EVALUATION OF UNITS IN PURE CUBIC NUMBER-FIELDS.

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CONTENTS

SUMMARY

A new algorithm, evolved by G. Szekeres, is described. In a manner similar to the continued fraction approximation of single irrationals, this algorithm gives a simultaneous approximation to an ordered pair of irrational numbers: (ξ,η).

A sequence of values is generated by the algorithm for the linear form

$$L(X,Y,Z) = X + Y\xi + Z\eta,$$

where X,Y,Z are rational integers.

It is shown that the values of $L(X,Y,Z)$ form a null sequence, but the order of the approximation is not established directly.

It is known that arbitrarily large X,Y,Z values may be found so that

$$|L(X,Y,Z)| = |X + Y\xi + Z\eta| < \frac{c}{M^2}, \qquad (1)$$

where $M = \max\{|X|,|Y|,|Z|\}$, and c is some constant depending on the values of ξ and η. It is conjectured that the algorithm generates an infinite sequence of $L(X,Y,Z)$ values for which (1) holds.

To test this conjecture, the algorithm is applied to a pair

$$(\bar{\xi},\bar{\eta})$$

chosen so that

$$1 > \bar{\xi} > \bar{\eta} > 0,$$

and the triplet

$$(\bar{\xi}, \bar{\eta}, 1)$$

forms an integral basis to the number-field,

$$R(\sqrt[3]{D}),$$

where D is a cube-free rational integer.

The norm of $X + Y\bar{\xi} + Z\bar{\eta} = L(X,Y,Z)$ is

$$N(X,Y,Z) = L(X,Y,Z)L'(X,Y,Z)L''(X,Y,Z),$$

where $L'(X,Y,Z)$, $L''(X,Y,Z)$ are the field-conjugates of $L(X,Y,Z)$.

It is easy to show that

$$L'(X,Y,Z)L''(X,Y,Z) = Q(X,Y,Z)$$

is a positive definite quadratic form, thus the values of $Q(X,Y,Z)$ form a sequence of the order

$$M^2 = (\mathrm{Max}\{X^2, Y^2, Z^2\}).$$

If $L(X,Y,Z)$ is a unit in the given number-field, its norm has the value ± 1, thus $L(X,Y,Z)$ must be of the order M^{-2}. Hence, if the algorithm produces for every D the units (numerically smaller than 1) of the field $R(\sqrt[3]{D})$, then the conjecture gains strong support.

This thesis presents a computational method based on the algorithm, to detect the units of the number-field $R(\sqrt[3]{D})$. The computations were carried out partly to test the order of the approximation and partly to explore a practical, systematic method for finding the units of cubic number fields.

In the calculations completed D was allowed to range over all cub-free integers from 1 to 200, and as a result, a table of units was prepared. Calculations were carried out to support the evidence that the units listed are fundamental.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and to the best of the author's knowledge and belief the thesis contains no material previously published or written by another person except when due reference is made in the text of the thesis.

Signed. ....

1. Pure cubic fields.

a. Introduction

A brief summary of results from the theory of algebraic number-fields, in particular on pure cubic fields is given in the following. Proofs of the results may be found e.g. in the book by Hecke [(a)]. Notations introduced here will be used in subsequent chapters.

Let D be a positive cube-free rational integer, i.e. an integer not divisible by the cube of any rational integer different from ±1. D may be presented in the factorised form

$$D = ab^2 \qquad (1-1)$$

where a,b are positive integers, and a is square-free. Since D is cube-free, b must also be square-free and a and b must be relatively prime.

Denote by β the real cube root of D, i.e. the real root of the cubic equation

$$F(x) = x^3 - D = 0 \qquad (1-2)$$

The pure cubic number-field R(β) may be defined as the field obtained by adjoining β to the field of the rational numbers R, i.e. R(β) is the set $\{\alpha\}$ where

$$\alpha = \frac{f(\beta)}{g(\beta)} \qquad (1-3)$$

Here f(x), g(x) are polynomials with rational coefficients and $g(\beta) \neq 0$.

It can be shown that (1-3) reduces <u>uniquely</u> to the form

$$\alpha = X + Y\beta + Z_1\beta^2 \qquad (1\text{-}4)$$

where X, Y, Z_1, are rational numbers. Thus the numbers

$$1,\ \beta,\ \beta^2$$

form a basis for the field $R(\beta)$.

For an alternative representation we may define

$$\gamma = \sqrt[3]{a^2 b} \qquad (1\text{-}5)$$

Clearly

$$\gamma = \frac{1}{b}\,\beta^2$$

Writing $Z = Z_1 b$ in (1-4) we obtain

$$\alpha = X + Y\beta + Z\gamma \qquad (1\text{-}6)$$

This representation is also unique, hence

$$1,\ \beta,\ \gamma$$

may be taken as an alternative basis of $R(\beta)$.

Let β', β'' be the conjugates of β, i.e. the other roots of equation (1-2).

Then

$$\beta' = \beta\omega$$

$$\beta'' = \beta\omega^2$$

where ω, ω^2 are the complex cube-roots of 1, i.e.

$$\omega = -\tfrac{1}{2} + \tfrac{i}{2}\sqrt{3}, \qquad \omega^2 = -\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3}.$$

The <u>field-conjugates</u> of α are derived from (1-4):

$$\left.\begin{aligned} \alpha' &= X + Y\beta' + Z_1\beta'^2 = X + Y\beta\omega + Z_1\beta^2\omega^2 \\ \text{and}\quad \alpha'' &= X + Y\beta'' + Z_1\beta''^2 = X + Y\beta\omega^2 + Z_1\beta^2\omega \end{aligned}\right\} \qquad (1\text{-}7)$$

Alternatively, we may use γ as in (1-6), and obtain

$$\left.\begin{aligned} \alpha' &= X + Y\beta\omega + Z\gamma\omega^2 \\ \alpha'' &= X + Y\beta\omega^2 + Z\gamma\omega \end{aligned}\right\} . \qquad (1\text{-}8)$$

α and its conjugates α', α'' are roots of a monic cubic polynomial

$$\phi(x) = x^3 + Px^2 + Qx + R \qquad (1\text{-}9)$$

where P, Q and R are polynomial functions of X, Y, Z, with integral coefficients.

The number α is rational if and only if the coefficients Y and Z_1 in (1-7) or Y and Z in (1-8) vanish.

In this case

$$\alpha = \alpha' = \alpha'' = X$$

and

$$\phi(x) = (x - \alpha)^3 .$$

Disregarding this special case, $\phi(x)$ is always irreducible and it is then the defining polynomial of α.

Of particular interest in the following is the <u>norm</u> of the irrational number α, defined as

$$N(X,Y,Z) = \alpha\,\alpha'\,\alpha'' = -R.$$

Using (1-8), the norm may be expressed in terms of β and γ, and using the relations

$$\beta^3 = ab^2$$
$$\gamma^3 = a^2b$$

and hence

$$\beta\gamma = ab,$$

we obtain for the norm the expression

$$N(X,Y,Z) = X^3 + Y^3ab^2 + Z^3a^2b - 3XYZab. \qquad (1\text{-}10)$$

If the defining (monic) polynomial of an algebraic number (e.g. $\phi(x)$ in (1-9) in the case of cubic irrationals) has rational integral coefficients, the number is called an algebraic integer. It can be shown that the set of algebraic integers is closed to addition and multiplication. Thus, since β,γ are algebraic integers, the number α in (1-6) is also an algebraic integer. The converse is not necessarily true, i.e. the form (1-6) does not always represent all the integers in $R(\beta)$. In other words, $(1,\beta,\gamma)$ is not necessarily an integral basis of $R(\beta)$. Dedekind[(b)] has determined integral bases for all pure cubic fields and his result is given in the following (based on the presentation of Leveque)[(c)].

b. Integral basis.

Assume that α and hence its conjugates α' and α'' are algebraic integers of degree 3, i.e. that they are the roots of a monic polynomial $\phi(x)$, of the third degree, with integral coefficients. From (1-6) and (1-8) we obtain the equations

$$\left.\begin{aligned} 3X &= \alpha + \alpha' + \alpha'' \\ 3Yab &= \gamma(\alpha + \alpha'\omega^2 + \alpha''\omega) \\ 3Zab &= \beta(\alpha + \alpha'\omega + \alpha''\omega^2) \end{aligned}\right\}. \qquad (1\text{-}11)$$

Since the right hand sides of the equations (1-11) represent algebraic integers and the left hand sides are rational, it follows that

$$3X,\ 3Yab,\ 3Zab$$

are rational integers. Thus (1-6) may be written as

$$3ab\alpha = 3abX + 3abY\beta + 3abZ\gamma,$$

or writing $\overline{X} = 3abX$, $\overline{Y} = 3abY$, $\overline{Z} = 3abZ$,

we have

$$3ab\alpha = \overline{X} + \overline{Y}\beta + \overline{Z}\gamma \qquad (1\text{-}12)$$

where $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are rational integers.

Using results concerning prime-ideal factors in algebraic number-fields, Dedekind showed first that each of the numbers $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ is divisible by every prime factor of a or b, hence by the product ab, (since a and b are square-free and relatively prime.)

Let

$$X^* = \frac{\overline{X}}{ab},\quad Y^* = \frac{\overline{Y}}{ab},\quad Z^* = \frac{\overline{Z}}{ab},$$

then

$$\alpha = \frac{X^*}{3} + \frac{Y^*}{3}\beta + \frac{Z^*}{3}\gamma, \qquad (1\text{-}13)$$

where X^*, Y^*, Z^* are rational integers.

The coefficients of the defining polynomial $\phi(x)$ of α may now be expressed in terms of X^*, Y^*, Z^* to obtain

$$\phi(x) = x^3 - X^*x^2 + \frac{X^{*2} - Y^*Z^*ab}{3}x - N(X^*,Y^*,Z^*) = 0, \qquad (1\text{-}14)$$

where

$$N(X^*,Y^*,Z^*) = \frac{1}{27}(X^{*3} + Y^{*3}ab^2 + Z^{*3}a^2b - 3X^*Y^*Z^*ab). \qquad (1\text{-}15)$$

There are two cases to be considered:

(i) One of a, b is divisible by 3,

(ii) a, b are both relatively prime to 3.

From the condition that all the coefficients in (1-14) must be rational integers, it follows by elementary considerations that in case (i) each of the numbers X^*, Y^* and Z^* is a multiple of 3, and so by (1-13),

$1, \beta, \gamma$ form an integral basis in case (i).

In case (ii) it may be shown easily that if any one of the numbers X^*, Y^*, Z^* is a multiple of 3, then all three of them are multiples of 3. Thus to establish the conditions for an integer base other than $1, \beta, \gamma$, it is sufficient to consider the case when

a, b, X^*, Y^*, Z^* are all relatively prime to 3, and thus

$$a^2 \equiv b^2 \equiv X^{*2} \equiv Y^{*2} \equiv Z^{*2} \equiv 1 \pmod 3 \qquad (1\text{-}16).$$

Referring to (1-15), it follows that

$$X^{*3} + Y^{*3}ab^2 + Z^{*3}a^2b \equiv X^* + aY^* + bZ^* \equiv 0 \pmod 3 \quad (1\text{-}17).$$

Each term on the left hand side of the last congruence is relatively prime to 3, thus

$$X^* \equiv aY^* \equiv bZ^* \pmod 3,$$

and using (1-16) again, it follows that

$$\left.\begin{aligned} Y^* &\equiv a^2Y^* \equiv aX^* \pmod 3 \\ \text{and} \quad Z^* &\equiv b^2Z^* \equiv bX^* \pmod 3 \end{aligned}\right\} \qquad (1\text{-}18)$$

It follows also from (1-16) that

$$a^2 - b^2 = 3j \qquad (1\text{-}19)$$

where j is some rational integer.

Making use of (1-18) and (1-19) it is easy to show that the coefficients of x in (1-14) are rational integers if and only if

$$j \equiv 0 \pmod 3, \quad \text{i.e. if}$$

and only if

$$a^2 - b^2 \equiv 0 \pmod 9 \qquad (1\text{-}20).$$

In this case the integral base may be found by using (1-18) and writing

$$Y^* = aX^* + 3m, \qquad Z^* = bX^* + 3n.$$

We then obtain from (1-13) that

$$\alpha = \frac{X^*}{3} + \frac{aX^* + 3m}{3}\beta + \frac{bX^* + 3n}{3}\gamma$$

i.e.

$$\alpha = X^*\left(\frac{1 + a\beta + b\gamma}{3}\right) + m\beta + n\gamma \qquad (1\text{-}21)$$

where X^*, m, n are arbitrary rational integers.

Following Dedekind's notation, a pure cubic field

$$R(\sqrt[3]{ab^2})$$

is called a field of the <u>first kind</u> if

$$9 \nmid (a^2 - b^2).$$

In this case, as seen in both situations (i) and (ii), α is an algebraic integer if and only if X^*, Y^* and Z^* are multiples of 3, i.e.

$1, \beta, \gamma$ form an integral basis for fields of the first kind.

If (1-20) holds, i.e.

$$9|(a^2 - b^2)$$

the field is called a field of the second kind.

It follows from (1-21) that in this case the numbers

$$\frac{1 + a\beta + b\gamma}{3}, \beta, \gamma \qquad (1\text{-}22)$$

form an integral basis.

It should be noted that in this case all integers of form (1-6) may certainly be represented in terms of basis (1-22),

$3X$, $Y - Xa$, $Z - Xb$ being the respective coefficients; the converse does not hold: e.g.

$$\alpha = \frac{1}{3}(1 + a\beta + b\gamma)$$

is not of form (1-6).

c. Units

Let $R(\vartheta)$ be an algebraic number-field of degree n, i.e. the field obtained by adjoining ϑ, an algebraic number with defining equation of degree n, to the field R of rational numbers, and denote by $R[\vartheta]$ the integral domain of the field, i.e. the subset of algebraic integers in $R(\vartheta)$.

As in the case of cubic fields, the (relative) norm, $N(\mu)$ of any number in $R(\vartheta)$ is defined as the product of μ and all its field conjugates. The norm is then some power of the constant term of the defining (monic) equation of μ.

Thus if μ is an algebraic integer, then $N(\mu)$ is a rational integer. Furthermore it is easy to show that for any two numbers μ, ν in $R(\vartheta)$

$$N(\mu\nu) = N(\mu)N(\nu). \qquad (1-23)$$

ϵ is called a unit in $R[\vartheta]$ if

$$\epsilon | 1.$$

It follows from the definition that

$$\epsilon | \mu$$

for all μ in $R[\vartheta]$, and by (1-23) ϵ is a unit in $R[\vartheta]$ if and only if

$$N(\epsilon) = \pm 1.$$

Clearly, the units of the integral domain $R[\vartheta]$ form a multiplicative Abelian group.

The question arises firstly whether every integral domain $R[\vartheta]$ has units different from the trivial units ± 1, and secondly, if such units exist, how many of them can be <u>independent</u> in the sense that no relation

$$\epsilon_1^{a_1}\, \epsilon_2^{a_2} \ldots \epsilon_s^{a_s} = 1$$

exists where the indices $a_1, a_2, \ldots a_s$ are rational integers, not all equal 0.

A full answer for algebraic number-fields of degree n is given by Dirichlet's[(d)] theorem on the group of units. As a special case of this theorem it follows that the positive units in $R(\beta)$ form an <u>infinite</u> <u>cyclic</u> group, i.e. each unit ϵ may be represented as

$$\epsilon = \pm \eta^{a},$$

where η is the <u>fundamental unit</u> and a is a positive or negative integer.

(It may be noted that this result still holds for the units of a field generated by an algebraic integer of degree 3, if the defining polynomial has a negative discriminant, i.e. the polynomial has one real and two imaginary roots. If the defining cubic has three real roots then the units are of the form

$$\epsilon = \pm \eta_1^{a_1}\eta_2^{a_2}$$

where η_1 and η_2 are independent.)

For the fundamental unit of the pure cubic number-field we may choose either the smallest unit which is greater than 1, or the greatest unit which is less than 1.

No direct way of calculating η is known for the general field $R(\beta)$, but its value has been determined in special cases, when $\beta^3 = D$ is not too large. A short survey of available results is given in the next chapter.

References

(a) Hecke E. Theorie der algebraischen Zahlen (Chapters IV - V).

(b) Dedekind R. Über die anzahl des Idealklassen in reinen kubischen Zahl-körpern. (Journal f. reine und angewandte Mathematik B.121 540-123) 1900.

(c) Leveque W.J. Topics in number theory. Vol.II ch. 3-7. Page 104-108.

(d) Dirichlet, P.G. Lejenne Monatsbericht der Berliner Akademie vom März 1846.

2. Earlier computations of units in cubic fields.

In the following we shall survey the results of previous workers. Tables of units of cubic number-fields, mostly of pure cubic fields, have been given before. For pure cubic fields, units have been evaluated for all values of D up to 70, some values between 70 and 100, and a few non-pure cubic fields have been also explored.

The earliest extensive table is by Markoff[(a)] dated in 1891, for fields of the form $R(\sqrt[3]{D})$, for D up to 70. The table is given as an appendix to a fundamental paper on cubic irrationalities in which the earlier works of Zolotareff and Markoff himself are surveyed. (This table is reproduced in the book by Delone and Faddeev[(b)].) The units listed are used as means of discovering the ideal factors of $R(\sqrt[3]{D})$ and only a short reference is made to his trial and error methods of obtaining them: "Ne nous arrêtant pas aux méthodes sûres mais fatigantes pour déterminer l'unité complexe fondamentale nous remarquons, que pour les valeurs petites de a et de b il est facile de trouver les unités complexes par le tâtonnement en considérant plusieurs nombres ξ composés des mêmes facteurs premiers."

A probably identical trial and error method is used by Dedekind who used his own computations to determine the units for $D \leqslant 23$. This work is described in more detail in his paper on cubic fields published in 1900[c]. (In this same paper can be found his theorem on the integral basis of the pure cubic field.)

The essence of the method is finding by trial a few algebraic integers $X + Y\beta + Z\gamma$ with reasonably small (real) coordinates X, Y, Z and having some prime ideals as their common factors, and then building new algebraic integers by multiplying together various combinations of the prime-ideals previously found amongst the factors. Amongst these products are some which have different absolute values while having identical prime-ideal divisors. The ratio of two such products must give a unit. The fundamental unit of the field is then found by selecting it from the set of units found, e.g. when considering units greater than 1, selecting the smallest one and checking whether the other units are exact powers of it.

These methods proved to be laborious and with an element of uncertainty in finding the fundamental unit. Dedekind observes that the unit listed for $D = 28$ by Markoff is the square of the fundamental unit found by himself.

It should be mentioned here that there is one theorem due to Delone and Nagell[d] which in certain special cases gives a criterion for fundamental units in $R(\sqrt[3]{D})$:

Consider the equation

$$x^3 + Dy^3 = 1 \qquad (2\text{-}1)$$

This equation has at most one solution in integers x,y different from zero. If (x_1,y_1) is such a solution, then the number

$$x_1 + y_1 \sqrt[3]{D} \qquad (2\text{-}2)$$

is either the fundamental unit of $R(\sqrt[3]{D})$ or its square. It can be the square of the fundamental unit for only finitely many values of D.

If the field $R(\sqrt[3]{D})$ is of the first kind, (following Dedekind's definition), and

$\eta = x_1 + y_1\beta + z_1\gamma$ is a unit,

then the square of η cannot be of the form (2-2), with one exception, namely

$$\eta = 1 + \sqrt[3]{20} - \sqrt[3]{50}$$

$$\eta^2 = -19 + 7\sqrt[3]{20}.$$

If the field is of the second kind, and a unit of the form

$\eta = \frac{1}{3}(x_1 + y_1\beta + z_1\gamma)$ exists

where $3 \nmid x_1y_1z_1$, then η^2 can be of form (2-2) only in finitely many cases. In particular when x_1 is even, there is only one case: $D = 19$

$$\eta^2 = \left(\frac{2 + 2\sqrt[3]{19} - \sqrt[3]{19}}{3}\right) = -8 + 3\sqrt[3]{19}.$$

There are more cases (but still a finite number of them), when x_1 is odd.

An entirely different approach to the problem of unit finding is suggested by the analogous problem in quadratic fields.

Let D be a square-free integer. An integral basis for the field $R(\sqrt{D})$ can be readily found. It is not hard to prove that

$$\text{if } D \equiv 2 \text{ or } \equiv 3 (\text{mod } 4)$$

then

$(1, \sqrt{D})$ is such a basis, i.e. all the integers of the field may be written in the form

$$a + b\sqrt{D}$$

where a,b are rational integers.

If $D \equiv 1 \ (\text{mod } 4)$,

then the integers of the field are of the form

$$\frac{a + b\sqrt{D}}{2},$$

where a and b are rational integers and <u>either both are even</u>, or <u>both are odd</u>.

If ϵ is a unit, and ϵ' its conjugate then

$$\epsilon\epsilon' = \pm 1,$$

thus $(a + b\sqrt{D})(a - b\sqrt{D}) = \pm 1$ in the first case,

and $\left(\frac{a + b\sqrt{D}}{2}\right)\left(\frac{a - bD}{2}\right) = \pm 1$ in the second case.

Hence if $D \equiv 2$ or $\equiv 3 (\text{mod } 4)$, the units of the field are given by the solution of Pell's equation

$$x^2 - Dy^2 = \pm 1, \qquad (2\text{-}3)$$

and if

$$D \equiv 1 \pmod 4$$

the units are given in the form

$$\frac{x + y\sqrt{D}}{2}$$

where (x,y) is some (integer) solution of

$$x^2 - Dy^2 = \pm 4.$$

(Clearly, x and y are both even or both odd for each solution.)

The solution of the general problem presented by Pell's equation was partly accomplished by Euler and completed by Lagrange.

The result can be stated in terms of the continued fraction expansion of the number $\sqrt{D}$, briefly described in the following.[e]

Let the symbol

$$[b_0, b_1, \dots b_n] \tag{2-4}$$

stand for the (terminating) continued fraction

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \dots + \cfrac{1}{b_n}}} \ .$$

Then (2-4) represents a rational number and we may write

$$\frac{p_n}{q_n} = [b_0, b_1, \dots b_n],$$

where p_n, q_n are relatively prime integers.

We may next define the expansion of an irrational number ξ_0.

For $\nu = 0,1,2...$ the number $\xi_{\nu+1}$ is found recursively by

$$\xi_\nu = b_\nu + \frac{1}{\xi_{\nu+1}} \qquad (2\text{-}5)$$

where $b_\nu = [\xi_\nu]$, hence $\xi_{\nu+1} > 1$.

Since ξ_0 is not rational, $\xi_{\nu+1}$ in (2-5) cannot be an integer and so the procedure can be continued indefinitely.

The <u>convergent fractions</u> $\frac{p_\nu}{q_\nu}$ may be defined next by

$$\frac{p_\nu}{q_\nu} = [b_0, b_1, \dots b_\nu]. \qquad (2\text{-}6)$$

Defining

$$p_{-2} = 0 \qquad p_{-1} = 1$$

$$q_{-2} = 1 \qquad q_{-1} = 0$$

it is easy to verify that for $\nu \geqslant 0$

$$\left.\begin{aligned} p_\nu &= b_\nu p_{\nu-1} + p_{\nu-2} \\ q_\nu &= b_\nu q_{\nu-1} + q_{\nu-2} \end{aligned}\right\} . \qquad (2\text{-}7)$$

The relations (2-7) may be then used to evaluate the values of p_ν and q_ν recursively.

It can be shown that for all $\nu \geqslant 0$

$$|\xi_0 - \frac{p_\nu}{q_\nu}| < \frac{1}{q_\nu^2} \leqslant \frac{1}{\nu^2}$$

hence the sequence

$$\{\frac{p_\nu}{q_\nu}\} \quad \text{converges to } \xi_0 .$$

It can be shown easily that if the expansion is periodic, i.e. if for some (positive integer) k,

$$b_{k+n} = b_n \quad \text{for all} \quad n,$$

then ξ_0 is a quadratic surd.
The converse was shown by Lagrange, and as a special case we may state that

if D is a positive integer, but not a perfect square, then the continued fraction expansion of

$$\xi_0 = \sqrt{D},$$

is periodic.
It may be shown next by induction that the numbers ξ_ν defined by the relations (2-5) may be written uniquely in the form

$$\xi_\nu = \frac{\sqrt{D} + P_\nu}{Q_\nu} \qquad (2\text{-}8)$$

where P_ν, Q_ν are rational integers. Moreover, it follows from the periodicity of the expansion that

$$\xi_k = \xi_0$$

and hence that the sequences $\{P_\nu\}$, $\{Q_\nu\}$ are also periodic, k being the length of their period.

