

Gauge Fixing, the Gribov Problem and BRST Symmetry

M.R. Stanford

*Department of Physics and Mathematical Physics
University of Adelaide*

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Abstract

The Gribov problem has been around for some thirty years and still no general solution has been found. After a detailed examination of gauge fixing in QCD and the Gribov problem in the continuum and in the lattice setting, we introduce BRST symmetry and how gauge fixing is performed within this framework. Then we investigate the problem, seemingly a consequence of the Gribov ambiguity, that expectation values of observables in a BRST invariant lattice theory are not normalisable. This is known as the Neuberger problem. We discuss a simple model, which clearly displays this problem and provide a solution to the Gribov problem in this case.

Declaration of Originality

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

M.R. Stanford

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Chapter 1

Introduction

The standard model has emerged from high energy physics research to be the best theory we have and which appears to successfully include three out of four of the forces of nature: electromagnetism and the strong and weak forces. Although it is almost certainly not a complete theory, it has been very successful in describing the interaction of the the most fundamental particles in nature. The part of the standard model which concerns the strong interaction is quantum chromodynamics(QCD), it tells us how quarks interact via the exchange of gluons. QCD, as with the rest of the standard model, has a fundamental symmetry: gauge symmetry. However, this symmetry is not observable in nature, as out of all such symmetric states, only one should be included when calculating physical observables, although it doesn't matter which one. Interesting issues arise as a consequence of this, which will be the focus of this thesis.

As just mentioned, the very existence of gauge symmetry in QCD means that there are unphysical degrees of freedom and so in order to perform calculations we must constrain the theory to the physical degrees of freedom only, that is we must gauge fix the theory. The standard gauge fixing method, the Fadeev-Popov procedure, does not completely fix the gauge and unphysical degrees of freedom

remain in general. This was first discovered by Gribov in 1978 and is known as the Gribov problem. It still remains an unsolved problem. All is not lost however since it is still possible to carry out perturbative calculations and the results will not be affected. However, if we wish to consider nonperturbative, long range phenomena, such as quark confinement, further consideration of this problem is necessary. It may be that the Gribov problem has no effect on the physical predictions of the theory, but this may not be the case and it could be that it is necessary to find a method which completely fixes the gauge symmetry. One particular framework for performing nonperturbative calculations is by Dyson-Schwinger equations, but here the Gribov problem is not addressed and the calculations are done without removing all the gauge freedom. Therefore this method makes the assumption that it is not necessary to completely fix the gauge in order to get accurate results. Computer simulations, where spacetime consists of a discrete, finite lattice are another popular technique for nonperturbative studies. Here it is possible to find a gauge fixing scheme which is Gribov copy free, the trade off being that of its nonlocality. Therefore, there seems to be two distinct approaches to the problem of gauge fixing, the first leaves it alone and includes the Gribov copies in the calculations and the second eradicates them. This brings up the question of how we define QCD, since the two approaches seemingly lead to distinct theories.

BRST symmetry is an alternative way of tackling the gauge fixing problem. It can be seen as a more general method since the Fadeev-Popov procedure arises as a special case. In fact BRST symmetry can even be used instead of gauge symmetry for the quantisation process, thus called BRST quantisation. An immediate question would be whether we can define this BRST formalism in a lattice model. This was indeed attempted by Neuberger with disastrous results. He found that within a BRST invariant theory in a lattice model, any physical observable took the indeterminate form $\frac{0}{0}$ under very general conditions. This is referred to the

Neuberger problem. Here this problem will be examined in detail and reasons for the occurrence of the $\frac{0}{0}$ will be investigated.

This thesis will begin with the basics of QCD as a non-abelian gauge theory, where we will define quantities and concepts that will be used throughout. We will also discuss the standard gauge fixing method, the Fadeev-Popov procedure, in addition to various choices of gauge. In the third chapter, we discuss the Gribov problem and its implications for the Fadeev-Popov procedure, we will also see that it is irrelevant within perturbation theory. In the fourth chapter, we will more closely examine the Landau gauge condition and then attempt to find a more restrictive condition which does in fact eliminate all the gauge freedom. In chapter five, we turn to the problem of gauge fixing in a lattice model, where we will first discuss the general ideas of lattice gauge theory. In chapter six we will examine BRST symmetry and how gauge fixing is carried out in this framework, as well as some general features of a BRST invariant theory which will be useful later. Chapter seven will begin with the formulation of BRST symmetry in a lattice model and then the Neuberger problem will be derived explicitly. We will also discuss here a simple lattice model, which displays clearly the mechanism of the Neuberger problem, as well as making some general comments about this problem. The final chapter will contain the summary and conclusions.

Part I

Gauge Fixing and the Gribov Problem

Chapter 2

QCD and Gauge Fixing

This chapter introduces the basic ideas of Quantum Chromodynamics (QCD) and gauge fixing. Firstly a brief overview of QCD will be provided to give some idea of how it came to be the theory describing the strong interaction and to also introduce some important concepts. Then non-abelian gauge invariance in the context of QCD will be discussed. It continues with an explanation and motivation as to why gauge fixing is an essential part of QCD, followed by a derivation of the Fadeev-Popov procedure, the standard gauge fixing method in QCD.

2.1 Introduction to QCD

Quantum electrodynamics (QED), the quantum field theory of electromagnetism, has been remarkably successful in describing the interaction of electrons and photons. It is an abelian gauge theory since it has an abelian gauge symmetry with spin $\frac{1}{2}$ matter fields and a force carrying gauge boson: the photon. The most notable successes of QED are its predictions for the magnetic moment of the electron and the Lamb shift in the hydrogen atom, both of which are in excellent agreement with experiment. This prompted the question of whether it would

be possible to formulate a gauge theory of the strong interaction, making use of the ideas which were so successful in QED. Early progress was made by Yang and Mills, who generalised QED to a non-abelian gauge theory, however their theory did not correspond to the physical world because it predicted unphysical particles.

The advent of the quark model radically changed the world of particle physics since it made it possible to understand the vast number of recently discovered elementary particles in terms of a small number of quarks. The proton, previously considered to be a fundamental point-like particle, became a composite object consisting of three quarks. In this model, quarks took over the role of the fundamental point-like particles. They have spin $\frac{1}{2}$, carry a fractional electric charge and also come in one of six flavours: up, down, strange, charm, bottom and top. They interact via the strong interaction by exchange of gluons, which are massless spin one particles. Predictions made by the quark model were found to agree with experimental results reasonably well and in fact elementary particles could now be explained simply in terms of either a three quark bound state or a quark antiquark bound state, referred to as baryons and mesons respectively.

However, there was a problem with the quark model. A particle consisting of three up quarks with identical spins was known to exist. This was seemingly in violation of the Pauli exclusion principle which does not allow two quarks to occupy the same state. The solution was to introduce a new degree of freedom, colour, analogous to electric charge in QED. Thus each quark was thought to carry a colour charge: red, blue or green. This made it possible for three quarks of the same flavour to have identical spins, thus making the conflict between the quark model and the exclusion principle avoidable.

Another problem was that no individual quarks had been observed, they were always found as baryons and mesons and never alone. This property of quarks is called confinement. Although individual quarks are predicted at sufficiently

high energy and density, where normal matter undergoes a phase transition to a so called quark-gluon plasma, concrete experimental evidence for this is still absent. The establishment of a detailed explanation and mechanism for quark confinement remains an outstanding problem. This is largely due to the fact that there are few methods which can be applied to large distance phenomena such as confinement, since the usual perturbative calculations break down in this case.

Quarks also displayed another interesting property. The force between them reduce at small distances, in contrast to gravity or the electromagnetic force, which increase at smaller distances. This property of the strong interaction is referred to as asymptotic freedom. When it was discovered that non-abelian gauge theories also had this property, it was then thought that such theories could indeed be successful in describing the strong interaction. QCD was thus constructed, the gauge symmetry consisting of an arbitrary redefinition of colour at any point in spacetime and with the gauge bosons being the gluons.

2.2 Non-Abelian Local Gauge Invariance

After the development of QED it was realised that the theory could be formulated by simply constructing the most general $U(1)$ gauge invariant renormalisable Lagrangian. This is the approach that we will adopt here to derive the QCD Lagrangian, the important difference being that we will require $SU(3)$ colour gauge invariance rather than $U(1)$ invariance. This leads to multiple gauge bosons, corresponding to gluons with different colour charges, also giving rise to gluon self interactions. Thus QCD is fundamentally different to QED, where the photon does not carry any electric charge and therefore does not interact with other photons. These gluon self interactions are known to be responsible for asymptotic freedom. There are many text books where this material can be found (e.g. [1] [2] [3] [4]) and more details can be found there.

Let us denote the generators of the Lie algebra of $SU(3)$ as t^a where a takes the values $1, 2, \dots, 8$. They satisfy the usual commutation relations for a Lie algebra,

$$[t^a, t^b] = if^{abc}t^c, \quad (2.1)$$

where f^{abc} are the totally antisymmetric structure constants. The normalisation of the generator matrices is given by,

$$\text{Tr}(t^a t^b) = C\delta^{ab}, \quad (2.2)$$

where the constant C depends on the representation of the generators. It is sometimes useful to choose the adjoint representation of $SU(3)$, in this case the generator matrices are given by,

$$(t^b)^{ac} = if^{abc}. \quad (2.3)$$

Keeping in mind that $SU(3)$ in the fundamental representation, is the group of 3 by 3 unitary matrices with unit determinant, therefore locally, an arbitrary gauge transformation can be written as,

$$g = \exp [ig_s \omega^a(x)t^a], \quad (2.4)$$

where ω^a are spacetime dependent gauge transformation parameters and g_s is the QCD coupling constant, inserted here purely for convenience. The above transformation acts on the matter fields, ψ and $\bar{\psi}$, in the following way,

$$\psi^g = g\psi, \quad (2.5)$$

$$\bar{\psi}^g = \bar{\psi}g^\dagger, \quad (2.6)$$

where the Dirac and colour indices have been omitted. In order to define a dynamical theory we need to define the derivative of the matter fields, which we require to satisfy the same transformation property as the fields themselves. Let

us try to define the derivative in the direction μ of the matter fields in the obvious way,

$$\partial_\mu \psi(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\psi(x + \epsilon \hat{\mu}) - \psi(x)), \quad (2.7)$$

where $\hat{\mu}$ is the unit vector in the direction μ . Thus it is clear that we need to consider the fields at two spacetime points. But since we have a local gauge symmetry which specifies a different gauge transformation at each point in spacetime, it means that fields at different spacetime points will transform differently, that is their transformation properties of the above fields are:

$$\psi^g(x) = g(x)\psi(x), \quad (2.8)$$

$$\psi^g(x + \epsilon \hat{\mu}) = g(x + \epsilon \hat{\mu})\psi(x + \epsilon \hat{\mu}), \quad (2.9)$$

where $g(x)$ is in general different to $g(x + \epsilon \hat{\mu})$. Therefore we can define the quantity $U(y, x)$, taking values in the Lie algebra of $SU(3)$, which accounts for the difference in the transformation properties at different spacetime points, so it transforms as

$$(U(y, x)\psi(x))^g = g(y)U(y, x)\psi(x). \quad (2.10)$$

Hence $U(y, x)\psi(x)$ has the same transformation property as $\psi(y)$. This means that $U(y, x)$ itself transforms as

$$U(y, x)^g = g(y)U(y, x)g^\dagger(x). \quad (2.11)$$

In general $U(y, x)$ will be dependent on the path taken from x to y , therefore we should also specify a path associated with $U(y, x)$. However, when defining the derivative we shall consider the limit as the two spacetime points approach each other, thus there will be no need to specify a path. Let us again attempt to define a derivative of ψ , this time using $U(y, x)$ to ensure it has a well defined transformation property. We make the following definition,

$$D_\mu \psi(x) \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\psi(x + \epsilon \hat{\mu}) - U(x + \epsilon \hat{\mu}, x)\psi(x)). \quad (2.12)$$

The transformation property of the above can be easily found from that of $U(y, x)$ and $\psi(x)$, it is:

$$(D_\mu \psi(x))^g = g(x) D_\mu \psi(x). \quad (2.13)$$

Therefore, D_μ transforms covariantly and is thus referred to as the covariant derivative. It can be written in a more convenient form if we expand $U(x + \epsilon \hat{\mu}, x)$ about x to first order in ϵ . Recall that $U(x + \epsilon \hat{\mu}, x)$ takes values in the Lie algebra of $SU(3)$ and therefore can be written as an exponential of a linear combination of the generators of $SU(3)$ in the same way as we wrote an arbitrary gauge transformation, hence

$$\begin{aligned} U(x + \epsilon \hat{\mu}, x) &= \exp[i\epsilon g_s A_\mu^a(x) t^a] \\ &= 1 + \epsilon A_\mu(x) + \mathcal{O}(\epsilon^2), \end{aligned} \quad (2.14)$$

where

$$A_\mu(x) \equiv ig_s A_\mu^a(x) t^a. \quad (2.15)$$

The fields A_μ^a are called gauge fields and they are interpreted as the gluons of QCD. If this is substituted into the definition for D_μ , we find that

$$D_\mu \psi(x) = \partial_\mu \psi(x) - A_\mu(x) \psi(x), \quad (2.16)$$

or equivalently

$$D_\mu = \partial_\mu - A_\mu(x). \quad (2.17)$$

This will be now considered as the definition of the covariant derivative. Expressing D_μ in the adjoint representation (as defined in (2.3)) allows its components to be written simply as

$$D_\mu^{ac} = \delta^{ac} \partial_\mu + g_s f^{abc} A_\mu^b(x). \quad (2.18)$$

The transformation of the A_μ is implicitly given by that of $D_\mu \psi$, let us now find

an explicit expression for this,

$$\begin{aligned}
(D_\mu \psi)^g &= (\partial_\mu - A_\mu^g) \psi^g \\
&= (\partial_\mu - A_\mu^g) g \psi \\
&= g \partial_\mu \psi + (\partial_\mu g) \psi - A_\mu^g g \psi \\
&= g (\partial_\mu + g^\dagger (\partial_\mu g) - g^\dagger A_\mu^g) \psi \\
&= g (\partial_\mu + g^\dagger (\partial_\mu - A_\mu^g) g) \psi.
\end{aligned} \tag{2.19}$$

Note that it should be understood in the above and the following that ∂_μ does not act outside the parentheses. Using the transformation (2.13) we have

$$A_\mu = -g^\dagger ((\partial_\mu - A_\mu^g) g). \tag{2.20}$$

After some rearranging we get

$$A_\mu^g = g A_\mu g^\dagger - g (\partial_\mu g^\dagger) = -g (D_\mu g^\dagger), \tag{2.21}$$

where we have used

$$0 = \partial_\mu (g g^\dagger) = g (\partial_\mu g^\dagger) + (\partial_\mu g) g^\dagger. \tag{2.22}$$

We now consider an infinitesimal gauge transformation, which is given by

$$g = 1 + i g_s \omega^a t^a + \mathcal{O}(\omega^2). \tag{2.23}$$

Using the transformation (2.21) just derived we can find an expression for the infinitesimal gauge transformation of A_μ^a ,

$$A_\mu^\omega = -(1 + i g_s \omega^a t^a) (D_\mu (1 - i g_s \omega^a t^a)). \tag{2.24}$$

Now expanding and writing in terms of $A_\mu^{\omega a}$, we arrive at the result

$$A_\mu^{\omega a} = A_\mu^a + \partial_\mu \omega^a + g_s f^{abc} A_\mu^b \omega^c + \mathcal{O}(\omega^2). \tag{2.25}$$

In order to describe the propagation and interaction of gluons, we will need to use the non-abelian version of the Maxwell field strength tensor, which can be defined as

$$F_{\mu\nu} \equiv [D_\mu, D_\nu] = -ig_s F_{\mu\nu}^a t^a, \quad (2.26)$$

where the expression for the covariant derivative in (2.17) has been inserted to get the above result. Notice that in the above D_μ is a differential operator but $F_{\mu\nu}$ is not. We have also introduced $F_{\mu\nu}^a$, defined as

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c. \quad (2.27)$$

We want to know how $F_{\mu\nu}$ transforms under gauge transformations. From (2.13) and (2.6) we can deduce that

$$D_\mu^g = g D_\mu g^\dagger. \quad (2.28)$$

It is important to note that D_μ on the right hand side acts on everything after the g^\dagger , this should be obvious since D_μ is indeed a differential operator. Now it is straightforward to see that $F_{\mu\nu}$ will have the same transformation property as D_μ , directly from its definition,

$$F_{\mu\nu}^g = g F_{\mu\nu} g^\dagger. \quad (2.29)$$

Let us summarise the transformation properties we have found:

$$\psi^g = g\psi, \quad (2.30)$$

$$\bar{\psi}^g = \bar{\psi}g^\dagger, \quad (2.31)$$

$$A_\mu^g = -g(D_\mu g^\dagger), \quad (2.32)$$

$$D_\mu^g = gD_\mu g^\dagger, \quad (2.33)$$

$$F_{\mu\nu}^g = gF_{\mu\nu} g^\dagger. \quad (2.34)$$

The overall aim was to construct the most general gauge invariant renormalisable Lagrangian. The renormalisable constraint boils down to only allowing terms

of dimension four or less. We shall also assume that the Lagrangian can only depend on the gauge fields via the field strength tensor. If we also assume parity conservation then the gauge field part of the Lagrangian must take the form¹

$$\mathcal{L}_G = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}. \quad (2.35)$$

When looking at the problem of gauge fixing we will not need to consider the matter part of the Lagrangian density and so we merely state the form of Lagrangian[2],

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \sum_f \bar{\psi}_f (i \not{D} - m) \psi_f, \quad (2.36)$$

where the sum is over all quark flavours. The gauge invariance of the first term is easily seen by first expressing it as a trace,

$$F_{\mu\nu}^a F^{a\mu\nu} = -\frac{1}{Cg_s^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}), \quad (2.37)$$

where we have used the definition of $F_{\mu\nu}$ and Eq.(2.2). It is now clear that this term is gauge invariant using the transformation property above and the cyclicity of the trace. The gauge invariance of the other terms immediately follows from Eq.(2.30).

