(t_0), t_0)$ if and only if there exists a continuously differentiable value function $V_j(x, t)$ for each player $j \in M$ such that the Hamilton-Jacobi-Bellman (HJB) equations hold for any state x and time t

$$-\frac{\partial}{\partial t} V_j(x, t) = \max_{u_j \in U_j} \{g_j(x, [u_j, \gamma_{-j}^*], t) + \frac{\partial}{\partial x} V_j(x, t) f(x, [u_j, \gamma_{-j}^*], t)\} \tag{2.9}$$

$$= g_j(x, \gamma^*, t) + \frac{\partial}{\partial x} V_j(x, t) f(x, \gamma^*, t), \tag{2.10}$$

and the boundary condition also holds

$$V_j(x, t_f) = S_j(x). \tag{2.11}$$

In applications, the main issue regarding the use of the HJB equations to compute the SFNE is known as the curse of dimensionality. Analysing differential games with a large n, or p is problematic.

The next section details the specific differential games that are the focus of this thesis.

2.4 Related work

Pursuit-evasion differential game theory plays an important role in applications in aerospace guidance and control. Various pursuit-evasion games have been studied in the literature, here we cite Liu et al. (2013), who studied the single-pursuer multiple-evader pursuit-evasion game, in which a fast pursuer aims to capture all evaders in minimum time and the evader team cooperate to maximise this time. Another example is the cooperative football differential game Garcia, Casbeer & Pachter (2021) in which an attacker aims to reach as close as possible to the goal line before it's intercepted by one of two defenders, and the defenders cooperate to achieve the exact opposite goal. More closely related to the current work, Boyell (1976) considered a 1-agent engagement scenario in which this single agent was tasked with intercepting a target moving at a constant speed in a straight line; here it was proved that $\frac{V_T}{V_A} \sin \angle \mathbf{x}_A(t)\mathbf{x}_T(t)\mathbf{x}_T(t+\Delta t) \leq 1$, where V_T is the speed of the moving target, and V_A is the maximum speed of the agent, and $\mathbf{x}_T(t)$ and $\mathbf{x}_A(t)$ denote their positions respectively; is necessary and sufficient for there to exist a strategy for the agent to capture the target. The relationship that the work of Boyell (1976) had with our research, is that it answered the question of how a slow agent could capture a fast agent if that fast agent could not change direction.

A common assumption among the various differentials games cited here is that every agent has *simple motion*. An agent g is said to have *simple motion* if its state at time t can be completely specified by its position vector $\mathbf{x}_g(t)$, and its dynamics is given by

$\dot{\mathbf{x}}_g(t) = V_g\mathbf{u}_g(t)$, where $\mathbf{u}_g(t)$ is an arbitrary vector with magnitude no greater than 1, chosen by the agent at every time t, and V_g is the maximum speed of agent g. This assumption essentially assumes that every agent has a fixed maximum speed but infinite acceleration/turn-rate. This accurately models engagement scenarios in which the time to accelerate towards maximum speed is small relative to the time at the maximum speed.

The differential game that is the focus of the present thesis, is named the Target-Attacker-Defender (TAD) pursuit-evasion game, also known as the Cooperative/Active Target Defence differential game. This is a continuous-time, zero-sum differential game consisting of two teams, team A and team T/D, and three agents, the Target, Attacker and Defender modelled with *simple motion*. The differential game terminates at the first time t_f the Attacker collides with one of the other two agents. The Attacker's goal is to minimise the distance between itself and the Target at time t_f, and the Target and Defender work as a team to maximise the aforementioned distance at time t_f. The TAD pursuit-evasion game is commonly motivated by visualising the Attacker as an explosive carrying aerial vehicle tasked with neutralising an evasive aerial Target; and the Target or another asset in its defence launches another drone (Defender) to intercept the Attacker.

The most general setting in which this pursuit-evasion game can be studied, is under the assumption that $V_T \leq V_A \leq V_D$, here the Target is no faster than the Attacker, which in turn is no faster than the Defender. Clearly, if V_T were greater than V_A then the Target can easily escape capture from the Attacker, and the game may never terminate; therefore this case is always neglected. The reason for why the case $V_A > V_D$ is dismissed is more complex. If $V_A > V_D$ and $V_A > V_T$, then the Target and Defender may cooperate as a team to delay the capture of the Target, but they cannot prevent it, that is under optimal play, at termination time t_f, $\mathbf{x}_A(t_f) = \mathbf{x}_T(t_f)$. Since the payoff/reward function is defined as the distance between the Target and Attacker at termination time, there is no incentive for team T/D to do anything, therefore this case is degenerate.

To provide evidence that this is indeed the case, here we quote a paragraph from the paper Garcia, Casbeer & Pachter (2021):

"*Remark. When $V_A > V_D$ (slower Defender) and point capture is required then A always captures T irrespective of the initial conditions. The slower agent, D in this case, is* *incapable of achieving point interception of the faster agent, A. Player A can always exploit its speed advantage to circumvent a slowly moving point and capture T . Additionally, since* $\frac{V_T}{V_A} < 1$ *(T is slower than A) the T/D team is not able to indefinitely keep A away from T or, in terms of [22], keep the state such that* $R > 0$ *and* $r > 0$ *where R is the* $A - T$ *separation and r is the* $A - D$ *separation. Hence, a rendezvous strategy between T and D always results in T being captured by A. Therefore, in this paper we focus on the case* $V_A = V_D$*. The results can also be extended to the case* $V_D > V_A$*.*"

Thus, in the case $V_A > V_D$, the Attacker always captures the Target, moreover the state-feedback Nash equilibrium is not uniquely defined in this region.

2.5 Winning region of team T/D

The works in Garcia, Casbeer & Pachter (2019) and all other works in this topic exclusively considered the case in which all agents move in 2-dimensional space, as the complexity of the analysis is prohibitive in higher dimensions. In these works, the complete state of the TAD differential game is specified by $\mathbf{x} := (x_T, y_T, x_A, y_A, x_D, y_D) \in \mathbb{R}^6$, where $(x_T(t), y_T(t))$, $(x_A(t), y_A(t))$ and $(x_D(t), y_D(t))$ denotes the position of the Target, Attacker and Defender respectively. The dynamics $\mathbf{f}(\mathbf{x}, \chi, \psi, \phi)$ are given by the differential equations

$$
\begin{aligned}
\dot{x}_A &= \cos\chi, & x_A(0) &= x_{A_0} \\
\dot{y}_A &= \sin\chi, & y_A(0) &= y_{A_0} \\
\dot{x}_D &= \cos\psi, & x_D(0) &= x_{D_0} \\
\dot{y}_D &= \sin\psi, & y_D(0) &= y_{D_0} \\
\dot{x}_T &= \alpha\cos\phi, & x_T(0) &= x_{T_0} \\
\dot{y}_T &= \alpha\sin\phi, & y_T(0) &= y_{T_0}
\end{aligned}
$$

where $\mathbf{u}_{T,D} = \{\phi, \psi\}$ are the control inputs for team T/D, $\mathbf{u}_A = \{\chi\}$ is the control input for team A, and $x(0)$ is the initial state. Here $0 < \alpha < 1$; that is, the Target is slower than the Attacker, and the Attacker and Defender have equal speed. The control constraints are given by $\chi, \phi, \psi \in (-\pi, \pi]$. Termination time t_f is defined endogenously as the first time either $(x_A, y_A) = (x_D, y_D)$ or $(x_A, y_A) = (x_T, y_T)$ holds.

Let $\mathbf{x}_0 = (x_{T_0}, y_{T_0}, x_{A_0}, y_{A_0}, x_{D_0}, y_{D_0})$ denote the initial state at the starting time $t_0 = 0$, and $\mathbf{x}_f = (x_T(t_f), y_T(t_f), x_A(t_f), y_A(t_f), x_D(t_f), y_D(t_f))$ denote the final state at the endogenously defined termination time t_f. Moreover let $\mathbf{u} = (\mathbf{u}_{T,D}, \mathbf{u}_A)$ denote the combination of all inputs. The payoff function is defined by

$$J(\mathbf{u}(\cdot), \mathbf{x}_0, 0) = S(\mathbf{x}_f, t_f), \tag{2.12}$$

where

$$S(\mathbf{x}_f, t_f) = \sqrt{(x_A(t_f) - x_T(t_f))^2 + (y_A(t_f) - y_T(t_f))^2}. \tag{2.13}$$

Here team A aims to minimise the payoff function, whereas team T/D cooperate to maximise it. In addition, let $\overline{DT} = \sqrt{(x_D(t) - x_T(t))^2 + (y_D(t) - y_T(t))^2}$ and $\overline{AT} = \sqrt{(x_A(t) - x_T(t))^2 + (y_A(t) - y_T(t))^2}$ denote the distance between the Defender and Target, and the distance between the Attacker and Target respectively.

Using the two-sided Pontryagin maximum principle, it was deduced in Garcia et al. (2019) that the state-feedback Nash equilibrium is given by the two following theorems, since the analysis of the differential game is separated into two distinct cases.

- Case 1: $\overline{DT} < \overline{AT}$
- Case 2: $\overline{DT} > \overline{AT}$

Also the boundary case $\overline{DT} = \overline{AT}$ was considered in Garcia et al. (2019), but it is not covered in this chapter.

Theorem 2.5.1 (Garcia et al. (2019)). *For any state satisfying $\overline{DT} < \overline{AT}$ the optimal state-feedback strategies are given by*

$$\cos\phi^* = \frac{x_T - x}{\sqrt{(x_T - x)^2 + (y_T - y)^2}}$$
$$\sin\phi^* = \frac{y_T - y}{\sqrt{(x_T - x)^2 + (y_T - y)^2}}$$
$$\cos\psi^* = \frac{x - x_D}{\sqrt{(x - x_D)^2 + (y - y_D)^2}}$$
$$\sin\psi^* = \frac{y - y_D}{\sqrt{(x - x_D)^2 + (y - y_D)^2}}$$
$$\cos\chi^* = \frac{x - x_A}{\sqrt{(x - x_A)^2 + (y - y_A)^2}}$$
$$\sin\chi^* = \frac{y - y_A}{\sqrt{(x - x_A)^2 + (y - y_A)^2}} \tag{2.14}$$

and the value function is given by

$$V(\mathbf{x}) = \alpha\sqrt{\frac{1}{4}((x_A - x_D)^2 + (y_A - y_D)^2) + (x - \frac{1}{2}(x_A + x_D))^2 + (y - \frac{1}{2}(y_A + y_D))^2}$$
$$+ \sqrt{(x_T - x)^2 + (y_T - y)^2} \tag{2.15}$$

where the coordinates (x, y) denote the point of interception of agent A and D under optimal play; it is given by

$$y = mx + n \tag{2.16}$$

where $m = -\frac{x_A - x_D}{y_A - y_D}$, $n = \frac{1}{2}(y_A + y_D) - \frac{m}{2}(x_A + x_D)$, and x is a real solution to the quartic

$$(1-\alpha^2)(m^2+1)^3x^4 + (1-\alpha^2)(m^2+1)^2(k_1 + 2k_2)x^3$$
$$+ \Big((m^2+1)^2(k_3 - \alpha^2 k_4) + 2(1-\alpha^2)(m^2+1)k_1k_2 + (m^2+1)(k_2^2 - \frac{\alpha^2}{4}k_1^2)\Big)x^2$$
$$+ \Big((m^2+1)(2k_2k_3 - \alpha^2 k_1k_4) + k_1k_2(k_2 - \frac{\alpha^2}{2}k_1)\Big)x$$
$$+ k_2^2k_3 - \frac{\alpha^2}{4}k_1^2k_4 = 0 \tag{2.17}$$

and k_1, k_2, k_3, k_4 is given by

$$
\begin{aligned}
k_1 &= 2mn - (x_A + x_D + m(y_A + y_D)) \\
k_2 &= mn - x_T - my_T \\
k_3 &= \frac{1}{2}(x_A^2 + x_D^2 + y_A^2 + y_D^2) + n^2 - n(y_A + y_D) \\
k_4 &= x_T^2 + (y_T - n)^2.
\end{aligned} \tag{2.18}
$$

Using the same techniques, the state-feedback Nash equilibrium was also characterised for the case $\overline{DT} > \overline{AT}$.

Theorem 2.5.2 (Garcia et al. (2019))**.** *For any state satisfying $\overline{DT} > \overline{AT}$ and $V(\mathbf{x}) > 0$, the optimal state-feedback strategies are given by*

$$
\begin{aligned}
\cos\phi^* &= \frac{x - x_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}} \\
\sin\phi^* &= \frac{y - y_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}} \\
\cos\psi^* &= \frac{x - x_D}{\sqrt{(x - x_D)^2 + (y - y_D)^2}} \\
\sin\psi^* &= \frac{y - y_D}{\sqrt{(x - x_D)^2 + (y - y_D)^2}} \\
\cos\chi^* &= \frac{x - x_A}{\sqrt{(x - x_A)^2 + (y - y_A)^2}} \\
\sin\chi^* &= \frac{y - y_A}{\sqrt{(x - x_A)^2 + (y - y_A)^2}}
\end{aligned} \tag{2.19}
$$

and the value function is given by

$$
\begin{aligned}
V(\mathbf{x}) = \alpha &\sqrt{\frac{1}{4}((x_A - x_D)^2 + (y_A - y_D)^2) + (x - \frac{1}{2}(x_A + x_D))^2 + (y - \frac{1}{2}(y_A + y_D))^2} \\
&- \sqrt{(x_T - x)^2 + (y_T - y)^2}
\end{aligned} \tag{2.20}
$$

where the coordinates (x, y) denote the point of interception of agent A and D under optimal play; it is given by (2.16)-(2.18)*.*

There are two main limitations from the works in Garcia et al. (2019).

1. The differential game is only studied in 2-dimensions, since the computation of the HJB equations quickly becomes intractable in higher dimensions.

2. The maximum speed of the Attacker is assumed to be exactly equal to the Defender ($V_A = V_D$); whereas the most general case for which this differential game can be studied in is where $V_T \leq V_A \leq V_D$.

The works of Garcia, Casbeer & Pachter (2017) analysed the TAD pursuit-evasion game to the more general setting of $V_A < V_D$. In this manuscript Pontryagin's maximum principle was applied to uncover a fundamental symmetry in the differential game. Under optimal play all agents move in straight line motion. But many open problems still remained, the derivatives of the value function were not found, and the HJB equations were not shown to hold. Nevertheless, it was argued that the value function for the general case $V_T < V_A < V_D$ can be characterised by the following optimisation criteria. Let $\mathcal{C}_{\mathcal{AD}}$ and $\mathcal{C}_{\mathcal{AT}}$ denote the set of all points in the interior of the AD and AT based Apollonius circles respectively (see Garcia et al. (2017) and references therein). Although without rigorous proof (which is the reason why they were able to publish a paper for the more complicated case $V_A < V_D$, before the work in Garcia et al. (2019), which contained a rigorous proof), it was reasoned that the value function should obey

$$V(\mathbf{x}(t)) = \begin{cases} \max\limits_{P \in \mathcal{C}_{\mathcal{AD}}} -\overline{TP} + \dfrac{V_T}{V_A}\overline{AP} & \text{if } \ (x_T, y_T) \in \mathcal{C}_{\mathcal{AD}} \\ \min\limits_{P \in \mathcal{C}_{\mathcal{AD}}} \overline{TP} + \dfrac{V_T}{V_A}\overline{AP} & \text{if } \ (x_T, y_T) \notin \mathcal{C}_{\mathcal{AD}} \end{cases} \tag{2.21}$$

At this point we should take note that, both in the works of Garcia et al. (2019) and Garcia et al. (2017), the value function and the optimal state-feedback strategies were separated into two cases. (More precisely, it was actually split into a total of three cases since there are singularities in some formulas when the Target sits exactly on the perimeter of the AD-based Apollonius circle. This case requires its own separate formulas to define the optimal headings.) In our third journal paper, we introduce a new unifying paradigm given by the Critical Escape Trajectories Theorem; which simultaneously characterises the Target's escape set and value function of the game in all regions, highlighting a deep underlying geometric connection between these two concepts.

2.6 Game of kind

On the topic of the Target's escape set, in Theorem 2.5.2, it was assumed that under optimal play, the Target would escape capture (i.e. $\overline{AT} > 0$ at termination time, or equivalently $V(\mathbf{x}) > 0$). The set of all states satisfying this condition is named the winning region of team T/D; and the problem of unearthing necessary and sufficient conditions of the current state $\mathbf{x}$ that yields $V(\mathbf{x}) > 0$ is named the game of kind. (As opposed to the game of degree which is the problem of finding the optimal state-feedback strategies.)

Liang et al. (2019) made a significant contribution in this regard. Let $\overline{AD} = \sqrt{(x_A(t) - x_D(t))^2 + (y_A(t) - y_D(t))^2}$ etc. They proved that under optimal play, the condition

$$V_A\overline{DT} < V_D\overline{AT} + V_T\overline{AD} \tag{2.22}$$

is necessary and sufficient for the Target to escape capture from the Attacker, that is, at termination time t_f, $\sqrt{(x_A(t_f) - x_T(t_f))^2 + (y_A(t_f) - y_T(t_f))^2} > 0$.

Similarly, the set of all states $\mathbf{x}$ which yield $V(\mathbf{x}) = 0$ is named the winning region of team A, where the Attacker captures the Target under optimal play. This occurs whenever $V_A\overline{DT} \geq V_D\overline{AT} + V_T\overline{AD}$ holds, but the optimal state-feedback strategies are not unique in this region. Unlike in the winning region of team T/D, where the optimal strategies are uniquely defined by Theorems 2.5.1 and 2.5.2. This is because if the Target 'knows' it is going to be captured anyway, there is no incentive to 'run away', since the objective function was defined as $S(\mathbf{x}_f, t_f) = \sqrt{(x_A(t_f) - x_T(t_f))^2 + (y_A(t_f) - y_T(t_f))^2}$.

To extend the analysis of the TAD pursuit-evasion game to the winning region of team A, it is necessary to redefine/generalise the payoff function. An appropriate generalisation of the reward function is given by

$$S(\mathbf{x}_f, t_f) = \sqrt{(x_A(t_f) - x_T(t_f))^2 + (y_A(t_f) - y_T(t_f))^2} - \sqrt{(x_A(t_f) - x_D(t_f))^2 + (y_A(t_f) - y_D(t_f))^2}. \tag{2.23}$$

This reward function has the property that it is equivalent to the reward function defined earlier in (2.13) for any state satisfying $V(\mathbf{x}) \geq 0$, since $\sqrt{(x_A(t_f) - x_D(t_f))^2 + (y_A(t_f) - y_D(t_f))^2} = 0$ in this region; But both teams still have an incentive to minimise/maximise the distance between the Attacker and Defender at termination time. Therefore in this formulation of the TAD pursuit-evasion game, the state-feedback optimal strategies are uniquely defined for any initial state.

2.7 Winning region of team A

Indeed the reward function (2.23) can be seen as a generalisation of (2.13) to encompass negative values. This was the approach taken in the works of Garcia et al. (2021), which exclusively focused on the case in which $V(\mathbf{x}) \leq 0$. Although technically they defined the reward function to be equal to the distance between the Target and Defender at termination time, this is equivalent to (2.23) in the winning region of team A since the Target and Attacker have the some position at termination time.

The results of Garcia et al. (2021) are given by the following theorem. Note that this is different to the earlier theorems 2.5.1-2.5.2, which considered a different reward function.

Theorem 2.7.1 (Garcia et al. (2021))**.** *For any state satisfying $V(\mathbf{x}) < 0$, the optimal state-feedback strategies are given by*

$$
\begin{aligned}
\cos\phi^* &= \frac{x - x_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}} \\
\sin\phi^* &= \frac{y - y_T}{\sqrt{(x - x_T)^2 + (y - y_T)^2}} \\
\cos\psi^* &= \frac{x - x_D}{\sqrt{(x - x_D)^2 + (y - y_D)^2}} \\
\sin\psi^* &= \frac{y - y_D}{\sqrt{(x - x_D)^2 + (y - y_D)^2}} \\
\cos\chi^* &= \frac{x - x_A}{\sqrt{(x - x_A)^2 + (y - y_A)^2}} \\
\sin\chi^* &= \frac{y - y_A}{\sqrt{(x - x_A)^2 + (y - y_A)^2}}
\end{aligned}
\tag{2.24}
$$

and the value function is given by

$$V(\mathbf{x}) = \sqrt{(x - x_D)^2 + (y - y_D)^2} - \frac{1}{\alpha}\sqrt{(x - x_T)^2 + (y - y_T)^2} \tag{2.25}$$

where the coordinates (x, y) denote the point of interception of agent A and T under optimal play; it is a solution of the system of two equations

$$\begin{aligned}&\big((x - x_T)^2 + (y - y_T)^2\big)\Bigg((x - x_D)(y - y_T) - (x - x_T)(y - y_D) - \alpha^2(x - x_D)(y - y_A)\\ &\quad + \alpha^2(x - x_A)(y - y_D)\Bigg)^2 - \alpha\big((x - x_D)^2\\ &\quad + (y - y_D)^2\big)\Bigg((x - x_A)(y - y_T) - (x - x_T)(y - y_A)\Bigg)^2 = 0\end{aligned} \tag{2.26}$$

$$(x - x_c)^2 + (y - y_c)^2 = r^2 \tag{2.27}$$

where

$$\begin{aligned}x_c &= \frac{1}{1 - \alpha^2}(x_T - \alpha^2 x_A)\\ y_c &= \frac{1}{1 - \alpha^2}(y_T - \alpha^2 y_A)\\ r &= \frac{\alpha}{1 - \alpha^2}\sqrt{(x_T - x_A)^2 + (y_T - y_A)^2}.\end{aligned}$$

The results of Garcia et al. (2021) suffer the same limitations. The analysis is only in 2 dimensions, and the speeds of the Attacker and Defender are assumed to be identical.

2.8 Summary of new results

The original results of our thesis are given by the following five manuscripts.

1. Mammadov, K., Lim, C., & Shi, P. (2020). State-feedback optimal strategies for the differential game of cooperative target defence: a geometric approach. *International Journal of Control*, *94*(10), 2615–2622.

2. Mammadov, K., Lim, C., & Shi, P. (2021). A state-feedback Nash equilibrium for the general Target-Attacker-Defender differential game of degree in arbitrary dimensions. *International Journal of Control*, *95*(1), 93–103.

3. Mammadov, K., Lim, C., & Shi, P. (2022). Generalising the capture the flag scenario to active target defence. *Australian and New Zealand Control Conference 2022*, accepted for publication.

4. Mammadov, K., Lim, C., & Shi, P. (2022). Unified optimality criteria for the Target–Attacker–Defender pursuit-evasion game. *European Journal of Control*, under review.

5. Mammadov, K., Lim, C., & Shi, P. (2022). The holographic principle for the differential game of active target defence. *International Journal of Control*, doi:10.1080/00207179.2022.2111369.

The first publication tackled with the curse of dimensionality by studying the differential game in n–dimensional euclidean space. Here a novel geometric approach is introduced as an alternative to the standard algebraic approach of Garcia et al. (2019), where the pursuit-evasion game is reformulated as a discrete-time turn-based dynamic game, in which the corresponding strategies are proved to be a Nash equilibrium. Via a recursive application of the main results, in view of the fact that no limits were placed on the infinitesimalness of the time increment, it is argued that the strategies also constitute a Nash equilibrium in the original continuous time formulation. Manuscript number 1 also demonstrated the robustness of the state-feedback Nash equilibrium, by conducting simulations of the TAD game in the case where the Attacker uses PN guidance. Furthermore, the computation of the SFNE was straight forward, as it involved minimising a convex function at every time increment.

The second publication is the first paper to have rigorously examined the Target–Attacker–Defender pursuit-evasion game in the general setting where $V_A < V_D$. Here the TAD differential game of degree is recast in a discrete-time turn-based variant with arbitrarily small time increment. In this formulation it is proven that the corresponding discretized optimal strategy profile constitutes a Nash equilibrium, thereby giving strong theoretical support to the claimed Nash equilibrium for the original continuous-time formulation.

The conference paper, number 3, improved upon the work given in 2. The second paper was the first to give a rigorous proof of the state-feedback Nash equilibrium in the general case $V_A < V_D$; however, at a single point in the proof it made the assumption of $V_T = 0$, as the machinery required to prove it for the general case $0 < V_T < V_A < V_D$ was not known at the time. The conference paper completes the missing proof so that the results now hold generally for any $V_T < V_A < V_D$.

Manuscript number 4, currently under preparation for journal submission, unveils a new unifying paradigm given by the Critical Escape Trajectories Theorem, based on the discovery of a symmetry named *Target Symmetry*; to express the value, the optimal strategies, and reveal a simple analytical technique to solve for the state-feedback Nash equilibrium of the Target-Attacker-Defender pursuit-evasion game. This new paradigm reveals a deep underlying geometric connection between the game of kind and the game of degree; and the methods developed in this manuscript are not a disjoint collection of techniques, unlike Garcia et al. (2017) and Garcia et al. (2019) which have different methods for different regions in the Target's escape set.

Journal manuscript number 5, is the result of a successful research project to develop a grand unifying optimality principle. Although the earlier manuscript was successful in characterising the state-feedback Nash equilibrium for any state in the winning region of team T/D, it only examined the game with the reward function defined by (2.13), thus it did not allow for negative values. This manuscript studies the TAD pursuit-evasion game in its most complete formulation, with $V_T < V_A < V_D$, reward function defined by (2.23), in n–dimensional euclidean space. Here we introduce and verify the most elegant characterisation of the state-feedback Nash equilibrium given by a one-inch formula that holds throughout the entire state space of the game. Whereas previous methods in the literature would segregate the state space into four separate regions, each containing a different method and proof, this manuscript introduces a new unifying principle given by the *Holographic Principle*, which contains within it *Target Symmetry* as a special case. Furthermore, it's conjectured that the *Holographic Principle* holds more broadly in a large class of simple motion pursuit-evasion games.

To discuss the relationship between the five papers; the first manuscript studies the TAD differential game in the case $V_T < V_A = V_D$, the next two journal manuscripts and the conference paper examines the TAD differential game for $V_T < V_A < V_D$, and the final journal manuscript studies the TAD game for the case $V_T < V_A < V_D$ in the winning region of team A.

Chapter 3

State-feedback optimal strategies for the differential game of cooperative target defence: A geometric approach

3.1 Contextual statement

Our first publication examines the Target-Attacker-Defender differential game in the simple case $V_T < V_A = V_D$. The novelty here is that this is the publication that establishes how the state-feedback Nash equilibrium can be identified and proved by transforming the continuous-time differential game into a discrete-time turn-based game with arbitrarily small time increment. This publication provides new techniques that drastically simplifies the analysis of the differential game, completely bypassing the complex and cumbersome methods in Garcia et al. (2019), and leads to an elegant proof that holds in n spatial dimensions.

Statement of Authorship

Title of Paper	State-feedback optimal strategies for the differential game of cooperative target defence: a geometric approach.
Publication Status	☑ Published ☐ Accepted for Publication ☐ Submitted for Publication ☐ Unpublished and Unsubmitted work written in manuscript style
Publication Details	Mammadov, K., Lim, C., & Shi, P. (2021). State-feedback optimal strategies for the differential game of cooperative target defence: a geometric approach. International Journal of Control, 94(10), 2615-2622. doi:10.1080/00207179.2020.1727016

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Overall percentage (%)	95%		
Certification:	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.		
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Co-Author Contributions

By signing the Statement of Authorship, each author certifies that:

i. the candidate's stated contribution to the publication is accurate (as detailed above);

ii. permission is granted for the candidate in include the publication in the thesis; and

iii. the sum of all co-author contributions is equal to 100% less the candidate's stated contribution.

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Signature		Date	8/04/2022

INTERNATIONAL JOURNAL OF CONTROL
2021, VOL. 94, NO. 10, 2615–2622
https://doi.org/10.1080/00207179.2020.1727016

State-feedback optimal strategies for the differential game of cooperative target defence: a geometric approach

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ABSTRACT
The three agent zero-sum differential game of active defence is investigated; in this pursuit-evader game, agent A attempts to capture agent T, and agents T and D coordinate to achieve the opposite goal. A novel geometric approach is introduced as an alternative to the standard algebraic approach where the pursuit-evader game is reformulated as a discrete-time turn-based dynamic game, in which the corresponding strategies are proved to be a Nash equilibrium. Via a recursive application of the main results, in view of the fact that no limits were placed on the infinitesimalness of the time increment, it is argued that the strategies also constitute a Nash equilibrium in the original continuous time formulation. Finally we simulate the Nash equilibrium strategies to verify its optimality and its robustness against other guidance laws.

ARTICLE HISTORY
Received 6 June 2019
Accepted 2 February 2020

KEYWORDS
Differential game theory; dynamic game theory; optimal state-feedback strategies; state-feedback Nash equilibrium; pursuit-evasion games

1. Introduction

Multiagent pursuit evader games present highly applicable but theoretically challenging problems in aerospace guidance and control. One particular class of pursuit evader games that has been given significant scholarly attention is known as the Target–Attacker–Defender (TAD) game. This class of games were first presented by Boyell (1976), who considered a moving naval ship launching a torpedo in its defence against an attacking submarine, and subsequently more applications were found by Rusnak (2005), where a bodyguard aims to protect a potential victim from a bandit, and (Li & Cruz, 2011) where an autonomous vehicle is deployed to defend an asset against an attacker.

Across these applications, a common feature is that there are two teams, Team A and Team T/D, and three agents, the Attacker, Defender and Target. Team A, with a single agent named the Attacker, agent A or just A, its goals are twofold. Its first priority is to capture the Target whilst evading collision with the Defender, if possible. If this objective is not possible, its secondary objective to minimise the terminal distance between the Target and its position at collision, thereby maximising the harm inflicted on the Target if agent A is visualised as a bomb. On the other hand, agents D and T coordinate their movements in such a manner as to achieve the opposite goal, namely to prevent the capture of Agent T, if possible, and do so in a manner as to maximise the aforementioned terminal distance.

There are many variations in the literature of how this three-agent engagement scenario is modelled. In Prokopov and Shima (2013), the Attacker is unaware of the Defender and employs a known linear one-on-one guidance law to catch agent T. Here it is also assumed the Defender and Target start at the same position (since it models a target aircraft launching a missile in its defence), and the agents dynamics are linearised along the initial lines of sight. Shima (2011) also considered the scenario in which both the Attacker and Defender's strategies were constrained to the most well-known guidance laws PN, APN and OGL. Under those assumptions on agent A's strategy and agent D's strategy, agent T solves a one-sided optimal control problem.

Another variation of this problem is commonly called the TAD game of kind, here the goal is to partition the state space into disjoint regions which indicate the winning teams; works on this problem include Bhattacharya, Basar, Hovakimyan (2016) and Zha, Chen, Peng, Gu (2017). In Liang, Deng, Peng, Li, and Zha (2019), the whole state space is separated into regions in which Team A wins (captures the target) and Team T/D wins (prevents its capture), moreover in the different regions optimal strategies are analysed using the winning time as the cost functional.

The focus of this paper is on the differential game formulation first introduced in Garcia, Casbeer, and Pachter (2015), where the Attacker is aware of the Defender and agents T and D do not necessarily start at the same position, but the starting position of agent T is assumed to be strictly closer to agent D than to agent A. Here the Attacker and Target–Defender compete in a zero-sum game to minimise and maximise respectively the final separation distance between agents A and T.

In this point in question, Garcia, Casbeer, and Pachter (2019) and Garcia, Casbeer, and Pachter (2018) derived and proved a state-feedback Nash equilibrium by demonstrating that the strategies defined by the Nash equilibrium satisfies the Hamilton–Jacobi–Bellman equations. However these results had two main limitations, first the speed of the Target was assumed to be strictly less than that of the Defender, and the results were

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not generalizable to dimensions greater than 2, suggesting a better approach is needed. These limitations are addressed in the current manuscript.

In fact all the results forenamed assume the agents move in two-dimensional space, hence with three agents in the TAD game, the state-space is six-dimensional to denote the coordinates of every agent. The focus of this paper is to explore the TAD game of degree in arbitrary dimensions where the agents move in n-dimensional space. In Section 2, it is explored how the traditional algebraic approach developed by Isaacs (1965), where the Hamilton–Jacobi–Bellman equations are verified using calculus and algebra suffers from an almost intractable complexity as the dimension of the state-space increases.

The main contribution of this article is given in Section 3, where a new geometric mechanism is developed to verify that the conjectured Nash equilibrium satisfies the Hamilton–Jacobi–Bellman equations; critically this new technique avoids the complexities that arise from a large state-space and is easily generalisable to arbitrary dimensions in pursuit-evader games.

2. TAD game of degree formulation and preliminary analysis

The following section recasts the TAD game of degree found in Garcia et al. (2019) in n-dimensional space. Throughout the manuscript, the notation $\overline{\mathbf{ab}}$ denotes the euclidean distance between two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

2.1 Problem definition of TAD game of degree

The TAD game of degree is a two player (two team) zero-sum differential game. The complete state of the differential game is specified by $\mathbf{x}(t) = (\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t))$ where $\mathbf{x}_A(t) = (x_{A_1}(t), \dots, x_{A_n}(t)) \in \mathbb{R}^n$, $\mathbf{x}_D(t)=(x_{D_1}(t), \dots, x_{D_n}(t)) \in \mathbb{R}^n$ and $\mathbf{x}_T(t) = (x_{T_1}(t), \dots, x_{T_n}(t)) \in \mathbb{R}^n$ denotes the position of the Attacker, Defender and Target respectively. Regarding the information structure of the game, both teams have access to the current state of the system $\mathbf{x}(t)$ at time t. Using that information, Team A must choose an instantaneous heading for agent A, and Team T/D for the headings of agents D and T. The controls of agents A, D and T are given by

$$\mathbf{u}_A(t), \mathbf{u}_D(t), \mathbf{u}_T(t) \in \text{unit}(n-1)\text{−sphere}. \tag{1}$$

The dynamics $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}_A(t), \mathbf{u}_D(t), \mathbf{u}_T(t))$ from time t_0 to t_f is given by

$$(\dot{\mathbf{x}}_A(t), \dot{\mathbf{x}}_D(t), \dot{\mathbf{x}}_T(t)) = (\mathbf{u}_A(t), \mathbf{u}_D(t), \alpha \mathbf{u}_T(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where $0 \leq \alpha \leq 1$. That is, the speed of the Attacker and Defender is equal and normalised to 1, and the Target is no faster than the Attacker. Here it is assumed the initial state $\mathbf{x}_0$ satisfies

$$\overline{\mathbf{x}_T(t_0)\mathbf{x}_D(t_0)} < \overline{\mathbf{x}_T(t_0)\mathbf{x}_A(t_0)}, \tag{3}$$

and the termination time t_f is defined endogenously as the first time t_f satisfying the following termination condition

$$\mathbf{x}_A(t_f) = \mathbf{x}_D(t_f). \tag{4}$$

Over the time horizon $[t_0, t_f]$, each team receives the following payoff

$$J(\mathbf{u}_A(t), \mathbf{u}_D(t), \mathbf{u}_T(t), \mathbf{x}_0) = \Phi(\mathbf{x}_f), \tag{5}$$

where

$$\Phi(\mathbf{x}_f) = \overline{\mathbf{x}_T(t_f)\mathbf{x}_A(t_f)}. \tag{6}$$

The objective of Team A is to minimise $J(\mathbf{u}_A(t), \mathbf{u}_D(t), \mathbf{u}_T(t), \mathbf{x}_0)$, whereas the objective of Team T/D is to maximise it. This differential game is later referred to in the article as Game 2.1.

2.2 Optimal state feedback strategies

In this section, the optimal state-feedback strategies are displayed, but first some preliminary notation is introduced.

The value functional $V : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [t_0, t_f]$, which should indicate the payoff-to-go at every state, is defined below

$$V(\mathbf{x}(t)) = \overline{\mathbf{x}_T(t) I(\mathbf{x}(t))} + \alpha \overline{\mathbf{x}_A(t) I(\mathbf{x}(t))}, \tag{7}$$

where the *predicted* interception point $I(\mathbf{x}(t))$ is given by

$$I(\mathbf{x}(t)) = \arg\min_{\mathbf{I}} \overline{\mathbf{x}_T(t)\mathbf{I}} + \alpha \overline{\mathbf{x}_A(t)\mathbf{I}} \tag{8a}$$

$$\text{s.t.} \quad \mathbf{I} \in \chi \tag{8b}$$

and where the $(n-1)$-dimensional plane χ is the midway plane between the Attacker and Defender[1]

$$\chi = \{p \in \mathbb{R}^n \mid \overline{p\mathbf{x}_A(t)} = \overline{p\mathbf{x}_D(t)}\}. \tag{9}$$

The following conjecture is a generalisation of the Nash equilibrium proved in Garcia et al. (2019) from 2D space to arbitrary dimensions.

Conjecture 2.1: A state-feedback Nash equilibrium of Game 2.1 is given by

$$\mathbf{u}_A^*(t) = \hat{\mathbf{A}}, \tag{10a}$$

$$\mathbf{u}_D^*(t) = \hat{\mathbf{D}}, \tag{10b}$$

$$\mathbf{u}_T^*(t) = -\hat{\mathbf{T}}, \tag{10c}$$

where

$$(\hat{\mathbf{A}}, \hat{\mathbf{D}}, \hat{\mathbf{T}}) = \left(\frac{I(\mathbf{x}(t)) - \mathbf{x}_A(t)}{\|I(\mathbf{x}(t)) - \mathbf{x}_A(t)\|}, \frac{I(\mathbf{x}(t)) - \mathbf{x}_D(t)}{\|I(\mathbf{x}(t)) - \mathbf{x}_D(t)\|}, \frac{I(\mathbf{x}(t)) - \mathbf{x}_T(t)}{\|I(\mathbf{x}(t)) - \mathbf{x}_T(t)\|} \right). \tag{11}$$

At termination time, $I(\mathbf{x}(t_f)) = \mathbf{x}_A(t_f)$, hence $V(\mathbf{x}(t_f)) = \overline{\mathbf{x}_T(t_f)\mathbf{x}_A(t_f)} = \Phi(\mathbf{x}_f)$, that is the boundary condition is satisfied. Therefore provided that the value functional is continuously differentiable, it is sufficient to prove that the Nash equilibrium strategies defined by (10) satisfies the Hamilton–Jacobi–Bellman equations provided below from t_0 to t_f.

$$\mathbf{u}_A^*(t) = \arg\min_{\mathbf{u}_A(t)} \frac{\partial V}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}_A(t), \mathbf{u}_D^*(t), \mathbf{u}_T^*(t)) \tag{12a}$$

$$\text{s.t.} \quad \forall t \quad \mathbf{u}_A(t) \in \text{unit } (n-1)\text{−sphere} \tag{12b}$$

$$(\mathbf{u}_D^*(t), \mathbf{u}_T^*(t)) = \arg\max_{(\mathbf{u}_D(t), \mathbf{u}_T(t))} \frac{\partial V}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}_A^*(t), \mathbf{u}_D(t), \mathbf{u}_T(t)) \tag{13a}$$

$$\text{s.t.} \quad \forall t \quad \mathbf{u}_D(t), \mathbf{u}_T(t) \in \text{unit } (n-1)\text{−sphere} \tag{13b}$$

$$\frac{\partial V}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}_A^*(t), \mathbf{u}_D^*(t), \mathbf{u}_T^*(t)) = 0. \tag{14}$$

The approach taken in the works of Garcia et al. (2019) used a similar expression to (7) to analytically calculate the gradient of the value function and used that result to verify the above equations. However that approach, even in 2D led to excruciatingly long calculations, since the Jacobian of the interception point had to be computed, which measures the change in $I(\mathbf{x}(t))$ with respect to the change in the state $\mathbf{x}(t)$. In the case of arbitrary dimensions that approach is no longer feasible. The next section details the introduction of a new geometric method of verifying the HJB equations, by adopting the discrete-time variant of the TAD game of degree.

3. Discrete-time turn-based approximation

3.1 Problem formulation of sequential game

Consider the following sequential game. The game starts at any time t from any initial state $\mathbf{x}(t) = (\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t))$ satisfying $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} < \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}$.

Team A with complete information of the full state $\mathbf{x}(t)$ moves first, it chooses a heading for agent A to move in a straight line from $\mathbf{x}_A(t)$ to $\mathbf{x}_A(t+\Delta t)$ of length Δt, where $\mathbf{x}_A(t+\Delta t)$ depends on its choice of direction.

In response, the T/D team with complete information of the full state $\mathbf{x}(t)$ and the path of agent A $\mathbf{x}_A(t) \to \mathbf{x}_A(t+\Delta t)$; it chooses headings for agents T and D to move in straight lines $\mathbf{x}_T(t) \to \mathbf{x}_T(t+\Delta t)$ and $\mathbf{x}_D(t) \to \mathbf{x}_D(t+\Delta t)$ of lengths $\alpha\Delta t$ and Δt respectively, where $\mathbf{x}_T(t+\Delta t)$ and $\mathbf{x}_D(t+\Delta t)$ depend on its choice of direction for agent T and D respectively.

The payoff received by the A team and T/D team is $V(\mathbf{x}(t+\Delta t))$, where for the A team the payoff is a cost to minimise, and for the T/D team the payoff is a reward to maximise.

Here

- $V(\mathbf{x}(t))$ is the value function defined in Equation (7),
- $I(\mathbf{x}(t))$ is the *predicted* interception point defined in Equation (8),
- χ is the midway plane defined in Equation (9),
- $\hat{\mathbf{A}}$, $\hat{\mathbf{D}}$ and $\hat{\mathbf{T}}$ are defined in (11),
- $\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t) \in \mathbb{R}^n$ for all $t \in \mathbb{R}$,
- $\overline{\mathbf{ab}}$ denotes the euclidean distance between two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,
- $0 \leq \alpha \leq 1$.

This sequential game is later referred to in the article as Game 3.1.

3.2 Nash equilibrium

The following lemmas are utilised to prove that regarding the strategies (10), its corresponding discrete-time turn-based duplicate constitutes a Nash equilibrium in Game 3.1, provided that the time increment Δt is sufficiently small.[2]

The following notation is used throughout this section. Let I_1 and χ_1 or just χ denote the interception point and the midway plane at time t respectively, and I_2 and χ_2 denote the interception point and the midway plane at time $t + \Delta t$ respectively.

In the analysis below, the γ strategy is defined as the following T/D team game plan; agent T moves away from the interception point $I(\mathbf{x}(t))$ and agent D mirrors the movement of agent A in such a manner as to ensure the midway plane χ remains unchanged.[3]

The first two lemmas establish the non-decreasing property of the value function.

Lemma 3.1: *For any A team strategy ζ, provided the response γ of the T/D team, $V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t))$.*

Proof: Proof by contradiction. There exists some A strategy η such that $V(\mathbf{x}(t+\Delta t)) < V(\mathbf{x}(t))$ provided the response γ of the T/D team. Expressing this statement mathematically

$$V(\mathbf{x}(t+\Delta t)) < V(\mathbf{x}(t))$$
$$\overline{\mathbf{x}_T(t+\Delta t)I_2} + \alpha\overline{\mathbf{x}_A(t+\Delta t)I_2}, \quad < \overline{\mathbf{x}_T(t)I_1} + \alpha\overline{\mathbf{x}_A(t)I_1},$$

where $I_1 = I(\mathbf{x}(t))$, $I_2 = I(\mathbf{x}(t+\Delta t))$; and due to T/D strategy γ, the plane χ remains unchanged, hence $I_1, I_2 \in \chi$.[4] Adding both sides of the above inequality by $\alpha\Delta t$, and using the fact that $\overline{\mathbf{x}_T(t+\Delta t)I_1} = \overline{\mathbf{x}_T(t)I_1} + \alpha\Delta t$:

$$\overline{\mathbf{x}_T(t+\Delta t)I_2} + \alpha\overline{\mathbf{x}_A(t+\Delta t)I_2} + \alpha\Delta t$$
$$< \overline{\mathbf{x}_T(t)I_1} + \alpha\overline{\mathbf{x}_A(t)I_1} + \alpha\Delta t$$
$$\overline{\mathbf{x}_T(t+\Delta t)I_2} + \alpha(\overline{\mathbf{x}_A(t+\Delta t)I_2} + \Delta t)$$
$$< \overline{\mathbf{x}_T(t+\Delta t)I_1} + \alpha\overline{\mathbf{x}_A(t)I_1},$$

and since A moves with speed 1 in some direction, $\overline{\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)} = \Delta t$, hence

$$\overline{\mathbf{x}_T(t+\Delta t)I_2} + \alpha(\overline{\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)} + \overline{\mathbf{x}_A(t+\Delta t)I_2})$$
$$< \overline{\mathbf{x}_T(t+\Delta t)I_1} + \alpha\overline{\mathbf{x}_A(t)I_1}. \tag{15}$$

Since the addition of $\alpha\Delta t$ only introduced a term that is constant, the point I_1 on the plane χ is also an optimal point which minimises $\overline{\mathbf{x}_T(t+\Delta t)I_1} + \alpha\overline{\mathbf{x}_A(t)I_1}$ under the constraint $I_1 \in \chi$.

This establishes that formula (15) is false, on the grounds that I_1 is an optimal solution that minimises α times the distance from $\mathbf{x}_A(t)$ to a point in χ plus the distance from that point in χ to $\mathbf{x}_T(t+\Delta t)$, but formula (15) proposes that it found a *strictly* better solution with the path $\mathbf{x}_A(t) \to \mathbf{x}_A(t+\Delta t) \to I_2 \to \mathbf{x}_T(t+\Delta t)$, where $I_2 \in \chi$.

Thus for any A team strategy, provided the response γ of the T/D team, $V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t))$. ■

Lemma 3.2: *For any A team strategy ζ, provided a best response of the T/D team, $V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t))$.*

Proof: The γ strategy may or may not be the best response of the T/D team to a strategy from A. Hence the value function $V(\mathbf{x}(t+\Delta t))$ given a best response of the T/D team is greater than or equal to the value function obtained given the T/D team strategy γ. Invoking Lemma 3.1, the later payoff is no less than $V(\mathbf{x}(t))$.

Consequently, for any A team strategy, provided a best response of the T/D team, $V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t))$. ■

The next set of results rely on the following bounds on the size of the time increment

$$\Delta t \leq \overline{\mathbf{x}_A(t)I(\mathbf{x}(t))} \tag{16}$$

$$\Delta t \leq \frac{1}{2}\left(\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}\right). \tag{17}$$

The immediate implication of bound (17) is specified below.

Lemma 3.3: *If* $\Delta t \leq \frac{1}{2}(\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)})$ *then* $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)} \leq \overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}$.

Proof: The distance $\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}$ is at a minimum if agent A moves towards agent T, hence the lower bound

$$\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)} \geq \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} - \Delta t \tag{18}$$

holds. Similarly, the distance $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)}$ is at a maximum if agent D moves away from agent T, thus

$$\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)} \leq \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} + \Delta t. \tag{19}$$

Manipulating equation (18),

$$\begin{aligned}
&\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} \geq -\Delta t \\
&-\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)} + \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} \leq \Delta t \\
&-\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)} + \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} \leq \frac{1}{2}\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} \\
&\quad - \frac{1}{2}\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} \quad \text{substituting (17)} \\
&-\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)} \leq -\frac{1}{2}\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} - \frac{1}{2}\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} \\
&\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)} \geq \frac{1}{2}(\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} + \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}).
\end{aligned} \tag{20}$$

Similarly, manipulating formula (19),

$$\begin{aligned}
&\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} \leq \Delta t \\
&\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} \leq \frac{1}{2}\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} \\
&\quad - \frac{1}{2}\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} \quad \text{applying (17)} \\
&\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)} \leq \frac{1}{2}(\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} + \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}).
\end{aligned} \tag{21}$$

Therefore combining formulas (20) and (21), it follows that $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)} \leq \overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}$. ■

The next two lemmas establish the non-increasing property of the value function.

Lemma 3.4: *If* $\Delta t \leq \overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}$ *and* $\Delta t \leq \frac{1}{2}(\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)})$, *then for any T/D team strategy* κ *in response to A moving towards the interception point,* $V(\mathbf{x}(t+\Delta t)) \leq V(\mathbf{x}(t))$.

Proof: Invoking Lemma 3.3, $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)} \leq \overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}$, in other words $\mathbf{x}_T(t)$ remains in the Defender's side of the midway plane at time $t+\Delta t$.

Moreover, since $\Delta t \leq \overline{\mathbf{x}_A(t)I_1}$ and A is moving in a straight line with speed 1 towards the point I_1 on χ_1, $\overline{I_1\mathbf{x}_D(t+\Delta t)} \geq \overline{I_1\mathbf{x}_A(t+\Delta t)}$, hence I_1 remains on the Attacker's side of the midway plane at time $t+\Delta t$.