Using the above notations, the following general result may be stated (without proof).

The equation

$$x^2 - Dy^2 = L$$

where D,L are integers and D is positive and is not a perfect square, and furthermore

$$\sqrt{D} > |L| > 0 ,$$

is soluble if and only if L occurs amongst the quantities

$$(-1)^{\nu} Q_{\nu} \qquad \text{(defined by (2-8)).}$$

From the periodicity of Q_{ν} it follows that in this case the equation has an infinite number of solutions.

For each ν satisfying the relation

$$(-1)^{\nu} Q_{\nu} = L$$

we obtain the solution

$$x = p_{\nu-1}, \qquad y = q_{\nu-1}.$$

If k is the number of terms in the recurring period, then for each positive integer n

$$\xi_{nk} = \xi_0, \quad \text{hence}$$

$$P_{nk} = 0, \qquad Q_{nk} = 1 \text{ (referring to (2-8)).}$$

Thus we have as a particular case of the general result, the theorem of Lagrange on the solution of equation (2-3). It may be stated as follows.

The equation

$$x^2 - Dy^2 = 1$$

is always soluble and it has an infinite number of solutions. If

$$\frac{p_{\nu}}{q_{\nu}}$$

is the ν^{th} convergent of $\sqrt{D}$ and k is the number of terms in the (smallest) recurring period, then all the solutions are given by

$x = p_{nk-1}$ $\quad n = 1,2,3,\ldots$ if k is even

$y = q_{nk-1}$ $\quad n = 2,4,6,\ldots$ if k is odd.

The equation

$x^2 - Dy^2 = -1$ is soluble if

and only if k is odd and then the solutions are

$$x = p_{nk-1},\ y = q_{nk-1}, \qquad n = 1,3,5,\ldots$$

Thus the problem of determining all the units of the quadratic number-field generated by $\sqrt{D}$ is accomplished by the use of a continued fraction algorithm. It should be noted that this algorithm has two characteristic features: its periodicity and the fact that it provides the best approximation of the irrational number ξ, in the sense that, if

$$\left|\frac{p}{q} - \xi_0\right| < \left|\frac{p_n}{q_n} - \xi_0\right|, \text{ then}$$

$$q \geqslant q_{n+1}.$$

Also

$$\left|\frac{p_n}{q_n} - \xi_0\right| < \frac{1}{q_n^2} .$$

It is possible to extend this later result to the case of simultaneous approximations.

Let $\theta_1, \theta_2, \ldots \theta_n$ be a set of numbers and let

$$\frac{p_1}{q}, \frac{p_2}{q}, \ldots \frac{p_n}{q}$$

be fractions approximating simultaneously the given set and with a common denominator q.

For each i, $(i = 1,\dots n)$

$$\frac{p_i}{q} < \theta_i < \frac{p_i+1}{q}$$

It is then possible to find an infinite number of positive integers q so that

$$\left|\frac{p_i}{q} - \theta_i\right| < \frac{1}{q\sqrt[n]{q}} \qquad (2\text{-}9)$$

This result may be proved by Dirichlet's box-principle[f].

We may write this in a slightly different form by defining for a number <u>a</u> the symbol

$$||a||$$

as the difference between <u>a</u> and the nearest integer.

We have then

$$\max(||q\theta_1||, ||q\theta_2||, \dots ||q\theta_n||) < q^{-\frac{1}{n}}.$$

Instead of considering n numbers depending on the single variable q, we may consider <u>n linear forms</u> with m variables.

The above theorem can then be generalised in the following manner[g].

Let

$$L_j(x) = \sum_i \theta_{ji} x_i \quad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n),$$

be n linear forms in m variables. To every real $X > 1$ there is an integral (vector) $\underset{\sim}{x} \neq 0$ such that

$$||L_j(\underset{\sim}{x})|| < X^{-\frac{m}{n}}, \quad |x_i| \leqslant X \quad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n).(2\text{-}10)$$

We may regard the approximation theorem referring to a single linear form as another special case of this general theorem. (cf. Introduction (1)).

The proof of the general theorem can be given by using Minkowski's linear form theorem from his geometry of numbers:

There are integers x_j not all 0 such that

$$\left| \sum_{j=1}^{n} a_{ij} x_j \right| \leqslant \rho_i \qquad (i = 1, 2, \ldots n)$$

provided that

$$\rho_1 \ldots \rho_n \geqslant |\Delta|$$

where Δ is the determinant of the matrix $[a_{ij}]$.

Generalisations of the continued fraction algorithm for two or more dimensions have been evolved by various workers, aiming to preserve either the periodicity of the one dimensional process, or its property of best approximation. Mahler(h) has proved that the two cannot (in general) be preserved simultaneously.

Simultaneous rational approximation by means of a periodic expansion was attempted and partially solved by Jacobi in 1868(i). He devised an extension of the continued fraction process which enabled him to determine rational approximations to the mutual ratios of three numbers and showed that every approximation could be expressed in terms of the three preceding ones and the coefficients of the expansion. He further showed that if the three numbers are taken to be $1, \theta, p + q\theta + r\theta^2$, where θ is the real root of a rational cubic equation of negative discriminant, the expansion may become periodic, and in certain numerical cases he actually demonstrated the periodicity. It has been found by Bachmann[j] however, that this periodicity exists only if a certain limiting inequality is satisfied.

Minkowski's[k] approach is different. A somewhat more detailed account is given of his algorithm, because it shows a certain similarity to the algorithm described in the next chapter, inasmuch that the approach is geometrical.

The inequalities

$$|L_i(x,y,z)| \leq \rho_i \qquad 1 \leq i \leq 3$$

where all the coefficients are real, ρ_1, ρ_2, ρ_3 are positive, and the determinant Δ of the matrix

$$L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

does not vanish, determine the interior and the surface of a parallelepiped.

By the theorem quoted earlier, this domain can be free of points of integer coordinates, (other than the origin), only if

$$\rho_1\rho_2\rho_3 < |\Delta|.$$

An "extremal" parallelepiped is defined as one which is <u>free of internal lattice-points</u>, i.e. for which the inequalities

$$L_i(x,y,z) < \rho_i \qquad 1 \leqslant i \leqslant 3$$

have no integral solution apart from

$$x = y = z = 0,$$

but which <u>ceases</u> to be <u>free</u> as soon as any positive number, however small, is added to any one of the parameters ρ_1, ρ_2, ρ_3.

Minkowski proves the existence of transformation-matrix associated with such an extremal parallelepiped,

$$P = \begin{bmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{bmatrix},$$

having the following properties:

each of the (r_i, s_i, t_i) $\qquad (i = 1,2,3)$,

means a point with integer coordinates on one of the planes

$$L_i(x,y,z) = \pm \rho_i,$$

and no two of the (r_i, s_i, t_i) are on opposite planes;

the determinant of P is equal to 1;

the entries of the matrix LP satisfy a certain prescribed set of inequalities.

He then shows that it is possible to construct a new set of parameters (σ_i) from the old set (ρ_i), so that

$$(LP)_i(x,y,z) \leqslant \sigma_i$$

represents again an extremal parallelepiped.

An algorithm to obtain uniquely a chain of extremal parallelepipeds is constructed in this manner. He proposes this algorithm to be applied for the determination of the units in the cubic field $R(\vartheta)$, where the field has a positive discriminant, i.e. the defining equation of ϑ has three real roots.

Let α, β, γ be an integer basis of $R(\vartheta)$. Form the initial matrix L of the algorithm by using the linear forms

$$L_1 = \alpha x + \beta y + \gamma z$$

and its conjugates L_1' and L_1''.

The discriminant of the field is D, where $D = \Delta^2$,

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} .$$

The norm of $L(x,y,z)$,

$$N(L) = LL'L'',$$

is a cubic form in (x,y,z) with rational integral coefficients.

It follows from the inequalities imposed on the coefficients, that if the algorithm is started with a certain matrix L, there are only a finite number of forms $N(L)$ possible, so that after a finite numbermof steps some form must recur. Let (ρ_i), (σ_i) represent the parameters belonging to the "extremal" parallelepipeds associated with identical forms. It can be proved then that whenever the ratios $\frac{\rho_i}{\sigma_i}$ are (algebraic) integers, they represent a set of conjugate units. In this way he proposes to find all the units of the field, amongst them the fundamental ones. (By Dirichlet's theorem there are two independent units in the case of positive discriminants.)

The algorithm is described in full detail, but it appears to be a very laborious procedure, and it seems that it has not been adopted by the various workers who have evaluated units, although in a paper by Berwick(1) a reference is made to the work of L. Kollros, supposed to be based on Minkowski's algorithm. Berwick in his own paper takes up the idea of a periodic algorithm, and devises one to find ideal classes in cubic number fields.

Earlier, Voronoi suggested an algorithm for finding the fundamental units of cubic fields with negative discriminants. Tables of fundamental units based on Voronoi's algorithm were worked out by B. Delone and K. Latyseva(m) for some 50 cubic fields with negative discriminants not greater in value than 379.

In 1923 a paper appeared by C. Wolfe(n) in which he made use of a mixture of methods of previous workers together with some theorems to short cut the calculations and he gave a complete table of the minimum positive solutions of the indeterminate cubic equation

$$x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$$

for $1 \leqslant D \leqslant 100$.

The solutions are identical with the fundamental units of $R(\sqrt[3]{D})$ whenever the field is of the first kind and D is square-free.

In a table given at the end of a paper by J.W. Cassels(o) in 1950 units are listed for D = 2 to D = 50 with some uncertainty whether the units given for D = 29, 41, 46, 47 are fundamental.

In the next chapter an algorithm proposed by G. Szekeres (as yet unpublished) will be described in detail.

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3. The simultaneous approximation algorithm.

The algorithm evolved by G. Szekeres and described in this chapter is designed for the simultaneous approximation of the irrational components $(\overline{\xi},\overline{\eta})$ of a two dimensional vector. It may be regarded as a two dimensional generalisation of the simple continued fraction process.

In approximating a single irrational number α by continued fractions, we may consider two facets of the problem:

(i) finding a sequence of rationals $\{a_n\}$ to approximate the number α,

(ii) finding a sequence of linear forms $\{p_n x - q_n\}$, such that p_n, q_n are relatively prime integers and

$$\{p_n x - q_n\} \to 0 \quad \text{for} \quad x = \alpha.$$

Correspondingly, the simultaneous approximation algorithm is designed to give

(i) a sequence of ordered pairs of rational numbers $\{\xi_k, \eta_k\}$ to approximate $\{\overline{\xi},\overline{\eta}\}$,

(ii) a sequence of linear forms with integral coefficients a_k, b_k, c_k so that

$$\{a_k\overline{\xi} + b_k\overline{\eta} + c_k\} \to 0.$$

No proof will be given that the sequences give "good" approximations in the sense of relations (2-9) and (2-10). However, the geometrical illustration of the algorithm will attempt to demonstrate the plausibility of the conjecture.

Without loss of generality we may restrict the values of $\bar{\xi},\bar{\eta}$ to

$$1 > \bar{\xi} > \bar{\eta} > 0,$$

and we may also assume that $\bar{\xi}$ and $\bar{\eta}$ are not rationally dependent, i.e. no rational numbers a,b,c other than 0 exist such that

$$a\bar{\xi} + b\bar{\eta} + c = 0.$$

The above restrictions place the point $P(\bar{\xi},\bar{\eta})$ inside the triangle $A_0B_0C_0$ as shown on diagram 3a, where the coordinates of the vertices are

$$A_0(1,0),\ B_0(1,1),\ C_0(0,0) \qquad (3\text{-}1)$$

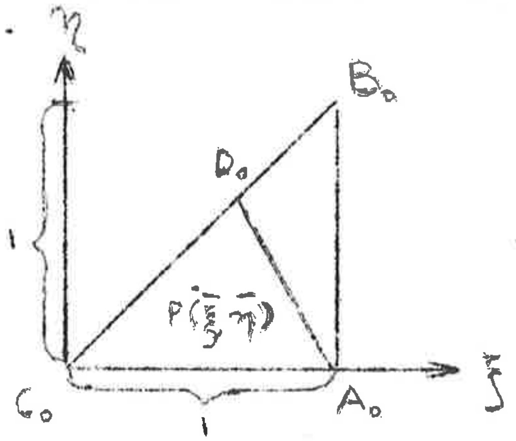

Fig.3a

The algorithm will generate a sequence of approximating triangles

$$(ABC)_K$$

and the sequences (i) and (ii) will be defined in terms of the approximating triangles.

The initial terms of sequences (i) and (ii) will be given by the coordinates of A_0, B_0, C_0 and by the equation of the side A_0B_0 respectively.

The linear forms representing the left hand sides of the equations of A_0B_0, A_0C_0 and B_0C_0 are:

$$\left.\begin{aligned} \left(L_{AB}\right)_0 \left(\xi,\eta\right) &= 1 - \xi \\ \left(L_{AC}\right)_0 \left(\xi,\eta\right) &= \eta \\ \left(L_{BC}\right)_0 \left(\xi,\eta\right) &= \xi - \eta \end{aligned}\right\} \qquad (3\text{-}2)$$

The coefficients of the forms are relatively prime rational integers and furthermore they are chosen so that

$$\left(L_{AB}\right)_0\left(\bar{\xi},\bar{\eta}\right), \left(L_{BC}\right)_0\left(\bar{\xi},\bar{\eta}\right) \text{ and } \left(L_{AC}\right)_0\left(\bar{\xi},\bar{\eta}\right)$$

are positive.

It will be shown that these properties of the coefficients are preserved by the forms

$$L_{AB}\left(\bar{\xi},\bar{\eta}\right),\ L_{BC}\left(\bar{\xi},\bar{\eta}\right),\ L_{AC}\left(\bar{\xi},\bar{\eta}\right)$$

defined by the subsequent algorithm.

We next express the Cartesian coordinates (ξ,η) in terms of homogenous coordinates (x,y,z) so that

$$\frac{x}{z} = \xi, \quad \frac{y}{z} = \eta \ . \qquad (3\text{-}3)$$

The representation is made unique by choosing

$$(z_A)_0 = (z_B)_0 = (z_C)_0 = 1$$

for the three initial points A_0, B_0, C_0, and by fixing the subsequent values of the homogeneous coordinates recursively as shown in the following.

Let A,B,C be the vertices of any approximating triangle. The coordinates of the vertices are

$$\left.\begin{aligned} \xi_A &= \frac{x_A}{z_A}, & \eta_A &= \frac{y_A}{z_A} \\ \xi_B &= \frac{x_B}{z_B}, & \eta_B &= \frac{y_B}{z_B} \\ \xi_C &= \frac{x_C}{z_C}, & \eta_C &= \frac{y_C}{z_C} \end{aligned}\right\} \qquad (3\text{-}4)$$

We now define a new point D, by prescribing its homogeneous coordinates as

$$\left.\begin{aligned} x_D &= x_B + x_C \\ y_D &= y_B + y_C \\ z_D &= z_B + z_C \end{aligned}\right\} \qquad (3\text{-}5)$$

It is easily seen that D is on the line-segment BC, dividing it in the ratio $\frac{z_C}{z_B}$. Assuming that the homogeneous coordinates of A,B and C are rational integers, it follows from (3-5) that the homogeneous coordinates of D are also integers.

The line AD subdivides the triangle ABC into two smaller triangles, and the point P must be <u>inside</u> one of the triangles. This follows from the fact that the equation of the line AD is of the form

$$a\xi + b\eta + c = 0, \qquad (3\text{-}6)$$

where a,b,c can be chosen to be integers, since the coordinates of A and D are rational. Since $\bar{\xi}$ and $\bar{\eta}$ are rationally independent, they cannot satisfy (3-6), thus P is not <u>on</u> AD.

Assume now that the equations of the sides of the triangle ABC are given by

$$L_{AB} = 0, \quad L_{BC} = 0, \quad L_{AC} = 0$$

where L_{AB}, L_{BC}, L_{AC} are linear forms in (ξ,η) with coprime integral coefficients, and furthermore that

$$L_{AB}(\bar{\xi},\bar{\eta}) > 0, \quad L_{BC}(\bar{\xi},\bar{\eta}) > 0, \quad L_{AC}(\bar{\xi},\bar{\eta}) > 0 \qquad (3\text{-}7)$$

(Thus the forms L_{AB}, L_{BC}, L_{AC} are uniquely defined.)

Clearly

$$\left.\begin{aligned} L_{AB}(\xi_A,\eta_A) &= L_{AB}(\xi_B,\eta_B) = L_{BC}(\xi_B,\eta_B) = L_{BC}(\xi_C,\eta_C) = \\ &= L_{AC}(\xi_A,\eta_A) = L_{AC}(\xi_C,\eta_C) = 0 \end{aligned}\right\} \qquad (3\text{-}8)$$

We note also that for the initial triangle $A_0B_0C_0$

$$z_C L_{AB}(\xi_C,\eta_C) = z_B L_{AC}(\xi_B,\eta_B) = z_A L_{BC}(\xi_A,\eta_A) = 1, \qquad (3\text{-}9)$$

and for the purposes of induction we assume that (3-9) is valid for the k^{th} approximating triangle.

We now define the new linear form, $L_{AD}(\xi,\eta)$ by the relation

$$|L_{AD}(\xi,\eta)| = |L_{AB}(\xi,\eta) - L_{AC}(\xi,\eta)| \qquad (3\text{-}10)$$

so that

$$L_{AD}(\bar{\xi},\bar{\eta}) > 0, \qquad (3\text{-}11)$$

i.e.

$$L_{AD}(\xi,\eta) = L_{AB}(\xi,\eta) - L_{AC}(\xi,\eta) \quad \text{if}$$
$$L_{AB}(\bar{\xi},\bar{\eta}) > L_{AC}(\bar{\xi},\bar{\eta}) > 0 \qquad (3\text{-}12)$$

and

$$L_{AD}(\xi,\eta) = L_{AC}(\xi,\eta) - L_{AB}(\xi,\eta) \quad \text{if}$$

$$L_{AC}(\bar{\xi},\bar{\eta}) > L_{AB}(\bar{\xi},\bar{\eta}) > 0 \ . \qquad (3\text{-}13)$$

(The situation $L_{AB}(\bar{\xi},\bar{\eta}) = L_{AC}(\bar{\xi},\bar{\eta})$ cannot occur because of the rational independence of $\bar{\xi}$ and $\bar{\eta}$).

The situations (3-12) and (3-13) can be illustrated geometrically.

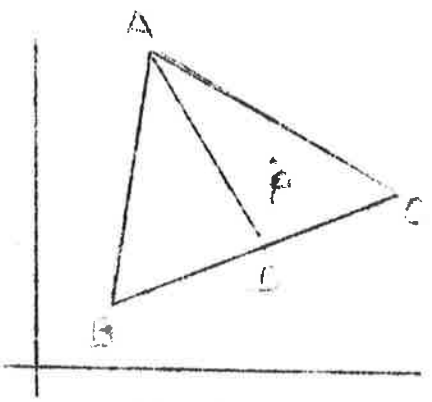

Fig.3b.

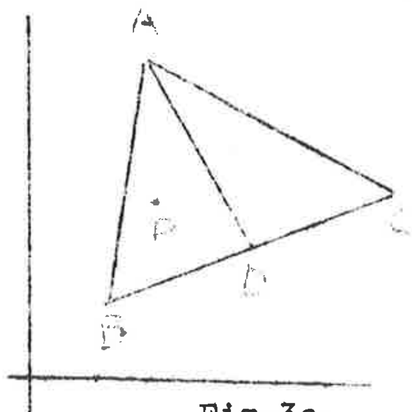

Fig.3c.

In situation (3-12) the point P is inside the triangle ACD (fig.3b), and in case (3-13), P is inside the triangle ABD (fig.3c).

It will be shown next that the equation of the line AD is represented by

$$L_{AD}(\xi_A,\eta_A) = 0.$$

From (3-8) and (3-10) it follows that

$$L_{AD}(\xi_A,\eta_A) = 0. \qquad (3\text{-}14)$$

To show that

$$L_{AD}(\xi_D,\eta_D) = 0 \qquad (3\text{-}15)$$

is also true, write

$$L_{AD}(\xi,\eta) = a\xi + by + c = \frac{1}{z}(ax + by + cz),$$

thus, using relations (3-5),

$$L_{AD}(\xi_D,\eta_D) = \frac{1}{z_B+z_C}\,(a(x_B+x_C) + b(y_B+y_C) + c(z_B+z_C))$$

$$= \frac{1}{z_B+z_C}(z_B L_{AD}(\xi_B,\eta_B) + z_C L_{AD}(\xi_C,\eta_C)).$$

It follows from (3-10) that

$$|L_{AD}(\xi_D,\eta_D)| = \left|\frac{1}{z_B+z_C}(z_B L_{AB}(\xi_B,\eta_B) - z_B L_{AC}(\xi_B,\eta_B) + z_C L_{AB}(\xi_C,\eta_C) - z_C L_{AC}(\xi_C,\eta_C))\right|$$

$$= \left|\frac{1}{z_B+z_C}(z_C L_{AB}(\xi_C,\eta_C) - z_B L_{AC}(\xi_B,\eta_B)\right|$$

(using (3-8).

Finally, assuming the validity of (3-9), the truth of (3-15) follows.

Like the forms L_{AB}, L_{AC}, L_{BC}, the form L_{AD} has rational integral coefficients, and a positive sign for $(\bar{\xi},\bar{\eta})$. This follows from the defining relations (3-10) and (3-11).

Furthermore, it follows from the defining relation (3-10), together with relations (3-8) and (3-9) that

$$|z_B L_{AD}(\xi_B,\eta_B)| = |z_C L_{AD}(\xi_C,\eta_C)| = 1. \qquad (3\text{-}16)$$

From here it follows immediately that the coefficients (a,b,c) in (3-6) are relatively prime, since

$$|z_B L_{AD}(\xi_B,\eta_B)| = |ax_B + by_B + cz_B| = 1,$$

and the coefficients (a,b,c) and the homogeneous coordinates (x_B, y_B, z_B) are all rational integers. It follows at the same time that x_B, y_B and z_B are also coprime, and by similar considerations applied to the relations (3-9) the same is true for (x_A, y_A, z_A) and (x_C, y_C, z_C) respectively.

The algorithm will be now continued by relabelling the vertices. In the situation (3-12) ADC will be used as the new approximating triangle, hence vertex B is discarded, and similarly in situation (3-13) the algorithm is continued on triangle ABD, and C discarded. Letting L_{AB} be the linear form associated with the k^{th} triangle, we choose L_{AD} as the linear form associated with the $k+1^{st}$ triangle, noting that L_{AD} has coprime integral coefficients as proved above and that assuming that (3-9) holds for the k^{th} triangle, relation (3-16) implies that it also holds for the $(k+1)^{st}$ triangle. (The modulus sign may be discarded, since in case (3-12), $L_{AD}(\xi_C, \eta_C) > 0$, and in case (3-13) $L_{AD}(\xi_B, \eta_B) > 0$.)

However, the transition from the k^{th} to the $k+1^{st}$ approximating triangle is still not fully defined. Remembering that the side

$$(AB)_{k+1} \text{ is \underline{fixed} as } (AD)_k ,$$

the choice is to be made between two procedures, illustrated by the diagrams 3d, 3e, and 3f.

Let us assume that P is inside the triangle ADC. To make the transition-procedure clear, we use subscripts to distinguish between the k^{th} and $k+1^{st}$ triangle. The situation is shown on fig.3d.

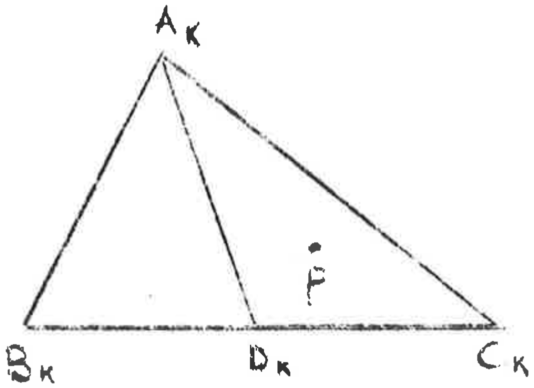

Fig.3d.