2.3 Gauge Fixing

In order to derive the QCD Lagrangian density, gauge invariance was used as the guiding principle. In this section we shall see that this gauge invariance leads directly to ill-defined expectation values, and it needs to be broken in order to quantise the theory. This breaking of gauge symmetry is referred to as gauge fixing. In the path integral approach, this corresponds to reducing the space of integration to avoid summing over physically identical gauge configurations.

¹For more details see [1] or [2]

Gauge fixing will be discussed in general terms here and a particular method will be described in the next section.

In the following, we will consider the QCD action with gauge fields only, that is

$$S_G = -\frac{1}{4} \int d^4x F_{\mu\nu}^a F_a^{\mu\nu}. \quad (2.38)$$

The Euclidean space expectation value of some observable, which depends only on gauge fields, can be defined in the path integral approach as a functional integral over all fields in the following way,

$$\langle \mathcal{O} \rangle \equiv \frac{\int \mathcal{D}A \mathcal{O}(A) \exp[-S_G[A]]}{\int \mathcal{D}A \exp[-S_G[A]]}, \quad (2.39)$$

where

$$S_G \equiv -\frac{1}{4} \int d^4x F_{\mu\nu}^a F_a^{\mu\nu}. \quad (2.40)$$

The above consists of an integration over all gauge fields, but since the QCD action is gauge invariant, it includes an infinite number of gauge fields corresponding to identical actions. Therefore it is infinite in both the numerator and the denominator. These infinities need to be removed before meaningful values can be obtained and thus the region of integration or gauge field configuration space needs to be examined more carefully.

Two gauge fields are said to be equivalent if they are related by a gauge transformation, otherwise they are inequivalent. The set of all gauge fields which are equivalent to some field A is called the gauge orbit of A , denoted $[A]$. In order to get a better idea of these concepts, it is helpful to represent gauge orbits as closed curves in the space of gauge fields, as shown in Fig. 2.1. It immediately follows that the action S_G is constant around any given gauge orbit. The functional integral over all gauge fields in $\langle \mathcal{O} \rangle$ can then be thought of as an integration over all gauge orbits and an integration of the gauge fields in each

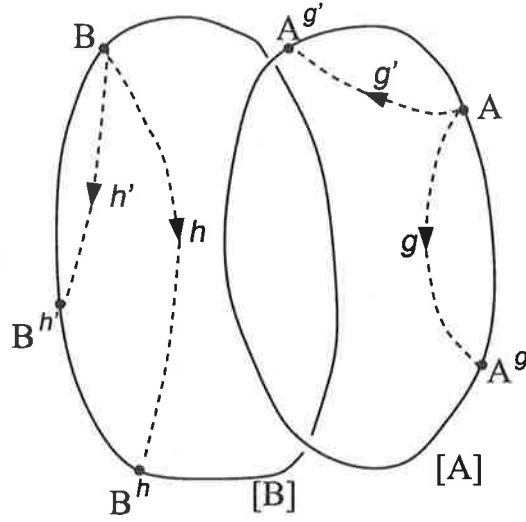


Figure 2.1: The gauge transformations g and h act on the two gauge fields, A and B respectively, producing the corresponding gauge transformed fields, A^g and B^h . When all gauge transformations are considered the gauge orbits, $[A]$ and $[B]$, are constructed. Note that by definition there is no gauge transformation which will transform a gauge field on one orbit into a gauge field on the other.

orbit. This can be written as,

$$\int \mathcal{D}A = \int \mathcal{D}A^{\text{ineq}} \int \mathcal{D}g, \quad (2.41)$$

where A^{ineq} are gauge inequivalent gauge fields and g is an element from $SU(3)$, which specifies the gauge fields within a particular gauge orbit. It is the integration around the gauge orbits that produces the infinity, so our aim is to remove the integration around the gauge orbit. In order to do this a gauge fixing functional, F , is introduced to constrain the integration to the gauge fields satisfying $F = 0$, which is called the gauge fixing condition.

The idea is to integrate over one and only one gauge field per gauge orbit. This can be thought about geometrically in the following way. The gauge fixing condition specifies a surface in gauge field configuration space, which intersects

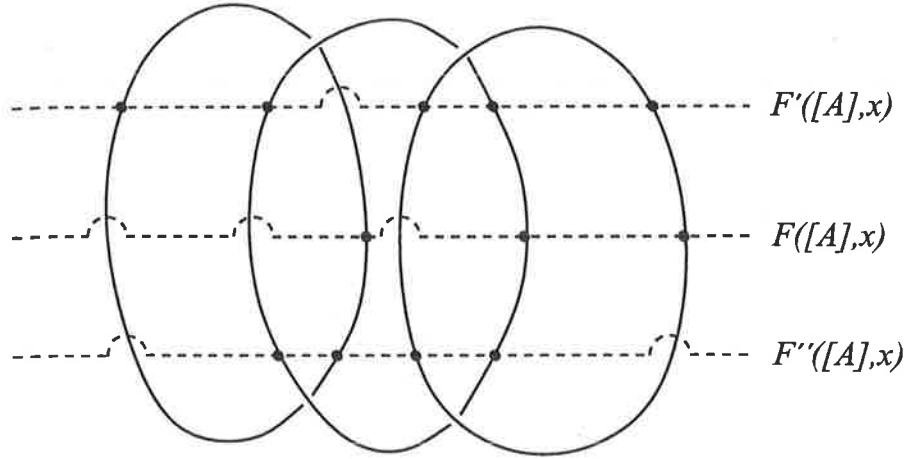


Figure 2.2: Three gauge orbits are shown with three different gauge fixing functions, F , F' and F'' . However, F is the only ideal gauge fixing function since it intersects every gauge orbit once and only once.

each gauge orbit once and only once. This is represented pictorially in Fig. 2.2. If this is the case, then we say that $F = 0$ is an ideal gauge fixing condition. The region in the space of gauge fields that satisfies an ideal gauge fixing condition is called the fundamental modular region. Therefore if we have an ideal gauge fixing condition, we can integrate over the fundamental modular region and the infinities discussed earlier would no longer be present in the numerator and denominator.

We can think about this from a physical point of view. Two equivalent gauge fields are physically indistinguishable because they correspond to identical actions. So the physical gauge field configuration space is the space of gauge inequivalent gauge fields or the space of gauge orbits, which is exactly the fundamental modular region just defined. Therefore, although gauge fixing was motivated as a way to get well defined expectation values of observables, it can be seen now to be a necessary and physically relevant process which is essential to properly define observables in QCD.

2.4 Standard Gauge Fixing Methods

The Fadeev-Popov procedure[5] is the standard gauge fixing method. It consists of inserting a delta function of an ideal gauge fixing function, F , into the expectation value so that the region of integration is restricted to only those gauge fields for which the gauge fixing function vanishes. Since an ideal gauge fixing function has, by definition, one zero per gauge orbit then, this delta function will indeed have the desired effect to reduce the region of integration to one gauge field per gauge orbit. However, when this is done we must compensate for this change by multiplying by the Fadeev-Popov determinant, the functional generalisation of the Jacobian determinant in multivariable calculus.

First the Fadeev-Popov procedure will be described in general and then common choices for the gauge fixing function will be examined.

2.4.1 The Fadeev-Popov Procedure

First recall some basic results from multivariable calculus, which is a good starting point for the Fadeev-Popov procedure since we will consider the functional generalisation of these results. Recall the identity,

$$\int d^n f \delta^{(n)}(\vec{f}) = 1, \quad (2.42)$$

where \vec{f} is a vector of real numbers with n components. Now consider the situation where \vec{f} is specified in terms of n variables \vec{x} , so that $\vec{f} = \vec{f}(\vec{x})$. We can make a change of integration variables from f to x in the following way,

$$\int d^n x \delta^{(n)}(\vec{f}(\vec{x})) \left| \det \left(\frac{\partial f_i}{\partial x_j} \right) \right|_{f=0} = 1, \quad (2.43)$$

where we have inserted the Jacobian determinant for this change of variables. The above is true so long as \vec{f} has one zero within the integration range, or equivalently, if this change of variables corresponds to a change of basis. Now we

want to generalise this to functional integration. So we begin with the following identity

$$\int \mathcal{D}F \delta[F([A], x)] = 1, \quad (2.44)$$

where A is some gauge field and $\mathcal{D}F$ is the functional integration measure. The notation $F([A], x)$ means that F is a functional of A and a function of x . F will be our gauge fixing function implementing the constraint on the functional integral, we will assume that it is an ideal gauge fixing function. Therefore we can write A in the form $(A^{\text{ineq}})^g$, where A^{ineq} is the unique gauge field satisfying $F(A^{\text{ineq}}) = 0$ and g gives the position of A on the gauge orbit. This also means that A^{ineq} can be considered our choice of origin of the gauge orbit and the set of all A^{ineq} specifies a fundamental modular region. Now we make a change of variables from F to g , analogously to what was done above. So we have

$$\int \mathcal{D}g \delta[F([A], x)] \Delta_{\text{FP}}[A] = 1, \quad (2.45)$$

where

$$\Delta_{\text{FP}}[A] \equiv \left| \det \left(M([A], x, y) \right) \right|_{F=0} \quad (2.46)$$

and

$$M([A], x, y) \equiv \frac{\delta F([A], x)}{\delta g(y)}. \quad (2.47)$$

$\Delta_{\text{FP}}[A]$ is the functional version of the Jacobian determinant and is called the Fadeev-Popov determinant. In this case however, $\Delta_{\text{FP}}[A]$ is a matrix in colour space as well as coordinate space because the gauge field, A , is itself a matrix in colour space.

Recall that the space of integration can be divided into an integration over gauge inequivalent gauge fields and an integration around each gauge orbit as explained previously

$$\int \mathcal{D}A = \int \mathcal{D}A^{\text{ineq}} \int \mathcal{D}g. \quad (2.48)$$

Using the above and also (2.45) we find that

$$\begin{aligned}
\int \mathcal{D}A^{\text{ineq}} &= \int \mathcal{D}A^{\text{ineq}} \int \mathcal{D}g \delta[F([(A^{\text{ineq}})^g], x)] \Delta_{\text{FP}}[(A^{\text{ineq}})^g] \\
&= \int \mathcal{D}A \delta[F([(A^{\text{ineq}})^g], x)] \Delta_{\text{FP}}[(A^{\text{ineq}})^g] \\
&= \int \mathcal{D}A \delta[F([A], x)] \Delta_{\text{FP}}[A].
\end{aligned} \tag{2.49}$$

So we see here that the integration over gauge inequivalent gauge fields can be expressed in terms of an integration over all gauge fields via the Fadeev-Popov determinant and a delta function constraint. We have argued in the previous section that we should not integrate over all gauge fields but only over gauge inequivalent gauge fields in order to produce the correct result and avoid over counting. Therefore, in the above notation we redefine $\langle \mathcal{O} \rangle$ to be

$$\langle \mathcal{O} \rangle \equiv \frac{\int \mathcal{D}A^{\text{ineq}} \mathcal{O}(A^{\text{ineq}}) \exp[-S_G[A^{\text{ineq}}]]}{\int \mathcal{D}A^{\text{ineq}} \exp[-S_G[A^{\text{ineq}}]]}. \tag{2.50}$$

Now following the same steps as above we get

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \mathcal{O}(A) \delta[F([A], x)] \Delta_{\text{FP}}[A] \exp[-S_G[A]]}{\int \mathcal{D}A \delta[F([A], x)] \Delta_{\text{FP}}[A] \exp[-S_G[A]]}. \tag{2.51}$$

Although the gauge fixing procedure is seemingly complete, ideally we would like to express the expectation value in the form (2.39) with a different action. This is useful because the Feynman rules can then be easily found and calculations can more readily be carried out. We can do this if we write the Fadeev-Popov determinant as a Grassmann functional integral. Recall that the Fadeev-Popov determinant is defined by

$$\Delta_{\text{FP}}[A] \equiv \left| \det \left(M([A], x, y) \right) \right|, \tag{2.52}$$

where A lies in the fundamental modular region. The following is a general result but we only need to consider it in the case of the matrix $M^{ab}([A], x, y)$,

$$\int \mathcal{D}\bar{c} \mathcal{D}c \exp \left[- \int d^4x d^4y \bar{c}^a(x) M^{ac}([A], x, y) c^c(y) \right] = \det \left(M([A], x, y) \right), \tag{2.53}$$

where c and \bar{c} are Lorentz scalar, anti-commuting fields called Fadeev-Popov ghosts. There is no violation of the spin-statistics theorem here because these fields do not correspond to physical particles and are never found in initial or final states of any interaction. However, before this result can be used the modulus around the Fadeev-Popov determinant must be removed. If we again consider Δ_{FP} to be the generalisation of the Jacobian for a change of variables in multivariable calculus we can see that this is possible because F is an ideal gauge fixing function. The analogous scenario in multivariable calculus to having an ideal gauge fixing function is simply a change of basis vectors. In this case the Jacobian determinant is necessarily non zero. Hence the Fadeev-Popov determinant will also be non zero in the fundamental modular region. Given that any two gauge fields in the fundamental modular region are connected by a smooth path², it follows that Δ_{FP} will not change sign in the fundamental modular region[6]. Therefore the modulus is irrelevant and we can use (2.53). Thus we can write the expectation value in (2.51) as

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \exp[-S_{\text{G}}[A] - S_{\text{M}}[A, c, \bar{c}] \delta[F([A], x)] \mathcal{O}(A)]}{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \exp[-S_{\text{G}}[A] - S_{\text{M}}[A, c, \bar{c}] \delta[F([A], x)]]}, \quad (2.54)$$

where

$$S_{\text{M}}[A, c, \bar{c}] \equiv \int d^4x d^4y \bar{c}^a(x) M^{ac}([A^g], x, y) c^c(y). \quad (2.55)$$

The above now consists of an integral over all gauge fields with the extra ghost terms and a delta function. The presence of the delta function means that only the gauge fields which lie on the gauge fixed surface in gauge field configuration space, defined by $F[A] = 0$, will contribute to the functional integral. Due to the assumption that $F = 0$ is an ideal gauge fixing condition one and only one gauge field per gauge orbit will be integrated over. Thus we have accomplished what we set out to do. Given a choice a gauge fixing function we can evaluate the matrix $M^{ab}([A], x, y)$ and express the delta function in a convenient manner.

²This will be proved in chapter 4

Thus we can write $\langle O \rangle$ as the functional integral over all gauge and ghost fields of $O(A)$, weighted by the exponential of an effective action. We shall examine various choices for F in what follows, where we will derive explicit expressions for this effective action.