It therefore follows that there exists some point on the line connecting $\mathbf{x}_T(t)$ to I_1 that is in the plane χ_2. Let Γ denote such a point.

Utilising the triangle inequality, it is true that

$$\begin{aligned}
&\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq \overline{\Gamma I_1} + \overline{\mathbf{x}_A(t+\Delta t)I_1} \\
&\alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq \alpha\overline{\Gamma I_1} + \alpha\overline{\mathbf{x}_A(t+\Delta t)I_1} \quad \text{since } \alpha \geq 0 \\
&\alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq \overline{\Gamma I_1} + \alpha\overline{\mathbf{x}_A(t+\Delta t)I_1} \quad \text{since } \alpha \leq 1.
\end{aligned}$$

Adding both sides of the above inequality by $\overline{\mathbf{x}_T(t)\Gamma} + \alpha\Delta t$:

$$\begin{aligned}
&\overline{\mathbf{x}_T(t)\Gamma} + \alpha\Delta t + \alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \\
&\quad \leq \overline{\mathbf{x}_T(t)\Gamma} + \overline{\Gamma I_1} + \alpha(\overline{\mathbf{x}_A(t+\Delta t)I_1} + \Delta t).
\end{aligned}$$

Since A moves in a straight line towards I_1 with speed 1 and (16) holds, $\overline{\mathbf{x}_A(t)I_1} = \overline{\mathbf{x}_A(t+\Delta t)I_1} + \Delta t$. Moreover, since $\overline{\mathbf{x}_T(t)\Gamma} + \overline{\Gamma I_1} = \overline{\mathbf{x}_T(t)I_1}$, applying to the inequality above:

$$\begin{aligned}
&\overline{\mathbf{x}_T(t)\Gamma} + \alpha\Delta t + \alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq \overline{\mathbf{x}_T(t)I_1} + \alpha\overline{\mathbf{x}_A(t)I_1} \\
&\overline{\mathbf{x}_T(t)\Gamma} + \alpha\Delta t + \alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq V(\mathbf{x}(t)).
\end{aligned} \tag{22}$$

Using the triangle inequality $\overline{\mathbf{x}_T(t+\Delta t)\Gamma} \leq \overline{\mathbf{x}_T(t)\Gamma} + \overline{\mathbf{x}_T(t)\mathbf{x}_T(t+\Delta t)}$, and since T moves with speed α, $\overline{\mathbf{x}_T(t)\mathbf{x}_T(t+\Delta t)} = \alpha\Delta t$, thus $\overline{\mathbf{x}_T(t+\Delta t)\Gamma} \leq \overline{\mathbf{x}_T(t)\Gamma} + \alpha\Delta t$. It follows that the left-hand side of the inequality (22) can be bounded below:

$$\begin{aligned}
&\overline{\mathbf{x}_T(t+\Delta t)\Gamma} + \alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \\
&\quad \leq \overline{\mathbf{x}_T(t)\Gamma} + \alpha\Delta t + \alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma}.
\end{aligned}$$

As a result, the following lower bound on $V(\mathbf{x}(t))$ holds:

$$\overline{\mathbf{x}_T(t+\Delta t)\Gamma} + \alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq V(\mathbf{x}(t)).$$

Hence it is sufficient to verify that

$$V(\mathbf{x}(t+\Delta t)) \leq \overline{\mathbf{x}_T(t+\Delta t)\Gamma} + \alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma}. \tag{23}$$

To confirm this, recall the definition of $V(\mathbf{x}(t+\Delta t))$

$$V(\mathbf{x}(t+\Delta t)) = \overline{\mathbf{x}_T(t+\Delta t)I_2} + \alpha\overline{\mathbf{x}_A(t+\Delta t)I_2}.$$

Γ may or may not be the optimal point in χ_2 that minimises $\overline{\mathbf{x}_T(t+\Delta t)I_2} + \alpha\overline{\mathbf{x}_A(t+\Delta t)I_2}$, hence

$$V(\mathbf{x}(t+\Delta t)) \leq \overline{\mathbf{x}_T(t+\Delta t)\Gamma} + \alpha\overline{\mathbf{x}_A(t+\Delta t)\Gamma},$$

which completes the proof. ■

Lemma 3.5: *If* $\Delta t \leq \overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}$ *and* $\Delta t \leq \frac{1}{2}(\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)})$, *then for any T/D team strategy* κ *in response to an optimal A strategy,* $V(\mathbf{x}(t+\Delta t)) \leq V(\mathbf{x}(t))$.

Proof: Follows from Lemma 3.4, since the upper bound $V(\mathbf{x}(t+\Delta t)) \leq V(\mathbf{x}(t))$ on $V(\mathbf{x}(t+\Delta t))$ holds given a potentially sub-optimal strategy from A, the upper bound also holds for an optimal A strategy. ■

The final lemma confirms that the value function remains unchanged in any Nash equilibrium.

Lemma 3.6: *If* $\Delta t \leq \overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}$ *and* $\Delta t \leq \frac{1}{2}(\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)})$, *then in any Nash equilibrium of Game 3.1,* $V(\mathbf{x}(t+\Delta t)) = V(\mathbf{x}(t))$.

Proof: As a result of Lemmas 3.2 and 3.5; given an optimal A strategy, the value function is bounded above $V(\mathbf{x}(t+\Delta t)) \leq V(\mathbf{x}(t))$, and given a best response from the T/D team, the value function is bounded below $V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t))$, hence Lemma 3.6 holds. ■

The main result of the paper is given below.

Theorem 3.7: *In the sequential game 3.1, at any starting state* $\mathbf{x}(t)$ *and time t satisfying* $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} < \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}$, *if*

$$\Delta t \leq \overline{\mathbf{x}_A(t)I(\mathbf{x}(t))} \tag{24}$$

and

$$\Delta t \leq \frac{1}{2}\left(\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}\right), \tag{25}$$

then the trajectory

$$\mathbf{x}_A(t+\Delta t) = \mathbf{x}_A(t) + \Delta t\hat{\mathbf{A}} \tag{26a}$$

$$\mathbf{x}_D(t+\Delta t) = \mathbf{x}_D(t) + \Delta t\hat{\mathbf{D}} \tag{26b}$$

$$\mathbf{x}_T(t+\Delta t) = \mathbf{x}_T(t) - \alpha\Delta t\hat{\mathbf{T}} \tag{26c}$$

is a Nash equilibrium with payoff

$$V(\mathbf{x}(t+\Delta t)) = V(\mathbf{x}(t)). \tag{27}$$

Proof: Since the initial state satisfies $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} < \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}$, $\hat{\mathbf{A}}$, $\hat{\mathbf{D}}$ and $\hat{\mathbf{T}}$ are well defined.

Due to the lower bound given in Lemma 3.2, the best payoff the A team can possibly achieve provided a best response from the T/D team is $V(\mathbf{x}(t))$. In Lemma 3.4, it was established that the A strategy to move towards the interception point (26a) guarantees no more than that number regardless of the response of the T/D team, hence that is an optimal A strategy, and provided a best response of the T/D team, $V(\mathbf{x}(t+\Delta t)) = V(\mathbf{x}(t))$.

Furthermore in Lemma 3.1, it was demonstrated that the response γ from the T/D team achieves $V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t))$ regardless of the strategy from A, hence is a best response of the T/D team to A moving toward the interception point. The T/D team response (26b)–(26c) is the γ strategy in this case. Therefore (26) is a Nash equilibrium trajectory.

Moreover, Lemma 3.6 confirmed that the value function remains unchanged in any Nash equilibrium, hence Theorem 3.7 holds. ■

3.3 Recursive application of Theorem 3.7 in game 2.1

This subsection makes evident the practicality of the results in the previous section with regard to the original formulation 2.1 and provides a preliminary outline of a proof of Conjecture 2.1 in arbitrary dimensions.

Recall that Theorem 3.7 established that at every state $\mathbf{x}(t)$ and time t satisfying $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} < \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}$, the Nash equilibrium (26) defines optimal strategies of the A team and T/D team to minimise and maximise the value function respectively; moreover, the value function remains unchanged under this optimal trajectory.

For Δt sufficiently small, formula (24) holds, consequently the Nash equilibrium strategies (26) preserves the condition $\overline{\mathbf{x}_T(t+\Delta t)\mathbf{x}_D(t+\Delta t)} < \overline{\mathbf{x}_T(t+\Delta t)\mathbf{x}_A(t+\Delta t)}$ in the next time step, unless the game terminated at time $t+\Delta t$.

Therefore Theorem 3.7 can be applied recursively from any initial state $\mathbf{x}_0$ and time t_0 satisfying $\overline{\mathbf{x}_T(t_0)\mathbf{x}_D(t_0)} < \overline{\mathbf{x}_T(t_0)\mathbf{x}_A(t_0)}$ to terminal state $\mathbf{x}_f$ and time t_f, where the time increment Δt at each time step must be sufficiently small to adhere to (24) and (25).

Since constraints (24) and (25) provide no limits to the infinitesimalness of Δt; as Δt approaches zero, the foresight the T/D team possess vanishes, and hence the Hamilton–Jacobi–Bellman equations (12), (13) and (14) hold from time t_0 to t_f.

Despite the fact that the basic argument above falls short of the standard of proof needed to establish Conjecture 2.1, it is possible that future research would prove its sufficiency using arguments along those lines.

4. 3D simulation of Nash equilibrium strategies

In the final section, the Nash equilibrium strategies are implemented in 3D on MATLAB 2019a and are compared against the baseline guidance algorithm Proportional Navigation, or PN for short. All the simulations described below use the time increment $\Delta t = 0.1$, and the game is terminated at the first time condition (24) or condition (25) is violated. The results of the simulations are displayed in Table 1. The MATLAB codes used in this section are available at: https://www.dropbox.com/s/djjnmco621w0uqa/TAD.zip?dl=0.

4.1 Battle scenario

In the simulated battle scenario, the starting positions of the three agents are:

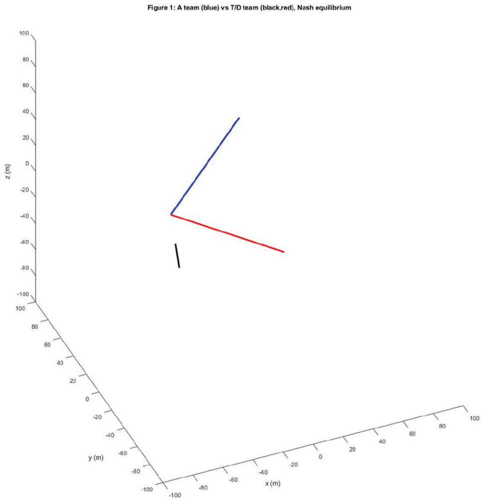
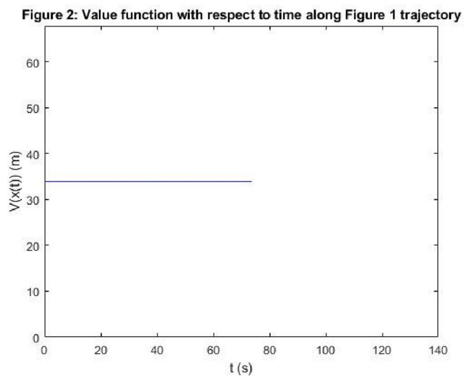
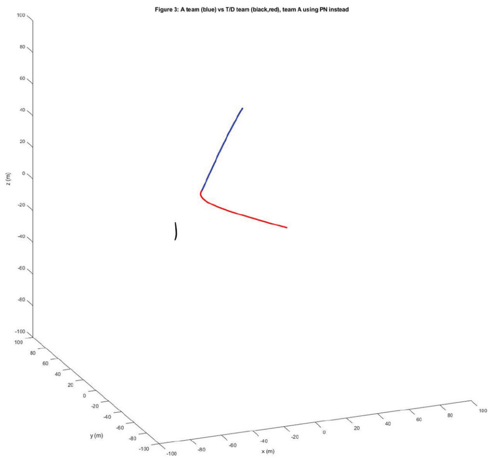
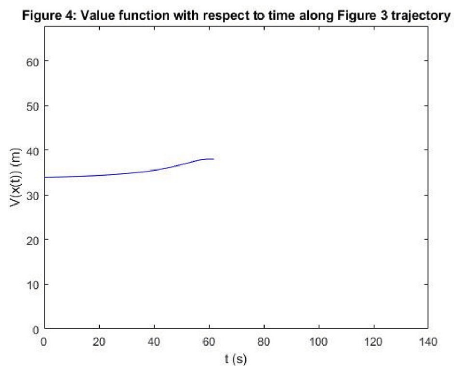
$$(\mathbf{x}_A(0), \mathbf{x}_D(0), \mathbf{x}_T(0)) = \left(\begin{bmatrix}10\\40\\50\end{bmatrix}, \begin{bmatrix}8\\-35\\0\end{bmatrix}, \begin{bmatrix}-50\\-2\\0\end{bmatrix}\right) \tag{28}$$

and the speed of the Target is set to $\alpha = 0.2$ m/s.

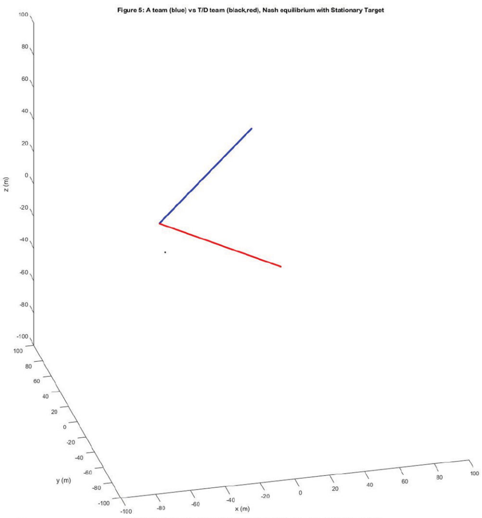
4.1.1 Nash equilibrium trajectory

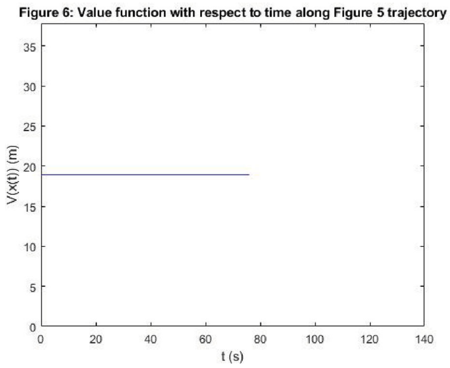
Figure 1 in Table 1 plots the trajectories of the Attacker (blue), Defender (red) and Target (black) from starting time to termination time, when all three agents play their Nash equilibrium strategies (26), and Figure 2 plots the value function $V(\mathbf{x}(t))$

Table 1. Trajectory and value function plots of Game 2.1 with T/D team using Nash equilibrium strategies versus different A team strategies and different Target speeds.

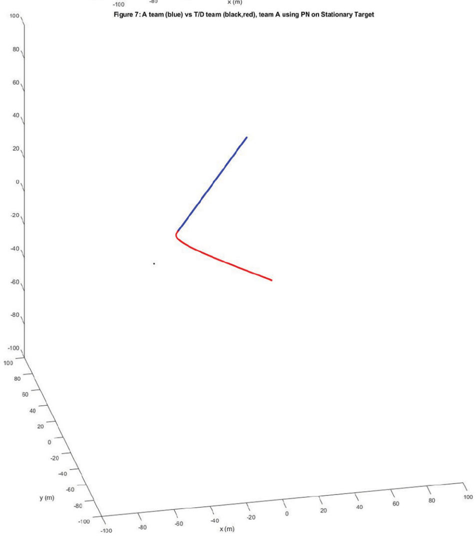
	Trajectory plot	Value function plot
Target speed $\alpha = 0.2$ and Attacker using Nash equilibrium strategy.		
Target speed $\alpha = 0.2$ and Attacker using Proportional Navigation.		

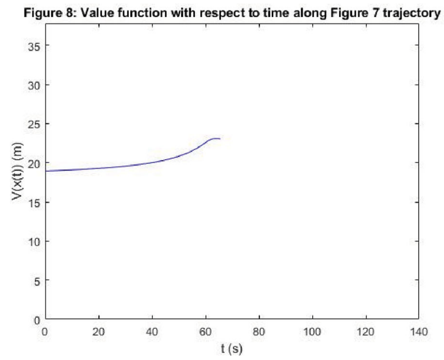
Target speed $\alpha = 0$ and Attacker using Nash equilibrium strategy.





Target speed $\alpha = 0$ and Attacker using Proportional Navigation.





across time along that trajectory. The results show that the distance between the Target and Attacker at termination time is 33.91 m.

4.1.2 Attacker using PN

In the next simulation, we examine the performance of the T/D team strategy (26b)–(26c) against an Attacker using the most common guidance law, Proportional Navigation.[5] Figure 3 plots the trajectories of the Attacker (blue), Defender (red) and Target (black) from starting time to termination time in this scenario, and Figure 4 plots the value function $V(\mathbf{x}(t))$ across time along that trajectory. In this case, the distance between the Target and Attacker at termination time is 37.96 m, which is larger than the previous terminal distance.

4.1.3 Stationary target

The above two simulations are repeated in the case of a stationary Target, that is $\alpha = 0$; the results are displayed in Table 1. Just as before, we observe a rise in the terminal distance between the Target and Attacker from 18.92 to 22.97 metres as the Attacker switches from their Nash equilibrium strategy to PN guidance. This demonstrates that the T/D team defence strategy given by the Nash equilibrium (26) is robust against missiles using PN guidance laws, and from the point of view of the Attacker, these simulations also provide evidence to validate the advantages of the guidance law given by (26a) in comparison to PN navigation in the TAD game of degree.

5. Conclusion

A new geometric method of verifying the Hamilton–Jacobi–Bellman equations was developed. This approach at first formulates a discrete-time turn-based variant of the differential game, then proves that the conjectured optimal strategies constitute a Nash equilibrium at each time step; finally in view of the fact that no limits were placed on the infinitesimalness of the time increment, those results are applied recursively in the original continuous-time formulation from starting time to termination time to verify the HJB equations.

This new approach was utilised in the TAD game of degree to generalise the results of Garcia et al. (2019) from 2D to arbitrary dimensions. In Section 4, the T/D team Nash equilibrium strategies were simulated against threats using various guidance laws, the results verified its optimality and robustness against missiles using PN guidance, authenticating its applicability in defence applications.

Future research directions could involve generalising the TAD game of degree scenario to include obstacles, and applying the new approach to other pursuit-evader games such as the fast-pursuer multiple-evader scenario found in Liu, Zhou, Tomlin, and Hedrick (2013).

Notes

1. In the degenerate case where $\mathbf{x}_A(t) = \mathbf{x}_D(t)$, the following definition applies $\chi = \{\mathbf{x}_A(t)\}$.
2. The initial condition $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} < \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}$ is an implicit assumption used throughout Section 3.2.
3. In the degenerate case where $\mathbf{x}_A(t+\Delta t) \in \chi_1$, the following definition of the γ strategy applies $\mathbf{x}_D(t+\Delta t) = \mathbf{x}_A(t+\Delta t)$, hence $\chi_2 = \{\mathbf{x}_A(t+\Delta t)\}$.
4. In the degenerate case where $\mathbf{x}_A(t+\Delta t) \in \chi_1$, $\chi_2 = \{\mathbf{x}_A(t+\Delta t)\}$ hence $I_2 = \mathbf{x}_A(t+\Delta t) \in \chi_1$, therefore the statement $I_1, I_2 \in \chi$ still holds.
5. In the implementation of PN, the initial velocity vector was set to point directly towards the Target.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Chapter 4

A state-feedback Nash equilibrium for the general Target-Attacker-Defender differential game of degree in arbitrary dimensions

4.1 Contextual statement

Our next publication is the first manuscript to rigorously examine the differential game of active target defence in the fast defender case $V_A < V_D$. The complexity herein lies in establishing that a fast defender can contain the attacker within the confines of the AD-based Apollonius circle. The perimeter of the Attacker-Defender based Apollonius circles defines the set of all points in space in which the Attacker and Defender are equally separated by time-to-reach. This paper identifies and proves the state-feedback optimal strategies for any $V_A < V_D$.

Statement of Authorship

Title of Paper	A state-feedback Nash equilibrium for the general Target–Attacker–Defender differential game of degree in arbitrary dimensions.
Publication Status	☑ Published ☐ Accepted for Publication ☐ Submitted for Publication ☐ Unpublished and Unsubmitted work written in manuscript style
Publication Details	Mammadov, K., Lim, C., & Shi, P. (2022). A state-feedback Nash equilibrium for the general Target–Attacker–Defender differential game of degree in arbitrary dimensions. International Journal of Control, 95(1), 93-103. doi:10.1080/00207179.2020.1779958.

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Contribution to the Paper	Selected research topic, conducted research, wrote manuscript, and acted as corresponding author.		
Overall percentage (%)	95%		
Certification:	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.		
Signature		Date	8/04/2022

Co-Author Contributions

By signing the Statement of Authorship, each author certifies that:

i. the candidate's stated contribution to the publication is accurate (as detailed above);

ii. permission is granted for the candidate in include the publication in the thesis; and

iii. the sum of all co-author contributions is equal to 100% less the candidate's stated contribution.

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Contribution to the Paper	Helped examine the manuscript.		
Signature		Date	8/04/2022

INTERNATIONAL JOURNAL OF CONTROL
2022, VOL. 95, NO. 1, 93–103
https://doi.org/10.1080/00207179.2020.1779958

A state-feedback Nash equilibrium for the general Target–Attacker–Defender differential game of degree in arbitrary dimensions

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ABSTRACT
In this manuscript we formulate the general Target–Attacker–Defender differential game of degree in both its continuous-time and discrete-time turn-based variants in n-dimensional euclidean space. In this three-agent engagement, the Attackers goal is to get as close as possible to the Target before collision with the Defender, whilst the Target and Defender coordinate to achieve the opposite. The most general setting for this zero-sum differential game is considered, where the agents move at speeds not necessarily equal, and the Nash equilibrium strategies are proven in the discrete-time turn-based variant.

ARTICLE HISTORY
Received 25 November 2019
Accepted 3 June 2020

KEYWORDS
Differential game theory; dynamic game theory; optimal state-feedback strategies; state-feedback Nash equilibrium; pursuit-evasion games

1. Introduction

In aerospace guidance and control, multi-agent pursuit evader differential games are often used to model maritime and aerospace engagement scenarios for defence applications. The focus of this manuscript is on one particular class of pursuit evader games, namely the three-agent Target–Attacker–Defender (TAD) differential game. This class of pursuit evader games were motivated by Li and Cruz (2011), who considered an autonomous vehicle tasked with defending a valuable asset against an attacker; and by Rusnak (2005) where a bodyguard's mission is to protect a potential victim against a bandit; and by Boyell (1976) in which a naval ship on the move encounters an attacking submarine, and must launch a torpedo in its defence.

More precisely, the Target–Attacker–Defender differential game is described as follows. There are three agents, named the Attacker or agent A, the Defender or agent D, and the Target or agent T; and two teams, team A and team T/D; where team A controls the movement of agent A and team T/D coordinates the movement of agents T and D. The primary goal of team A is to capture the Target whilst evading collision with the Defender, and the primary goal of team T/D is to achieve the opposite goal, namely for the Defender to intercept the Attacker before it reaches agent T.

Using the winning time as the cost functional, this formulates the TAD differential game of kind. Works on this problem include Bhattacharya et al. (2016) and Zha et al. (2017). The most comprehensive analysis of this game was made by Liang et al. (2019), in which the whole state space was separated into the winning regions, namely into regions in which team A wins (capture the target), and where team T/D wins (prevents its capture). Moreover in each region optimal strategies were derived in which the winning team minimised the termination time and the losing team worked to maximise it.

In cases where the Target–Attacker–Defender differential game is motivated from an aerospace engagement scenario for a defence application, normally the Attacker models a guided bomb being used in an attempt to destroy agent T. Clearly in this case it is not necessary for agent A to capture agent T, but only to get close enough to be within its blast radius. This engagement scenario is normally modelled as a zero-sum differential game in which the objective of team A is to minimise the distance between the Target and agent A at its time of collision, and team T/D coordinate their movements in such a manner as to achieve the opposite goal. This is named the TAD differential game of degree and the focus of the current manuscript.

Other variations of the engagement scenario are considered in Prokopov and Shima (2013), Shima (2011) and Shaferman and Shima (2010). Here it is assumed the Target and Defender start at the same position (since it models a target aircraft launching a missile in its defence), and the Attacker employs a known linear one-on-one guidance law to capture agent. Shima (2011) also considered the scenario in which both the Attacker and Defender's strategies were constrained to the most well-know guidance laws PN, APN and OGL. Under those assumptions on agent A's strategy and agent D's strategy, agent T solves a one-sided optimal control problem.

The present paper is specifically focused on the formulation of the TAD differential game of degree presented in Garcia et al. (2019). Here each agent moves in 2-dimensional euclidean space (has an x and y position), and it is assumed that the speeds of agents A and D are equal, and the speed of the Target is no greater than the speed of agent A. In this work a state-feedback Nash equilibrium was derived and proved by demonstrating that the strategy profile defined by the Nash equilibrium satisfies the Hamilton-Jacobi-Bellman equations. In the work of Mammadov et al. (2020), the aforementioned results were

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extended to the more general setting where each agent moves in n-dimensional space.

In the present manuscript, we consider the most general case of the Target–Attacker–Defender differential game, in which the speeds of the three agents are not equal. More specifically where the speed of the Target is less than the speed of the Attacker which in turn is less than the speed of the Defender. Note that the cases in which the order of this inequality is switched, it leads to degeneracies were only one team wins, hence this is the most general non-degenerate formulation.

This general setting was first considered in Garcia et al. (2017) and in an earlier conference version, in $\mathbb{R}^2$. Here a two-person extension of Pontryagin's maximum principle was used to prove that provided the co-state variables are non-zero, under optimal play the trajectories of all the agents are straight lines. Then, under the constraint that every agent must move in a straight line for all time (cannot move in curved paths), the optimal headings of every agent was proved. This constraint however is limiting since there are examples of state-feedback Nash equilibria, such as (14), in which the optimal state-feedback strategies are not necessarily held constant for all time, but rather only held constant when all teams play their optimal strategies (thus satisfying the implication of the maximum principle). In the present manuscript, the aforementioned constraint is dispensed with and the results hold in $\mathbb{R}^n$. The main contribution of this paper is to establish a state-feedback Nash equilibrium of the TAD differential game of the degree in this most general setting in arbitrary dimensions.

The remainder of the article is organised as follows. In Section 1.1, the notation used throughout the manuscript is introduced, and an algebraic inequality used in the subsequent sections is presented. In Section 2 the mathematical formulation of the TAD differential game of degree is given, and a Nash equilibrium of the game is proposed. In Section 3, the TAD differential game of degree is recast in a discrete-time turn based variant with arbitrarily small time increment. In this formulation it is proven that the corresponding discretised optimal strategy profile constitutes a Nash equilibrium, thereby giving strong theoretical support to the claimed Nash equilibrium for the original continuous-time formulation.

1.1 Preliminaries

The notation used throughout the manuscript is listed as follows. Given any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

- $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ denotes the set of all positive real numbers,
- $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ denotes the set of all non-negative real numbers,
- $\mathbf{u} \cdot \mathbf{v}$ denotes the dot product,
- $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$,
- $\overrightarrow{\mathbf{uv}} = \mathbf{v} - \mathbf{u}$,
- $\overline{\mathbf{uv}} = ||\overrightarrow{\mathbf{uv}}||$ denotes the euclidean distance between $\mathbf{u}$ and $\mathbf{v}$,
- $\angle\mathbf{uvw}$ denotes the angle between vectors $\overrightarrow{\mathbf{vu}}$ and $\overrightarrow{\mathbf{vw}}$,
- $\dot{\mathbf{u}}(t)$ denotes the time derivative $\frac{d}{dt}\mathbf{u}(t)$.

Next we provide an inequality which is used in the subsequent sections.

Proposition 1: *For all $\theta \in \mathbb{R}, \phi \in \mathbb{R}, \delta \in \mathbb{R}_0^+$*

$$\sqrt{\cos^2\theta + \delta\sin^2\theta}\sqrt{\cos^2\phi + \delta\sin^2\phi} \geq |\cos\theta||\cos\phi| + \delta|\sin\theta||\sin\phi|. \tag{1}$$

Proof: For any θ and ϕ in $\mathbb{R}$,

$$(|\cos\theta||\sin\phi| - |\sin\theta||\cos\phi|)^2 \geq 0,$$

expanding the left-hand side:

$$\cos^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi - 2|\sin\theta||\cos\theta||\sin\phi||\cos\phi| \geq 0,$$

which remains true if it is multiplied by any $\delta \in \mathbb{R}_0^+$

$$\delta\cos^2\theta\sin^2\phi + \delta\sin^2\theta\cos^2\phi \geq 2\delta|\sin\theta||\cos\theta||\sin\phi||\cos\phi|.$$

Adding both sides by $\cos^2\theta\cos^2\phi + \delta^2\sin^2\theta\sin^2\phi$,

$$\begin{aligned}\cos^2\theta\cos^2\phi &+ \delta\cos^2\theta\sin^2\phi + \delta\sin^2\theta\cos^2\phi + \delta^2\sin^2\theta\sin^2\phi \\ &\geq \cos^2\theta\cos^2\phi + 2\delta|\sin\theta||\cos\theta||\sin\phi||\cos\phi| + \delta^2\sin^2\theta\sin^2\phi,\end{aligned}$$

factorising both sides we obtain:

$$(\cos^2\theta + \delta\sin^2\theta)(\cos^2\phi + \delta\sin^2\phi) \geq (|\cos\theta||\cos\phi| + \delta|\sin\theta||\sin\phi|)^2.$$

Since both sides of the above inequality are non-negative, and the square root function is monotonically increasing in the interval $[0, \infty)$, taking the square root maintains the order of the inequality, thus

$$\sqrt{\cos^2\theta + \delta\sin^2\theta}\sqrt{\cos^2\phi + \delta\sin^2\phi} \geq |\cos\theta||\cos\phi| + \delta|\sin\theta||\sin\phi|. \qquad \blacksquare$$

2. TAD game of degree formulation and preliminary analysis

2.1 Problem definition of TAD game of degree

The TAD game of degree is a two player (two team) zero-sum differential game. The complete state of the differential game is specified by $\mathbf{x}(t) = (\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t))$ where $\mathbf{x}_A(t) = (x_{A_1}(t), \ldots, x_{A_n}(t)) \in \mathbb{R}^n$, $\mathbf{x}_D(t) = (x_{D_1}(t), \ldots, x_{D_n}(t)) \in \mathbb{R}^n$ and $\mathbf{x}_T(t) = (x_{T_1}(t), \ldots, x_{T_n}(t)) \in \mathbb{R}^n$ denotes the position of the Attacker, Defender and Target respectively. Regarding the information structure of the game, both teams have access to the current state of the system $\mathbf{x}(t)$ at time t. Using that information, team A must choose an instantaneous heading for agent A, and team T/D for the headings of agents D and T. The controls of agents A, D and T are given by

$$\mathbf{u}_A(t), \mathbf{u}_D(t), \mathbf{u}_T(t) \in \text{unit } (n-1)\text{−sphere}, \tag{2}$$

where a unit $(n-1)$-sphere is the set of all points in $\mathbb{R}^n$ that are a unit distance from the origin. The dynamics $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}_A(t), \mathbf{u}_D(t), \mathbf{u}_T(t))$ from time t_0 to t_f is given by

$$(\dot{\mathbf{x}}_A(t), \dot{\mathbf{x}}_D(t), \dot{\mathbf{x}}_T(t)) = (V_A\mathbf{u}_A(t), V_D\mathbf{u}_D(t), V_T\mathbf{u}_T(t)),$$
$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{3}$$

where $V_A, V_D, V_T \in \mathbb{R}_0^+$ denotes the speed of the Attacker, Defender and Target respectively. The termination time t_f is defined endogenously as the first time t_f satisfying at least one of the termination conditions (4) or (5).

$$\mathbf{x}_A(t_f) = \mathbf{x}_D(t_f), \tag{4}$$
$$\mathbf{x}_A(t_f) = \mathbf{x}_T(t_f). \tag{5}$$

Over the time horizon $[t_0, t_f]$, each team receives the following payoff

$$J(\mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot), \mathbf{u}_T(\cdot), \mathbf{x}_0) = \Phi(\mathbf{x}_f) \tag{6}$$

where

$$\Phi(\mathbf{x}_f) = \overline{\mathbf{x}_T(t_f)\mathbf{x}_A(t_f)}. \tag{7}$$

The objective of team A is to minimise $J(\mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot), \mathbf{u}_T(\cdot), \mathbf{x}_0)$, whereas the objective of team T/D is to maximise it. We also make the following three assumptions. The speed of the agents satisfies the inequality

$$V_T < V_A < V_D \tag{8}$$

and the initial state $\mathbf{x}_0$ satisfies

$$\frac{\overline{\mathbf{x}_T(t_0)\mathbf{x}_A(t_0)}}{V_A} > \frac{\overline{\mathbf{x}_T(t_0)\mathbf{x}_D(t_0)}}{V_D}, \tag{9}$$

and

$$\mathbf{x}_A(t_0) \neq \mathbf{x}_D(t_0). \tag{10}$$

This differential game is later referred to in the article as Game 2.1. Due to the constraint (9) on the initial state of the game, and the relative speeds (8), under optimal play agent A will always fail to capture agent T, hence the termination condition (5) is discarded in the proceeding analysis.

2.2 Optimal state-feedback strategies

In the present subsection, the optimal state-feedback strategies are displayed, but first some preliminary notation is introduced.

The value functional $V : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [t_0, t_f]$, which should indicate the payoff-to-go at every state, is defined below

$$V(\mathbf{x}(t)) = \overline{\mathbf{x}_T(t)I(\mathbf{x}(t))} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}, \tag{11}$$

where the *predicted* interception point $I(\mathbf{x}(t))$ is the unique solution to

$$I(\mathbf{x}(t)) = \arg\min_{\mathbf{I}} \overline{\mathbf{x}_T(t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}} \tag{12a}$$

$$\text{s.t.} \quad \mathbf{I} \in \chi(\mathbf{x}(t)) \tag{12b}$$

and where the $(n-1)$-dimensional boundary $\chi(\mathbf{x}(t))$ is the containment boundary defined by

$$\chi(\mathbf{x}(t)) = \left\{ p \in \mathbb{R}^n \mid \frac{\overline{p\mathbf{x}_A(t)}}{V_A} = \frac{\overline{p\mathbf{x}_D(t)}}{V_D} \right\}. \tag{13}$$

The containment boundary is an Apollonius circle, that is a set of all points whose distances from two fixed points are in a constant ratio. The containment boundary given by (13) at time t is the Appollonius circle in which the two fixed points are the current positions of agents A and D, and the distance to their respective positions are in the constant ratio $V_A : V_D$.

Note that in several instances in this manuscript, we denote χ as shorthand for $\chi(\mathbf{x}(t))$ when it is clear we are referring to the containment boundary at the current state.

The optimal state-feedback strategies for both teams are given by Proposition 2.

Proposition 2: *A state-feedback Nash equilibrium of Game 2.1 is given by*

$$\mathbf{u}_A^*(t) = \frac{I(\mathbf{x}(t)) - \mathbf{x}_A(t)}{\|I(\mathbf{x}(t)) - \mathbf{x}_A(t)\|}, \tag{14a}$$
$$\mathbf{u}_D^*(t) = \frac{I(\mathbf{x}(t)) - \mathbf{x}_D(t)}{\|I(\mathbf{x}(t)) - \mathbf{x}_D(t)\|}, \tag{14b}$$
$$\mathbf{u}_T^*(t) = -\left(\frac{I(\mathbf{x}(t)) - \mathbf{x}_T(t)}{\|I(\mathbf{x}(t)) - \mathbf{x}_T(t)\|}\right). \tag{14c}$$

To prove Proposition 2, we would have to verify that the strategy profile given by Equation (14) satisfies the Hamilton-Jacobi-Bellman equations provided below from time t_0 to t_f.

$$\mathbf{u}_A^*(t) = \arg\min_{\mathbf{u}_A(t)} \frac{\partial V}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}_A(t), \mathbf{u}_D^*(t), \mathbf{u}_T^*(t)) \tag{15a}$$

$$\text{s.t.} \quad \mathbf{u}_A(t) \in \text{unit } (n-1)\text{−sphere} \tag{15b}$$

$$(\mathbf{u}_D^*(t), \mathbf{u}_T^*(t)) = \arg\max_{(\mathbf{u}_D(t), \mathbf{u}_T(t))} \frac{\partial V}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}_A^*(t), \mathbf{u}_D(t), \mathbf{u}_T(t)) \tag{16a}$$

$$\text{s.t.} \quad \mathbf{u}_D(t), \mathbf{u}_T(t) \in \text{unit } (n-1)\text{−sphere} \tag{16b}$$

$$\frac{\partial V}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}_A^*(t), \mathbf{u}_D^*(t), \mathbf{u}_T^*(t)) = 0. \tag{17}$$

The approach taken in the works of Garcia et al. (2019) used a similar expression to (11) to analytically calculate the gradient of the value function, and used that result to verify Equations (15)–(17). However that approach, even in the simpler case where $V_A = V_D$ led to excruciatingly long calculations, since the Jacobian of the interception point had to be computed, which measures the change in $I(\mathbf{x}(t))$ with respect to the change in the state $\mathbf{x}(t)$.

The approach taken in this manuscript is to instead reformulate Game 2.1 as a discrete-time turn-based game. The advantage of a discrete-time formulation is that in the time interval t to $t + \Delta t$, the paths of all the agents are constrained to be straight lines; and the advantage of a turn-based formulation, specifically where team A moves first followed by team T/D, is that agent D can ensure that agent A remains within the confines of the containment boundary χ, regardless of team A's strategy.

In the next section we define the Run and Contain strategy for team T/D and the Interception Attack strategy for team A; these are the discrete-time turn-based equivalents of the strategies defined in (14), and show that they constitute a Nash equilibrium at every discrete-time interval t to $t + \Delta t$. Since the time increment Δt can be set arbitrarily small, this provides strong theoretical support for the optimality of (14) in the continuous-time formulation.

3. Discrete-time turn-based variant

3.1 Problem formulation of sequential game

Consider the following sequential game. The game starts at any time t from any initial state $\mathbf{x}(t) = (\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t))$.

Team A with complete information of the full state $\mathbf{x}(t)$ moves first, it chooses a heading for agent A to move in a straight line from $\mathbf{x}_A(t)$ to $\mathbf{x}_A(t + \Delta t)$ of length $V_A \Delta t$.

In response, team T/D with complete information of the full state $\mathbf{x}(t)$ and path of agent A $\mathbf{x}_A(t) \to \mathbf{x}_A(t + \Delta t)$; it chooses headings for agents T and D to move in straight lines of lengths $V_T \Delta t$ and $V_D \Delta t$ respectively.

The payoff received by the A team and T/D team is $V(\mathbf{x}(t + \Delta t))$, where for the A team the payoff is a cost to minimise, and for the T/D team the payoff is a reward to maximise.

Below we list the notation used:

- $V(\mathbf{x}(t))$ is the value function defined in (11),
- $I(\mathbf{x}(t))$ is the *predicted* interception point defined in (12),
- χ is the containment boundary defined in (13),
- $\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t) \in \mathbb{R}^n$ denotes the position of the three agents for all $t \in \mathbb{R}$,
- $V_A, V_D, V_T \in \mathbb{R}_0^+$ denotes the speed of the Attacker, Defender and Target respectively,
- $\Delta t \in \mathbb{R}^+$ is the time increment of the game.

In addition, we make the following three assumptions in this game

$$V_T < V_A < V_D, \tag{18}$$

$$\Delta t \le \frac{1}{V_A + V_D} \overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}, \tag{19}$$

$$\Delta t \le \frac{1}{2} \left(\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{V_A} - \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}}{V_D} \right), \tag{20}$$

as well as the following two assumptions on the state $\mathbf{x}(t)$

$$\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}}{V_D} < \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{V_A}, \tag{21}$$

$$\mathbf{x}_A(t) \ne \mathbf{x}_D(t). \tag{22}$$

This sequential game is later referred to in the article as Game 3.1.

3.2 Nash equilibrium

Below we define the Run and Contain strategy for the T/D team and the Interception Attack strategy for team A. It is proved that these strategies constitute a Nash equilibrium in Game 3.1.

Definition 1 (Containment Strategy): The Containment strategy for agent D; a function of the current state $\mathbf{x}(t)$ and path of agent A $\mathbf{x}_A(t) \to \mathbf{x}_A(t + \Delta t)$ is defined as follows. Whichever direction agent A moves in, extend the line $\mathbf{x}_A(t) \to \mathbf{x}_A(t + \Delta t)$ until it intercepts a point on the containment boundary χ, agent D moves in a straight line towards that point on the containment boundary.

Note that the Containment strategy is well defined since (19) holds, hence agent A remains inside the boundary χ at time $t + \Delta t$.

Definition 2 (Run and Contain Strategy): The Run and Contain strategy for the T/D team is defined as follows. Agent D plays the Containment strategy defined in Definition 1 and agent T moves in a straight line away from the predicted interception point $I(\mathbf{x}(t))$.

Definition 3 (Interception Attack Strategy): The Interception Attack strategy for team A is defined as follows. Agent A moves in a straight line towards the predicted interception point $I(\mathbf{x}(t))$.

Figures 1 and 2 display the Run and Contain strategy for team T/D and the Interception Attack strategy for team A respectively. The first result proves that the Containment strategy for agent D does indeed contain agent A within the confines of the boundary χ; that is, the time it takes for agent D to reach any point on the boundary χ is less than or equal to the time it takes for agent A to reach that point.

Theorem 3.1: *In Game 3.1, if agent D plays the Containment strategy, then for any point* $\mathbf{P} \in \chi$, $\frac{\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}}{V_D} \le \frac{\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}}{V_A}$.

Proof: Let the function $\mathrm{proj}(\mathbf{v}) = \frac{\mathbf{v} \cdot \overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)} \cdot \overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}} \overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}$ denote the vector projection of any vector $\mathbf{v} \in \mathbb{R}^n$ onto

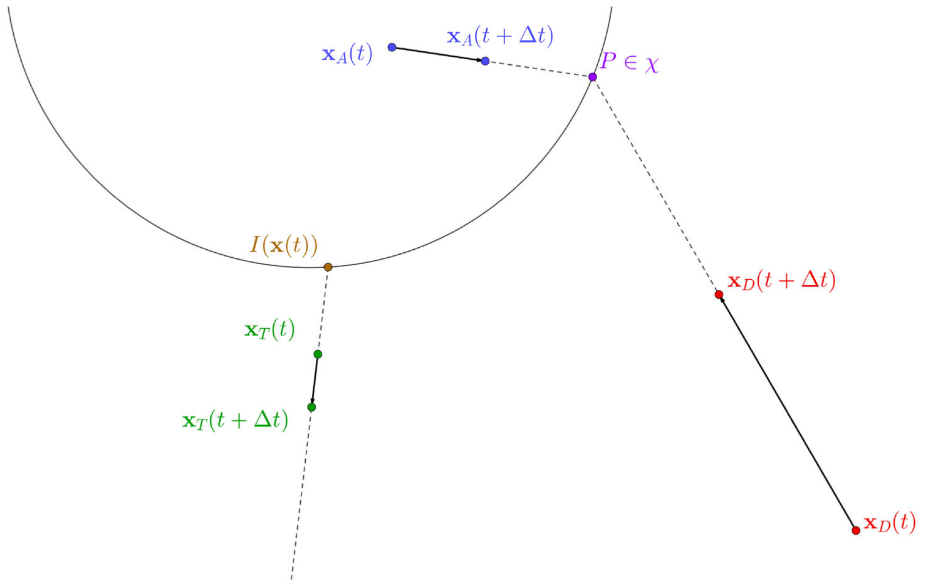

Figure 1. Example of Run and Contain strategy for team T/D

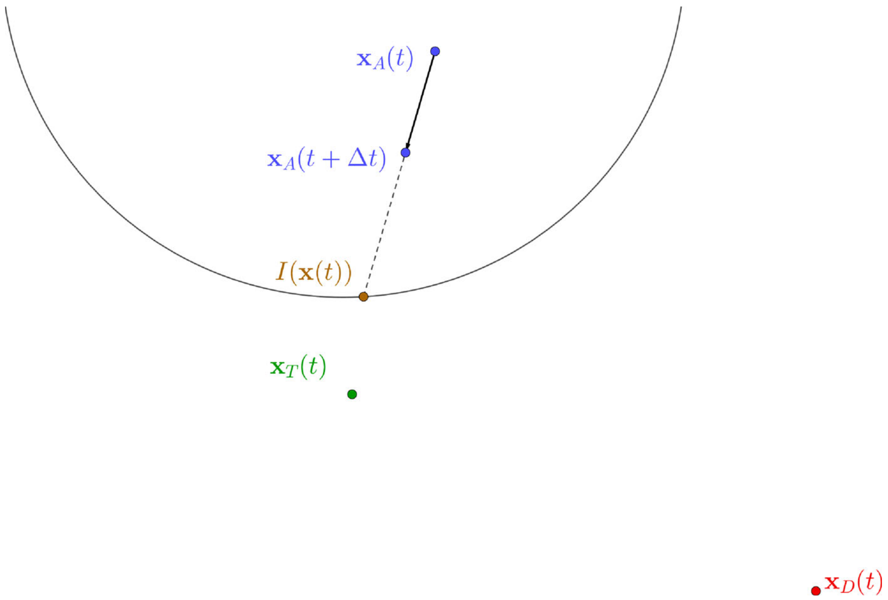

Figure 2. Interception Attack strategy for team A

$\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}$. Since (22) holds, the following four vectors in $\mathbb{R}^n$ are well defined.

$$\mathbf{A} = \mathbf{x}_A(t) + \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)}), \tag{23}$$

$$\mathbf{D} = \mathbf{x}_D(t) + \mathrm{proj}(\overrightarrow{\mathbf{x}_D(t)\mathbf{x}_D(t+\Delta t)}), \tag{24}$$

$$\mathbf{M_1} = \mathbf{x}_A(t) + \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}), \tag{25}$$

$$\mathbf{M_2} = \mathbf{M_1} + \overrightarrow{\mathbf{A}\mathbf{x}_A(t+\Delta t)}. \tag{26}$$

Here vectors $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{M_1}$ are the projections of $\mathbf{x}_A(t+\Delta t)$, $\mathbf{x}_D(t+\Delta t)$ and $\mathbf{P}$ respectively onto the 1-d plane defined by containing the line from $\mathbf{x}_A(t)$ to $\mathbf{x}_D(t)$. Figure 3 plots an example of this geometry in 2d.

The first preliminary result is given below

$$\overrightarrow{\mathbf{A}\mathbf{x}_A(t+\Delta t)} = \overrightarrow{\mathbf{D}\mathbf{x}_D(t+\Delta t)}. \tag{27}$$

Formula (27) follows trivially from the following geometric argument. In the case where Δt was set equal to the time required for $\mathbf{x}_A(t+\Delta t) = \mathbf{x}_D(t+\Delta t)$ (such a Δt exists since $V_A < V_D$ and agent D plays the Containment strategy), $\mathbf{A} = \mathbf{D}$ and hence $\overrightarrow{\mathbf{A}\mathbf{x}_A(t+\Delta t)} = \overrightarrow{\mathbf{D}\mathbf{x}_D(t+\Delta t)}$.