Let the relabelling

$$\left.\begin{array}{l} A_k \rightarrow A_{k+1} \\ D_k \rightarrow B_{k+1} \\ C_k \rightarrow C_{k+1} \end{array}\right\} \qquad (3\text{-}17)$$

define the new vertices. The new triangle is shown on fig. 3e. The subsequent subdivision is now defined by D_{k+1} being on side $B_{k+1}C_{k+1}$, (formerly $D_k\ C_k$).

The alternate relabelling is

$$\left.\begin{array}{l} A_k \rightarrow B_{k+1} \\ D_k \rightarrow A_{k+1} \\ C_k \rightarrow C_{k+1} \end{array}\right\} \qquad (3\text{-}18)$$

(fig. 3f.)

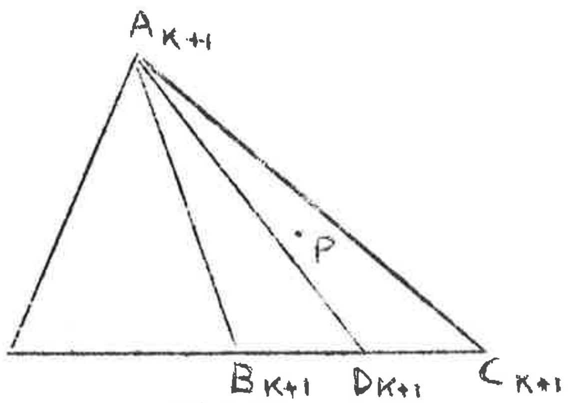

Fig.3e

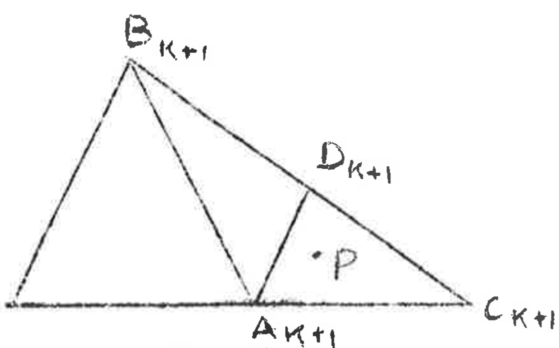

Fig.3f

In both cases

$$(AD)_k \to (AB)_{k+1},\ C_k \to C_{k+1},$$

and the $(k+1)^{st}$ approximating triangles are the same apart from the naming of two of the vertices. However, the two procedures lead to different triangles in the $(k+2)^{nd}$ step, and in case (3-17) $A_{k+1}D_{k+1}C_{k+1}$ is of a more elongated shape than the corresponding triangle in case (3-18).

It appears intuitively that in this particular case the second choice is more advantageous. Generally it is desirable to make the choice so that in the subsequent step of finding the new D point it should be the longer side which is subdivided to prevent the occurrence of approximating triangles of elongated shape. (Some justification will be given later.) Guided by these considerations, (3-17) seems to be the reasonable choice if

$$|D_k C_k| > |A_k C_k| \quad \text{and}$$

(3-18) in the case when

$$|D_k C_k| < |A_k C_k|.$$

It is possible that P is inside triangle ABD instead of triangle ACD. To make the relabelling uniformly (3-17) or (3-18), we may in this case rename first B_k as C_k (after discarding the original vertex C_k).

The computation of lengths for the purposes of comparison is cumbersome. Instead of lengths, we may define the "<u>span</u>"-s of the sides in the following manner:

If (ξ_A, η_A) and (ξ_B, η_B) are the Cartesian coordinates of the points A and B, then the <u>span</u> of the line-segment AB is defined as:

$$S_{AB} = \max\{|\xi_A - \xi_B|, |\eta_A - \eta_B|\}.$$

We modify the algorithm slightly by making the choice

(3-17) if $S_{CD} > S_{AC}$

and (3-18) if $S_{CD} < S_{AC}$.

(It may happen that $S_{CD} = S_{AC}$, but in this case it does not matter which choice is taken, for it does not affect appreciably the shape of the subsequent approximating triangle. For definiteness we choose in this case the relation (3-18).)

To carry further the analogy with ordinary continued fractions, we may define the "digits" of the algorithm, (corresponding to the numbers $b_0, b_1, \ldots b_k$ in (2-4)).

Assume that we have a run of r_k steps so that in each cycle

$$L_{AB}(\bar{\xi},\bar{\eta}) > L_{AC}(\bar{\xi},\bar{\eta}),$$

but in the $r_k + 1^{st}$ step

$$L_{AB}(\bar{\xi},\bar{\eta}) < L_{AC}(\bar{\xi},\bar{\eta}) \qquad (r_k \geq 0) .$$

We define

$$r_k + 1 = k^{th} \text{ digit.} \tag{3-19}$$

For the main purpose of this work (unit-finding), the digits are of no importance, but their computation is incorporated in the program evolved, partly because of their intrinsic interest, (e.g."almost periodic" runs in many cases), partly because of their value in comparing and checking the computations.

It follows from (3-5) that the sequence

$$\{z_A\}$$

i.e. the sequence of the denominators of the rational coordinates of the approximating vertices A, increases, (though not necessarily with termwise monotonousness), and hence goes to infinity (z_A being integer).(*)

Thus it follows from (3-9) that

$$\{L_{AB}(\bar{\xi},\bar{\eta})\}$$

is a null-sequence.

Consider now the area of the approximating triangle ABC.

$$\text{Area} = \frac{1}{2} \begin{vmatrix} \xi_A & \eta_A & 1 \\ \xi_B & \eta_B & 1 \\ \xi_C & \eta_C & 1 \end{vmatrix} = \frac{1}{2z_A z_B z_C} \Delta_{ABC},$$

$$\text{where} \quad \Delta_{ABC} = \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix} .$$

Clearly, $|\Delta_{ABC}|$ remains unchanged through the algorithm, since the next value of the determinant is that of

---

(*) Since $z_D = z_B+z_C$, the triple (z_A,z_B,z_C) is replaced in each step by a new triple with the sum $z_A+z_B+z_C$ being greater than before. Furthermore no term of the triple remains constant, since the choice procedure makes it certain than in a finite number of steps each vertex is replaced.

$$\Delta_{ABD} = \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_B+x_C & y_B+y_C & z_B+z_C \end{vmatrix}$$

or of

$$\Delta_{ACD} = \begin{vmatrix} x_A & y_A & z_A \\ x_C & y_C & z_C \\ x_B+x_C & y_B+y_C & z_B+z_C \end{vmatrix} .$$

Since for the initial triangle $A_0 B_0 C_0$,

$$\Delta_{ABC} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

it follows that for the general approximation triangle ABC,

$$\text{Area} = \tfrac{1}{2} \cdot \frac{1}{z_A z_B z_C} .$$

If throughout the algorithm the shape of the triangle is kept from becoming too elongated, then the sides or altitudes vary roughly as the square-root of the area and so it can be expected that the distance of P from any of the three approximating vertices varies roughly as

$$\frac{1}{z^{\frac{3}{2}}} .$$

Comparing this with the result (2-9) (previous chapter), applied for $n = 2$, it is not unreasonable to expect that the simultaneous approximation achieved by the algorithm described is of the desired order.

Thus it can be hoped that if the algorithm is applied to a number pair $(\bar{\xi},\bar{\eta})$, where the triple

$$(1,\bar{\xi},\bar{\eta})$$

forms a basis in the pure cubic field determined by the equation

$$x^3 = D \qquad \text{(D square-free rational integer)},$$

then the units of the field will occur amongst the terms of the sequence

$$\{L_{AD}(\bar{\xi},\bar{\eta})\},$$

by the argument brought forward in the summary.

To detect the presence of a unit, the norm of each $L_{AB}(\bar{\xi},\bar{\eta})$ must be computed. Then

$$L_{AB} = a\bar{\xi} + b\bar{\eta} + c \quad \text{is a unit if}$$

and only if

$$N(L_{AB}) = (a\bar{\xi} + b\bar{\eta} + c)(a\bar{\xi}' + b\bar{\eta} + c)(a\bar{\xi}'' + b\bar{\eta}'' + c) = \pm 1.$$

Here $a\bar{\xi}' + b\bar{\eta}' + c$ and $a\bar{\xi}'' + b\bar{y}'' + c$ are the field-conjugates of L_{AB}.

In the following chapters the solution of the computational problems presented by this method of unit-finding will be described. The first requirement for a satisfactory development of the proposed algorithm is a high accuracy decimal approximation of the coordinates $(\bar{\xi},\bar{\eta})$. These, in the case of the cubic field, depend on the cubic root of some integer (not a perfect cube.) Ordinary computational routines give cubic roots at accuracies of 10 or at most 20 decimal places. This is inadequate in all but a very few cases.

To develop the algorithm for finding units in the field

$$R(\sqrt[3]{D}),$$

when the value of D was taken from 2 to 200, it was generally necessary to use in the first place an accuracy of 100 decimal places, and this accuracy was raised to 320 places to find the fundamental units in the fields

$$R(\sqrt[3]{167}) \text{ and } R(\sqrt[3]{177}),$$

also to check higher powers of the fundamental units in other fields.

A technique was developed therefore

a) for computing cubic roots to an unlimited accuracy, and

b) to carry out computations with numerals consisting of several hundreds of digits.

4. The cubic root procedure.

Computer programs for evaluating the roots of cubic equations and more specially cubic roots of numbers, are based on Newton's method. In principle, this gives a rapidly converging process for finding the roots. In practice, however, when a really high accuracy is desired, the method fails, because the machines do not normally handle the required multiplications and divisions to more than 8-10 significant figures, and even with double precision routines available, the rounding errors swamp the results after a few iterations and so a limit is put to the number of digits which can reliably be obtained.

It was necessary therefore to evolve a method which - while considerably slower than Newton's method where only ordinary accuracy is needed - is suitable for computing roots to any required precision. It turned out in the course of this work that in some cases accuracies up to 320 decimal places were needed. The method described below has no restricting upper limit of accuracy other than the increase in computing time which is roughly proportional to the square of the number of digits required. With the maximum accuracy used, the computing time on the IBM 7090 computer was of the order of one minute.

Let x be the real cubic root of D, where $1 \leq D < 1000$, i.e. we want the solution of

$$x^3 = D \qquad (4-1)$$

Since we need the roots of numbers greater than 1000, we do not restrict D to integer values.

The function

$$f(\xi) = D - \xi^3$$

is a monotonously decreasing function of the real variable ξ, hence if σ_n, τ_n are two numbers for which

$$\left.\begin{aligned} &f(\sigma_n) > 0 \\ \text{and} \quad &f(\tau_n) < 0 \end{aligned}\right\} \qquad (4-2)$$

and x the solution of (4-1), then

$$\sigma_n < x < \tau_n. \qquad (4-3)$$

We define the sequences $\{\sigma_n\}$ and $\{\tau_n\}$ to give the lower and upper decimal approximations of the real root x to n digits.

Thus

$$\left.\begin{aligned} \sigma_n &= \frac{a_n}{10^{n-1}} \\ \text{and} \quad & \\ \tau_n &= \frac{a_n+1}{10^{n-1}} \end{aligned}\right\}. \qquad (4-4)$$

Here a_n is a positive integer and

$$10^{n-1} < a_n < 10^n. \qquad (4-5)$$

It follows from (4-2) and (4-4) that

$$a_n^3 < 10^{3(n-1)}D < (a_n+1)^3.$$

This inequality may be rewritten by defining

$$D_n = 10^{3(n-1)}D,$$

obtaining

$$a_n^3 < D_n < (a_n+1)^3 . \qquad (4\text{-}6)$$

Let $x_n = 10^{n-1}\xi$, and define $f_n(x_n)$ as

$$f_n(x_n) = 10^{3(n-1)}f(\xi) = 10^{3(n-1)}(D-\xi^3) = D_n - x_n^3 . \quad (4\text{-}7)$$

The computational process consists of a series of trials. We try to determine $x_n = a_n$ so that the inequalities (4-5) and (4-6) are satisfied. Like $f(\xi)$, $f_n(x_n)$ is a monotonously decreasing function, and the value of a_n is restricted to positive integers in a limited range, so only a few trials are necessary. When a_n is found, we may step up the accuracy, introducing in a similar manner the variable x_{n+1} and proceeding to find a_{n+1}. We establish recursion formulae for computing the values of x_n^3, x_{n+1}^3, etc.

To achieve this, we introduce the subscripted difference functions $\nabla_n(x_n)$ and $\nabla_n^2(x_n)$, defined as

$$\nabla_n(x_n) = x_n^3 - (x_n - 1)^3 = 3x_n^2 - 3x_n + 1 \qquad (4\text{-}8)$$

and

$$\nabla_n^2(x_n) = \nabla_n(x_n) - \nabla_n(x_n - 1) = 6x_n - 6 \qquad (4\text{-}9)$$

noting that

$$\nabla_n^3(x_n) = \nabla_n^2(x_n) - \nabla_n^2(x_n - 1) = 6$$

is constant for all n and all x_n.

Assume now that a_n is known, i.e. that the cubic root has been already computed to n digits (counting the units), i.e. $(n - 1)$ decimal places. Furthermore we have the values of $\nabla_n(a_n)$, $\nabla_n^2(a_n + 1)$ and $f_n(a_n)$ listed.

We may then take the following steps to find a_{n+1}.

Let

$$a_{n+1} = 10a_n + b_{n+1} \qquad (4\text{-}10)$$

where b_{n+1} is the $(n+1)^{th}$ digit of the cubic root.

Correspondingly the variable x_{n+1} is given by

$$x_{n+1} = 10a_n + y_{n+1} \qquad (4\text{-}11)$$

where y_{n+1} is the $(n+1)^{st}$ "trial digit", for which we try

$y_{n+1} = 1, \ldots b_{n+1}, b_{n+1} + 1$ where $0 \leqslant b_{n+1} \leqslant 9$.

Using (4-8) and (4-11) and writing $y_{n+1} = 1$ we next have

$$\nabla_{n+1}(x_{n+1}) = 300a_n^2 + 30a_n + 1 \qquad (4\text{-}12)$$

Substituting $x_n = a_n$ in (4-8) and multiplying by 100, we obtain

$$100\nabla_n(a_n) = 300a_n^2 - 300a_n + 300.$$

Subtracting this from (4-12), we obtain

$$\nabla_{n+1}(10a_n + 1) = 100\nabla_n(a_n) + 330a_n - 99. \qquad (4\text{-}13)$$

Also from relation (4-9)

$$\nabla_{n+1}^2(10a_n + 1) = 60a_n = 10\nabla_n^2(a_n + 1). \qquad (4\text{-}14)$$

The recursion formulae (4-13) and (4-14) give the initial values for the subscripted difference-functions $\nabla_{n+1}, \nabla_{n+1}^2$.

Since

$$x_{n+1}^3 = (10a_n + 1)^3 = 10^3a_n^3 + \nabla_{n+1}(10a_n + 1)$$

we have in the first place

$$f_{n+1}(x_{n+1}) = D_{n+1} - x_{n+1}^3 = D_{n+1} - 10^3a_n^3 - \nabla_{n+1}(10a_n + 1),$$

when $x_{n+1} = 10a_n + 1$.

Thus $f_{n+1}(10a_n + 1) = 10^3(D_n - a_n^3) - \nabla_{n+1}(10a_n + 1)$,

i.e.

$$f_{n+1}(10a_n + 1) = 10^3 f_n(a_n) - \nabla_{n+1}(10a_n + 1). \quad (4\text{-}15)$$

If $f_{n+1}(10a_n + 1) > 0$, we may increase y_{n+1} by 1,

i.e. calculate

$$\nabla_{n+1}(10a_n + 2) = \nabla_{n+1}(10a_n + 1) + \nabla^2_{n+1}(10a_n + 2) \quad (4\text{-}16)$$

where $\nabla_{n+1}^2(10a_n + 2) = \nabla_{n+1}^2(10a_n + 1) + 6$, (4-17)

whence

$$f_{n+1}(10a_n + 2) = D_{n+1} - (10a_n + 2)^3 = D_{n+1} - \\ -\{(10a_n + 1)^3 + \nabla_{n+1}(10a_n + 2) \\ = f_{n+1}(10a_n + 1) - \nabla_{n+1}(10a_n + 2) \quad (4\text{-}18)$$

We repeat this procedure, increasing y_{n+1} by 1 and, whenever

$$f_{n+1}(x_{n+1}) = f_{n+1}(10a_n + y_n) > 0.$$

At some stage, which may be as early as $y_{n+1} = 1$, we may find that

$$f_{n+1}(10a_n + y_{n+1}) \leqslant 0. \quad (4\text{-}19)$$

If the equality sign holds, the root is exact, and the procedure is terminated with $b_{n+1} = y_{n+1}$ as the last digit.

In the general case, however, (4-19) is an inequality, and in that case

$$b_{n+1} = y_{n+1} - 1. \quad \text{(Note that } y_{n+1} \geqslant 1.\text{)}$$

It may also happen that for

$$y_{n+1} = 9 \quad \text{we have still}$$

$$f_{n+1}(x_n) = f_{n+1}(10a_n + 9) > 0.$$

In this case we conclude that

$$b_{n+1} = 9.$$

This is justified, since

$$f_{n+1}(10a_n + 10) = D_{n+1} - (10(a_n + 1))^3 = 10^3(D_n - (a_n + 1)^3) < 0$$
$$\text{(using (4-6)).}$$

Thus, in at most 9 trials, b_{n+1} can be determined, and we may proceed to the next digit.

Note that once $\nabla_{n+1}(10a_n + 1)$ and $\nabla^2_{n+1}(10a_n + 1)$ are evaluated by the recursion formulae (4-13) and (4-14), the remaining trial-steps, illustrated by the relations (4-15) to (4-18), only involve additions and subtractions.

The recursion formulae (4-13) and (4-14) contain multiplications by powers of 10 and by 330, apart from additions and subtractions.

These simple operations, however, have to be carried out on the numbers a_n, D_n, etc, which may contain hundreds of digits.

The calculations are therefore arranged by splitting all the variables x_n, f_n, ∇_n, etc. into N blocks where N represents the number of digits required in the final answer.

Let $\sigma_N = \dfrac{a_N}{10^{N-1}}$ be the final approximation desired for the cubic root.

Then

$$10^{N-1}\sigma_N = a_N = 10^{N-1}b_1 + 10^{N-2}b_2 + \dots + b_N \quad (4\text{-}20)$$

$$10^{3(N-1)}D = D_N = 10^{3(N-1)}D_N^{(1)} + 10^{3(N-2)}D_N^{(2)} + \dots + D_N^{(N)} \quad (4\text{-}21)$$

$$\nabla_N = 10^{3(N-1)}\nabla_N^{(1)} + 10^{3(N-2)}\nabla_N^{(2)} + \dots + \nabla_N^{(N)} \quad (4\text{-}22)$$

$$\nabla_N^2 = 10^{3(N-1)}\nabla_N^{2\,(1)} + \dots \qquad + \nabla_N^{2\,(N)} \quad (4\text{-}23)$$

As seen before, $b_1, b_2 \dots b_N$ are the digits, i.e. non-negative integers ranging from 0 to 9, while

$$D_N^{(i)}, \quad \nabla_N^{(i)}, \quad \nabla_N^{2\,(i)} \qquad (i = 1,2\dots N)$$

are integers ranging from 0 to 999.

The arithmetical operations needed in the procedure may then be carried out on the individual blocks, e.g. $\nabla_N^{(i)}$, $D_N^{(i)}$, etc. with the provision of "carrying" the overflow digits obtained during the calculation in each block.

To illustrate the procedure, we rewrite the equation (4-16) as

$$\sum_{i=1}^{n+1} 10^{3(n+1-i)}\nabla_{n+1}^{(i)}(10a_n+2) = \sum_{i=1}^{n+1} 10^{3(n+1-i)}(\nabla_{n+1}^{(i)}(10a_n+1) + \\ + \nabla_{n+1}^{2\,(i)}(10a_n+2)).$$

Assuming that c_{i+1} is the "overflow-digit" carried when $\nabla_{n+1}^{(i+1)}$ and $\nabla_{n+1}^{2\,(i+1)}$ are added, (in this particular operation $c_{i+1} = 0$ or $c_{i+1} = 1$ are the only possibilities), we may write out the addition of the i^{th} block as

$$\nabla_{n+1}^{(i)}(10a_n + 2) = \nabla_{n+1}^{(i)}(10a_n + 1) + \nabla_{n+1}^{2(i)}(10a_n + 2) + c_{i+1} - 1000c_i$$

where

$$c_i = \left[\frac{1}{1000}(\nabla_{n+1}^{(i)}(10a_n + 1) + \nabla_{n+1}^{2(i)}(10a_n + 2) + c_{i+1})\right],$$

(the [ ] symbol is used in the usual manner for "integer part of").

The block-additions, (and subtractions) are all carried out in this manner.

Multiplication by 10^{3k} means a simple shift by k blocks when applied to the variables ∇_n, D_n, etc.

The multiplication by 10 or 100 requires more thought.

Regard, e.g. the relation (4-14) which we rewrite as

$$\sum_{i=1}^{n+1} 10^{3(n+1-i)}\nabla_{n+1}^{2(i)}(10a_n + 1) = 10\sum_{i=1}^{n} 10^{3(n-1)}\nabla_n^{2(i)}(a_n + 1) =$$

$$= \sum_{i=1}^{n} \frac{1}{100}\cdot 10^{3(n+1-i)}\nabla_n^{2(i)}(a_n + 1).$$

Hence where the transition from accuracy n to accuracy $n+1$ is made, each block of the old ∇_n^2 must be divided by 100 to obtain the corresponding block of the new ∇_{n+1}^2, and this will result in shifting some digits over into the next block.

This is best illustrated by a numerical example:

Let $\nabla_4^2(a_n + 1) = 32|895|751|216$,

then

$$\nabla_5^2(10a_n + 1) = 10\nabla_4^2(a_n + 1) = 328957512160 =$$

$$= \frac{1}{100}(32|895|751|216|000) =$$

$$|\ \ 0|328|957|512|160|.$$

Thus we may write down the relation for the

i^{th} block $(i = 1,2,3,4,5)$,

$$\nabla_5^{(i)}(10a_n + 1) = [\tfrac{1}{100}\nabla_4^{(i)}(a_n + 1)] + \nabla_4^{(i-1)}(a_n + 1) -$$

$$- 100\,[\tfrac{1}{100}\nabla_4^{(i-1)}(a_n + 1)].$$

Equation (4-13) involves a multiplication by 330, which may be arranged as follows.

$$330a_n = 330(10a_{n-1} + b_n) = 10.330a_{n-1} + 330b_n.$$

Here b_n has only one digit, hence the number $330b_n$ may be either fitted in the last block, or have just one "carry" - digit for the (n-1)st block. The other part of the product may be evaluated recursively, defining

$$R_n = 330a_n.$$

Thus (4-13) may be rewritten as

$$\nabla_{n+1}(10a_n + 1) = 100\nabla_n(a_n) + R_n - 99 \qquad (4\text{-}24)$$

where

$$R_n = 10R_{n-1} + 330b_n. \qquad (4\text{-}25)$$

The evaluation of R_n and $\nabla_{n+1}(10a_n + 1)$ by relation (4-24) and (4-25) can then be carried out in the manner described earlier.

Before summarising the whole procedure, we add one slight modification:

Define

$$T_n(x_n) = [f_n(x_n)] = [D_n - x_n^3] = [D_n] - x_n^3,$$

(since x_n is integer).

Here

$$0 \leq f_n - T_n < 1, \quad \text{hence}$$

if $T_n(x_n)$ is positive or negative, the same is true for $f_n(x_n)$. In the exceptional case when $T_n(x_n) = 0$, x_n is an exact root if and only if also $f_n(x_n) = 0$, hence $f_n(x_n)$ must also be tested. In practice this testing is easy, it merely means ascertaining whether the radicand D has any non-zero digits to the right of the section representing D_n. If there are any non-zero digits left, the computation may be carried on in the same manner as before, to determine the later digits of the cube-root.