2.4.2 Generalised Covariant Gauge

Landau gauge is a common choice of gauge, where we consider only those gauge fields that satisfy $\partial^\mu A_\mu = 0$. So the gauge fixing function is given by

$$F([A], x) = \partial^\mu A_\mu(x). \quad (2.56)$$

Let us define for later convenience

$$f([A], x) \equiv \partial^\mu A_\mu(x). \quad (2.57)$$

We can generalise Landau gauge by introducing an arbitrary function, k , such that

$$F([A], x) = f([A], x) - k(x). \quad (2.58)$$

It is now possible to explicitly calculate the Fadeev-Popov determinant for this generalised Landau gauge. We will do this now, but we first note that since it is evaluated at the gauge field for each orbit which lies in the fundamental modular region, then we will only need to consider a neighbourhood of this gauge field, this means that only infinitesimal gauge transformations need to be considered. Recall from (2.25) that an infinitesimal gauge transformation can be written as,

$$A_\mu^{\omega^a} = A_\mu^a + \partial_\mu \omega^a + g_s f^{abc} A_\mu^b \omega^c + \mathcal{O}(\omega^2). \quad (2.59)$$

Now recall the following functional identity,

$$\frac{\delta}{\delta \omega^b(y)} \partial_\mu^x \omega^a(x) = \delta^{ab} \partial_\mu^x \delta^{(4)}(x - y). \quad (2.60)$$

Now we can calculate the matrix, $M^{ac}([A'], x, y)$ in Landau gauge,

$$\begin{aligned}
M^{ac}([A^\omega], x, y) &\equiv \frac{\delta F^a([A^\omega], x)}{\delta \omega^c(y)} \\
&= \frac{\delta(\partial_x^\mu A_\mu^{\omega a} - k^a(x))}{\delta \omega^c(y)} \\
&= \partial_x^\mu \left(\frac{\delta A_\mu^{\omega a}(x)}{\delta \omega^c(y)} \right) \\
&= \partial_x^\mu \left(\frac{\delta}{\delta \omega^c(y)} \left(A_\mu^a(x) + \partial_\mu^x \omega^a(x) + g_s f^{abc} A_\mu^b(x) \omega^c(x) \right) \right) \\
&= \partial_x^\mu \left(\left(\partial_\mu^x \delta^{ac} + g_s f^{abc} A_\mu^b(x) \right) \delta^{(4)}(x - y) \right). \tag{2.61}
\end{aligned}$$

So the Fadeev-Popov determinant is,

$$\begin{aligned}
\Delta_{\text{FP}}[A] &\equiv \left| \det \left(\partial_x^\mu \left(\left(\partial_\mu^x \delta^{ac} + g_s f^{abc} A_\mu^b(x) \right) \delta^{(4)}(x - y) \right) \right) \right|_{F=0} \\
&= \left| \det \left(\partial^\mu \left(D_\mu^{ac} \delta^{(4)}(x - y) \right) \right) \right|_{F=0}, \tag{2.62}
\end{aligned}$$

where the covariant derivative is in the adjoint representation. So if we assume that this generalised Landau gauge is an ideal gauge fixing condition we can drop the modulus in the Fadeev-Popov determinant and we have a functional integral representation for the Fadeev-Popov determinant in this generalised Landau gauge,

$$\begin{aligned}
\Delta_{\text{FP}}[A] &= \int \mathcal{D}\bar{c}\mathcal{D}c \exp \left[- \int d^4x d^4y \bar{c}^a(x) (\partial^\mu D_\mu^{ac} \delta(x - y)) c^c(y) \right] \\
&= \int \mathcal{D}\bar{c}\mathcal{D}c \exp \left[- \int d^4x \bar{c}^a(x) \partial^\mu D_\mu^{ac} c^c(x) \right]. \tag{2.63}
\end{aligned}$$

We can perform a Gaussian weighted average over these generalised Landau gauges, that is multiply by

$$\exp \left[- \frac{1}{2\xi} \int d^4x k^2(x) \right], \tag{2.64}$$

where ξ is a constant, then integrate over k . The width of the Gaussian is specified by ξ , which we will refer to as the gauge fixing parameter for reasons that will

become clear. After this Gaussian weighted average over these generalised Landau gauges is done the expectation value (2.51) is given by

$$\begin{aligned} \langle \mathcal{O} \rangle &= \frac{\int \mathcal{D}A \mathcal{D}k \exp \left[-S_G[A] - \frac{1}{2\xi} \int d^4x k^2 \right] \delta[\partial^\mu A_\mu(x) - k(x)] \Delta_{\text{FP}} \mathcal{O}(A)}{\int \mathcal{D}A \mathcal{D}k \exp \left[-S_G[A] - \frac{1}{2\xi} \int d^4x k^2 \right] \delta[\partial^\mu A_\mu(x) - k(x)] \Delta_{\text{FP}}} \\ &= \frac{\int \mathcal{D}A \exp \left[-S_G[A] - \frac{1}{2\xi} \int d^4x (\partial^\mu A_\mu^a)^2 \right] \Delta_{\text{FP}} \mathcal{O}(A)}{\int \mathcal{D}A \exp \left[-S_G[A] - \frac{1}{2\xi} \int d^4x (\partial^\mu A_\mu^a)^2 \right] \Delta_{\text{FP}}}, \end{aligned} \quad (2.65)$$

where the delta function has eliminated the integration over the scalar function k . When the expression for the Fadeev-Popov determinant is also inserted, we get

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \exp [-S[A, c, \bar{c}]] \mathcal{O}(A)}{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \exp [-S[A, c, \bar{c}]]}, \quad (2.66)$$

where

$$S[A, c, \bar{c}] = S_G[A] + \int d^4x \frac{1}{2\xi} (\partial^\mu A_\mu^a)^2 + \bar{c}^a(x) \partial^\mu D_\mu^{ac} c^c(x). \quad (2.67)$$

The Feynman rules for QCD can be derived from this action. Therefore the definition of perturbative QCD in arbitrary covariant gauge arises from the Fadeev-Popov gauge fixing method based on a Gaussian averaged generalisation of Landau gauge. In the limit $\xi \rightarrow 0$, the Gaussian width vanishes and we recover Landau gauge, $\partial^\mu A_\mu = 0$. Also $\xi = 1$ is called Feynman gauge.

2.4.3 Axial Gauge

Axial gauge is a common choice in canonical quantisation but it can also be used in the path integral approach. Here we will chose to write it in the general form,

$$n^\mu A_\mu = 0, \quad (2.68)$$

where n is some unit four vector, which we are free to choose. If we take $n = (1, 0, 0, 0)$ this corresponds to temporal gauge,

$$A_0 = 0. \quad (2.69)$$

So axial gauge refers to a whole class of gauges which can be written in the form (2.68). The gauge fixing function for axial gauge is given by,

$$F([A], x) = n^\mu A_\mu(x). \quad (2.70)$$

This gauge fixing scheme is not Lorentz covariant because n specifies a preferred direction in spacetime. This means that any calculation done in axial gauge will be reference frame dependent for gauge dependent quantities, which may not be the desired outcome. All the steps leading to the arbitrary covariant gauge can be followed again here by defining

$$\begin{aligned} F([A], x) &= F([A], x) - k(x) \\ &= n^\mu A_\mu(x) - k(x). \end{aligned} \quad (2.71)$$

We now calculate the matrix $M^{ac}([A^\omega], x, y)$ for this generalised axial gauge and find

$$\begin{aligned} M^{ac}([A^\omega], x, y) &= n^\mu \frac{\delta A_\mu^{\omega a}(x)}{\delta \omega^c(y)} \\ &= n^\mu \left((\partial_\mu \delta^{ac} + g_s f^{abc} A_\mu^b(x)) \delta^{(4)}(x - y) \right) \\ &= n^\mu \delta^{ac} \partial_\mu \delta^{(4)}(x - y). \end{aligned} \quad (2.72)$$

Hence the Fadeev-Popov determinant for axial gauge is

$$\Delta_{\text{FP}}[A] = \left| \det \left(n^\mu \delta^{ac} \partial_\mu \delta^{(4)}(x - y) \right) \right|_{F=0} \quad (2.73)$$

So we find that the expectation value in axial gauge is given by

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \exp[-S[A, c, \bar{c}]] \mathcal{O}(A)}{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \exp[-S[A, c, \bar{c}]]}. \quad (2.74)$$

where

$$S[A, c, \bar{c}] = S_G[A] + \int d^4x \frac{1}{2\xi} (n^\mu A_\mu)^2 + \bar{c}^a(x) n^\mu \partial_\mu c^a(x). \quad (2.75)$$

Notice that Δ_{FP} is independent of A and thus it can be factored out of an expectation value. This means that there are no need for ghosts in axial gauge or

equivalently that the ghosts decouple from the gauge fields. Just as exact Landau gauge, $\partial^\mu A_\mu(x) = 0$ is recovered as the $\xi \rightarrow 0$ limit of generalised covariant gauge, we see that exact axial gauge, $n^\mu A_\mu(x) = 0$, is the $\xi \rightarrow 0$ limit of the Gaussian weighted averaged axial gauge. It should be noted that axial gauge has the problem that singularities arise in the propagators and it is not clear how to deal with these beyond one loop in perturbation theory.

Chapter 3

The Gribov Problem

Gauge fixing has been described as an essential part of QCD, where the physical configuration space was identified as the space of gauge orbits, a subset of the space of all gauge fields. We discussed the Fadeev-Popov gauge fixing scheme as one method of gauge fixing. However, the key assumption of this method was that the local gauge fixing condition considered was an ideal gauge fixing condition. In this chapter we shall examine this assumption further and in fact show that for the case of Coulomb gauge this is not the case. In addition, under quite general circumstances an ideal local gauge fixing condition has been shown not to exist. The multiple zeros of a non ideal gauge fixing condition in a particular gauge orbit are referred to as Gribov copies. Gribov copies will be examined in the lattice formulation of QCD and will also be seen to be irrelevant in perturbative calculations, therefore establishing the Gribov ambiguity as a nonperturbative problem.

3.1 Gribov Copies in Coulomb Gauge

Gribov[7] showed that Gribov copies exist in Coulomb gauge for $SU(2)$. We shall discuss this now. We want to find the solutions of the Coulomb gauge fixing

condition for the gauge transformed A . The assumption in the Fadeev-Popov procedure was that there is only one solution per gauge orbit. We will now demonstrate that this is not the case for $SU(2)$ in Coulomb gauge. Consider the following general element from a spherically symmetric subset of $SU(2)$,

$$\tilde{g} = \cos\left(\frac{\alpha(r)}{2}\right) + i\sigma \cdot \mathbf{n} \sin\left(\frac{\alpha(r)}{2}\right). \quad (3.1)$$

This gauge transformation acts on the gauge field, A , in the following way

$$A_\mu \rightarrow A_\mu^g = -\tilde{g}D_\mu\tilde{g}^\dagger. \quad (3.2)$$

Consider the orbit equivalent to the trivial gauge field, $A = 0$

$$A_\mu^g = -\tilde{g}\partial_\mu\tilde{g}^\dagger. \quad (3.3)$$

Coulomb gauge is

$$\partial_i A_i^g = 0. \quad (3.4)$$

Now find solutions to the Coulomb gauge fixing condition, they are solutions to the equation

$$\partial_i (\tilde{g}\partial_i\tilde{g}^\dagger) = 0. \quad (3.5)$$

It can be shown that the above equation is equivalent to the second order differential equation

$$\alpha''(t) + \alpha'(t) - 2\sin(\alpha(t)) = 0, \quad (3.6)$$

where $t = \log r$. This is the classical equation for a damped pendulum in a gravitational field. One should envisage a pendulum initially in a position of unstable equilibrium therefore it would most likely not remain in this position. Depending on the initial conditions it will fall either clockwise or anticlockwise until it comes to rest in the stable equilibrium. The important fact is that there are many ways it can do this, for example it could simply go directly to its final position with minimal oscillations, or it could make any number of complete

revolutions before it does this. Therefore there are in principle an infinite number of solutions to the above differential equation, which also means that there are an infinite number of gauge fields that satisfy the coulomb gauge condition and are related by a gauge transformation. This implies for our problem that there are an infinite number of Gribov copies. Note that we have restricted ourselves to considering only spherically symmetric $SU(2)$ gauge transformations and also only considered one gauge orbit, so there could be many more Gribov copies in this case. This scenario is expected to be unchanged for $SU(3)$ gauge transformations in Landau gauge.

3.2 Gribov Copies in Axial Gauge

We have seen that Coulomb gauge is not a complete gauge fixing condition, this is also the case for generalised axial gauges. Recall that the generalised axial gauge condition is given by

$$n^\mu A_\mu = 0. \quad (3.7)$$

Gribov copies will be present when there are multiple solutions to the following equation

$$n^\mu A_\mu^g = 0, \quad (3.8)$$

$$n^\mu g \partial_\mu g^\dagger = 0, \quad (3.9)$$

where we are again considering only the trivial orbit. Let us first consider temporal gauge, that is

$$A_0 = 0. \quad (3.10)$$

In this case (3.9) becomes

$$g \partial_0 g^\dagger = 0. \quad (3.11)$$

It is obvious that any g independent of x_0 will satisfy this equation and hence there are infinitely many solutions to the temporal gauge condition. If we now

consider the generalised axial gauge and look for solutions to (3.9) restricted to infinitesimal gauge transformations, then we have using (2.25),

$$n^\mu A^{\omega^a} = 0, \quad (3.12)$$

$$n^\mu \partial_\mu \omega^a = 0. \quad (3.13)$$

It is clear that the above is satisfied whenever n is orthogonal to $\partial_\mu \omega^a$. Therefore generalised axial gauge also has an infinite number of Gribov copies.

3.3 The Fadeev-Popov Procedure Revisited

We have just seen that the choices that we made for the gauge fixing function in the previous chapter in order to explicitly carry out the Fadeev-Popov procedure do not satisfy the required property that they have one and only one zero for every gauge orbit. Indeed no such function is known. It has been proven that this is true under quite general conditions[8].

Therefore it seems that the assumption made in the Fadeev-Popov procedure is not able to be made, so here we will examine how the Fadeev-Popov procedure would change if we do not make this assumption.

In the case when the gauge fixing function has multiple zeroes we have in general

$$\int \mathcal{D}g \delta[F(A^g)] \Delta_{\text{FP}}[A] \neq 1, \quad (3.14)$$

where

$$\Delta_{\text{FP}}[A] = \left| \det \left[\frac{\delta F(A^g)}{\delta g} \right] \right|_{F=0}. \quad (3.15)$$

We shall proceed by redefining the Fadeev-Popov determinant as a sum over the zeros of F , that is

$$\tilde{\Delta}_{\text{FP}}[A] = \sum_n \left| \det \left[\frac{\delta F(A^g)}{\delta g} \right] \right|_{g=g_n}, \quad (3.16)$$

where n labels the zeros of F . However if we were to use this definition in the Fadeev-Popov procedure, it would not be possible to introduce the ghost fields to represent the Fadeev-Popov determinant, making it difficult to derive the Feynman rules for this theory. As an alternative we make the definition

$$n(A) \equiv \int \mathcal{D}g \delta[F(A^g)] \det \left[\frac{\delta F(A^g)}{\delta g} \right]. \quad (3.17)$$

Here we have dropped the modulus around the Fadeev-Popov determinant and so it can now take positive and negative values. In the case where F has only one zero per gauge orbit, the sign of the Fadeev-Popov determinant is irrelevant and $n(A)$ will be unity, and so the equality in (3.14) will be restored. Thus we see that the above reduces to the identity used in the Fadeev-Popov procedure in the absence of Gribov copies.

The Fadeev-Popov determinant will contribute the same magnitude for each Gribov copy but possibly with a different sign, so $n(A)$ consists of a sum of $+1$'s and -1 's, representing the contributions from the Gribov copies. The plus and minus signs can be interpreted as the way in which the gauge fixing surface intersects the fundamental modular region.

Evidence has been put forward[9][10] that Gribov copies have no effect on expectation values of observables. Of course this is the preferred outcome because then the whole issue of the Gribov problem could be ignored, however these arguments are not conclusive and it is yet to be seen that the Gribov problem has no effect. There is also the prospect that there might be as many positive as negative contributions from the Gribov copies, meaning that $n(A)$ vanishes. This would be a disastrous scenario, leaving expectation values ill-defined. We shall return to this possibility later.

3.4 The Gribov Problem in Perturbative QCD

Perturbation theory is very useful in calculations in QED, since the coupling constant in QED is small. Such techniques have not been found to be as useful in QCD since here the coupling is large. However, due to the property of asymptotic freedom, the coupling is small at high momenta and perturbative calculations can be used.