For any other time increment Δt, since both agents move in a straight line with constant speed, $\overrightarrow{\mathbf{A}\mathbf{x}_A(t+\Delta t)}$ and $\overrightarrow{\mathbf{D}\mathbf{x}_D(t+\Delta t)}$ point in the same direction for all Δt; moreover the sizes of $\overrightarrow{\mathbf{A}\mathbf{x}_A(t+\Delta t)}$ and $\overrightarrow{\mathbf{D}\mathbf{x}_D(t+\Delta t)}$ increase linearly

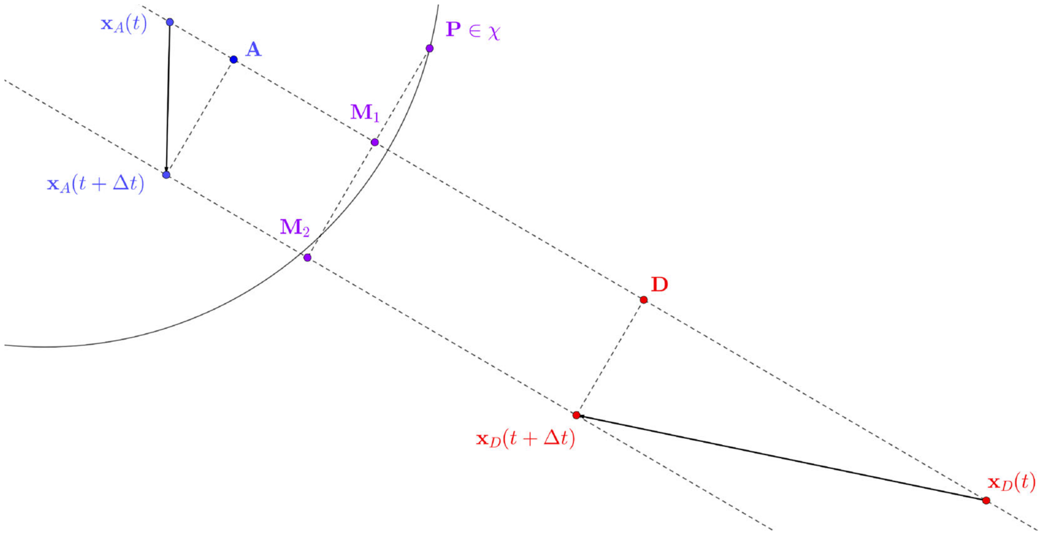

Figure 3. Diagram in 2d of the vectors defined in Theorem 3.1

with respect to time, and since the sizes are equal for one non-zero Δt, it is also true for all Δt. Hence (27) holds for all Δt.

It follows from (27) that the line from $\mathbf{x}_A(t+\Delta t)$ to $\mathbf{x}_D(t+\Delta t)$ is parallel to the line from $\mathbf{x}_A(t)$ to $\mathbf{x}_D(t)$. We can deduce from this the following properties:

$$\overline{\mathbf{M_1M_2}} = \overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}} = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{D}}, \tag{28}$$

$$\overline{\mathbf{AM_1}} = \overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}, \tag{29}$$

$$\overline{\mathbf{DM_1}} = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}. \tag{30}$$

Let T denote the time required for agents A and D to reach point $\mathbf{P}$ from their initial positions $\mathbf{x}_A(t)$, $\mathbf{x}_D(t)$ respectively. Next we note all the right-angle triangles formed by the geometry.

$$\overline{\mathbf{x}_A(t)\mathbf{A}}^2 + \overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}}^2 = V_A^2\Delta t^2, \tag{31}$$

$$\overline{\mathbf{x}_D(t)\mathbf{D}}^2 + \overline{\mathbf{x}_D(t+\Delta t)\mathbf{D}}^2 = V_D^2\Delta t^2, \tag{32}$$

$$\overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 + \overline{\mathbf{M_1P}}^2 = V_A^2T^2, \tag{33}$$

$$\overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 + \overline{\mathbf{M_1P}}^2 = V_D^2T^2, \tag{34}$$

$$\overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}^2 + \overline{\mathbf{M_2P}}^2 = \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2, \tag{35}$$

$$\overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}^2 + \overline{\mathbf{M_2P}}^2 = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2. \tag{36}$$

Note that the algebraic proofs of formulas (28)–(36) are given in the Appendix, where the formulas are deduced from Equations (23)–(27).

Applying the triangle inequality, we have that $\overline{\mathbf{x}_D(t)\mathbf{M_1}} \le \overline{\mathbf{x}_D(t)\mathbf{D}} + \overline{\mathbf{DM_1}}$. In the case where $\overline{\mathbf{x}_D(t)\mathbf{M_1}} < \overline{\mathbf{x}_D(t)\mathbf{D}} + \overline{\mathbf{DM_1}}$, agent D is no further to point $\mathbf{P}$ than agent A at time $t+\Delta t$, and since agent D is at least as fast as agent A, Theorem 3.1 is trivially satisfied. The remainder of the proof is concerned with the case

$$\overline{\mathbf{x}_D(t)\mathbf{M_1}} = \overline{\mathbf{x}_D(t)\mathbf{D}} + \overline{\mathbf{DM_1}}. \tag{37}$$

Utilising the above properties, the following two results, which are essential to the proof, are derived.

$$\begin{aligned}&\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_D(t)\mathbf{D}}\,\overline{\mathbf{x}_D(t)\mathbf{M_1}}\\ &\quad= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - V_D^2\Delta t^2,\end{aligned} \tag{38}$$

$$\begin{aligned}&\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}}\\ &\quad\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - V_A^2\Delta t^2.\end{aligned} \tag{39}$$

To derive (38), first we verify that $-2\overline{\mathbf{x}_D(t)\mathbf{D}}\ \overline{\mathbf{x}_D(t)\mathbf{M_1}} = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2$.

$$\begin{aligned}&-2\overline{\mathbf{x}_D(t)\mathbf{D}}\,\overline{\mathbf{x}_D(t)\mathbf{M_1}}\\ &\quad= -2\overline{\mathbf{x}_D(t)\mathbf{D}}\,\overline{\mathbf{x}_D(t)\mathbf{M_1}} + \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 + \overline{\mathbf{x}_D(t)\mathbf{D}}^2\\ &\qquad - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2\\ &\quad= (\overline{\mathbf{x}_D(t)\mathbf{M_1}} - \overline{\mathbf{x}_D(t)\mathbf{D}})^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2\\ &\quad= \overline{\mathbf{DM_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 \quad \text{applying (37)}\\ &\quad= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2.\\ &\qquad \text{substituting (30)}\end{aligned}$$

Therefore formula (38) is derived as follows:

$$\begin{aligned}&\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_D(t)\mathbf{D}}\,\overline{\mathbf{x}_D(t)\mathbf{M_1}}\\ &\quad= \overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 + \overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}^2\\ &\qquad - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2\\ &\quad= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2\\ &\qquad - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 \quad \text{applying (36)}\end{aligned}$$

$$= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 \quad \text{applying (34)}$$
$$= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - \overline{\mathbf{x}_D(t+\Delta t)\mathbf{D}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 \quad \text{substituting (28)}$$
$$= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - V_D^2\Delta t^2. \quad \text{applying (32)}$$

To derive formula (39), the following inequality is proved

$$-2\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}} \le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2. \tag{40}$$

Applying the triangle inequality twice, we have that $\overline{\mathbf{x}_A(t)\mathbf{M_1}} \le \overline{\mathbf{AM_1}} + \overline{\mathbf{x}_A(t)\mathbf{A}}$ and $\overline{\mathbf{x}_A(t)\mathbf{A}} \le \overline{\mathbf{AM_1}} + \overline{\mathbf{x}_A(t)\mathbf{M_1}}$, consequently

$$\overline{\mathbf{AM_1}} \ge \left|\overline{\mathbf{x}_A(t)\mathbf{M_1}} - \overline{\mathbf{x}_A(t)\mathbf{A}}\right| \text{ holds,}$$
$$\text{thus } \overline{\mathbf{AM_1}}^2 \ge (\overline{\mathbf{x}_A(t)\mathbf{M_1}} - \overline{\mathbf{x}_A(t)\mathbf{A}})^2. \tag{41}$$

Using (41) we deduce (40) as follows:

$$-2\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}}$$
$$= -2\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}} + \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 + \overline{\mathbf{x}_A(t)\mathbf{A}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2$$
$$= (\overline{\mathbf{x}_A(t)\mathbf{M_1}} - \overline{\mathbf{x}_A(t)\mathbf{A}})^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2$$
$$\le \overline{\mathbf{AM_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 \quad \text{applying(41)}$$
$$\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2. \quad \text{substituting (29)}$$

Thus by applying (40), inequality (39) is proven as follows:

$$\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}}$$
$$\le \overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 + \overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2$$
$$\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 \text{applying (35)}$$
$$\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 \quad \text{applying (33)}$$
$$\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - \overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 \quad \text{substituting (28)}$$
$$\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - V_A^2\Delta t^2. \quad \text{applying (31)}$$

With formulas (38) and (39) verified, the proof of Theorem 3.1 can commence. The starting point of the proof is the inequality given by Proposition 1.

Invoking Proposition 1; If $\delta \ge 0$ then the inequality

$$\sqrt{\cos^2\theta + \delta\sin^2\theta}\sqrt{\cos^2\phi + \delta\sin^2\phi} \ge |\cos\theta||\cos\phi| + \delta|\sin\theta||\sin\phi| \tag{42}$$

holds for all $\theta, \phi \in \mathbb{R}$.

Let $\theta = \angle \mathbf{A}\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)$ and $\phi = \angle \mathbf{M_1}\mathbf{x}_A(t)\mathbf{P}$. Since the points $\mathbf{x}_A(t)$, $\mathbf{A}$ and $\mathbf{x}_A(t+\Delta t)$ form a right-angle triangle:

$$\overline{\mathbf{x}_A(t)\mathbf{A}} = V_A\Delta t|\cos\theta|, \tag{43}$$
$$\overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}} = \overline{\mathbf{M_1M_2}} = V_A\Delta t|\sin\theta|. \tag{44}$$

Also $\mathbf{x}_A(t)$, $\mathbf{M_1}$ and $\mathbf{P}$ form a right-angle triangle:

$$\overline{\mathbf{x}_A(t)\mathbf{M_1}} = V_AT|\cos\phi|, \tag{45}$$
$$\overline{\mathbf{M_1P}} = V_AT|\sin\phi|. \tag{46}$$

We can deduce $\overline{\mathbf{x}_D(t)\mathbf{D}}$ using Pythagoras' theorem:

$$\overline{\mathbf{x}_D(t)\mathbf{D}} = \sqrt{V_D^2\Delta t^2 - \overline{\mathbf{x}_D(t+\Delta t)\mathbf{D}}^2}$$
$$= \sqrt{V_D^2\Delta t^2 - \overline{\mathbf{M_1M_2}}^2} \quad \text{substituting (28)}$$
$$= \sqrt{V_D^2\Delta t^2 - V_A^2\Delta t^2\sin^2\theta} \quad \text{substituting (44)}$$
$$= V_D\Delta t\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\theta}. \tag{47}$$

Similarly it can be deduced that

$$\overline{\mathbf{x}_D(t)\mathbf{M_1}} = V_DT\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\phi}. \tag{48}$$

Let $\delta = 1 - \frac{V_A^2}{V_D^2}$. Since $0 < \frac{V_A}{V_D} < 1$, we have $\delta \ge 0$, hence we may utilise Proposition 1 for this choice of θ, ϕ, δ. Re-expressing (42) with $\delta = 1 - \frac{V_A^2}{V_D^2}$ we obtain

$$\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\theta}\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\phi} \ge |\cos\theta||\cos\phi| + \left(1 - \frac{V_A^2}{V_D^2}\right)|\sin\theta||\sin\phi|.$$

Let $\psi = \angle \mathbf{M_2M_1P}$. The above inequality also holds if we multiply the term $|\sin\theta||\sin\phi|$ with $\cos\psi$, since this adjustment does not increase the right-hand side of the inequality:

$$\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\theta}\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\phi} \ge |\cos\theta||\cos\phi| + \left(1 - \frac{V_A^2}{V_D^2}\right)|\sin\theta||\sin\phi|\cos\psi.$$

Substituting formulas (43)–(48) into the inequality above, and then multiplying both sides by $2\Delta tT$, we have:

$$\frac{2}{V_D^2}\overline{\mathbf{x}_D(t)\mathbf{D}}\,\overline{\mathbf{x}_D(t)\mathbf{M_1}} \geq \frac{2}{V_A^2}\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}} + 2\left(\frac{1}{V_A^2} - \frac{1}{V_D^2}\right)\overline{\mathbf{M_1M_2}}\,\overline{\mathbf{M_1P}}\cos\psi.$$

Applying the law of cosines $\overline{\mathbf{M_2P}}^2 = \overline{\mathbf{M_1M_2}}^2 + \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{M_1M_2}}\,\overline{\mathbf{M_1P}}\cos\psi$:

$$\frac{2}{V_D^2}\overline{\mathbf{x}_D(t)\mathbf{D}}\,\overline{\mathbf{x}_D(t)\mathbf{M_1}} \geq \frac{2}{V_A^2}\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}} + \left(\frac{1}{V_A^2} - \frac{1}{V_D^2}\right) \times (\overline{\mathbf{M_1M_2}}^2 + \overline{\mathbf{M_1P}}^2 - \overline{\mathbf{M_2P}}^2).$$

Re-arranging the above inequality we obtain

$$\frac{1}{V_A^2}\left(\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}}\right) \geq \frac{1}{V_D^2}\left(\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_D(t)\mathbf{D}}\,\overline{\mathbf{x}_D(t)\mathbf{M_1}}\right). \tag{49}$$

Substituting Equation (38) into the right-hand side:

$$\frac{1}{V_A^2}\left(\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}}\right) \geq \frac{1}{V_D^2}\left(\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - V_D^2\Delta t^2\right).$$

Applying inequality (39), it is also true that

$$\frac{1}{V_A^2}\left(\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - V_A^2\Delta t^2\right) \geq \frac{1}{V_A^2}\left(\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\,\overline{\mathbf{x}_A(t)\mathbf{M_1}}\right),$$

and hence

$$\frac{1}{V_A^2}\left(\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - V_A^2\Delta t^2\right) \geq \frac{1}{V_D^2}\left(\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - V_D^2\Delta t^2\right).$$

Thus

$$\frac{\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2}{V_A^2} \geq \frac{\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2}{V_D^2},$$

or $\frac{\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}}{V_A} \geq \frac{\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}}{V_D}$; which completes the proof. ■

Using Theorem 3.1, we now prove that the Run and Contain strategy ensures the value function defined in (11) is non-decreasing. The result we have however assumes that the Target does not move, hence a capture the flag scenario. It remains an open problem to generalise the theorem given below to arbitrary V_T satisfying (18).

Theorem 3.2: *In Game 3.1, if $V_T = 0$ and team T/D play the Run and Contain strategy, then $V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t))$.*

Proof: Proof by contradiction. Assume team T/D play the Run and Contain strategy and $V(\mathbf{x}(t+\Delta t)) < V(\mathbf{x}(t))$. This implies there exists $\mathbf{I_2} \in \chi(\mathbf{x}(t+\Delta t))$ such that

$$\overline{\mathbf{x}_T(t+\Delta t)\mathbf{I_2}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\mathbf{I_2}} < \overline{\mathbf{x}_T(t)I(\mathbf{x}(t))} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))},$$

and since $V_T = 0$

$$\overline{\mathbf{x}_T(t)\mathbf{I_2}} < \overline{\mathbf{x}_T(t)I(\mathbf{x}(t))}. \tag{50}$$

Since agent D plays the Containment strategy, we can invoke Theorem 3.1. On the line connecting $\mathbf{I_2}$ to $\mathbf{x}_T(t)$, it must pass through a point in $\chi(\mathbf{x}(t))$ since $\mathbf{x}_T(t)$ lies outside the boundary $\chi(\mathbf{x}(t))$ due to (21) and $\mathbf{I_2}$ is inside the boundary $\chi(\mathbf{x}(t))$ due to Theorem 3.1; denote such a point as $\mathbf{I_1}$. Thus

$$\overline{\mathbf{x}_T(t)\mathbf{I_1}} \leq \overline{\mathbf{x}_T(t)\mathbf{I_2}}. \tag{51}$$

Since $\mathbf{I_1} \in \chi(\mathbf{x}(t))$ and $I(\mathbf{x}(t)) = \arg\min_{\mathbf{I}\in\chi(\mathbf{x}(t))}\overline{\mathbf{x}_T(t)\mathbf{I}}$, we also have that

$$\overline{\mathbf{x}_T(t)I(\mathbf{x}(t))} \leq \overline{\mathbf{x}_T(t)\mathbf{I_1}}. \tag{52}$$

Combining (50), (51) and (52) we have

$$\overline{\mathbf{x}_T(t)I(\mathbf{x}(t))} < \overline{\mathbf{x}_T(t)I(\mathbf{x}(t))},$$

which is false. Hence in Game 3.1, it cannot be the case that $V(\mathbf{x}(t+\Delta t)) < V(\mathbf{x}(t))$ if team T/D play the Run and Contain strategy and $V_T = 0$. ■

The next set of results rely on the implications of bounds (19) and (20), and are given below.

Lemma 3.3: *If $\Delta t \leq \frac{1}{V_A+V_D}\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}$ then $\Delta t \leq \frac{\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}}{V_A}$.*

Proof: The $(n-1)$-dimensional boundary χ is the surface in which if agents A and D moved in a straight line towards a point on that surface, both agents would reach that point at the same time. Obviously the point in χ in which agents A and D would reach in the shortest time is the point at which the agents would move straight towards each other, and this shortest time is $\frac{1}{V_A+V_D}\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}$. On the other-hand, $\frac{\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}}{V_A}$ is the time it takes for agent A to reach some other point in χ, and hence $\frac{\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}}{V_A} \geq \frac{1}{V_A+V_D}\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}$. ■

Lemma 3.4: *If $\Delta t \leq \frac{1}{2}\left(\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{V_A} - \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}}{V_D}\right)$ then $\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)}}{V_D} \leq \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}}{V_A}$.*

Proof: The distance $\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}$ is at a minimum if agent A moves towards agent T, hence the lower bound

$$\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)} \geq \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} - V_A\Delta t \quad (53)$$

holds. Similarly, the distance $\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)}$ is at a maximum if agent D moves away from agent T, thus

$$\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)} \leq \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} + V_D\Delta t. \quad (54)$$

Manipulating Equation (53),

$$\begin{aligned}
&\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)} - \overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} \geq -V_A\Delta t\\
&-\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}}{V_A} + \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{V_A} \leq \Delta t\\
&-\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}}{V_A} + \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{V_A}\\
&\leq \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{2V_A} - \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}}{2V_D} \quad \text{substituting (20)}\\
&-\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}}{V_A}\\
&\leq -\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{2V_A} - \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}}{2V_D} - \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}}{V_A}\\
&\geq \frac{1}{2}\left(\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{V_A} + \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t))}}{V_D}\right).
\end{aligned} \quad (55)$$

Similarly, manipulating formula (54),

$$\begin{aligned}
&\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)}}{V_D} - \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}}{V_D} \leq \Delta t\\
&\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)}}{V_D} - \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}}{V_D}\\
&\leq \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{2V_A} - \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}}{2V_D} \quad \text{applying (20)}\\
&\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)}}{V_D}\\
&\leq \frac{1}{2}\left(\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{V_A} + \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t))}}{V_D}\right).
\end{aligned} \quad (56)$$

Therefore combining formulas (55) and (56), it follows that $\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)}}{V_D} \leq \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}}{V_A}$. ■

The next theorem establishes the non-increasing property of the value function.

Theorem 3.5: *In Game 3.1, if team A plays the Interception Attack strategy, then $V(\mathbf{x}(t+\Delta t)) \leq V(\mathbf{x}(t))$.*

Proof: In Game 3.1, since it is assumed the time increment Δt satisfies (19) and (20), we may invoke Lemmas 3.3 and 3.4.

Team A plays the Interception Attack strategy, meaning that agent A moves in a straight line with speed V_A towards $I(\mathbf{x}(t))$. Moreover from Lemma 3.3, $\Delta t \leq \frac{\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}}{V_A}$, hence $I(\mathbf{x}(t))$ remains within the Attacker's side of the containment boundary $\chi(\mathbf{x}(t+\Delta t))$ at time $t+\Delta t$.

Invoking Lemma 3.4, $\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t+\Delta t)}}{V_D} \leq \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t+\Delta t)}}{V_A}$, in other words $\mathbf{x}_T(t)$ remains in the Defender's side of the containment boundary $\chi(\mathbf{x}(t+\Delta t))$ at time $t+\Delta t$.

It therefore follows that there exists some point on the line connecting $\mathbf{x}_T(t)$ to $I(\mathbf{x}(t))$ that is in $\chi(\mathbf{x}(t+\Delta t))$. Let Γ denote such a point.

Utilising the triangle inequality, it is true that

$$\begin{aligned}
\overline{\mathbf{x}_A(t+\Delta t)\Gamma} &\leq \overline{\Gamma I(\mathbf{x}(t))} + \overline{\mathbf{x}_A(t+\Delta t)I(\mathbf{x}(t))}\\
\frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma} &\leq \frac{V_T}{V_A}\overline{\Gamma I(\mathbf{x}(t))} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)I(\mathbf{x}(t))}\\
&\text{since } \frac{V_T}{V_A} \geq 0\\
\frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma} &\leq \overline{\Gamma I(\mathbf{x}(t))} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)I(\mathbf{x}(t))}.\\
&\text{since } \frac{V_T}{V_A} \leq 1
\end{aligned}$$

Adding both sides of the above inequality by $\overline{\mathbf{x}_T(t)\Gamma} + V_T\Delta t$:

$$\begin{aligned}
&\overline{\mathbf{x}_T(t)\Gamma} + V_T\Delta t + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq \overline{\mathbf{x}_T(t)\Gamma} + \overline{\Gamma I(\mathbf{x}(t))}\\
&+ \frac{V_T}{V_A}(\overline{\mathbf{x}_A(t+\Delta t)I(\mathbf{x}(t))} + V_A\Delta t).
\end{aligned}$$

Since agent A moves in a straight line towards $I(\mathbf{x}(t))$ with speed V_A and (19) holds, $\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))} = \overline{\mathbf{x}_A(t+\Delta t)I(\mathbf{x}(t))} + V_A\Delta t$. Moreover, since $\overline{\mathbf{x}_T(t)\Gamma} + \overline{\Gamma I(\mathbf{x}(t))} = \overline{\mathbf{x}_T(t)I(\mathbf{x}(t))}$, applying to the inequality above:

$$\begin{aligned}
&\overline{\mathbf{x}_T(t)\Gamma} + V_T\Delta t + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq \overline{\mathbf{x}_T(t)I(\mathbf{x}(t))}\\
&+ \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}\\
&\overline{\mathbf{x}_T(t)\Gamma} + V_T\Delta t + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq V(\mathbf{x}(t)).
\end{aligned} \quad (57)$$

Using the triangle inequality $\overline{\mathbf{x}_T(t+\Delta t)\Gamma} \leq \overline{\mathbf{x}_T(t)\Gamma} + \overline{\mathbf{x}_T(t)\mathbf{x}_T(t+\Delta t)}$, and since agent T moves with speed V_T, $\overline{\mathbf{x}_T(t)\mathbf{x}_T(t+\Delta t)} = V_T\Delta t$, thus $\overline{\mathbf{x}_T(t+\Delta t)\Gamma} \leq \overline{\mathbf{x}_T(t)\Gamma} + V_T\Delta t$. It follows that the left-hand side of the inequality (57) can be bounded below:

$$\begin{aligned}
&\overline{\mathbf{x}_T(t+\Delta t)\Gamma} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq \overline{\mathbf{x}_T(t)\Gamma}\\
&+ V_T\Delta t + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma}.
\end{aligned}$$

As a result, the following lower bound on $V(\mathbf{x}(t))$ holds:

$$\overline{\mathbf{x}_T(t+\Delta t)\Gamma} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq V(\mathbf{x}(t)).$$

Hence it is sufficient to verify that

$$V(\mathbf{x}(t+\Delta t)) \leq \overline{\mathbf{x}_T(t+\Delta t)\Gamma} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma}, \quad (58)$$

where $\Gamma \in \chi(\mathbf{x}(t+\Delta t))$. To confirm this, recall the definition of $V(\mathbf{x}(t+\Delta t))$

$$V(\mathbf{x}(t+\Delta t)) = \overline{\mathbf{x}_T(t+\Delta t)I(\mathbf{x}(t+\Delta t))} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)I(\mathbf{x}(t+\Delta t))},$$

where

$$I(\mathbf{x}(t+\Delta t)) = \arg\min_{\mathbf{I}} \overline{\mathbf{x}_T(t+\Delta t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\mathbf{I}},$$
$$\text{s.t.} \quad \mathbf{I} \in \chi(\mathbf{x}(t+\Delta t)).$$

Γ may or may not be the optimal point in $\chi(\mathbf{x}(t+\Delta t))$ that minimises $\overline{\mathbf{x}_T(t+\Delta t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\mathbf{I}}$, but $I(\mathbf{x}(t+\Delta t))$ is, hence

$$V(\mathbf{x}(t+\Delta t)) \leq \overline{\mathbf{x}_T(t+\Delta t)\Gamma} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\Gamma} \leq V(\mathbf{x}(t)),$$

which completes the proof. ■

Consolidating all the propositions, lemmas and theorems proved earlier in the paper, we now present the main result of this manuscript.

Theorem 3.6: *In Game 3.1, if $V_T = 0$ then the strategy profile of team A playing the Interception Attack strategy and team T/D playing the Run and Contain strategy constitutes a Nash equilibrium, with corresponding payoff $V(\mathbf{x}(t+\Delta t)) = V(\mathbf{x}(t))$.*

Proof: From Theorem 3.5, we know that if team A plays the Interception Attack strategy, the value function is non-increasing, that is

$$V(\mathbf{x}(t+\Delta t)) \leq V(\mathbf{x}(t)).$$

On the other-hand, since $V_T = 0$, we may invoke Theorem 3.2 to deduce that if team T/D play the Run and Contain strategy, the value function is non-decreasing

$$V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t)).$$

It follows from Theorems 3.2 and 3.5 that the payoff from this strategy profile is $V(\mathbf{x}(t+\Delta t)) = V(\mathbf{x}(t))$.

The goal of team T/D is to maximise the value function, and given that team A is playing the Interception Attack strategy, team T/D cannot theoretically obtain a value function higher than its current state, hence team T/D has nothing to gain by changing only their own strategy.

The objective of team A is to minimise the value function, and with team T/D playing the Run and Contain strategy, the value function is non-decreasing, hence team A cannot benefit by changing its strategy while team T/D keeps theirs unchanged. Thus this strategy profile is a Nash equilibrium. ■

4. Conclusion

In summary, Theorem 3.6 establishes the optimal strategies for teams A and T/D in the discrete-time turn-based variant of the Target–Attacker–Defender (TAD) differential game of degree. This in turn provides strong theoretical support to the claims made in Proposition 3 regarding the Nash equilibrium strategy profile of the original continuous-time formulation.

A future research direction could involve generalising Theorem 3.2 to arbitrary speeds of the Target V_T satisfying (8), since it was only proved in the special case $V_T = 0$.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix 1. Containment theorem formula derivations

Here we provide some algebraic derivations that were omitted in Theorem 3.1. In Game 3.1, Equations (23)–(27) are utilised to derive formulas (28)–(36).

First the derivation of formulas (28) to (30) are given. Directly from (27), we have that $\overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}} = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{D}}$, thus

$$\begin{aligned}
\overline{\mathbf{M_1M_2}} &= ||\overrightarrow{\mathbf{M_1M_2}}|| \\
&= ||\mathbf{M_2} - \mathbf{M_1}|| \\
&= ||\mathbf{M_1} + \overrightarrow{\mathbf{Ax}_A(t+\Delta t)} - \mathbf{M_1}|| \\
&= ||\overrightarrow{\mathbf{Ax}_A(t+\Delta t)}|| \\
&= \overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}}.
\end{aligned}$$

$$\begin{aligned}
\overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}} &= ||\overrightarrow{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}|| \\
&= ||\mathbf{M_2} - \mathbf{x}_A(t+\Delta t)|| \\
&= ||\mathbf{M_1} + \overrightarrow{\mathbf{Ax}_A(t+\Delta t)} - \mathbf{x}_A(t+\Delta t)|| \\
&= ||\mathbf{M_1} + \mathbf{x}_A(t+\Delta t) - \mathbf{A} - \mathbf{x}_A(t+\Delta t)|| \\
&= ||\mathbf{M_1} - \mathbf{A}|| \\
&= ||\overrightarrow{\mathbf{AM_1}}||
\end{aligned}$$

$$\begin{aligned}
= \overline{\mathbf{AM_1}}.\overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}} &= ||\overrightarrow{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}|| \\
&= ||\mathbf{M_2} - \mathbf{x}_D(t+\Delta t)|| \\
&= ||\mathbf{M_1} + \overrightarrow{\mathbf{Ax}_A(t+\Delta t)} - \mathbf{x}_D(t+\Delta t)|| \\
&= ||\mathbf{M_1} + \overrightarrow{\mathbf{Dx}_D(t+\Delta t)} - \mathbf{x}_D(t+\Delta t)|| \\
&= ||\mathbf{M_1} + \mathbf{x}_D(t+\Delta t) - \mathbf{D} - \mathbf{x}_D(t+\Delta t)|| \\
&= ||\mathbf{M_1} - \mathbf{D}|| \\
&= ||\overrightarrow{\mathbf{DM_1}}|| \\
&= \overline{\mathbf{DM_1}}.
\end{aligned}$$

To verify formulas (31)–(36), it is sufficient to show that in each case, the two legs of the triangle are orthogonal, and hence the Pythagorean theorem is applicable. Since the derivations are similar, only the validation of (32), (34) and (36) are displayed.

$$\begin{aligned}
&\overrightarrow{\mathbf{x}_D(t)\mathbf{D}} \cdot \overrightarrow{\mathbf{Dx}_D(t+\Delta t)} \\
&\quad = \big(\mathbf{D} - \mathbf{x}_D(t)\big) \cdot \big(\mathbf{x}_D(t+\Delta t) - \mathbf{D}\big) \\
&\quad = \big(\mathbf{x}_D(t) + \mathrm{proj}(\overrightarrow{\mathbf{x}_D(t)\mathbf{x}_D(t+\Delta t)}) - \mathbf{x}_D(t)\big) \cdot \big(\mathbf{x}_D(t+\Delta t) - \mathbf{x}_D(t) \\
&\qquad - \mathrm{proj}(\overrightarrow{\mathbf{x}_D(t)\mathbf{x}_D(t+\Delta t)})\big) \\
&\quad = \mathrm{proj}(\overrightarrow{\mathbf{x}_D(t)\mathbf{x}_D(t+\Delta t)}) \cdot \big(\overrightarrow{\mathbf{x}_D(t)\mathbf{x}_D(t+\Delta t)} \\
&\qquad - \mathrm{proj}(\overrightarrow{\mathbf{x}_D(t)\mathbf{x}_D(t+\Delta t)})\big) \\
&\quad = 0.
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\mathbf{x}_D(t)\mathbf{M_1}} \cdot \overrightarrow{\mathbf{M_1P}} &= \big(\mathbf{M_1} - \mathbf{x}_D(t)\big) \cdot \big(\mathbf{P} - \mathbf{M_1}\big) \\
&= \big(\mathbf{x}_A(t) + \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}) - \mathbf{x}_D(t)\big) \\
&\quad \cdot \big(\mathbf{P} - \mathbf{x}_A(t) - \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}})\big) \\
&= \big(\mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}) - \overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}\big) \cdot \big(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}} - \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}})\big) \\
&= 0.
\end{aligned}$$

To verify that $\overrightarrow{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}$ and $\overrightarrow{\mathbf{M_2P}}$ are orthogonal, first we deduce simplified expressions for these terms:

$$\begin{aligned}
\overrightarrow{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}} &= \mathbf{M_2} - \mathbf{x}_D(t+\Delta t) \\
&= \mathbf{M_1} + \overrightarrow{\mathbf{Ax}_A(t+\Delta t)} - \mathbf{x}_D(t+\Delta t) \\
&= \mathbf{x}_A(t) + \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}) + \overrightarrow{\mathbf{Ax}_A(t+\Delta t)} - \mathbf{x}_D(t+\Delta t) \\
&= \mathbf{x}_A(t) + \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}) + \overrightarrow{\mathbf{Dx}_D(t+\Delta t)} - \mathbf{x}_D(t+\Delta t) \\
&= \mathbf{x}_A(t) + \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}) - \mathbf{x}_D(t) - \mathrm{proj}(\overrightarrow{\mathbf{x}_D(t)\mathbf{x}_D(t+\Delta t)}).
\end{aligned}$$

Based on the definition of the projection function, we have that $\mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}) - \mathrm{proj}(\overrightarrow{\mathbf{x}_D(t)\mathbf{x}_D(t+\Delta t)}) = \mathrm{proj}(\overrightarrow{\mathbf{x}_D(t+\Delta t)\mathbf{P}}) + \mathbf{x}_D(t) - \mathbf{x}_A(t)$, hence

$$\overrightarrow{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}} = \mathrm{proj}(\overrightarrow{\mathbf{x}_D(t+\Delta t)\mathbf{P}}).$$

The find an expression for $\overrightarrow{\mathbf{M_2P}}$:

$$\begin{aligned}
\overrightarrow{\mathbf{M_2P}} &= \mathbf{P} - \mathbf{M_2} \\
&= \mathbf{P} - \mathbf{M_1} - \overrightarrow{\mathbf{Ax}_A(t+\Delta t)} \\
&= \mathbf{P} - \mathbf{x}_A(t) - \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}) - \overrightarrow{\mathbf{Ax}_A(t+\Delta t)} \\
&= \mathbf{P} - \mathbf{x}_A(t) - \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}) - \mathbf{x}_A(t+\Delta t) + \mathbf{x}_A(t) \\
&\quad + \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)}) \\
&= \mathbf{P} - \mathbf{x}_A(t+\Delta t) - \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}) + \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)}) \\
&= \overrightarrow{\mathbf{x}_A(t+\Delta t)\mathbf{P}} - \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t+\Delta t)\mathbf{P}}).
\end{aligned}$$

It can be shown that $\mathrm{proj}(\overrightarrow{\mathbf{x}_D(t+\Delta t)\mathbf{P}}) \cdot (\overrightarrow{\mathbf{x}_A(t+\Delta t)\mathbf{P}} - \mathrm{proj}(\overrightarrow{\mathbf{x}_A(t+\Delta t)\mathbf{P}})) = 0$, thus $\overrightarrow{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}} \cdot \overrightarrow{\mathbf{M_2P}} = 0$.

Chapter 5

Generalising the capture the flag scenario to active target defence

5.1 Contextual statement

A conference paper accepted for publication at ANZCC 2022, is an improvement upon the earlier publication titled '*A state-feedback Nash equilibrium for the general TargetAttackerDefender differential game of degree in arbitrary dimensions*'. The earlier publication successfully identified and proved the state-feedback optimal strategies in the general case $V_A < V_D$, and most of the proof held for any $V_T < V_A < V_D$, but there contained a component within the proof that made the restrictive assumption of $V_T = 0$. This assumption corresponds with the much simpler capture-the-flag differential game. The final manuscript uncovered the machinery required to complete the proof for any $0 \leq V_T < V_A < V_D$.

Statement of Authorship

Title of Paper	Generalising the capture the flag scenario to active target defence.
Publication Status	☐ Published ☑ Accepted for Publication ☑ Submitted for Publication ☐ Unpublished and Unsubmitted work written in manuscript style
Publication Details	Mammadov, K., Lim, C., & Shi, P. (2022). Generalising the capture the flag scenario to active target defence. Australian and New Zealand Control Conference 2022.

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Contribution to the Paper	Selected research topic, conducted research, wrote manuscript, and acted as corresponding author.		
Overall percentage (%)	95%		
Certification:	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.		
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Co-Author Contributions

By signing the Statement of Authorship, each author certifies that:

i. the candidate's stated contribution to the publication is accurate (as detailed above);

ii. permission is granted for the candidate in include the publication in the thesis; and

iii. the sum of all co-author contributions is equal to 100% less the candidate's stated contribution.

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Generalising the capture the flag scenario to active target defence

Kamal Mammadov, Cheng-Chew Lim and Peng Shi

***Abstract*—This manuscript examines the TAD differential game. Here there are three drones, the Drone A, Drone T and Drone D, all obeying simple-motion. This game doesn't terminate at some predefined time t_f, rather t_f is the first time in which Drone A collides with either of the other two drones. The objective of Drone A is to minimise the distance between itself and Drone T at termination time; Drone T and D on the other-hand work together as a team to maximise the aforementioned distance. The present manuscript expands upon the analysis previously given in the work of [1], here we study the game in the general case where $V_T < V_A < V_D$ (denoting the speeds of Drone T, Drone A and Drone D respectively), and the drones move in n-dimensional space. The previous work identified and rigorously proved the SFNE. Most of the proofs given in that work held for any $V_T < V_A < V_D$, however the proof of the non-decreasing property of the value function made the restrictive assumption of $V_T = 0$, as the machinery required to prove it for the general case $V_T \geq 0$ was not known at the time. $V_T = 0$ corresponds with the capture the flag scenario since Drone T cannot move. The present manuscript brings to light new symmetries in the value function, which are used to complete the missing proof so that the results now hold generally for any $V_T < V_A < V_D$.**

***Index Terms*—Differential games, target defence, capture the flag, state-feedback Nash equilibrium.**

I. Introduction

The theory of differential games was first explored in-depth by Rufus Isaacs in the late 1950's to 60's, the book [2], followed by [3] and [4] primarily explored the theoretical underpinnings of two-player zero-sum differential games, driven by real-world applications in defence.

In two-player zero-sum differential games, the problem of the existence of a state-feedback Nash equilibrium (SFNE) was explored in [5]. Here two discrete-time turn-based counterparts of a continuous-time differential game was proposed; one in which the player whose objective is to maximise the reward function moves first, followed by the player that aims to minimise the reward function; the other turn-based counterpart was the exact opposite. Clearly in the first variant, the player whose goal is to minimise, obtains an advantage in comparison to the original continuous-time formulation, thus the value of this dynamic game is named the lower value; similarly the value of the other turn-based game is named the upper value. As the time increment of these two discrete-time turn-based variants approach zero, we should expect the upper value and lower value to converge to a single number (if a SFNE exists), and that number is the value of the original differential game. The task of finding general conditions of a differential game in which the upper and lower values always converge is named the convergence problem [6].

The present manuscript is focused on the study of simple-motion, two-team, zero-sum pursuit-evasion games. A simple-motion game is a differential game in which the complete state is given by the position of N agents $\mathbf{x}_i(t)$, each governed by $\dot{\mathbf{x}}_i(t) = V_i \mathbf{u}_i(t)$, where $\mathbf{u}_i(t)$ is a unit vector controlled by agent i. An example of one such differential game is given in the works of [7], here a single fast pursuer was tasked with capturing a team of evaders in minimum time, and the team of evaders aspired to maximise the capture time.

The current manuscript examines a simple-motion pursuit-evasion game known as the active target defence, or Target-Attacker-Defender, differential game [8]. This differential game emulates a common aerospace engagement scenario in defence applications, in which an explosive carrying drone (Drone A) aims to destroy Drone T by getting as near as possible to it. Drone T has in its defence Drone D which can neutralise Drone A upon impact. A complete mathematical description of this differential can be found in Section II.

Two questions naturally arise when examining this engagement scenario. Firstly, under what conditions can Drone T escape capture. This problem is know as the game of kind, and the most succinct result on this issue is given by [9]; They have shown that under optimal play, Drone T escapes point capture from Drone A if and only if

$$V_A \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} < V_D \overline{\mathbf{x}_A(t)\mathbf{x}_T(t)} + V_T \overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}.$$

The game of degree on the other-hand is the problem of identifying and proving the SFNE. In the case where the speed of Drone A and D are equal, [10] identified the SFNE, and rigoursly validated the result by demonstrating they obey the Hamilton-Jacobi-Bellman equations. The more complex case where the speed of Drone D is faster than Drone A was explored in [11], here the Nash equilibrium strategies were identified but a rigours proof remained elusive.

The work of [1] on the other-hand built upon the earlier work of [12], here we analyse the discrete-time turn-based variant of this differential game. Theses works wrote proofs that the identified optimal strategies constitute a Nash equilibrium, with novel geometric methods. Specifically, the work of [1] was the first to rigorously establish the Nash equilibrium

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in the fast Defender case $V_A < V_D$. However, although most of the proof written held generally for any $V_T < V_A < V_D$, a critical piece of the proof put the additional assumption of $V_T = 0$, that is, Drone T must be stationary. This corresponds with a capture the flag scenario, a much simpler game than the game of active target defence. The present manuscript completes the work of [1], by providing that missing piece.

Thus with this paper, we rigoursly prove the SFNE of the TAD differential game for any $V_T < V_A < V_D$, for any position of Drone T outside the AD-based Apollonius circle, which is a sufficient (but not necessary) condition for Drone T to escape capture.

II. TAD PURSUIT-EVASION GAME FORMULATION

A. Preliminaries

Throughout the manuscript the following notation is used. Given any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

- $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$
- $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$
- $\mathbf{u} \cdot \mathbf{v}$ is the dot product
- $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$,
- $\overrightarrow{\mathbf{uv}} = \mathbf{v} - \mathbf{u}$,
- $\overline{\mathbf{uv}} = ||\overrightarrow{\mathbf{uv}}||$
- $\angle \mathbf{uvw}$ denotes the angle between vectors $\overrightarrow{\mathbf{vu}}$ and $\overrightarrow{\mathbf{vw}}$; that is $\overrightarrow{\mathbf{vu}} \cdot \overrightarrow{\mathbf{vw}} = \overline{\mathbf{vu}}\ \overline{\mathbf{vw}} \cos \angle \mathbf{uvw}$, where $0 \leq \angle \mathbf{uvw} \leq \pi$.
- $\dot{\mathbf{u}}(t)$ is the time derivative $\dfrac{d}{dt}\mathbf{u}(t)$.

Furthermore, we list the acronyms used:

- SFNE - State-feedback Nash equilibrium
- SFOS - State-feedback optimal strategies
- ADAC - AD-based Apollonius circle
- HJI - Hamilton-Jacobi-Isaacs
- TAD - Target-Attacker-Defender
- NE - Nash equilibrium

B. Problem definition

In TAD game consists of two sides. The A side, also known as Drone A; and the T/D side, consisting of two drones, Drone T and Drone D. Every drone obeys simple-motion, thus the state of the differential game can be completely characterised by $\mathbf{x}(t) = (\mathbf{x}_T(t), \mathbf{x}_A(t), \mathbf{x}_D(t))$, and the dynamics are governed by the differential equations

$$(\dot{\mathbf{x}}_T(t), \dot{\mathbf{x}}_A(t), \dot{\mathbf{x}}_D(t)) = (V_T\mathbf{u}_T(t), V_A\mathbf{u}_A(t), V_D\mathbf{u}_D(t)). \tag{1}$$

Here subscript A, D and T denotes Drone A, D and T respectively. The game is analysed most generally in n-dimensional space thus $\mathbf{x}_T(t), \mathbf{x}_A(t), \mathbf{x}_D(t) \in \mathbb{R}^n$. At every time t, the A side selects a control input for Drone A satisfying (2); similarly the T/D side selects headings for Drone T and Drone D obeying (2).

$$\|\mathbf{u}_T(t)\|, \|\mathbf{u}_A(t)\|, \|\mathbf{u}_D(t)\| = 1. \tag{2}$$

The starting time t_0 and starting state $\mathbf{x}(t_0) = \mathbf{x}_0$ obeys

$$\frac{\overline{\mathbf{x}_T(t_0)\mathbf{x}_A(t_0)}}{V_A} > \frac{\overline{\mathbf{x}_T(t_0)\mathbf{x}_D(t_0)}}{V_D}, \tag{3a}$$

$$\mathbf{x}_A(t_0) \neq \mathbf{x}_D(t_0). \tag{3b}$$

This condition ensures Drone T escapes capture under optimal play. The final time t_f on the other-hand is not a predefined number, rather it's the first time satisfying either (4) or (5).

$$\mathbf{x}_A(t_f) = \mathbf{x}_D(t_f), \tag{4}$$

$$\mathbf{x}_A(t_f) = \mathbf{x}_T(t_f). \tag{5}$$

In the time interval $[t_0, t_f]$, the sides are awarded the payoff $\Phi(\mathbf{x}_f)$ given by

$$\Phi(\mathbf{x}_f) = \overline{\mathbf{x}_T(t_f)\mathbf{x}_A(t_f)}. \tag{6}$$

The A side aims to minimise $\Phi(\mathbf{x}_f)$, whereas the T/D side aims to maximise $\Phi(\mathbf{x}_f)$. The speed of the drones are related by

$$V_T < V_A < V_D. \tag{7}$$

In the article we refer to this differential game as Game II-B.

C. Nash equilibrium strategies

Here in this subsection, the SFOS are displayed. The value function is given by

$$V(\mathbf{x}(t)) = \overline{\mathbf{x}_T(t)I(\mathbf{x}(t))} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))} \tag{8}$$

where the I vector $I(\mathbf{x}(t))$ is the unique solution to

$$I(\mathbf{x}(t)) = \arg\min_{\mathbf{I}} \overline{\mathbf{x}_T(t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}} \tag{9a}$$

$$\text{s.t.} \quad \mathbf{I} \in \mathcal{C}_{AD}(\mathbf{x}(t)) \tag{9b}$$

and $\mathcal{C}_{AD}(\mathbf{x}(t))$ is the ADAC, defined by

$$\mathcal{C}_{AD}(\mathbf{x}(t)) = \{\mathbf{p} \in \mathbb{R}^n \mid \frac{\overline{\mathbf{p}\mathbf{x}_A(t)}}{V_A} = \frac{\overline{\mathbf{p}\mathbf{x}_D(t)}}{V_D}\}. \tag{10}$$

Let $\mathcal{C}_{AD}$ be shorthand for $\mathcal{C}_{AD}(\mathbf{x}(t))$ when it's obvious it refers to $\mathcal{C}_{AD}(\mathbf{x}(t))$.

The SFNE of Game II-B is given by Proposition 1.

Proposition 1. The SFOS of Game II-B are given by

$$\mathbf{u}_A^*(t) = \frac{I(\mathbf{x}(t)) - \mathbf{x}_A(t)}{\|I(\mathbf{x}(t)) - \mathbf{x}_A(t)\|}, \tag{11a}$$

$$\mathbf{u}_D^*(t) = \frac{I(\mathbf{x}(t)) - \mathbf{x}_D(t)}{\|I(\mathbf{x}(t)) - \mathbf{x}_D(t)\|}, \tag{11b}$$

$$\mathbf{u}_T^*(t) = -\Big(\frac{I(\mathbf{x}(t)) - \mathbf{x}_T(t)}{\|I(\mathbf{x}(t)) - \mathbf{x}_T(t)\|}\Big). \tag{11c}$$

To establish Proposition 1, we must show that (11) satisfies the HJI formulas given below from t_0 to t_f.

$$\mathbf{u}_A^*(t) = \arg\min_{\mathbf{u}_A(t)} \frac{\partial V}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}_A(t), \mathbf{u}_D^*(t), \mathbf{u}_T^*(t))$$
$$\text{s.t.} \quad \|\mathbf{u}_A(t)\| = 1 \tag{12}$$

$$(\mathbf{u}_D^*(t), \mathbf{u}_T^*(t)) = \arg\max_{(\mathbf{u}_D(t), \mathbf{u}_T(t))} \frac{\partial V}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}_A^*(t), \mathbf{u}_D(t), \mathbf{u}_T(t))$$
$$\text{s.t.} \quad \|\mathbf{u}_D(t)\|, \|\mathbf{u}_T(t)\| = 1 \tag{13}$$

$$\frac{\partial V}{\partial \mathbf{x}} \cdot f(\mathbf{x}(t), \mathbf{u}_A^*(t), \mathbf{u}_D^*(t), \mathbf{u}_T^*(t)) = 0 \tag{14}$$

The works of [10] used the expression (8) to analytically derive the gradient of the value function, and with that verified the HJI formulas (12)-(14) hold. The limitation in their analysis however is that they assumed the much simpler case where $V_A = V_D$, and each agent moves in 2-dimensional space ($n = 2$).

Their method involved the calculation of the Jacobian of the I vector, which measures the change in $I(\mathbf{x}(t))$ with respect to the change in the state $\mathbf{x}(t)$; this becomes prohibitively complex in the more general formulation given in II-B.

III. Discretized variant

A. Formulation of discretized game

Here we define a discrete-time turn-based variant of Game II-B. The game starts at any time t and any state $\mathbf{x}(t)$ satisfying (3).

Side A moves first, using the information of the current state $\mathbf{x}(t)$, it picks a direction for Drone A to move in a straight line from $\mathbf{x}_A(t)$ to $\mathbf{x}_A(t+\Delta t)$, with speed V_A.

After observing the movement of Drone A, the T/D side, informed of the current state $\mathbf{x}(t)$ and Drone A's next position $\mathbf{x}_A(t+\Delta t)$, it picks directions for Drone T and D to move in straight lines $\mathbf{x}_T(t)$ to $\mathbf{x}_T(t+\Delta t)$, and $\mathbf{x}_D(t)$ to $\mathbf{x}_D(t+\Delta t)$ with speeds V_T and V_D respectively.

The goal of the A side is to minimise the value function at the next time increment, that is to minimise $V(\mathbf{x}(t+\Delta t))$. On the other-hand the T/D side aims to maximise it.