The only relations governing the computation and containing the variable $f_n(x_n)$ are (4-15) and (4-18), or the relations obtained from (4-18) by substituting $10a_n + y_n$ in the place of $10a_n + 1$. It is easy to see that in (4-18) or the more general relations, f_{n+1} may be replaced by

$$[f_{n+1}] = T_{n+1}, \quad \text{since} \quad \nabla_{n+1}(10a_n + y_n) \quad \text{is an integer.}$$

Thus we obtain

$$T_{n+1}(10a_n + y_n + 1) = T_{n+1}(10a_n + y_n) - \nabla_{n+1}(10a_n + y_n + 1) \tag{4-26}$$

Relation (4-15), i.e., the transition from the n^{th} to the $n+1^{st}$ digit of the root has to be modified:

Introduce the notation $D^{(n)}$ for the n^{th} 3-digit "block" of the radicand, i.e.

$$D^{(n)} = [D_n] - 1000[D_{n-1}]. \qquad (4\text{-}27)$$

Taking integer parts on both sides of (4-15), we obtain

$$T_{n+1}(10a_n + 1) = [10^3 f_n(a_n)] - \nabla_{n+1}(10a_n + 1) .$$

Using (4-7),

$$10^3 f_n(a_n) = 10^3(D_n - a_n{}^3) = 10^{3n}D - (10a_n)^3 =$$
$$= D_{n+1} - (10a_n)^3 .$$

Making use of notation (4-27) and taking integer parts:

$$[10^3 f_n(a_n)] = [D_{n+1}] - (10a_n)^3 = D^{(n+1)} + 10^3([D_n] - a_n^3) =$$
$$= D^{(n+1)} + 10^3 T_n(a_n).$$

Thus, (4-15) becomes

$$T_{n+1}(10a_n + 1) = D^{(n+1)} + 10^3 T_n(a_n) - \nabla_{n+1}(10a_n + 1).$$

It is convenient to regard this relation a special case of (4-26) and write

$$T_{n+1}(10a_n) = D^{(n+1)} + 10^3 T_n(a_n) \qquad (4\text{-}28).$$

The procedure may now be summarised by a flow-diagram. For greater clarity, each of the variables is denoted as a single entity, but it should be remembered that on the actual computation the variables are treated as N-dimensional vectors, e.g. $\{y_i\}$, $\{T_n{}^{(i)}\}$, $\{\nabla_n{}^{(i)}\}$ etc. and the calculations are carried out on the "components", i.e. on blocks of 3 digits, in the manner described earlier.

Summary:

Tabulate initial values:

$D^{(1)}, D^{(2)}, \ldots D^{(N)}$, i.e. divide radicand into N sections (as defined by (4-27))

$T_1(0) = [D] = D^{(1)}$

$y_1 = 0$

$\nabla_1 = 1$ (Using (4-8) and (4-9)).

$\nabla_1^2 = 0$

$\nabla_1^3 = \nabla_2^3 = \ldots = \nabla_N^3 = 6$

Flow diagram for finding the digits:

1st digit: Repeat the following cycle:

→ **Calculate**: $T_1(y_1+1) = T_1(y_1) - \nabla_1(y_1+1)$ (by (4-26))

Test the **sign** of T_1

$T_1 > 0$
- $y_1 \to y_1+1$
- $\nabla_1^2 \to \nabla_1^2 + 6$
- $\nabla_1 \to \nabla_1 + \nabla_1^2$
- Return to beginning

$T_1 = 0$
- If $D-[D]=0$, process terminates.
- **Exact** cubic root: $b_1 = y_1$
- If $D-[D] \neq 0$, treat same as $T_1 > 0$

$T_1 < 0$
- **First digit** is found: $b_1 = y_1 - 1$
- $T_1 \to T_1 + \nabla_1$ (i.e. recover previous value)
- $\nabla_1 \to \nabla_1 - \nabla_1^2$ (" ")
- **Store** values of $b_1, \nabla_1, \nabla_1^2$. Define $R_0 = 0$.
- Proceed to next digit

↓

2nd digit

↓

⋮

n^{th} digit

↓

↓

Stored values: $a_n=10a_{n-1}+b_n, T_n, \nabla_n, \nabla_n^2, R_{n-1}$.

Computation of $n+1^{st}$ digit:

Initial values: $R_n = 10R_{n-1}+330b_n$ (by (4-25))

$\nabla_{n+1}=100\nabla_n+R_n-99$ (by (4-24))

$\nabla_{n+1}^2=10\nabla_n^2$ (by (4-14))

$T_{n+1}=1000T_n+D^{(n+1)}$ (by (4-28))

$y_{n+1}=0$

Cycle: Calculate: $T_{n+1}\to T_{n+1}-\nabla_{n+1}$ (i.e. $T_{n+1}(y_{n+1}+1) = T_{n+1}(y_{n+1})-\nabla_{n+1}(y_{n+1}+1)$)

Test the sign of T_{n+1}

$T_{n+1}>0$:
↓
$y_{n+1}\to y_{n+1}+1$
$\nabla_{n+1}^2\to\nabla_{n+1}^2+6$
$\nabla_{n+1}\to\nabla_{n+1}+\nabla_{n+1}^2$
Return to beginning

$T_{n+1}=0$:
If $D^{(k)}=0$ for all $k>n+1$, process terminates.
Exact root:
Last digit:
$b_{n+1}=y_{n+1}$
otherwise same as $T_{n+1}>0$

$T_{n+1}<0$:
↓
$T_{n+1}\to T_{n+1}+\nabla_{n+1}$ (restore previous value)
$\nabla_{n+1}\to\nabla_{n+1}-\nabla_{n+1}^2$
The n+1st digit:
$b_{n+1}=y_{n+1}-1$
Store
$b_{n+1}, T_{n+1}, \nabla_{n+1}, \nabla_{n+1}^2, R_n$
↓
Proceed to next digit → until N^{th} digit.

For illustrating the procedure the calculation of the first 4 digits of $\sqrt[3]{29\cdot929}$ is shown on the table attached, giving 3·104 as answer.

$D = 29 \cdot 929$ }

Blocks: $D^{(1)} = 29$ $\quad D^{(n)} = 0$

$D^{(2)} = 929$ $\quad (n \geq 3)$

y_1	T_1	∇_1	∇_1^2		
	29		0		
		1			
1	28		6		
		7			
2	21	19	12		
			18		
3	2	37			
4	neg				
STORE: b_1	T_1	∇_1	∇_1^2		
|3|	2	19	18		
y_2	T_2	∇_2	∇_2^2	R_1	
	2|929	2|791	180	990	$T_2 = D^{(2)} + 10^3 T_1 = 2929$
1	0|138		186		$R_1 = 330 b_1 = 990$
2	neg	2|977			$\nabla_2 = 100\nabla_1 + R_1 - 99 = 2791$
					$\nabla_2^2 = 10\nabla_1^2 = 180$
STORE: b_2	T_2	∇_2	∇_2^2	R_1	
|1|	138	2791	186	990	
y_3	T_3	∇_3	∇_3^2	R_2	$T_3 = 1000 T_2 + D^{(3)}$
	138|000	289|231	1860	10|230	$R_2 = 10R_1 + 330b_2 = 9900 + 330$
1	neg				$\nabla_3 = 100\nabla_2 + R_2 - 99 = 279100 + 10230 - 99 = 289231$
					$\nabla_3^2 = 10\nabla_2^2 = 1860$
STORE: b_3	T_3	∇_3	∇_3^2		
|0|	138 000	287371	1860		
y_4	T_4	∇_4	∇_4^2	R_3	$T_4 = 1000\, T_3 + D^{(4)}$
	138|000|000	28|839|301	18|600	102300	$R_3 = 10R_2 + 330b_3 = 102300$
1	109|160|699	28|857|907	18|606		$\nabla_4 = 100\nabla_3 + R_3 - 99 = 28737100 + 102300 - 99$
2	80|302|792	28|876|519	18|612		$\nabla_4^2 = 10\nabla_3^2 = 18600$
3	51|426|273	28|895|137	18|618		
|4|	22|531|136	8|913|761	18|624		
$b_4 = 4$	neg				

To conclude the description of this computational procedure, it may be added that the method generalises readily for the case of <u>mixed</u> cubic equations, i.e. the real solutions of equations of the form

$$px^3 + qx + r = 0 \quad \text{(integral coefficients)}$$

may be found with unlimited accuracy.

A program has been worked out and tested, (so far for somewhat limited values of the coefficients (p,q,r). The computation time was of the same order as for the pure cubic equations.

5. Computation of units in fields of the first kind.

Assume that

$$D = ab^2 \qquad (5\text{-}1)$$

is a cube-free number, and <u>a</u> a square-free factor.

Following Dedekind's notation (cf. Chapter (1)) the field

$R(\sqrt[3]{D})$ is called of the first kind, if

$$9 \nmid (a^2 - b^2).$$

In this case the numbers $1, \beta, \gamma,$ where

$$\beta = \sqrt[3]{ab^2}, \qquad \gamma = \sqrt[3]{a^2b} \qquad (5\text{-}2)$$

form an integral basis, i.e. all the integers of the field, and in particular all the units of the field may be written in the form:

$$X + Y\beta + Z\gamma \qquad (5\text{-}3)$$

where X,Y,Z are rational integers, (and of course all numbers of form (5-3) are integers of the field.)

Using the algorithm described in Chapter (3), we obtain a sequence of linear forms (5-3), and we may then test each by finding its norm. Whenever the norm is equal to 1, (it will be seen that the norms of the numbers found by this algorithm are always positive,) the number tested is a unit.

Using the notations of Chapter (3), we choose now the numbers $\bar{\xi}, \bar{\eta}$ as follows:

$$\bar{\xi} = \max\{\beta - [\beta], \gamma - [\gamma]\} \qquad (5\text{-}4)$$
$$\bar{\eta} = \min\{\beta - [\beta], \gamma - [\gamma]\}$$

It is obvious that

$$1 > \bar{\xi} > \bar{\eta} > 0.$$

The inequalities hold in the strict sense, since β and γ are irrational, also rationally independent, since they form an integral basis. It follows also that $\bar{\xi}$ and $\bar{\eta}$ must be rationally independent.

The successive linear forms and their norms will be computed following the algorithm given in Chapter (3).

The work throughout involves accuracies of 50-300 decimal places, hence the technique of working in blocks of digits as indicated in Chapter (4), is used here. In the program described, the length of the blocks is 4 digits. It does not contain operations other than comparisons, additions and subtractions and few multiplications and divisions by integers, which themselves do not exceed one block.

As an illustration of the method, consider as an example the addition of the numbers

$$a = |387|9581|2345|1679\cdot|2910$$

and $$b = |1859|3780|7529\cdot|9915|3200.$$

The blocks are always arranged so that the decimal-point should be at the end of a block. The index of each block gives the necessary information about the place-value.

Here $\underline{a}$ consists of the blocks:

$a_1 = 387$, $a_2 = 9581$, $a_3 = 2345$, $a_4 = 1679$, $a_5 = 2910$,

and $\underline{b}$ of:

$b_1 = 0$, $b_2 = 1859$, $b_3 = 3780$, $b_4 = 7529$, $b_5 = 9915$, $b_6 = 3200$.

The index of the blocks ending with the decimal point must be the same, and to have the same number of blocks in each of the terms to be added, we complete the number a by

$$a_6 = 0.$$

The addition is started at the highest index, adding blocks of the same index. If there is an "overflow", it is "carried" to the next box,

e.g. $a_5 + b_5 = 2910 + 9915 = 12825.$

Here we "carry"

$$[\frac{12825}{10000}] = 1,$$

i.e. we add 1 to the sum of a_4 and b_4, while retaining

$$2825 = 12825 - 10000 \times [\frac{12815}{10000}] \text{ in the } 5^{th} \text{ block.}$$

Multiplication by integers not too large does not present any further complication, and may be carried out in blocks in a similar manner.

As an example for subtraction regard the evaluation of

$$c = a - b,$$

where $a = 3\cdot|6812|7813|2910$

and $b = 17\cdot|3178|4326|3459,$

i.e. $a_1 = 3$, $a_2 = 6812, \ldots$, $b_1 = 17, \ldots$ etc.

To keep all the blocks positive, (with the possible exception of block 1, we add 10000 to each but the first block, and subtract 1 from each but the highest index block.

Thus $c_4 = 10000 + 2910 - 3459 = 9451$

$c_3' = 10000 + 7813 - 4326 - 1 = 13986,$

retaining

$c_3 = 3486$ and "carry"-ing 1.

$c_2' = 10000 + 6812 - 3178 - 1 + 1 = 13634$

whence

$c_2 = 3634$, 1 carried:

$c_1 = 3 - 17 - 1 + 1 = -14.$

Answer: $c = -14 + \cdot 363434869451.$

Finally consider as an example for division

$$17|4182|3120 \div 16$$

$a_1 = 17, \quad a_2 = 4182, \quad a_3 = 3120.$

The blocks of the quotient will be q_1, q_2, q_3,

where

$q_1 = [\frac{a_1}{16}] = 1$ and the remainder: $a_1 - 16\,[\frac{a_1}{16}] = 1$ is carried.

$q_2 = [\frac{1 \times 10000 + a_2}{16}] = [\frac{14182}{16}] = 886,$
the remainder being: $a_2 + 10000 - 16q_2 = 6$, which is carried.

$q_3 = [\frac{6 \times 10000 + a_3}{16}] = [\frac{63120}{16}] = 3820$

i.e. $q = 1|0886|3820.$

No further mention will be made in the description to follow of this block-arithmetic. Apart from a few operations where ordinary accuracies are sufficient, and which will be indicated, the full accuracy of the numbers computed is retained, and the block-routine is followed.

To start the algorithm, the initial values of $L_{AB}(\bar{\xi},\bar{\eta})$ and $L_{BC}(\bar{\xi},\bar{\eta})$, as given by (3-2) are calculated from the values of

$$\beta = \sqrt[3]{ab^2} \quad \text{and} \quad \gamma = \sqrt[3]{a^2b},$$ by the

procedure in Chapter 4, and collected into blocks,

e.g.

$$L_{AC}(\bar{\xi},\bar{\eta})_j = 10^3\bar{\eta}_{4j-3} + 10^2\bar{\eta}_{4j-2} + 10\bar{\eta}_{4j-1} + \bar{\eta}_{4j},$$ where $\bar{\eta}_i$ is the i^{th} digit to the right of the decimal point of $\bar{\eta}$.

Clearly, the values of L_{AB}, L_{AC}, L_{BC} are less than 1. The initial values of the (homogeneous) coordinates of the vertices of the approximating triangles are given by

$$\left.\begin{array}{lll} x_A = 1 & y_A = 0 & z_A = 1 \\ x_B = 1 & y_B = 1 & z_B = 1 \\ x_C = 0 & y_C = 0 & z_C = 1 \end{array}\right\} \qquad (5\text{-}5)$$

The values of x_D, y_D, z_D and of $L_{AD}(\bar{\xi},\bar{\eta})$ are found next by (3-5), (3-10) and (3-11).

To continue the algorithm, a choice must be made between (3-17) and (3-18) and for this purpose the spans of sides CD and AC must be calculated. To make these successive calculations practicable throughout the algorithm, consider the definition of the span of a distance PQ:

$$S_{PQ} = \max\{|\xi_P-\xi_Q|,|\eta_P-\eta_Q|\} = \max\{|\frac{x_P}{z_P} - \frac{x_Q}{z_Q}|,|\frac{y_P}{z_P} - \frac{y_Q}{z_Q}|\} \quad (5\text{-}6),$$

where (ξ_P,η_P) and (x_P,y_P,z_P) represent the Cartesian and the homogeneous coordinates of P, respectively.

Consider the expression $\frac{x_P}{z_P} - \frac{x_Q}{z_Q} = \frac{\begin{vmatrix} x_P & x_Q \\ z_P & z_Q \end{vmatrix}}{z_P z_Q}$, (here $\begin{vmatrix} x_P & x_Q \\ z_P & z_Q \end{vmatrix}$ stands for the determinant-value

$$x_P z_Q - z_P x_Q).$$

Denote

$$(K_{PQ})_\xi = \begin{vmatrix} x_P & x_Q \\ z_P & z_Q \end{vmatrix} \quad \text{and} \quad (K_{PQ})_\eta = \begin{vmatrix} y_P & y_Q \\ z_P & z_Q \end{vmatrix} \qquad (5\text{-}7),$$

i.e. K_{PQ} may be regarded as a vector of two components. Let D be defined according to (3-5), i.e.

$$x_D = x_B + x_C$$

$$y_D = y_B + y_C$$

$$z_D = z_B + z_C .$$

It is easy to check that the vectors K_{BC}, etc. as defined by (5-7) have the following properties:

$$\left.\begin{array}{ll} (a) & K_{BC} = -K_{CB} \\ (b) & K_{DC} = K_{BC} \\ & K_{DB} = K_{CB} = -K_{BC} \\ (c) & K_{AD} = K_{AB} + K_{AC} \end{array}\right\} \qquad (5\text{-}8)$$

(a) is clear.

(Proof of (b). Taking x component:

$$(K_{DC})_x = \begin{vmatrix} x_D & x_C \\ z_D & z_C \end{vmatrix} = \begin{vmatrix} x_B + x_C & x_C \\ z_B + z_C & z_C \end{vmatrix} = \begin{vmatrix} x_B & x_C \\ z_B & z_C \end{vmatrix} = (K_{BC})_x .$$

Similarly for $(K_{DB})_x$.

Proof of (c) $(K_{AD})_x = \begin{vmatrix} x_A & x_D \\ z_A & z_D \end{vmatrix} = \begin{vmatrix} x_A & x_B + x_C \\ z_A & z_B + z_C \end{vmatrix} = \begin{vmatrix} x_A & x_B \\ z_A & z_B \end{vmatrix} + \begin{vmatrix} x_A & x_C \\ z_A & z_C \end{vmatrix} =$

$$= (K_{AB})_x + (K_{AC})_x).$$

Using relations (5-8), the integers $(K_{AB})_x$, $(K_{AB})_y$, etc. are computed recursively throughout the algorithm, whenever the approximating triangle $A_nB_nC_n$ is replaced by $A_{n+1}B_{n+1}C_{n+1}$. The K vector belonging to each (directed) side of the new triangle is defined by these relations, beginning with the initial values, which are found using (5-5) and (5-7). The initial values are:

$$(K_{AB})_o = \begin{pmatrix} 0 \\ -1 \end{pmatrix}; \quad (K_{AC})_o = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad (K_{BC})_o = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The spans of the sides CD and AC required in the algorithm may be expressed, (using 5-6 and 5-7) in the form:

$$S_{CD} = \max\left\{ \left|\frac{(K_{CD})_x}{z_C z_D}\right|, \left|\frac{(K_{CD})_y}{z_C z_D}\right| \right\},$$
$$S_{AC} = \max\left\{ \left|\frac{(K_{AC})_x}{z_A z_C}\right|, \left|\frac{(K_{AC})_y}{z_A z_C}\right| \right\}; \qquad (5\text{-}9)$$

hence the choice-procedure required involves

a) determining the component greater in absolute value of (K_{CD}) and (K_{AC}) respectively,

b) evaluating

$$R_{CD} = \frac{\max\{|(K_{CD})_x|, |(K_{CD})_y|\}}{z_D}$$

and

$$R_{AC} = \frac{\max\{|(K_{AC})_x|, |(K_{AC})_y|\}}{z_A}, \text{ and}$$

c) comparing R_{CD} and R_{AC}.

It follows from (5-9) that

$$R_{CD} > R_{AC} \iff S_{CD} > S_{AC}, \quad \text{since}$$

z_C is always positive.

It should be noted however that the orders of magnitude and accuracies involved require special care in these computations.

The integer quantities such as $(K_{CD})_x$, z_A, etc. are evaluated exactly throughout the algorithm, i.e. the block-method is used and no digits are lost. The values of R_{CD} and R_{AC} however are used only for comparison on a single occasion, and no recursion depends on them, so that standard accuracy is adequate here. In the course of calculation the true values of the integers $(K_{AC})_x$, z_D etc. exceed the range of the computer, on the other hand the quantity $|R_{CD} - R_{AC}|$, even if R_{CD}, R_{AC} are calculated with a satisfactory accuracy, may be of too small order of magnitude for reliable comparison with 0.

Noting that R_{CD}, R_{AC} are positive quantities, the expression

$$\frac{R_{AC} - R_{CD}}{R_{AC} + R_{CD}} \qquad (5\text{-}10)$$

has the same sign as $R_{AC} - R_{CD}$, and its order of magnitude is in the range normally handled by the computer.

To evaluate the ratio (5-10), the numbers R_{AC}, R_{CD} are found first in the following manner.

Assume e.g. that

$$z_A = 759|8910|1234|2915|7283 = 7598910\cdot\ldots \times 10^{12}$$

and that

$$\max\{(K_{AC})_x,(K_{AC})_y\} = 1822|4183|7238|1111 = 18224183\cdot\ldots \times 10^{8}$$

Then R_{AC} is calculated as

$$\frac{18224183}{7598910} \times 10^{8-12} = a \times 10^{j}$$

where <u>a</u> is the ratio of the leading blocks of K_{AC} and z_A. (The first two blocks are taken in each to ensure an accuracy of at least 5 sign. figures), and the index j determines the decimal order.

Let

$R_{AC} = a.10^{j}$ and $R_{CD} = b.10^{k}$, where

a, b, j and k are computed separately.

Then

$$\frac{R_{AC} - R_{CD}}{R_{AC} + R_{CD}} = \frac{a.10^{j} - b.10^{k}}{a.10^{j} + b.10^{k}} = \frac{a.10^{j-k} - b}{a.10^{j-k} + b} \quad (5\text{-}11).$$

The expression (5-11) contains only numbers within the ordinary working range of the computer, and hence can be easily evaluated, no matter how large the numbers z_A, (K_{AC}), etc. By this simple device the accuracy of the computations could be indefinitely increased.

The choice-procedure governs the algorithm as described in Chapter (3). On each transition the coordinates of the vertices, the expressions $L_{AB}, (K_{AB})_x$ etc. belonging to each side are reassigned.

Recalling the description of the algorithm in Chapter (3), the reassignment of these values can be summarised in the following:

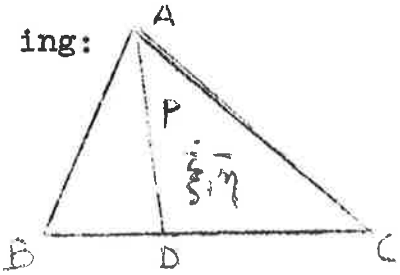

Fig 5(a)

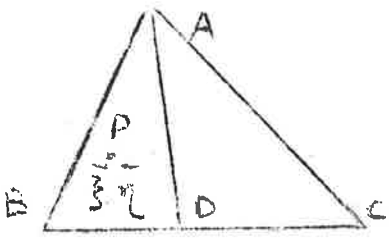

Fig 5(b)

Firstly, we reassign, if necessary, the vertex C and the corresponding variables to make $P(\bar{\xi},\bar{\eta})$ internal to triangle ACD,

i.e.

if

$$L_{AB}(\bar{\xi},\bar{\eta}) > L_{AC}(\bar{\xi},\bar{\eta}) \quad \text{(fig. 5(a))}$$

i.e., when

$$L_{AB}(\bar{\xi},\bar{\eta}) - L_{AC}(\bar{\xi},\bar{\eta}) \rightarrow L_{AD}(\bar{\xi},\bar{\eta}),$$

the coordinates of the vertices remain unchanged,

$$L_{BC}(\bar{\xi}\ \bar{\eta}) \rightarrow L_{CD}(\bar{\xi},\bar{\eta})$$

$$\text{and } - K_{BC}(\bar{\xi},\bar{\eta}) \rightarrow K_{CD}(\bar{\xi},\bar{\eta}).$$

In the situation illustrated in 5(b), i.e. when

$$L_{AB}(\bar{\xi},\bar{\eta}) < L_{AC}(\bar{\xi},\bar{\eta}), \text{ hence}$$

$$L_{AC}(\bar{\xi},\bar{\eta}) - L_{AB}(\bar{\xi},\bar{\eta}) \rightarrow L_{AD}(\bar{\xi},\bar{\eta}),$$ then we reassign B,

so that

$$B \rightarrow C \quad (\text{i.e. } x_B \rightarrow x_C \text{ etc.}),$$

and with this:

$$L_{AB}(\bar{\xi},\bar{\eta}) \rightarrow L_{AC}(\bar{\xi},\bar{\eta}),$$

$$K_{AB}(\bar{\xi},\bar{\eta}) \rightarrow K_{AC}(\bar{\xi},\bar{\eta})$$

$$L_{BC}(\bar{\xi},\bar{\eta}) \rightarrow L_{CD}(\bar{\xi},\bar{\eta}) \quad \text{(as in the other case)}$$

and $$K_{BC}(\bar{\xi},\bar{\eta}) \rightarrow K_{CD}(\bar{\xi},\bar{\eta}).$$

The span-comparison is then always applied to triangle ACD, and if

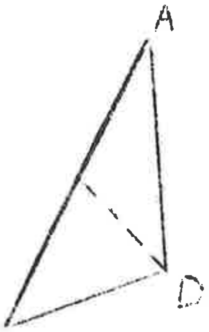

Fig 5(c)

$$S_{AC} > S_{CD}, \qquad \text{(fig. 5(c))},$$

then

$$D \to A$$
$$A \to B$$
$$C \to C$$

hence

$$L_{CD}(\bar{\xi},\bar{\eta}) \to L_{AC}(\bar{\xi},\bar{\eta})$$
$$L_{AC}(\bar{\xi},\bar{\eta}) \to L_{BC}(\bar{\xi},\bar{\eta})$$
$$-K_{CD}(\bar{\xi},\bar{\eta}) \to K_{AC}(\bar{\xi},\bar{\eta})$$
$$K_{AC}(\bar{\xi},\bar{\eta}) \to K_{BC}(\bar{\xi},\bar{\eta})$$
$$-K_{AD}(\bar{\xi},\bar{\eta}) \to K_{AB}(\bar{\xi},\bar{\eta}).$$

In the other case, when

$$S_{CD} > S_{AC} \qquad \text{(fig. (5(d))},$$

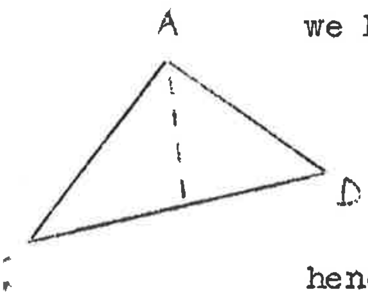

Fig 5(d)

we have

$$A \to A$$
$$C \to C$$
$$D \to B$$

hence

$$\left.\begin{array}{l} L_{AC}(\bar{\xi},\bar{\eta}) \to L_{AC}(\bar{\xi},\bar{\eta}) \\ K_{AC}(\bar{\xi},\bar{\eta}) \to K_{AC}(\bar{\xi},\bar{\eta}) \end{array}\right\} \quad \text{i.e. no change}$$
$$L_{CD}(\bar{\xi},\bar{\eta}) \to L_{BC}(\bar{\xi},\bar{\eta})$$
$$-K_{CD}(\bar{\xi},\bar{\eta}) \to K_{BC}(\bar{\xi},\bar{\eta})$$
$$K_{AD}(\bar{\xi},\bar{\eta}) \to K_{AB}(\bar{\xi},\bar{\eta}).$$

In each case we have

$$L_{AD}(\bar{\xi},\bar{\eta}) \to L_{AB}(\bar{\xi},\bar{\eta}) \qquad (5\text{-}12).$$

The sequence of linear forms with integral coefficients a,b,c and with the property

$$\{a\overline{\xi} + b\overline{\eta} + c\} \to 0$$

required by the algorithm is defined by the transitions (5-12).