Let us remind ourselves about the relevant objects and try to show that Gribov copies do not contribute to perturbative calculations. The action is given by

$$\begin{aligned} S_G[A] &= -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} \\ &= \frac{1}{4Cg_s^2} \int d^4x \text{Tr}(F_{\mu\nu} F^{\mu\nu}), \end{aligned} \quad (3.18)$$

which is written in terms of the gauge invariant quantity

$$\begin{aligned} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) &\equiv \text{Tr}([D_\mu, D_\nu][D^\mu, D^\nu]) \\ &= \text{Tr}((\partial_\nu A_\mu - \partial_\mu A_\nu - [A_\mu, A_\nu])(\partial^\nu A^\mu - \partial^\mu A^\nu - [A^\mu, A^\nu])) \end{aligned} \quad (3.19)$$

where

$$A_\mu \equiv ig_s A_\mu^a t^a. \quad (3.20)$$

Also recall

$$\langle \mathcal{O} \rangle \equiv \frac{\int \mathcal{D}A \exp[-S_G[A]] \mathcal{O}(A)}{\int \mathcal{D}A \exp[-S_G[A]]}. \quad (3.21)$$

For a perturbative calculation we consider the limit as g_s goes to zero, thus $\langle \mathcal{O} \rangle$ will be dominated by gauge field configurations which give small values of $\text{Tr}(F_{\mu\nu} F^{\mu\nu})$. This quantity is positive definite since it is a square and it has the value 0 only on the trivial gauge orbit, ie. the orbit containing $A = 0$. So in the limit $g_s \rightarrow 0$, $\langle \mathcal{O} \rangle$ is dominated by gauge fields close to the trivial orbit. We want to show that the Landau gauge condition has only one zero near $A = 0$ along its gauge orbit. This is easily seen if we make use of the fact that the stationary

points of the functional

$$\|A\|^2 \equiv - \int d^4x \operatorname{Tr}(A_\mu^2) \quad (3.22)$$

are the gauge fields satisfying Landau gauge (this will be shown explicitly in the next chapter). $A = 0$ is clearly a nondegenerate absolute minimum of (3.22) and so near $A = 0$ it must be increasing and therefore $A = 0$ is the only solution to the Landau gauge condition near $A = 0$. Hence in a perturbative calculation there are no Gribov copies in Landau gauge.

3.5 The Non-Perturbative Regime

We know now that we do not need to worry about the Gribov problem in perturbative QCD, but what about QCD in the nonperturbative regime? Is the use of the QCD action derived from the Fadeev-Popov procedure justified in this case? It seems there are some quite distinct arguments of how one should handle this situation. One could argue that the gauge equivalent gauge fields are unphysical and one must integrate over the physical configuration space, that is, a fundamental modular region. This necessarily excludes all Gribov copies, but exactly how one would try to implement this is a non trivial task. The other point of view one could take is if one says QCD should be defined non-perturbatively with a non-ideal gauge fixing condition, such as Landau gauge. This means that either the effects of Gribov copies are ignored or that calculations are assumed to not be affected by them. This is the point of view generally accepted when considering non perturbative issues. In fact the renormalisability of QCD is based on the action derived from the Fadeev-Popov procedure in Landau gauge. In addition, it is the basis for the Dyson-Schwinger formalism, a common nonperturbative calculational tool. Another possibility, which has found to be quite useful in lattice QCD simulations, is a non local gauge fixing condition which ensures that Gribov copies do not affect the calculations.

Chapter 4

The Fundamental Modular Region of Landau Gauge

It has been shown in the previous chapter that Landau gauge is not an ideal gauge fixing condition. This immediately invalidates the Fadeev–Popov procedure since it was a key assumption in the derivation. In the current chapter we shall attempt to identify the fundamental modular region that lies within the gauge fixing surface defined by Landau gauge. In order to do this we will find the local minima of some functional and show that this defines a subset of the gauge fields in Landau gauge. However it will be found that it does not constitute a fundamental modular region and therefore we shall consider absolute minima. We shall see that for this region, Gribov copies are only present on the boundary. We proceed in much the same way as others have previously[11][12][13].

4.1 Minimising the Norm Functional

Some time ago it was realised[13][14] that gauge fields corresponding to local minima over each gauge orbit of the norm functional also satisfied the Landau

gauge condition. We define the norm functional by

$$\|A\|^2 \equiv - \int d^4x \operatorname{Tr}(A_\mu^2). \quad (4.1)$$

Considering one gauge orbit at a time, we wish to find those gauge fields out of all the gauge fields in each gauge orbit which are local minima of this functional. Therefore we consider the norm of the gauge transformed field, A^g , and find its local minima as we vary the gauge parameter of the gauge transformation g . So the object we are interested in is,

$$\|A^g\|^2 = - \int d^4x \operatorname{Tr} \left((gA_\mu g^\dagger - g\partial_\mu g^\dagger)^2 \right), \quad (4.2)$$

where g is a gauge transformation, which can be written locally in the same way as it was previously,

$$g = \exp[ig_s \omega^a(x)t^a]. \quad (4.3)$$

We shall now explicitly show that the local minima of the above does indeed satisfy the Landau gauge condition, that is they satisfy

$$\partial^\mu A_\mu = 0. \quad (4.4)$$

Expanding (4.2) locally about the gauge field A , we get

$$\begin{aligned} \|A^g\|^2 &= - \int d^4x \operatorname{Tr} (A_\mu^2 + (g\partial^\mu g^\dagger)^2 - 2(\partial^\mu g^\dagger)gA_\mu) \\ &= \|A\|^2 - \int d^4x \operatorname{Tr} ((g\partial^\mu g^\dagger)^2 + 2g^\dagger(\partial^\mu g)A_\mu). \end{aligned} \quad (4.5)$$

We need to calculate expressions for the two integrands above. By writing the gauge transformations, g , as Taylor series and keeping only second order terms,

we have

$$\begin{aligned}
g\partial_\mu g^\dagger &= \left(1 + ig_s\omega^a t^a + \frac{1}{2}(ig_s\omega^a t^a)^2\right) \partial_\mu \left(1 - ig_s\omega^a t^a + \frac{1}{2}(-ig_s\omega^a t^a)^2\right) \\
&= (1 + ig_s\omega^a t^a) \left(-ig_s\partial_\mu\omega^a t^a - \frac{1}{2}g_s^2 t^a t^b ((\partial_\mu\omega^a)\omega^b + \omega^a\partial_\mu(\omega^b))\right) \\
&= (1 + ig_s\omega^a t^a) \left(-ig_s\partial_\mu\omega^a t^a - \frac{1}{2}g_s^2\omega^a(\partial_\mu\omega^b)(t^a t^b + t^b t^a)\right) \\
&= -ig_s\partial_\mu\omega^a t^a - \frac{1}{2}g_s^2\omega^a(\partial_\mu\omega^b)(t^a t^b + t^b t^a) + g_s^2\omega^a(\partial_\mu\omega^b)t^a t^b \\
&= -ig_s\partial_\mu\omega^a t^a + \frac{1}{2}g_s^2\omega^a(\partial_\mu\omega^b)(t^a t^b - t^b t^a) \\
&= -ig_s(\partial_\mu\omega^a)t^a + \frac{1}{2}g_s^2 if^{abc}\omega^a(\partial_\mu\omega^b)t^c. \tag{4.6}
\end{aligned}$$

In the same way it can be shown that

$$g^\dagger\partial_\mu g = ig_s(\partial_\mu\omega^a)t^a + \frac{1}{2}g_s^2 if^{abc}\omega^a(\partial_\mu\omega^b)t^c. \tag{4.7}$$

We proceed, making use of these expressions, and so (4.5) becomes

$$\begin{aligned}
\|A^\omega\|^2 = \|A\|^2 &- \int d^4x \operatorname{Tr} \left(\left(-ig_s(\partial_\mu\omega^a)t^a + \frac{1}{2}g_s^2 if^{abc}\omega^a(\partial_\mu\omega^b)t^c \right)^2 \right. \\
&\left. + \left(2ig_s(\partial^\mu\omega^a)t^a + ig_s^2 f^{abc}\omega^a(\partial^\mu\omega^b)t^c \right) A_\mu \right). \tag{4.8}
\end{aligned}$$

The above is easily simplified by discarding terms cubic in ω , expressing A_μ in its components A_μ^a and evaluating the trace using (2.2), thus expanding the above we get

$$\begin{aligned}
\|A^\omega\|^2 = \|A\|^2 &- \int d^4x \operatorname{Tr}(-g_s^2(\partial_\mu\omega^a)t^a(\partial_\mu\omega^b)t^b) \\
&- \int d^4x 2g_s^2 \operatorname{Tr}((\partial^\mu\omega^a)A_\mu^b t^a t^b - g_s^3 f^{abc}\omega^a(\partial^\mu\omega^b)t^c A_\mu^d t^d) \\
= \|A\|^2 &- \int d^4x -Cg_s^2(\partial_\mu\omega^a)(\partial_\mu\omega^b)\delta^{ab} \\
&- \int d^4x 2Cg_s^2(\partial^\mu\omega^a)A_\mu^b\delta^{ab} - Cg_s^3 f^{abc}\omega^a(\partial^\mu\omega^b)A_\mu^d\delta^{cd} \tag{4.9} \\
= \|A\|^2 &- Cg_s^2 \int d^4x 2\omega^a(\partial^\mu A_\mu^a) + \left(\omega^a(\partial^2\omega^a) + g_s f^{abc}\omega^a A_\mu^b(\partial^\mu\omega^c)\right).
\end{aligned}$$

Therefore, the entire result reads

$$\|A^\omega\|^2 = \|A\|^2 - Cg_s^2 \int d^4x \, 2\omega^a (\partial^\mu A_\mu^a) + \left(\omega^a (\partial^2 \omega^a) + g_s f^{abc} \omega^a A_\mu^b (\partial^\mu \omega^c) \right). \quad (4.10)$$

We now have the expression for $\|A^\omega\|^2$ in a neighbourhood of A , containing only gauge fields in the gauge orbit of A . It is necessary to express this result in a slightly different form, introducing an integral over y and a delta function, so that functional derivatives can be extracted directly. Proceeding in this way we get

$$\begin{aligned} \|A^\omega\|^2 &= \|A\|^2 - 2Cg_s^2 \int d^4x \, \omega^a(x) \partial^\mu A_\mu^a(x) \\ &\quad - Cg_s^2 \int d^4x d^4y \, \delta^{(4)}(x-y) \left(\omega^a(y) \partial_x^2 \omega^a(x) + g_s f^{abc} \omega^a(x) A_\mu^b(x) \partial_y^\mu \omega^c(y) \right) \\ &= \|A\|^2 - 2Cg_s^2 \int d^4x \, \omega^a(x) \partial^\mu A_\mu^a(x) \\ &\quad - Cg_s^2 \int d^4x d^4y \, \omega^a(x) \omega^c(y) \left(\delta^{ac} \partial_x^2 \delta^{(4)}(x-y) - g_s f^{abc} A_\mu^b(x) \partial_y^\mu \delta^{(4)}(x-y) \right) \\ &= \|A\|^2 - 2Cg_s^2 \int d^4x \, \omega^a(x) \partial^\mu A_\mu^a(x) \\ &\quad - Cg_s^2 \int d^4x d^4y \, \omega^a(x) \omega^c(y) \left(\delta^{ac} \partial_x^2 \delta^{(4)}(x-y) + g_s f^{abc} A_\mu^b(x) \partial_x^\mu \delta^{(4)}(x-y) \right). \end{aligned} \quad (4.11)$$

Now that $\|A^\omega\|^2$ has been expressed in the above form it is not even necessary to carry out the functional differentiation, since we may merely read off the functional derivatives that we need if we remember the functional version of the Taylor series expansion of the functional $\|A^\omega\|^2$ about A or equivalently about $\omega = 0$. Recall

$$\begin{aligned} \|A^\omega\|^2 = \|A\|^2 &+ \int d^4x \, \omega^a(x) \left. \frac{\delta \|A^\omega\|^2}{\delta \omega^a(x)} \right|_{\omega=0} \\ &+ \frac{1}{2} \int d^4x d^4y \, \omega^a(x) \omega^c(y) \left. \frac{\delta^2 \|A^\omega\|^2}{\delta \omega^a(x) \delta \omega^c(y)} \right|_{\omega=0} + \dots \end{aligned} \quad (4.12)$$

Hence we find the first functional derivative to be

$$\left. \frac{\delta \|A^\omega\|^2}{\delta \omega^a(x)} \right|_{\omega=0} = -2Cg_s^2 \partial^\mu A_\mu^a(x), \quad (4.13)$$

and the second functional derivative is

$$\left. \frac{\delta^2 \|A^\omega\|^2}{\delta\omega^a(x)\delta\omega^c(y)} \right|_{\omega=0} = -2Cg_s^2 \left(\delta^{ac} \partial_x^2 + g_s f^{abc} A_\mu^b(x) \partial_x^\mu \right) \delta^{(4)}(x-y). \quad (4.14)$$

We know that the gauge field, A , is a local minimum of the norm functional if the first functional derivative vanishes at A_0 and if the second derivative is positive at A . It is possible that A could also be a local minimum when the second derivative vanishes but this is not generally the case. Therefore, for the gauge field A to be a local minimum it must satisfy

$$\partial^\mu A_\mu^a = 0, \quad (4.15)$$

$$-\partial_x^\mu \left(\left(\delta^{ac} \partial_\mu^x + g_s f^{abc} A_\mu^b(x) \right) \delta^{(4)}(x-y) \right) \geq 0. \quad (4.16)$$

Notice that this is more simply written when we choose the adjoint representation for the covariant derivative and so

$$\partial^\mu A_\mu^a = 0, \quad (4.17)$$

$$-\partial_x^\mu \left(D_\mu^{ac} \delta^{(4)}(x-y) \right) \geq 0. \quad (4.18)$$

The object on the left hand side of (4.18) is called the Fadeev-Popov operator. The Gribov region for Landau gauge, Ω , is defined as the gauge fields for which the above conditions hold. On the boundary of Ω , called the Gribov horizon (denoted $\partial\Omega$), the Fadeev-Popov determinant vanishes (or equivalently the lowest eigenvalue of the Fadeev-Popov operator vanishes).

It is obvious that any gauge field in Ω will also satisfy the Landau gauge condition used in the Fadeev-Popov procedure. We know that Landau gauge contains Gribov copies and so in order to identify a true fundamental modular region we would need to make further restrictions on the gauge fields to eliminate the Gribov copies. It should be noted that a fundamental modular region must contain one gauge field per gauge orbit so we must ensure that we do not exclude entire gauge orbits when introducing restrictions on the gauge fields. It has

been shown elsewhere[15] that the above gauge fixing does in fact include every gauge orbit. So let us now proceed to investigate as to whether Ω is indeed a fundamental modular region.

4.2 Gribov copies within the Gribov Horizon

Here we consider the possibility that there are Gribov copies within the Gribov horizon, in this case the Gribov region would not form a fundamental modular region. This will be done by first assuming that the Gribov region contains no Gribov copies and then finding a contradiction.

Let us begin by assuming that Ω is a fundamental modular region, then in Ω there is one and only one gauge field per gauge orbit. It has been shown that the elements of Ω are local minima of the norm functional, this implies that gauge fields in Ω are in fact absolute minima since there is only one local minimum per gauge orbit it must be the absolute minimum. In [13] it was shown that if Ω consisted of absolute minima then the boundary of Ω must also contain absolute minima by a simple continuity argument. Now take a generic gauge field, A_0 , from the Gribov horizon, it will be an absolute minimum and will also satisfy the following,

$$\left. \frac{\delta \|A^\omega\|^2}{\delta \omega^a(x)} \right|_{A=A_0} = 0, \quad (4.19)$$

$$\left. \frac{\delta^2 \|A^\omega\|^2}{\delta \omega^a(x) \delta \omega^c(y)} \right|_{A=A_0} = 0. \quad (4.20)$$

Since the second functional derivative is zero, it does not tell us anything about the nature of the critical point. However if we look at the third functional derivative at A_0 and if we find that it is non zero it means that A_0 is not a local extremum. Therefore for the argument just described to be consistent, the third derivative must be zero at A_0 . Dell'Antonio and Zwanziger[13] have explicitly shown, in the case of gauge group $SU(2)$ with spacetimes R^3 or T^n , that there

are points on $\partial\Omega$ which have a non vanishing third derivative. If we take A_0 to be such a point then it is not an absolute minimum and therefore by continuity, the gauge fields in some neighbourhood of A_0 cannot be absolute minima. Since A_0 is on the boundary of Ω then the gauge fields in this neighbourhood, which are also in Ω are not absolute minima. Therefore they are necessarily local minima and are Gribov copies of the gauge field in Ω corresponding to the absolute minima on that same gauge orbit. Thus we see that, at least in the case considered, there are Gribov copies inside the Gribov horizon. This is believed to be the general case and we shall continue under this assumption.

4.3 The Absolute Minima of the Norm Functional

Under the assumption that Ω is not a fundamental modular region, we may then ask whether we can identify a subset of Ω that does constitute a fundamental modular region. We shall investigate this now.