As before we assume the speed of the agents satisfy (7). In addition, we assume the time increment Δt of the sequential game is sufficiently small such that

$$\Delta t \le \frac{1}{V_A+V_D}\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}, \tag{15}$$

$$\Delta t \le \frac{1}{2}\left(\frac{\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)}}{V_A} - \frac{\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}}{V_D}\right), \tag{16}$$

hold. This discrete-time turn-based game is referred to in the manuscript as Game III-A.

B. The Nash equilibrium

Underneath we define the SFOS for Game III-A.

Definition 1 (I Vector Tactic). A strategy for the A side defined by Drone A moving straight towards $I(\mathbf{x}(t))$.

Definition 2 (Confinement Tactic). A strategy for Drone D, defined by Drone D moving straight towards some point on the Apollonius circle $\mathcal{C}_{AD}$; that point on the circle being the same point Drone A travelled to.

Definition 3 (Gallop and Confine Tactic). A strategy for the T/D side, defined by Drone D playing the Confinement tactic, and Drone T moving straight away from $I(\mathbf{x}(t))$.

Next we show that the Confinement strategy for Drone D ensures that $\mathcal{C}_{AD}(\mathbf{x}(t+\Delta t))$ is completely encapsulated within $\mathcal{C}_{AD}(\mathbf{x}(t))$.

Theorem III.1. *In Game III-A, if Drone D plays the Confinement Tactic, then for any point* $\mathbf{P}\in\mathcal{C}_{AD}(\mathbf{x}(t))$, $\frac{\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}}{V_D} \le \frac{\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}}{V_A}$.

Proof. Let $\text{proj}(\mathbf{v}) = \frac{\mathbf{v}\cdot\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}\cdot\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}}\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}$ denote the vector projection of $\mathbf{v}\in\mathbb{R}^n$ onto $\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_D(t)}$. Since (3) holds, the following four vectors in $\mathbb{R}^n$ are well defined.

$$\mathbf{A} = \mathbf{x}_A(t) + \text{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)}), \tag{17}$$

$$\mathbf{D} = \mathbf{x}_D(t) + \text{proj}(\overrightarrow{\mathbf{x}_D(t)\mathbf{x}_D(t+\Delta t)}), \tag{18}$$

$$\mathbf{M_1} = \mathbf{x}_A(t) + \text{proj}(\overrightarrow{\mathbf{x}_A(t)\mathbf{P}}), \tag{19}$$

$$\mathbf{M_2} = \mathbf{M_1} + \overrightarrow{\mathbf{A}\mathbf{x}_A(t+\Delta t)}. \tag{20}$$

The vectors $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{M_1}$ denote the projections of $\mathbf{x}_A(t+\Delta t)$, $\mathbf{x}_D(t+\Delta t)$ and $\mathbf{P}$ respectively onto the 1-d plane defined by containing the line from $\mathbf{x}_A(t)$ to $\mathbf{x}_D(t)$.

With some geometric reasoning, it is trivial to show that

$$\overrightarrow{\mathbf{A}\mathbf{x}_A(t+\Delta t)} = \overrightarrow{\mathbf{D}\mathbf{x}_D(t+\Delta t)}. \tag{21}$$

holds. Moreover, from (21) it follows that the lines $\mathbf{x}_A(t+\Delta t)$ to $\mathbf{x}_D(t+\Delta t)$, and $\mathbf{x}_A(t)$ to $\mathbf{x}_D(t)$ are parallel. From this we deduce the following properties:

$$\overline{\mathbf{M_1}\mathbf{M_2}} = \overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}} = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{D}}, \tag{22}$$

$$\overline{\mathbf{A}\mathbf{M_1}} = \overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}, \tag{23}$$

$$\overline{\mathbf{D}\mathbf{M_1}} = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}. \tag{24}$$

Define T as the time required for drones A and D to reach point $\mathbf{P}$ from their initial positions $\mathbf{x}_A(t)$, $\mathbf{x}_D(t)$. Listing all the right-angle triangles formed by the geometry we get:

$$\overline{\mathbf{x}_A(t)\mathbf{A}}^2 + \overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}}^2 = V_A^2\Delta t^2, \tag{25}$$

$$\overline{\mathbf{x}_D(t)\mathbf{D}}^2 + \overline{\mathbf{x}_D(t+\Delta t)\mathbf{D}}^2 = V_D^2\Delta t^2, \tag{26}$$

$$\overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 + \overline{\mathbf{M_1}\mathbf{P}}^2 = V_A^2T^2, \tag{27}$$

$$\overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 + \overline{\mathbf{M_1}\mathbf{P}}^2 = V_D^2T^2, \tag{28}$$

$$\overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}^2 + \overline{\mathbf{M_2}\mathbf{P}}^2 = \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2, \tag{29}$$

$$\overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}^2 + \overline{\mathbf{M_2}\mathbf{P}}^2 = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2. \tag{30}$$

From the triangle inequality, we have that $\overline{\mathbf{x}_D(t)\mathbf{M_1}} \le \overline{\mathbf{x}_D(t)\mathbf{D}} + \overline{\mathbf{D}\mathbf{M_1}}$ holds. Theorem III.1 is trivially satisfied for the case where $\overline{\mathbf{x}_D(t)\mathbf{M_1}} < \overline{\mathbf{x}_D(t)\mathbf{D}} + \overline{\mathbf{D}\mathbf{M_1}}$, since Drone D is no further to point $\mathbf{P}$ than Drone A at time $t+\Delta t$, and Drone A is no faster than Drone D. The remainder of the proof is concerned with the case

$$\overline{\mathbf{x}_D(t)\mathbf{M_1}} = \overline{\mathbf{x}_D(t)\mathbf{D}} + \overline{\mathbf{D}\mathbf{M_1}}. \tag{31}$$

The following two results are derived with the above properties:

$$\begin{aligned}\overline{\mathbf{M_2}\mathbf{P}}^2 - \overline{\mathbf{M_1}\mathbf{M_2}}^2 - \overline{\mathbf{M_1}\mathbf{P}}^2 - 2\overline{\mathbf{x}_D(t)\mathbf{D}}\;\overline{\mathbf{x}_D(t)\mathbf{M_1}} \\ = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - V_D^2\Delta t^2,\end{aligned} \tag{32}$$

$$\begin{aligned}\overline{\mathbf{M_2}\mathbf{P}}^2 - \overline{\mathbf{M_1}\mathbf{M_2}}^2 - \overline{\mathbf{M_1}\mathbf{P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\;\overline{\mathbf{x}_A(t)\mathbf{M_1}} \\ \le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - V_A^2\Delta t^2.\end{aligned} \tag{33}$$

We derive (32) by first verifying that $-2\overline{\mathbf{x}_D(t)\mathbf{D}}\ \overline{\mathbf{x}_D(t)\mathbf{M_1}} = \overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2$.

$$\begin{aligned}
&-2\overline{\mathbf{x}_D(t)\mathbf{D}}\ \overline{\mathbf{x}_D(t)\mathbf{M_1}} \\
&=-2\overline{\mathbf{x}_D(t)\mathbf{D}}\ \overline{\mathbf{x}_D(t)\mathbf{M_1}} + \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 + \overline{\mathbf{x}_D(t)\mathbf{D}}^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 \\
&= (\overline{\mathbf{x}_D(t)\mathbf{M_1}} - \overline{\mathbf{x}_D(t)\mathbf{D}})^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 \\
&= \overline{\mathbf{DM_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 && \text{applying (31)} \\
&= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2. && \text{substituting (24)}
\end{aligned}$$

Thus formula (32) can be derived with:

$$\begin{aligned}
&\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_D(t)\mathbf{D}}\ \overline{\mathbf{x}_D(t)\mathbf{M_1}} \\
&= \overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 + \overline{\mathbf{x}_D(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 \\
&= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - \overline{\mathbf{x}_D(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 && \text{apply (30)} \\
&= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 && \text{apply (28)} \\
&= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - \overline{\mathbf{x}_D(t+\Delta t)\mathbf{D}}^2 - \overline{\mathbf{x}_D(t)\mathbf{D}}^2 && \text{apply (22)} \\
&= \overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - V_D^2\Delta t^2. && \text{apply (26)}
\end{aligned}$$

To demonstrate formula (33) holds, first the inequality (34) is verified.

$$-2\overline{\mathbf{x}_A(t)\mathbf{A}}\ \overline{\mathbf{x}_A(t)\mathbf{M_1}} \le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2. \quad (34)$$

From the triangle inequality we have $\overline{\mathbf{x}_A(t)\mathbf{M_1}} \le \overline{\mathbf{AM_1}} + \overline{\mathbf{x}_A(t)\mathbf{A}}$ and $\overline{\mathbf{x}_A(t)\mathbf{A}} \le \overline{\mathbf{AM_1}} + \overline{\mathbf{x}_A(t)\mathbf{M_1}}$, consequently

$$\begin{aligned}
&\overline{\mathbf{AM_1}} \ge |\overline{\mathbf{x}_A(t)\mathbf{M_1}} - \overline{\mathbf{x}_A(t)\mathbf{A}}| \quad \text{holds,} \\
\text{thus}\quad &\overline{\mathbf{AM_1}}^2 \ge (\overline{\mathbf{x}_A(t)\mathbf{M_1}} - \overline{\mathbf{x}_A(t)\mathbf{A}})^2. \quad (35)
\end{aligned}$$

Applying (35) we derive (34) as follows:

$$\begin{aligned}
&-2\overline{\mathbf{x}_A(t)\mathbf{A}}\ \overline{\mathbf{x}_A(t)\mathbf{M_1}} \\
&= -2\overline{\mathbf{x}_A(t)\mathbf{A}}\ \overline{\mathbf{x}_A(t)\mathbf{M_1}} + \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 + \overline{\mathbf{x}_A(t)\mathbf{A}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 \\
&= (\overline{\mathbf{x}_A(t)\mathbf{M_1}} - \overline{\mathbf{x}_A(t)\mathbf{A}})^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 \\
&\le \overline{\mathbf{AM_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 && \text{applying (35)} \\
&\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2. && \text{substituting (23)}
\end{aligned}$$

Hence by using (34), inequality (33) is proven as follows:

$$\begin{aligned}
&\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\ \overline{\mathbf{x}_A(t)\mathbf{M_1}} \\
&\le \overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 + \overline{\mathbf{x}_A(t+\Delta t)\mathbf{M_2}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 \\
&\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - \overline{\mathbf{x}_A(t)\mathbf{M_1}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 && \text{apply (29)} \\
&\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 && \text{apply (27)} \\
&\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - \overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}}^2 - \overline{\mathbf{x}_A(t)\mathbf{A}}^2 && \text{apply (22)} \\
&\le \overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - V_A^2\Delta t^2. && \text{apply (25)}
\end{aligned}$$

Using the results (32) and (33), the proof of Theorem III.1 can now commence. The starting point of the proof is the inequality:

$$\sqrt{\cos^2\theta + \delta\sin^2\theta}\sqrt{\cos^2\phi + \delta\sin^2\phi} \ge |\cos\theta||\cos\phi| + \delta|\sin\theta||\sin\phi|, \quad (36)$$

which holds for all $\delta \ge 0$, $\delta, \theta, \phi \in \mathbb{R}$. Let $\theta = \angle \mathbf{A}\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)$ and $\phi = \angle \mathbf{M_1}\mathbf{x}_A(t)\mathbf{P}$. Since $\mathbf{x}_A(t)$, $\mathbf{A}$ and $\mathbf{x}_A(t+\Delta t)$ form a right angle triangle:

$$\overline{\mathbf{x}_A(t)\mathbf{A}} = V_A\Delta t|\cos\theta|, \quad (37)$$

$$\overline{\mathbf{x}_A(t+\Delta t)\mathbf{A}} = \overline{\mathbf{M_1M_2}} = V_A\Delta t|\sin\theta|. \quad (38)$$

Moreover $\mathbf{x}_A(t)$, $\mathbf{M_1}$ and $\mathbf{P}$ form a right angle triangle:

$$\overline{\mathbf{x}_A(t)\mathbf{M_1}} = V_AT|\cos\phi|, \quad (39)$$

$$\overline{\mathbf{M_1P}} = V_AT|\sin\phi|. \quad (40)$$

Thus we deduce $\overline{\mathbf{x}_D(t)\mathbf{D}}$ using Pythagoras' theorem:

$$\begin{aligned}
\overline{\mathbf{x}_D(t)\mathbf{D}} &= \sqrt{V_D^2\Delta t^2 - \overline{\mathbf{x}_D(t+\Delta t)\mathbf{D}}^2} \\
&= \sqrt{V_D^2\Delta t^2 - \overline{\mathbf{M_1M_2}}^2} && \text{substituting (22)} \\
&= \sqrt{V_D^2\Delta t^2 - V_A^2\Delta t^2\sin^2\theta} && \text{substituting (38)} \\
&= V_D\Delta t\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\theta}. && (41)
\end{aligned}$$

Similarly it can be shown that

$$\overline{\mathbf{x}_D(t)\mathbf{M_1}} = V_DT\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\phi}. \quad (42)$$

Let $\delta = 1 - \frac{V_A^2}{V_D^2}$. Since $0 < \frac{V_A}{V_D} < 1$, we have $\delta \ge 0$, hence we may apply inequality (36) for this choice of θ, ϕ, δ. Re-expressing (36) with $\delta = 1 - \frac{V_A^2}{V_D^2}$ gives:

$$\begin{aligned}
&\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\theta}\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\phi} \\
&\ge |\cos\theta||\cos\phi| + (1 - \frac{V_A^2}{V_D^2})|\sin\theta||\sin\phi|.
\end{aligned}$$

Let $\psi = \angle \mathbf{M_2M_1P}$. Multiplying the term $|\sin\theta||\sin\phi|$ with $\cos\psi$ on the right-hand side, is an adjustment that preserves the inequality. Thus:

$$\begin{aligned}
&\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\theta}\sqrt{1 - \frac{V_A^2}{V_D^2}\sin^2\phi} \\
&\ge |\cos\theta||\cos\phi| + (1 - \frac{V_A^2}{V_D^2})|\sin\theta||\sin\phi|\cos\psi.
\end{aligned}$$

Applying formulas (37)-(42) into the inequality gives us:

$$\begin{aligned}
&\frac{2}{V_D^2}\overline{\mathbf{x}_D(t)\mathbf{D}}\ \overline{\mathbf{x}_D(t)\mathbf{M_1}} \ge \frac{2}{V_A^2}\overline{\mathbf{x}_A(t)\mathbf{A}}\ \overline{\mathbf{x}_A(t)\mathbf{M_1}} \\
&+2(\frac{1}{V_A^2} - \frac{1}{V_D^2})\overline{\mathbf{M_1M_2}}\ \overline{\mathbf{M_1P}}\cos\psi.
\end{aligned}$$

From law of cosines $\overline{\mathbf{M_2P}}^2 = \overline{\mathbf{M_1M_2}}^2 + \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{M_1M_2}}\ \overline{\mathbf{M_1P}}\cos\psi$ holds, hence:

$$\begin{aligned}
&\frac{2}{V_D^2}\overline{\mathbf{x}_D(t)\mathbf{D}}\ \overline{\mathbf{x}_D(t)\mathbf{M_1}} \ge \frac{2}{V_A^2}\overline{\mathbf{x}_A(t)\mathbf{A}}\ \overline{\mathbf{x}_A(t)\mathbf{M_1}} \\
&+(\frac{1}{V_A^2} - \frac{1}{V_D^2})(\overline{\mathbf{M_1M_2}}^2 + \overline{\mathbf{M_1P}}^2 - \overline{\mathbf{M_2P}}^2).
\end{aligned}$$

Re-arranging the inequality gives us

$$\begin{aligned}
&\frac{1}{V_A^2}\Big(\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\ \overline{\mathbf{x}_A(t)\mathbf{M_1}}\Big) \\
&\ge \frac{1}{V_D^2}\Big(\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_D(t)\mathbf{D}}\ \overline{\mathbf{x}_D(t)\mathbf{M_1}}\Big).
\end{aligned}$$

Applying (32) on the right-hand side:

$$\frac{1}{V_A^2}\Big(\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\ \overline{\mathbf{x}_A(t)\mathbf{M_1}}\Big) \geq \frac{1}{V_D^2}\Big(\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - V_D^2\Delta t^2\Big).$$

Substituting inequality (33), it follows that

$$\frac{1}{V_A^2}\Big(\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - V_A^2\Delta t^2\Big) \geq \frac{1}{V_A^2}\Big(\overline{\mathbf{M_2P}}^2 - \overline{\mathbf{M_1M_2}}^2 - \overline{\mathbf{M_1P}}^2 - 2\overline{\mathbf{x}_A(t)\mathbf{A}}\ \overline{\mathbf{x}_A(t)\mathbf{M_1}}\Big),$$

and hence

$$\frac{1}{V_A^2}\Big(\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2 - V_A^2T^2 - V_A^2\Delta t^2\Big) \geq \frac{1}{V_D^2}\Big(\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2 - V_D^2T^2 - V_D^2\Delta t^2\Big).$$

Thus

$$\frac{\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}^2}{V_A^2} \geq \frac{\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}^2}{V_D^2},$$

or $\frac{\overline{\mathbf{x}_A(t+\Delta t)\mathbf{P}}}{V_A} \geq \frac{\overline{\mathbf{x}_D(t+\Delta t)\mathbf{P}}}{V_D}$; which completes the proof. □

The next theorem utilises the results from Theorem III.1 to prove that the value function (8) is non-decreasing. The result here differs from the earlier publication [1] which assumed the capture the flag scenario where $V_T = 0$. The results given below hold generally for any $V_T < V_A < V_D$.

Theorem III.2. *In Game III-A, if the T/D side plays the Gallop and Confine Tactic, then* $V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t))$.

Proof. Recall the definition of the value function

$$V(\mathbf{x}(t)) = \min_{\mathbf{I}} \overline{\mathbf{x}_T(t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}} \quad \text{s.t.} \quad \mathbf{I} \in \mathcal{C}_{AD}(\mathbf{x}(t))$$

and the corresponding optimal $\mathbf{I}$ is denoted as $I(\mathbf{x}(t))$. We may add the term $V_T\Delta t$ to the expression:

$$\begin{aligned} V(\mathbf{x}(t)) + V_T\Delta t &= \min_{\mathbf{I}\in\mathcal{C}_{AD}(\mathbf{x}(t))} \overline{\mathbf{x}_T(t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}} + V_T\Delta t, \\ &= \overline{\mathbf{x}_T(t)I(\mathbf{x}(t))} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))} + V_T\Delta t. \end{aligned}$$

Since Drone T moves in a straight line away from $I(\mathbf{x}(t))$ with speed V_T, $\overline{\mathbf{x}_T(t+\Delta t)I(\mathbf{x}(t))} = \overline{\mathbf{x}_T(t)I(\mathbf{x}(t))} + V_T\Delta t$, thus

$$V(\mathbf{x}(t)) + V_T\Delta t = \overline{\mathbf{x}_T(t+\Delta t)I(\mathbf{x}(t))} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)I(\mathbf{x}(t))}. \tag{43}$$

At this point we have to establish the relationship between $I(\mathbf{x}(t))$ and $I(\mathbf{x}(t+\Delta t))$. Recall the definition of $I(\mathbf{x}(t))$, we may interpret (9) as the problem of finding the path for a particle to travel from $\mathbf{x}_T(t)$ to $\mathbf{x}_A(t)$ in minimum time, where inside the circle $\mathcal{C}_{AD}(\mathbf{x}(t))$ the particle moves at speed $\frac{V_A}{V_T}$, and outside the circle the particle moves with speed 1. Clearly the optimal path for the particle would be to travel in some straight line from $\mathbf{x}_T(t)$ towards the point $I(\mathbf{x}(t))$ on the perimeter of $\mathcal{C}_{AD}(\mathbf{x}(t))$, and then to travel in a straight line towards $\mathbf{x}_A(t)$.

Since Drone T moved in a straight line away from $I(\mathbf{x}(t))$, clearly the optimal point $I(\mathbf{x}(t))$ would not change, thus

$$I(\mathbf{x}(t)) = \arg\min_{\mathbf{I}\in\mathcal{C}_{AD}(\mathbf{x}(t))} \overline{\mathbf{x}_T(t+\Delta t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}}. \tag{44}$$

Consequently we may deduce from formulas (43) and (44) that

$$V(\mathbf{x}(t)) + V_T\Delta t = \min_{\mathbf{I}\in\mathcal{C}_{AD}(\mathbf{x}(t))} \overline{\mathbf{x}_T(t+\Delta t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}}. \tag{45}$$

Using the same interpretation of the optimisation problem (45), clearly if the optimisation variable $\mathbf{I}$ was not only free to choose any point on the perimeter of $\mathcal{C}_{AD}(\mathbf{x}(t))$, but also any point in the interior of that circle, this would not change the optimal point $I(\mathbf{x}(t))$, since the optimal point will always be on the perimeter, seeing as that the particle travels faster in the circle than outside the circle (since $\frac{V_A}{V_T} > 1$).

Moreover, the implication of Theorem III.1 is that the set of all points in the interior of $\mathcal{C}_{AD}(\mathbf{x}(t+\Delta t))$ is a subset of all the points in the interior of $\mathcal{C}_{AD}(\mathbf{x}(t))$. Hence it follows that

$$V(\mathbf{x}(t)) + V_T\Delta t \leq \min_{\mathbf{I}\in\mathcal{C}_{AD}(\mathbf{x}(t+\Delta t))} \overline{\mathbf{x}_T(t+\Delta t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}}. \tag{46}$$

Finally, all that remains is to establish the relationship between $V(\mathbf{x}(t+\Delta t)) = \min_{\mathbf{I}\in\mathcal{C}_{AD}(\mathbf{x}(t+\Delta t))} \overline{\mathbf{x}_T(t+\Delta t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t+\Delta t)\mathbf{I}}$, and the right-hand side of inequality (46). Once again using our interpretation, $V(\mathbf{x}(t+\Delta t))$ is the minimum time it would take a particle to traverse from $\mathbf{x}_T(t+\Delta t)$ to $\mathbf{x}_A(t+\Delta t)$. If instead the particle must traverse from $\mathbf{x}_T(t+\Delta t)$ to $\mathbf{x}_A(t)$, this would at worst take $\frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)}$ units of time longer. Thus

$$\min_{\mathbf{I}\in\mathcal{C}_{AD}(\mathbf{x}(t+\Delta t))} \overline{\mathbf{x}_T(t+\Delta t)\mathbf{I}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}} \leq V(\mathbf{x}(t+\Delta t)) + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)}. \tag{47}$$

Moreover due to the speed of Drone A, $\overline{\mathbf{x}_A(t)\mathbf{x}_A(t+\Delta t)} = V_A\Delta t$, therefore combining (46) and (47) we obtain

$$V(\mathbf{x}(t)) \leq V(\mathbf{x}(t+\Delta t))$$

as desired. □

It was proven in the earlier publication of [1], that provided the A side plays the I Vector Tactic, the payoff is guaranteed to be no higher than $V(\mathbf{x}(t))$. Moreover this result was proven to hold generally for any $V_T < V_A < V_D$, thus we refer the reader to the original publication for the proof of the next theorem.

Theorem III.3. *In Game III-A, if the A side plays the I Vector Tactic, then* $V(\mathbf{x}(t+\Delta t)) \leq V(\mathbf{x}(t))$.

Now that Theorems III.2 and III.3 hold for any $V_T < V_A < V_D$, we can proceed to prove that the discretized analogue of (11) is a NE. Moreover, since the time increment Δt can be infinitesimally small, the next theorem also proves that (11) is a SFNE of the continuous-time differential game II-B.

Theorem III.4. *The I Vector Tactic for the A side, with the Gallop and Confine Tactic for the T/D side, is a Nash equilibrium with payoff* $V(\mathbf{x}(t+\Delta t)) = V(\mathbf{x}(t))$.

Proof. Invoking Theorem III.3, if the A side plays the I Vector Tactic, the payoff $V(\mathbf{x}(t+\Delta t))$ is bounded above by

$$V(\mathbf{x}(t+\Delta t)) \leq V(\mathbf{x}(t)).$$

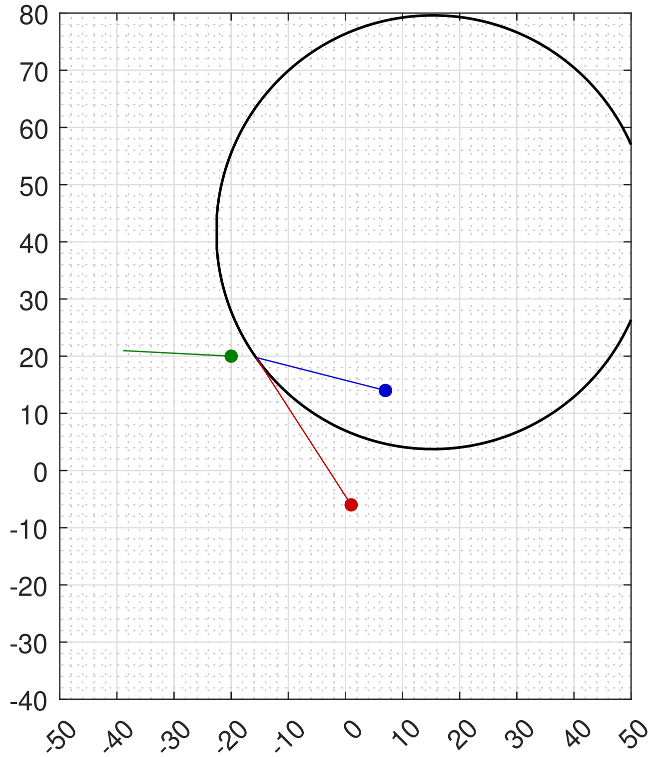

Fig. 1: The SFNE. The green, blue and red dots denote the starting positions of Drone T, A and D respectively, and the black circle denotes the ADAC at t_0.

Moreover Theorem III.2 established that if the T/D side plays the Gallop and Confine Tactic, the payoff $V(\mathbf{x}(t+\Delta t))$ is bounded below by

$$V(\mathbf{x}(t+\Delta t)) \geq V(\mathbf{x}(t)).$$

Thus this yields a payoff equal to: $V(\mathbf{x}(t+\Delta t)) = V(\mathbf{x}(t))$.

The T/D side aims to maximise the value function $V(\mathbf{x}(t+\Delta t))$, but provided the A side plays the I Vector Tactic, the T/D side cannot obtain a payoff larger than the current value $V(\mathbf{x}(t))$. Similarly, the A side tries to minimise $V(\mathbf{x}(t+\Delta t))$, but with the T/D side playing the Gallop and Confine Tactic, cannot obtain a payoff smaller than $V(\mathbf{x}(t))$. Thus this constitutes a NE. □

IV. Simulation of the SFNE

Consider the following numerical example of a TAD game, where the initial state and relative speeds are given by

$$\mathbf{x}(t_0) = (\begin{bmatrix}-20\\20\end{bmatrix}, \begin{bmatrix}7\\14\end{bmatrix}, \begin{bmatrix}1\\-6\end{bmatrix}), \quad (V_T, V_A, V_D) = (1.3, 1.6, 2.1) \tag{48}$$

Figure 1 plots a simulation of the SFNE in scenario (48). Here $V(\mathbf{x}(t_0)) = V(\mathbf{x}(t_f)) = 22.36$

V. In Summary

The present manuscript successfully generalised the results of [1]. Here we expanded the analysis from the capture the flag scenario $V_T = 0$, to the general three agent engagement scenario where $V_T < V_A < V_D$. To this end, new novel mechanisms were found to prove Theorem III.2, uncovering symmetries that would later be used in the developments in [13].

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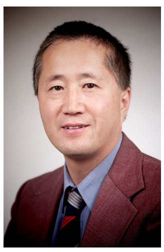

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Chapter 6

Unified optimality criteria for the Target-Attacker-Defender pursuit-evasion game

6.1 Contextual statement

This manuscript, under review in the European Journal of Control, discovers a surprising symmetry in the differential game of active target defence. We name this in the manuscript as *Target Symmetry*. A symmetry refers to a mapping/transformation of a state/object that preserves some quantity or property. In the context of this differential game, a symmetry refers to a transformation of the state that preserves the optimal headings of all agents. The symmetry uncovered in this manuscript is that if we where to change the position of the Target to be anywhere in-front of its optimal heading, the optimal headings of all three agents remain unchanged. This symmetry is utilised to characterise the state-feedback Nash equilibrium everywhere in the Target's escape set, unifying what was previously several disjoint methods.

Statement of Authorship

Title of Paper	Unified optimality criteria for the Target-Attacker-Defender pursuit-evasion game.
Publication Status	☐ Published ☐ Accepted for Publication ☑ Submitted for Publication ☐ Unpublished and Unsubmitted work written in manuscript style
Publication Details	Mammadov, K., Lim, C., & Shi, P. (2022). Unified optimality criteria for the Target-Attacker-Defender pursuit-evasion game. European Journal of Control, under review. [redacted]

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Contribution to the Paper	Selected research topic, conducted research, wrote manuscript, and acted as corresponding author.		
Overall percentage (%)	95%		
Certification:	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.		
Signature		Date	8/04/2022

Co-Author Contributions

By signing the Statement of Authorship, each author certifies that:

i. the candidate's stated contribution to the publication is accurate (as detailed above);

ii. permission is granted for the candidate in include the publication in the thesis; and

iii. the sum of all co-author contributions is equal to 100% less the candidate's stated contribution.

Name of Co-Author	Cheng-Chew Lim		
Contribution to the Paper	Helped examine and correct the manuscript.		
Signature		Date	8/04/2022

Name of Co-Author	Peng Shi		
Contribution to the Paper	Helped examine the manuscript.		
Signature		Date	8/04/2022

Unified optimality criteria for the Target-Attacker-Defender pursuit-evasion game

Kamal Mammadov, Cheng-Chew Lim and Peng Shi

Abstract

Here we study the TAD pursuit-evasion game. The game consists of two teams, team A (Attacker), and team T/D (Target and Defender). Team A, also known as the Attacker, or agent A, aims to minimise the separation between itself and the Target at time t_f; and team T/D, consisting of agents T and D work to maximise the separation distance. Time t_f is defined as first time agent A achieves point capture of either of the other two agents. Previous works in the literature identified three distinct optimality criteria to characterise the state-feedback Nash equilibrium of the TAD differential game of degree, depending upon the starting position of the Target relative to the AD-based Apollonius circle. The main contribution of the present manuscript is to introduce a new unifying paradigm given by the Critical Escape Trajectories Theorem; which simultaneously characterises the Target's escape set and the value function of the game in all regions, highlighting a deep underlying geometric connection between the TAD differential game of kind and the TAD differential game of degree. Leveraging previous results in the literature, the Critical Escape Trajectories Theorem is proved to be equivalent; and is utilised to develop an efficient algorithm for the computation of the state-feedback Nash equilibrium strategies.

Index Terms

Differential game theory, dynamic game theory, optimal state-feedback strategies, state-feedback Nash equilibrium, pursuit-evasion games.

I. INTRODUCTION

Pursuit-evasion differential game theory plays an important role in applications in aerospace guidance and control. Various pursuit-evasion games have been studied in the literature, here we cite [1], who studied the single-pursuer multiple-evader pursuit-evasion game, in which a fast pursuer aims to capture all evaders in minimum time and the evader team cooperate to maximise this time. Another example is the cooperative football differential game in which an attacker aims to reach as close as possible to the goal line before it's intercepted by one of two defenders, and the defenders cooperate to achieve the exact opposite goal. More closely related to the current work, [2] considered a 1-agent engagement scenario in which this single agent was tasked with intercepting a target moving at a constant speed in a straight line; Here it was proved that $\frac{V_T}{V_A} \sin \angle \mathbf{x}_A(t)\mathbf{x}_T(t)\mathbf{x}_T(t+\Delta t) \leq 1$, where V_T is the speed of the moving target, and V_A is the maximum speed of the agent, and $\mathbf{x}_T(t)$ and $\mathbf{x}_A(t)$ denote their positions respectively; is necessary and sufficient for there to exist a strategy for the agent to capture the target.

The focus of the present manuscript is the pursuit-evasion game most commonly named the Target-Attacker-Defender (TAD), or the Cooperative/Active Target Defence differential game. This is a continuous-time, zero-sum differential game consisting of two teams, team A and team T/D, and three agents, the Target, Attacker and Defender modelled with *simple motion*. The differential game terminates at the first time t_f the Attacker collides with one of the other two agents. The Attacker's goal is to minimise the distance between itself and the Target at time t_f, and the Target and Defender work as a team to maximise the aforementioned distance at time t_f. The TAD pursuit-evasion game is commonly motivated by visualising the Attacker as an explosive carrying aerial vehicle tasked with neutralising an evasive aerial Target; and the Target or another asset in its defence launches another drone (Defender) to intercept the Attacker.

The most general setting in which this pursuit-evasion game can be studied is under the assumption that $V_T \leq V_A \leq V_D$, here the Target is no faster than the Attacker, which in turn is no faster than the Defender. Clearly, if V_T were greater than V_A then the Target can easily escape capture from the Attacker, and the game may never terminate; therefore this case is always neglected. The reason for why the case $V_A > V_D$ is dismissed is more complex. If $V_A > V_D$ and $V_A > V_T$, then the Target and Defender may cooperate as a team to delay the capture of the Target, but they cannot prevent it, that is under optimal play, at termination time t_f, $\mathbf{x}_A(t_f) = \mathbf{x}_T(t_f)$. Since the payoff/reward function is defined as the distance between the Target and Attacker at termination time, there is no incentive for team T/D to do anything, therefore this case is degenerate.

A closely related, but different problem to the one considered in the present manuscript, is given in the work of [3]. Here the reward function was not defined as the distance between the Attacker and Target at termination time; rather it was defined as the distance between the Attacker and Defender at termination time. Here the Attacker seeks to maximise the distance, and the Target and Defender cooperative to minimise it. In this work the state-feedback Nash equilibrium was derived and the Attacker's winning region was determined.

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More directly related to the present manuscript, in the field of study of the TAD pursuit-evasion game, [4] made a significant contribution. They proved that under optimal play, the condition

$$V_A\overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} < V_D\overline{\mathbf{x}_A(t)\mathbf{x}_T(t)} + V_T\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)} \tag{1}$$

is necessary and sufficient for the Target to escape capture from the Attacker, that is, at termination time t_f, $\overline{\mathbf{x}_A(t_f)\mathbf{x}_T(t_f)} > 0$. Here the over-line notation denotes the euclidean distance between any two points.

Another prominent work on this topic includes [5]. In this manuscript three separate optimality principles were proposed to solve for the value of this game, depending upon the position of the Target. The value $V(\mathbf{x}(t))$ as a function of the state of the game $\mathbf{x}(t)$, denotes the expected payoff $\overline{\mathbf{x}_A(t_f)\mathbf{x}_T(t_f)}$ under the state-feedback Nash equilibrium (SFNE). In the case where the Target is located strictly inside the AD-based Apollonius circle, the following maximisation problem characterises the SFNE

$$V(\mathbf{x}(t)) = \max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} -\overline{\mathbf{x}_T(t)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}}, \tag{2}$$

whereas in the case where the Target is located strictly outside the AD-based Apollonius circle, the value is given by

$$V(\mathbf{x}(t)) = \min_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} \overline{\mathbf{x}_T(t)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}}. \tag{3}$$

Here $\mathcal{C}_{\mathcal{AD}}$ denotes the set of all points located on the circumference of the AD-based Apollonius circle. The third optimality principle identified in [5] defines the SFNE in the case where the Target is located on the boundary of the AD-based Apollonius circle. The description of the third optimality principle is lengthy, so we refer the reader to [5] for details.

In the case where the maximisation in (2) yields a negative number, we say that the Target cannot escape capture, i.e. $V(\mathbf{x}(t)) = 0$. Thus the combination of all three optimality principles fully characterises the SFNE of the TAD pursuit-evasion game.

It was proven in [6], that in two dimensions, in the case $V_A = V_D$, the value function is continuous and continuously differentiable over the Target's escape set, and that it satisfies the Hamilton-Jacobi-Isaacs equation everywhere in this set. Although the value function was not defined by (2)-(3), but rather by applying the two-sided Pontryagin's maximum principle (see [7] and [8]) to synthesise the state feedback strategies; in the process the value function is obtained. Just as in [5], the techniques used in [6] are split into the cases $\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} > \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}$ and $\overline{\mathbf{x}_T(t)\mathbf{x}_A(t)} < \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)}$; and involve heavy calculus.

Nonetheless the validity of these three separate optimality principles are quite well established. In [5], quite convincing arguments were given, and in the works of [9] and [10], direct rigours proofs of formula (3) were given, based on the theory of upper and lower values (see [11] and [12]); first in the case $V_A = V_D$, then in the general case $V_A < V_D$. These proofs are also valid in arbitrary dimensions.

Using Pontryagin's maximum principle, in the work of [5], it was proven that in the Target's escape set, under optimal play, the headings of the Attacker, the Target, and the Defender are constant. Therefore under optimal play every agent moves in straight lines with constant speed from the start until termination. However the SFNE cannot be uniquely determined from this principle alone, because there still remains too many degrees of freedom (i.e. which direction does each agent move in). In [5], it was argued that formulas (2)-(3) provide the correct angles for which each agent must move in.

The present manuscript brings to light a fundamental symmetry previously unacknowledged in the literature. We name it *Target Symmetry*; *Target Symmetry* elucidates the property that in the Target's escape set, between any time increment t to $t + \Delta t$, if the Attacker and Defender for whatever reason decided not to move, and the Target moved in its optimal heading, then the optimal headings of all three agents at time $t + \Delta t$ are the same as they were before at time t.

Target Symmetry is best illustrated in Figure 4 in Section III. Here the blue and red dots indicate the position of the Attacker and Defender, respectively. To highlight *Target Symmetry*, the position of the Target is not given. Rather, it is considered 'variable' in this diagram. The region inside the blue curve is named the Non-escape Region, if the Target's position were anywhere inside here then $V = 0$; outside this region $V > 0$. Emanating from the boundary of the Non-escape Region are rays. If the Target were located at the initial point of one of these rays, then the ray reveals the optimal path for the Target, and the intersection between the ray and the red circle pinpoints the location were all three agents would move towards until collision.

Thus the value of the TAD pursuit-evasion game can be given by

$$V(\mathbf{x}(t)) = \overline{\mathbf{x}_T(t)\mathbf{E}_c(\mathbf{x}(t))} \tag{4}$$

where $\mathbf{E}_c(\mathbf{x}(t))$ denotes the initial point of the ray containing $\mathbf{x}_T(t)$. Formula (4) is valid everywhere in the Target's escape set, thereby unifying (2)-(3). The contribution of the present manuscript is to introduce and prove a new unifying paradigm, based on *Target Symmetry*, to express the value, the optimal strategies, and reveal a simple analytical technique to solve for the state-feedback Nash equilibrium of the Target-Attacker-Defender pursuit-evasion game. This paradigm reveals a deep underlying geometric connection between the game of kind and the game of degree, and the techniques used here are much simpler than the ones developed in [6] using the two-sided Pontryagin's maximum principle. The methods developed in the present manuscript are not a disjoint collection of techniques, unlike [5] and [6] which have different methods for different regions in the Target's escape set.

The remainder of the paper is organised as follows. Section 2 lists the notation and terminology used throughout the manuscript, provides the mathematical definition of the TAD pursuit-evasion game, and presents some preliminary results on the game of kind. Section 3 contains three subsections. The opening of Section 3 chronicles the unified optimality principle given by the Critical Escape Trajectories Theorem, the next subsection reviews the disjointed optimality principles already well established in the literature, the final subsection proves the Critical Escape Trajectories Theorem by verifying its equivalence to the aforementioned disjointed optimality principles in each of their separate regions. Based on this new unified paradigm, Section 4 describes explicitly the mathematical formulas used to calculate the state-feedback optimal strategies and value of the TAD pursuit-evasion game.

II. Preliminaries

A. Notation and terminology

The notation used throughout the manuscript is listed as follows. Given any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

- $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ denotes the set of all positive real numbers.
- $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ denotes the set of all non-negative real numbers.
- $\mathbf{u} \cdot \mathbf{v}$ denotes the dot product.
- $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.
- $\overrightarrow{\mathbf{uv}} = \mathbf{v} - \mathbf{u}$.
- $\overline{\mathbf{uv}} = ||\overrightarrow{\mathbf{uv}}||$ denotes the euclidean distance between $\mathbf{u}$ and $\mathbf{v}$.
- $\angle\mathbf{uvw}$ denotes the angle between vectors $\overrightarrow{\mathbf{vu}}$ and $\overrightarrow{\mathbf{vw}}$; that is $\overrightarrow{\mathbf{vu}} \cdot \overrightarrow{\mathbf{vw}} = \overline{\mathbf{vu}}\ \overline{\mathbf{vw}} \cos\angle\mathbf{uvw}$, where $0 \leq \angle\mathbf{uvw} \leq \pi$.
- $\dot{\mathbf{u}}(t)$ denotes the time derivative $\frac{d}{dt}\mathbf{u}(t)$.
- $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real part and complex part, respectively, of any complex number z.
- z^* denotes the complex conjugate of z.

Team A, and agent A both refer to the Attacker. Agent T and agent D denotes the Target and Defender, respectively, and team T/D denotes the team comprising of those two agents. SFNE is the abbreviation for state-feedback Nash equilibrium.

B. Problem formulation

This section provides a rigorous mathematical formulation of the TAD game. The TAD pursuit-evasion game is a zero-sum differential game consisting of team A, and team T/D. The state of the differential game is specified by $\mathbf{x}(t) = (\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t))$, where $\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t) \in \mathbb{R}^n$ for any integer $n \geq 2$, specifies the location of agents A, D and T respectively. Provided an initial state $\mathbf{x}_0$, the dynamics from starting time t_0 to final time t_f is given by

$$(\dot{\mathbf{x}}_A(t), \dot{\mathbf{x}}_D(t), \dot{\mathbf{x}}_T(t)) = (V_A\mathbf{u}_A(t), V_D\mathbf{u}_D(t), V_T\mathbf{u}_T(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{5}$$

where $V_A, V_D, V_T \in \mathbb{R}_0^+$; $\mathbf{u}_D(t), \mathbf{u}_T(t)$ are the control inputs of team T/D, and $\mathbf{u}_A(t)$ is the control input of team A. At every time t, both teams make choices for their control inputs with knowledge of the current state $\mathbf{x}(t)$ of the game. The control vectors are constrained to be no larger than unit vectors:

$$\|\mathbf{u}_A(t)\|, \|\mathbf{u}_D(t)\|, \|\mathbf{u}_T(t)\| \leq 1 \tag{6}$$

The final time t_f is the earliest time obeying either (7) or (8)

$$\mathbf{x}_A(t_f) = \mathbf{x}_D(t_f), \tag{7}$$

$$\mathbf{x}_A(t_f) = \mathbf{x}_T(t_f). \tag{8}$$

In that time horizon $[t_0, t_f]$, the reward function is given by

$$J(\mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot), \mathbf{u}_T(\cdot), \mathbf{x}_0) = \overline{\mathbf{x}_T(t_f)\mathbf{x}_A(t_f)}. \tag{9}$$

Team A selects control laws to minimise J, and team T/D selects control inputs to maximise J. Let $\alpha = \frac{V_T}{V_A}$, $\gamma = \frac{V_D}{V_A}$. We assume the relative speeds of the agents satisfy

$$V_T < V_A < V_D. \tag{10}$$

Denote $\mathbf{c}_{AD}$ and r_{AD} as the centre and radius of the AD-based Apollonius circle

$$\mathbf{c}_{AD} = \frac{V_D^2}{V_D^2 - V_A^2}\mathbf{x}_A(t) + \frac{V_A^2}{V_A^2 - V_D^2}\mathbf{x}_D(t), \tag{11a}$$

$$r_{AD} = \frac{V_A V_D}{|V_A^2 - V_D^2|}\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}. \tag{11b}$$

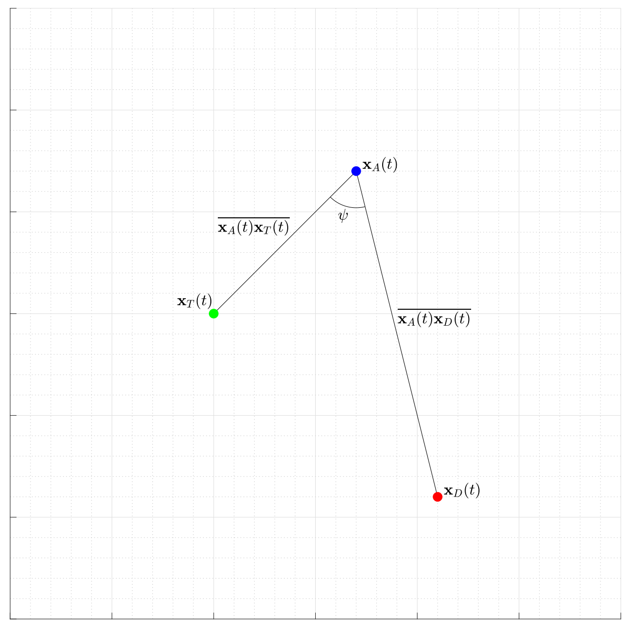

Fig. 1: Distances $\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}$, $\overline{\mathbf{x}_A(t)\mathbf{x}_T(t)}$ and angle ψ define the state of the game.

The interior of this circle represents the set of all points in $\mathbb{R}^n$ the Attacker can reach before the Defender. Similarly, let $\mathbf{c}_{AT}(\mathbf{p})$ and $r_{AT}(\mathbf{p})$ denote the centre and radius of the AT-based Apollonius circle, as a function of the position $\mathbf{p}$ of the Target

$$\mathbf{c}_{AT}(\mathbf{p}) = \frac{V_T^2}{V_T^2 - V_A^2}\mathbf{x}_A(t) + \frac{V_A^2}{V_A^2 - V_T^2}\mathbf{p}, \tag{12a}$$

$$r_{AT}(\mathbf{p}) = \frac{V_A V_T}{|V_A^2 - V_T^2|}\overline{\mathbf{x}_A(t)\mathbf{p}}. \tag{12b}$$

Thus the centre and radius of the AT-based Apollonius circle is given by $\mathbf{c}_{AT}(\mathbf{x}_T(t))$ and $r_{AT}(\mathbf{x}_T(t))$, respectively. To illustrate *Target Symmetry*, it is best to depict the position of the Target as 'variable'. Throughout the manuscript the following numerical example:

$$\mathbf{x}_A(t) = \begin{bmatrix} 2 \\ 12 \end{bmatrix}, \quad \mathbf{x}_D(t) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \quad (V_T, V_A, V_D) = (0.5, 1, 1.2), \tag{13}$$

is used frequently to display diagrams of key concepts.[1]

C. Game of kind

The Target-Attacker-Defender differential game of kind is the puzzle of unearthing necessary and sufficient conditions under which the value of the TAD differential game is equal to zero; in other words to find the Target's escape set. [5] determined a formula for the critical speed ratio α_c so that the Target can escape capture if and only if $\frac{V_T}{V_A} > \alpha_c$. [4] completely characterised the solution to the game of kind with the simple linear formula (1).

However for the purposes of illustrating the unified optimality criteria given in the subsequent section, it is more useful to characterise the Target's escape region as a function of the starting position of the Target. To that end formulas (14) and (15) parameterises the escape region as a function of two quantities $\overline{\mathbf{x}_A(t)\mathbf{p}}$ and $\angle \mathbf{p}\mathbf{x}_A(t)\mathbf{x}_D(t)$, where $\mathbf{p}$ is the arbitrary position of the Target.

Definition II.1 (Non-escape Region)**.** The Non-escape Region $\mathcal{N}(\mathbf{x}(t))$ is the set of all points $\mathbf{p}$ in $\mathbb{R}^n$ satisfying (14). On the limit $V_A = V_D$ the Non-escape Region is rather defined by (15).

$$\overline{\mathbf{x}_A(t)\mathbf{p}} \leq \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{1-\gamma^2}\left(\cos\psi + \alpha\gamma - \sqrt{(\cos\psi + \alpha\gamma)^2 - (1-\alpha^2)(1-\gamma^2)}\right) \tag{14}$$

$$\overline{\mathbf{x}_A(t)\mathbf{p}} \leq \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{2}\,\frac{1-\alpha^2}{\cos\psi + \alpha} \tag{15}$$

[1]Note that the position of the Target is not specified as these diagrams consider its position to be variable.

where $\psi = \angle \mathbf{p}\mathbf{x}_A(t)\mathbf{x}_D(t)$.