As indicated in Chapter (3), the numbers $L_{AB}(\overline{\xi},\overline{\eta})$ are tested whether they are units of the field, by computing their norms.

It follows from relations (5-4) that

$$L_{AB}(\overline{\xi},\overline{\eta}) = a\overline{\xi} + b\overline{\eta} + c = X + Y\beta + Z\gamma, \qquad (5\text{-}13)$$

where

β,γ are defined according to (5-2) and X,Y,Z are rational integers. Every unit of the field is represented uniquely in the form (5-13) and in particular there exists a triplet of rational integers X^*, Y^*, Z^* such that

$X^* + Y^*\beta + Z^*\gamma$ gives the fundamental unit of the field, i.e. all the units of the field are (positive integral) powers of the fundamental unit.

The norm of $L_{AB}(\overline{\xi},\overline{\eta})$ is by definition

$$N(X,Y,Z) = (X + Y\beta + Z\gamma)(X + Y\beta + Z\gamma)'(X + Y\beta + Z\gamma)'',$$

where $(X + Y\beta + Z\gamma)'$ and $(X + Y\beta + Z\gamma)''$ are the field-conjugates of L_{AB}.

Recalling (1-8) and (1-10) in Chapter (1) the norm may be written as

$$N(X,Y,Z) = (X + Y\beta + Z\gamma)(X + Y\beta\omega + Z\gamma\omega^2)(X + Y\beta\omega^2 + Z\gamma\omega), \qquad (5\text{-}14)$$

where ω,ω^2 are the imaginary cubic-roots of 1; or as

$$N(X,Y,Z) = X^3 + ab^2Y^3 + a^2bZ^3 - 3abXYZ. \qquad (5\text{-}15)$$

Clearly, $N(X,Y,Z)$ is a real rational integer.

$L_{AB}(X,Y,Z)$ is a unit, if and only if

$$N(X,Y,Z) = \pm 1.$$

However, form (5-15) is unsuited for computational purposes, (in view of the very large values which are obtained in most cases for X,Y,Z), and while the expression supplies a valuable check in many instances, it is necessary to evolve a recursion method for the computation of $N(X,Y,Z)$, independently of (5-15).

Consider the product

$$\frac{N(X,Y,Z)}{L_{AB}(X,Y,Z)} = (X + Y\beta + Z\gamma\omega^2)(X + Y\beta\omega^2 + Z\gamma\omega) \quad (\text{using } (5\text{-}14)).$$

Substituting

$$\omega = \frac{-1 + i\sqrt{3}}{2} \quad \text{and} \quad \omega^2 = \frac{-1 + i\sqrt{3}}{2}, \quad \text{we obtain}$$

$$\frac{N(X,Y,Z)}{L_{AB}(X,Y,Z)} = (X - Y\tfrac{\beta}{2} - Z\tfrac{\gamma}{2} + i\sqrt{3}(Y\tfrac{\beta}{2} - Z\tfrac{\gamma}{2}))(X - Y\tfrac{\beta}{2} - Z\tfrac{\gamma}{2} - i\sqrt{3}(Y\tfrac{\beta}{2} - Z\tfrac{\gamma}{2}))$$

$$= (L_R(X,Y,Z))^2 + 3(L_I(X,Y,Z))^2$$

where

$$\left.\begin{aligned} L_R(X,Y,Z) &= X - \tfrac{\beta}{2}Y - \tfrac{\gamma}{2}Z \\ L_I(X,Y,Z) &= \tfrac{\beta}{2}Y - \tfrac{\gamma}{2}Z \end{aligned}\right\} \qquad (5\text{-}16)$$

i.e.

$$N = L_{AB}(L_R^2 + 3L_I^2). \qquad (5\text{-}17)$$

It follows from (5-17), that N remains positive throughout the calculations, since the algorithm is arranged to keep

$L_{AB}(X,Y,Z)$ always positive, and since L_R, and L_I are real. Hence if a unit occurs, its norm must be $+1$.

Secondly it follows from the fact that the forms $L_{AB}(X,Y,Z)$, $L_R(X,Y,Z)$ and $L_I(X,Y,Z)$ are linear, that if

$$X = X_1 \pm X_2, \quad Y = Y_1 \pm Y_2, \quad Z = Z_1 \pm Z_2$$

then

$$\left.\begin{aligned} L_{AB}(X,Y,Z) &= L_{AB}(X_1,Y_1,Z_1) \pm L_{AB}(X_2,Y_2,Z_2) \\ L_R(X,Y,Z) &= L_R(X_1,Y_1,Z_1) \pm L_R(X_2,Y_2,Z_2) \\ L_I(X,Y,Z) &= L_I(X_1,Y_1,Z_1) \pm L_I(X_2,Y_2,Z_2) \end{aligned}\right\} . \qquad (5\text{-}18)$$

In each step of the algorithm the value of $L_{AD}(\bar{\xi},\bar{\eta})$ is computed first by (3-10), and later reassigned as the new $L_{AB}(\bar{\xi},\bar{\eta})$ value according to (5-12). Considering (3-10), and remembering that except for $(L_{AC})_o$, all the other $L_{AC}(\bar{\xi}\ \bar{\eta})$ values occurred in the sequence labelled as $L_{AB}(\bar{\xi},\bar{\eta})$; and were reassigned later, it is clear that each $L_{AB}(\bar{\xi},\bar{\eta})$ value is a difference of two previous $L_{AB}(\bar{\xi},\bar{\eta})$ values. It follows clearly from the relations (5-18), that the X,Y,Z values and the L_R and L_I values associated with the current L_{AB} value of the sequence, can be obtained by a corresponding subtraction from the respective X,Y,Z,L_R,L_I values associated with the previous L_{AB} values (or L_{AB} value and L_{AC} value).

Thus X,Y,Z,L,L_R,L_I may be treated simply as the components of a single vector-quantity, associated with a line AB, AC etc., and the algorithm must be applied to the whole of this vector.

The initial values of L_{AB}, L_{AC}, L_{BC} are given by (3-2), The associated initial values of $X_{AB}, Y_{AB}, L_{(R)AB}, L_{(I)AB}$ etc. are found from these, in the following manner.

By denoting suitably a and b in (5-2), we may write without loss of generality:

$$\beta - [\beta] < \gamma - [\gamma], \qquad (5\text{-}19)$$

and hence it follows from (5-4) that

$$\bar{\xi} = \gamma - [\gamma], \qquad \bar{\eta} = \beta - [\beta].$$

We construct the table of initial values, using (3-2) and (5-16).

Since

$$L_{AB} = 1 - \bar{\xi} = 1 + [\gamma] - \gamma$$
$$L_{AC} = \bar{\eta} = -[\beta] + \beta$$
$$L_{BC} = \bar{\xi} - \bar{\eta} = [\beta] - [\gamma] - \beta + \gamma$$

we obtain

$$X_{AB} = 1 + [\gamma], \quad Y_{AB} = 0, \quad Z_{AB} = -1$$
$$X_{AC} = -[\beta] \qquad Y_{AC} = 1, \quad Z_{AC} = 0$$
$$X_{BC} = [\beta]-[\gamma], \quad Y_{BC} = -1, \quad Z_{BC} = 1,$$

and from these:

$$L_{(R)AB} = X_{AB} - Y_{AB}\frac{\beta}{2} - Z_{AB}\frac{\gamma}{2} = 1 + [\gamma] + \tfrac{1}{2}\gamma$$
$$L_{(I)AB} = Y_{AB}\frac{\beta}{2} - Z_{AB}\frac{\gamma}{2} = \tfrac{1}{2}\gamma$$
$$L_{(R)AC} = X_{AC} - Y_{AC}\frac{\beta}{2} - Z_{AC}\frac{\gamma}{2} = -[\beta] - \frac{\beta}{2}$$
$$L_{(I)AC} = Y_{AC}\frac{\beta}{2} - Z_{AC}\frac{\gamma}{2} = \tfrac{1}{2}\beta$$
$$L_{(R)BC} = X_{BC} - Y_{BC}\frac{\beta}{2} - Z_{BC}\frac{\gamma}{2} = [\beta] - [\gamma] + \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma$$
$$L_{(I)BC} = Y_{BC}\frac{\beta}{2} - Z_{BC}\frac{\gamma}{2} = -\tfrac{1}{2}\beta - \tfrac{1}{2}\gamma.$$

The expressions on the right hand side are all easily computable by the block-method, and the computations can be arranged conveniently by first finding $L_{(I)AB}$ and $L_{(I)AC}$ and then finding the other expressions in terms of these.

Thus the program consists of two parts: the initial values are computed in the first part as described above and the algorithm is then executed in the second part. In each step of the algorithm all the components of the AB,AC etc. vectors, i.e. (L, L_R, L_I, X, Y, Z) are computed or reassigned simultaneously.

The computation of the L_R and L_I components presents the need for some further manipulation. It can be seen from (5-17) and the fact that $L \to 0$ that the L_R and L_I values form generally increasing, unbounded sequences, hence at certain stages of the calculation the leading block of digits in at least one of $L_{(R)AD}$ or $L_{(I)AD}$ will "overflow", i.e. exceed the allowed block-maximum (10^4 in this program). When this happens it is necessary to carry out a "block-shifting" operation, not only in the $L_{(R)}$ or $L_{(I)}$ quantities in which the overflow actually occurs, but in all the other $L_{(R)BC}$, $L_{(I)AC}$, etc. quantities which are stored at that stage. This means that the last block(of highest index) has to be discarded and the index of the remaining blocks is raised by 1, i.e. the place-value of each index is increased by a 10^4 factor.

Since the norm is known to be a (positive) integer, formula (5-17) does not require greater precision than what is available in ordinary computer (floating-point) arithmetic. It is sufficient to take the leading blocks of L, L_R, L_I, (at their true place-values.)

In the program (given in the appendix), the integer coefficients X_{AB}, Y_{AB}, Z_{AC} are calculated only to the point where the first unit is found. To check however whether the square, cube etc. of the first $L_{AB}(X^*, Y^* Z^*)$ which is assumed to be the fundamental unit, occurs in the algorithm, the rest of the computation is carried on. The results are listed in the tables, but it may be mentioned here that in two instances the square was missed out, namely in the case of $D = 167$ and $D = 177$ even the accuracy of 320 decimal places was insufficient to yield the square of the first unit found, though the indication is that a further stepping up the accuracy, (which becomes too costly in computing time) would give the desired result.

In the first place the program was only developed to find units in fields of the first kind. To extend this for the computation of units in fields of the second kind further considerations are necessary. The description of these is given in the next chapter.

6. Computation of units in fields of the second kind.

Following Dedekind's notation, the field $R(\sqrt[3]{D})$, where

$$D = ab^2 \quad (D \text{ cube-free, } \underline{a} \text{ square-free}),$$

is called a pure cubic field of the second kind, if

$$9 | (a^2 - b^2) \qquad (6\text{-}1).$$

In this case, recalling Dedekind's theorem (cf. Chapter 1), the triplet

$$\alpha = \frac{1}{3}(1 + a\sqrt[3]{ab^2} + b\sqrt[3]{a^2b}), \quad \beta = \sqrt[3]{ab^2}, \quad \gamma = \sqrt[3]{a^2b} \quad (6\text{-}2)$$

forms an integer basis of the field.

To simplify the calculations required by the algorithm, this basis will be replaced by another one:

$$1, \beta, \delta,$$

where

$$\delta = p + q\beta + r\gamma \qquad (6\text{-}3)$$

and p, q and r are suitably chosen rational numbers.

Consider the transformation

$$T\begin{pmatrix} 1 \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \qquad (6\text{-}4)$$

where the matrix

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

has rational integers for all its entries. If furthermore the determinant

$$|T| = \pm 1, \qquad (6\text{-}5)$$

then the numbers

$$1,\ \beta,\ \delta \text{ form an integer basis.}$$

(Clearly T^{-1} has also integers for all its entries, thus $1, \beta, \delta$ are integers of $R(\sqrt[3]{D})$, and since α, β, γ are linear functions of integer coefficients of $(1, \beta, \delta)$, all integers of $R(\sqrt[3]{D})$ may be represented in terms of this new basis.)

Using (6-2) and (6-3) we may write (6-4) as follows:

$$t_{11} + t_{12}\beta + t_{13}(p + q\beta + r\gamma) = \frac{1}{3} + \frac{a}{3}\beta + \frac{b}{3}\gamma$$

$$t_{21} + t_{22}\beta + t_{23}(p + q\beta + r\gamma) = \beta$$

$$t_{31} + t_{32}\beta + t_{33}(p + q\beta + r\gamma) = \gamma.$$

By equating the coefficients of 1, β and γ, we obtain for the matrix T:

$$T = \begin{bmatrix} \frac{1}{3} - \frac{pb}{3r} & \frac{a}{3} - \frac{bq}{3r} & \frac{b}{3r} \\ 0 & 1 & 0 \\ -\frac{p}{r} & -\frac{q}{r} & \frac{1}{r} \end{bmatrix}$$

Since the determinant of this matrix,

$$|T| = \frac{1}{3r},$$

we must choose

$$r = \frac{1}{3} \text{ or } -\frac{1}{3} \text{ to fulfil requirement (6-5).}$$

It should be noted that a and b are relatively prime, otherwise D could not be cube-free. It follows from this that 3 cannot be a common divisor of a and b, and by (6-1) it cannot be the divisor of one and not the other. This leaves four possibilities for $\underline{a}$ and $\underline{b}$ (mod 3.)

(i) $a \equiv 1$, $b \equiv 1$

(ii) $a \equiv -1$, $b \equiv 1$

(iii) $a \equiv -1$, $b \equiv -1$

(iv) $a \equiv 1$, $b \equiv -1$.

We may now choose the values of p and q in each case so that all the entries of the matrix T are integers. Furthermore we can choose the sign of r to make δ positive in each case, the other elements of the basis, namely 1 and β being positive.

Without loss of generality, we may assume that

$$a > b,$$

since the field may be regarded either as

$$R(\sqrt[3]{a^2b}) \text{ or as } R(\sqrt[3]{ab^2}).$$

Thus in case (i): $r = p = q = \frac{1}{3}$, hence $\delta = \frac{1}{3}(1 + \beta + \gamma)$,

in case (ii) $r = p = \frac{1}{3}, q = -\frac{1}{3}$ " $\delta = \frac{1}{3}(1 - \beta + \gamma)$,

in case (iii) $r = q = \frac{1}{3}, p = -\frac{1}{3}$ " $\delta = \frac{1}{3}(-1 + \beta + \gamma)$,

in case (iv) $r = p = -\frac{1}{3}, q = \frac{1}{3}$ " $\delta = \frac{1}{3}(-1 + \beta - \gamma)$.

(6-6)

The algorithm can now be applied in a manner similar to that described in Chapter 5, δ replacing γ in the basis.

The coordinates of the points to be approximated are given by

$$\bar{\xi}_\circ = \max(\beta - [\beta], \delta - [\delta])$$
$$\bar{\eta} = \min(\beta - [\beta], \delta - [\delta]) \, . \qquad (6\text{-}7)$$

The linear expressions evolved in the course of the algorithm will be of the form:

$$L(X,Y,Z) = X + Y\beta + Z\delta \qquad (6\text{-}8).$$

Once the initial values are defined, the algorithm procedure differs from the algorithm for fields of first kind, only in the computation of the norm of $L(X,Y,Z)$.

Consider e.g. case (i).

$$\left.\begin{aligned} X + Y\beta + Z\delta &= X + Yp + \frac{Z}{3}(1 + \beta + \gamma) = \\ &= (X + \frac{Z}{3}) + (Y + \frac{Z}{3})\beta + \frac{Z}{3}\gamma = \\ &= X' + Y'\beta + Z'\gamma \end{aligned}\right\} \qquad (6\text{-}9)$$

where the relation between (X,Y,Z) and (X', Y', Z') is given by

$$X' = X + \frac{Z}{3}, \quad Y' = Y + \frac{Z}{3}, \quad Z' = \frac{Z}{3} . \qquad (6\text{-}10)$$

The norm of $L(X,Y,Z)$ may now be found, by computing the expression (5-17), but replacing X,Y,Z by X',Y',Z'.

It follows however from (6-10) and from corresponding relations obtained for cases (ii), (iii) and (iv), that after finding the initial values for X', Y', Z', the successive values of these, and also those of

$$L_{(R)}(X',Y',Z') \quad \text{and} \quad L_{(I)}(X',Y',Z')$$

can be obtained by the same recursion as the $L(X,Y,Z)$ forms and the (X,Y,Z) triples.

It should be noted that now

$$L_{(R)}(X',Y',Z') = X' - \frac{\beta}{2}Y' - \frac{\gamma}{2}Z'$$
$$L_{(I)}(X',Y',Z') = \frac{\beta}{2}Y' - \frac{\gamma}{2}Z'.$$

The variables which are calculated recursively are:

$L(X,Y,Z)$, $L_{(R)}(X',Y',Z')$, $L_{(I)}(X',Y',Z')$, X,Y,Z, where X, Y, Z, i.e. the coefficients of $1, \beta, \delta$ are rational integers, but X', Y', Z' are not necessarily integral, as can be seen from (6-10).

The extension of the program from fields of the first kind to those of the second kind is then effected by inserting a branch-program for the calculation of the necessary initial values corresponding to AB and AC. Once L, $L_{(R)}(X', Y', Z')$, X, Y, Z etc. are found for AB the corresponding quantities for BC are found the same way as in the unextended program, hence the calculation of $L_{BC}, \ldots$ etc. is not part of the branch-program.

In constructing the table for the computation of the initial values, it must be remembered that whereas in the case of fields of the first kind the expressions a^2b and ab^2 allowed symmetrical roles for a and b, and thus they could be assigned freely to make

$$\beta - [\beta] < \gamma - [\gamma],$$

this is no longer the case when determining the relation between

$$\delta - [\delta] \quad \text{and} \quad \beta - [\beta].$$

A choice has already been made to make $a > b$, and so in each of cases (i), (ii), (iii) and (iv) the program must include a test to decide between the possibilities

I) $\delta - [\delta] > \beta - [\beta]$ and

II) $\delta - [\delta] < \beta - [\beta]$. (cf. (6-1))

Thus there are altogether 8 alternative cases for the required initial values. The table for these is shown in the following.