Define Λ to be the set gauge fields in Ω which are absolute minima of the norm functional on each gauge orbit. Therefore by definition Λ consists of one gauge field per gauge orbit so long as there are no degenerate absolute minima. Therefore we make the additional definition, let Λ_0 be the set of gauge fields which are unique absolute minima of the norm functional. Clearly Λ_0 is contained in Λ . Let us express these ideas more precisely and we shall return later to the question of degenerate absolute minima. Consider an arbitrary gauge field, A , which is acted on by a gauge transformation g , producing A^g . So we can define Λ in the following way

$$\Lambda \equiv \{A \mid \|A^g\|^2 \geq \|A\|^2, \forall g \in SU(3)\}. \quad (4.21)$$

The arguments here rely on the fact that Λ is convex, that is, there is a continuous

path between any two gauge fields in Λ , which is itself contained in Λ . So given A_1, A_2 in Λ , we want to show that

$$A = sA_1 + (1-s)A_2 \in \Lambda, \quad (4.22)$$

for $s \in [0, 1]$, or equivalently that

$$\|(sA_1 + (1-s)A_2)^g\|^2 - \|sA_1 + (1-s)A_2\|^2 \geq 0, \quad (4.23)$$

for all s and all g . Gauge transforming and expanding this we get

$$\begin{aligned} & \|(sA_1 + (1-s)A_2)^g\|^2 - \|sA_1 + (1-s)A_2\|^2 \\ &= \int \text{Tr}((sA_1 + (1-s)A_2)^g)^2 - (sA_1 + (1-s)A_2)^2 \\ &= \int \text{Tr}((g(sA_1 + (1-s)A_2)g^\dagger - g\partial g^\dagger)^2 - (sA_1 + (1-s)A_2)^2) \\ &= \int \text{Tr}(-2g(sA_1 + (1-s)A_2)\partial g^\dagger + (g\partial g^\dagger)^2) \\ &= \int \text{Tr}\left(s(A_1^2 - 2gA_1g^\dagger g\partial g^\dagger + (g\partial g^\dagger)^2) - sA_1^2 \right. \\ & \quad \left. + (1-s)(A_2^2 - 2gA_2g^\dagger g\partial g^\dagger + (g\partial g^\dagger)^2) - (1-s)A_2^2\right) \\ &= \int \text{Tr}\left(s(A_1^{g^2} - sA_1^2) - (1-s)(A_2^{g^2} - A_2^2)\right) \\ &= s\left(\|(A_1^g\|^2 - \|(A_1\|^2)\right) + (1-s)\left(\|(A_2^g\|^2 - \|(A_2\|^2)\right). \end{aligned} \quad (4.24)$$

Since both A_1 and A_2 are in Λ then both the above quantities are greater or equal to zero therefore we have proved (4.23). Notice that we can similarly show that Λ_0 is convex.

Now take any gauge field A in $\Lambda - \Lambda_0$ and apply (4.24) with $A_1 = A$ and $A_2 = 0$. Hence we have

$$\|sA^g\|^2 - \|sA\|^2 = s\left(\|(A^g\|^2 - \|(A\|^2)\right) + (1-s)\|0^g\|^2. \quad (4.25)$$

Since A is in Λ then the first term is non negative and the second term is positive for all values of s (since $A = 0$ is in Λ_0), except $s = 1$, therefore

$$\|sA^g\|^2 - \|sA\|^2 > 0, \quad s \neq 1. \quad (4.26)$$

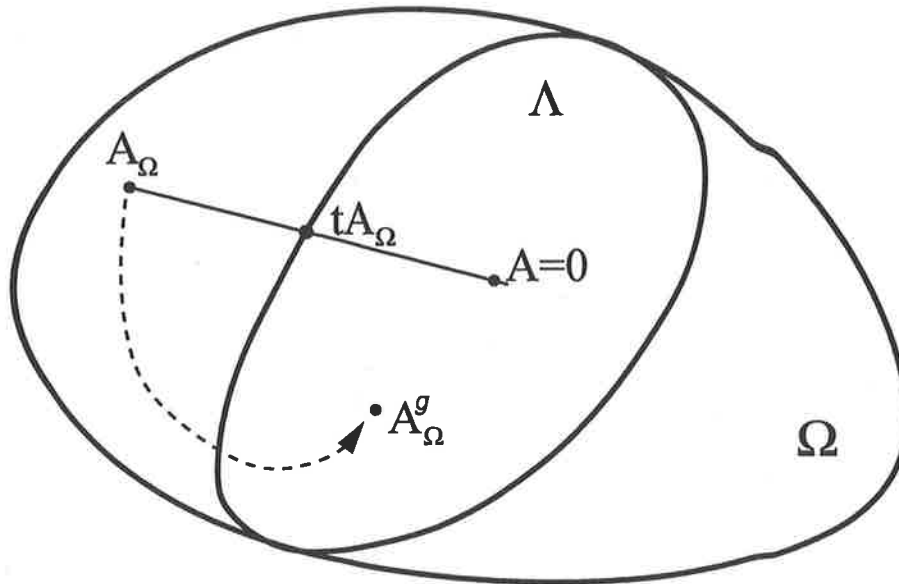


Figure 4.1: The linear path from $A = 0$ to A_Ω is shown above. It crosses the boundary of Λ at $s = t$. A_Ω^g is the gauge field which corresponds to the absolute minima of the gauge orbit containing A_Ω .

Thus sA is in Λ_0 for all s in $[0, 1)$, meaning that A is in the boundary of Λ_0 . Since A was arbitrary, it follows that $\Lambda = \Lambda_0 \cup \delta\Lambda_0$.

Thus, so far we have found that degenerate minima lie on the boundary of Λ , if these degenerate minima are never on the same gauge orbit then Λ would constitute a fundamental modular region, but as we shall see this is not the case. We now consider this problem geometrically[12]. Construct a linear path from $A = 0$ to some arbitrary gauge field A_Ω , in Ω but not in Λ . An arbitrary gauge field on this path can be written as sA_Ω , where s goes from 0 to 1. Therefore there is some value of s , say t , for which tA_Ω lies on the boundary of Λ . There is also a gauge field in Λ , say A_Ω^g , which is related to A_Ω by a gauge transformation and has a norm strictly less than that of A_Ω . In general A_Ω^g will not lie on the path. This scenario is shown in Figure 4.1 Let us consider what happens when

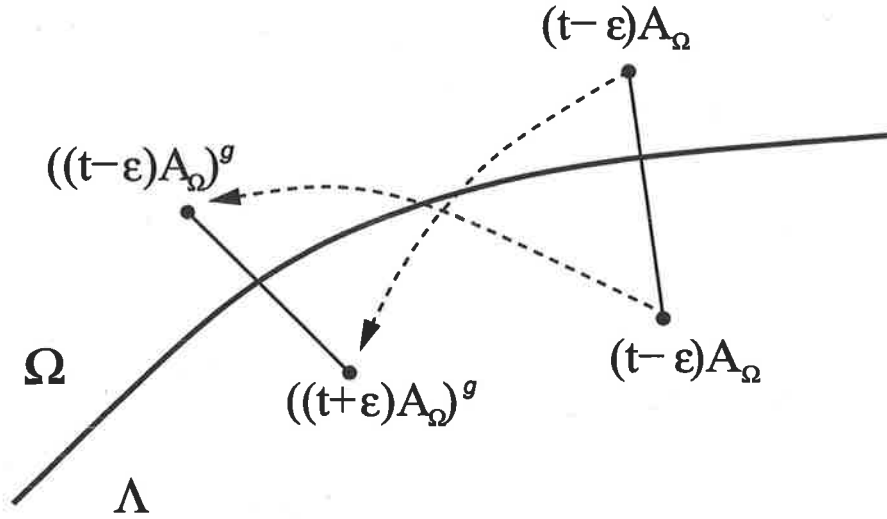


Figure 4.2: The behaviour of the gauge fields sA_Ω and $(sA_\Omega)^g$ near $s = t$ is shown above. When sA_Ω passes inside Λ , $(sA_\Omega)^g$ must move outside and therefore at $s = t$ they will both lie on the boundary

we start from $s = 1$ and move along the path towards $s = t$. What happens to the gauge copy of sA_Ω as s changes? $(sA_\Omega)^g$ is given by (dropping the index μ)

$$(sA_\Omega)^g = sgA_\Omega g^\dagger - g\partial g^\dagger. \quad (4.27)$$

So we can see that $(sA_\Omega)^g$ lies on a linear path from $-g\partial g^\dagger$ to $gA_\Omega g^\dagger - g\partial g^\dagger$. Therefore as s changes the gauge transformed gauge field $(sA_\Omega)^g$ follows its own continuous one parameter path. This will be useful in the following.

We shall now examine what happens near $s = t$. Consider $s = t + \epsilon$, the gauge field $(t + \epsilon)A_\Omega$ will still have a gauge copy of itself inside Λ but when $s < t$, say $t - \epsilon$, then there can be no gauge copy inside Λ because it is now in the interior of Λ and thus it is a non degenerate absolute minimum. Therefore the gauge transformed gauge field $((t - \epsilon)A_\Omega)^g$ must now lie outside of Λ . This is shown in Figure 4.2. Using the result above that $(sA_\Omega)^g$ follows a continuous path, we find that at $s = t$ there are, in general, two distinct gauge fields which are related by a

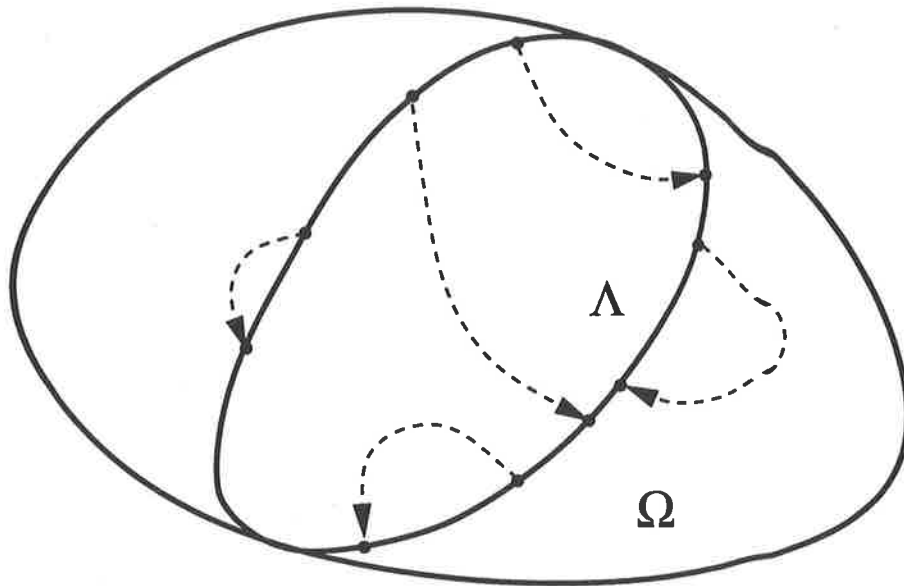


Figure 4.3: The general scenario with Gribov copies all around the boundary of Λ

gauge transformation: they are Gribov copies. These gauge fields will obviously have the same norm since they are both absolute minima on the same gauge orbit. Since the gauge field A_Ω was arbitrary then this is the general scenario, as shown in Figure 4.3. Thus we have found that the fundamental modular region of Landau gauge is a non trivial region since we necessarily have to choose only one of these degenerate minima on the boundary of Λ so that there is only one contribution from each gauge orbit.

Chapter 5

Gauge Fixing in Lattice QCD

In this chapter we shall examine gauge fixing and the Gribov problem for QCD defined on a spacetime lattice. Firstly, we will make some general comments about lattice QCD, which will be followed by the definition of the lattice gauge fields and lattice expectation values. We will then investigate the Gribov problem on the lattice and then examine some lattice gauge fixing methods.

5.1 Lattice QCD

In order to study long range phenomena, such as confinement, non-perturbative methods for QCD calculations are required. The formulation of QCD on a finite spacetime lattice is one such method. The overall aim is to simulate QCD on a computer so that physical quantities can be calculated numerically. This makes many more calculations possible, giving useful insights into QCD, which may not have been otherwise possible. The main restriction on the possible calculations is the computer power available and since the computational speed is ever increasing, the future looks good for lattice QCD. But with the current computational speed, calculations at physical quark masses cannot be done. Therefore, unphysical heavy quarks are used and then the results are extrapolated to their physical

values. It is clear that this extrapolation must be done correctly in order that correct results are obtained, exactly how to do this is still an open question.

There is an alternative motivation for studying QCD on a lattice. In the lattice formulation of QCD all the quantities are finite and well defined in contrast to the continuum path integral formulation, where the functional integration measure lacks a rigorous definition. Therefore lattice QCD also provides a useful framework to carry out rigorous calculations.

Here we shall consider a four dimensional finite spacetime lattice, with a volume V , where adjacent lattice sites are separated by a distance a . We shall label the lattice sites with the discrete variable x . The matter field, $\psi(x)$, are defined only for each lattice site. We shall see in the next section that lattice gauge fields are represented by the directed links of the lattice, exactly the parallel transport operators defined earlier. For a complete introduction to lattice gauge theory see [16][17].

5.2 Lattice Gauge Fields

Gauge fields are written in terms of the parallel transport operator from each lattice site to each of its neighbouring sites.

Recall that in the continuum, we define the covariant derivative in the following way,

$$D_\mu \psi(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\psi(x + \epsilon \hat{\mu}) - U(x + \epsilon \hat{\mu}, x) \psi(x)), \quad (5.1)$$

where we introduce the quantities $U(x + \epsilon \hat{\mu}, x)$ to ensure the correct transformation properties. In the continuum we expanded $U(x + \epsilon n, x)$ about x , introducing $A_\mu(x)$,

$$U(x + \epsilon \hat{\mu}, x) = \exp[i\epsilon g_s A_\mu^a(x) t^a] = 1 + \epsilon A_\mu(x) + \mathcal{O}(\epsilon^2), \quad (5.2)$$

where

$$A_\mu(x) \equiv ig_s A_\mu^a(x) t^a. \quad (5.3)$$

But we shall not do this to derive the lattice formulation. In the lattice setting we realise that when defining the covariant derivative our infinitesimal parameter ϵ corresponds to the lattice spacing a and we drop the limit because in the lattice setting a is the smallest distance greater than zero, so

$$D_\mu \psi(x) = \frac{1}{a}(\psi(x + a\hat{\mu}) - U_\mu(x)\psi(x)), \quad (5.4)$$

where

$$U_\mu(x) \equiv U(x + a\hat{\mu}, x). \quad (5.5)$$

It is clear that the quantities $U_\mu(x)$ correspond to the lattice link from the site x to the site $x + a\hat{\mu}$. Thus lattice gauge fields are directed link of the lattice.

5.3 The Lattice Expectation Values

In the continuum, the ungauged fixed expectation value of \mathcal{O} for pure gauge QCD, is given by

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \exp[-S[A]] \mathcal{O}(A)}{\int \mathcal{D}A \exp[-S[A]]}. \quad (5.6)$$

On the lattice we express the gauge fields in terms of the links $U_\mu(x)$ so we need to integrate over all possible links. We immediately realise that in any finite lattice there is a finite number of links, hence the functional integral reduces to a finite product of one dimensional integrals. Therefore the expectation value on a lattice is a well defined quantity unlike the functional integral in the continuum. In the formulation of QCD on the lattice we ensure that gauge invariance is preserved, so the integration over the links should be done in a gauge invariant way, using a gauge invariant measure. We will use the group invariant measure dU , which satisfies

$$\int dU = 1, \quad (5.7)$$

$$\int f(U) dU = \int f(gU) dU, \quad (5.8)$$

for all g in the group. So we define the lattice expectation value of \mathcal{O} as,

$$\langle \mathcal{O} \rangle_L = \frac{\int \prod_{x\mu} dU_\mu(x) \exp[-S(U_\mu(x))] \mathcal{O}(U_\mu(x))}{\int \prod_{x\mu} dU_\mu(x) \exp[-S(U_\mu(x))]} \quad (5.9)$$

Now the action depends on the set of all links on the lattice, that is $\{U_\mu(x)\}$ and thus it is no longer a functional. In practice, a finite number of link variables are generated with a probability distribution of $\exp[-S(U)]$ and then $\langle \mathcal{O} \rangle$ is just the average value of \mathcal{O} over these configurations, so we have

$$\langle \mathcal{O} \rangle_L = \frac{1}{N} \sum_{i=1}^N \mathcal{O}(U_i), \quad (5.10)$$

where $\{U_i\}$ is an ensemble of N , configurations with the probability distribution $\exp[-S(U)]$.