In actuality, the value of the game can be parameterised in a coordinate-independent manner by the distances $\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}$, $\overline{\mathbf{x}_A(t)\mathbf{x}_T(t)}$ and angle ψ, as shown in Figure 1. Assuming that the differential game is well posed, meaning that under optimal play the Target is captured by the Attacker if and only if the AT-based Apollonius circle is completely encapsulated within the AD-based Apollonius circle, we may prove the following.

Lemma II.2. *Under optimal play the Target escapes capture if and only if* $\mathbf{x}_T(t) \notin \mathcal{N}(\mathbf{x}(t))$.

Proof. The proof is split into the two cases $V_A < V_D$ and $V_A = V_D$.

If $V_T < V_A < V_D$ and supposing the differential game is well posed, the threshold between the Target's escape set and the Target's non-escape set is given by

$$\overline{\mathbf{c}_{AD}\mathbf{c}_{AT}(\mathbf{p})} + r_{AT}(\mathbf{p}) = r_{AD}. \tag{16}$$

Here the set of points $\mathbf{p} \in \mathbb{R}^n$ satisfying (16) define the boundary of the Non-escape Region. Using the property that $\mathbf{c}_{AD} - \mathbf{x}_A = \frac{V_A^2}{V_A^2 - V_D^2}\overrightarrow{\mathbf{x}_A\mathbf{x}_D}$ and $\mathbf{c}_{AT}(\mathbf{p}) - \mathbf{x}_A = \frac{V_A^2}{V_A^2 - V_T^2}\overrightarrow{\mathbf{x}_A\mathbf{p}}$, we may express $\overline{\mathbf{c}_{AD}\mathbf{c}_{AT}(\mathbf{p})}$ as a function of ψ as follows:

$$\begin{aligned}
\overline{\mathbf{c}_{AD}\mathbf{c}_{AT}(\mathbf{p})}^2 &= \big(\mathbf{c}_{AD} - \mathbf{c}_{AT}(\mathbf{p})\big) \cdot \big(\mathbf{c}_{AD} - \mathbf{c}_{AT}(\mathbf{p})\big) \\
&= \big(\mathbf{c}_{AD} - \mathbf{x}_A - \mathbf{c}_{AT}(\mathbf{p}) + \mathbf{x}_A\big) \cdot \big(\mathbf{c}_{AD} - \mathbf{x}_A - \mathbf{c}_{AT}(\mathbf{p}) + \mathbf{x}_A\big) \\
&= \big(\frac{V_A^2}{V_A^2 - V_D^2}\overrightarrow{\mathbf{x}_A\mathbf{x}_D} - \frac{V_A^2}{V_A^2 - V_T^2}\overrightarrow{\mathbf{x}_A\mathbf{p}}\big) \cdot \big(\frac{V_A^2}{V_A^2 - V_D^2}\overrightarrow{\mathbf{x}_A\mathbf{x}_D} - \frac{V_A^2}{V_A^2 - V_T^2}\overrightarrow{\mathbf{x}_A\mathbf{p}}\big) \\
&= \big(\frac{V_A^2}{V_A^2 - V_D^2}\big)^2\overline{\mathbf{x}_A\mathbf{x}_D}^2 - 2\big(\frac{V_A^2}{V_A^2 - V_D^2}\big)\big(\frac{V_A^2}{V_A^2 - V_T^2}\big)\overline{\mathbf{x}_A\mathbf{x}_D}\ \overline{\mathbf{x}_A\mathbf{p}}\cos\psi + \big(\frac{V_A^2}{V_A^2 - V_T^2}\big)^2\overline{\mathbf{x}_A\mathbf{p}}^2.
\end{aligned}$$

Re-arranging (16) we obtain

$$\overline{\mathbf{c}_{AD}\mathbf{c}_{AT}(\mathbf{p})}^2 - (r_{AD} - r_{AT}(\mathbf{p}))^2 = 0 \tag{17}$$

where

$$\begin{aligned}
(r_{AD} - r_{AT}(\mathbf{p}))^2 &= \big(\frac{V_A V_D}{V_D^2 - V_A^2}\overline{\mathbf{x}_A\mathbf{x}_D} - \frac{V_A V_T}{V_A^2 - V_T^2}\overline{\mathbf{x}_A\mathbf{p}}\big)^2 \\
&= \big(\frac{V_A V_D}{V_D^2 - V_A^2}\big)^2\overline{\mathbf{x}_A\mathbf{x}_D}^2 - 2\big(\frac{V_A V_D}{V_D^2 - V_A^2}\big)\big(\frac{V_A V_T}{V_A^2 - V_T^2}\big)\overline{\mathbf{x}_A\mathbf{x}_D}\ \overline{\mathbf{x}_A\mathbf{p}} + \big(\frac{V_A V_T}{V_A^2 - V_T^2}\big)^2\overline{\mathbf{x}_A\mathbf{p}}^2 \\
&= \frac{V_D^2}{V_A^2}\big(\frac{V_A^2}{V_A^2 - V_D^2}\big)^2\overline{\mathbf{x}_A\mathbf{x}_D}^2 + 2\frac{V_T V_D}{V_A^2}\big(\frac{V_A^2}{V_A^2 - V_D^2}\big)\big(\frac{V_A^2}{V_A^2 - V_T^2}\big)\overline{\mathbf{x}_A\mathbf{x}_D}\ \overline{\mathbf{x}_A\mathbf{p}} + \frac{V_T^2}{V_A^2}\big(\frac{V_A^2}{V_A^2 - V_T^2}\big)^2\overline{\mathbf{x}_A\mathbf{p}}^2.
\end{aligned}$$

Substituting the expressions derived for $\overline{\mathbf{c}_{AD}\mathbf{c}_{AT}(\mathbf{p})}^2$ and $(r_{AD} - r_{AT}(\mathbf{p}))^2$ into (17) we obtain

$$\big(\frac{V_A^2}{V_A^2 - V_D^2}\big)^2\overline{\mathbf{x}_A\mathbf{x}_D}^2\big(\frac{V_A^2 - V_D^2}{V_A^2}\big) - 2\big(\frac{V_A^2}{V_A^2 - V_D^2}\big)\big(\frac{V_A^2}{V_A^2 - V_T^2}\big)\overline{\mathbf{x}_A\mathbf{x}_D}\ \overline{\mathbf{x}_A\mathbf{p}}\big(\cos\psi + \frac{V_T V_D}{V_A^2}\big) + \big(\frac{V_A^2}{V_A^2 - V_T^2}\big)^2\overline{\mathbf{x}_A\mathbf{p}}^2\big(\frac{V_A^2 - V_T^2}{V_A^2}\big) = 0,$$

which simplifies to

$$\frac{V_A^2}{V_A^2 - V_D^2}\overline{\mathbf{x}_A\mathbf{x}_D}^2 - 2\big(\frac{V_A^2}{V_A^2 - V_D^2}\big)\big(\frac{V_A^2}{V_A^2 - V_T^2}\big)\overline{\mathbf{x}_A\mathbf{x}_D}\ \overline{\mathbf{x}_A\mathbf{p}}\big(\cos\psi + \frac{V_T V_D}{V_A^2}\big) + \frac{V_A^2}{V_A^2 - V_T^2}\overline{\mathbf{x}_A\mathbf{p}}^2 = 0,$$

and further simplifies to

$$\overline{\mathbf{x}_A\mathbf{p}}^2 - 2\Big(\frac{\cos\psi + \frac{V_T}{V_A}\frac{V_D}{V_A}}{1 - (\frac{V_D}{V_A})^2}\Big)\overline{\mathbf{x}_A\mathbf{x}_D}\ \overline{\mathbf{x}_A\mathbf{p}} + \Big(\frac{1 - (\frac{V_T}{V_A})^2}{1 - (\frac{V_D}{V_A})^2}\Big)\overline{\mathbf{x}_A\mathbf{x}_D}^2 = 0.$$

This is a quadratic equation for $\overline{\mathbf{x}_A\mathbf{p}}$. Since $\frac{1-(\frac{V_T}{V_A})^2}{1-(\frac{V_D}{V_A})^2} < 0$, the only non-negative solution for $\overline{\mathbf{x}_A\mathbf{p}}$ is given by

$$\overline{\mathbf{x}_A\mathbf{p}} = \overline{\mathbf{x}_A\mathbf{x}_D}\left(\frac{\cos\psi + \frac{V_T}{V_A}\frac{V_D}{V_A}}{1 - (\frac{V_D}{V_A})^2} + \sqrt{\Big(\frac{\cos\psi + \frac{V_T}{V_A}\frac{V_D}{V_A}}{1 - (\frac{V_D}{V_A})^2}\Big)^2 - \frac{1 - (\frac{V_T}{V_A})^2}{1 - (\frac{V_D}{V_A})^2}}\right). \tag{18}$$

Equation (18) determines the boundary of the Non-escape Region; cases where the distance between the Target and Attacker is less than the right-hand side is where the Target cannot escape and vice-versa. With $\alpha = \frac{V_T}{V_A}$ and $\gamma = \frac{V_D}{V_A}$, this is also equivalent to

$$\overline{\mathbf{x}_A(t)\mathbf{p}} = \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{1 - \gamma^2}\left(\cos\psi + \alpha\gamma - \sqrt{(\cos\psi + \alpha\gamma)^2 - (1 - \alpha^2)(1 - \gamma^2)}\right).$$

This completes the proof for the case $V_T < V_A < V_D$. In the case where $V_T = V_A < V_D$, clearly the Target can escape capture by simply moving away from the Attacker, thus $\overline{\mathbf{x}_A(t)\mathbf{p}} = 0$. Indeed if we substitute $\alpha = 1$ into (14), this gives us 0. Thus the formula also holds more generally in $V_T \leq V_A < V_D$.

In the case where $V_T < V_A = V_D$ and supposing the differential game is well posed, the threshold between the Target's escape set and the Target's non-escape set is given by

$$\overline{\mathbf{x}_A\mathbf{c}_{AT}(\mathbf{p})}\cos\psi + r_{AT}(\mathbf{p}) = \frac{1}{2}\overline{\mathbf{x}_A\mathbf{x}_D}. \tag{19}$$

Here the set of points $\mathbf{p} \in \mathbb{R}^n$ satisfying (19) define the boundary of the Non-escape Region. We may express $\overline{\mathbf{x}_A\mathbf{c}_{AT}(\mathbf{p})}$ as a function of ψ as follows:

$$\begin{aligned}
\overline{\mathbf{x}_A\mathbf{c}_{AT}(\mathbf{p})} &= \sqrt{\big(\mathbf{c}_{AT}(\mathbf{p}) - \mathbf{x}_A\big)\cdot\big(\mathbf{c}_{AT}(\mathbf{p}) - \mathbf{x}_A\big)\big)} \\
&= \sqrt{\big(\frac{V_A^2}{V_A^2 - V_T^2}\overrightarrow{\mathbf{x}_A\mathbf{p}}\big)\cdot\big(\frac{V_A^2}{V_A^2 - V_T^2}\overrightarrow{\mathbf{x}_A\mathbf{p}}\big)} \\
&= \big(\frac{V_A^2}{V_A^2 - V_T^2}\big)\overline{\mathbf{x}_A\mathbf{p}}.
\end{aligned}$$

Thus the threshold (19) is given by

$$\big(\frac{V_A^2}{V_A^2 - V_T^2}\big)\overline{\mathbf{x}_A\mathbf{p}}\cos\psi + \frac{V_AV_T}{V_A^2 - V_T^2}\overline{\mathbf{x}_A\mathbf{p}} = \frac{1}{2}\overline{\mathbf{x}_A\mathbf{x}_D},$$

solving for $\overline{\mathbf{x}_A\mathbf{p}}$ we obtain

$$\overline{\mathbf{x}_A\mathbf{p}} = \frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{2}\frac{1 - (\frac{V_T}{V_A})^2}{\cos\psi + \frac{V_T}{V_A}}.$$

Which is equivalent to the expression given in (15) with $\alpha = \frac{V_T}{V_A}$. This completes the proof for the case $V_T < V_A = V_D$. In the case where $V_T = V_A$, so long as $\mathbf{x}_T(t) \neq \mathbf{x}_A(t)$, the Target can escape capture by simply moving in a straight line away from the Attacker with speed V_T; thus in this case $\overline{\mathbf{x}_A(t)\mathbf{p}} = 0$. Plugging the value $V_T = V_A$ into (15), we obtain the same value of zero, thus (15) holds more generally in $V_T \leq V_A = V_D$. □

Later in the manuscript, Lemma II.2 will be proved as a simple corollary of the Critical Escape Trajectories Theorem, without having to assume the differential game is well posed.

Figure 2 displays the AD-based Apollonius circle in red and the boundary of the Non-escape Region $\mathcal{N}(\mathbf{x}(t))$ in blue for the example given by (13). The blue teardrop shape is defined by inequality (14) holding with equality; and the blue and red dots represent the position of the Attacker and Defender respectively. Under optimal play, if the Target's starting position is anywhere inside the blue teardrop shape shown in Figure 2, then the Target will be captured by the Attacker; otherwise the Target escapes capture.

In Figure 2, the interior of the AD-based Apollonius circle defines all the points in which if the Attacker and Defender where to move in straight lines at there respective maximum speeds towards that point, the Attacker would reach there first; and vice-versa. Note however the Target can still escape if it were located between the red and blue curves; this is because the Target's speed $V_T = 0.5 > 0$ in example (13).

Figure 3 displays the AD-based Apollonius circle in red and the boundary of the Non-escape Region $\mathcal{N}(\mathbf{x}(t))$ in green for the example given by (13), but instead of $V_A = 1$, we have $V_A = 1.2 = V_D$. In this case the green curve is defined by inequality (15) holding with equality; and the AD-based Apollonius circle on the limit $V_A = V_D$ is just a plane defining the halfway point between the Attacker and Defender.

The next lemma shows that formula (15) can be derived as a special case of formula (14) using L'hôpital's rule.

Lemma II.3.

$$\begin{aligned}
\lim_{\gamma\to 1}\frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{1-\gamma^2}&\Big(\cos\psi + \alpha\gamma - \sqrt{(\cos\psi + \alpha\gamma)^2 - (1-\alpha^2)(1-\gamma^2)}\Big) \\
&= \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{2}\frac{1-\alpha^2}{\cos\psi + \alpha}
\end{aligned}$$

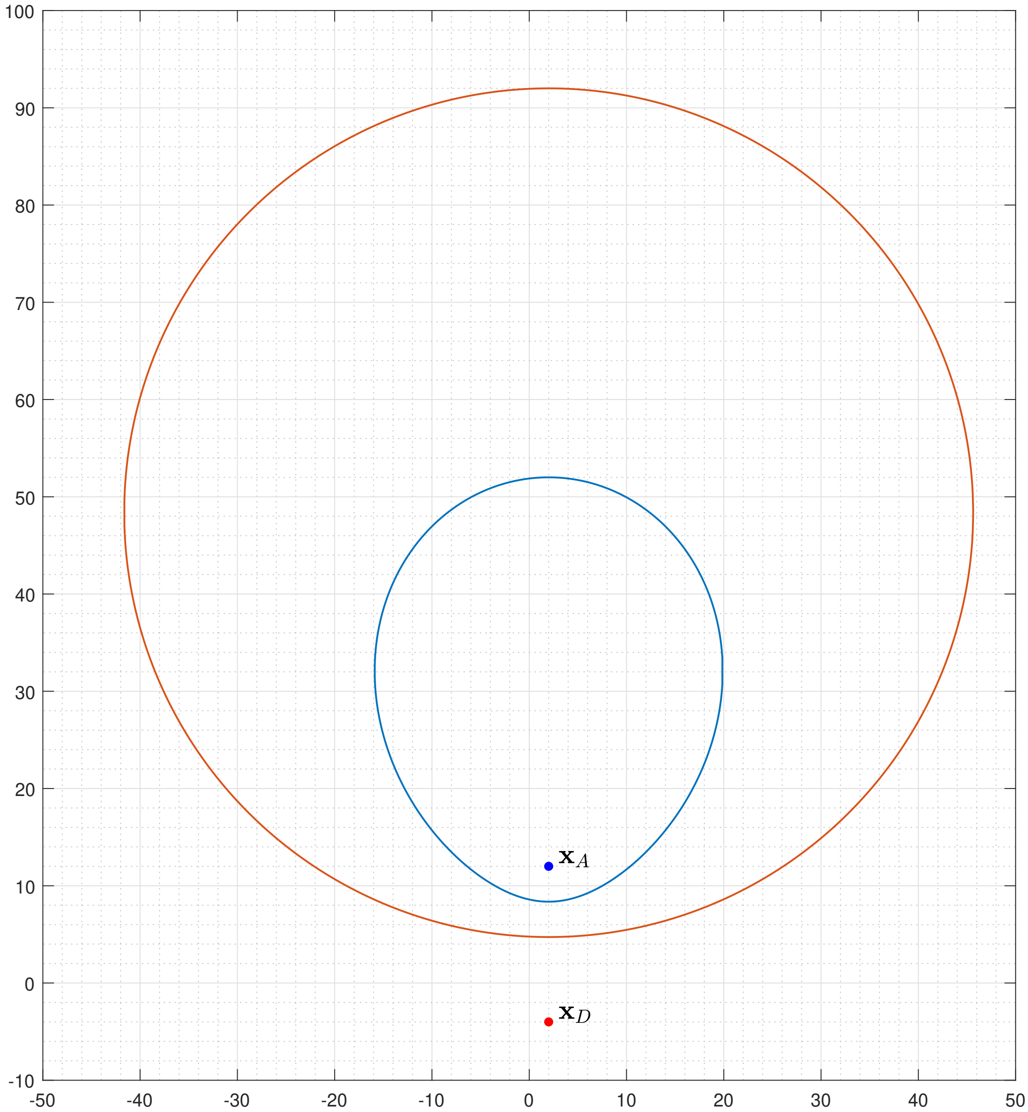

Fig. 2: Blue Non-escape Region and red AD-based Apollonius circle.

Proof. Note that at $\gamma = 1$, formula (14) yields $\frac{0}{0}$. To resolve this case we apply L'hôpital's rule:

$$\begin{aligned}
&\lim_{\gamma\to 1} \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{1-\gamma^2}\left(\cos\psi + \alpha\gamma - \sqrt{(\cos\psi+\alpha\gamma)^2 - (1-\alpha^2)(1-\gamma^2)}\right) \\
&= \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{-2\gamma}\left(\alpha - \frac{1}{2}\frac{2\alpha(\cos\psi+\alpha\gamma)+2\gamma(1-\alpha^2)}{\sqrt{(\cos\psi+\alpha\gamma)^2-(1-\alpha^2)(1-\gamma^2)}}\right)\Bigg|_{\gamma=1} \\
&= \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{-2}\left(\alpha - \frac{\alpha\cos\psi + \alpha^2 + 1 - \alpha^2}{\sqrt{(\cos\psi+\alpha)^2 - 0}}\right) \\
&= \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{2}\left(\frac{-\alpha(\cos\psi+\alpha)}{\cos\psi+\alpha} + \frac{\alpha\cos\psi+1}{\cos\psi+\alpha}\right) \\
&= \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{2}\frac{1-\alpha^2}{\cos\psi+\alpha}.
\end{aligned}$$

□

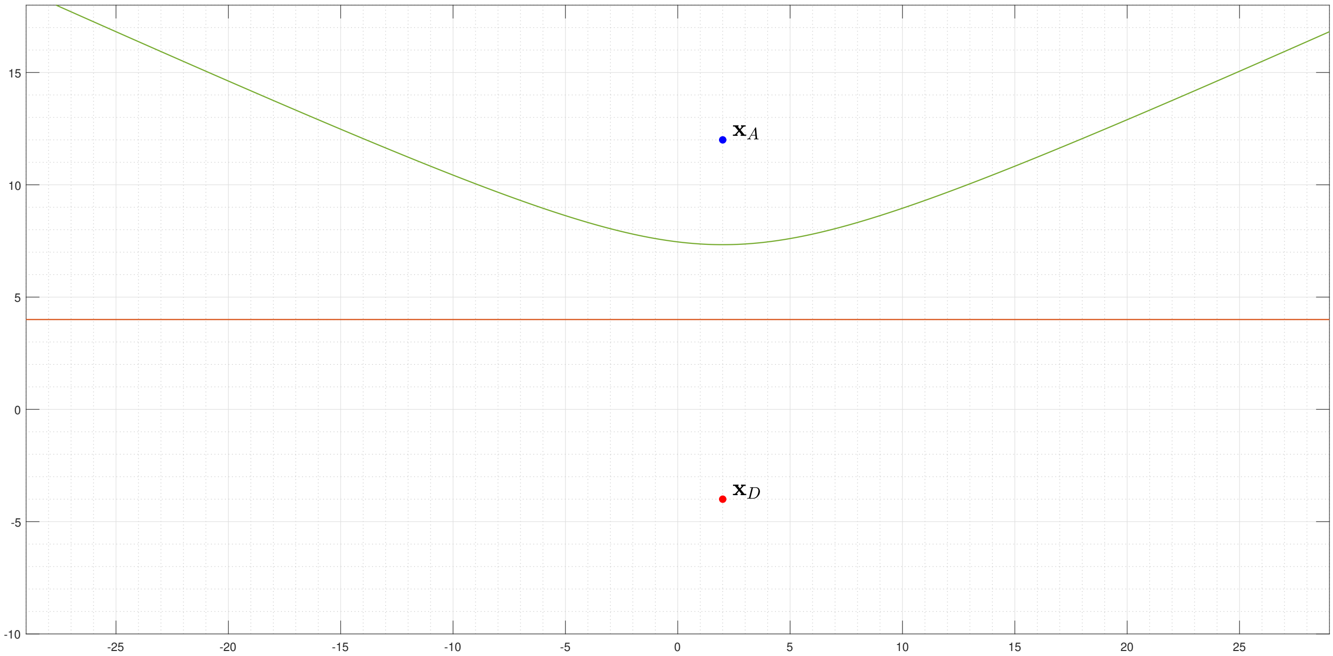

Fig. 3: Green Non-escape Region and red AD-based Apollonius circle if $V_A = V_D$.

For the remainder of the manuscript we define the function $l(\psi)$ given by

$$l(\psi) = \begin{cases} \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{2} \frac{1-\alpha^2}{\cos\psi + \alpha} & \text{if } \; V_T < V_A = V_D \\ \frac{\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}}{1-\gamma^2} \left(\cos\psi + \alpha\gamma - \sqrt{(\cos\psi + \alpha\gamma)^2 - (1-\alpha^2)(1-\gamma^2)} \right) & \text{else } \; V_T < V_A < V_D \end{cases} \tag{20}$$

to parameterise the boundary of the Non-escape Region. In the next section we state the unified optimality criteria, and prove its equivalence to the three disjointed optimality principles identified in the literature.

III. Unified optimality criteria

The unified optimality criteria completely characterises the state-feedback Nash equilibrium of the Target-Attacker-Defender pursuit evasion game; furthermore, it provides necessary and sufficient conditions under which the Target cannot escape capture from the Attacker under optimal play; thereby simultaneously solving the differential game of degree and kind.

To construct the unified optimality criteria, the definition of a Critical Escape Trajectory must be provided. Below the terminology *boundary of the Non-Escape Region* refers to the Non-Escape Region defined in Definition II.1 but where the inequalities hold with equality.

Definition III.1 (Critical Escape Trajectory)**.** A Critical Escape Trajectory $\zeta(\mathbf{E}_c)$, defined for any point $\mathbf{E}_c$ on the boundary of the Non-escape Region, is the following ray

$$\zeta(\mathbf{E}_c) = \left\{ \mathbf{p} \in \mathbb{R}^n \,\middle|\, \mathbf{p} = \mathbf{E}_c + \delta\big(\mathbf{I}_c(\mathbf{E}_c) - \mathbf{E}_c\big) \;\text{ for some }\; \delta \geq 0 \right\} \tag{21}$$

where $\mathbf{I}_c(\mathbf{E}_c)$ is the corresponding Critical Collision Point defined by (22). On the limit $V_A = V_D$ the Critical Collision Point is rather defined by (23).

$$\mathbf{I}_c(\mathbf{E}_c) = \mathbf{c}_{AT}(\mathbf{E}_c) + r_{AT}(\mathbf{E}_c) \frac{\mathbf{c}_{AT}(\mathbf{E}_c) - \mathbf{c}_{AD}}{\|\mathbf{c}_{AT}(\mathbf{E}_c) - \mathbf{c}_{AD}\|}, \tag{22}$$

$$\mathbf{I}_c(\mathbf{E}_c) = \mathbf{c}_{AT}(\mathbf{E}_c) + r_{AT}(\mathbf{E}_c) \frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|}. \tag{23}$$

Consider again the numerical example of (13). Figure 4 depicts roughly eighty examples of Critical Escape Trajectories as dashed lines, superimposed upon Figure 2. The meaning behind a Critical Escape Trajectory is as follows. If the Target were located on the boundary of the Non-escape Region, there exists a unique path for the Target to follow to escape capture; this path defines the ray ζ for that point on the boundary. Hence the name *Critical Escape Trajectory*, *Escape* since the Target does

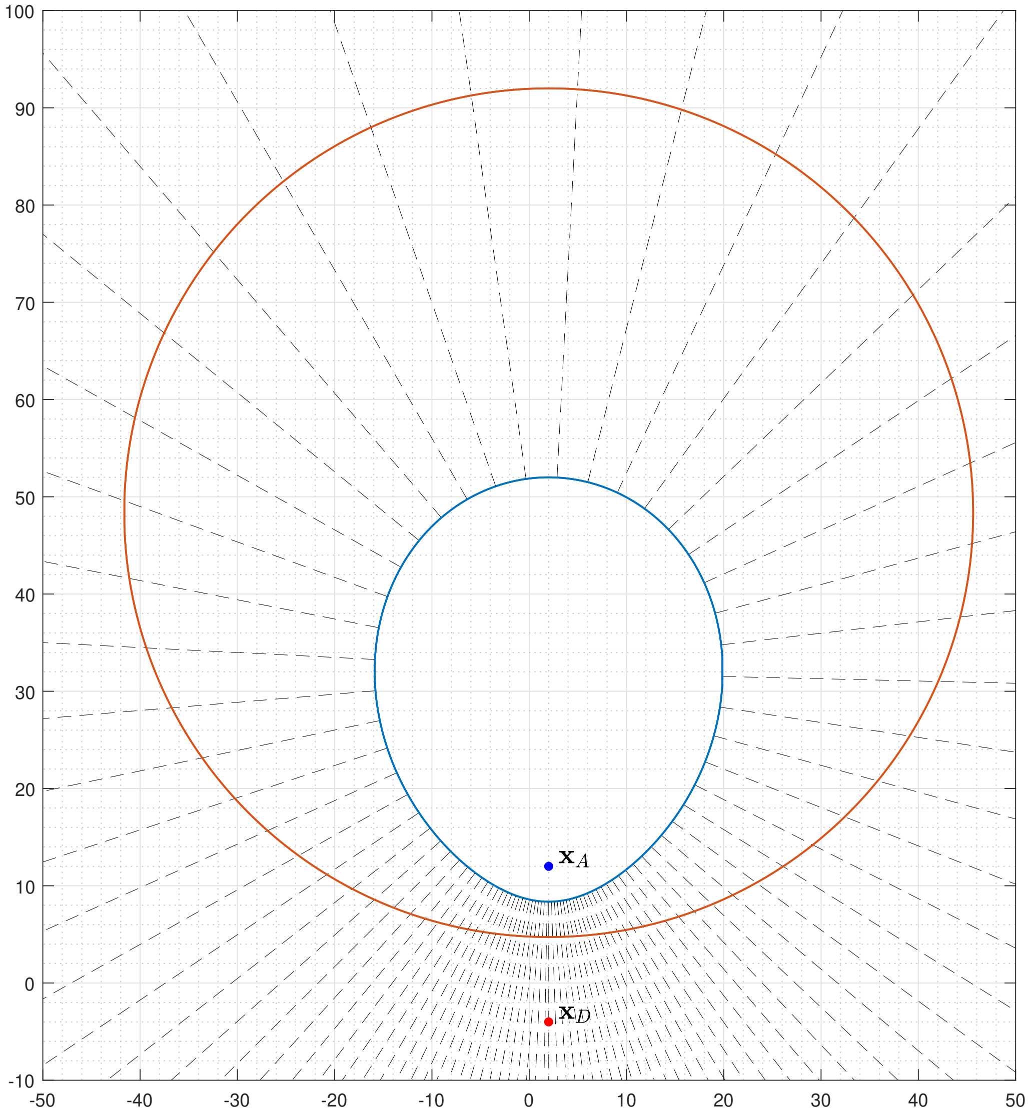

Fig. 4: Blue Non-escape Region, red AD-based Apollonius circle and dashed lines depict Critical Escape Trajectories.

escape, but *Critical* since the Target escapes capture by only an infinitesimal distance. The Critical Collision Point $\mathbf{I}_c(\mathbf{E}_c)$ is the point on the AD-based Apollonius circle in which all three agents collide if the Target's starting position is $\mathbf{E}_c$.

Similarly for the case $V_A = V_D$, Figure 5 depicts roughly fifty examples of Critical Escape Trajectories as dashed lines, superimposed upon Figure 3. The initial point of each ray on the boundary of the Non-escape Region is a Critical Escape Point $\mathbf{E}_c$, and the intersection of the ray with the red curve is the corresponding Critical Collision Point $\mathbf{I}_c(\mathbf{E}_c)$.

Before proceeding with the unified optimality criteria, we present a lemma remarking the connection between formulas (22) and (23).

Lemma III.2.

$$\lim_{\gamma \to 1^+} \mathbf{c}_{AT}(\mathbf{E}_c) + r_{AT}(\mathbf{E}_c)\frac{\mathbf{c}_{AT}(\mathbf{E}_c) - \mathbf{c}_{AD}}{\|\mathbf{c}_{AT}(\mathbf{E}_c) - \mathbf{c}_{AD}\|} = \mathbf{c}_{AT}(\mathbf{E}_c) + r_{AT}(\mathbf{E}_c)\frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|}.$$

Proof. Substituting formulas (11) and (12) for the AD and AT based Apollonius circles respectively we obtain

$$\mathbf{c}_{AT}(\mathbf{E}_c) - \mathbf{c}_{AD} = \frac{V_T^2}{V_T^2 - V_A^2}\mathbf{x}_A - \frac{V_A^2}{V_T^2 - V_A^2}\mathbf{E}_c - \frac{V_D^2}{V_D^2 - V_A^2}\mathbf{x}_A - \frac{V_A^2}{V_A^2 - V_D^2}\mathbf{x}_D.$$

We may parameterise $\mathbf{E}_c$ in terms of an angle $\psi = \angle \mathbf{E}_c \mathbf{x}_A(t) \mathbf{x}_D(t)$ and a unit vector $\hat{\mathbf{n}}$ as follows:

$$\mathbf{E}_c = \mathbf{x}_A(t) + l(\psi)\cos\psi \frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|} + l(\psi)\sin\psi\ \hat{\mathbf{n}},$$

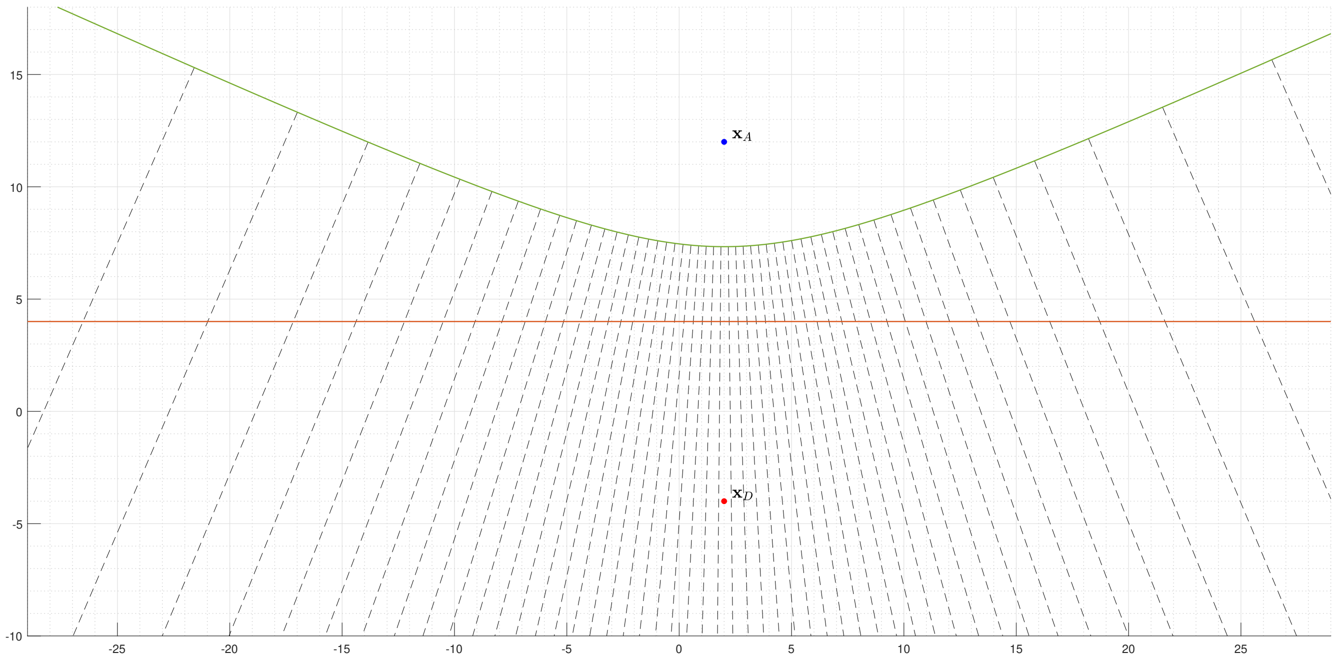

Fig. 5: Green Non-escape Region, red AD-based Apollonius circle and dashed lines depict Critical Escape Trajectories if $V_A = V_D$.

where $\hat{\mathbf{n}} \cdot (\mathbf{x}_D(t) - \mathbf{x}_A(t)) = 0$. This gives us:

$$\begin{aligned}
&\mathbf{c}_{AT}(\mathbf{E}_c) - \mathbf{c}_{AD} \\
&\quad = \frac{V_A^2}{V_D^2 - V_A^2}(\mathbf{x}_D - \mathbf{x}_A) + \frac{V_A^2}{V_A^2 - V_T^2}\Big(l(\psi)\cos\psi \frac{\mathbf{x}_D - \mathbf{x}_A}{\overline{\mathbf{x}_A\mathbf{x}_D}} + l(\psi)\sin\psi\ \hat{\mathbf{n}} \Big) \\
&\quad = \Big(\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{\gamma^2 - 1} + \frac{l(\psi)\cos\psi}{1-\alpha^2} \Big) \frac{\mathbf{x}_D - \mathbf{x}_A}{\overline{\mathbf{x}_A\mathbf{x}_D}} + \frac{l(\psi)\sin\psi}{1-\alpha^2}\ \hat{\mathbf{n}}. \qquad (24)
\end{aligned}$$

Clearly as $\gamma \to 1^+$, $\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{\gamma^2-1} \to \infty$; thus $\frac{\mathbf{c}_{AT}(\mathbf{E}_c)-\mathbf{c}_{AD}}{\|\mathbf{c}_{AT}(\mathbf{E}_c)-\mathbf{c}_{AD}\|} \to \frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|}$. □

The unified optimality criteria is given by the following theorem, which specifies the state-feedback Nash equilibrium of the Target-Attacker-Defender pursuit-evasion game.

Theorem III.3 (Critical Escape Trajectories Theorem (SFNE))**.** *If $\mathbf{x}_T(t) \in \zeta(\mathbf{E}_c)$, then the state-feedback Nash equilibrium is given by the Attacker and Defender moving towards the Critical Collision Point $\mathbf{I}_c(\mathbf{E}_c)$, and the Target moving along the ray $\zeta(\mathbf{E}_c)$ outwardly.*[2]

It holds that any point outside the Non-escape Region $\mathcal{N}(\mathbf{x}(t))$ is an element of a unique Critical Escape Trajectory $\zeta(\mathbf{E}_c)$. Thus for any position of the Target $\mathbf{x}_T(t) \notin \mathcal{N}(\mathbf{x}(t))$, the Critical Escape Trajectories Theorem can be applied to uniquely determine the state-feedback Nash equilibrium.

The Critical Escape Trajectories Theorem may also be equivalently expressed as a statement on the value of the differential game.

Theorem III.4 (Critical Escape Trajectories Theorem (Value))**.** *If $\mathbf{x}_T(t) \in \zeta(\mathbf{E}_c)$, then the value function is given by $V(\mathbf{x}(t)) = \overline{\mathbf{x}_T(t)\mathbf{E}_c}$.*

Using this expression of the unified optimality criteria we provide a simple proof of Lemma II.2.

Corollary III.5. *Under optimal play the Target escapes capture if and only if $\mathbf{x}_T(t) \notin \mathcal{N}(\mathbf{x}(t))$.*

Proof. At the boundary of the Non-escape Region $\mathcal{N}(\mathbf{x}(t))$, $\mathbf{x}_T(t) \in \zeta(\mathbf{E}_c)$ if and only if $\mathbf{E}_c = \mathbf{x}_T(t)$. Thus applying Theorem III.4, we have that for all positions of the Target on the boundary of $\mathcal{N}(\mathbf{x}(t))$, $V(\mathbf{x}(t)) = 0$ holds.

The value function $V(\mathbf{x}(t))$ cannot be greater than zero in the interior of the Non-escape Region since it would be disadvantageous for the Target to start closer to the Attacker.

[2]In Theorem III.3 all three agents must move at there respective maximum speeds.

On the other-hand $V(\mathbf{x}(t)) > 0$ for all $\mathbf{x}_T(t) \notin \mathcal{N}(\mathbf{x}(t))$ since $V(\mathbf{x}(t)) = \overline{\mathbf{x}_T(t)\mathbf{E}_c}$ and $\mathbf{E}_c \in \mathcal{N}(\mathbf{x}(t))$. □

Combining Theorems III.3, III.4 and III.5, we express the unified optimality criteria in full with the following theorem.

Theorem III.6 (Critical Escape Trajectories Theorem)**.** *If $\mathbf{x}_T(t) \in \mathcal{N}(\mathbf{x}(t))$, then $V(\mathbf{x}(t)) = 0$. Otherwise the value function is given by*

$$V(\mathbf{x}(t)) = \overline{\mathbf{x}_T(t)\mathbf{E}_c(\mathbf{x}(t))}, \tag{25}$$

and the state-feedback Nash equilibrium is given by

$$\mathbf{u}_T(t) = \frac{\mathbf{x}_T(t) - \mathbf{E}_c(\mathbf{x}(t))}{\|\mathbf{x}_T(t) - \mathbf{E}_c(\mathbf{x}(t))\|}, \tag{26a}$$

$$\mathbf{u}_A(t) = \frac{\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{x}_A(t)}{\|\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{x}_A(t)\|}, \tag{26b}$$

$$\mathbf{u}_D(t) = \frac{\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{x}_D(t)}{\|\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{x}_D(t)\|}, \tag{26c}$$

where $\mathbf{E}_c(\cdot)$ denotes a function mapping any state of the differential game $\mathbf{x}(t)$ to a unique Critical Escape Point $\mathbf{E}_c$ on the boundary of the Non-escape Region satisfying $\mathbf{x}_T(t) \in \zeta(\mathbf{E}_c)$.

Theorem III.6 provides a unified framework for the analysis of the differential game of degree and the differential game of kind; previously thought to have been separate problems. This theme is further expanded upon in Section IV, where concepts introduced here are expressed concretely.

The Critical Escape Trajectories Theorem brings to light a fundamental symmetry previously unacknowledged. In [5], applying Pontryagin's maximum principle gave the following result.

Theorem III.7 ([5])**.** *The optimal headings of the Attacker, the Target, and the Defender are constant under optimal play.*

However the state-feedback Nash equilibrium cannot be deduced from Pontryagin's maximum principle, as it gives too many degrees of freedom. The Critical Escape Trajectories Theorem imposes an additional symmetry.

Theorem III.8 (Target symmetry)**.** *The optimal headings of the Attacker, the Target, and the Defender are constant under optimal play of the Target with Attacker and Defender frozen.*

In other-words, if the Attacker's and Defender's position were static, the Target moving in its optimal heading would not change the optimal headings of any of the three agents. Theorem III.8 posits considerable structure to any state-feedback Nash equilibrium, so much so that the state-feedback Nash equilibrium can be uniquely determined.

Proof. Theorem III.8 follows as a simple corollary of Theorem III.3. If the Target were to move in its optimal heading, that is to move along the ray $\zeta(\mathbf{E}_c)$ outwardly, the Target will still remain within the same ray. Thus the optimal heading of the Target remains unchanged, and the optimal headings of the Attacker and Defender $\mathbf{I}_c(\mathbf{E}_c)$ also remain unchanged. □

In the next section we summarise the three disjointed optimality principles currently known in the literature. Then in Section III-B a proof is given to validate their equivalence. Target symmetry is the main concept used to prove the equivalence.

A. Three separate optimality criterion

In the literature, it is known that the state-feedback Nash equilibrium can be characterised by some point $\mathbf{I}$ on the surface of the AD-based Apollonius circle, which denotes the collision point between the Attacker and Defender under optimal play.

However, the criteria which determines the optimal interception point $\mathbf{I}$ is split into three cases; target located in the interior, boundary or exterior of the AD-based Apollonius circle. Let

$$\mathcal{C_{AD}} = \big\{\mathbf{p} \in \mathbb{R}^n \,\big|\, \|\mathbf{c}_{AD} - \mathbf{p}\| = r_{AD}\big\} \tag{27}$$

denote the set of all points on the circumference of the AD-based Apollonius circle.

Theorem III.9 ([5])**.** *If $\max_{\mathbf{p}\in\mathcal{C_{AD}}} -\overline{\mathbf{x}_T(t)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}} \leq 0$, then $V(\mathbf{x}(t)) = 0$. Otherwise the value function is given by*

$$V(\mathbf{x}(t)) = \begin{cases} \max_{\mathbf{p}\in\mathcal{C_{AD}}} -\overline{\mathbf{x}_T(t)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}} & \textit{if } \|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| < r_{AD} \\ \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{x}_T(t)} & \textit{if } \|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| = r_{AD} \\ \min_{\mathbf{p}\in\mathcal{C_{AD}}} \overline{\mathbf{x}_T(t)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}} & \textit{if } \|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| > r_{AD} \end{cases} \tag{28}$$

and the state-feedback Nash equilibrium is given by

$$\mathbf{u}_T(t) = \begin{cases} \dfrac{\mathbf{I}(\mathbf{x}(t)) - \mathbf{x}_T(t)}{\|\mathbf{I}(\mathbf{x}(t)) - \mathbf{x}_T(t)\|} & \text{if } \|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| < r_{AD} \\ \dfrac{\mathbf{x}_T(t) - \mathbf{I}(\mathbf{x}(t))}{\|\mathbf{x}_T(t) - \mathbf{I}(\mathbf{x}(t))\|} & \text{if } \|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| > r_{AD} \end{cases} \tag{29a}$$

$$\mathbf{u}_A(t) = \frac{\mathbf{I}(\mathbf{x}(t)) - \mathbf{x}_A(t)}{\|\mathbf{I}(\mathbf{x}(t)) - \mathbf{x}_A(t)\|}, \tag{29b}$$

$$\mathbf{u}_D(t) = \frac{\mathbf{I}(\mathbf{x}(t)) - \mathbf{x}_D(t)}{\|\mathbf{I}(\mathbf{x}(t)) - \mathbf{x}_D(t)\|}, \tag{29c}$$

where the optimal interception point $\mathbf{I}(\mathbf{x}(t))$ *is given by*

$$\mathbf{I}(\mathbf{x}(t)) = \begin{cases} \arg\max\limits_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} -\overline{\mathbf{x}_T(t)\mathbf{p}} + \dfrac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}} & \text{if } \|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| < r_{AD} \\ \mathbf{x}_T(t) & \text{if } \|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| = r_{AD} \\ \arg\min\limits_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} \overline{\mathbf{x}_T(t)\mathbf{p}} + \dfrac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}} & \text{if } \|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| > r_{AD} \end{cases} \tag{30}$$

In all cases, the Attacker and Defender move in straight lines towards the optimal interception point $\mathbf{I}(\mathbf{x}(t))$, but the criteria to calculate $\mathbf{I}(\mathbf{x}(t))$ differs. On the other-hand, the Critical Escape Trajectories Theorem is valid in all three cases; thereby unifying these disjointed principles into a single framework. The next section proves the equivalence between Theorem III.9 and the Critical Escape Trajectories Theorem.

Clearly Theorem III.6 is a much more elegant and compact representation of the value and state-feedback Nash equilibrium of the pursuit-evasion game than Theorem III.9. Furthermore we have opted to omit the case for $\|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| = r_{AD}$ in (29a); this has quite a complicated derivation in [5] and would greatly increase the complexity of the statement of Theorem III.9 if it were included.

B. Equivalence proof

Using Target symmetry, the state-feedback Nash equilibrium can be uniquely determined by the optimal headings on the boundary separating the regions $V(\mathbf{x}(t)) = 0$ and $V(\mathbf{x}(t)) > 0$. The equivalence proof utilises this fact.

In the proof we assume the disjointed optimality principles hold true. Using them three lemmas are established; in combination they verify that the state-feedback Nash equilibrium provided by the disjointed optimality principles (29) are equivalent to (26).

Lemma III.10. $\max\limits_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} -\overline{\mathbf{x}_T(t)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}} \le 0$ *if and only if* $\mathbf{x}_T(t) \in \mathcal{N}(\mathbf{x}(t))$.

Proof. Under the disjointed optimality principles, the value function equals zero if and only if

$$\max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} -\overline{\mathbf{x}_T\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A\mathbf{p}} \le 0,$$

which is equivalent to

$$\max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} -\frac{1}{V_T}\overline{\mathbf{x}_T\mathbf{p}} + \frac{1}{V_A}\overline{\mathbf{x}_A\mathbf{p}} \le 0.$$

The above maximisation problem can be interpreted as the problem of finding a point on the surface of the AD-based Apollonius circle such that if both the Target and Attacker where to move in a straight line at their respective maximum speeds towards that point, the time difference between the Target arriving at that point versus the Attacker arriving at that point is maximised.

Under this interpretation, the maximum is less than or equal to zero if and only if the AT-based Apollonius circle is completely encapsulated within the AD-based Apollonius circle. Recall from Section II that this was the definition of the Non-escape Region $\mathcal{N}(\mathbf{x}(t))$. Thus $V(\mathbf{x}(t)) = 0$ if and only if $\mathbf{x}_T(t) \in \mathcal{N}(\mathbf{x}(t))$. □

Lemma III.11. *If* $\max\limits_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} -\overline{\mathbf{x}_T(t)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}} = 0$ *then* $\mathbf{I}(\mathbf{x}(t)) = \mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big)$.

That is, on the boundary separating the regions $V(\mathbf{x}(t)) = 0$ and $V(\mathbf{x}(t)) > 0$ the optimal headings given by Theorem III.9 are the same as the optimal headings given by Theorem III.6.

Proof. Earlier in Lemma III.10 it was deduced that the AT-based Apollonius circle is completely encapsulated within the AD-based Apollonius circle if and only if

$$\max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} -\overline{\mathbf{x}_T(t)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}} \leq 0.$$

If the above inequality holds with equality this implies the following properties:

(L1) The Target is located at some point on the boundary of the Non-escape Region.

(L2) As a result of (L1) and $V_T < V_A$, the Target is located in the interior of the AD-based Apollonius.

(L3) The surface of the AT-based Apollonius circle intersects the surface of the AD-based Apollonius circle at some unique point denoted $\mathbf{Q}$.

Due to property (L2), the state-feedback Nash equilibrium provided by Theorem III.9 is given by all agents moving towards the point

$$\mathbf{I}(\mathbf{x}(t)) = \arg\max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} -\overline{\mathbf{x}_T(t)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{p}};$$

Moreover $\mathbf{I}(\mathbf{x}(t)) = \mathbf{Q}$, that is the above maximisation problem yields the unique point at which the surface of the AT-based Apollonius circle and AD-based Apollonius circle intersect (this follows trivially from the interpretation of the maximisation problem stated earlier in Lemma III.10). Thus it suffices to show that $\mathbf{Q} = \mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big)$.