		CASE (i) $a=1$ $b=1$ $\delta=\frac{1}{3}(1+\beta+\gamma)$ I	II	CASE (ii) $a=-1$ $b=1$ $\delta=\frac{1}{3}(1-\beta+\gamma)$ I	II	CASE (iii) $a=-1$ $b=-1$ $\delta=\frac{1}{3}(\beta+\gamma-1)$ I	II	CASE (iv) $a=1$ $b=-1$ $\delta=\frac{1}{3}(-1+\beta-\gamma)$ I	II
		$\delta-[\delta]>\beta-[\beta]$	$\delta-[\delta]<\beta-[\beta]$	$\delta-[\delta]>\beta-[\beta]$	$\beta-[\beta]>\delta-[\delta]$	$\delta-[\delta]>\beta-[\beta]$	$\delta-[\delta]<\beta-[\beta]$	$\delta-[\delta]>\beta-[\beta]$	$\delta-[\delta]<\beta-[\beta]$
AB	$L(\beta,\delta)=1-\bar{\xi}$	$1+[\delta]-\delta$	$1+[\beta]-\beta$	$1+[\delta]-\delta$	$1+[\beta]-\beta$	$1+[\delta]-\delta$	$1+[\beta]-\beta$	$1+[\delta]-\delta$	$1+[\beta]-\beta$
	$L(\beta,\gamma)$	$\frac{2}{3}+[\delta]-\frac{1}{3}\beta-\frac{1}{3}\gamma$	$1+[\beta]-\beta$	$\frac{2}{3}+[\delta]+\frac{1}{3}\beta-\frac{1}{3}\gamma$	$1+[\beta]-\beta$	$\frac{4}{3}+[\delta]-\frac{1}{3}\beta-\frac{1}{3}\gamma$	$1+[\beta]-\beta$	$\frac{4}{3}+\delta-\frac{1}{3}\beta+\frac{1}{3}\gamma$	$1+[\beta]-\beta$
	X,Y,Z	$1+[\delta],0,-1$	$1+[\beta],-1,0$	$1+[\delta],0,-1$	$1+[\beta],-1,0$	$1+[\delta],0,-1$	$1+[\beta],-1,0$	$1+[\delta],0,-1$	$1+[\beta],-1,0$
	X',Y',Z'	$\frac{2}{3}+[\delta],-\frac{1}{3},-\frac{1}{3}$	$1+[\beta],-1,0$	$[\delta]+\frac{2}{3},\frac{1}{3},-\frac{1}{3}$	$1+[\beta],-1,0$	$[\delta]+\frac{4}{3},-\frac{1}{3},-\frac{1}{3}$	$1+[\beta],-1,0$	$[\delta]+\frac{4}{3},-\frac{1}{3},\frac{1}{3}$	$1+[\beta],-1,0$
	$L_{(R)}(\beta,\gamma)=X'-\frac{\beta}{2}Y'-\frac{\gamma}{2}Z'$	$\frac{2}{3}+[\delta]+\frac{1}{6}\beta+\frac{1}{6}\gamma$	$1+[\beta]+\frac{1}{2}\beta$	$[\delta]+\frac{2}{3}-\frac{1}{6}\beta+\frac{1}{6}\gamma$	$1+[\beta]+\frac{1}{2}\beta$	$[\delta]+\frac{4}{3}+\frac{1}{6}\beta+\frac{1}{6}\gamma$	$1+[\beta]+\frac{1}{2}\beta$	$[\delta]+\frac{4}{3}+\frac{1}{6}\beta-\frac{1}{6}\gamma$	$1+[\beta]+\frac{1}{2}\beta$
	$L_{(R)}(\beta,\delta)$	$[\delta]+\frac{1}{2}+\frac{1}{2}\delta$	$1+[\beta]+\frac{1}{2}\beta$	$[\delta]+\frac{1}{2}+\frac{1}{2}\delta$	$1+[\beta]+\frac{1}{2}\beta$	$[\delta]+\frac{3}{2}+\frac{1}{2}\delta$	$1+[\beta]+\frac{1}{2}\beta$	$[\delta]+\frac{3}{2}+\frac{1}{2}\delta$	$1+[\beta]+\frac{1}{2}\beta$
	$L_{(I)}(\beta,\gamma)=\frac{\beta}{2}Y'-\frac{\gamma}{2}Z'$	$-\frac{1}{6}\beta+\frac{1}{6}\gamma$	$-\frac{1}{2}\beta$	$\frac{1}{6}\beta+\frac{1}{6}\gamma$	$-\frac{1}{2}\beta$	$-\frac{1}{6}\beta+\frac{1}{6}\gamma$	$-\frac{1}{2}\beta$	$-\frac{1}{6}\beta-\frac{1}{6}\gamma$	$-\frac{1}{2}\beta$
	$L_{(I)}(\beta,\delta)$	$-\frac{1}{6}-\frac{1}{3}\beta+\frac{1}{2}\delta$	$-\frac{1}{2}\beta$	$-\frac{1}{6}+\frac{1}{3}\beta+\frac{1}{2}\delta$	$-\frac{1}{2}\beta$	$\frac{1}{6}-\frac{1}{3}\beta+\frac{1}{2}\delta$	$-\frac{1}{2}\beta$	$\frac{1}{6}-\frac{1}{3}\beta+\frac{1}{2}\delta$	$-\frac{1}{2}\beta$
AC	$L(\beta,\delta)=\bar{\eta}$	$-[\beta]+\beta$	$-[\delta]+\delta$	$-[\beta]+\beta$	$+[\delta]+\delta$	$-[\beta]+\beta$	$-[\delta]+\delta$	$-[\beta]+\beta$	$-[\delta]+\delta$
	$L(\beta,\gamma)$	$-[\beta]+\beta$	$-[\delta]+\frac{1}{3}+\frac{1}{3}\beta+\frac{1}{3}\gamma$	$-[\beta]+[\beta]$	$-[\delta]+\frac{1}{3}-\frac{1}{3}\beta+\frac{1}{3}\gamma$	$-[\beta]+\beta$	$-[\delta]-\frac{1}{3}+\frac{1}{3}\beta+\frac{1}{3}\gamma$	$-[\beta]+\beta$	$-[\delta]-\frac{1}{3}+\frac{1}{3}\beta-\frac{1}{3}\gamma$
	X,Y,Z	$-[\beta],1,0$	$-[\delta],0,1$	$-[\beta],1,0$	$-[\delta],0,1$	$-[\beta],1,0$	$-[\delta],0,1$	$-[\beta],1,0$	$-[\delta],0,1$
	X',Y',Z'	$-[\beta],1,0$	$-[\delta]+\frac{1}{3},\frac{1}{3},\frac{1}{3}$	$-[\beta],1,0$	$-[\delta]+\frac{1}{3},-\frac{1}{3},\frac{1}{3}$	$-[\beta],1,0$	$-[\delta]-\frac{1}{3},\frac{1}{3},\frac{1}{3}$	$-[\beta],1,0$	$-[\delta]-\frac{1}{3},\frac{1}{3},-\frac{1}{3}$
	$L_{(R)}(\beta,\gamma)=X'-\frac{\beta}{2}Y'-\frac{\gamma}{2}Z'$	$-[\beta]-\frac{1}{2}\beta$	$-[\delta]+\frac{1}{3}-\frac{1}{6}\beta-\frac{1}{6}\gamma$	$-[\beta]-\frac{1}{2}\beta$	$-[\delta]+\frac{1}{3}+\frac{1}{6}\beta-\frac{1}{6}\gamma$	$-[\beta]-\frac{1}{2}\beta$	$-[\delta]-\frac{1}{3}-\frac{1}{6}\beta-\frac{1}{6}\gamma$	$-[\beta]-\frac{1}{2}\beta$	$-[\delta]-\frac{1}{3}-\frac{1}{6}\beta+\frac{1}{6}\gamma$
	$L_{(R)}(\beta,\delta)$	$-[\beta]-\frac{1}{2}\beta$	$-[\delta]+\frac{1}{2}-\frac{1}{2}\delta$	$-[\beta]-\frac{1}{2}\beta$	$-[\delta]+\frac{1}{2}-\frac{1}{2}\delta$	$-[\beta]-\frac{1}{2}\beta$	$-[\delta]-\frac{1}{2}-\frac{1}{2}\delta$	$-[\beta]-\frac{1}{2}\beta$	$-[\delta]-\frac{1}{2}-\frac{1}{2}\delta$
	$L_{(I)}(\beta,\gamma)=\frac{\beta}{2}Y'-\frac{\gamma}{2}Z'$	$\frac{1}{2}\beta$	$\frac{1}{6}\beta-\frac{1}{6}\gamma$	$\frac{1}{2}\beta$	$-\frac{1}{6}\beta-\frac{1}{6}\gamma$	$\frac{1}{2}\beta$	$\frac{1}{6}\beta-\frac{1}{6}\gamma$	$\frac{1}{2}\beta$	$\frac{1}{6}\beta+\frac{1}{6}\gamma$
	$L_{(I)}(\beta,\delta)$	$\frac{1}{2}\beta$	$\frac{1}{6}+\frac{1}{3}\beta-\frac{1}{2}\delta$	$\frac{1}{2}\beta$	$\frac{1}{6}-\frac{1}{3}\beta-\frac{1}{2}\delta$	$\frac{1}{2}\beta$	$-\frac{1}{6}+\frac{1}{3}\beta-\frac{1}{2}\delta$	$\frac{1}{2}\beta$	$-\frac{1}{6}+\frac{1}{3}\beta-\frac{1}{2}\delta$

The program, given in the appendix, incorporates the computation for all cases, i.e. fields of first kind, and the four cases listed for fields of the second kind.

Before summarising the results of the computations listed on the following tables, it may be mentioned that this work is being extended for fields which are generated by the roots of equations of type

$$x^3 + x = D \qquad (D > 0).$$

These equations, like the pure cubics, have one real and two imaginary roots, hence the units in the fields to be examined, form cyclic groups as before. The algorithm to be used is the same as the one described, but modifications are necessary in the cubic-root subroutine, and in the calculation of the norms. The general principle to determine sets of basis elements has been established, but there are considerably more cases for computation than in the pure cubic fields, particularly if D is an even number.

7. Description of the computer program.

The full print-out of the program to evaluate the units of pure cubic fields is shown in the appendix. It is coded in FORTRAN for the IBM 7090 computer. The execution time taken to evaluate the first two units of a certain field is roughly proportional to the square of the number of decimal places needed in the cubic root for an accuracy adequate for the purpose. If more units are calculated, the execution time depends linearly on the number of units, but it is not proportional to it, because a considerable part of the time is used to evaluate the necessary cubic roots and the initial values of the variables associated with them. For an accuracy of 96 decimal places in the cubic root which proved to be adequate or more than adequate for the majority of the fields investigated, the computing time was about 1·4 minute for each field and it went up to almost 15 minutes for $D = 167$ and $D = 177$ when 320 places were used and the second unit not quite reached.

The program incorporates the procedures for fields of the first kind and the four types of fields of the second kind. The cubic-root computation is not part of the main program, but is carried out as a subroutine. The program, a result of long experimentations to overcome the difficulties inherent in the range of numbers handled, is dimensioned to a maximum accuracy of 320 decimal digits in the cube-root.

This accuracy could be stepped ~~up~~ without changing the program but the computation time becomes prohibitive.

Four items are read in from the data-cards of the main program:

(i) the accuracy parameter (I)

(ii) the maximum number of units to be computed (MU)

(iii) the type of field (INST)

(iv) a correction parameter (KEN).

The values of $D = ab^2$ and $D' = a^2b$ together with the accuracy parameter are on the data-cards belonging to the subroutine of root-finding (CROOT).

(i) The accuracy parameter, I, regulates the number of decimal digits to be obtained in the cubic-root process and with it the numbers of digit-blocks making up the other variables.

(ii) Since the values of L_{AB} are essentially decreasing, the first unit expected is the fundamental unit. Further units are calculated to check whether they are the second, third, etc. powers of the first unit found.

(iii) The value of this parameter is 1,2,3 or 4, when the field is of the second kind, as shown on the table at the end of Chapter (6), and it is 5 when the field is of the first kind.

(iv) When this parameter has the value 1, the program is executed in the ordinary manner. When it takes the value 2, the program tests at the end of each cycle the norm of L_{AB}, corresponding to the alternative approximating triangle which is discarded after the span-test. This is done to discover units which may have been lost in the choice-procedure.

Where high precision is needed, the variables are broken up into blocks of 3 or 4 digits as described in Chapters (4) and (5). These variables appear with subscripts, and some of the variables have two subscripts, the second subscript signifying the number of decimal-block.

The following table connects the variables described in the previous chapters and their corresponding FORTRAN symbols:

The cubic root subroutine (CRØØT):

Radicand D (2 blocks of 3 digits): LØT(1),LØT(2).

T_n (ℓ^{th} block of 3 digits): NUM(L)

∇_n (ℓ^{th} " " " "): MØRE(L)

∇^2_n (ℓ^{th} " " " "): INCR(L)

R_n (ℓ^{th} " " " "): JUMP(L)

The current (n^{th}) place is marked with J,

where J = 1,2,... M,

(i.e. M is the total number of digits required in the cubic-root.)

"Carry"-digit: NIP.

Output:

$\beta = \sqrt[3]{ab^2}$ (j^{th} digit): KNØW(J); J = 1,2,...M.

Number of digits to the left of the decimal point: } INTK.

$\gamma = \sqrt[3]{a^2b}$ (j^{th} digit): LEAVE(J); J = 1,2,...M.

Number of digits to the left of the decimal point: } INTL.

The main program:

Vertices:

Homogenous coordinates of A:

x_A(ℓ^{th} block of 4 digits):	NA(1,L);	L = 1,2,...NN
y_A(ℓ^{th} block of 4 digits)	NA(2,L);	L = 1,2,...NN
z_A(ℓ^{th} block of 4 digits)	NA(3,L);	L = 1,2,...NN.

Similarly for vertices B,C and D:

(x_B,y_B,z_B):	NB(M,L);	{M = 1,2,3 {L = 1,2,...NN
(x_C,y_C,z_C)	NC(M,L)	" "
(x_D,y_D,z_D)	ND(M,L)	" "

Sides:

Variables connected with AB (following the notations of chapter 4).

a) L_{AB}	(ℓ^{th} block of 4 digits):	LAB(1,L),	L = 1,2,...N
L_R	(ℓ^{th} block of 4 digits):	LAB(2,L)	" "
L_I	(ℓ^{th} block of 4 digits):	LAB(3,L)	" "

b) <u>Scaling</u> (shift)-factor: 10^{4s}

value of <u>s</u> for L_R MAB(2)

" " " L_I MAB(3).

c) Coefficients X,Y,Z of the elements of the <u>integral basis</u> (i.e. $1,\beta,\gamma$ or $1,\beta,\delta$ resp.)

X(ℓ^{th} block of 4 digits) JAB(1,L) L = 1,2,...NK

Y(ℓ^{th} " " ") JAB(2,L) " "

Z(ℓ^{th} " " ") JAB(3,L) " "

d) $(K_{AB})_x$ (ℓ^{th} block of 4 digits) KAB(1,L), L = 1,2,...NK

$(K_{AB})_y$ (ℓ^{th} block of 4 digits) KAB(2,L), " "

The variables corresponding to the other sides are denoted similarly, e.g.

a) LAD (M,L) M = 1,2,3; L = 1,2,...N.

b) MAC (M) M = 2,3

c) JBC (M,L) M = 1,2,3; L = 1,2,...NK.

d) KCD (M,L) M = 1,2; L = 1,2,...NK.

Auxiliary (subscripted) variables used in program:

(i) NAME(K); K = 1,2,...NK.

This stands (in order) for KCD(M,K); M = 1,2

and KAC(M,K); M = 1,2,

and it is introduced for the purpose of avoiding repetition of instructions.

(ii) VAL(M)
(iii) IND(M) } M = 1,2.

These two numbers characterise the true values of the variables $(K_{CD})_x$ etc., VAL(M) representing the leading digits (collected from the two leading blocks) to maximum accuracy available in floating point, while IND(M) fixes the required decimal place.

(iv) KEEP(J) $J = 1,2,\ldots 30.$

This gives a "digit" of the algorithm according to the definition (3-19). For convenience, the output is arranged so that 30 digits are printed out at a time.

Other non-subscripted variables are defined within the program and require no special explanation.

The subroutine and the main program are executed according to the description in chapters 4,5 and 6. The results of the computations are tabulated. An explanation of the tables and a survey of the results is given in the next chapter.

8. Survey of the results

The results of the calculations are summarised in the table of units attached, for values of D between 2 and 199 (inclusively). The computations were carried out over a period of about two years and most of the units were computed several times. It was necessary to carry out many tests during the development to eliminate initial program errors. (These arose mainly when the accuracy was raised from the originally projected accuracy of about 150 decimal digits to accuracies limited only by the dimension-statement; at this stage the program had to be rewritten.) The most important independent check on the coefficients X,Y, and Z was provided by the formula (1-10) for the value of the norm.

In the case of each field the accuracy parameter was adjusted to find at least two successive units of the field. However, in two cases, namely for D = 167 and D = 177 only one unit was found. It would have been possible to increase the accuracy further, without altering the program and raising the dimensions only. The indication is that an accuracy of 360 digits would have probably yielded the missing square of the first unit found, but this would have required excessive computation times. However, the computations were carried far enough to show up the third power of the fundamental unit, if the calculated unit η_1 would have been the square of the fundamental unit. Thus there is still some justification in regarding η_1 as the fundamental unit in both cases.

The tables attached consist of six columns. The first column gives the value of D. As seen before, D is assumed to be cube-free. Furthermore, since the numbers $D = ab^2$ and $D' = a^2b$ (where a,b are square-free and relatively prime) generate the same field, it is sufficient to list one of the numbers D and D'. Thus all perfect squares, and more generally, all numbers of form $D' = a^2b$ $(a > b)$, have been omitted. (The fields corresponding to 18, 50, 75, 98, 147 and 180 are those represented by 12, 20, 45, 28, 63 and 150 respectively.)

The second and third columns give η_1 and η_2, the first and second unit found by the computations, listed to accuracies of 5 and 4 significant figures respectively. In all but two cases η_2 is the square of η_1. For $D = 28$ and $D = 123$ we find that

$$\eta_2 = \eta_1^3,$$

i.e. it appears that the square of the fundamental unit is missed out.

In the majority of the cases the computations have produced higher powers of η_1, i.e. the cube, and often the fourth, fifth etc. power in succession. A "YES" in the fourth column indicates the occurrence of higher powers of η_1.

The fifth column gives the three elements of the integer basis used. For convenience of tabulation the conventional cubic-root signs are replaced here by Fortran symbols, e.g.

$\sqrt[3]{5}$ is written as 5**1/3.

Finally, the last column contains in order the integer coefficients X,Y,Z of the basis elements $(1,\beta,\gamma)$ or $(1,\beta,\delta)$, in the expressions

$$\eta_1 = X + Y\beta + Z\gamma \quad \text{(first kind)}$$

or

$$\eta_1 = X + Y\beta + Z\delta \quad \text{(second kind)}.$$

Whenever possible, the results have been compared with other available results, notably those found in tables by Cassels[(o)], Markoff[(b)] and Wolfe[(n)]. (Cf. references at the end of Chapter 2.)

Cassels lists fundamental units for values of D up to 50. Except for D = 46, this table gives the fundamental units less than 1 in the form $X + Y\beta + Z\gamma$. The results tabulated here agree with all his results and for D = 46 his result is the reciprocal of the value of η_1 listed here.

All the units listed by Markoff and Wolfe are greater than 1. The reciprocals of the units, believed to be fundamental by these authors, have been computed and compared with the values listed here. There is agreement with Markoff for all values of D (up to 70) except for D = 28 and D = 55. The reciprocals of Markoff's turn out to be in each case equal to the <u>square</u> of η_1 hence in these two cases the units given by Markoff are not fundamental.

Wolfe's table, listing fundamental units for D up to 100 (for square-free D where the field is of the first kind), agrees with these tables for all values of D, except D = 85. Here again the reciprocal of Wolfe's result is equal to the square of η_1 given here.

These findings, together with the evidence given by the tables themselves, support the belief that all the η_1 values listed here are fundamental units.