To recover the continuum, we take the infinite configuration limit, the zero lattice spacing limit and the infinite volume limit in that order

$$\langle \mathcal{O} \rangle = \lim_{V \rightarrow \infty} \lim_{a \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathcal{O}(U_i). \quad (5.11)$$

5.4 The Gribov Problem on the Lattice

Now let us examine the lattice version of QCD in terms of what we know about the Gribov problem. Recall that the physical space of gauge fields corresponds to the space of gauge orbits. Gauge fixing was described as limiting the functional integration over gauge fields to this physical space so that one and only one gauge field per gauge orbit was integrated over.

An expectation value in lattice QCD is calculated by first taking a finite ensemble of gauge configurations with a weighting of $\exp[-S[U]]$. So QCD is approximated on the lattice by taking a finite number of configurations rather than all possible configurations which is what is done in continuum QCD. So the Gribov problem will manifest itself in the lattice setting if two configurations in

the chosen ensemble lie in the same gauge orbit. When calculating the expectation value of an observable, we can sample from anywhere on the gauge orbit since an observable is gauge invariant and is therefore constant on each gauge orbit. So we just have to make sure that we do not sample twice from the same gauge orbit. Remember that even on the lattice there are an infinite number of gauge orbits. Therefore if we choose N gauge fields the likelihood of two of them lying in the same orbit is vanishingly small. Hence on the lattice when calculating a physical observable there is no need for gauge fixing and there is also no Gribov problem. However, when considering gauge dependent quantities, such as the gluon propagator, the above argument does not hold and gauge fixing is necessary. In order to avoid the Gribov problem we need to find an ideal gauge fixing. We now examine the implementation of Landau gauge on the lattice.

5.5 Gauge Fixing on the Lattice

In the continuum the Landau gauge fixing condition is given by $\partial_\mu A^\mu = 0$, now we want to know what the lattice analog of this would be. Recall in the previous chapter that it was shown that Landau gauge could be reinterpreted in terms of the local minima of the norm functional

$$\|A\|^2 \equiv - \int d^4x \operatorname{Tr}(A_\mu^2). \quad (5.12)$$

It was also found that this set of local minima did not constitute a fundamental modular region and absolute minima were needed to be considered. This definition of Landau gauge is able to be implemented on the lattice. The quantity that we want to minimise on the lattice is

$$\mathcal{F} = \frac{1}{2} \sum_{\mu x} \operatorname{Tr}(U_\mu(x) + U_\mu^\dagger(x)). \quad (5.13)$$

The problem with this minimisation process is that most algorithms find local minima, which are the Gribov copies. In fact finding the absolute minima of this

functional is a numerical problem of non-polynomial time complexity. Therefore the gauge choice with this gauge fixing method is a random choice from the Gribov copies in the gauge orbit. However, Laplacian gauge is an alternative which is Gribov copy free, but it relies on finding the eigenvector of the lattice Laplacian operator and thus is a non local gauge fixing scheme, it will not be discussed here, but details can be found elsewhere[18][19][20].

Part II

BRST Symmetry

Chapter 6

BRST Symmetry

Recall the Fadeev-Popov gauge fixed Lagrangian which consisted of the usual Lagrangian with the addition of gauge fixing terms deduced from the Fadeev-Popov procedure. This Lagrangian is no longer invariant under gauge transformations. This was precisely the aim of gauge fixing since the integration over gauge fields needed to be restricted to the physical configuration space, the space of gauge orbits. However when we recall that gauge invariance was the principle that initially led us to the form of the QCD Lagrangian, it seems undesirable to break this symmetry straight away to ensure that we make correct calculations of physical observables. It was discovered by Becchi, Rouet, Stora and Tyupkin [21] that the gauge fixed Lagrangian does have a residual symmetry, now known as BRST symmetry. This symmetry is closely related to gauge symmetry and in fact gauge invariance implies BRST invariance. We will see that the Fadeev-Popov gauge fixed Lagrangian (in Landau gauge) may be obtained from a particular choice of a BRST invariant Lagrangian and hence the Fadeev-Popov procedure can be thought to be in some sense contained within a BRST invariant gauge fixing method. We shall take this a step further and suggest that we may elevate BRST invariance to the guiding principle in determining the form of the

QCD Lagrangian, replacing gauge invariance. This requires the introduction of the unphysical ghosts fields at the classical level before the theory is quantised. This approach, referred to as BRST quantisation[22], is seen to be a far more elegant and direct method than the Fadeev-Popov procedure. We shall examine the differences between the two and also investigate some properties of our BRST invariant theory

6.1 BRST Invariance

Recall that the expectation value of \mathcal{O} in Landau gauge, which was found by the Fadeev-Popov procedure is given by

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \exp[-S[A, c, \bar{c}]] \mathcal{O}(A)}{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \exp[-S[A, c, \bar{c}]]}, \quad (6.1)$$

where

$$S[A, c, \bar{c}] = S_G[A] + \int d^4x \frac{1}{2\xi} (\partial^\mu A_\mu^a)^2 + \bar{c}^a(x) \partial^\mu D_\mu^{ac} c^c(x). \quad (6.2)$$

We express the above in a slightly different way. The gauge fixing term can be rewritten by the introduction of auxiliary fields b^a , using the following identity,

$$\exp \left[-\frac{1}{2\xi} \int (\partial^\mu A_\mu^a)^2 \right] = \mathcal{N} \int \mathcal{D}b \exp \left[\int d^4x \frac{\xi}{2} (b^a)^2 + b^a \partial^\mu A_\mu^a \right]. \quad (6.3)$$

So we have

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \mathcal{D}b \mathcal{D}\bar{c} \mathcal{D}c \exp[-S_G[A] - S_{GF}[A, c, \bar{c}, b]] \mathcal{O}(A)}{\int \mathcal{D}A \mathcal{D}b \mathcal{D}\bar{c} \mathcal{D}c \exp[-S_G[A] + S_{GF}[A, c, \bar{c}, b]]}, \quad (6.4)$$

where

$$S_{GF}[A, c, \bar{c}, b] = \int d^4x -\frac{\xi}{2} (b^a)^2 - b^a \partial^\mu A_\mu^a + \bar{c}^a(x) \partial^\mu D_\mu^{ac} c^c(x), \quad (6.5)$$

The Fadeev-Popov procedure produces the following gauge fixed Lagrangian

$$\mathcal{L} = -\frac{1}{4} (F_{\mu\nu}^a)^2 - \frac{\xi}{2} (b^a)^2 - b^a \partial^\mu A_\mu^a + \bar{c}^a \partial^\mu D_\mu^{ac} c^c. \quad (6.6)$$

This Lagrangian is no longer gauge invariant due to the addition of gauge fixing terms, which by construction break gauge invariance. However it is invariant under the so called BRST transformations[21]. The BRST transformations for the gauge fields are obtained from the infinitesimal gauge transformations by replacing the gauge parameter by the ghost field. In order to maintain consistency we therefore require that the infinitesimal parameter is Grassmann valued so that ϵc remains a commuting number. That is we make the identification

$$\omega(x) \equiv \epsilon c(x). \quad (6.7)$$

This automatically means that any gauge invariant object is BRST invariant. The ghosts and the auxiliary field do not have gauge transformation properties so we will simply define their BRST transformation properties, which we state now and will then show that (6.6) is indeed invariant. The infinitesimal BRST transformations are

$$\delta A_\mu^a = \epsilon D_\mu^{ac} c^c, \quad (6.8)$$

$$\delta c^a = -\frac{1}{2} g_s \epsilon f^{abc} c^b c^c, \quad (6.9)$$

$$\delta \bar{c}^a = \epsilon b^a, \quad (6.10)$$

$$\delta b^a = 0, \quad (6.11)$$

where the delta means the variation of the field under a BRST transformation, so if ϕ is any field, A_μ , c , \bar{c} or b then its BRST transformation is given by

$$\phi \rightarrow \phi + \delta\phi. \quad (6.12)$$

Note that the BRST variation of a product of any two of the above fields is given by the usual product rule

$$\delta(\phi_1\phi_2) = \delta(\phi_1)\phi_2 + \phi_1\delta(\phi_2). \quad (6.13)$$

We will first show that the quantity $D_\mu^{ac} c^c$ is BRST invariant.

$$\begin{aligned}
\delta(D_\mu^{ac} c^c) &= \delta(\partial_\mu c^a + g_s f^{abc} A_\mu^b c^c) \\
&= \partial_\mu \left(-\frac{1}{2} g_s \epsilon f^{abc} c^b c^c \right) + g_s f^{abc} \left(\epsilon (D_\mu^{bd} c^d) c^c - \frac{1}{2} g_s \epsilon f^{cde} A_\mu^b c^d c^e \right) \\
&= -\frac{1}{2} g_s \epsilon f^{abc} ((\partial_\mu c^b) c^c - (\partial_\mu c^c) c^b) + \epsilon g_s f^{abc} (\partial_\mu c^b) c^c \\
&\quad + \epsilon g_s^2 f^{abc} f^{bed} A_\mu^e c^d c^c - \frac{1}{2} \epsilon g_s^2 f^{abc} f^{cde} A_\mu^b c^d c^e \\
&= -\frac{1}{2} g_s \epsilon f^{abc} ((\partial_\mu c^b) c^c + (\partial_\mu c^c) c^b) + \epsilon g_s f^{abc} (\partial_\mu c^b) c^c \\
&\quad - \frac{1}{2} \epsilon g_s^2 A_\mu^b c^d c^e (f^{abc} f^{cde} - 2 f^{ace} f^{cbd}) \\
&= -\frac{1}{2} \epsilon g_s^2 A_\mu^b c^d c^e (f^{abc} f^{cde} - f^{ace} f^{cbd} + f^{acd} f^{cbe}) \\
&= 0,
\end{aligned} \tag{6.14}$$

where the we have used the Jacobi identity to get to the last line.

Now we show explicitly that (6.6) is invariant. The first term is clearly BRST invariant because it is gauge invariant, second term is trivial, so the transformation of the third is

$$\delta(b^a \partial^\mu A_\mu^a) = b^a \partial^\mu \delta(A_\mu^a) \tag{6.15}$$

$$= \epsilon b^a \partial^\mu D_\mu^{ac} c^c. \tag{6.16}$$

The BRST variation of the last term is easy if we use the fact that $\delta(D_\mu^{ac} c^c) = 0$,

$$\delta(\bar{c}^a \partial^\mu D_\mu^{ac} c^c) = \delta(\bar{c}^a) \partial^\mu D_\mu^{ac} c^c + \bar{c}^a \partial^\mu \delta(D_\mu^{ac} c^c) \tag{6.17}$$

$$= \epsilon b^a \partial^\mu D_\mu^{ac} c^c. \tag{6.18}$$

We immediately see that BRST variation of the above two terms cancel each other and so (6.6) is indeed invariant under BRST transformations. Now we define the BRST operator, s , as

$$\delta\phi \equiv \epsilon s\phi. \tag{6.19}$$

Since ϵ is Grassmann it implies that s is also Grassmann, in the sense that $s(\phi)$ has the opposite Grassmann grading to ϕ . The action of s on the fields follows directly from this definition,

$$sA_\mu^a = D_\mu^{ac} c^c, \quad (6.20)$$

$$sc^a = -\frac{1}{2} g_s f^{abc} c^b c^c, \quad (6.21)$$

$$s\bar{c}^a = b^a, \quad (6.22)$$

$$sb^a = 0. \quad (6.23)$$

The BRST operator acting on a product of fields is then given by

$$s(\phi_1\phi_2) \equiv \frac{1}{\epsilon}\delta(\phi_1\phi_2) \quad (6.24)$$

$$= \frac{1}{\epsilon}(\delta(\phi_1)\phi_2 + \phi_1\delta(\phi_2)) \quad (6.25)$$

$$= \frac{1}{\epsilon}(\epsilon s(\phi_1)\phi_2 + \phi_1\epsilon s(\phi_2)) \quad (6.26)$$

$$= s(\phi_1)\phi_2 \pm \phi_1 s(\phi_2), \quad (6.27)$$

where we take the plus(minus) sign when ϕ_1 is Grassmann even(odd). Now the statement that the (6.6) is invariant under BRST transformation in terms of s can be written as

$$s\mathcal{L}(A, c, \bar{c}, b) = 0. \quad (6.28)$$

The operator s is nilpotent, that is

$$s^2 = 0. \quad (6.29)$$

This means that when the BRST operator acts twice on any product of the fields A_μ , c , \bar{c} or b , the result is always zero. We will prove this by induction, so first

consider s^2 acting on one field

$$s^2 A_\mu^a = s(D_\mu^{ac} c^c) = 0, \quad (6.30)$$

$$\begin{aligned} s^2 c^a &= -\frac{1}{2} g_s f^{abc} s(c^b c^c) \\ &= \frac{1}{4} g_s^2 f^{abc} (f^{bde} c^d c^e c^c + f^{cde} c^c c^d c^e) = 0, \end{aligned} \quad (6.31)$$

$$s^2 \bar{c}^a = s(b^a) = 0, \quad (6.32)$$

$$s^2 b^a = 0, \quad (6.33)$$

where the first line follows from (6.14) and (6.19). Now assuming that s^2 acting on $(n-1)$ fields is zero, we prove that s^2 acting on n fields is zero,

$$\begin{aligned} s^2(\phi_1 \phi_2 \dots \phi_n) &= s(s(\phi_1) \phi_2 \dots \phi_n \pm \phi_1 s(\phi_2 \dots \phi_n)) \\ &= \mp s(\phi_1) s(\phi_2 \dots \phi_n) \pm s(\phi_1) s(\phi_2 \dots \phi_n) \\ &= 0, \end{aligned} \quad (6.34)$$

where we have used the fact that the Grassmann grading of $s(\phi_1)$ is the opposite to that of ϕ_1 . Thus the BRST operator is nilpotent.

6.2 The Fadeev-Popov Lagrangian from the BRST Approach

We now make the observation that we can obtain the Fadeev-Popov gauge fixed action from the original gauge invariant action in (2.38), by simply requiring BRST invariance. This method is a more natural way to derive the Fadeev-Popov Lagrangian rather than resorting to the mathematical trickery of the Fadeev-Popov procedure. The problems associated with gauge fixing as described in chapter 3 will still be present since they are independent of the method used, however we shall note that final result is not exactly the same as the that of the Fadeev-Popov procedure. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + s \left(-\frac{\xi}{2}\bar{c}^a b^a - \bar{c}^a \partial^\mu A_\mu^a \right). \quad (6.35)$$

The above is BRST invariant since the first term is gauge invariant and the second due to the nilpotency of s . Therefore,

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(F_{\mu\nu}^a)^2 - \frac{\xi}{2}s(\bar{c}^a)b^a - s(\bar{c}^a)\partial^\mu A_\mu^a + \bar{c}^a s(\partial^\mu A_\mu^a) \\ &= -\frac{1}{4}(F_{\mu\nu}^a)^2 - \frac{\xi}{2}(b^a)^2 - b^a \partial^\mu A_\mu^a + \bar{c}^a \partial^\mu D_\mu c^a. \end{aligned} \quad (6.36)$$

Therefore the Fadeev-Popov Lagrangian is recovered in the form that we derived earlier, in (6.4), with the auxiliary field, b , present. Note that in the Fadeev-Popov procedure, the functional determinant appears with a modulus since it is a Jacobian of a change of variables, then one has to justify dropping the modulus before expressing it as a functional integral over ghost fields. This was justified under the assumption of an ideal gauge fixing condition but as discussed previously we do not know of one. In the BRST formalism the ghost terms in the gauge fixed Lagrangian appear naturally and there is no need for the Fadeev-Popov procedure. The BRST approach is a more elegant way of achieving the identical result as the conventional gauge fixing methods, but as we shall see it provides a more general framework for gauge fixing.

6.3 BRST Quantisation

In the previous section it was shown that the Fadeev-Popov Lagrangian could be obtained from a BRST invariant Lagrangian. This prompts the idea that BRST symmetry could be elevated to the guiding principle, it can be thought of a generalisation of gauge invariance because BRST symmetry implies gauge symmetry but the reverse is not true. Therefore rather than finding the most general gauge invariant and renormalisable Lagrangian we should be looking for the most general BRST invariant and renormalisable Lagrangian. This means that the gauge

fixing is done when the Lagrangian is written down rather than having to add extra terms as in the Fadeev-Popov procedure. This means that the ghosts are introduced at the classical level and then quantised with the gauge and matter fields. When taking this approach we are not restricted to considering only one ghost and its anti-ghost it is possible to introduce more ghosts, but we shall not consider this here. So we have the fields A , c , \bar{c} and b . We now define a new property, ghost number by specifying that the ghost number of c and \bar{c} is $+1$ and -1 , respectively and the ghost number is zero for the other fields. The ghost number of a product of fields is the sum of the ghost numbers of the individual fields. For more details on BRST quantisation see [22][23].