Due to property (L1), we have that $\mathbf{x}_T(t) \in \zeta(\mathbf{x}_T(t))$ and thus $\mathbf{E}_c(\mathbf{x}(t)) = \mathbf{x}_T(t)$. Recalling the definition of the Critical Collision Point:

$$\mathbf{I}_c(\mathbf{E}_c) = \begin{cases} \mathbf{c}_{AT}(\mathbf{E}_c) + r_{AT}\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|} & \text{if } \ V_A = V_D \\ \mathbf{c}_{AT}(\mathbf{E}_c) + r_{AT}\frac{\mathbf{c}_{AT}(\mathbf{E}_c)-\mathbf{c}_{AD}}{\|\mathbf{c}_{AT}(\mathbf{E}_c)-\mathbf{c}_{AD}\|} & \text{else } \ V_T < V_A < V_D \end{cases}$$

Observing the above definition, clearly $\mathbf{I}_c(\mathbf{x}_T(t)) = \mathbf{Q}$. Thus $\mathbf{I}(\mathbf{x}(t)) = \mathbf{Q} = \mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big)$ where $\mathbf{E}_c(\mathbf{x}(t)) = \mathbf{x}_T(t)$. □

Lemma III.12. *The state-feedback Nash equilibrium provided by Theorem III.9 obeys Target Symmetry.*

Proof. Here *Target Symmetry* is defined by Theorem III.8. The optimal headings are determined by the optimal interception point $\mathbf{I}(\mathbf{x}(t))$, which is a function of the state $\mathbf{x}(t)$. Thus to prove *Target Symmetry* it must be shown that $\mathbf{I}(\mathbf{x}(t))$ remains constant if the Attacker and Defender are frozen and the Target moves in its optimal heading. But the formula for $\mathbf{I}(\mathbf{x}(t))$ is split into the three cases $\|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| < r_{AD}$, $\|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| = r_{AD}$ and $\|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| > r_{AD}$. It is obvious that there cannot be a discontinuous change in the optimal headings as the Target crosses the threshold $\|\mathbf{c}_{AD} - \mathbf{x}_T(t)\| = r_{AD}$, thus this case is omitted. In the remaining two cases, the following propositions prove that $\mathbf{I}(\mathbf{x}(t))$ is unchanged as the Target moves in its optimal heading.

Since the centre and radius of the AD-based Apollonius circle changes dynamically with time, for this subsection we adopt the notation $\mathbf{c}_{AD}(\mathbf{x}(t))$ to denote the centre of the AD-based Apollonius circle at time t, and $\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t))$ to denote the surface of the circle at time t (given by (11) and (27), respectively).

Proposition 1. For any state $\mathbf{x}(t_0)$ satisfying $\|\mathbf{c}_{AD}(\mathbf{x}(t_0)) - \mathbf{x}_T(t_0)\| < r_{AD}(\mathbf{x}(t_0))$, if

$$\begin{aligned} &\mathbf{x}_T(t_f) = \mathbf{x}_T(t_0) + \beta\big(\mathbf{I}(\mathbf{x}(t_0)) - \mathbf{x}_T(t_0)\big) \quad \text{for some } \ 0 \leq \beta < 1 \\ &\mathbf{x}_A(t_f) = \mathbf{x}_A(t_0) \\ &\mathbf{x}_D(t_f) = \mathbf{x}_D(t_0) \end{aligned}$$

then

$$\mathbf{I}(\mathbf{x}(t_f)) = \mathbf{I}(\mathbf{x}(t_0)).$$

Proof. First note that since $\beta < 1$ we have that $\|\mathbf{c}_{AD}(\mathbf{x}(t_f)) - \mathbf{x}_T(t_f)\| < r_{AD}(\mathbf{x}(t_f))$. Thus the optimal interception point at time t_f is given by

$$\mathbf{I}(\mathbf{x}(t_f)) = \arg\max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_f))} -\overline{\mathbf{x}_T(t_f)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_f)\mathbf{p}},$$

whereas at time t_0 it is given by

$$\mathbf{I}(\mathbf{x}(t_0)) = \arg\max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))} -\overline{\mathbf{x}_T(t_0)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}}.$$

Since $\mathbf{x}_A(t_f) = \mathbf{x}_A(t_0)$ and $\mathbf{x}_D(t_f) = \mathbf{x}_D(t_0)$ we have that $\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_f)) = \mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))$. Thus it would suffice to show

$$\arg\max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))} -\overline{\mathbf{x}_T(t_f)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}} = \arg\max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))} -\overline{\mathbf{x}_T(t_0)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}}.$$

To that end, recall the interpretation of the maximisation problem given earlier in Lemma III.10. The argument of the maximum is the point on the surface of the AD-based Apollonius circle in which the time difference between the Target arriving versus the Attacker arriving is maximised. Since the Attacker did not change its position, the time it takes for the Attacker to reach any point on the AD-based Apollonius circle remains the same. The Target moved towards but did not yet arrive at the optimal point $\mathbf{I}(\mathbf{x}(t_0))$. Thus the time it takes for the Target to reach $\mathbf{I}(\mathbf{x}(t_0))$ from its new position $\mathbf{x}_T(t_f)$ has decreased, and has decreased more for $\mathbf{I}(\mathbf{x}(t_0))$ than any other point on the AD-based Apollonius circle. Thus $\mathbf{I}(\mathbf{x}(t_0))$ remains as the optimal point at time t_f, and hence $\mathbf{I}(\mathbf{x}(t_0)) = \mathbf{I}(\mathbf{x}(t_f))$. □

Proposition 2. For any state $\mathbf{x}(t_0)$ satisfying $\|\mathbf{c}_{AD}(\mathbf{x}(t_0)) - \mathbf{x}_T(t_0)\| > r_{AD}(\mathbf{x}(t_0))$, if

$$\begin{aligned} \mathbf{x}_T(t_f) &= \mathbf{x}_T(t_0) + \beta\big(\mathbf{x}_T(t_0) - \mathbf{I}(\mathbf{x}(t_0))\big) \quad \text{for some } \beta \geq 0 \\ \mathbf{x}_A(t_f) &= \mathbf{x}_A(t_0) \\ \mathbf{x}_D(t_f) &= \mathbf{x}_D(t_0) \end{aligned}$$

then

$$\mathbf{I}(\mathbf{x}(t_f)) = \mathbf{I}(\mathbf{x}(t_0)).$$

Proof. In the case $\|\mathbf{c}_{AD}(\mathbf{x}(t_0)) - \mathbf{x}_T(t_0)\| > r_{AD}(\mathbf{x}(t_0))$ the optimal interception point is given by

$$\mathbf{I}(\mathbf{x}(t_0)) = \arg \min_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))} \overline{\mathbf{x}_T(t_0)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}},$$

whereas at time t_f it is given by

$$\mathbf{I}(\mathbf{x}(t_f)) = \arg \min_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_f))} \overline{\mathbf{x}_T(t_f)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_f)\mathbf{p}}.$$

Since $\mathbf{x}_A(t_f) = \mathbf{x}_A(t_0)$ and $\mathbf{x}_D(t_f) = \mathbf{x}_D(t_0)$ we have that $\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_f)) = \mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))$. Thus it would suffice to show

$$\arg \min_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))} \overline{\mathbf{x}_T(t_f)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}} = \arg \min_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))} \overline{\mathbf{x}_T(t_0)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}}.$$

We may interpret the expression $\overline{\mathbf{x}_T(t_0)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}}$ as the time it takes a particle to traverse from point $\mathbf{x}_T(t_0)$ to point $\mathbf{x}_A(t_0)$, where the particle moves with speed 1 whilst it is outside the AD-based Apollonius circle $\mathcal{C}_{\mathcal{AD}}$, but once it is inside the circle the particle may move at the faster speed of $\frac{V_A}{V_T}$.

Thus $\min_{\mathbf{p}\in\mathcal{C}_{AD}} \overline{\mathbf{x}_T(t_0)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}}$ is the minimum time it takes for the particle to go from $\mathbf{x}_T(t_0)$ to $\mathbf{x}_A(t_0)$, and $\arg\min_{\mathbf{p}\in\mathcal{C}_{AD}} \overline{\mathbf{x}_T(t_0)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}}$ denotes the corresponding unique optimal path $\mathbf{x}_T(t_0) \to \mathbf{I}(\mathbf{x}(t_0)) \to \mathbf{x}_A(t_0)$.

This geometric interpretation implies that if the particle were located anywhere on the closed line segment whose endpoints are $\mathbf{x}_T(t_0)$ and $\mathbf{I}(\mathbf{x}(t_0))$, the optimal interception point would not change. That is, for all $s \in [0, 1]$

$$\mathbf{I}(\mathbf{x}(t_0)) = \arg \min_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} \overline{\Big(\mathbf{I}(\mathbf{x}(t_0)) + s\big(\mathbf{x}_T(t_0) - \mathbf{I}(\mathbf{x}(t_0))\big)\Big)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}}, \tag{31}$$

holds. Formula (31) follows from the fact that if the particle were to move on the optimal path from its initial position $\mathbf{x}_T(t_0)$ towards $\mathbf{I}(\mathbf{x}(t_0))$, the optimal interception point would not change; or otherwise it would have found a path to reach $\mathbf{x}_A(t_0)$ in less time, which is a contradiction.

But since formula (31) holds for any $\mathbf{x}_T(t_0)$ outside the AD-based Apollonius circle, we can deduce that more generally, for all $s \in [0, \infty)$

$$\mathbf{I}(\mathbf{x}(t_0)) = \arg \min_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} \overline{\Big(\mathbf{I}(\mathbf{x}(t_0)) + s\big(\mathbf{x}_T(t_0) - \mathbf{I}(\mathbf{x}(t_0))\big)\Big)\mathbf{p}} + \frac{V_T}{V_A}\overline{\mathbf{x}_A(t_0)\mathbf{p}}. \tag{32}$$

Thus for $s = 1 + \beta$ formula (32) establishes that $\mathbf{I}(\mathbf{x}(t_f)) = \mathbf{I}(\mathbf{x}(t_0))$. □

Due to Propositions 1 and 2 Theorem III.9 obeys *Target Symmetry*. □

The value and state-feedback Nash equilibrium provided in the Critical Escape Trajectories Theorem is parameterised by a function $\mathbf{E}_c(\mathbf{x}(t))$ mapping any state to a point on the boundary of the Non-escape Region satisfying $\mathbf{x}_T(t) \in \zeta(\mathbf{E}_c(\mathbf{x}(t)))$. The next section details explicitly a method for computing such a function.

IV. IMPLEMENTATION

The theorem given below provides the fundamental equations that govern the Critical Escape Point $\mathbf{E}_c(\mathbf{x}(t))$; to be used to compute the state-feedback Nash equilibrium provided by Theorem III.6.

Theorem IV.1 (Critical Escape Point). *For any $\mathbf{x}_T(t) \notin \mathcal{N}(\mathbf{x}(t))$, the Critical Escape Point $\mathbf{E}_c(\mathbf{x}(t))$ is given by*

$$\mathbf{E}_c(\mathbf{x}(t)) = \mathbf{x}_A(t) + l(\psi)\cos\psi \frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|} + l(\psi)\sin\psi \ \hat{\mathbf{n}}(\mathbf{x}(t)), \tag{33}$$

where

$$\mathbf{n}(\mathbf{x}(t)) = \mathbf{x}_T(t) - \mathbf{x}_A(t) - \frac{\big(\mathbf{x}_T(t) - \mathbf{x}_A(t)\big)\cdot\big(\mathbf{x}_D(t) - \mathbf{x}_A(t)\big)}{\big(\mathbf{x}_D(t) - \mathbf{x}_A(t)\big)\cdot\big(\mathbf{x}_D(t) - \mathbf{x}_A(t)\big)}\big(\mathbf{x}_D(t) - \mathbf{x}_A(t)\big), \tag{34}$$

and $\hat{\mathbf{n}}(\mathbf{x}(t)) = \frac{\mathbf{n}(\mathbf{x}(t))}{\|\mathbf{n}(\mathbf{x}(t))\|}$ if $\mathbf{n}(\mathbf{x}(t)) \neq 0$, otherwise $\hat{\mathbf{n}}(\mathbf{x}(t)) = 0$. $\psi \in [0, \pi]$ is a solution to the sixth-order polynomial

$$\begin{aligned}
&\Big(\alpha\gamma H_{\mathbf{x}}^2 - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}\Big)e^{6i\psi} + \Big((\alpha^2+\gamma^2-1)H_{\mathbf{x}}^2 + \mathrm{Re}(\kappa_{\mathbf{x}})H_{\mathbf{x}} + \mathrm{Im}(\kappa_{\mathbf{x}})^2\Big)e^{5i\psi}\\
&+ \Big(\alpha\gamma H_{\mathbf{x}}^2 + 2\alpha\gamma H_{\mathbf{x}}H_{\mathbf{x}}^* - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}^* + 2i\,\mathrm{Re}(\kappa_{\mathbf{x}})\,\mathrm{Im}(\kappa_{\mathbf{x}})\Big)e^{4i\psi}\\
&+ \Big(2(\alpha^2+\gamma^2-1)H_{\mathbf{x}}H_{\mathbf{x}}^* + 2\,\mathrm{Re}(\kappa_{\mathbf{x}})\,\mathrm{Re}(H_{\mathbf{x}}) - \mathrm{Re}(\kappa_{\mathbf{x}})^2 - 2\,\mathrm{Im}(\kappa_{\mathbf{x}})^2\Big)e^{3i\psi}\\
&+ \Big(\alpha\gamma H_{\mathbf{x}}^2 + 2\alpha\gamma H_{\mathbf{x}}H_{\mathbf{x}}^* - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}^* + 2i\,\mathrm{Re}(\kappa_{\mathbf{x}})\,\mathrm{Im}(\kappa_{\mathbf{x}})\Big)^*e^{2i\psi}\\
&+ \Big((\alpha^2+\gamma^2-1)H_{\mathbf{x}}^2 + \mathrm{Re}(\kappa_{\mathbf{x}})H_{\mathbf{x}} + \mathrm{Im}(\kappa_{\mathbf{x}})^2\Big)^*e^{i\psi} + \Big(\alpha\gamma H_{\mathbf{x}}^2 - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}\Big)^* = 0.
\end{aligned} \tag{35}$$

At the limit $V_A = V_D$, $\psi \in [0, \pi]$ is instead given by the fourth-order polynomial

$$\alpha T_{\mathbf{x}_T}e^{4i\psi} + 2(\eta_{\mathbf{x}} + \alpha^2 T_{\mathbf{x}_T})e^{3i\psi} + 6\alpha\,\mathrm{Re}(T_{\mathbf{x}_T})e^{2i\psi} + 2(\eta_{\mathbf{x}} + \alpha^2 T_{\mathbf{x}_T})^*e^{i\psi} + \alpha T_{\mathbf{x}_T}^* = 0. \tag{36}$$

Here

$$T_{\mathbf{x}_T} = rej_{\mathbf{x}_T} + i\,proj_{\mathbf{x}_T}, \tag{37}$$

$$\eta_{\mathbf{x}} = rej_{\mathbf{x}_T} + i\,\frac{1-\alpha^2}{2}\overline{\mathbf{x}_A\mathbf{x}_D}, \tag{38}$$

$$H_{\mathbf{x}} = T_{\mathbf{x}_T} - i\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}, \tag{39}$$

$$\kappa_{\mathbf{x}} = 2rej_{\mathbf{x}_T} - i\alpha\gamma\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}, \tag{40}$$

where $proj_{\mathbf{x}_T}$ and $rej_{\mathbf{x}_T}$ are given by

$$proj_{\mathbf{x}_T} = \big(\mathbf{x}_T(t) - \mathbf{x}_A(t)\big)\cdot\frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|}, \tag{41}$$

$$rej_{\mathbf{x}_T} = \|\mathbf{n}(\mathbf{x}(t))\|. \tag{42}$$

Proof. Recall from Theorem III.6 that $\mathbf{E}_c(\mathbf{x}(t))$ need only be defined for all states $\mathbf{x}(t)$ satisfying $\mathbf{x}_T(t) \notin \mathcal{N}(\mathbf{x}(t))$, since (25)-(26) are only defined for $\mathbf{x}_T(t) \notin \mathcal{N}(\mathbf{x}(t))$. The function $\mathbf{E}_c(\mathbf{x}(t))$ must satisfy the following two properties:

$$\mathbf{E}_c(\mathbf{x}(t)) \in \partial\mathcal{N}(\mathbf{x}(t)), \tag{43}$$

$$\mathbf{x}_T(t) \in \zeta\big(\mathbf{E}_c(\mathbf{x}(t))\big). \tag{44}$$

Formula (43) specifies that $\mathbf{E}_c(\mathbf{x}(t))$ must lie somewhere on the boundary of $\mathcal{N}(\mathbf{x}(t))$, and (44) specifies that the Target must be located on the ray $\zeta\big(\mathbf{E}_c(\mathbf{x}(t))\big)$. The set of all points $\mathbf{E}_c(\mathbf{x}(t))$ in $\mathbb{R}^n$ satisfying (43) can be parameterised by an angle $\psi(\mathbf{x}(t))$ and a unit vector $\hat{\mathbf{n}}(\mathbf{x}(t))$ as follows: (here $\mathbf{E}_c$, ψ and $\hat{\mathbf{n}}$ are all functions of the state, but the notation is dropped occasionally for brevity)

$$\mathbf{E}_c = \mathbf{x}_A(t) + l(\psi)\cos\psi\frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|} + l(\psi)\sin\psi\ \hat{\mathbf{n}}, \tag{45}$$

where $\hat{\mathbf{n}}\cdot(\mathbf{x}_D(t) - \mathbf{x}_A(t)) = 0$ and $l(\psi)$ was defined earlier by (20). Clearly for $\mathbf{E}_c(\mathbf{x}(t))$ in formula (45) to satisfy the additional constraint (44), we must have

$$\mathbf{n}(\mathbf{x}(t)) = \mathbf{x}_T(t) - \mathbf{x}_A(t) - \frac{\big(\mathbf{x}_T(t) - \mathbf{x}_A(t)\big)\cdot\big(\mathbf{x}_D(t) - \mathbf{x}_A(t)\big)}{\big(\mathbf{x}_D(t) - \mathbf{x}_A(t)\big)\cdot\big(\mathbf{x}_D(t) - \mathbf{x}_A(t)\big)}\big(\mathbf{x}_D(t) - \mathbf{x}_A(t)\big), \tag{46}$$

where $\hat{\mathbf{n}}(\mathbf{x}(t)) = \frac{\mathbf{n}(\mathbf{x}(t))}{\|\mathbf{n}(\mathbf{x}(t))\|}$. In the case $\mathbf{n}(\mathbf{x}(t)) = 0$ then clearly $\sin\psi = 0$, thus we may define $\hat{\mathbf{n}}(\mathbf{x}(t)) = 0$ in that degenerate case. Formula (46) ensures that $\mathbf{E}_c(\mathbf{x}(t))$ lies somewhere on the 2-dimensional plane containing $\mathbf{x}_T(t)$, $\mathbf{x}_A(t)$, $\mathbf{x}_D(t)$.

All that remains is to determine the angle $\psi(\mathbf{x}(t))$ in formula (45) that satisfies constraint (44). To this end, let the Target's position be parameterised by

$$\mathbf{x}_T(t) = \mathbf{x}_A(t) + proj_{\mathbf{x}_T} \frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|} + rej_{\mathbf{x}_T}\ \hat{\mathbf{n}}(\mathbf{x}(t)), \tag{47}$$

where $proj_{\mathbf{x}_T}$ and $rej_{\mathbf{x}_T}$ can be computed with

$$\begin{aligned} proj_{\mathbf{x}_T} &= \big(\mathbf{x}_T(t) - \mathbf{x}_A(t)\big) \cdot \frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|}, \\ rej_{\mathbf{x}_T} &= \|\mathbf{n}(\mathbf{x}(t))\|. \end{aligned}$$

Recall the definition of a Critical Escape Trajectory:

$$\zeta(\mathbf{E}_c) = \big\{\mathbf{p} \in \mathbb{R}^n \,\big|\, \mathbf{p} = \mathbf{E}_c + \delta\big(\mathbf{I}_c(\mathbf{E}_c) - \mathbf{E}_c\big) \text{ for some } \delta \geq 0\big\},$$

we must have $\mathbf{x}_T(t) \in \zeta\big(\mathbf{E}_c(\mathbf{x}(t))\big)$, or

$$\mathbf{x}_T(t) - \mathbf{E}_c(\mathbf{x}(t)) = \delta\Big(\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{E}_c(\mathbf{x}(t))\Big) \text{ for some } \delta \geq 0. \tag{48}$$

Applying (45) and (47), the left-hand side of (48) is given by

$$\mathbf{x}_T(t) - \mathbf{E}_c(\mathbf{x}(t)) = \big(proj_{\mathbf{x}_T} - l(\psi)\cos\psi\big)\frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|} + \big(rej_{\mathbf{x}_T} - l(\psi)\sin\psi\big)\ \hat{\mathbf{n}}(\mathbf{x}(t)). \tag{49}$$

To determine the right-hand side of (48), $\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{E}_c(\mathbf{x}(t))$ must be parameterised in terms of $\frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|}$ and $\hat{\mathbf{n}}(\mathbf{x}(t))$. Applying formulas (22)-(23):

$$\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big) = \mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) + r_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)\begin{cases} \dfrac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|} & \text{if } V_A = V_D \\ \dfrac{\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{c}_{AD}}{\|\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{c}_{AD}\|} & \text{else } V_T < V_A < V_D \end{cases}$$

Here $\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)$, $r_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)$ and $\mathbf{c}_{AD}$ are given by:

$$\begin{aligned} \mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) &= \frac{V_T^2}{V_T^2 - V_A^2}\mathbf{x}_A(t) + \frac{V_A^2}{V_A^2 - V_T^2}\mathbf{E}_c(\mathbf{x}(t)), \\ &= \frac{-\alpha^2}{1-\alpha^2}\mathbf{x}_A(t) + \frac{1}{1-\alpha^2}\mathbf{E}_c(\mathbf{x}(t)). \end{aligned}$$

$$\begin{aligned} \mathbf{c}_{AD} &= \frac{V_D^2}{V_D^2 - V_A^2}\mathbf{x}_A(t) + \frac{V_A^2}{V_A^2 - V_D^2}\mathbf{x}_D(t), \\ &= \frac{-\gamma^2}{1-\gamma^2}\mathbf{x}_A(t) + \frac{1}{1-\gamma^2}\mathbf{x}_D(t). \end{aligned}$$

$$\begin{aligned} r_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) &= \frac{V_A V_T}{|V_A^2 - V_T^2|}\overline{\mathbf{x}_A \mathbf{E}_c(\mathbf{x}(t))}, \\ &= \frac{\alpha}{1-\alpha^2}l(\psi). \end{aligned}$$

To break $\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{E}_c(\mathbf{x}(t))$ into its constituent parts in $\frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|}$ and $\hat{\mathbf{n}}(\mathbf{x}(t))$, two intermediate formulas are first derived:

$$\begin{aligned} \mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{E}_c(\mathbf{x}(t)) &= \frac{-\alpha^2}{1-\alpha^2}\mathbf{x}_A(t) + \frac{1}{1-\alpha^2}\mathbf{E}_c(\mathbf{x}(t)) - \mathbf{E}_c(\mathbf{x}(t)), \\ &= \frac{-\alpha^2}{1-\alpha^2}\mathbf{x}_A(t) + \frac{\alpha^2}{1-\alpha^2}\mathbf{E}_c(\mathbf{x}(t)), \\ &= \frac{\alpha^2}{1-\alpha^2}\big(\mathbf{E}_c(\mathbf{x}(t)) - \mathbf{x}_A(t)\big), \\ &= \frac{\alpha^2}{1-\alpha^2}\big(l(\psi)\cos\psi\frac{\mathbf{x}_D(t) - \mathbf{x}_A(t)}{\|\mathbf{x}_D(t) - \mathbf{x}_A(t)\|} + l(\psi)\sin\psi\ \hat{\mathbf{n}}\big). \end{aligned} \tag{50}$$

$$
\begin{aligned}
\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{c}_{AD} &= \frac{-\alpha^2}{1-\alpha^2}\mathbf{x}_A(t) + \frac{1}{1-\alpha^2}\mathbf{E}_c(\mathbf{x}(t)) + \frac{\gamma^2}{1-\gamma^2}\mathbf{x}_A(t) - \frac{1}{1-\gamma^2}\mathbf{x}_D(t),\\
&= \frac{1}{1-\alpha^2}\big(\mathbf{E}_c(\mathbf{x}(t)) - \mathbf{x}_A(t)\big) + \frac{1}{1-\gamma^2}(\mathbf{x}_A(t) - \mathbf{x}_D(t)),\\
&= \frac{1}{1-\alpha^2}\big(\mathbf{E}_c(\mathbf{x}(t)) - \mathbf{x}_A(t)\big) - \frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|},\\
&= \left(\frac{l(\psi)\cos\psi}{1-\alpha^2} - \frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}\right)\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|} + \frac{l(\psi)\sin\psi}{1-\alpha^2}\ \hat{\mathbf{n}},\\
&= -\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}\Bigg(\left(1 - \frac{\cos\psi}{1-\alpha^2}\frac{(1-\gamma^2)l(\psi)}{\overline{\mathbf{x}_A\mathbf{x}_D}}\right)\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|}\\
&\qquad\qquad - \frac{\sin\psi}{1-\alpha^2}\frac{(1-\gamma^2)l(\psi)}{\overline{\mathbf{x}_A\mathbf{x}_D}}\ \hat{\mathbf{n}}\Bigg),\\
&= -\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{(1-\alpha^2)(1-\gamma^2)}\left(\left(1-\alpha^2+g(\psi)\cos\psi\right)\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|} + g(\psi)\sin\psi\ \hat{\mathbf{n}}\right),
\end{aligned}
$$

where $g(\psi) = -\frac{(1-\gamma^2)l(\psi)}{\overline{\mathbf{x}_A\mathbf{x}_D}} = -\cos\psi - \alpha\gamma + \sqrt{(\cos\psi+\alpha\gamma)^2 - (1-\alpha^2)(1-\gamma^2)}$. Thus

$$
\frac{\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{c}_{AD}}{\|\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{c}_{AD}\|} = \frac{\left(1-\alpha^2+g(\psi)\cos\psi\right)\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|} + g(\psi)\sin\psi\ \hat{\mathbf{n}}}{\sqrt{g(\psi)^2 + 2(1-\alpha^2)g(\psi)\cos\psi + (1-\alpha^2)^2}}.
$$

The formula above can be simplified further. It can easily be derived that $g(\psi)^2 = -2(\cos\psi+\alpha\gamma)g(\psi) - (1-\alpha^2)(1-\gamma^2)$. Hence

$$
\begin{aligned}
&\sqrt{g(\psi)^2 + 2(1-\alpha^2)g(\psi)\cos\psi + (1-\alpha^2)^2}\\
&= \sqrt{g(\psi)^2 + (1-\alpha^2)\big(-g(\psi)^2 - 2\alpha\gamma g(\psi) - (1-\alpha^2)(1-\gamma^2)\big) + (1-\alpha^2)^2}\\
&= \sqrt{\alpha^2 g(\psi)^2 - 2\alpha\gamma(1-\alpha^2)g(\psi) + \gamma^2(1-\alpha^2)^2}\\
&= \sqrt{\big(-\alpha g(\psi) + \gamma(1-\alpha^2)\big)^2}.
\end{aligned}
$$

The maximum value of $g(\psi)$ occurs at $\psi = \pi$, and $g(\pi) = 1 - \alpha\gamma + \gamma - \alpha$; thus $-\alpha g(\psi) + \gamma(1-\alpha^2) > 0$. Hence we obtain

$$
\frac{\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{c}_{AD}}{\|\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{c}_{AD}\|} = \frac{\left(1-\alpha^2+g(\psi)\cos\psi\right)\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|} + g(\psi)\sin\psi\ \hat{\mathbf{n}}}{-\alpha g(\psi) + \gamma(1-\alpha^2)}. \tag{51}
$$

From this point onwards we consider exclusively the simpler case for $V_A = V_D$, the more general case $V_T < V_A < V_D$ will be returned to later. The right-hand side of (48) is given by

$$
\begin{aligned}
&\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{E}_c(\mathbf{x}(t))\\
&= \mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big) - \mathbf{E}_c(\mathbf{x}(t)) + r_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|},\\
&= \frac{\alpha^2}{1-\alpha^2}\left(l(\psi)\cos\psi\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|} + l(\psi)\sin\psi\ \hat{\mathbf{n}}\right) + \frac{\alpha}{1-\alpha^2}l(\psi)\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|},\\
&= \frac{\alpha}{1-\alpha^2}l(\psi)\left((1+\alpha\cos\psi)\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|} + \alpha\sin\psi\ \hat{\mathbf{n}}\right).
\end{aligned} \tag{52}
$$

Thus using formulas (49) and (52), (48) holds iff for some $\delta \geq 0$

$$
\begin{aligned}
proj_{\mathbf{x}_T} - l(\psi)\cos\psi &= \delta(1+\alpha\cos\psi), \qquad \text{and}\\
rej_{\mathbf{x}_T} - l(\psi)\sin\psi &= \delta\alpha\sin\psi.
\end{aligned}
$$

Hence $\psi(\mathbf{x}(t))$ can be deduced from

$$
\frac{proj_{\mathbf{x}_T} - l(\psi)\cos\psi}{1+\alpha\cos\psi} = \frac{rej_{\mathbf{x}_T} - l(\psi)\sin\psi}{\alpha\sin\psi}.
$$

Re-arranging the above equation and substituting $l(\psi) = \frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{2}\frac{1-\alpha^2}{\cos\psi+\alpha}$ we obtain

$$
\frac{(1-\alpha^2)\overline{\mathbf{x}_A\mathbf{x}_D}}{2}\sin\psi = (\cos\psi+\alpha)\big(rej_{\mathbf{x}_T}(1+\alpha\cos\psi) - \alpha\ proj_{\mathbf{x}_T}\sin\psi\big),
$$

substituting the complex exponential formulas for sine and cosine:

$$\frac{(1-\alpha^2)\overline{\mathbf{x}_A\mathbf{x}_D}}{4i}(e^{i\psi}-e^{-i\psi})=\frac{1}{4}(e^{i\psi}+e^{-i\psi}+2\alpha)\big(rej_{\mathbf{x}_T}(2+\alpha(e^{i\psi}+e^{-i\psi}))+\alpha i\ proj_{\mathbf{x}_T}(e^{i\psi}-e^{-i\psi})\big),$$

multiplying both sides by $e^{2i\psi}$:

$$(1-\alpha^2)i\overline{\mathbf{x}_A\mathbf{x}_D}(e^{i\psi}-e^{3i\psi})=(e^{2i\psi}+1+2\alpha e^{i\psi})\big(rej_{\mathbf{x}_T}(2e^{i\psi}+\alpha(e^{2i\psi}+1))+\alpha i\ proj_{\mathbf{x}_T}(e^{2i\psi}-1)\big),$$

thus we obtain the fourth order polynomial governing $\psi(\mathbf{x}(t))$:

$$\begin{aligned}
&\alpha\Big(rej_{\mathbf{x}_T}+i\,proj_{\mathbf{x}_T}\Big)e^{4i\psi}+\Big(2\,rej_{\mathbf{x}_T}+2\alpha^2(rej_{\mathbf{x}_T}+i\,proj_{\mathbf{x}_T})+(1-\alpha^2)i\overline{\mathbf{x}_A\mathbf{x}_D}\Big)e^{3i\psi}\\
&+\Big(\alpha(rej_{\mathbf{x}_T}-i\,proj_{\mathbf{x}_T})+4\alpha\,rej_{\mathbf{x}_T}+\alpha(rej_{\mathbf{x}_T}+i\,proj_{\mathbf{x}_T})\Big)e^{2i\psi}\\
&+\Big(2\alpha^2(rej_{\mathbf{x}_T}-i\,proj_{\mathbf{x}_T})+2\,rej_{\mathbf{x}_T}-(1-\alpha^2)i\overline{\mathbf{x}_A\mathbf{x}_D}\Big)e^{i\psi}+\alpha(rej_{\mathbf{x}_T}-i\,proj_{\mathbf{x}_T})=0.
\end{aligned}$$

Let $T_{\mathbf{x}_T}=rej_{\mathbf{x}_T}+i\,proj_{\mathbf{x}_T}$ and let $\eta_{\mathbf{x}}=rej_{\mathbf{x}_T}+i\,\frac{1-\alpha^2}{2}\overline{\mathbf{x}_A\mathbf{x}_D}$. This gives the more succinct polynomial

$$\alpha T_{\mathbf{x}_T}e^{4i\psi}+2(\eta_{\mathbf{x}}+\alpha^2T_{\mathbf{x}_T})e^{3i\psi}+6\alpha\,\mathrm{Re}(T_{\mathbf{x}_T})e^{2i\psi}+2(\eta_{\mathbf{x}}+\alpha^2T_{\mathbf{x}_T})^*e^{i\psi}+\alpha T^*_{\mathbf{x}_T}=0.$$

Thus the proof for the simpler case $V_A=V_D$ is complete. Note that since $e^{-i\psi}={e^{i\psi}}^*$, the roots of the above fourth-order polynomial can also be expressed as the zeros of the real part of the second-order polynomial

$$\mathrm{Re}\left(\alpha T_{\mathbf{x}_T}e^{2i\psi}+2(\eta_{\mathbf{x}}+\alpha^2T_{\mathbf{x}_T})e^{i\psi}+3\alpha T_{\mathbf{x}_T}\right)=0.$$

Moving on to the general case $V_T<V_A<V_D$ the calculations become quite tedious, hence not all lines are shown. The right-hand side of (48) is given by

$$\begin{aligned}
&\mathbf{I}_c\big(\mathbf{E}_c(\mathbf{x}(t))\big)-\mathbf{E}_c(\mathbf{x}(t))\\
&=\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)-\mathbf{E}_c(\mathbf{x}(t))+r_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)\frac{\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)-\mathbf{c}_{AD}}{\|\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)-\mathbf{c}_{AD}\|},\\
&=\frac{\alpha^2}{1-\alpha^2}\left(l(\psi)\cos\psi\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|}+l(\psi)\sin\psi\ \hat{\mathbf{n}}\right)+\frac{\alpha}{1-\alpha^2}l(\psi)\frac{\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)-\mathbf{c}_{AD}}{\|\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)-\mathbf{c}_{AD}\|},\\
&=\frac{\alpha l(\psi)}{1-\alpha^2}\left(\alpha\cos\psi\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|}+\alpha\sin\psi\ \hat{\mathbf{n}}+\frac{\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)-\mathbf{c}_{AD}}{\|\mathbf{c}_{AT}\big(\mathbf{E}_c(\mathbf{x}(t))\big)-\mathbf{c}_{AD}\|}\right),\\
&=\frac{\alpha l(\psi)}{-\alpha g(\psi)+\gamma(1-\alpha^2)}\left(\big(1+(g(\psi)+\alpha\gamma)\cos\psi\big)\frac{\mathbf{x}_D(t)-\mathbf{x}_A(t)}{\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|}+(g(\psi)+\alpha\gamma)\sin\psi\ \hat{\mathbf{n}}\right),
\end{aligned}$$

where $\frac{\mathbf{c}_{AT}(\mathbf{E}_c(\mathbf{x}(t)))-\mathbf{c}_{AD}}{\|\mathbf{c}_{AT}(\mathbf{E}_c(\mathbf{x}(t)))-\mathbf{c}_{AD}\|}$ is given by formula (51). Thus (48) holds if and only if for some $\delta\geq 0$

$$\begin{aligned}
proj_{\mathbf{x}_T}-l(\psi)\cos\psi&=\delta\Big(1+(g(\psi)+\alpha\gamma)\cos\psi\Big),\qquad\text{and}\\
rej_{\mathbf{x}_T}-l(\psi)\sin\psi&=\delta\Big((g(\psi)+\alpha\gamma)\sin\psi\Big).
\end{aligned}$$

Hence $\psi(\mathbf{x}(t))$ can be deduced from

$$\frac{proj_{\mathbf{x}_T}-l(\psi)\cos\psi}{1+(g(\psi)+\alpha\gamma)\cos\psi}=\frac{rej_{\mathbf{x}_T}-l(\psi)\sin\psi}{(g(\psi)+\alpha\gamma)\sin\psi}.$$

Let $\mathcal{Q}=\sqrt{(\cos\psi+\alpha\gamma)^2-(1-\alpha^2)(1-\gamma^2)}$. It follows that $g(\psi)=\mathcal{Q}-(\cos\psi+\alpha\gamma)$ and $l(\psi)=-\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}\big(\mathcal{Q}-\cos\psi-\alpha\gamma\big)$. Hence

$$\frac{proj_{\mathbf{x}_T}+\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}\big(\mathcal{Q}-\cos\psi-\alpha\gamma\big)\cos\psi}{1+(\mathcal{Q}-\cos\psi)\cos\psi}=\frac{rej_{\mathbf{x}_T}+\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}\big(\mathcal{Q}-\cos\psi-\alpha\gamma\big)\sin\psi}{(\mathcal{Q}-\cos\psi)\sin\psi}.$$

Solving for $\mathcal{Q}$ we obtain

$$\begin{aligned}
\Big((proj_{\mathbf{x}_T}-\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2})\sin\psi-rej_{\mathbf{x}_T}\cos\psi\Big)\mathcal{Q}&=\Big((proj_{\mathbf{x}_T}-\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2})\sin\psi-rej_{\mathbf{x}_T}\cos\psi\Big)\cos\psi\\
&\quad+rej_{\mathbf{x}_T}-\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}\alpha\gamma\sin\psi,
\end{aligned}$$

and thus by taking the square on both sides we get

$$\Big((proj_{\mathbf{x}_T} - \frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2})\sin\psi - rej_{\mathbf{x}_T}\cos\psi\Big)^2(2\alpha\gamma\cos\psi - 1 + \alpha^2 + \gamma^2) =$$
$$2\cos\psi\Big(rej_{\mathbf{x}_T} - \frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}\alpha\gamma\sin\psi\Big)\Big((proj_{\mathbf{x}_T} - \frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2})\sin\psi - rej_{\mathbf{x}_T}\cos\psi\Big) + \Big(rej_{\mathbf{x}_T} - \frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}\alpha\gamma\sin\psi\Big)^2.$$

Let $T_{\mathbf{x}_T} = rej_{\mathbf{x}_T} + i\,proj_{\mathbf{x}_T}$, $H_{\mathbf{x}} = T_{\mathbf{x}_T} - i\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}$ and $\kappa_{\mathbf{x}} = 2rej_{\mathbf{x}_T} - i\alpha\gamma\frac{\overline{\mathbf{x}_A\mathbf{x}_D}}{1-\gamma^2}$. After substituting the complex exponential definition of sine and cosine we obtain the sixth-order polynomial

$$\begin{aligned}&\Big(\alpha\gamma H_{\mathbf{x}}^2 - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}\Big)e^{6i\psi} + \Big((\alpha^2+\gamma^2-1)H_{\mathbf{x}}^2 + \mathrm{Re}(\kappa_{\mathbf{x}})H_{\mathbf{x}} + \mathrm{Im}(\kappa_{\mathbf{x}})^2\Big)e^{5i\psi}\\
&+ \Big(\alpha\gamma H_{\mathbf{x}}^2 + 2\alpha\gamma H_{\mathbf{x}}H_{\mathbf{x}}^* - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}^* + 2i\,\mathrm{Re}(\kappa_{\mathbf{x}})\,\mathrm{Im}(\kappa_{\mathbf{x}})\Big)e^{4i\psi}\\
&+ \Big(2(\alpha^2+\gamma^2-1)H_{\mathbf{x}}H_{\mathbf{x}}^* + 2\,\mathrm{Re}(\kappa_{\mathbf{x}})\,\mathrm{Re}(H_{\mathbf{x}}) - \mathrm{Re}(\kappa_{\mathbf{x}})^2 - 2\,\mathrm{Im}(\kappa_{\mathbf{x}})^2\Big)e^{3i\psi}\\
&+ \Big(\alpha\gamma H_{\mathbf{x}}^2 + 2\alpha\gamma H_{\mathbf{x}}H_{\mathbf{x}}^* - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}^* + 2i\,\mathrm{Re}(\kappa_{\mathbf{x}})\,\mathrm{Im}(\kappa_{\mathbf{x}})\Big)^*e^{2i\psi}\\
&+ \Big((\alpha^2+\gamma^2-1)H_{\mathbf{x}}^2 + \mathrm{Re}(\kappa_{\mathbf{x}})H_{\mathbf{x}} + \mathrm{Im}(\kappa_{\mathbf{x}})^2\Big)^*e^{i\psi} + \Big(\alpha\gamma H_{\mathbf{x}}^2 - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}\Big)^* = 0.\end{aligned}$$

The roots of the above sixth-order polynomial may be expressed as the zeros of the real part of the third-order polynomial

$$\begin{aligned}&\mathrm{Re}\Bigg(\Big(\alpha\gamma H_{\mathbf{x}}^2 - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}\Big)e^{3i\psi} + \Big((\alpha^2+\gamma^2-1)H_{\mathbf{x}}^2 + \mathrm{Re}(\kappa_{\mathbf{x}})H_{\mathbf{x}} + \mathrm{Im}(\kappa_{\mathbf{x}})^2\Big)e^{2i\psi}\\
&+ \Big(\alpha\gamma H_{\mathbf{x}}^2 + 2\alpha\gamma H_{\mathbf{x}}H_{\mathbf{x}}^* - i\,\mathrm{Im}(\kappa_{\mathbf{x}})H_{\mathbf{x}}^* + 2i\,\mathrm{Re}(\kappa_{\mathbf{x}})\,\mathrm{Im}(\kappa_{\mathbf{x}})\Big)e^{i\psi}\\
&+ (\alpha^2+\gamma^2-1)H_{\mathbf{x}}H_{\mathbf{x}}^* + \mathrm{Re}(\kappa_{\mathbf{x}})\,\mathrm{Re}(H_{\mathbf{x}}) - \frac{1}{2}\,\mathrm{Re}(\kappa_{\mathbf{x}})^2 - \mathrm{Im}(\kappa_{\mathbf{x}})^2\Bigg) = 0.\end{aligned}$$

□

V. Conclusion

The present manuscript uncovered a fundamental symmetry in the TAD pursuit-evasion differential game, we named it *Target Symmetry*. This is the property that if the Target were to move in its optimal heading, but the Attacker and Defender remained at there current location, the optimal headings of all agents will remain unchanged. In the manuscript we also gave a solution to the game of kind as a function of the position of the Target; the Target can escape if and only if $\mathbf{x}_T(t) \notin \mathcal{N}(\mathbf{x}(t))$. From these two results, we uniquely identified the state-feedback Nash equilibrium, given by Theorem III.6, to be completely determined by the optimal headings at the boundary of $\mathcal{N}(\mathbf{x}(t))$. This is to say that the boundary of $\mathcal{N}(\mathbf{x}(t))$ encodes all of the information of the pursuit-evasion game, including the entire escape set and the entire state-feedback Nash equilibrium; in much the same way as the event horizon of a black hole.

For eloquence the state-feedback Nash equilibrium provided in Section III was presented more abstractly as a function of $\mathbf{E}_c(\mathbf{x}(t))$. In the last section we applied all the equations previously listed in Sections II and III to parameterise $\mathbf{E}_c(\mathbf{x}(t))$ as a solution to a sixth-order polynomial, providing an efficient method for the computation of the SFNE given by Theorem III.6.

Future works on this topic can prove that the Critical Escape Trajectories Theorem satisfies the Hamilton-Jacobi-Isaacs equation everywhere in the Target's escape set.

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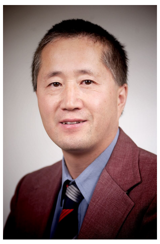

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Chapter 7

The holographic principle for the differential game of active target defence

7.1 Contextual statement

This manuscript extends the analysis of the differential game of active target defence to incorporate negative values. This is where if under optimal play the Target is captured by the Attacker, the Target and Defender work to minimise the separation between themselves at termination (see Section 2.7). It turns out that *Target Symmetry* does not hold outside the Target's escape set, rather a different symmetry named *Defender Symmetry* takes shape in this region. *Defender Symmetry* reveals that mapping the position of the Defender anywhere in-front of its optimal heading preserves the optimal headings of all agents. Analysing what these two symmetries have in-common, we conjecture the holographic principle for simple-motion pursuit-evasion games; and prove an application of it to the differential game of active target defence with the holographic theorem.

Statement of Authorship

Title of Paper	The holographic principle for the differential game of active target defence.
Publication Status	☑ Published ☐ Accepted for Publication ☑ Submitted for Publication ☐ Unpublished and Unsubmitted work written in manuscript style
Publication Details	Mammadov, K., Lim, C., & Shi, P. (2022). The holographic principle for the differential game of active target defence. International Journal of Control. [redacted]

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Name of Principal Author (Candidate)	Kamal Mammadov		
Contribution to the Paper	Selected research topic, conducted research, wrote manuscript, and acted as corresponding author.		
Overall percentage (%)	95%		
Certification:	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.		
Signature		Date	8/04/2022

Co-Author Contributions

By signing the Statement of Authorship, each author certifies that:

i. the candidate's stated contribution to the publication is accurate (as detailed above);

ii. permission is granted for the candidate in include the publication in the thesis; and

iii. the sum of all co-author contributions is equal to 100% less the candidate's stated contribution.

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Contribution to the Paper	Helped examine the manuscript.		
Signature		Date	8/04/2022

INTERNATIONAL JOURNAL OF CONTROL
https://doi.org/10.1080/00207179.2022.2111369

The holographic principle for the differential game of active target defence

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ABSTRACT

We examine the Target–Attacker–Defender (TAD) pursuit-evasion game. This is a two team, zero-sum differential game consisting of the aforementioned three agents obeying simple motion. The game terminates at the first time t_f when the Attacker achieves point capture of either the Target or Defender. The Attacker chases the Target in such a manner as to minimise the distance between itself and the Target, minus the distance between itself and the Defender at termination time; and the Target and Defender work as a team to achieve the exact opposite goal. This paper introduces and verifies the most elegant characterisation of the state-feedback Nash equilibrium. Whereas previous methods in the literature segregated the state space into at least three separate regions, we introduce a new unifying paradigm given by the Holographic Principle, and conjecture that this principle is more broadly applicable to a large class of simple motion pursuit-evasion games.

ARTICLE HISTORY
Received 6 April 2022
Accepted 2 August 2022

KEYWORDS
Differential game theory; dynamic game theory; optimal state-feedback strategies; state-feedback Nash equilibrium; pursuit-evasion games

1. Introduction

In aerospace engagement scenarios involving autonomous agents, the synthesis of intelligent actions must consider the potential strategies by the adversary. When analysing the possible outcomes of an engagement, unpredictability of the adversaries decisions presents the main challenge, the design of our strategies must be robust to a very broad set of possible counter strategies employed by the adversary. Differential game theory provides the framework to analyse and design optimal strategies for every team in these dynamic engagement scenarios. Here the goal is to find the state-feedback Nash equilibrium, the optimal outcome of the engagement scenario in which all parties with knowledge of the strategies deployed cannot increase their payoff by altering their decision-making process. Foundational works on the general theory of differential games are given in Basar and Olsder (1998) and Lewin (1994). The topic of pursuit-evasion differential games is a subcategory of this general theory focusing on applications of differential game theory to common aerospace engagement scenarios.

More specifically, the topic that is the focus of the present manuscript is named simple-motion pursuit-evasion games. This is a subcategory of pursuit-evasion differential games in which every agent has simple motion. An agent g is said to have *simple motion* if its state at time t can be completely specified by its position vector $\mathbf{x}_g(t)$, and its dynamics is given by $\dot{\mathbf{x}}_g(t) = V_g\mathbf{u}_g(t)$, where $\mathbf{u}_g(t)$ is an arbitrary vector with magnitude no greater than 1, chosen by the agent at every time t, and V_g is the maximum speed of agent g. This assumption essentially assumes that every agent has a fixed maximum speed but infinite acceleration/turn rate.

Simple-motion pursuit-evasion games are of particular interest in the literature to study aerospace engagements, because it gives a sufficiently accurate approximation to the dynamics of any platform in an engagement so long as the time taken to accelerate to maximum speed is small relative to the time the platform is at its maximum speed. Moreover, the assumption of simple motion more often then not leads to simple calculations for the state-feedback Nash equilibrium (SFNE), since the headings of all agents are constant under optimal play and the optimal trajectories are straight lines. Although this property of the SFNE is not true for every simple-motion game, but in the cases in which it is true, the point captures in the SFNE lie on the Apollonius circles determined by their instantaneous positions and their speed ratios (Isaacs, 1965).

One of the most famous examples of a simple-motion pursuit-evasion game is the 1-Pursuer 2-Evader differential game in Breakwell and Hagedorn (1979). This paper considered the problem of point capture of two successive evaders of identical speed in minimum total time. It turns out that, in some cases, the nearer evader is captured first at a specific position on her Apollonius circle, while the second evader runs directly away from that position. That position is that point on the Apollonius circle which maximises the sum of the distances from pursuer and second evader. This geometric solution, however, is valid only if the second evader remains further from the pursuer than does the first evader at all times prior to capture of the first pursuer. Otherwise, it turns out that the solution must be modified to include a phase involving curved motions by all three players, during which the pursuer remains equidistant

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from both evaders, and during which the pursuer can change her mind at any time as to which evader she will capture first.