A P P E N D I X

a. MAIN PROGRAM

```
  UNITS IN PURE CUBIC FIELDS
 DIMENSIONKNOW(322),LEAVE(322),LAB(3,80),LAC(3,80),LBC(3,80),LAD(3,
180),LCD(3,80),MAB(3),MAC(3),MBC(3),MAD(3),MCD(3),JAB(3(40),JAC(3,4
20),JBC(3,40),JAD(3,40),JCD(3,40),NA(3,50),NB(3,50),NC(3,50),ND(3,
350),KAB(2,40),KAC(2,40),KBC(2,40),KCD(2,40),KAD(2,40),NAME(40),VAL
4(2),KEEP(30),NIPS(3),IND(2)
  IT=10
  IH=100
  KEY=1000
  MUCH=10000
  NITH=9999
1 READINPUTTAPE2,102,I,MU,INST,KEN
2 FORMAT(I2,I5,I3,I2)
  NM=8*I
  N=2*I
  NN=(5*I+1)/4
  NK=I-1
  NG=NN-1
  DO100M=1,3
  DO100L=1,NG
  NA(M,L)=0
  NB(M,L)=0
0 NC(M,L)=0
  NA(1,NN)=1
  NA(2,NN)=0
  NA(3,NN)=1
  NB(1,NN)=1
  NB(2,NN)=1
  NB(3,NN)=1
  NC(1,NN)=0
  NC(2,NN)=0
  NC(3,NN)=1
  NG=NK-1
  DO289M=1,3
  DO289L=1,NG
  JAB(M,L)=0
  JAC(M,L)=0
9 JBC(M,L)=0
  DO256M=1,2
  DO256L=1,NG
  KAB(M,L)=0
  KBC(M,L)=0
6 KAC(M,L)=0
  KAB(1,NK)=0
  KAB(2,NK)=-1
  KBC(1,NK)=1
  KBC(2,NK)=1
  KAC(1,NK)=1
```

```
KAC(2,NK)=0
CALLCROOT(INTK,KNOW,INTL,LEAVE)
IF(KNOW(1))216,218,103
NUMK=0
DO301L=1,INTK
LI=INTK-L
NUMK=NUMK+KNOW(L)*IT**LI
NUML=0
DO302L=1,INTL
LI=INTL-L
NUML=NUML+LEAVE(L)*IT**LI
DO104K=1,NM
KK=K+INTK
KL=K+INTL
KNOW(K)=KNOW(KK)
LEAVE(K)=LEAVE(KL)
GOTO(601,611,601,631,641),INST
LEAVE(NM)=LEAVE(NM)+KNOW(NM)
NIP=LEAVE(NM)/IT
LEAVE(NM)=LEAVE(NM)-IT*NIP
DO602K=2,NM
L=NM+1-K
LEAVE(L)=LEAVE(L)+KNOW(L)+NIP
NIP=LEAVE(L)/IT
LEAVE(L)=LEAVE(L)-IT*NIP
IF(INST-3)603,621,216
NUMT=NUML+NUMK+NIP+1
GOTO604
NUMT=NUML+NUMK+NIP-1
NUML=NUMT/3
NIP=NUMT-3*NUML
DO605K=1,NM
LEFT=IT*NIP+LEAVE(K)
LEAVE(K)=LEFT/3
NIP=LEFT-3*LEAVE(K)
GOTO641
LEAVE(NM)=IT+LEAVE(NM)-KNOW(NM)
NIP=LEAVE(NM)/IT
LEAVE(NM)=LEAVE(NM)-IT*NIP
DO612K=2,NM
L=NM+1-K
LEAVE(L)=IT+LEAVE(L)-KNOW(L)+NIP-1
NIP=LEAVE(L)/IT
LEAVE(L)=LEAVE(L)-IT*NIP
NUMT=NUML-NUMK+NIP
GOTO604
LEAVE(NM)=IT+KNOW(NM)-LEAVE(NM)
NIP=LEAVE(NM)/IT
```

```
  LEAVE(NM)=LEAVE(NM)-IT*NIP
  DO632K=2,NM
  L=NM+1-K
  LEAVE(L)=KNOW(L)+IT-LEAVE(L)+NIP-1
  NIP=LEAVE(L)/IT
2 LEAVE(L)=LEAVE(L)-IT*NIP
  NUMT=NUMK-NUML+NIP-2
  GOTO604
1 DO105K=1,NM
  IF(KNOW(K)-LEAVE(K))106,105,108
5 CONTINUE
5 INTB=NUMK
  INTG=NUML
  MIND=1
  DO107K=1,N
  LCD(1,K)=KEY*LEAVE(4*K-3)+IH*LEAVE(4*K-2)+IT*LEAVE(4*K-1)+LEAVE(4*
 1K)
7 LAC(1,K)=KEY*KNOW(4*K-3)+IH*KNOW(4*K-2)+IT*KNOW(4*K-1)+KNOW(4*K)
  GOTO110
8 INTB=NUML
  INTG=NUMK
  MIND=2
  DO109K=1,N
  LAC(1,K)=KEY*LEAVE(4*K-3)+IH*LEAVE(4*K-2)+IT*LEAVE(4*K-1)+LEAVE(4*
 1K)
9 LCD(1,K)=KEY*KNOW(4*K-3)+IH*KNOW(4*K-2)+IT*KNOW(4*K-1)+KNOW(4*K)
0 MAB(2)=1
  MAB(3)=1
  MAC(2)=1
  MAC(3)=1
  MBC(2)=1
  MBC(3)=1
  MAD(2)=1
  MAD(3)=1
  MCD(2)=1
  MCD(3)=1
  NIP=1
  DO111K=2,N
  L=N+2-K
  LBC(1,L)=MUCH+LCD(1,L)-LAC(1,L)+NIP-1
  NIP=LBC(1,L)/MUCH
1 LBC(1,L)=LBC(1,L)-MUCH*NIP
  LBC(1,1)=LCD(1,1)-LAC(1,1)+NIP-1
  DO112K=1,N
  L=N+1-K
2 LAB(1,L)=NITH-LCD(1,L)
  GOTO(650,650,650,650,691),INST
0 GOTO(651,652),MIND
```

```
ITEM=(INTG+1)/2
NIPB=INTG+1-2*ITEM
GOTO(653,655,655,655,691),INST
JOT=(INTB-1)/3
NIPC=INTB-1-3*JOT
GOTO657
JOT=(INTB+1)/3
NIPC=INTB+1-3*JOT
DO659K=2,N
LCD(2,K)=(NIPB*MUCH+LCD(1,K-1))/2
NIPB=NIPB*MUCH+LCD(1,K-1)-2*LCD(2,K)
LCD(3,K)=(NIPC*MUCH+LAC(1,K-1))/3
NIPC=NIPC*MUCH+LAC(1,K-1)-3*LCD(3,K)
GOTO(661,663,661,661,691),INST
LAB(3,N)=MUCH+LCD(2,N)-LCD(3,N)
NIP=LAB(3,N)/MUCH
LAB(3,N)=LAB(3,N)-MUCH*NIP
DO665K=3,N
L=N+2-K
LAB(3,L)=MUCH+LCD(2,L)-LCD(3,L)+NIP-1
NIP=LAB(3,L)/MUCH
LAB(3,L)=LAB(3,L)-MUCH*NIP
GOTO(667,663,669,669,691),INST
LAB(3,1)=ITEM+NIP-(JOT+2)
GOTO673
LAB(3,1)=ITEM+NIP-(JOT+1)
GOTO673
NIP=0
DO671K=2,N
L=N+2-K
LAB(3,L)=LCD(2,L)+LCD(3,L)+NIP
NIP=LAB(3,L)/MUCH
LAB(3,L)=LAB(3,L)-MUCH*NIP
LAB(3,1)=ITEM+JOT+NIP-1
LAB(2,1)=(3*INTG+1)/2
NIP=3*INTG+1-2*LAB(2,1)
DO675K=2,N
LAB(2,K)=(MUCH*NIP+LCD(1,K-1))/2
NIP=MUCH*NIP+LCD(1,K-1)-2*LAB(2,K)
IF(INST-3)679,677,677
LAB(2,1)=LAB(2,1)+1
LAC(3,1)=INTB/2
NIP=INTB-2*LAC(3,1)
DO681K=2,N
LAC(3,K)=(MUCH*NIP+LAC(1,K-1))/2
NIP=MUCH*NIP+LAC(1,K-1)-2*LAC(3,K)
LAC(2,1)=(3*INTB)/2
NIP=3*INTB-2*LAC(2,1)
```

```
DO683K=2,N
LAC(2,K)=(MUCH*NIP+LAC(1,K-1))/2
NIP=MUCH*NIP+LAC(1,K-1)-2*LAC(2,K)
DO685K=2,N
LAC(2,K)=NITH-LAC(2,K)
LAC(2,1)=-LAC(2,1)-1
GOTO686
ITEM=(INTB+1)/2
NIPB=INTB+1-2*ITEM
GOTO(654,656,656,656,691),INST
JOT=(INTG-1)/3
NIPC=INTG-1-3*JOT
GOTO658
JOT=(INTG+1)/3
NIPC=INTG+1-3*JOT
DO660K=2,N
LCD(2,K)=(NIPB*MUCH+LAC(1,K-1))/2
NIPB=NIPB*MUCH+LAC(1,K-1)-2*LCD(2,K)
LCD(3,K)=(NIPC*MUCH+LCD(1,K-1))/3
NIPC=NIPC*MUCH+LCD(1,K-1)-3*LCD(3,K)
NIP=LAC(3,N)/MUCH
IF(INST-2)666,662,666
LAC(3,N)=LCD(2,N)+LCD(3,N)
LAC(3,N)=MUCH*NIP+NITH-LAC(3,N)
DO664K=3,N
L=N+2-K
LAC(3,L)=LCD(2,L)+LCD(3,L)+NIP
NIP=LAC(3,L)/MUCH
LAC(3,L)=MUCH*NIP+NITH-LAC(3,L)
LAC(3,1)=-(NIP+ITEM+JOT)
GOTO674
LAC(3,N)=MUCH+LCD(3,N)-LCD(2,N)
NIP=LAC(3,N)/MUCH
LAC(3,N)=LAC(3,N)-MUCH*NIP
DO668K=3,N
L=N+2-K
LAC(3,L)=MUCH+LCD(3,L)-LCD(2,L)+NIP-1
NIP=LAC(3,L)/MUCH
LAC(3,L)=LAC(3,L)-MUCH*NIP
IF(INST-3)672,670,670
LAC(3,1)=JOT-ITEM+NIP-1
GOTO674
LAC(3,1)=JOT-ITEM+NIP
LAB(2,1)=(3*INTG)/2
NIPB=3*INTG-2*LAB(2,1)
LAB(2,1)=LAB(2,1)+1
LAB(3,1)=INTG/2
NIPC=INTG-2*LAB(3,1)
```

```
  LAB(3,1)=-(LAB(3,1)+1)
  DO676K=2,N
  LAB(2,K)=(MUCH*NIPB+LCD(1,K-1))/2
  NIPB=MUCH*NIPB+LCD(1,K-1)-2*LAB(2,K)
  LAB(3,K)=(MUCH*NIPC+LCD(1,K-1))/2
  NIPC=MUCH*NIPC+LCD(1,K-1)-2*LAB(3,K)
5 LAB(3,K)=NITH-LAB(3,K)
  LAC(2,1)=(3*INTB+1)/2
  NIP=3*INTB+1-2*LAC(2,1)
  IF(INST-3)678,680,680
3 LAC(2,1)=-LAC(2,1)
  GOTO682
) LAC(2,1)=-(LAC(2,1)+1)
2 DO684K=2,N
  LAC(2,K)=(MUCH*NIP+LAC(1,K-1))/2
  NIP=MUCH*NIP+LAC(1,K-1)-2*LAC(2,K)
4 LAC(2,K)=NITH-LAC(2,K)
5 LBC(2,N)=LAB(2,N)+LAC(2,N)
  NIPB=LBC(2,N)/MUCH
  LBC(2,N)=MUCH*NIPB+NITH-LBC(2,N)
  LBC(3,N)=LAB(3,N)+LAC(3,N)
  NIPC=LBC(3,N)/MUCH
  LBC(3,N)=NITH+MUCH*NIP-LBC(3,N)
  DO687K=3,N
  L=N+2-K
  LBC(2,L)=NIPB+LAB(2,L)+LAC(2,L)
  LBC(3,L)=NIPC+LAB(3,L)+LAC(3,L)
  NIPB=LBC(2,L)/MUCH
  NIPC=LBC(3,L)/MUCH
  LBC(2,L)=NITH+MUCH*NIPB-LBC(2,L)
7 LBC(3,L)=NITH+MUCH*NIPC-LBC(3,L)
  LBC(2,1)=-LAB(2,1)-LAC(2,1)-NIPB
  LBC(3,1)=-LAB(3,1)-LAC(3,1)-NIPC-1
  GOTO697
1 LAB(3,1)=INTG/2
  NIPB=INTG-2*LAB(3,1)
  LAC(3,1)=INTB/2
  NIPC=INTB-2*LAC(3,1)
  DO113K=2,N
  LAB(3,K)=(NIPB*MUCH+LCD(1,K-1))/2
  NIPB=NIPB*MUCH+LCD(1,K-1)-2*LAB(3,K)
  LAC(3,K)=(NIPC*MUCH+LAC(1,K-1))/2
3 NIPC=NIPC*MUCH+LAC(1,K-1)-2*LAC(3,K)
  DO114K=2,N
  L=N+2-K
4 LAB(2,L)=LAB(3,L)
  LAB(2,1)=INTG+LAB(3,1)+1
  DO115K=2,N
```

```
L=N+2-K
LAC(2,L)=NITH-LAC(3,L)
LAC(2,1)=-LAC(3,1)-INTB-1
NIP=0
DO116K=1,N
L=N+1-K
LBC(3,L)=LAB(3,L)+LAC(3,L)+NIP
NIP=LBC(3,L)/MUCH
LBC(3,L)=LBC(3,L)-MUCH*NIP
DO117K=2,N
L=N+2-K
LBC(3,L)=NITH-LBC(3,L)
LBC(3,1)=-1-LBC(3,1)
NIP=1
DO118K=2,N
L=2+N-K
LBC(2,L)=MUCH+LAC(3,L)-LAB(3,L)+NIP-1
NIP=LBC(2,L)/MUCH
LBC(2,L)=LBC(2,L)-MUCH*NIP
LBC(2,1)=LAC(3,1)-LAB(3,1)+NIP+INTB-INTG-1
GOTO(801,811),MIND
JAB(1,NK)=1+INTG
JAB(2,NK)=0
JAB(3,NK)=-1
JAC(1,NK)=-INTB
JAC(2,NK)=1
JAC(3,NK)=0
JBC(1,NK)=INTB-INTG
JBC(2,NK)=-1
JBC(3,NK)=1
GOTO821
JAB(1,NK)=1+INTG
JAB(2,NK)=-1
JAB(3,NK)=0
JAC(1,NK)=-INTB
JAC(2,NK)=0
JAC(3,NK)=1
JBC(1,NK)=INTB-INTG
JBC(2,NK)=1
JBC(3,NK)=-1
MARK=1
MARKU=0
DO502M=1,3
JCD(M,NK)=JAB(M,NK)
DO502K=1,N
LCD(M,K)=LAB(M,K)
DO517NO=1,3
GOTO(501,503,505),NO
```

```
1 MOT=4
  GOTO200
3 DO504M=1,3
  JAB(M,NK)=JAC(M,NK)
  DO504K=1,N
4 LAB(M,K)=LAC(M,K)
  MOT=5
  GOTO200
5 DO506M=1,3
  JAB(M,NK)=JBC(M,NK)
  DO506K=1,N
6 LAB(M,K)=LBC(M,K)
  MOT=6
  GOTO200
7 CONTINUE
  DO508M=1,3
  JAB(M,NK)=JCD(M,NK)
  DO508K=1,N
8 LAB(M,K)=LCD(M,K)
9 DO210J=1,30
  NOM=1
0 DO122M=1,3
  NIP=0
  DO121K=2,NN
  L=NN+2-K
  ND(M,L)=NB(M,L)+NC(M,L)+NIP
  NIP=ND(M,L)/MUCH
1 ND(M,L)=ND(M,L)-MUCH*NIP
2 ND(M,1)=NB(M,1)+NC(M,1)+NIP
  DO124M=1,2
  NIP=0
  DO123K=2,NK
  L=NK+2-K
  KAD(M,L)=KAB(M,L)+KAC(M,L)+NIP
  NIP=KAD(M,L)/MUCH
3 KAD(M,L)=KAD(M,L)-MUCH*NIP
4 KAD(M,1)=KAB(M,1)+KAC(M,1)+NIP
  L=N-1
  DO125K=1,L
  IF(LAB(1,K)-LAC(1,K))139,125,126
5 CONTINUE
  IF(LAB(1,N)-LAC(1,N))139,101,126
6 NOTE=0
  DO128M=1,3
  NIPS(M)=1
  DO127K=2,N
  L=N+2-K
  LAD(M,L)=MUCH+LAB(M,L)-LAC(M,L)+NIPS(M)-1
```

```
NIPS(M)=LAD(M,L)/MUCH
LAD(M,L)=LAD(M,L)-MUCH*NIPS(M)
LAD(M,1)=LAB(M,1)-LAC(M,1)+NIPS(M)-1
DO132M=2,3
NIPS(M)=LAD(M,1)/MUCH
LAD(M,1)=LAD(M,1)-MUCH*NIPS(M)
MOVE=NIPS(M)
IF(NIPS(M))130,132,130
DO131K=3,N
L=N+3-K
LAD(M,L)=LAD(M,L-1)
LAD(M,2)=MUCH+LAD(M,1)
NIPS(M)=LAD(M,2)/MUCH
LAD(M,2)=LAD(M,2)-MUCH*NIPS(M)
LAD(M,1)=MOVE+NIPS(M)-1
DO288K=2,N
L=N+2-K
LAB(M,L)=LAB(M,L-1)
LAC(M,L)=LAC(M,L-1)
LBC(M,L)=LBC(M,L-1)
LCD(M,L)=LCD(M,L-1)
LAB(M,1)=0
LAC(M,1)=0
LBC(M,1)=0
LCD(M,1)=0
MAB(M)=MAB(M)+1
MBC(M)=MBC(M)+1
MAC(M)=MAC(M)+1
MCD(M)=MCD(M)+1
MAD(M)=MAD(M)+1
CONTINUE
IF(NOTE)134,134,142
IF(MARKU)135,135,137
DO136M=1,3
NIPS(M)=0
DO136K=1,NK
L=NK+1-K
JAD(M,L)=JAB(M,L)-JAC(M,L)+NIPS(M)
NIPS(M)=JAD(M,L)/MUCH
JAD(M,L)=JAD(M,L)-MUCH*NIPS(M)
NOM=NOM+1
DO138M=1,2
DO138K=1,NK
KCD(M,K)=-KBC(M,K)
GOTO151
NOTE=1
DO141M=1,3
NIPS(M)=1
```

```
  DO140K=2,N
  L=N+2-K
  LAD(M,L)=MUCH+LAC(M,L)-LAB(M,L)+NIPS(M)-1
  NIPS(M)=LAD(M,L)/MUCH
0 LAD(M,L)=LAD(M,L)-MUCH*NIPS(M)
1 LAD(M,1)=LAC(M,1)-LAB(M,1)+NIPS(M)-1
  GOTO129
2 IF(MARKU)143,143,145
3 DO144M=1,3
  NIPS(M)=0
  DO144K=1,NK
  L=NK+1-K
  JAD(M,L)=JAC(M,L)-JAB(M,L)+NIPS(M)
  NIPS(M)=JAD(M,L)/MUCH
4 JAD(M,L)=JAD(M,L)-MUCH*NIPS(M)
5 DO146M=1,3
  DO146K=1,NN
6 NC(M,K)=NB(M,K)
  DO147M=1,2
  DO147K=1,NK
  KAC(M,K)=KAB(M,K)
7 KCD(M,K)=KBC(M,K)
  DO148M=1,3
  MAC(M)=MAB(M)
  DO148K=1,N
8 LAC(M,K)=LAB(M,K)
  IF(MARKU)149,149,151
9 DO150M=1,3
  DO150K=1,NK
0 JAC(M,K)=JAB(M,K)
1 DO152M=1,3
  MCD(M)=MBC(M)
  DO152K=1,N
2 LCD(M,K)=LBC(M,K)
  IF(MARKU)153,153,155
3 DO154M=1,3
  DO154K=1,NK
4 JCD(M,K)=JBC(M,K)
5 DO176NO=1,2
  GOTO(156,159),NO
6 M=1
7 DO158K=1,NK
8 NAME(K)=KCD(M,K)
  GOTO162
9 M=1
0 DO161K=1,NK
1 NAME(K)=KAC(M,K)
2 L=NK-1
```

```
DO163K=1,L
IF(NAME(K))166,163,166
CONTINUE
IND(M)=-4
T=NAME(NK)
VAL(M)=T*10000.
GOTO167
IND(M)=4*(NK-K-1)
T=NAME(K)
U=NAME(K+1)
VAL(M)=T*10000.+U
IF(M-1)168,168,169
M=2
GOTO(157,160),NO
VALA=ABSF(VAL(1))
VALB=ABSF(VAL(2))
IF(IND(1)-IND(2))172,20,170
IF(VALA-VALB)172,170,170
GOTO(171,174),NO
SCD=VALA
TCD=VALB
INCD=IND(1)
ICD=IND(2)
GOTO176
GOTO(173,175),NO
SCD=VALB
TCD=VALA
INCD=IND(2)
ICD=IND(1)
GOTO176
SAC=VALA
TAC=VALB
INAC=IND(1)
IAC=IND(2)
GOTO176
SAC=VALB
TAC=VALA
INAC=IND(2)
IAC=IND(1)
CONTINUE
L=NN-1
DO177K=1,L
IF(ND(3,K))216,177,179
CONTINUE
IF(ND(3,NN))216,216,178
INDQ=-4
T=ND(3,NN)
DIVD=T*10000.
```

```
GOTO180
INDQ=4*(NN-K-1)
T=ND(3,K)
U=ND(3,K+1)
DIVD=T*10000.+U
RCD=SCD/DIVD
INCD=INCD-INDQ
ICD=ICD-INDQ
L=NN-1
DO181K=1,L
IF(NA(3,K))216,181,183
CONTINUE
IF(NA(3,NN))216,216,182
T=NA(3,NN)
DIVA=T*10000.
INDQ=-4
GOTO184
INDQ=4*(NN-K-1)
T=NA(3,K)
U=NA(3,K+1)
DIVA=T*10000.+U
RACO=SAC/DIVA
INAC=INAC-INDQ
IAC=IAC-INDQ
IND=INAC-INCD
RAC=RACO*10.0**IND
IF(ABSF((RAC-RCD)/(RAC+RCD))-1.E-7)931,931,932
IF(RAC-RCD)185,931,193
PACO=TAC/DIVA
IND=IAC-ICD
PCD=TCD/DIVD
PAC=PACO*10.0**IND
IF(ABSF((PAC-PCD)/(PAC+PCD))-1.E-7)193,193,933
IF(PAC-PCD)185,193,193
GOTO(480,430),KEN
DO431K=1,I
IF(LAD(1,K)-LCD(1,K))432,431,434
CONTINUE
GOTO480
LAB(1,K+1)=MUCH+LCD(1,K+1)-LAD(1,K+1)
NIP=LAD(1,K+1)/MUCH
LAB(1,K+1)=LAB(1,K+1)-MUCH*NIP
LAB(1,K)=LCD(1,K)-LAD(1,K)+NIP-1
DO433M=2,3
LAB(M,2)=MUCH+LCD(M,2)-LAD(M,2)
NIPS(M)=LAB(M,2)/MUCH
LAB(M,2)=LAB(M,2)-MUCH*NIPS(M)
LAB(M,1)=LCD(M,1)-LAD(M,1)+NIPS(M)-1
```

```
MAB(M)=MCD(M)
MOT=2
GOTO202
LAB(1,K+1)=MUCH+LAD(1,K+1)-LCD(1,K+1)
NIP=LAB(1,K+1)/MUCH
LAB(1,K+1)=LAB(1,K+1)-MUCH*NIP
LAB(1,K)=LAD(1,K)-LCD(1,K)+NIP-1
DO435M=2,3
LAB(M,2)=MUCH+LAD(M,2)-LCD(M,2)
NIPS(M)=LAB(M,2)/MUCH
LAB(M,2)=LAB(M,2)-MUCH*NIPS(M)
LAB(M,1)=LAD(M,1)-LCD(M,1)+NIPS(M)-1
MAB(M)=MAD(M)
MOT=2
GOTO202
DO188M=1,3
DO186K=1,NN
NB(M,K)=ND(M,K)
DO187K=1,N
LBC(M,K)=LCD(M,K)
LAB(M,K)=LAD(M,K)
MBC(M)=MCD(M)
MAB(M)=MAD(M)
MOT=1
IF(MARKU)189,189,191
DO190M=1,3
DO190K=1,NK
JBC(M,K)=JCD(M,K)
JAB(M,K)=JAD(M,K)
DO192M=1,2
DO192K=1,NK
KAB(M,K)=KAD(M,K)
KBC(M,K)=-KCD(M,K)
GOTO200
GOTO(490,440),KEN
DO441K=1,I
IF(LAD(1,K)-LAC(1,K))442,441,444
CONTINUE
GOTO490
LAB(1,K+1)=MUCH+LAC(1,K+1)-LAD(1,K+1)
NIP=LAB(1,K+1)/MUCH
LAB(1,K+1)=LAB(1,K+1)-MUCH*NIP
LAB(1,K)=LAC(1,K)-LAD(1,K)+NIP-1
DO443M=2,3
LAB(M,2)=MUCH+LAC(M,2)-LAD(M,2)
NIPS(M)=LAB(M,2)/MUCH
LAB(M,2)=LAB(M,2)-MUCH*NIPS(M)
LAB(M,1)=LAC(M,1)-LAD(M,1)+NIPS(M)-1
```

```
MAB(M)=MAC(M)
MOT=3
GOTO202
LAB(1,K+1)=MUCH+LAD(1,K+1)-LAC(1,K+1)
NIP=LAB(1,K+1)/MUCH
LAB(1,K+1)=LAB(1,K+1)-MUCH*NIP
LAB(1,K)=LAD(1,K)-LAC(1,K)+NIP-1
DO445M=2,3
LAB(M,2)=MUCH+LAD(M,2)-LAC(M,2)
NIPS(M)=LAB(M,2)/MUCH
LAB(M,2)=LAB(M,2)-MUCH*NIPS(M)
LAB(M,1)=LAD(M,1)-LAC(M,1)+NIPS(M)-1
MAB(M)=MAD(M)
MOT=3
GOTO202
DO195M=1,3
DO194K=1,NN
NB(M,K)=NA(M,K)
NA(M,K)=ND(M,K)
MBC(M)=MAC(M)
MAB(M)=MAD(M)
MAC(M)=MCD(M)
DO195K=1,N
LBC(M,K)=LAC(M,K)
LAB(M,K)=LAD(M,K)
LAC(M,K)=LCD(M,K)
MOT=1
IF(MARKU)196,196,198
DO197M=1,3
DO197K=1,NK
JBC(M,K)=JAC(M,K)
JAB(M,K)=JAD(M,K)
JAC(M,K)=JCD(M,K)
DO199M=1,2
DO199K=1,NK
KBC(M,K)=KAC(M,K)
KAB(M,K)=-KAD(M,K)
KAC(M,K)=-KCD(M,K)
DO201K=1,N
IF(LAB(1,K))216,201,202
CONTINUE
KS=((K-1)/6)+1
GOTO(730,731,732,733,734,735,736),KS
J1=4*(K+1)
J2=4*(MAB(2)-2)
J3=4*(MAB(3)-2)
GOTO751
J1=4*(K-5)
```

```
  J2=4*(MAB(2)-5)
  J3=4*(MAB(3)-5)
  GOTO751
  J1=4*(K-11)
  J2=4*(MAB(2)-8)
  J3=4*(MAB(3)-8)
  GOTO751
  J1=4*(K-17)
  J2=4*(MAB(2)-11)
  J3=4*(MAB(3)-11)
  GOTO751
  J1=4*(K-23)
  J2=4*(MAB(2)-14)
  J3=4*(MAB(3)-14)
  GOTO751
  J1=4*(K-29)
  J2=4*(MAB(2)-17)
  J3=4*(MAB(3)-17)
  GOTO751
  J1=4*(K-35)
  J2=4*(MAB(2)-20)
  J3=4*(MAB(3)-20)
  T=LAB(1,K)
  U=LAB(1,K+1)
  R=T*10000.+U
  TR=LAB(2,1)
  UR=LAB(2,2)
  SR=TR*10000.+UR
  TI=LAB(3,1)
  UI=LAB(3,2)
  SI=TI*10000.+UI
  A=R*10.**(-J1)
  B=SR*10.**J2
  C=SI*10.**J3
  FORM=A*(B**2+3.*C**2)
  IF(FORM-1.5)203,203,902
2 GOTO(209,480,490,517,517,517),MOT
3 WRITEOUTPUTTAPE3,204,MARK,MOT
4 FORMAT(6H MARK=I5,5H CASEI2)
  KIND=J1+(KS-1)*24
  WRITEOUTPUTTAPE3,840,R,KIND
0 FORMAT(5H LAB=F11.1,6H EXP=-I3)
  MARKU=MARKU+1
  IF(MARKU-1)205,205,209
5 WRITEOUTPUTTAPE3,206,(JAB(1,K),K=1,NK)
6 FORMAT(3H X=20I5/20I5)
  WRITEOUTPUTTAPE3,207,(JAB(2,K),K=1,NK)
7 FORMAT(3H Y=20I5/20I5)
```

```
WRITEOUTPUTTAPE3,208,(JAB(3,K),K=1,NK)
FORMAT(3H Z=20I5/20I5)
GOTO(850,480,490,517,517,517),MOT
MARK=MARK+1
IF(MARKU-MU)405,405,216
IF(NOTE)120,120,210
KEEP(J)=NOM
WRITEOUTPUTTAPE3,211,KEEP
FORMAT(30I4)
WRITEOUTPUTTAPE3,212,(LAB(1,K),K=1,I)
FORMAT(5H LAB=20I5/20I5)
DO213K=1,I
IF(LAB(1,K))213,213,119
CONTINUE
GOTO101
CALLEXIT
END
```

b. SUBROUTINE

```
      SUBROUTINECROOT(INTK,KNOW,INTL,LEAVE)
      DIMENSIONLOT(2),NUM(322),MORE(322),NEW(322),INCR(322),JUMP(322),
     1KNOW(322),LEAVE(322)
      IS=6
      IT=10
      IN=99
      IH=100
      MULT=330
      KEY=1000
      MUCH=10000
5     NOTE=0
    9 READINPUTTAPE2,1,I,INT,LOT
    1 FORMAT(I3,I2,2I4)
      M=8*I+2
      IF(LOT(1))90,90,3
3     NUM(1)=LOT(1)
      JUMP(1)=0
      INCR(1)=0
      NEW(1)=0
      MORE(1)=1
      DO16N=1,10
      KNOW(1)=N-1
      IF(NUM(1)-MORE(1))22,17,17
17    NUM(1)=NUM(1)-MORE(1)
      INCR(1)=INCR(1)+IS
6     MORE(1)=MORE(1)+INCR(1)
22    MORE(1)=MORE(1)-INCR(1)
      DO 80 J=2,M
      IF(J-2)4,4,5
4     NUM(J)=LOT(J)
      GOTO7
5     NUM(J)=0
7     MORE(J)=0
      NEW(1)=MORE(1)/IT
      DO28 K=2,J
28    NEW(K)=IH*(MORE(K-1)-IT*(MORE(K-1)/IT))+MORE(K)/IT
      JUMP(J)=MULT*KNOW(J-1)+IT*(JUMP(J-1)-IH*(JUMP(J-1)/IH))
      NIP=JUMP(J)/KEY
      JUMP(J)=JUMP(J)-KEY*NIP
      IF(J-2)37,37,33
33    DO36 K=3,J
      L=J+2-K
      JUMP(L)=IT*(JUMP(L-1)-IH*(JUMP(L-1)/IH))+JUMP(L)/IH+NIP
      NIP=JUMP(L)/KEY
36    JUMP(L)=JUMP(L)-KEY*NIP
37    JUMP(1)=JUMP(1)/IH+NIP
      INCR(J)=IT*(INCR(J-1)-IH*(INCR(J-1)/IH))
      IF(J-2)42,42,40
```

```
40   DO41K=3,J
     L=J+2-K
41   INCR(L)=INCR(L)/IH+IT*(INCR(L-1)-IH*(INCR(L-1)/IH))
42   INCR(1)=INCR(1)/IH
     MORE(J)=NEW(J)+JUMP(J)-IN
     NIP=MORE(J)/KEY
     MORE(J)=MORE(J)-KEY*NIP
     DO49K=2,J
     L=J+1-K
     MORE(L)=NEW(L)+JUMP(L)+NIP
     NIP=MORE(L)/KEY
49   MORE(L)=MORE(L)-KEY*NIP
     DO73N=1,10
     KNOW(J)=N-1
50   DO52K=1,J
     IF(NUM(K)-MORE(K))82,52,53
52   CONTINUE
53   NIP=1
     DO57K=2,J
     L=J+2-K
     NUM(L)=NUM(L)+NIP-1+KEY-MORE(L)
     NIP=NUM(L)/KEY
57   NUM(L)=NUM(L)-KEY*NIP
     NUM(1)=NUM(1)-1+NIP-MORE(1)
     INCR(J)=INCR(J)+IS
     NIP=INCR(J)/KEY
     INCR(J)=INCR(J)-KEY*NIP
     DO66K=2,J
     L=J+1-K
     INCR(L)=INCR(L)+NIP
     NIP=INCR(L)/KEY
66   INCR(L)=INCR(L)-KEY*NIP
     MORE(J)=MORE(J)+INCR(J)
     NIP=MORE(J)/KEY
     MORE(J)=MORE(J)-KEY*NIP
     DO73K=2,J
     L=J+1-K
     MORE(L)=MORE(L)+INCR(L)+NIP
     NIP=MORE(L)/KEY
73   MORE(L)=MORE(L)-KEY*NIP
2    NIP=1
     DO79K=2,J
     L=J+2-K
     MORE(L)=KEY+MORE(L)+NIP-1-INCR(L)
     NIP=MORE(L)/KEY
79   MORE(L)=MORE(L)-KEY*NIP
30   MORE(1)=MORE(1)+NIP-1-INCR(1)
     IF(NOTE)83,83,85
```

```
3    DO84K=1,M
4    LEAVE(K)=KNOW(K)
     INTL=INT
     NOTE=1
     GOTO9
 85  WRITEOUTPUTTAPE3,2,LOT
  2  FORMAT(2I4)
     INTK=INT
     WRITEOUTPUTTAPE3,86,INTK,KNOW
 86  FORMAT(5H INT=I1,6H KNOW=102I1/1X110I1/1X110I1)
     WRITEOUTPUTTAPE3,87,INTL,LEAVE
 87  FORMAT(5H INT=I1,7H LEAVE=102I1/1X110I1/1X110I1)
     GOTO92
 90  DO91K=1,M
 91  KNOW(K)=0
 92  RETURN
     END
```