We begin the BRST quantisation procedure by specifying BRST transformations, which we will use as the guiding principle in developing the theory, they are exactly as were given previously but here they are to be interpreted as the transformations of classical fields,

$$sA_\mu^a = D_\mu^{ac}c^c, \quad (6.37)$$

$$sc^a = -\frac{1}{2}g_s f^{abc}c^b c^c, \quad (6.38)$$

$$s\bar{c}^a = b^a, \quad (6.39)$$

$$sb^a = 0. \quad (6.40)$$

Now notice that when the BRST operator, s , acts on the individual fields above it raises the ghost number by one and this will also hold for any product of fields since s obeys the product rule above. We say that s has ghost number $+1$.

6.3.1 The General form of a BRST Invariant Lagrangian

The aim here is to use BRST symmetry to find the most general renormalisable BRST invariant Lagrangian just as we did in the first chapter when constructing the QCD Lagrangian from gauge invariance. We can immediately say that the

action must have ghost number zero since ghosts are unphysical particles and can not appear in any physical state. Now we claim that the Lagrangian must take the form

$$\mathcal{L}(A, c, \bar{c}, b) = \mathcal{L}_0(A) + s\Psi(A, c, \bar{c}, b), \quad (6.41)$$

where Ψ is called the gauge fixing fermion, it must have ghost number -1 so that $s\Psi$ has the required ghost number zero. We now prove this statement in much the same way as done in [2]. First we define a new operator t , which is defined to give zero when acting on any field except b , in this case we have

$$tb^a = \bar{c}^a. \quad (6.42)$$

Notice that t lowers the ghost number by one and therefore itself has ghost number -1 . Consider the operator

$$\{s, t\} \equiv st + ts. \quad (6.43)$$

It is easily deduced that this operator has the property that when it acts on any field it gives zero except for the two cases

$$\{s, t\}\bar{c}^a = \bar{c}^a, \quad (6.44)$$

$$\{s, t\}b^a = b^a. \quad (6.45)$$

Note also that when acting on a product of fields it merely gives the total number of \bar{c} and b fields in the product, that is

$$\{s, t\}\phi_1 \dots \phi_m = n\phi_1 \dots \phi_m, \quad (6.46)$$

where the product $\phi_1 \dots \phi_m$ consists of a total of n \bar{c} and b fields. The motivation for introducing this new operator t will become clear when we expand an arbitrary Lagrangian as a sum of particular total numbers of \bar{c} and b fields, that is

$$\mathcal{L}(A, c, \bar{c}, b) = \sum_{n=0}^{\infty} \mathcal{L}_n(A, c, \bar{c}, b), \quad (6.47)$$

where $\mathcal{L}_n(A, c, \bar{c}, b)$ has a total of n \bar{c} and b fields. We notice that the BRST transformations conserve the total number of \bar{c} and b fields. Therefore the requirement that the Lagrangian be BRST invariant means that each term in the sum has to be BRST invariant since there will be no cancellation between terms with different total numbers of \bar{c} and b fields. Now consider the operator $\{s, t\}$ acting on each term \mathcal{L}_n in the sum. We have just found that this operator will just give the same result back again multiplied by the total number of \bar{c} and b fields, therefore

$$\{s, t\}\mathcal{L}_n = n\mathcal{L}_n. \quad (6.48)$$

But we also realise that

$$\{s, t\}\mathcal{L}_n = st\mathcal{L}_n, \quad (6.49)$$

since each \mathcal{L}_n is BRST invariant. Therefore we have

$$st\mathcal{L}_n = n\mathcal{L}_n. \quad (6.50)$$

This tells us that for $n \neq 0$ each term \mathcal{L}_n can be written as s acting on some product of fields.

$$\mathcal{L}(A, c, \bar{c}, b) = \mathcal{L}_0(A, c) + st \sum_{n=1}^{\infty} \frac{1}{n} \mathcal{L}_n(A, c, \bar{c}, b). \quad (6.51)$$

We know that \mathcal{L}_0 is independent of \bar{c} and b since it has $n = 0$ and if we also require that the Lagrangian have ghost number zero then we can also deduce that \mathcal{L}_0 must be independent of c . Hence we can write an arbitrary Lagrangian in the form

$$\mathcal{L}(A, c, \bar{c}, b) = \mathcal{L}_0(A) + s\Psi(A, c, \bar{c}, b), \quad (6.52)$$

which will have ghost number zero so long as Ψ has ghost number -1.

6.3.2 Expectation Values and Physical Observables

Up to now, everything has been classical and now we quantise the theory by the path integral method by expressing expectation values as a functional integral

weighted by the exponential of the action. Therefore we write the expectation value of some operator as

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \mathcal{D}b \exp \left[-S_G[A] - \int d^4x s\Psi(A, c, \bar{c}, b) \right] \mathcal{O}(A, c, \bar{c}, b)}{\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \mathcal{D}b \exp \left[-S_G[A] - \int d^4x s\Psi(A, c, \bar{c}, b) \right]}. \quad (6.53)$$

Unlike the case in chapter 2, the gauge fixing has already been done and the above does not contain infinite integrals around the gauge orbits. We have generalised the Fadeev-Popov procedure since we can easily recover the results obtained in the Fadeev-Popov procedure by making a particular choice for Ψ . However the Fadeev-Popov procedure is restricted to gauge fixing terms quadratic in the ghosts, but here there is no such restriction because we are free to choose Ψ to contain higher order terms in the ghosts (eg. we could include a 4 ghost interaction, which is not present in the Landau gauge fixed theory).

Let us now examine some properties of the physical observables in the BRST formalism. A BRST exact quantity is one which can be written as s acting on something else, so if \mathcal{O} is BRST exact then there exists some \mathcal{O}' such that

$$\mathcal{O} = s\mathcal{O}'. \quad (6.54)$$

We now show that the integral of a BRST exact quantity vanishes, that is

$$\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \mathcal{D}b s\mathcal{O}(A, c, \bar{c}, b) = 0, \quad (6.55)$$

where \mathcal{O} is some arbitrary product of the fields A, c, \bar{c}, b . First consider the quantity

$$\int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \mathcal{D}b \mathcal{O}(A, c, \bar{c}, b). \quad (6.56)$$

Now make a change of integration variables corresponding to a BRST transformation, that is we change each field by the action of the field under a BRST transformation

$$\phi \rightarrow \phi + \delta\phi \equiv \phi + \epsilon s\phi, \quad (6.57)$$

where ϕ is any of the fields. The integration measure is BRST invariant so it does not change under this transformation, but \mathcal{O} does change since it is not necessarily BRST invariant

$$\mathcal{O} \rightarrow \mathcal{O} + \epsilon s\mathcal{O}. \quad (6.58)$$

Since we are merely performing a change of variables the actual value of the integral will be the same so we get the following

$$\int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}b \mathcal{O} = \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}b (\mathcal{O} + \epsilon s\mathcal{O}). \quad (6.59)$$

Therefore

$$\int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}b s\mathcal{O} = 0. \quad (6.60)$$

This result is useful because it implies that the same result would apply to expectation values because the same change of variables could be done for $\langle \mathcal{O} \rangle$ and since the action is BRST invariant we would get the analogous result that

$$\langle s\mathcal{O}(A, c, \bar{c}, b) \rangle = 0. \quad (6.61)$$

And so we find that expectation values remain unchanged when we add a BRST exact term and so

$$\langle \mathcal{O} + s\mathcal{O}' \rangle = \langle \mathcal{O} \rangle + \langle s\mathcal{O}' \rangle = \langle \mathcal{O} \rangle. \quad (6.62)$$

Therefore two quantities are physically equivalent if their difference is a BRST exact object. Hence quantities are only defined up to the addition of a BRST exact quantity. Now consider \mathcal{O} to be a physical observable. It is gauge invariant and therefore it is BRST invariant, which means that

$$s\mathcal{O} = 0. \quad (6.63)$$

This condition is trivially satisfied if \mathcal{O} is itself BRST exact since

$$s\mathcal{O} = s(s\mathcal{O}') = s^2\mathcal{O}' = 0. \quad (6.64)$$

This is true for all \mathcal{O}' since the BRST operator is nilpotent. So we now see that any observable that is BRST exact is physically equivalent to 0. Therefore the physical observables are in fact those quantities which are BRST invariant but are not BRST exact. This set of objects is referred to as the BRST cohomology.

Chapter 7

The Neuberger Problem

In the last two chapters we have discussed lattice gauge fixing and the BRST formalism, now the aim is to combine these two and define BRST symmetry in the lattice setting. This was attempted some time ago by Neuberger[24], but it was also found that this lead to expectation values of observables taking the indeterminate form $\frac{0}{0}$. This is what is referred to as the Neuberger problem. Here we shall discuss this problem and work through a simple model, which clearly displays the problem.

7.1 Lattice BRST Symmetry

First we want to find the lattice BRST transformations. Since on the lattice we express the gauge fields as the links $U_\mu(x)$ and not the fields $A_\mu(x)$, we want to define the BRST transformation of $U_\mu(x)$. Recall that the BRST transformations for the gauge fields were merely the gauge transformations with the gauge parameter replaced by the ghost field, so we can define the BRST transformation on $U_\mu(x)$ in the same way. Recall that the gauge transformation of $U_\mu(x)$ is

$$U_\mu(x)' = g(x + a\hat{\mu})U_\mu(x)g^\dagger(x), \quad (7.1)$$

where

$$g(x) = \exp[i\omega^a(x)t^a]. \quad (7.2)$$

Expanding the gauge transformations in powers of the lattice spacing we get,

$$g(x) = 1 + i\omega^a(x)t^a + \mathcal{O}(a^2). \quad (7.3)$$

Therefore,

$$\begin{aligned} U_\mu(x)' &\equiv U_\mu(x)' = (1 + i\omega^a(x + a\hat{\mu})t^a)U_\mu(x)(1 - i\omega^a(x)t^a) \\ &= U_\mu(x) - iU_\mu(x)\omega^a(x)t^a + i\omega^a(x + a\hat{\mu})t^a U_\mu(x). \end{aligned} \quad (7.4)$$

Hence the gauge variation of $U_\mu(x)$ is given by

$$\delta_g U_\mu(x) = i(\omega^a(x + a\hat{\mu})t^a U_\mu(x) - U_\mu(x)\omega^a(x)t^a). \quad (7.5)$$

We obtain the BRST variation by replacing the gauge parameter with the ghost field multiplied by ϵ , a Grassmann parameter, that is

$$\omega(x) = \epsilon c(x). \quad (7.6)$$

So the BRST variation is

$$\delta U_\mu(x) = \epsilon i(c^a(x + a\hat{\mu})t^a U_\mu(x) - U_\mu(x)c^a(x)t^a). \quad (7.7)$$

Using

$$\delta \equiv \epsilon s, \quad (7.8)$$

we can find the BRST transformation of $U_\mu(x)$, we get

$$sU_\mu(x) = i\left(c(x + a\hat{\mu})U_\mu(x) - U_\mu(x)c(x)\right), \quad (7.9)$$

where

$$c(x) \equiv c^a(x)t^a. \quad (7.10)$$

The transformation of the lattice ghost field follows from the constraint $s^2 U_\mu(x) = 0$, the rest are the same as the continuum transformations except that the fields are only defined on lattice sites. So the lattice BRST transformations are

$$sU_\mu(x) = i(c(x + a\hat{\mu})U_\mu(x) - U_\mu(x)c(x)), \quad (7.11)$$

$$sc^a(x) = -\frac{1}{2}f^{abc}c^b(x)c^c(x), \quad (7.12)$$

$$s\bar{c}^a(x) = b^a(x), \quad (7.13)$$

$$sb^a(x) = 0. \quad (7.14)$$

Now define a BRST invariant measure on the lattice so that the lattice can be integrated over in a BRST invariant way.

$$d\mu \equiv \prod_{ax} \left(dc^a(x) d\bar{c}^a(x) db^a(x) \right) \prod_{\mu x} dU_\mu(x), \quad (7.15)$$

where the measure $dU_\mu(x)$ is the invariant group measure as used previously (see section 5.3).

It has been shown[24] that the most general BRST invariant lattice action takes the form

$$S(U, c, \bar{c}, b) = S_0(U) + s\Psi(U, c, \bar{c}, b), \quad (7.16)$$

where S_0 is a QCD lattice action with gauge fields only and Ψ is the lattice version of the gauge fixing fermion. This is proved in much the same way as was done previously for the continuum case by counting the number of \bar{c} and b fields.

Here the action is not a functional any longer since it depends on a finite number of variables, one for each link and three for each lattice site. This is the same result which was found in the continuum and so it is no surprise. Now we can write down the lattice expectation value in a BRST invariant theory on a lattice

$$\langle \mathcal{O} \rangle = \frac{\int d\mu \exp[-S_0(U) - s\Psi(U, c, \bar{c}, b)] \mathcal{O}(U, c, \bar{c}, b)}{\int d\mu \exp[-S_0(U) - s\Psi(U, c, \bar{c}, b)]}. \quad (7.17)$$

7.2 The Neuberger Problem

We shall now explicitly derive the Neuberger problem, that is that the lattice expectation value of an observable has the indeterminate form $\frac{0}{0}$ within the BRST framework. We begin by defining[25]

$$F_{\mathcal{O}}(t) \equiv \int d\mu \exp[-S_0(U) - ts\Psi(U, c, \bar{c}, b)] \mathcal{O}(U, c, \bar{c}, b), \quad (7.18)$$

where t is some real number. $F_{\mathcal{O}}(t)$ is defined in this way so that $F_{\mathcal{O}}(1)$ is the numerator of $\langle \mathcal{O} \rangle$. Let us first take a look at $F_{\mathcal{O}}(0)$,

$$F_{\mathcal{O}}(0) = \int d\mu \exp[-S_0(U)] \mathcal{O}(U, c, \bar{c}, b). \quad (7.19)$$

Notice that the integrand does not contain any ghost fields so by the rules of Grassmannian integration it vanishes. So we have

$$F_{\mathcal{O}}(0) = 0. \quad (7.20)$$

Now we differentiate $F_{\mathcal{O}}$ with respect to t ,

$$\frac{dF_{\mathcal{O}}(t)}{dt} = \int d\mu (-s\Psi(U, c, \bar{c}, b)) \exp[-S_0(U) - ts\Psi(U, c, \bar{c}, b)] \mathcal{O}(U, c, \bar{c}, b). \quad (7.21)$$

Since S_0 is gauge invariant and therefore BRST invariant, \mathcal{O} is BRST invariant and also due to the nilpotency of s , we can write the above as

$$\frac{dF_{\mathcal{O}}(t)}{dt} = \int d\mu s \left(-\Psi(U, c, \bar{c}, b) \exp[-S_0(U) - ts\Psi(U, c, \bar{c}, b)] \mathcal{O}(U, c, \bar{c}, b) \right). \quad (7.22)$$

The measure $d\mu$ is BRST invariant and integral of a BRST exact object vanishes, as given in(6.55). This implies that

$$\frac{dF_{\mathcal{O}}(t)}{dt} = 0. \quad (7.23)$$

Therefore $F_{\mathcal{O}}$ is independent of t and so it follows that

$$F_{\mathcal{O}}(1) = F_{\mathcal{O}}(0) = 0. \quad (7.24)$$

So the numerator of the expectation value of \mathcal{O} vanishes. It is easily seen that the denominator also vanishes by simply considering the case when the operator, \mathcal{O} is the identity. Therefore the expectation value of a BRST invariant operator has the indeterminate form $\frac{0}{0}$. Since all observables are BRST invariant then we have shown that every observable has the value $\frac{0}{0}$.