More specifically relevant to the current manuscript, the work of Garcia et al. (2019), among many others, considered the Target–Attacker–Defender pursuit-evasion game. Although there are variations in the literature, what is common among them is that there are three agents, the Target, Attacker and Defender, each satisfying simple motion with maximum speeds V_{T}, V_{A} and V_{D} respectively. The termination time t_f is defined endogenously as the first time the Attacker collides with either of the other two agents. The most widely examined variant, and the variant considered in the current manuscript, is where the objective function is defined by

$$J(\mathbf{x}(t_f)) = \overline{\mathbf{x}_{\mathrm{A}}(t_f)\mathbf{x}_{\mathrm{T}}(t_f)} - \overline{\mathbf{x}_{\mathrm{A}}(t_f)\mathbf{x}_{\mathrm{D}}(t_f)},$$

where the over-line notation denotes the euclidean distance between two agents in $\mathbb{R}^n$. Team A, consisting of a single agent, the Attacker, aims to minimise $J(\mathbf{x}(t_f))$, whilst team T/D, consisting of two agents, the Target and Defender, cooperate to maximise $J(\mathbf{x}(t_f))$. Under optimal play, the set of all initial states that results in the outcome $J(\mathbf{x}(t_f)) > 0$, is named the winning region of team T/D, similarly all starting states which result in $J(\mathbf{x}(t_f)) < 0$ under optimal play, is named the winning region of team A. The task of unearthing necessary and sufficient conditions under which $J(\mathbf{x}(t_f)) = 0$ is named the TAD game of kind. The most elegant result answering this question is provided in the work of Liang et al. (2019), who proved that under optimal play, $J(\mathbf{x}(t_f)) = 0$ if and only if

$$V_{\mathrm{A}}\overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{x}_{\mathrm{D}}(t)} = V_{\mathrm{D}}\overline{\mathbf{x}_{\mathrm{A}}(t)\mathbf{x}_{\mathrm{T}}(t)} + V_{\mathrm{T}}\overline{\mathbf{x}_{\mathrm{A}}(t)\mathbf{x}_{\mathrm{D}}(t)}. \quad (1)$$

The TAD game of degree is the challenge of providing a method to find the state-feedback Nash equilibrium. To this end, a prominent work on this question includes Garcia et al. (2017). This manuscript considered the winning region of team T/D, here separate optimality principles were proposed to solve for the value of this game, depending upon the position of the Target. The value $V(\mathbf{x}(t))$ as a function of the state of the game $\mathbf{x}(t) = (\mathbf{x}_{\mathrm{A}}(t), \mathbf{x}_{\mathrm{D}}(t), \mathbf{x}_{\mathrm{T}}(t))$, denotes the expected payoff $J(\mathbf{x}(t_f))$ under the state-feedback Nash equilibrium (SFNE). In the case where the Target is located inside the AD-based Apollonius circle, the maximisation

$$V(\mathbf{x}(t)) = \max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} -\overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{p}} + \frac{V_{\mathrm{T}}}{V_{\mathrm{A}}}\overline{\mathbf{x}_{\mathrm{A}}(t)\mathbf{p}}, \quad (2)$$

yields the SFNE, whereas in the case where the Target is located outside the AD-based Apollonius circle, the value is given by

$$V(\mathbf{x}(t)) = \min_{\mathbf{p}\in\mathcal{C}_{\mathcal{AD}}} \overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{p}} + \frac{V_{\mathrm{T}}}{V_{\mathrm{A}}}\overline{\mathbf{x}_{\mathrm{A}}(t)\mathbf{p}}. \quad (3)$$

Here $\mathcal{C}_{\mathcal{AD}}$ denotes the perimeter of the AD-based Apollonius circle. In the case where the maximisation in (2) yields a negative number, we say that the Target cannot escape capture, i.e. $V(\mathbf{x}(t)) = 0$. Thus the combination of (2)–(3) characterises the SFNE of the TAD pursuit-evasion game for $V(\mathbf{x}(t)) \geq 0$.

It was proven in Garcia et al. (2019) that in two dimensions, in the case $V_{\mathrm{A}} = V_{\mathrm{D}}$, the value function is continuous and continuously differentiable in the winning region of team T/D, and that it satisfies the Hamilton–Jacobi–Isaacs equation everywhere in this set. Although the value function was not defined by (2)–(3), but rather by applying the two-sided Pontryagin's maximum principle (see Basar & Olsder, 1998; Lewin, 1994) to synthesise the state feedback strategies; in the process the value function is obtained. Just as in Garcia et al. (2017), the techniques used in Garcia et al. (2019) are split into the cases $\overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{x}_{\mathrm{A}}(t)} > \overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{x}_{\mathrm{D}}(t)}$ and $\overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{x}_{\mathrm{A}}(t)} < \overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{x}_{\mathrm{D}}(t)}$; and involve heavy calculus.

Later, the work of Garcia et al. (2021) studied the winning region of team A. The main conclusion from this manuscript [1] is that the value can be found using

$$V(\mathbf{x}(t)) = \max_{\mathbf{p}\in\mathcal{C}_{\mathcal{AT}}} -\overline{\mathbf{x}_{\mathrm{D}}(t)\mathbf{p}} + \frac{V_{\mathrm{D}}}{V_{\mathrm{T}}}\overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{p}}. \quad (4)$$

Thus taken in combination, the optimality principles (2), (3) and (4) provide a complete characterisation of the value of the TAD pursuit-evasion game.

Using Pontryagin's maximum principle, in the work of Garcia et al. (2017, 2018), it was proven that under optimal play, the headings of the Attacker, the Target and the Defender are constant. Therefore under optimal play every agent moves in straight lines with constant speed from the start until termination. However the SFNE cannot be uniquely determined from this principle alone, because there still remains too many degrees of freedom (i.e. which direction does each agent move in). In Garcia et al. (2017), it was argued that formulas (2)–(3) provide the correct angles for which each agent must move in. The works of Mammadov et al. (2020, 2021) provided a rigorous proof of (3) using the theory of upper and lower values, and generalised the results to $\mathbb{R}^n$. Finally the work of Garcia et al. (2021) provided a proof for (4), in the case $V_{\mathrm{A}} = V_{\mathrm{D}}$.

A major unification of the previous works was accomplished in the paper (Mammadov et al., 2022). This paper discovered a fundamental symmetry named Target Symmetry, eliciting an invariance in the state-feedback Nash equilibrium. This was used to develop a unified optimality principle given by

$$V(\mathbf{x}(t)) = \overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{E}_{\mathrm{T}}(\mathbf{x}(t))}, \quad (5)$$

which is valid in the entire winning region of team T/D, that is, it unifies (2) and (3). However, Target Symmetry does not hold in the winning region of team A, which is why the work of Mammadov et al. (2022) only studied the winning region of team T/D.

Thus, to further the goal of unification, it was found by the present authors that a different symmetry holds in the winning region of team A, that is named Defender Symmetry. It can be expressed by

$$V(\mathbf{x}(t)) = -\overline{\mathbf{x}_{\mathrm{D}}(t)\mathbf{E}_{\mathrm{D}}(\mathbf{x}(t))}. \quad (6)$$

Moreover, after an investigation into the similarities and differences between Target and Defender Symmetry, it was discovered that they could be unified by a much grander optimality principle we named the Holographic Principle, the main focus of this manuscript. The Holographic Principle holds in both the winning region of team T/D and the winning region of team A.

It unifies all the previous aforementioned optimality principles with

$$V(\mathbf{x}(t)) = \overline{\mathbf{x}_T(t)\mathbf{E}_T(\mathbf{x}(t))} - \overline{\mathbf{x}_D(t)\mathbf{E}_D(\mathbf{x}(t))}. \tag{7}$$

Moreover, we conjecture that the Holographic Principle holds not just for the TAD pursuit-evasion differential game, but a wide variety of simple-motion pursuit-evasion differential games. Although not in all simple-motion games, since the present authors are already aware of two counter examples in Breakwell and Hagedorn (1979) and Liang et al. (2019). Normally the Holographic Principle does not hold whenever in the SFNE the agents do not move in straight lines.

Thus the three main contributions of the present manuscript are: the first manuscript to study the TAD pursuit-evasion game in the winning region of team A in the more general setting $V_T < V_A < V_D$ (the previous publications only considered $V_A = V_D$), the development of a grand unified optimality principle with (7) and more broadly a conjecture that the methods developed in this manuscript can be applied to other simple-motion pursuit-evasion games.

The remainder of the manuscript is organised as follows. Section 2 first introduces the notation and terminology used throughout the manuscript, and provides a complete mathematical definition of the TAD pursuit-evasion game; finally it also provides some preliminary results on the game of kind that are used in subsequent sections. Section 3 provides a general description of the Holographic Principle and details its application to the TAD pursuit-evasion game. Note , however, that the general description is obtuse at the present time since the exact mathematical formulation would require a general direct proof; the current manuscript only verifies that it holds for the TAD game. Finally Section 4 mathematically proves the symmetry breaking of the Holographic Principle into the disjointed formulas of (2), (3) and (4).

2. Preliminaries

2.1 Notation and terminology

The notation used throughout the manuscript is listed as follows. Given any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

- $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ denotes the set of all positive real numbers.
- $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ denotes the set of all non-negative real numbers.
- $\mathbf{u} \cdot \mathbf{v}$ denotes the dot product.
- $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.
- $\overrightarrow{\mathbf{uv}} = \mathbf{v} - \mathbf{u}$.
- $\overline{\mathbf{uv}} = \|\overrightarrow{\mathbf{uv}}\|$ denotes the euclidean distance between $\mathbf{u}$ and $\mathbf{v}$.
- $\angle\mathbf{uvw}$ denotes the angle between vectors $\overrightarrow{\mathbf{vu}}$ and $\overrightarrow{\mathbf{vw}}$; that is $\overrightarrow{\mathbf{vu}} \cdot \overrightarrow{\mathbf{vw}} = \overline{\mathbf{vu}}\,\overline{\mathbf{vw}} \cos\angle\mathbf{uvw}$, where $0 \leq \angle\mathbf{uvw} \leq \pi$.
- $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ denotes the vector projection of any vector $\mathbf{v}$ onto $\mathbf{u}$.
- $\dot{\mathbf{u}}(t)$ denotes the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(t)$.

Team A and agent A both refer to the Attacker. Agent T and agent D denote the Target and Defender respectively, and team T/D denotes the team comprising of those two agents. SFNE is the abbreviation for state-feedback Nash equilibrium.

2.2 Problem formulation

In this section, we present the problem formulation of the Target–Attacker–Defender (TAD) pursuit-evasion game. This is a continuous-time, zero-sum differential game consisting of two teams, team A and team T/D. Team A consists of a single agent, agent A (also named the Attacker), and team T/D comprises of two agents, agent T and agent D (also named the Target and Defender respectively). The complete state of the TAD pursuit-evasion game is given by $\mathbf{x}(t) = (\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t))$ denoting the position of the Attacker, Defender and Target respectively. Here $\mathbf{x}_A(t), \mathbf{x}_D(t), \mathbf{x}_T(t) \in \mathbb{R}^n$ for any integer $n \geq 2$. The dynamics $\dot{\mathbf{x}}(t)$ from time t_0 to t_f is given by

$$(\dot{\mathbf{x}}_A(t), \dot{\mathbf{x}}_D(t), \dot{\mathbf{x}}_T(t)) = (V_A\mathbf{u}_A(t), V_D\mathbf{u}_D(t), V_T\mathbf{u}_T(t)),$$
$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{8}$$

where $V_A, V_D, V_T \in \mathbb{R}_0^+$ denotes the maximum speed of the Attacker, Defender and Target respectively. Both teams have access to the information of the current state $\mathbf{x}(t)$ at time t. Using that information, team A must choose an instantaneous heading for agent A; and team T/D must choose headings for agents D and T.[2] The controls of agents A, D and T must satisfy

$$\|\mathbf{u}_A(t)\|, \quad \|\mathbf{u}_D(t)\|, \quad \|\mathbf{u}_T(t)\| \leq 1 \tag{9}$$

for all time t. The termination time t_f is defined endogenously as the first time t_f satisfying at least one of the termination conditions (10) or (11)

$$\mathbf{x}_A(t_f) = \mathbf{x}_D(t_f), \tag{10}$$
$$\mathbf{x}_A(t_f) = \mathbf{x}_T(t_f). \tag{11}$$

Over the time horizon $[t_0, t_f]$, each team receives the following payoff:

$$J(\mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot), \mathbf{u}_T(\cdot), \mathbf{x}_0) = \overline{\mathbf{x}_A(t_f)\mathbf{x}_T(t_f)} - \overline{\mathbf{x}_A(t_f)\mathbf{x}_D(t_f)}. \tag{12}$$

The goal of team A is to minimise $J(\mathbf{u}_A(\cdot), \mathbf{u}_D(\cdot), \mathbf{u}_T(\cdot), \mathbf{x}_0)$, whereas the aim of team T/D is to maximise it. The maximum speed of the agents satisfies the inequality

$$V_T < V_A < V_D. \tag{13}$$

We denote $\mathbf{u}_A(t)$, $\mathbf{u}_D(t)$ and $\mathbf{u}_T(t)$ as any permissible control input at time t (any input satisfying (9)), whereas we use the notation $\mathbf{u}_A(\mathbf{x}(t)), \mathbf{u}_D(\mathbf{x}(t)), \mathbf{u}_T(\mathbf{x}(t))$ to denote the state-feedback optimal strategies, at state $\mathbf{x}(t)$. Moreover, let $\alpha = \frac{V_T}{V_A}$ and $\gamma = \frac{V_D}{V_A}$ denote the speed of the Target and Defender relative to the speed of the Attacker. Throughout the manuscript, we reference the following circles. Let $\mathbf{c}_{AD}$ and r_{AD} denote the centre and radius of the AD-based Apollonius circle, as a function of the state $\mathbf{x}(t)$

$$\mathbf{c}_{AD}(\mathbf{x}(t)) = \frac{V_D^2}{V_D^2 - V_A^2}\mathbf{x}_A(t) + \frac{V_A^2}{V_A^2 - V_D^2}\mathbf{x}_D(t), \tag{14a}$$

$$r_{AD}(\mathbf{x}(t)) = \frac{V_A V_D}{|V_A^2 - V_D^2|}\overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}. \tag{14b}$$

Similarly, let $\mathbf{c}_{\mathrm{AT}}$ and r_{AT} denote the centre and radius of the AT-based Apollonius circle

$$\mathbf{c}_{\mathrm{AT}}(\mathbf{x}(t)) = \frac{V_{\mathrm{T}}^2}{V_{\mathrm{T}}^2 - V_{\mathrm{A}}^2}\mathbf{x}_{\mathrm{A}}(t) + \frac{V_{\mathrm{A}}^2}{V_{\mathrm{A}}^2 - V_{\mathrm{T}}^2}\mathbf{x}_{\mathrm{T}}(t), \tag{15a}$$

$$r_{\mathrm{AT}}(\mathbf{x}(t)) = \frac{V_{\mathrm{A}}V_{\mathrm{T}}}{|V_{\mathrm{A}}^2 - V_{\mathrm{T}}^2|}\overline{\mathbf{x}_{\mathrm{A}}(t)\mathbf{x}_{\mathrm{T}}(t)}. \tag{15b}$$

Moreover, let $\mathcal{C}_{\mathcal{AD}}$ and $\mathcal{C}_{\mathcal{AT}}$ denote the set of all points on the surface of the AD and AT-based Apollonius circles, respectively. It is defined by

$$\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t)) = \{\mathbf{p} \in \mathbb{R}^n \mid \|\mathbf{c}_{\mathrm{AD}}(\mathbf{x}(t)) - \mathbf{p}\| = r_{\mathrm{AD}}(\mathbf{x}(t))\}, \tag{16}$$

$$\mathcal{C}_{\mathcal{AT}}(\mathbf{x}(t)) = \{\mathbf{p} \in \mathbb{R}^n \mid \|\mathbf{c}_{\mathrm{AT}}(\mathbf{x}(t)) - \mathbf{p}\| = r_{\mathrm{AT}}(\mathbf{x}(t))\}. \tag{17}$$

The interior of the AD-based (Attacker–Defender) Apollonius circle represents all the points in $\mathbb{R}^n$ the Attacker can reach before the Defender, whereas the points outside the circle the Defender would reach first. Similarly the interior of the AT-based Apollonius circle is the set of all points the Target can reach before the Attacker.

The central goal of studying differential games is to characterise the state-feedback Nash equilibrium (SFNE). That is, what are the optimal state-feedback strategies $\mathbf{u}_{\mathrm{A}}(\mathbf{x}(t)), \mathbf{u}_{\mathrm{D}}(\mathbf{x}(t)), \mathbf{u}_{\mathrm{T}}(\mathbf{x}(t))$, and what is the value function $V(\mathbf{x}(t)) = J(\mathbf{u}_{\mathrm{A}}(\mathbf{x}(t)), \mathbf{u}_{\mathrm{D}}(\mathbf{x}(t)), \mathbf{u}_{\mathrm{T}}(\mathbf{x}(t)), \mathbf{x}_0)$. The main contribution of the present manuscript is to conjecture a holographic principle for some large class of simple motion pursuit-evasion games. The holographic principle reveals that in the case of the TAD pursuit-evasion game, it would suffice to find the optimal strategies in the special case that the starting state $\mathbf{x}_0$ satisfies $V(\mathbf{x}_0) = 0$.

To that end, the next subsection characterises the set of all starting states $\mathbf{x}_0$ satisfying $V(\mathbf{x}_0) = 0$, and the optimal strategies $\mathbf{u}_{\mathrm{A}}(\mathbf{x}_0), \mathbf{u}_{\mathrm{D}}(\mathbf{x}_0), \mathbf{u}_{\mathrm{T}}(\mathbf{x}_0)$ in this special case. Section 3 reveals how any arbitrary state $\mathbf{x}(t) = (\mathbf{x}_{\mathrm{A}}(t), \mathbf{x}_{\mathrm{D}}(t), \mathbf{x}_{\mathrm{T}}(t))$ can be mapped into the alternative state $(\mathbf{x}_{\mathrm{A}}(t), \mathbf{E}_{\mathrm{D}}, \mathbf{E}_{\mathrm{T}})$ satisfying $V(\mathbf{x}_{\mathrm{A}}(t), \mathbf{E}_{\mathrm{D}}, \mathbf{E}_{\mathrm{T}}) = 0$; and how this mapping can be used to deduce the SFNE for any state $\mathbf{x}(t)$.

2.3 Game of kind

In pursuit-evasion differential games, the game of kind normally refers to the puzzle of unearthing necessary and sufficient conditions under which the value function is equal to zero. For the TAD pursuit-evasion game considered in this manuscript, this occurs whenever under optimal play, the state at termination time satisfies

$$\mathbf{x}_{\mathrm{A}}(t_f) = \mathbf{x}_{\mathrm{D}}(t_f) = \mathbf{x}_{\mathrm{T}}(t_f). \tag{18}$$

In the prominent work of Liang et al. (2019), it was determined that condition (18) eventualises under optimal play if and only if

$$V_{\mathrm{A}}\overline{\mathbf{x}_{\mathrm{T}}(t)\mathbf{x}_{\mathrm{D}}(t)} = V_{\mathrm{D}}\overline{\mathbf{x}_{\mathrm{A}}(t)\mathbf{x}_{\mathrm{T}}(t)} + V_{\mathrm{T}}\overline{\mathbf{x}_{\mathrm{A}}(t)\mathbf{x}_{\mathrm{D}}(t)}. \tag{19}$$

In the work of Mammadov et al. (2022), it was deduced that formula (19) is equivalent to

$$\frac{\overline{\mathbf{x}_{\mathrm{A}}(t)\mathbf{x}_{\mathrm{T}}(t)}}{\overline{\mathbf{x}_{\mathrm{A}}(t)\mathbf{x}_{\mathrm{D}}(t)}} = \frac{1}{1-\gamma^2}(\cos\psi + \alpha\gamma - \sqrt{(\cos\psi + \alpha\gamma)^2 - (1-\alpha^2)(1-\gamma^2)}), \tag{20}$$

where $\psi = \angle \mathbf{x}_{\mathrm{T}}(t)\mathbf{x}_{\mathrm{A}}(t)\mathbf{x}_{\mathrm{D}}(t)$. Formulae (19) and (20) can be derived by determining the set of all states $\mathbf{x}(t)$ satisfying

$$\overline{\mathbf{c}_{\mathrm{AD}}(\mathbf{x}(t))\mathbf{c}_{\mathrm{AT}}(\mathbf{x}(t))} + r_{\mathrm{AT}}(\mathbf{x}(t)) = r_{\mathrm{AD}}(\mathbf{x}(t)); \tag{21}$$

That is, the Target cannot escape capture from the Attacker if and only if the AT-based Apollonius circle is completely encapsulated within the AD-based Apollonius circle. Formulas (19), (20) and (21) are all equivalent. Consolidating the results listed, the following Lemma summarises the conclusion.

Lemma 2.1: *The following are equivalent conditions for any state $\boldsymbol{x}(t)$ of the differential game defined in Section 2.2.*

(1) $V(\boldsymbol{x}(t)) \geq 0$,
(2) $V_{\mathrm{A}}\overline{\boldsymbol{x}_{\mathrm{T}}(t)\boldsymbol{x}_{\mathrm{D}}(t)} \leq V_{\mathrm{D}}\overline{\boldsymbol{x}_{\mathrm{A}}(t)\boldsymbol{x}_{\mathrm{T}}(t)} + V_{\mathrm{T}}\overline{\boldsymbol{x}_{\mathrm{A}}(t)\boldsymbol{x}_{\mathrm{D}}(t)}$,
(3) $\frac{\overline{\boldsymbol{x}_{\mathrm{A}}(t)\boldsymbol{x}_{\mathrm{T}}(t)}}{\overline{\boldsymbol{x}_{\mathrm{A}}(t)\boldsymbol{x}_{\mathrm{D}}(t)}} \geq \frac{1}{1-\gamma^2}(\cos\psi + \alpha\gamma - \sqrt{(\cos\psi + \alpha\gamma)^2 - (1-\alpha^2)(1-\gamma^2)})$,
(4) $\overline{\boldsymbol{c}_{\mathrm{AD}}(\boldsymbol{x}(t))\boldsymbol{c}_{\mathrm{AT}}(\boldsymbol{x}(t))} + r_{\mathrm{AT}}(\boldsymbol{x}(t)) \geq r_{\mathrm{AD}}(\boldsymbol{x}(t))$.

And conditions (1), (2), (3) and (4) all remain equivalent when replacing the inequalities with equalities.

2.4 Necessary conditions for optimality in simple motion games

The results of (19)–(21) determine the set of all states $\mathbf{x}_0$ satisfying $V(\mathbf{x}_0) = 0$. The question remains, how can we deduce the optimal strategies $\mathbf{u}_{\mathrm{A}}(\mathbf{x}(t)), \mathbf{u}_{\mathrm{D}}(\mathbf{x}(t)), \mathbf{u}_{\mathrm{T}}(\mathbf{x}(t))$. To this end, we cite the following well-known result.

Theorem 2.2 (Isaacs): *The optimal headings of all agents remain constant under optimal play, not including termination.*

This essentially carriers the meaning that every agent moves in straight lines at maximum speed; the exemption occurs at times in which the termination conditions are triggered, such as (10) and (11) in the TAD pursuit-evasion game. The meaning here will become more clear as more examples are given in Section 4.

Theorem 2.2 is attributed to the works of Isaacs (1965). It holds for a large class of simple motion pursuit-evasion games and has been proven explicitly to hold in the TAD pursuit-evasion game, in the work of Garcia et al. (2017). However it should be noted that Theorem 2.2 does not hold for all simple motion pursuit-evasion games. For example, in the work of Liang et al. (2019), there are cases in which agents move in curved paths in the state-feedback Nash equilibrium.

In general, the state-feedback Nash equilibrium cannot be deduced from this optimality principle, as it gives too many degrees of freedom (that is, we know that every agent must move in straight lines under optimal play, but which direction are they going). Thus we name it a necessary condition of optimality. Nevertheless there is one special case for which the optimal headings $\mathbf{u}_{\mathrm{A}}, \mathbf{u}_{\mathrm{D}}, \mathbf{u}_{\mathrm{T}}$ can be uniquely derived from Theorem 2.2, that is the case in which $V(\mathbf{x}_0) = 0$.

Recall the earlier result (21). Under optimal play, the state at termination time satisfies $\mathbf{x}_A(t_f) = \mathbf{x}_D(t_f) = \mathbf{x}_T(t_f)$ if and only if the AT-based Apollonius circle is completely in the interior of the AD-based Apollonius circle, except for the unique point in which they intersect. Denote this unique point $\mathbf{I}_c$, throughout the manuscript it is known as the Critical Collision Point, and it is given by the formula

$$\mathbf{I}_c(\mathbf{x}(t)) = \mathbf{c}_{AT}(\mathbf{x}(t)) + r_{AT}(\mathbf{x}(t))\frac{\mathbf{c}_{AT}(\mathbf{x}(t)) - \mathbf{c}_{AD}(\mathbf{x}(t))}{\|\mathbf{c}_{AT}(\mathbf{x}(t)) - \mathbf{c}_{AD}(\mathbf{x}(t))\|}. \tag{22}$$

Thus in the case where (19) holds, due to the following three reasons:

(1) Under optimal play every agent collides at a single point.
(2) The Critical Collision Point $\mathbf{I}_c$ is the only point that exists in which if every agent was to move at their respective maximum speeds towards, they would all arrive at the same time.
(3) Invoking Theorem 2.2, under optimal play every agent moves in straight lines at their respective maximum speeds.

It follows that the optimal strategies are given by the following Lemma.

Lemma 2.3: *For any state $\mathbf{x}(t)$ satisfying $V(\mathbf{x}(t)) = 0$, the optimal strategies are given by*

$$\mathbf{u}_A(\mathbf{x}(t)) = \frac{\mathbf{I}_c(\mathbf{x}(t)) - \mathbf{x}_A(t)}{\|\mathbf{I}_c(\mathbf{x}(t)) - \mathbf{x}_A(t)\|}, \tag{23a}$$

$$\mathbf{u}_D(\mathbf{x}(t)) = \frac{\mathbf{I}_c(\mathbf{x}(t)) - \mathbf{x}_D(t)}{\|\mathbf{I}_c(\mathbf{x}(t)) - \mathbf{x}_D(t)\|}, \tag{23b}$$

$$\mathbf{u}_T(\mathbf{x}(t)) = \frac{\mathbf{I}_c(\mathbf{x}(t)) - \mathbf{x}_D(t)}{\|\mathbf{I}_c(\mathbf{x}(t)) - \mathbf{x}_D(t)\|}, \tag{23c}$$

where the Critical Collision Point $\mathbf{I}_c(\mathbf{x}(t))$ is given by formula (22).

Be that as it may, Lemma 2.3 only reveals the optimal state-feedback strategies $\mathbf{u}_A(\mathbf{x}(t)), \mathbf{u}_D(\mathbf{x}(t)), \mathbf{u}_T(\mathbf{x}(t))$ in the special case where the state $\mathbf{x}(t)$ satisfies (19). The main issue here is that Theorem 2.2 does not provide enough constraints on the set of all possible optimal strategies to uniquely determine the state-feedback Nash equilibrium. In the next section, we introduce with the holographic principle a new method to uniquely determine everywhere the optimal state-feedback strategies.

3. The holographic principle

3.1 General statement of the holographic principle

The holographic principle reveals a fundamental symmetry in simple motion pursuit-evasion games previously unacknowledged. A symmetry, also known as an invariance, is a transformation or mapping of a certain type which preserves some quantity or property. For example, consider a polynomial equation with only real coefficients; if r_1 is a root, then the complex conjugate of r_1 is also a root. Thus complex conjugation is a symmetry of the roots of a polynomial with real coefficients.

In the topic of pursuit-evasion games, a symmetry or invariance of the state-feedback Nash equilibrium refers to a transformation $\mathcal{F}$ of the state which preserves the optimal headings of all agents. Thus for the TAD pursuit-evasion game this means that the optimal headings obey $\mathbf{u}_A(\mathbf{x}(t)) = \mathbf{u}_A(\mathcal{F}(\mathbf{x}(t))), \mathbf{u}_D(\mathbf{x}(t)) = \mathbf{u}_D(\mathcal{F}(\mathbf{x}(t))), \mathbf{u}_T(\mathbf{x}(t)) = \mathbf{u}_T$ $(\mathcal{F}(\mathbf{x}(t)))$. Theorem 2.2 specifies one such transformation $\mathcal{F}$. The holographic principle elicits another transformation $\mathcal{F}$ which has this property. It can be expressed by the following conjecture.

Conjecture 3.1 (Holographic Principle): For some large class of simple-motion pursuit-evasion games, under optimal play of a single agent with all other agents frozen in-place, the optimal strategies remain constant, so long as this single agent is not the next agent to terminate.

The conjecture is purposefully obtuse where it states 'For some large class of simple-motion pursuit-evasion games', because as of writing it yet remains unclear just how broadly it applies to simple motion pursuit-evasion games. The present manuscript proves that Conjecture 3.1 holds for the TAD pursuit-evasion game described in Section 2.2, and it has been verified to hold for the 1-Pursuer 2-Evader game in certain cases. There already has been identified a differential game where all agents obey simple motion, but nonetheless does not comply with 3.1, for example in Liang et al. (2019), where a fast moving Attacker must capture a Target in minimum time. Thus it remains an open problem to classify an exact criteria determining the 'large class' of games satisfying Conjecture 3.1.

To describe with more detail the consequences and implications of the Holographic Principle, it is best to illustrate with an example. The sentence 'Under optimal play of a single agent with all other agents frozen in-place, the optimal strategies remain constant'; describes a crucial symmetry that the state-feedback optimal strategies must satisfy in simple motion games. Let $\mathbf{x}(t) = (\mathbf{x}_1(t), \ldots, \mathbf{x}_N(t))$ denote the state at time t of any simple motion pursuit-evasion game containing N agents, where $\mathbf{x}_i(t)$ denotes the position of the ith agent at time t; and let $\mathbf{u}_1(\mathbf{x}(t)), \ldots, \mathbf{u}_N(\mathbf{x}(t))$ denote the state-feedback optimal strategies, that is under optimal play $\dot{\mathbf{x}}_i(t) = \mathbf{u}_i(\mathbf{x}(t))$ for $i = 1, \ldots, N$. For any agent j not the next agent to terminate; if from time t to $t + \Delta t$, the agents move according to $\dot{\mathbf{x}}_j(t) = \mathbf{u}_j(\mathbf{x}(t))$, and $\dot{\mathbf{x}}_i(t) = 0$ for all $i \neq j$, then $\mathbf{u}_i(\mathbf{x}(t + \Delta t)) = \mathbf{u}_i(\mathbf{x}(t))$ for all $i = 1, \ldots, N$.[3]

Linking Conjecture 3.1 directly to the pursuit-evasion game specified in Section 2.2. If $V(\mathbf{x}(t)) > 0$, under optimal play the Attacker collides with the Defender at termination time t_f, hence termination condition (10) is triggered. Therefore in this case we say that both agent A and agent D are next to terminate, whereas the Target does not terminate since termination condition (11) is not triggered. As a consequence, the Holographic Principle can only be applied to the Target in the case $V(\mathbf{x}(t)) > 0$. On the other hand, if $V(\mathbf{x}(t)) < 0$, under optimal play the Attacker collides with the Target at termination time, thus termination condition (11) is triggered. As a result, in this case agent A and agent T are next to terminate, whereas the Defender does not terminate. Consequently Conjecture 3.1 may only be applied to the Defender when $V(\mathbf{x}(t)) < 0$. We describe this splitting, the division of methods and procedures for calculating the state-feedback Nash equilibrium, despite all these methods originating from a single unified principle, as symmetry breaking.

The Holographic Principle provides significant constraints that the state-feedback Nash equilibrium must obey, so much so that taken together with Theorem 2.2, can be used to uniquely deduce the SFNE. Thus the Holographic Principle, with Theorem 2.2, constitutes a set of necessary and sufficient conditions for optimality. The remainder of Section 3 is organised as follows. Section 3.2 applies Conjecture 3.1 to uniquely deduce the state-feedback Nash equilibrium for the differential game of active target defence, Section 3.3 provides explicitly the equations used to calculate the value and state-feedback optimal strategies, and finally Section 3.4 gives a numerical example.

3.2 Holographic principle for TAD pursuit-evasion game

In this section, we apply Conjecture 3.1 to uniquely deduce the state-feedback Nash equilibrium of the TAD pursuit-evasion game defined in Section 2.2. The Holographic Principle undergoes symmetry breaking at the threshold $V(\mathbf{x}(t)) = 0$, which is given by the following two theorems.

Theorem 3.2 (Target Symmetry): *If $V(\boldsymbol{x}(t)) > 0$, under optimal play of the Target with the Attacker and Defender frozen in-place, the optimal strategies remain constant.*

Theorem 3.3 (Defender Symmetry): *If $V(\boldsymbol{x}(t)) < 0$, under optimal play of the Defender with the Attacker and Target frozen in-place, the optimal strategies remain constant.*

In other words, in the case $V > 0$, if the Attacker's and Defender's position were static, the Target moving in its optimal heading would not change the optimal headings of any of the three agents. Whereas in the case $V < 0$, if the Target and Defender were frozen in place, the Defender moving in its optimal heading would not change the optimal strategies.

Target Symmetry and Defender Symmetry can be used to uniquely determine the state-feedback Nash equilibrium for all $\mathbf{x}(t)$ as follows. We already know what the optimal strategies are for the special case $V(\mathbf{x}(t)) = 0$, this was easily derived using Theorem 2.2. The central hypothesis of the Holographic Principle is that there exists a mapping between any state $\mathbf{x}(t) = (\mathbf{x}_\mathrm{A}(t), \mathbf{x}_\mathrm{D}(t), \mathbf{x}_\mathrm{T}(t))$ to another state $\mathbf{x}_\partial(t) = (\mathbf{x}_\mathrm{A}(t), \mathbf{E}_\mathrm{D}, \mathbf{E}_\mathrm{T})$ satisfying $V(\mathbf{x}_\partial(t)) = 0$, such that $\mathbf{u}_\mathrm{A}(\mathbf{x}_\partial(t)) = \mathbf{u}_\mathrm{A}(\mathbf{x}(t))$, $\mathbf{u}_\mathrm{D}(\mathbf{x}_\partial(t)) = \mathbf{u}_\mathrm{D}(\mathbf{x}(t))$ and $\mathbf{u}_\mathrm{T}(\mathbf{x}_\partial(t)) = \mathbf{u}_\mathrm{T}(\mathbf{x}(t))$. Note that obviously the value at $\mathbf{x}(t)$ is not the same as the value at $\mathbf{x}_\partial(t)$, but the optimal strategies are the same between these two states, thus can be used to deduce the SFNE. To that end, we define the following rays.

Definition 3.4: A Target Ray $\zeta_\mathrm{T}(\mathbf{x}(t))$, defined for any state $\mathbf{x}(t)$ satisfying $V(\mathbf{x}(t)) = 0$, is described by

$$\zeta_\mathrm{T}(\mathbf{x}(t)) = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} = \mathbf{x}_\mathrm{T}(t) + \delta(\mathbf{I}_c(\mathbf{x}(t)) - \mathbf{x}_\mathrm{T}(t)) \text{ for some } \delta > 0\}. \tag{24}$$

Similarly, a Defender Ray $\zeta_\mathrm{D}(\mathbf{x}(t))$, defined for any state $\mathbf{x}(t)$ satisfying $V(\mathbf{x}(t)) = 0$, is described by

$$\zeta_\mathrm{D}(\mathbf{x}(t)) = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} = \mathbf{x}_\mathrm{D}(t) - \delta(\mathbf{I}_c(\mathbf{x}(t)) - \mathbf{x}_\mathrm{D}(t)) \text{ for some } \delta > 0\}. \tag{25}$$

Here $\mathbf{I}_c(\mathbf{x}(t))$ is the corresponding Critical Collision Point defined by (22), and recall that $V(\mathbf{x}(t)) = 0$ if and only if formula (19) holds. We also need to define the functions $\mathbf{E}_\mathrm{T}(\mathbf{x}(t))$ and $\mathbf{E}_\mathrm{D}(\mathbf{x}(t))$.

Definition 3.5: If $V(\mathbf{x}(t)) > 0$, then the function $\mathbf{E}_\mathrm{T}(\cdot)$ is defined by mapping any state of the differential game $\mathbf{x}(t)$ to the unique point $\mathbf{E}_\mathrm{T}$ satisfying

$$V(\mathbf{x}_\mathrm{A}(t), \mathbf{x}_\mathrm{D}(t), \mathbf{E}_\mathrm{T}) = 0 \tag{26}$$

and

$$\mathbf{x}_\mathrm{T}(t) \in \zeta_\mathrm{T}(\mathbf{x}_\mathrm{A}(t), \mathbf{x}_\mathrm{D}(t), \mathbf{E}_\mathrm{T}). \tag{27}$$

Otherwise if $V(\mathbf{x}(t)) \le 0$, then $\mathbf{E}_\mathrm{T}(\mathbf{x}(t)) = \mathbf{x}_\mathrm{T}(t)$.

Definition 3.6: If $V(\mathbf{x}(t)) < 0$ then the function $\mathbf{E}_\mathrm{D}(\cdot)$ is defined by mapping any state of the differential game $\mathbf{x}(t)$ to a point $\mathbf{E}_\mathrm{D}$ satisfying

$$V(\mathbf{x}_\mathrm{A}(t), \mathbf{E}_\mathrm{D}, \mathbf{x}_\mathrm{T}(t)) = 0, \tag{28}$$

and

$$\mathbf{x}_\mathrm{D}(t) \in \zeta_\mathrm{D}(\mathbf{x}_\mathrm{A}(t), \mathbf{E}_\mathrm{D}, \mathbf{x}_\mathrm{T}(t)). \tag{29}$$

In the instances in which a solution $\mathbf{E}_\mathrm{D}$ to (28) and (29) is not unique, pick the solution for $\mathbf{E}_\mathrm{D}$ that minimises $\overline{\mathbf{x}_\mathrm{D}(t)\mathbf{E}_\mathrm{D}}$. Otherwise in the case $V(\mathbf{x}(t)) \ge 0$ then $\mathbf{E}_\mathrm{D}(\mathbf{x}(t)) = \mathbf{x}_\mathrm{D}(t)$.

Here Lemma 2.1 should be used to determine given any state, whether the value is less than, equal to, or greater than zero. The substitute state $\mathbf{x}_\partial(t) = (\mathbf{x}_\mathrm{A}(t), \mathbf{E}_\mathrm{D}(\mathbf{x}(t)), \mathbf{E}_\mathrm{T}(\mathbf{x}(t)))$ obeys $V(\mathbf{x}_\partial(t)) = 0$, thus Lemma 2.3 can be used to determine the optimal strategies for $\mathbf{x}_\partial(t)$. The Holographic Principle states that the optimal strategies at $\mathbf{x}(t)$ are the same for $\mathbf{x}_\partial(t)$.

Using the terminology defined above, the state-feedback Nash equilibrium of the TAD pursuit-evasion game can be fully characterised by the following theorem.

Theorem 3.7 (Holographic Theorem for TAD differential game): *The value of the pursuit-evasion game defined in Section 2.2 is given by*

$$V(\mathbf{x}(t)) = \overline{\mathbf{x}_\mathrm{T}(t)\mathbf{E}_\mathrm{T}(\mathbf{x}(t))} - \overline{\mathbf{x}_\mathrm{D}(t)\mathbf{E}_\mathrm{D}(\mathbf{x}(t))}, \tag{30}$$

and the optimal state-feedback strategies are given by

$$\mathbf{u}_\mathrm{A}(\mathbf{x}(t)) = \mathbf{u}_\mathrm{A}(\mathbf{x}_\partial(t)), \tag{31a}$$

$$\mathbf{u}_\mathrm{D}(\mathbf{x}(t)) = \mathbf{u}_\mathrm{D}(\mathbf{x}_\partial(t)), \tag{31b}$$

$$\mathbf{u}_\mathrm{T}(\mathbf{x}(t)) = \mathbf{u}_\mathrm{T}(\mathbf{x}_\partial(t)), \tag{31c}$$

where $\boldsymbol{x}_\partial(t) = (\boldsymbol{x}_\mathrm{A}(t), \boldsymbol{E}_\mathrm{D}(\boldsymbol{x}(t)), \boldsymbol{E}_\mathrm{T}(\boldsymbol{x}(t)))$ is the substitute state satisfying $V(\boldsymbol{x}_\partial(t)) = 0$. Thus the optimal controls $\boldsymbol{u}_\mathrm{A}(\boldsymbol{x}_\partial(t))$, $\boldsymbol{u}_\mathrm{D}(\boldsymbol{x}_\partial(t)), \boldsymbol{u}_\mathrm{T}(\boldsymbol{x}_\partial(t))$ can be determined from Lemma 2.3.

The proof of Theorem 3.7 is given in Section 4. The proof leverages previous results in the literature which characterised the SFNE using formulas (43)–(44); the proof of Theorem 4.1 verifies that Theorem 3.7 provides an equivalent characterisation of the SFNE.