C TABLES

D	η_1	η_2	Higher units	Integer basis	X	Y	Z
2	2.5922E- 1	6.756E- 2	YES	1, 2**1/3, 4**1/3	-1	1	0
3	8.0083E- 2	6.413E- 3	YES	1, 3**1/3, 9**1/3	-2	0	1
5	8.1317E- 3	6.612E- 5	YES	1, 5**1/3, 25**1/3	1	-4	2
6	3.0582E- 3	9.353E- 6	YES	1, 6**1/3, 36**1/3	1	-6	3
7	8.7069E- 2	7.581E- 3	YES	1, 7**1/3, 49**1/3	2	-1	0
10	4.2914E- 2	1.842E- 3	YES	1, 10**1/3, (1+10**1/3+100*1/3)/3	-3	-1	2
11	3.7455E- 3	1.403E- 5	YES	1, 11**1/3, 121**1/3	1	4	-2
12	6.0613E- 3	3.674E- 5	YES	1, 12**1/3, 18**1/3	1	3	-3
13	3.5456E- 3	1.257E- 5	YES	1, 13**1/3, 169**1/3	-4	-3	2
14	1.1499E- 2	1.322E- 4	YES	1, 14**1/3, 196**1/3	1	2	-1
15	6.1717E- 5	3.809E- 9	YES	1, 15**1/3, 225**1/3	1	-30	12
17	1.0289E- 3	1.059E- 6	YES	1, 17**1/3, (1-17**1/3+289**1/3)/3	18	-7	0
19	7.2145E- 2	5.205E- 3	YES	1, 19**1/3, (1+19**1/3+361**1/3)/3	1	1	-1
20	3.0386E- 2	9.233E- 4	YES	1, 20**1/3, 50**1/3	1	1	-1
21	1.9550E- 4	3.822E- 8	YES	1, 21**1/3, 441**1/3	-47	6	4
22	4.2035E- 4	1.767E- 7	YES	1, 22**1/3, 484**1/3	23	3	-4
23	1.5385E-10	2.367E- 20	YES	1, 23**1/3, 529**1/3	-41 399	-3 160	6 230
26	3.7504E- 2	1.407E- 3	YES	1, 26**1/3, (1-26**1/3+676**1/3)/3	3	-1	0
28	1.9128E- 1*	6.999E- 3	YES	1, 98**1/3, (-1+98**1/3-28**1/3)/3	0	0	1
29	3.2404E-18	1.050E- 35	YES	1, 29**1/3, 841**1/3	-322 461 439	103 819 462	370 284
30	4.1101E- 4	1.689E- 7	YES	1, 30**1/3, 900**1/3	1	9	-3

D	η_1	η_2	Higher units	Integer basis	X	Y	Z
31	3.2935E- 6	1.085E- 11	YES	1, 31**1/3, 961**1/3	-367	54	20
33	2.1828E-14	4.765E- 28	YES	1, 33**1/3, 1089**1/3	3 742 201	97 392	-394 098
34	9.9755E- 7	9.951E- 13	YES	1, 34**1/3, 1156**1/3	613	-24	-51
35	3.5961E- 3	1.293E- 5	YES	1, 35**1/3, (1-35**1/3+1225**1/3)/3	-7	3	-1
37	3.3344E- 3	1.112E- 5	YES	1, 37**1/3, (1+37**1/3+1369**1/3)/3	10	-3	0
38	1.1466E- 5	1.315E- 10	YES	1, 38**1/3, 1444**1/3	-151	55	-3
39	6.3010E- 4	3.970E- 7	YES	1, 39**1/3, 1521**1/3	-23	0	2
41	3.5796E-25	1.281E- 49		1, 41**1/3, 1681**1/3	-211 991 370 839	305 478 475 184	-70 761 183 382
42	1.5746E- 5	2.479E- 10	YES	1, 42**1/3, 1764**1/3	1	-42	12
43	6.7961E- 3	4.619E- 5	YES	1, 43**1/3, 1849**1/3	-7	2	0
44	2.4956E- 4	6.228E- 8	YES	1, 44**1/3, (-1+44**1/3+242**1/3)/3	32	5	-17
45	2.2561E- 7	5.090E- 14	YES	1, 45**1/3, 75**1/3	1 081	66	-312
46	2.0265E- 8	4.107E- 16	YES	1, 46**1/3, (1+46**1/3+2116**1/3)/3	-4 448	-261	927
47	5.9212E-13	3.506E- 25		1, 47**1/3, 2209**1/3	-592 199	-69 704	64 786
51	3.0908E- 9	9.553E- 18	YES	1, 51**1/3, 2601**1/3	-11 015	2 592	102
52	1.5949E- 3	2.544E- 6	YES	1, 52**1/3, 338**1/3	1	-4	2
53	2.9494E-12	8.699E- 24	YES	1, 53**1/3, (1-53**1/3+2809**1/3)/3	-367 542	27 719	69 606
55	1.0117E-14	1.024E- 28	YES	1, 55**1/3, (1+55**1/3+3025**1/3)/3	6 787 357	-822 258	-569 988
57	2.2816E- 7	5.206E- 14	YES	1, 57**1/3, 3249**1/3	1 084	57	-88
58	3.5881E- 4	1.287E- 7	YES	1, 58**1/3, 3364**1/3	1	-8	2
59	1.5419E-23	2.37 E- 46		1, 59**1/3, 3481**1/3	46 334 227 393	-42 285 555 004	7 804 684 934

D	η_1	η_2	Higher units	Integer basis	X	Y	Z
60	1.5424E- 4	2.379E- 8	YES	1, 60**1/3, 450**1/3	1	-12	6
61	8.5361E- 5	7.286E- 9	YES	1, 61**1/3, 3721**1/3	1	-16	4
62	3.7332E- 5	1.393E- 9	YES	1, 62**1/3, (1-62**1/3+3844**1/3)/3	-5	-18	18
63	2.0943E- 2	4.386E- 4	YES	1, 63**1/3, 147**1/3	4	-1	0
65	2.0726E- 2	4.296E- 4	YES	1, 65**1/3, 4225**1/3	-4	1	0
66	3.5069E- 5	1.230E- 9	YES	1, 66**1/3, 4356**1/3	1	24	-6
67	7.7718E- 5	6.040E- 9	YES	1, 67**1/3, 4489**1/3	1	16	-4
68	1.3611E- 4	1.853E- 8	YES	1, 68**1/3, 578**1/3	1	12	-6
69	8.2328E-46	6.778E- 91		1, 69**1/3, 4761**1/3	13 753 628 523 894 008 059 401	-5 630 668 308 465 438 120 720 301	555 266 459 615 284 770
70	2.9735E- 4	8.842E- 8	YES	1, 70**1/3, 4900**1/3	1	8	-2
71	1.8639E-19	3.474E- 38	YES	1, 71**1/3, (1-71**1/3+5041**1/3)/3	1 392 287 490	-315 733 795	-18 183 798
73	3.3357E- 6	1.113E- 11	YES	1, 73**1/3, (1+73**1/3+5329**1/3)/3	142	-99	36
74	3.2500E- 7	1.056E- 13	YES	1, 74**1/3, 5476**1/3	-961	-23	60
76	1.0929E- 3	1.194E- 6	YES	1, 76**1/3, 722**1/3	1	4	-2
77	8.8838E-17	7.892E- 33	YES	1, 77**1/3, 5929**1/3	-40 232 807	-7 113 592	3 894 986
78	3.3951E-14	1.153E- 27	YES	1, 78**1/3, 6084**1/3	-2 134 079	841 944	-80 154
79	8.9980E- 7	8.096E- 13	YES	1, 79**1/3, 6241**1/3	292	95	-38
82	1.3081E-13	1.711E- 26	YES	1, 82**1/3, (1+82**1/3+6724**1/3)/3	1 521 461	158 445	-273 730
83	4.4373E-32	1.969E- 63		1, 83**1/3, 6889**1/3	1 146 072 952 401 913	454 318 149 188 752-164	383 888 363 874
84	2.0042E- 6	4.017E- 12	YES	1, 84**1/3, 882**1/3	379	12	-45

	η_1	η_2	Higher units	Integer basis	X	Y	Z
85	5.0590E-19	2.559E- 37		1, 85**1/3, 7225**1/3			
86	5.8997E- 4	3.481E- 7	YES	1, 86**1/3, 7396**1/3	445 707 361	-213 091 974	25 409 628
87	3.8329E-39	1.469E- 77		1, 87**1/3, 7569**1/3	-7	6	-1
89	8.8960E-10	7.914E- 19	YES	1, 89**1/3, (1-89**1/3+7921**1/3)/3	-2 025 487 074 437 153 495	2 295 582 499 903 407 744	-414 906 405 399 771 342
90	5.7155E- 6	3.267E- 11	YES	1, 90**1/3, 300**1/3	21 729	-1 649	-2 617
91	4.1171E- 3	1.695E- 5	YES	1, 91**1/3, (1+91**1/3+8281**1/3)/3	1	-54	36
92	3.1108E- 9	9.677E- 18	YES	1, 92**1/3, 1058**1/3	9	-2	0
93	3.6980E-12	1.367E- 23	YES	1, 93**1/3, 8649**1/3	-8 279	-737	1 139
94	3.4675E-13	1.202E- 25	YES	1, 94**1/3, 8836**1/3	-16 022	-64 428	15 001
95	9.3204E-14	8.629E- 27		1, 95**1/3, 9025**1/3	-1 128 751	107 457	30 965
97	3.2049E-22	1.027E- 43		1, 97**1/3, 9409**1/3	1 867 321	-419 488	2 246
99	1.4062E-22	1.977E- 44		1, 99**1/3, 363**1/3	-12 891 251 368	7 987 833 890	-1 127 854 887
01	8.8012E-22	7.746E- 43		1, 101**1/3, 10201**1/3	-50 708 057 399	10 025 456 082	606 946 152
02	3.8518E-12	1.484E- 23	YES	1, 102**1/3, 10404**1/3	-7 591 749 839	4 748 284 228	-669 551 396
03	1.4615E-19	2.136E- 38	YES	1, 103**1/3, 10609**1/3	-123 929	71 883	-9 708
05	3.9641E-10	1.571E- 19	YES	1, 105**1/3, 11025**1/3	-239 972 695	-293 511 054	73 536 248
06	1.9529E- 9	3.814E- 18	YES	1, 106**1/3, 11236**1/3	30 241	-5 844	-120
07	8.8302E-24	7.797E- 47		1, 107**1/3, (1-107**1/3+11449**1/3)/3	-8 585	3 177	-288
09	3.2524E- 9	1.058E- 17	YES	1, 109**1/3, (1+109**1/3+11881**1/3)/3	220 355 178 972	-16 965 946 451	-22 320 903 094
10	6.0496E- 4	3.660E- 7	YES	1, 110*1/3, 12100**1/3	11 029	-1 806	-252
					1	-5	1

)	η_1	η_2	Higher units	Integer basis	X	Y	Z
11	1.1344E-22	1.287E- 44		1, 111**1/3, 12321**1/3	43 525 986 334	-12 634 237 836	744 387 361
13	6.9199E-20	4.789E- 39		1, 113**1/3, 12769**1/3	2 462 929 921	-147 738 928	-74 815 168
14	8.0647E-11	6.504E- 21	YES	1, 114**1/3, 12996**1/3	61 561	-13 758	219
15	1.0989E-15	1.208E- 30	YES	1, 115**1/3, 13225**1/3	-19 825 999	1 437 980	542 670
16	4.0092E-11	1.607E- 21	YES	1, 116**1/3, (-1+116**1/3+1682**1/3)/3	66 640	13 914	-25 587
17	2.5881E- 6	6.698E- 12	YES	1, 117**1/3, 507**1/3	412	-50	-21
18	1.4312E-15	2.049E- 30	YES	1, 118**1/3, (1+118**1/3+13924**1/3)/3	16 170 822	500 588	-1 864 939
19	1.4357E-19	2.061E- 38		1, 119**1/3, 14161**1/3	1 712 946 929	-244 884 818	-21 015 420
22	2.1856E- 5	4.777E- 10	YES	1, 122**1/3, 14884**1/3	1	-25	5
23	1.0377E-24	1.118E- 72*		1, 123**1/3, 15129**1/3	-589 540 519 295	9 801 566 106	21 865 679 148
24	1.3369E- 2	1.787E- 4	YES	1, 124**1/3, 1922**1/3	5	-1	0
26	1.3298E- 2	1.768E- 4	YES	1, 126**1/3, 588**1/3	-5	1	0
27	1.5333E-12	2.351E- 24	YES	1, 127**1/3, (1+127**1/3+16129**1/3)/3	-431 171	-39 054	60 156
29	6.9415E-11	4.819E- 21	YES	1, 129**1/3, 16641**1/3	25 978	10 401	-3 076
30	5.6970E- 5	3.246E- 9	YES	1, 130**1/3, 16900**1/3	1	15	-3
31	4.8220E-34	2.325E- 67		1, 131**1/3, 17161**1/3	-30 023 631 038 413 999	3 723 639 648 485 850	430 809 483 281 540
32	3.6902E-12	1.362E- 23	YES	1, 132**1/3, 2178**1/3	-224 531	-22 959	26 340
133	7.2568E- 8	5.266E- 15	YES	1, 133**1/3, 17689**1/3	64	-426	81
134	1.5961E-11	2.548E- 22	YES	1, 134**1/3, (1-134**1/3+17956**1/3)/3	-87 315	29 653	-8 758
137	3.4081E-59	1.162E-117		1, 137**1/3, 18769**1/3	110 262 725 140 452 285 942 750 281 801	-5 702 964 008 527 234 123 102 976 810	-3 042 778 664 012 338 593 522 149 640

D	η_1	η_2	Higher units	Integer basis	X	Y	Z
138	4.4295E-17	1.962E- 33		1, 138**1/3, 19044**1/3	25 429 951	13 776 309	-3 618 144
139	3.3940E-15	1.152E- 29	YES	1, 139**1/3, 19321**1/3	9 403 114	182 707	-385 696
140	4.7551E- 4	2.261E- 7	YES	1, 140**1/3, 2450**1/3	1	5	-2
141	3.6653E-12	1.344E- 23	YES	1, 141**1/3, 19881**1/3	253 801	-12 306	15 288
142	6.0384E-10	3.646E- 19	YES	1, 142**1/3, 20164**1/3	14 059	-5 199	480
143	1.3689E-10	1.874E- 20	YES	1, 143**1/3, (1-143**1/3+20449**1/3)/3	57 935	-6 732	-2 950
145	7.2446E-32	5.248E- 63		1, 145**1/3, (1+145**1/3+21025**1/3)/3	-519 308 851 609 577	517 320 331 337 970	-194 821 322 359 266
146	5.0656E-22	2.566E- 43		1, 146**1/3, 21316**1/3	1 392 658 961	-4 998 461 411	899 032 809
148	1.7747E- 7	3.150E- 14	YES	1, 148**1/3, 2738**1/3	1	259	-98
149	3.8773E-31	1.503E- 61		1, 149**1/3, 22201**1/3	968 734 389 957 041	-16 895 323 473 828	-31 280 904 945 436
150	1.2300E- 3	1.513E- 6	YES	1, 150**1/3, 180**1/3	1	3	-3
151	5.6606E-15	3.204E- 29	YES	1, 151**1/3, 22801**1/3	-8 545 391	1 183 490	79 108
153	1.7655E- 4	3.117E- 8		1, 153**1/3, 867**1/3	-50	4	3
154	4.6399E-10	2.153E- 19	YES	1, 154**1/3, (1+154**1/3+23716**1/3)/3	-30 923	3 277	1 142
155	4.1293E-19	1.705E- 37	YES	1, 155**1/3, 24025**1/3	-1 036 060 919	87 752 416	19 569 658
156	1.7591E-10	3.095E- 20	YES	1, 156**1/3, 3042**1/3	-42 119	-501	3 093
157	8.3083E-26	6.903E- 51	YES	1, 157**1/3, 24649**1/3	1 866 152 503 433	46 378 481 778	-72 720 089 536
158	3.7263E-15	1.389E- 29	YES	1, 158**1/3, 24964**1/3	10 010 881	-1 625 176	-41 914
159	4.0149E-28	1.612E- 55	YES	1, 159**1/3, 25281**1/3	-17 294 170 949 567	-2 947 566 357 324	1 133 335 052 934
161	8.4870E-15	7.203E- 29		1, 161**1/3, (1-161**1/3+25921**1/3)/3	-1 158 729	1 070 123	-556 101
163	6.4985E-13	4.223E- 25	YES	1, 163**1/3, (1+163**1/3+26569**1/3)/3	-112 310	-147 747	75 978

D	η_1	η_2	Higher units	Integer basis	X	Y	Z
164	2.0530E- 6	4.215E- 12		1, 164**1/3, 3362**1/3	329		22 -30
165	2.5507E- 6	6.506E- 12		1, 165**1/3, 27225**1/3	1		-66 12
166	1.8858E- 7	3.556E- 14	YES	1, 166**1/3, 27556**1/3	1		-242 44
167	1.6101E-96			1, 167**1/3, 27889**1/3	-414 411 640 332 837 485 844 486 202 518 490 191 124 803 514 079	 411 858 685 575 211 051 461 889 145 880 700 684	88 -2 128 389 448 363 592 189 731 536 246 123 766 369 356 541 842 554 650 894
170	1.8982E-12	3.603E- 24	YES	1, 170**1/3, (1-170**1/3+28900**1/3)/3	466 507	-27	117 -36 288
171	5.4645E- 5	2.986E- 11		1, 171**1/3, 1083**1/3	58		-16 3
172	1.7049E- 5	2.907E- 10	YES	1, 3698**1/3, (-1+3698**1/3-172**1/3)/3	-167		25 -74
173	2.5664E-67	6.587E-134		1, 173**1/3, 29929**1/3	769 611 096 692 874 048 289 533 337 276 313	 967 686 855 369 565 128 259 329	-234 17 942 381 000 165 954 164 518 160 843 978 023 656
174	1.4158E- 9	2.004E- 18	YES	1, 174**1/3, 30276**1/3	15 661	-2	688 -21
175	4.4091E- 6	1.944E- 11		1, 175**1/3, 245**1/3	281		-48 -2
177	2.1786E-86			1, 177**1/3, 31329**1/3	4 515 784 771 330 872 029 175 376 073 843 773 853 415 073	-388 619 389 488 019 539 741 919 054 294 269	648 -74 031 651 038 577 223 302 112 488 966 092 095 838 395 821 054
178	7.9026E-20	6.245E- 39	YES	1, 178**1/3, 31684**1/3	1 558 656 289	-413 710	656 24 288 330
179	2.1287E-40	4.531E- 80		1, 179**1/3, (1-179**1/3+32041**1/3)/3	-8 137 124 021 611 676 250	6 334 337 680 932	619 -3 225 937 824 769 061 566 414
181	5.8939E-20	3.474E- 39	YES	1, 181**1/3, (1+181**1/3+32761**1/3)/3	-2 769 372 839	245 785	405 107 029 295
182	1.1533E- 3	1.330E- 6		1, 182**1/3, 33124**1/3	-17		3 0

D	η_1	η_2	Higher units	Integer basis	X	Y	Z
183	7.8737E-50	6.200E- 99		1, 183**1/3, 33489**1/3	-2 271 061 738 529 269 104 509 999	93 564 180 886 674 837 442 290	53 977 597 990 225 190 121 700
185	2.7777E-12	7.716E- 24	YES	1, 185**1/3, 34225**1/3	-118 399	68 460	-8 368
186	1.5489E-10	2.399E- 20	YES	1, 186**1/3, 34596**1/3	-36 827	-2 676	1 599
187	1.0485E-30	1.099E- 60		1, 187**1/3, 34969**1/3	-647 250 250 118 471	67 257 618 666 096	8 031 526 351 230
188	8.0418E-22	6.467E- 43	YES	1, 188**1/3, (-1+188**1/3+4418**1/3)/3	-20 726 135 762	-528 013 935	3 370 937 481
190	1.3820E-17	1.910E- 34		1, 190**1/3, (1+190**1/3+36100**1/3)/3	35 313 377	-33 767 139	11 971 022
191	2.3794E-45	5.661E- 90		1, 191**1/3, 36481**1/3	-3 717 874 551 082 228 301 439	2 300 546 605 848 762 774 884	-287 369 541 288 731 548 592
193	1.0727E-27	1.151E- 54		1, 193**1/3, 37249**1/3	-19 961 612 630 959	1 130 191 108 734	402 141 569 636
194	4.8923E-26	2.394E- 51		1, 194**1/3, 37636**1/3	2 807 954 442 181	-406 402 766 104	-13 586 300 107
195	3.2233E-21	1.039E- 41		1, 195**1/3, 38025**1/3	-5 072 427 359	2 018 952 168	-197 318 508
197	1.0223E-60	1.045E-120		1, 197**1/3, (1-197**1/3+38809**1/3)/3	-166 638 623 568 137 785 560 077 859 618	96 324 097 399 519 162 396 557 121 529	-40 688 460 813 520 807 488 452 596 644
198	2.8035E- 4	7.860E- 8		1, 198**1/3, 1452**1/3	1	-6	3
199	3.3415E- 5	1.117E- 9	YES	1, 199**1/3, (1+199**1/3+39601**1/3)/3	43	16	-10

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