3.3 Application of the Holographic Theorem

The Holographic Theorem completely characterises the state-feedback Nash equilibrium of the TAD pursuit-evasion game, by transforming any state $\mathbf{x}(t)$ into the substitute state $\mathbf{x}_\partial(t) = (\mathbf{x}_\text{A}(t), \mathbf{E}_\text{D}(\mathbf{x}(t)), \mathbf{E}_\text{T}(\mathbf{x}(t)))$ satisfying $V(\mathbf{x}_\partial(t)) = 0$. This section outlines explicitly the mathematical formulas used to calculate $\mathbf{E}_\text{D}(\mathbf{x}(t))$ and $\mathbf{E}_\text{T}(\mathbf{x}(t))$. To that end, let

$$\phi = \angle \mathbf{x}_\text{T}(t)\mathbf{x}_\text{A}(t)\mathbf{x}_\text{D}(t), \tag{32}$$

denote the angle between vectors $\mathbf{x}_\text{T}(t) - \mathbf{x}_\text{A}(t)$ and $\mathbf{x}_\text{D}(t) - \mathbf{x}_\text{A}(t)$, and let the function $g(\psi)$ be defined by

$$g(\psi) = \frac{1}{1-\gamma^2}(\cos\psi + \alpha\gamma - \sqrt{(\cos\psi + \alpha\gamma)^2 - (1-\alpha^2)(1-\gamma^2)}). \tag{33}$$

Theorem 3.8: *The substitute state $\boldsymbol{x}_\partial(t) = (\boldsymbol{x}_\text{A}(t), \boldsymbol{E}_\text{D}(\boldsymbol{x}(t)), \boldsymbol{E}_\text{T}(\boldsymbol{x}(t)))$ can be computed in the case $V(\boldsymbol{x}(t)) < 0$ with*

$$\mathbf{E}_\text{D}(\mathbf{x}(t)) = \mathbf{x}_\text{A}(t) + \frac{\overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{T}(t)}}{g(\psi(\mathbf{x}(t)))}\cos\psi(\mathbf{x}(t))\,\hat{\mathbf{n}}_1(\mathbf{x}(t)) + \frac{\overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{T}(t)}}{g(\psi(\mathbf{x}(t)))}\sin\psi(\mathbf{x}(t))\hat{\mathbf{n}}_2(\mathbf{x}(t)), \tag{34}$$

where

$$\hat{\mathbf{n}}_1(\mathbf{x}(t)) = \frac{\mathbf{x}_\text{T}(t) - \mathbf{x}_\text{A}(t)}{\|\mathbf{x}_\text{T}(t) - \mathbf{x}_\text{A}(t)\|},$$
$$\hat{\mathbf{n}}_2(\mathbf{x}(t)) = \frac{\mathbf{x}_\text{D}(t) - \mathbf{x}_\text{A}(t) - \text{proj}_{\hat{\mathbf{n}}_1(\mathbf{x}(t))}(\mathbf{x}_\text{D}(t) - \mathbf{x}_\text{A}(t))}{\mathbf{x}_\text{D}(t) - \mathbf{x}_\text{A}(t) - \text{proj}_{\hat{\mathbf{n}}_1(\mathbf{x}(t))}(\mathbf{x}_\text{D}(t) - \mathbf{x}_\text{A}(t))},$$

and where $\psi(\boldsymbol{x}(t)) \in [0, \pi]$ is a solution to

$$(\alpha\gamma - \frac{1-\alpha^2}{g(\psi)})\sin\psi\left(\overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{D}(t)}\cos\phi - \frac{\overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{T}(t)}}{g(\psi)}\cos\psi\right) = (1 + (\alpha\gamma - \frac{1-\alpha^2}{g(\psi)})\cos\psi) \times \left(\overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{D}(t)}\sin\phi - \frac{\overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{T}(t)}}{g(\psi)}\sin\psi\right). \tag{35}$$

In the instances in which there are multiple solutions ψ between 0 and π to formula (35), *pick the angle ψ generating $\boldsymbol{E}_\text{D}(\boldsymbol{x}(t))$ which yields the smallest value for $\overline{\boldsymbol{x}_\text{D}(t)\boldsymbol{E}_\text{D}(\boldsymbol{x}(t))}$.*

In the case $V(\boldsymbol{x}(t)) > 0$, the substitute state can be computed from

$$\mathbf{E}_\text{T}(\mathbf{x}(t)) = \mathbf{x}_\text{A}(t) + \overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{D}(t)}g(\psi(\mathbf{x}(t)))\cos\psi(\mathbf{x}(t))\hat{\mathbf{n}}_3(\mathbf{x}(t)) + \overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{D}(t)}g(\psi(\mathbf{x}(t)))\sin\psi(\mathbf{x}(t))\hat{\mathbf{n}}_4(\mathbf{x}(t)), \tag{36}$$

where

$$\hat{\mathbf{n}}_3(\mathbf{x}(t)) = \frac{\mathbf{x}_\text{D}(t) - \mathbf{x}_\text{A}(t)}{\|\mathbf{x}_\text{D}(t) - \mathbf{x}_\text{A}(t)\|},$$
$$\hat{\mathbf{n}}_4(\mathbf{x}(t)) = \frac{\mathbf{x}_\text{T}(t) - \mathbf{x}_\text{A}(t) - \text{proj}_{\hat{\mathbf{n}}_3(\mathbf{x}(t))}(\mathbf{x}_\text{T}(t) - \mathbf{x}_\text{A}(t))}{\mathbf{x}_\text{T}(t) - \mathbf{x}_\text{A}(t) - \text{proj}_{\hat{\mathbf{n}}_3(\mathbf{x}(t))}(\mathbf{x}_\text{T}(t) - \mathbf{x}_\text{A}(t))},$$

and in this case $\psi(\boldsymbol{x}(t)) \in [0, \pi]$ is the unique solution to

$$(\alpha\gamma - (1-\gamma^2)g(\psi))\sin\psi(\overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{T}(t)}\cos\phi - \overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{D}(t)}g(\psi)\cos\psi) = (1 + (\alpha\gamma - (1-\gamma^2)g(\psi))\cos\psi)(\overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{T}(t)}\sin\phi - \overline{\mathbf{x}_\text{A}(t)\mathbf{x}_\text{D}(t)}g(\psi)\sin\psi). \tag{37}$$

Proof: Let $\mathbf{x} = (\mathbf{x}_\text{A}, \mathbf{x}_\text{D}, \mathbf{x}_\text{T})$ denote any state of the differential game defined in Section 2.2. The proof for the case $V(\mathbf{x}) > 0$ has already been given in the earlier work of Mammadov et al. (2022), thus the proof for this case is omitted. In the case the state $\mathbf{x}$ is in the region $V(\mathbf{x}) < 0$, we must find the substitute state $\mathbf{x}_\partial = (\mathbf{x}_\text{A}, \mathbf{E}_\text{D}, \mathbf{x}_\text{T})$ satisfying (28) and (29), that is

$$V(\mathbf{x}_\partial) = 0,$$

and

$$\mathbf{x}_\text{D} = \mathbf{E}_\text{D} - \delta(\mathbf{I}_c(\mathbf{x}_\partial) - \mathbf{E}_\text{D}), \quad \text{for some } \delta > 0.$$

Recall from Lemma 2.1 that a state $\mathbf{x}_\partial = (\mathbf{x}_\text{A}, \mathbf{E}_\text{D}, \mathbf{x}_\text{T})$ obeys $V(\mathbf{x}_\partial) = 0$ if and only if

$$\frac{\overline{\mathbf{x}_\text{A}\mathbf{x}_\text{T}}}{\overline{\mathbf{x}_\text{A}\mathbf{E}_\text{D}}} = g(\psi), \quad \text{where}$$
$$g(\psi) = \frac{1}{1-\gamma^2}(\cos\psi + \alpha\gamma - \sqrt{(\cos\psi + \alpha\gamma)^2 - (1-\alpha^2)(1-\gamma^2)}),$$

and where $\psi = \angle \mathbf{x}_\text{T}\mathbf{x}_\text{A}\mathbf{E}_\text{D}$. As a result, clearly we can parameterise the set of all possible points $\mathbf{E}_\text{D}$ satisfying (28) and (29) by

$$\mathbf{E}_\text{D} = \mathbf{x}_\text{A} + \frac{\overline{\mathbf{x}_\text{A}\mathbf{x}_\text{T}}}{g(\psi)}\cos\psi\hat{\mathbf{n}}_1 + \frac{\overline{\mathbf{x}_\text{A}\mathbf{x}_\text{T}}}{g(\psi)}\sin\psi\hat{\mathbf{n}}_2, \tag{38}$$

where

$$\hat{\mathbf{n}}_1 = \frac{\mathbf{x}_\text{T} - \mathbf{x}_\text{A}}{\|\mathbf{x}_\text{T} - \mathbf{x}_\text{A}\|},$$
$$\hat{\mathbf{n}}_2 = \frac{\mathbf{x}_\text{D} - \mathbf{x}_\text{A} - \text{proj}_{\hat{\mathbf{n}}_1}(\mathbf{x}_\text{D} - \mathbf{x}_\text{A})}{\mathbf{x}_\text{D} - \mathbf{x}_\text{A} - \text{proj}_{\hat{\mathbf{n}}_1}(\mathbf{x}_\text{D} - \mathbf{x}_\text{A})}.$$

Here ψ is the as of yet unknown parameter we must compute. In comparison, the position of the Defender can be expressed as

$$\mathbf{x}_\text{D} = \mathbf{x}_\text{A} + \overline{\mathbf{x}_\text{A}\mathbf{x}_\text{D}}\cos\phi\hat{\mathbf{n}}_1 + \overline{\mathbf{x}_\text{A}\mathbf{x}_\text{D}}\sin\phi\hat{\mathbf{n}}_2,$$

where $\phi = \angle \mathbf{x}_\text{T}\mathbf{x}_\text{A}\mathbf{x}_\text{D}$. Thus the difference $\mathbf{x}_\text{D} - \mathbf{E}_\text{D}$ is

$$\mathbf{x}_\text{D} - \mathbf{E}_\text{D} = (\overline{\mathbf{x}_\text{A}\mathbf{x}_\text{D}}\cos\phi - \frac{\overline{\mathbf{x}_\text{A}\mathbf{x}_\text{T}}}{g(\psi)}\cos\psi)\hat{\mathbf{n}}_1$$

$$+ (\overline{\mathbf{x}_A\mathbf{x}_D}\sin\phi - \frac{\overline{\mathbf{x}_A\mathbf{x}_T}}{g(\psi)}\sin\psi)\hat{\mathbf{n}}_2. \tag{39}$$

On the other-hand, to obtain the difference $\mathbf{I}_c(\mathbf{x}_\partial) - \mathbf{E}_D$, we first must parameterise $\mathbf{I}_c(\mathbf{x}_\partial)$ with respect to $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$ and ψ. To that end, the Critical Collision Point can be expressed as

$$\mathbf{I}_c(\mathbf{x}_\partial) = \mathbf{c}_{AT}(\mathbf{x}_\partial) + r_{AT}(\mathbf{x}_\partial)\cos\rho\hat{\mathbf{n}}_1 + r_{AT}(\mathbf{x}_\partial)\sin\rho\hat{\mathbf{n}}_2, \tag{40}$$

for some angle ρ. Note that since in this case the substitute state only changes the position of the Defender, we have that $\mathbf{c}_{AT}(\mathbf{x}_\partial) = \mathbf{c}_{AT}(\mathbf{x})$ and $r_{AT}(\mathbf{x}_\partial) = r_{AT}(\mathbf{x})$, hence the only unknown in the above expression for $\mathbf{I}_c(\mathbf{x}_\partial)$ is ρ. We can find ρ by expressing it in terms of another angle $\theta = \angle\mathbf{c}_{AT}(\mathbf{x}_\partial)\mathbf{c}_{AD}(\mathbf{x}_\partial)\mathbf{E}_D$. After two simple steps it can be shown that these angles are related by

$$\rho = \psi - \theta.$$

Moreover, via geometry it can easily be shown that the angle θ can be found with

$$(r_{AD}(\mathbf{x}_\partial) - r_{AT}(\mathbf{x}_\partial))\sin\theta = (\overline{\mathbf{x}_A\mathbf{x}_T} + \overline{\mathbf{x}_T\mathbf{c}_{AT}(\mathbf{x}_\partial)})\sin\psi,$$

$$(r_{AD}(\mathbf{x}_\partial) - r_{AT}(\mathbf{x}_\partial))\cos\theta = (\overline{\mathbf{x}_A\mathbf{x}_T} + \overline{\mathbf{x}_T\mathbf{c}_{AT}(\mathbf{x}_\partial)})\cos\psi + \overline{\mathbf{x}_A\mathbf{c}_{AD}(\mathbf{x}_\partial)}.$$

Combining the above results the Critical Collision Point can be explicitly calculated as follows:

$$\begin{aligned}
\mathbf{I}_c(\mathbf{x}_\partial) &= \mathbf{c}_{AT}(\mathbf{x}_\partial) + r_{AT}(\mathbf{x}_\partial)\cos\rho\hat{\mathbf{n}}_1 + r_{AT}(\mathbf{x}_\partial)\sin\rho\hat{\mathbf{n}}_2 \\
&= \mathbf{x}_A + (\overline{\mathbf{x}_A\mathbf{x}_T} + \overline{\mathbf{x}_T\mathbf{c}_{AT}})\hat{\mathbf{n}}_1 + r_{AT}(\cos\rho\hat{\mathbf{n}}_1 + \sin\rho\hat{\mathbf{n}}_2) \\
&= \mathbf{x}_A + r_{AT}\left(\left(\frac{1}{\alpha} + \cos\rho\right)\hat{\mathbf{n}}_1 + \sin\rho\hat{\mathbf{n}}_2\right) \\
&= \mathbf{x}_A + r_{AT}\left(\left(\frac{1}{\alpha} + \cos\psi\cos\theta + \sin\psi\sin\theta\right)\hat{\mathbf{n}}_1\right. \\
&\quad \left. + (\sin\psi\cos\theta - \cos\psi\sin\theta)\,\hat{\mathbf{n}}_2\right) \\
&= \mathbf{x}_A + \frac{r_{AT}}{r_{AD} - r_{AT}}\left(\left(\frac{r_{AD} - r_{AT}}{\alpha}\right.\right. \\
&\quad + \cos\psi\left(\frac{r_{AT}}{\alpha}\cos\psi + \frac{r_{AD}}{\gamma}\right) \\
&\quad \left. + \sin\psi\left(\frac{r_{AT}}{\alpha}\sin\psi\right)\right)\hat{\mathbf{n}}_1 \\
&\quad + \left(\sin\psi\left(\frac{r_{AT}}{\alpha}\cos\psi + \frac{r_{AD}}{\gamma}\right)\right. \\
&\quad \left.\left. - \cos\psi\left(\frac{r_{AT}}{\alpha}\sin\psi\right)\right)\hat{\mathbf{n}}_2\right) \\
&= \mathbf{x}_A + \frac{r_{AT}^2}{\alpha(r_{AD} - r_{AT})} \\
&\quad \left(\left(\frac{r_{AD} - r_{AT}}{r_{AT}} + \cos^2\psi + \frac{\alpha r_{AD}}{\gamma r_{AT}}\cos\psi + \sin^2\psi\right)\hat{\mathbf{n}}_1\right. \\
&\quad \left. + \left(\sin\psi\cos\psi + \frac{\alpha r_{AD}}{\gamma r_{AT}}\sin\psi - \cos\psi\sin\psi\right)\hat{\mathbf{n}}_2\right) \\
&= \mathbf{x}_A + \frac{r_{AT}^2}{\alpha(r_{AD} - r_{AT})} \\
&\quad \times\left(\left(\frac{r_{AD}}{r_{AT}} + \frac{\alpha r_{AD}}{\gamma r_{AT}}\cos\psi\right)\hat{\mathbf{n}}_1 + \frac{\alpha r_{AD}}{\gamma r_{AT}}\sin\psi\hat{\mathbf{n}}_2\right) \\
&= \mathbf{x}_A + \frac{r_{AT}r_{AD}}{\gamma(r_{AD} - r_{AT})}\left(\left(\frac{\gamma}{\alpha} + \cos\psi\right)\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2\right).
\end{aligned}$$

Here $r_{AT} = \frac{\alpha}{1-\alpha^2}\overline{\mathbf{x}_A\mathbf{x}_T}$ and $r_{AD} = \frac{\gamma}{\gamma^2-1}\frac{\overline{\mathbf{x}_A\mathbf{x}_T}}{g(\psi)}$. Consequently, the difference $\mathbf{I}_c(\mathbf{x}_\partial) - \mathbf{E}_D$ is given by

$$\begin{aligned}
\mathbf{I}_c(\mathbf{x}_\partial) - \mathbf{E}_D &= \frac{r_{AT}r_{AD}}{\gamma(r_{AD} - r_{AT})}\left(\left(\frac{\gamma}{\alpha} + \cos\psi\right)\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2\right) \\
&\quad - \frac{\overline{\mathbf{x}_A\mathbf{x}_T}}{g(\psi)}(\cos\psi\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2) \\
&= \frac{r_{AT}r_{AD}}{\gamma(r_{AD} - r_{AT})}\left(\left(\frac{\gamma}{\alpha} + \cos\psi\right)\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2\right) \\
&\quad - \frac{(\gamma^2 - 1)r_{AD}}{\gamma}(\cos\psi\,\hat{\mathbf{n}}_1 + \sin\psi\,\hat{\mathbf{n}}_2) \\
&= \frac{r_{AT}r_{AD}}{\gamma(r_{AD} - r_{AT})}\left(\left(\frac{\gamma}{\alpha} + \cos\psi\right)\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2\right. \\
&\quad \left. - \frac{(\gamma^2 - 1)(r_{AD} - r_{AT})}{r_{AT}}(\cos\psi\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2)\right).
\end{aligned}$$

Here $\frac{r_{AT}}{r_{AD}-r_{AT}} = \frac{\alpha(\gamma^2-1)g(\psi)}{\gamma(1-\alpha^2)-\alpha(\gamma^2-1)g(\psi)}$. Since the maximum value of $g(\psi)$ occurs at $\psi = \pi$, and since $g(\pi) = \frac{1}{\gamma^2-1}(1 - \alpha\gamma + \gamma - \alpha)$, we have that $r_{AD} - r_{AT} > 0$. As a result, the difference $\delta(\mathbf{I}_c(\mathbf{x}_\partial) - \mathbf{E}_D)$, weighted by some positive scalar δ is given by

$$\begin{aligned}
\delta(\mathbf{I}_c(\mathbf{x}_\partial) - \mathbf{E}_D) &= \alpha(\gamma^2 - 1)g(\psi) \\
&\quad \times\left(\left(\frac{\gamma}{\alpha} + \cos\psi\right)\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2\right) \\
&\quad - (\gamma^2 - 1)(\gamma(1 - \alpha^2) - \alpha(\gamma^2 - 1)g(\psi)) \\
&\quad \times(\cos\psi\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2) \\
&= \gamma(\gamma^2 - 1)g(\psi)\hat{\mathbf{n}}_1 + (\gamma(1 - \alpha^2) - \gamma^3(1 - \alpha^2) \\
&\quad + \alpha\gamma^2(\gamma^2 - 1)g(\psi))(\cos\psi\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2) \\
&= \gamma(\gamma^2 - 1)g(\psi)\hat{\mathbf{n}}_1 + (\gamma(1 - \alpha^2)(1 - \gamma^2) \\
&\quad + \alpha\gamma^2(\gamma^2 - 1)g(\psi))(\cos\psi\,\hat{\mathbf{n}}_1 + \sin\psi\hat{\mathbf{n}}_2) \\
&= g(\psi)\hat{\mathbf{n}}_1 + (\alpha^2 - 1 + \alpha\gamma g(\psi))(\cos\psi\hat{\mathbf{n}}_1 + \sin\psi\,\hat{\mathbf{n}}_2) \\
&= (g(\psi) + (\alpha\gamma g(\psi) - 1 + \alpha^2)\cos\psi)\hat{\mathbf{n}}_1 \\
&\quad + (\alpha\gamma g(\psi) - 1 + \alpha^2)\sin\psi\hat{\mathbf{n}}_2.
\end{aligned} \tag{41}$$

Note that in the above calculation for (41), the terms are not necessarily equal, but proportional, since δ is an arbitrary positive scalar. Thus combining the results (39) and (41), for any state $\mathbf{x} = (\mathbf{x}_A, \mathbf{x}_D, \mathbf{x}_T)$ in the region $V(\mathbf{x}) < 0$, constraints (28)–(29) hold if and only if $\mathbf{E}_D$ is given by (38) and ψ satisfies

$$(\alpha\gamma g(\psi) - 1 + \alpha^2)\sin\psi\left(\overline{\mathbf{x}_A\mathbf{x}_D}\cos\phi - \frac{\overline{\mathbf{x}_A\mathbf{x}_T}}{g(\psi)}\cos\psi\right)$$

$$= (g(\psi) + (\alpha\gamma g(\psi) - 1 + \alpha^2)\cos\psi)$$

$$\times \left(\overline{\mathbf{x}_A \mathbf{x}_D} \sin \phi - \frac{\overline{\mathbf{x}_A \mathbf{x}_T}}{g(\psi)} \sin \psi \right).$$

■

3.4 Numerical example

Consider the numerical example:

$$\mathbf{x}_A(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x}_D(t) = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{x}_T(t) = \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix},$$
$$(V_T, V_A, V_D) = (0.8, 1.2, 1.7) \tag{42}$$

To find the SFNE, first we must determine whether or not the Target escapes capture under optimal play; that is, we must determine which of $V(\mathbf{x}(t)) > 0$, $V(\mathbf{x}(t)) = 0$, $V(\mathbf{x}(t)) < 0$ is true. Recall Lemma 2.1, $V(\mathbf{x}(t)) > 0$ if and only if $V_A \overline{\mathbf{x}_T(t)\mathbf{x}_D(t)} < V_D \overline{\mathbf{x}_A(t)\mathbf{x}_T(t)} + V_T \overline{\mathbf{x}_A(t)\mathbf{x}_D(t)}$. Computing the distance between the Target and Defender multiplied by V_A we get 7.6837, whereas the right-hand side computes to 13.1328, thus we now know that $V(\mathbf{x}(t)) > 0$.

The next step is to compute $\phi = \angle \mathbf{x}_T(t)\mathbf{x}_A(t)\mathbf{x}_D(t)$, which can be computed using Matlab's acos function with the command acos$\left(\frac{(\mathbf{x}_T(t)-\mathbf{x}_A(t))\cdot(\mathbf{x}_D(t)-\mathbf{x}_A(t))}{\|\mathbf{x}_T(t)-\mathbf{x}_A(t)\|\|\mathbf{x}_D(t)-\mathbf{x}_A(t)\|}\right)$, giving us $\phi = 1.3739$

Applying Theorem 3.8, the angle ψ is the unique solution to

$$\begin{aligned}(0.9444 + 1.0069g(\psi)) \sin \psi (1.2060 - 3.3166g(\psi) \cos \psi) \\ = (1 + (0.9444 + 1.0069g(\psi)) \cos \psi) \\ \times (6.0453 - 3.3166g(\psi) \sin \psi),\end{aligned}$$

where $g(\psi) = -0.9931(\cos \psi + 0.9444 - \sqrt{(\cos \psi + 0.9444)^2 + 0.5594})$. Using a non-linear equation solver, we get that the solution is $\psi = 2.0085$. Thus $\mathbf{E}_T(\mathbf{x}(t))$ is given by

$$\mathbf{E}_T(\mathbf{x}(t)) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 0.5455\hat{\mathbf{n}}_3(\mathbf{x}(t)) + 1.1655\hat{\mathbf{n}}_4(\mathbf{x}(t)),$$

where the vectors $\hat{\mathbf{n}}_3(\mathbf{x}(t))$ and $\hat{\mathbf{n}}_3(\mathbf{x}(t))$ can be found from the formulas just above (37), hence:

$$\hat{\mathbf{n}}_3(\mathbf{x}(t)) = \begin{bmatrix} -0.3015 \\ -0.9045 \\ -0.3015 \end{bmatrix}, \quad \hat{\mathbf{n}}_4(\mathbf{x}(t)) = \begin{bmatrix} -0.9324 \\ 0.3459 \\ -0.1053 \end{bmatrix},$$
$$\text{and} \quad \mathbf{E}_T(\mathbf{x}(t)) = \begin{bmatrix} 0.0778 \\ 0.8965 \\ 2.0418 \end{bmatrix}.$$

Thus we conclude that the substitute state is given by

$$\mathbf{x}_\partial(t) = (\mathbf{x}_A(t), \mathbf{E}_D(\mathbf{x}(t)), \mathbf{E}_T(\mathbf{x}(t))) = \left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.0778 \\ 0.8965 \\ 2.0418 \end{bmatrix} \right).$$

At the substitute state, the Critical Collision Point is given by $\mathbf{I}_c(\mathbf{x}_\partial(t)) = \begin{bmatrix} -1.9724 \\ 0.9383 \\ 1.6212 \end{bmatrix}$ from (22). Thus using Lemma 2.3 we deduce that the optimal headings at the substitute state is given by

$$\mathbf{u}_A(\mathbf{x}_\partial(t)) = \begin{bmatrix} -0.9466 \\ 0.2988 \\ -0.1207 \end{bmatrix}, \quad \mathbf{u}_D(\mathbf{x}_\partial(t)) = \begin{bmatrix} -0.4434 \\ 0.8854 \\ 0.1396 \end{bmatrix},$$
$$\mathbf{u}_T(\mathbf{x}_\partial(t)) = \begin{bmatrix} -0.9794 \\ 0.0200 \\ -0.2009 \end{bmatrix},$$

and since the transformation $\mathbf{x}(t) \mapsto \mathbf{x}_\partial(t)$ preserves the optimal headings, we deduce that the matrices above also denote the optimal headings for $\mathbf{u}_A(\mathbf{x}(t))$, $\mathbf{u}_D(\mathbf{x}(t))$ and $\mathbf{u}_T(\mathbf{x}(t))$ respectively. Finally the value at state $\mathbf{x}(t)$ is given by $V(\mathbf{x}(t)) = \overline{\mathbf{x}_T(t)\mathbf{E}_T(\mathbf{x}(t))} - \overline{\mathbf{x}_D(t)\mathbf{E}_D(\mathbf{x}(t))} = 5.1846$.

4. Symmetry breaking

4.1 Symmetry breaking theorem

Beyond the symmetry breaking described earlier where the Holographic Principle separates into Target Symmetry in the case where $V > 0$, and Defender Symmetry in the case where $V < 0$. Target Symmetry and Defender Symmetry itself undergo further symmetry breaking when the value function is characterised not by the points $\mathbf{E}_T(\mathbf{x}(t))$ and $\mathbf{E}_D(\mathbf{x}(t))$, but rather by the Optimal Collision Point $\mathbf{I}(\mathbf{x}(t))$ defined as the point at which the Attacker collides under optimal play; that is $\mathbf{I}(\mathbf{x}(t)) = \mathbf{x}_A(t_f)$ in the SFNE.

More specifically, Target Symmetry divides into two distinct optimality principles given by Case A and Case B; and Defender Symmetry devolves into another optimality principle in Case C. These cases are defined as follows.

Case A: $V(\mathbf{x}(t)) \geq 0$ and $\|\mathbf{c}_{AD}(\mathbf{x}(t)) - \mathbf{x}_T(t)\| \leq r_{AD}(\mathbf{x}(t))$.
Case B: $V(\mathbf{x}(t)) \geq 0$ and $\|\mathbf{c}_{AD}(\mathbf{x}(t)) - \mathbf{x}_T(t)\| \geq r_{AD}(\mathbf{x}(t))$.
Case C: $V(\mathbf{x}(t)) \leq 0$.

The following theorem reveals that Theorem 3.7 provides a ground-breaking unification of the methods used in the earlier works of Mammadov et al. (2020, 2021, 2022), Garcia et al. (2017, 2019, 2021) and others.

Theorem 4.1 (Symmetry Breaking): *Formula* (30) *is equivalent to*

$$V(\mathbf{x}(t)) = \begin{cases} -\overline{\mathbf{x}_T(t)\mathbf{I}(\mathbf{x}(t))} + \dfrac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}(\mathbf{x}(t))} & \textit{Case A} \\ \overline{\mathbf{x}_T(t)\mathbf{I}(\mathbf{x}(t))} + \dfrac{V_T}{V_A}\overline{\mathbf{x}_A(t)\mathbf{I}(\mathbf{x}(t))} & \textit{Case B} \\ -\overline{\mathbf{x}_D(t)\mathbf{I}(\mathbf{x}(t))} + \dfrac{V_D}{V_T}\overline{\mathbf{x}_T(t)\mathbf{I}(\mathbf{x}(t))} & \textit{Case C} \end{cases} \tag{43}$$

where $\mathbf{I}(\mathbf{x}(t))$ is the Optimal Collision Point defined by

$$\mathbf{I}(\mathbf{x}(t)) = \begin{cases} \arg \max\limits_{\mathbf{I}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t))} -\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}} + \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t)\mathbf{I}} & \textit{Case A} \\ \arg \min\limits_{\mathbf{I}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t))} \overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}} + \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t)\mathbf{I}} & \textit{Case B} \\ \arg \max\limits_{\mathbf{I}\in\mathcal{C}_{\mathcal{AT}}(\mathbf{x}(t))} -\overline{\mathbf{x}_\mathrm{D}(t)\mathbf{I}} + \frac{V_\mathrm{D}}{V_\mathrm{T}}\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}} & \textit{Case C} \end{cases} \tag{44}$$

Here $\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t))$ and $\mathcal{C}_{\mathcal{AT}}(\mathbf{x}(t))$ denotes the set of all points on the surface of the AD and AT-based Apollonius circles respectively, defined by (16) *and* (17).

Proof: We shall first prove formula (30) is equivalent to formula (43) in Case A and B. In the case $V(\mathbf{x}(t)) \geq 0$, we have $\mathbf{E}_\mathrm{D}(\mathbf{x}(t)) = \mathbf{x}_\mathrm{D}(t)$. Thus formula (30) simplifies to

$$V(\mathbf{x}(t)) = \overline{\mathbf{x}_\mathrm{T}(t)\mathbf{E}_\mathrm{T}(\mathbf{x}(t))},$$

and the substitute state is given by $\mathbf{x}_\partial(t) = (\mathbf{x}_\mathrm{A}(t), \mathbf{x}_\mathrm{D}(t), \mathbf{E}_\mathrm{T}(\mathbf{x}(t)))$. The Critical Collision Point $\mathbf{I}_c(\mathbf{x}_\partial(t))$ is at the intersection between the Target Ray $\zeta_\mathrm{T}(\mathbf{x}_\partial(t))$ and the AD-based Apollonius circle $\mathcal{C}_{\mathcal{AD}}(\mathbf{x}_\partial(t))$. Thus characterising the above formula in terms of $\mathbf{I}_c(\mathbf{x}_\partial(t))$ yields:

$$\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{E}_\mathrm{T}(\mathbf{x}(t))} = \begin{cases} -\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))} + \overline{\mathbf{E}_\mathrm{T}(\mathbf{x}(t))\mathbf{I}_c(\mathbf{x}_\partial(t))} & \text{if } \|\mathbf{c}_\mathrm{AD}(\mathbf{x}_\partial(t)) - \mathbf{x}_\mathrm{T}(t)\| \leq r_\mathrm{AD}(\mathbf{x}_\partial(t)) \\ \overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))} + \overline{\mathbf{E}_\mathrm{T}(\mathbf{x}(t))\mathbf{I}_c(\mathbf{x}_\partial(t))} & \text{if } \|\mathbf{c}_\mathrm{AD}(\mathbf{x}_\partial(t)) - \mathbf{x}_\mathrm{T}(t)\| \geq r_\mathrm{AD}(\mathbf{x}_\partial(t)) \end{cases}$$

We know that for any state $\mathbf{y} = (\mathbf{y}_A, \mathbf{y}_D, \mathbf{y}_T)$ satisfying $V(\mathbf{y}) = 0$,

$$V_\mathrm{A} V_\mathrm{D}\overline{\mathbf{y}_T\mathbf{I}_c(\mathbf{y})} = V_\mathrm{D} V_\mathrm{T}\overline{\mathbf{y}_A\mathbf{I}_c(\mathbf{y})} = V_\mathrm{A} V_\mathrm{T}\overline{\mathbf{y}_D\mathbf{I}_c(\mathbf{y})}, \tag{45}$$

holds, where $\mathbf{I}_c(\cdot)$ is the Critical Collision Point defined in (22). This is due to Lemma 2.1 and Theorem 2.2; since under optimal all agents arrive at $\mathbf{I}_c(\mathbf{y})$ at the same time, we have $\Delta t = \frac{1}{V_\mathrm{T}}\overline{\mathbf{y}_T\mathbf{I}_c(\mathbf{y})} = \frac{1}{V_\mathrm{A}}\overline{\mathbf{y}_A\mathbf{I}_c(\mathbf{y})} = \frac{1}{V_\mathrm{D}}\overline{\mathbf{y}_D\mathbf{I}_c(\mathbf{y})}$, where Δt is the time taken for either of the three agents to travel from their initial position to $\mathbf{I}_c(\mathbf{y})$. Thus (45) holds.

Since the substitute state $\mathbf{x}_\partial(t)$ satisfies $V(\mathbf{x}_\partial(t)) = 0$, applying (45) we have that

$$\overline{\mathbf{E}_\mathrm{T}(\mathbf{x}(t))\mathbf{I}_c(\mathbf{x}_\partial(t))} = \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))}. \tag{46}$$

Moreover, since the AD-based Apollonius circle only depends upon the position of the Attacker and Defender, we have that $\mathbf{c}_\mathrm{AD}(\mathbf{x}_\partial(t)) = \mathbf{c}_\mathrm{AD}(\mathbf{x}(t))$, $r_\mathrm{AD}(\mathbf{x}_\partial(t)) = r_\mathrm{AD}(\mathbf{x}(t))$ and $\mathcal{C}_{\mathcal{AD}}(\mathbf{x}_\partial(t)) = \mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t))$. Thus the value function is given by

$$V(\mathbf{x}(t)) = \begin{cases} -\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))} + \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))} & \text{if } \|\mathbf{c}_\mathrm{AD}(\mathbf{x}(t)) - \mathbf{x}_\mathrm{T}(t)\| \leq r_\mathrm{AD}(\mathbf{x}(t)) \\ \overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))} + \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))} & \text{if } \|\mathbf{c}_\mathrm{AD}(\mathbf{x}(t)) - \mathbf{x}_\mathrm{T}(t)\| \geq r_\mathrm{AD}(\mathbf{x}(t)) \end{cases} \tag{47}$$

for all states $V(\mathbf{x}(t)) \geq 0$. Therefore, to prove that (47) is equivalent to (43) in Cases A and B, we must verify that the Optimal Collision Point $\mathbf{I}(\mathbf{x}(t))$ defined by (44) is equivalent to $\mathbf{I}_c(\mathbf{x}_\partial(t))$. First let us consider Case A. Here $\mathbf{I}(\mathbf{x}(t))$ is given by

$$\begin{aligned} \mathbf{I}(\mathbf{x}(t)) &= \arg \max_{\mathbf{I}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t))} -\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}} + \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t)\mathbf{I}} \\ &= \arg \max_{\mathbf{I}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t))} -\frac{1}{V_\mathrm{T}}\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}} + \frac{1}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t)\mathbf{I}}. \end{aligned} \tag{48}$$

Formula (48) may be interpreted as finding the point on the AD-based Apollonius circle such that the time difference between the Target reaching that point versus the Attacker reaching that point is maximised, so that the Target can reach that point much sooner than the Attacker.

Consider the special case where $V(\mathbf{x}(t)) = 0$. In this case, it is obvious that $\mathbf{I}(\mathbf{x}(t)) = \mathbf{I}_c(\mathbf{x}_\partial(t))$, because the maximum time difference is zero since $V(\mathbf{x}(t)) = 0$, and $\mathbf{I}_c(\mathbf{x}_\partial(t))$ is the only point that achieves that time difference.

Moreover, due to the interpretation provided above, formula (48) obeys Target Symmetry. Seeing as if the Target was to move in its optimal heading defined by (48), then the remaining time to reach that point is minimised more than any other point on the AD-based Apollonius circle, and thus the optimal heading defined by (48) remains unchanged in the next time increment. Consequently, $\mathbf{I}(\mathbf{x}(t)) = \mathbf{I}_c(\mathbf{x}_\partial(t))$ for all states in Case A.

Next we consider Case B. Here we have

$$\begin{aligned} \mathbf{I}(\mathbf{x}(t)) &= \arg \min_{\mathbf{I}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t))} \overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}} + \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t)\mathbf{I}} \\ &= \arg \min_{\mathbf{I}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t))} \frac{1}{V_\mathrm{T}}\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}} + \frac{1}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t)\mathbf{I}}. \end{aligned} \tag{49}$$

We may interpret the formula above as the problem of finding the optimal path to go from point $\mathbf{x}_\mathrm{T}(t)$ to point $\mathbf{x}_\mathrm{A}(t)$ in minimum time, where whilst on the outside of the AD-based Apollonius circle we traverse at the speed V_T, whereas on the inside of the AD-based Apollonius circle we traverse at the faster speed of V_A.

Obviously in the special case where $\|\mathbf{c}_\mathrm{AD}(\mathbf{x}(t)) - \mathbf{x}_\mathrm{T}(t)\| = r_\mathrm{AD}(\mathbf{x}(t))$, $\mathbf{I}(\mathbf{x}(t)) = \mathbf{x}_\mathrm{T}(t)$ in both formulas (48) and (49). Since we already proved that $\mathbf{I}(\mathbf{x}(t)) = \mathbf{I}_c(\mathbf{x}_\partial(t))$ for formula (48), this implies that $\mathbf{I}_c(\mathbf{x}_\partial(t)) = \mathbf{x}_\mathrm{T}(t)$. Thus in this special case, $\mathbf{I}_c(\mathbf{x}_\partial(t)) = \mathbf{I}(\mathbf{x}(t)) = \mathbf{x}_\mathrm{T}(t)$.

Moreover, due to the interpretation given to formula (49), $\mathbf{I}(\mathbf{x}(t))$ defined in (49) obeys Target Symmetry; seeing as though if the Target was to move in the optimal path from $\mathbf{x}_\mathrm{T}(t)$ to $\mathbf{x}_\mathrm{A}(t)$, the optimal path would not change midway. Geometrically, this also implies the reverse result, that heading in the exact opposite direction to the optimal path would not change it. As a consequence, $\mathbf{I}(\mathbf{x}(t)) = \mathbf{I}_c(\mathbf{x}_\partial(t))$ for all states $V(\mathbf{x}(t)) \geq 0$.

Finally we consider Case C, in which $V(\mathbf{x}(t)) \leq 0$. Here $\mathbf{E}_\mathrm{T}(\mathbf{x}(t)) = \mathbf{x}_\mathrm{T}(t)$, thus formula (30) simplifies to

$$V(\mathbf{x}(t)) = -\overline{\mathbf{x}_\mathrm{D}(t)\mathbf{E}_\mathrm{D}(\mathbf{x}(t))},$$

and the substitute state is given by $\mathbf{x}_\partial(t) = (\mathbf{x}_\mathrm{A}(t), \mathbf{E}_\mathrm{D}(\mathbf{x}(t)), \mathbf{x}_\mathrm{T}(t))$. The Critical Collision Point $\mathbf{I}_c(\mathbf{x}_\partial(t))$ is at the intersection between the Defender Ray $\zeta_\mathrm{D}(\mathbf{x}_\partial(t))$ (if the parameter δ was extended to negative values in formula (25)) and the AT-based

Apollonius circle $\mathcal{C}_{\mathcal{AT}}(\mathbf{x}_\partial(t))$. Thus characterising the above formula in terms of $\mathbf{I}_c(\mathbf{x}_\partial(t))$ yields

$$\overline{\mathbf{x}_\mathrm{D}(t)\mathbf{E}_\mathrm{D}(\mathbf{x}(t))} = \overline{\mathbf{x}_\mathrm{D}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))} - \overline{\mathbf{E}_\mathrm{D}(\mathbf{x}(t))\mathbf{I}_c(\mathbf{x}_\partial(t))}.$$

Moreover, since the substitute state $\mathbf{x}_\partial(t)$ satisfies $V(\mathbf{x}_\partial(t)) = 0$, applying (45) we have that $\overline{\mathbf{E}_\mathrm{D}(\mathbf{x}(t))\mathbf{I}_c(\mathbf{x}_\partial(t))} = \frac{V_\mathrm{D}}{V_\mathrm{T}}\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))}$. Thus the value function for all states $V(\mathbf{x}(t)) \leq 0$ is given by

$$V(\mathbf{x}(t)) = -\overline{\mathbf{x}_\mathrm{D}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))} + \frac{V_\mathrm{D}}{V_\mathrm{T}}\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}_c(\mathbf{x}_\partial(t))}.$$

Thus to establish that this formula for the value function is equivalent to (43), all that remains is to show that

$$\begin{aligned}\mathbf{I}(\mathbf{x}(t)) &= \arg\max_{\mathbf{I}\in\mathcal{C}_{\mathcal{AT}}(\mathbf{x}(t))} -\overline{\mathbf{x}_\mathrm{D}(t)\mathbf{I}} + \frac{V_\mathrm{D}}{V_\mathrm{T}}\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}}, \\ &= \arg\max_{\mathbf{I}\in\mathcal{C}_{\mathcal{AT}}(\mathbf{x}(t))} -\frac{1}{V_\mathrm{D}}\overline{\mathbf{x}_\mathrm{D}(t)\mathbf{I}} + \frac{1}{V_\mathrm{T}}\overline{\mathbf{x}_\mathrm{T}(t)\mathbf{I}}. \end{aligned} \tag{50}$$

is equivalent to $\mathbf{I}_c(\mathbf{x}_\partial(t))$. Formula (50) may be interpreted as an optimisation problem in which the goal is to find the point on the AT-based Apollonius circle that minimises the time difference between the Target reaching there and the Defender reaching there.

Clearly in the special case $V(\mathbf{x}(t)) = 0$, $\mathbf{I}_c(\mathbf{x}_\partial(t)) = \mathbf{I}(\mathbf{x}(t))$ since the minimum time difference is zero and that unique point is given by $\mathbf{I}_c(\mathbf{x}_\partial(t))$. Moreover due to the above interpretation of formula (50), it obeys Defender Symmetry. Therefore $\mathbf{I}(\mathbf{x}(t)) = \mathbf{I}_c(\mathbf{x}_\partial(t))$ for all $\mathbf{x}(t)$. This completes the proof that the formula for the value function given by (30) is equivalent (43) in each of the respective cases. ■

Clearly seeing as (30) unifies three different equations into a single one-inch formula, the Holographic Theorem provides a far more elegant characterisation of the state-feedback Nash equilibrium than prior works. The next section reviews the literature and provides some intuitive arguments for the validity of formulas (43)–(44).

4.2 Corroboration with past results

With Theorem 4.1 proven, the Holographic Theorem follows as a simple corollary from the next proposition, which is a well-established result from past publications.

Proposition 1: *The value function defined in Theorem 4.1 correctly characterises the state-feedback Nash equilibrium of the TAD pursuit-evasion game defined in Section 2.2.*

The proof that the value function defined in Theorem 4.1 characterises the state-feedback Nash equilibrium of the TAD pursuit-evasion game defined in Section 2.2, is dispersed in numerous past publications. This is due to the fact that most papers published on this topic do not cover the problem in full. For example, the papers Mammadov et al. (2020) and Garcia et al. (2019) study the TAD pursuit-evasion game for the case $V(\mathbf{x}(t)) > 0$ and $V_\mathrm{A} = V_\mathrm{D}$; the papers Mammadov et al. (2021, 2022) and Garcia et al. (2017) examines the case for $V(\mathbf{x}(t)) > 0$ and $V_\mathrm{A} < V_\mathrm{D}$, and finally Garcia et al. (2021) examines the case in which $V(\mathbf{x}(t)) < 0$ and $V_\mathrm{A} = V_\mathrm{D}$.

The present manuscript considers the TAD pursuit-evasion game in its entirety, in the most general setting where $V_\mathrm{T} < V_\mathrm{A} < V_\mathrm{D}$ and the value can be positive or negative. The only case that we do not cover is for $V_\mathrm{A} = V_\mathrm{D}$, since it induces additional nuances that get in the way of the main concepts and could be thought to be included anyhow by simply making the speed V_D infinitesimally larger than V_A. Nevertheless, since the proof of Theorem 1, in all its technical details, is already provided in past publications, we do not repeat it in this manuscript. Here we provide a more intuitive argument for its validity.

The results of Theorem 1 can be primarily attributed to Pontryagin's maximum principle. In Garcia et al. (2017), as a simple application of the maximum principle, it was proven that under optimal play all agents move in straight line trajectories at their respective maximum speeds. This immediately implies that if the Attacker and Defender collide under optimal play, that is if $V > 0$, this collision must occur on the surface of the AD-based Apollonius circle. Recall that the perimeter of the AD-based Apollonius defines the set of all points the Attacker and Defender are equidistant to with respect to time, that is, the set of all points $\mathbf{p} \in \mathbb{R}^n$ satisfying $\frac{\overline{\mathbf{p}\mathbf{x}_\mathrm{D}(t)}}{V_\mathrm{D}} = \frac{\overline{\mathbf{p}\mathbf{x}_\mathrm{A}(t)}}{V_\mathrm{A}}$. By the same logic, in the case where the Attacker and Target collide under optimal play, that is $V < 0$, the collision must occur on the surface of the AT-based Apollonius circle. What remains is to determine at which point on the respective Apollonius circles does the Attacker collide, we denote this point $\mathbf{I}(\mathbf{x}(t))$.

In Case B, where the Target starts outside the AD-based Apollonius circle, clearly under optimal play the Target would move directly away from $\mathbf{I}(\mathbf{x}(t))$, thus the Attacker must choose the optimal $\mathbf{I}$ minimising $\overline{\mathbf{x}_\mathrm{T}(t_0)\mathbf{I}} + V_\mathrm{T}(t_f - t_0)$, where t_0 is the starting time and t_f is the termination time under optimal play. Since the time elapsed is given by $t_f - t_0 = \frac{1}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t_0)\mathbf{I}}$, this gives us the optimisation problem given in (43) and (44) for Case B. Rigorously, this result was proven in the works in Mammadov et al. (2020) for $V_\mathrm{A} = V_\mathrm{D}$ and later in the works of Mammadov et al. (2021) for $V_\mathrm{A} > V_\mathrm{D}$.

In Case A, where $V \geq 0$ but the Target starts inside the AD-based Apollonius circle, we may deduce the SFNE based on the fact that under optimal play, the differential game would at some intermediate time t_I transition from Case A into Case B, since the Target escapes capture. Based on the previous result from Case B, the value at this intermediate time t_I is given by $V(\mathbf{x}(t_I)) = \min_{\mathbf{I}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_I))} \overline{\mathbf{x}_\mathrm{T}(t_I)\mathbf{I}} + \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t_I)\mathbf{I}} = \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t_I)\mathbf{x}_\mathrm{T}(t_I)}$, where $\mathbf{x}_\mathrm{T}(t_I) \in \mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))$. This is proportional to the time difference between the Target reaching the point on $\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))$ versus the Attacker reaching that point. Thus under optimal play the Target must select a heading in which the time difference between the Target intersecting the AD-based Apollonius circle versus the Attacker is maximised, therefore the value is given by $V(\mathbf{x}(t_0)) = \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t_I)\mathbf{x}_\mathrm{T}(t_I)} = \frac{V_\mathrm{T}}{V_\mathrm{A}}(-V_\mathrm{A}(t_I - t_0) + \overline{\mathbf{x}_\mathrm{A}(t_0)\mathbf{x}_\mathrm{T}(t_I)}) = -\overline{\mathbf{x}_\mathrm{T}(t_0)\mathbf{x}_\mathrm{T}(t_I)} + \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t_0)\mathbf{x}_\mathrm{T}(t_I)} = \max_{\mathbf{I}\in\mathcal{C}_{\mathcal{AD}}(\mathbf{x}(t_0))} -\overline{\mathbf{x}_\mathrm{T}(t_0)\mathbf{I}} + \frac{V_\mathrm{T}}{V_\mathrm{A}}\overline{\mathbf{x}_\mathrm{A}(t_0)\mathbf{I}}$. Similar arguments were made in Garcia et al. (2017).

Finally in Case C, the Target and Attacker collide at some point on the surface of the AT-based Apollonius circle under optimal, and the Defender starts at some point outside the AT-based Apollonius circle. Clearly, the optimal strategy is for the Defender to head straight towards the collision point between the Target and Attacker, and the Target to select a collision point on the AT-based Apollonius circle that minimises the distance between the Target and Defender at termination time (or maximise the negative of that distance). This is given by $V(\mathbf{x}(t_0)) = -\overline{\mathbf{x}_\mathrm{T}(t_f)\mathbf{x}_\mathrm{D}(t_f)} = -(\overline{\mathbf{x}_\mathrm{T}(t_f)\mathbf{x}_\mathrm{D}(t_0)} - V_\mathrm{D}(t_f - t_0)) = -\overline{\mathbf{x}_\mathrm{T}(t_f)\mathbf{x}_\mathrm{D}(t_0)} + \frac{V_\mathrm{D}}{V_\mathrm{T}} \overline{\mathbf{x}_\mathrm{T}(t_0)\mathbf{x}_\mathrm{T}(t_f)} = \max_{\mathbf{I} \in \mathcal{C}_{\mathcal{AT}}(\mathbf{x}(t_0))} -\overline{\mathbf{x}_\mathrm{D}(t_0)\mathbf{I}} + \frac{V_\mathrm{D}}{V_\mathrm{T}}\overline{\mathbf{x}_\mathrm{T}(t_0)\mathbf{I}}$. These properties held in the work of Garcia et al. (2021) in 2-dimensions in the case for $V_\mathrm{A} = V_\mathrm{D}$.

5. Conclusion

In summary, the current manuscript examined the TAD pursuit-evasion differential game in the most general setting where $V_\mathrm{T} < V_\mathrm{A} < V_\mathrm{D}$, and in both the winning region of team A and team T/D. The state-feedback Nash equilibrium can be characterised in the most elegant and unified manner with the Holographic Theorem, by transforming any state of the differential game $\mathbf{x}(t)$ into the substitute state $\mathbf{x}_\partial(t)$ satisfying $V(\mathbf{x}_\partial(t)) = 0$.

The Holographic Principle undergoes symmetry breaking at $V(\mathbf{x}(t)) = 0$, it devolves into Target Symmetry for all $V(\mathbf{x}(t)) > 0$ and Defender Symmetry for all $V(\mathbf{x}(t)) < 0$. Further symmetry breaking occurs when it is desired to express the SFNE as a function of the optimal collision point of the Attacker, $\mathbf{x}_\mathrm{A}(t_f)$. This would explain the diverse and disjointed results obtained in previous publications.

Obtaining a general direct proof of the Holographic Principle would be the most fruitful and also the most challenging task. Future works may also explore its applications to other pursuit-evasion differential games.

Notes

1. The results from Garcia et al. (2021) were given differently, but are deemed to be equivalent to (4).
2. In other words, the current position of all agents is known to both teams; that is, the controls $\mathbf{u}_\mathrm{A}, \mathbf{u}_\mathrm{D}, \mathbf{u}_\mathrm{T}$ are functions of the current state $\mathbf{x}(t)$.
3. So long as agent j remains as not the next agent to terminate at time $t + \Delta t$.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Chapter 8

Conclusion

The present thesis introduced the theory of differential games; and wrote a dedicated chapter on the past literature studying the differential game of active target defence. In this chapter several open problems were identified; firstly the proofs of the state-feedback Nash equilibrium only held in 2 spatial dimensions as the Hamilton-Jacobi-Bellman equations are difficult to compute in higher dimensions. Moreover, a rigorous proof of the SFNE in the general case $V_A < V_D$ was still missing, only the simpler case $V_A = V_D$ so far had been solved. The first two International Journal of Control publications, and the ANZCC 2022 conference paper, resolved these outstanding problems; via introducing novel geometric methods on the discrete-time turn-based variant of the differential game.

The next two journal manuscripts introduced unifying optimality principles based on *Target Symmetry* for any state with positive value, and *Defender Symmetry* for any state with negative value. Generally, a symmetry refers to a transformation that leaves some quantity or property unchanged. For example, an equilateral triangle that is rotated by 120 degrees keeps the shape unchanged, despite changing the location of each corner. In the case of *Target* and *Defender Symmetry* for the pursuit-evasion game of active target defence, a symmetry refers to a transformation of the state that does not change the optimal headings of any of the three agents. These symmetries that are discovered in the final two journal manuscripts are very surprising, as they are not an implication of Pontryagin's maximum principle (the maximum principle only implies that the SFNE should be straight-line motion).

Overall, we have six papers:

1. Mammadov, K. (2019). Pole placement parameterisation for full-state feedback with minimal dimensionality and range. *International Journal of Control*, *94*(2), 382–389.

2. Mammadov, K., Lim, C., & Shi, P. (2020). State-feedback optimal strategies for the differential game of cooperative target defence: a geometric approach. *International Journal of Control*, *94*(10), 2615–2622.

3. Mammadov, K., Lim, C., & Shi, P. (2021). A state-feedback Nash equilibrium for the general Target-Attacker-Defender differential game of degree in arbitrary dimensions. *International Journal of Control*, *95*(1), 93–103.

4. Mammadov, K., Lim, C., & Shi, P. (2022). Generalising the capture the flag scenario to active target defence. *Australian and New Zealand Control Conference 2022*, accepted for publication.

5. Mammadov, K., Lim, C., & Shi, P. (2022). Unified optimality criteria for the Target–Attacker–Defender pursuit-evasion game. *European Journal of Control*, under review.

6. Mammadov, K., Lim, C., & Shi, P. (2022). The holographic principle for the differential game of active target defence. *International Journal of Control*, doi:10.1080/00207179.2022.2111369.

Although my first publication '*Pole placement parameterisation for full-state feedback with minimal dimensionality and range*' was on the unrelated topic of pole placement in linear control theory, hence not included in the thesis.

Moreover, during my candidature I have given the following local presentations:

1. Autonomous navigation of UAV, guidance and control. *CIT GC Critical Design Review - The University of Adelaide*, May 15$^{\text{th}}$ 2019.

2. Pole Placement Parameterisation. *Systems and Control Group - The University of Adelaide*, June 12$^{\text{th}}$ 2019.

3. Analysis of optimal state-feedback strategies for pursuit-evader differential games using novel geometric methods. *Systems and Control Group - The University of Adelaide*, March 4^{th} 2020.

4. Unified optimality criteria for the target-attacker-defender pursuit-evader game. *EEE School Seminar - The University of Adelaide*, October 23^{rd} 2020.

5. Grand unified optimality principle in simple motion pursuit evader differential games. *Systems and Control Group - The University of Adelaide*, July 7^{th} 2021.

6. Goku & krillin vs frieza. *3MT Competition - The University of Adelaide*, July 16^{th} 2021.

There exists several avenues for further research in the differential game of active target defence, and also more broadly in simple-motion pursuit-evasion differential games.

Firstly, one could verify that the results of manuscript number 6 obeys the Hamilton-Jacobi-Isaacs equation. Another avenue for research is to apply the conjecture in manuscript number 6 to the 1-Pursuer, 2-Evader differential game Liu et al. (2013). Also, we could examine the TAD differential game in the case where the Defender and Attacker have a positive capture radius Liang et al. (2019), rather than point capture as considered in this thesis. Another interesting direction is to study the case in which all three agents have acceleration constraints. And the must fruitful, but also the most difficult avenue for further research, is to construct a general proof of the Holographic Principle for some large class of simple-motion pursuit-evasion differential games.

With that summary of achievements and accomplishments, that concludes the thesis.

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