7.3 A Simple Example of the Neuberger Problem

Testa proposed[26] a simple model which displayed the Gribov problem explicitly, here we shall examine this model in detail. Consider the simplest lattice model with gauge group $U(1)$ in one dimension and consider only one link variable as shown in Figure 7.1. The link variable can be written as

$$U = \exp [iaA]. \quad (7.25)$$

Recall the BRST transformation on the links as we derived earlier

$$sU_\mu(x) = i \left(c(x + a\hat{\mu})U_\mu(x) - U_\mu(x)c(x) \right). \quad (7.26)$$

In this case we have

$$sU = i(c_2 - c_1)U. \quad (7.27)$$

Note that the ghosts commute with the links in this $U(1)$ model. It will be more useful to work with the variable A and also we can derive its BRST transformation easily from the above, we get

$$sA = \frac{1}{a}(c_2 - c_1). \quad (7.28)$$

We could continue and derive the BRST transformation of the two ghost fields, but instead we make the simplification, which just amounts to a change of variables

$$c \equiv \frac{1}{a}(c_2 - c_1). \quad (7.29)$$

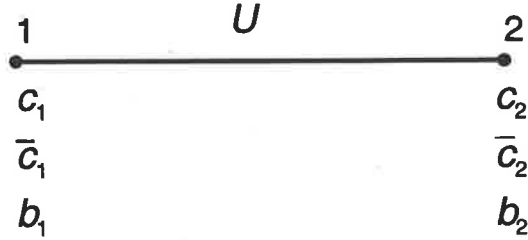


Figure 7.1: The simple lattice model

The transformation of the c field is zero due to the nilpotency of s . To complete the BRST algebra we need one anti-ghost and auxiliary field, which transform as we would expect. Therefore the BRST transformations for this model are

$$sA = c, \quad (7.30)$$

$$sc = 0, \quad (7.31)$$

$$s\bar{c} = b, \quad (7.32)$$

$$sb = 0. \quad (7.33)$$

In order to write down the action for this model we need to find the field strength tensor, but we see that it is trivial in this one dimensional model,

$$F_{xx} = \partial_x A - \partial_x A = 0. \quad (7.34)$$

The ungauged fixed expectation value of \mathcal{O} in this case is

$$\langle \mathcal{O} \rangle = \frac{\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \mathcal{O}(A)}{\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA}. \quad (7.35)$$

Now adding a BRST exact gauge fixing term we get

$$\langle \mathcal{O} \rangle = \frac{\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int_{-\infty}^{\infty} db \int d\bar{c}dc \exp[s\Psi(A, c, \bar{c}, b)] \mathcal{O}(A)}{\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int_{-\infty}^{\infty} db \int d\bar{c}dc \exp[s\Psi(A, c, \bar{c}, b)]}. \quad (7.36)$$

We shall choose Ψ so that we get the generalised covariant gauge, hence

$$\Psi(A, c, \bar{c}, b) = i\bar{c}f(A) - \frac{\xi}{2}\bar{c}b. \quad (7.37)$$

Therefore

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} dA \int_{-\infty}^{\infty} db \int d\bar{c}dc \exp \left[-\frac{\xi}{2}b^2 + ibf(A) - i\bar{c}f'(A)c \right] \mathcal{O}(A), \quad (7.38)$$

where

$$\mathcal{Z} \equiv \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} dA \int_{-\infty}^{\infty} db \int d\bar{c}dc \exp \left[-\frac{\xi}{2}b^2 + ibf(A) - i\bar{c}f'(A)c \right]. \quad (7.39)$$

We can constrain the possible forms for the gauge fixing function, f , by requiring that the expectation values of BRST exact quantities vanish as was found to be the case in any BRST symmetric theory. Therefore let

$$\Gamma \equiv i\bar{c}F(A, b), \quad (7.40)$$

and so

$$s\Gamma = ibF(A, b) - i\bar{c}F'(A, b)c, \quad (7.41)$$

where the prime denotes differentiation with respect to A . Evaluating the Grassmann integrals using a Taylor expansion for the exponential involving ghosts, we get

$$\begin{aligned} \langle s\Gamma \rangle &\equiv \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} dA \int_{-\infty}^{\infty} db \int d\bar{c}dc \exp \left[-\frac{\xi}{2}b^2 + ibf(A) - i\bar{c}f'(A)c \right] s\Gamma \\ &= \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} dA \int_{-\infty}^{\infty} db \int d\bar{c}dc \exp \left[-\frac{\xi}{2}b^2 + ibf(A) \right] \\ &\quad \cdot (1 - i\bar{c}f'(A)c) (ibF(A, b) - i\bar{c}F'(A, b)c) \\ &= \int_{-\infty}^{\infty} db \exp \left[-\frac{\xi}{2}b^2 \right] \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} dA \exp [ibf(A)] (i^2bf'(A)F(A, b) + iF'(A, b)) \\ &= i \int_{-\infty}^{\infty} db \exp \left[-\frac{\xi}{2}b^2 \right] \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} dA \frac{d}{dA} (\exp [ibf(A)] F(A, b)) \\ &= i \int_{-\infty}^{\infty} db \exp \left[-\frac{\xi}{2}b^2 + ibf(A) \right] F(A, b) \Big|_{A=-\frac{\pi}{\alpha}}^{A=\frac{\pi}{\alpha}}. \end{aligned} \quad (7.42)$$

The above is zero only if the functions F and f are periodic with period $\frac{2\pi}{a}$. We can now explicitly show that $\langle \mathcal{O} \rangle = \frac{0}{0}$, in agreement with the Neuberger problem.

The denominator of $\langle \mathcal{O} \rangle$ is

$$\begin{aligned} \mathcal{Z} &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \int_{-\infty}^{\infty} db \int d\bar{c}dc \exp \left[-\frac{\xi}{2} b^2 + ibf(A) - i\bar{c}f'(A)c \right] \\ &= -i\sqrt{\frac{2\pi}{\xi}} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \exp \left[-\frac{1}{2\xi} (f(A))^2 \right] f'(A) \\ &= -i\sqrt{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA f'(A) \delta(f(A)), \end{aligned} \quad (7.43)$$

where $\xi \rightarrow 0$ produced the delta function. Now recall the following identity

$$\delta(f(x)) = \sum_n \frac{\delta(x - x_n)}{|f'(x_n)|}, \quad (7.44)$$

where a_n are the zeros of $f(x)$. Using this the above becomes

$$\mathcal{Z} = -i\sqrt{2\pi} \sum_n \frac{f'(A_n)}{|f'(A_n)|}. \quad (7.45)$$

Clearly this is a constant times a sum of +1's and -1's, which is determined by the sign of the derivative at each zero. We know that f is periodic and thus it has an even number of zeros in $[-\frac{\pi}{a}, \frac{\pi}{a}]$. Also, there is necessarily as many zeros with positive derivative as those with negative derivative, meaning that overall everything will cancel, giving

$$\mathcal{Z} = 0. \quad (7.46)$$

We consider now the numerator of $\langle \mathcal{O} \rangle$, the only difference will be the presence of the function $\mathcal{O}(A)$. If this function is an observable then it will be constant around the gauge orbit, that is it will have the same value at every zero of $f(A)$ and so the numerator will be the same as the denominator up to a constant. Hence

$$\langle \mathcal{O} \rangle = \frac{0}{0}. \quad (7.47)$$

It is clear how this zero arises, we are using a non ideal gauge fixing function and when we naively calculate the expectation value of an observable we sum over all the contributions from the Gribov copies and in this case we get zero. This explicitly shows the Neuberger problem at work.

A solution proposed by Testa[26] is to redefine the delta function which imposes the constraint of the b field to a periodic delta function. The idea is to make the number of zeros of f odd and prevent the cancellation between the Gribov copies. Let us define

$$\delta(f(A)) \rightarrow \delta_P(f(A)) \equiv \sum_{n=-\infty}^{\infty} \delta\left(f(A) - n\frac{2\pi}{a}\right). \quad (7.48)$$

Using the Poisson summation formula, which is given by

$$\sum_{n=-\infty}^{\infty} \delta(x - n) = \sum_{n=-\infty}^{\infty} \exp[2\pi i x n], \quad (7.49)$$

we can write δ_P as

$$\begin{aligned} \delta_P(f(A)) &= \frac{a}{2\pi} \sum_{n=-\infty}^{\infty} \exp(i a n f(A)) \\ &= \frac{a}{2\pi} \sum_{n=-\infty}^{\infty} \exp(i b_n f(A)), \end{aligned} \quad (7.50)$$

where

$$b_n \equiv a n. \quad (7.51)$$

So the auxiliary fields take discrete values and the change to the BRST transformations is the obvious one

$$s\bar{c} = b_n. \quad (7.52)$$

Therefore the gauge fixed expectation value is now given by

$$\langle \tilde{\mathcal{O}} \rangle = \frac{1}{\tilde{\mathcal{Z}}} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \sum_{n=-\infty}^{\infty} \int d\bar{c}dc \exp \left[-\frac{\xi}{2} b_n^2 + i b_n f(A) - i \bar{c} f'(A) c \right], \quad (7.53)$$

where

$$\tilde{\mathcal{Z}} = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dA \sum_{n=-\infty}^{\infty} \int d\bar{c}dc \exp \left[-\frac{\xi}{2} b_n^2 + ib_n f(A) - i\bar{c}f'(A)c \right]. \quad (7.54)$$

Again we require that the expectation value of a BRST exact quantity vanishes

$$\tilde{\Gamma} \equiv i\bar{c}F(A, b_n), \quad (7.55)$$

$$s\tilde{\Gamma} = ib_n F(A, b_n) - i\bar{c}F'(A, b_n)c. \quad (7.56)$$

The same calculation can be carried out as before and we end up with a similar result

$$\langle s\tilde{\Gamma} \rangle = \sum_{n=-\infty}^{\infty} \exp \left[-\frac{\xi}{2} b_n^2 + ib_n f(A) \right] F(A, b_n) \Big|_{A=-\frac{\pi}{a}}^{A=\frac{\pi}{a}}. \quad (7.57)$$

The important difference this time is that the above will still remain zero if the gauge fixing function f is periodic up to a shift, but with F still strictly periodic, that is

$$f\left(A + \frac{2\pi}{a}\right) = f(A) + \frac{2\pi}{a}. \quad (7.58)$$

Using the above note that

$$\begin{aligned} \exp \left[ib_n f\left(-\frac{\pi}{a}\right) \right] &= \exp \left[ib_n f\left(\frac{\pi}{a} - \frac{2\pi}{a}\right) \right] \\ &= \exp \left[ib_n f\left(\frac{\pi}{a}\right) - ib_n \frac{2\pi}{a} \right] \\ &= \exp \left[ib_n f\left(\frac{\pi}{a}\right) - 2\pi i n \right] \\ &= \exp \left[ib_n f\left(\frac{\pi}{a}\right) \right]. \end{aligned} \quad (7.59)$$

Now it quickly follows that $\langle s\tilde{\Gamma} \rangle$ still vanishes

$$\begin{aligned} \langle s\tilde{\Gamma} \rangle &= \sum_{n=-\infty}^{\infty} \exp \left[-\frac{\xi}{2} b_n^2 \right] F\left(\frac{\pi}{a}, b_n\right) \left(\exp \left[ib_n f\left(\frac{\pi}{a}\right) \right] - \exp \left[ib_n f\left(-\frac{\pi}{a}\right) \right] \right) \\ &= 0. \end{aligned} \quad (7.60)$$

This means that f will now have an odd number of zeros in $[-\frac{\pi}{a}, \frac{\pi}{a}]$ and therefore \mathcal{Z} will not vanish and the Gribov problem will be avoided.

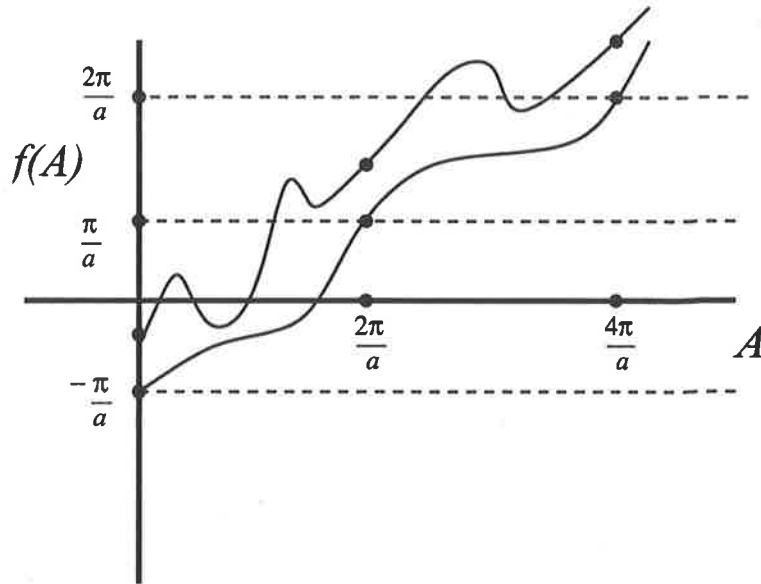


Figure 7.2: Two examples of functions periodic up to a shift of $\frac{2\pi}{a}$. The lower function has one zero and the other has three zeros and we can see that in general there will be an odd number of zeros

7.4 The Neuberger Problem Revisited

Let us return to examining the details of the Neuberger problem in order to understand better how it arises[27].

Standard lattice gauge fixing consists of adding the following term to the action

$$s\Gamma = \alpha s(\bar{c}f) + \beta s(\bar{c}b). \quad (7.61)$$

An expectation value can be written as

$$\langle \mathcal{O} \rangle = \frac{\int d\mu \exp[-S_0(U) - \alpha s(\bar{c}f) + \beta s(\bar{c}b)] \mathcal{O}(U)}{\int d\mu \exp[-S_0(U) - \alpha s(\bar{c}f) + \beta s(\bar{c}b)]}. \quad (7.62)$$

Normally, $s\bar{c} = b$ and $sb = 0$ and the second term is just βb^2 . So all the dependence on the ghosts lies in the first term. This is the key point because if we set

α to zero, we get

$$\langle \mathcal{O} \rangle = \frac{\int d\mu \exp[-S_0(U) - \beta b^2] \mathcal{O}(U)}{\int d\mu \exp[-S_0(U) - \beta b^2]}. \quad (7.63)$$

We see that all dependence on the ghost is gone and therefore the Grassmann integrals give zero. Therefore when $\alpha = 0$ an expectation value of an observable is zero over zero (we must ensure that the remaining Gaussian integral over b does not diverge because in that case we must consider 0 times ∞ , which can be a finite number). This will be the case in general if the expectation value does not depend on α , which is necessarily the case because any physical observable must be independent on the gauge fixing parameters.

The solution to this problem is to ensure that ghost terms remain when α is set to zero then the ghost integrals will not vanish. This will happen if the transformation of the b field contains ghosts. Since b has ghost number zero, sb has ghost number one, so sb could involve one ghost or two ghosts and an anti-ghost or any other combination with overall ghost number one. Schaden[28][29] has proposed a lattice BRST symmetry where the b field transforms non trivially. This situation arises naturally when the lattice gauge theory is gauge fixed equivariantly. In the case considered in [29] an $SU(2)$ lattice gauge theory was reduced to a $U(1)$ lattice gauge theory.

Chapter 8

Summary and Conclusions

Recall that we began by reviewing QCD as a non-abelian gauge theory and then discussed the fundamentals of gauge fixing, while introducing the key concepts that are necessary to consider this problem. We saw that a functional integral over the space of all gauge fields could be divided into an integral over the space of all gauge orbits and an integral over all gauge transformations. It was deduced that the physical configuration space is the space of all gauge orbits, also referred to as a fundamental modular region (FMR), and any functional integral should be constrained to this space only. In order to do this a gauge fixing function was introduced, which was required to satisfy the property that it had one zero for each and every gauge orbit, a so called ideal gauge fixing function. We then explicitly showed how standard gauge fixing is done via the Fadeev-Popov procedure, which of course relied on the assumption that we could find such an ideal gauge fixing function. Particular examples of such gauge fixing functions, like Landau gauge, axial gauge and a generalised covariant gauge were then discussed.

In the following chapter we found that our earlier assumption was not justified and we saw that at least in Coulomb gauge for an $SU(2)$ gauge theory this was not the case. The problem was seen to be equivalent to finding solutions to a damped

pendulum in a gravitational field. Then we showed that in perturbative QCD the Gribov problem is irrelevant and correct results can be obtained using the standard gauge fixing techniques. In chapter 4, the problem of finding the FMR that lies within the set of gauge fields that satisfies the Landau gauge condition. It was found that the local minima of the norm functional satisfied Landau gauge and that the Fadeev-Popov operator was positive. Therefore the Gribov region was defined to be all those gauge fields which satisfy these conditions. The Gribov problem says that there are gauge equivalent gauge fields inside the Gribov region and this was indeed seen to be the case. Thus it was necessary to constrain the Gribov region further in order to search for a FMR, therefore the absolute minima of the norm functional was considered as a possible FMR. After showing that this region was convex, it was seen that the Gribov copies lay on the boundary of this region and they corresponded to degenerate absolute minima. Therefore we found a FMR, which consisted of all nondegenerate absolute minima and includes only one of each of the gauge equivalent degenerate absolute minima. This gives the FMR a non trivial feature, since exactly how we are to choose from the degenerate minima is a non trivial task.

We then moved on to discussing gauge fixing in a lattice model, where we merely introduced the basic ideas of lattice gauge theory and gauge fixing on the lattice. We concluded that gauge fixing was only required on the lattice for gauge dependent quantities, such as the gluon propagator. It was also mentioned that there exists a Gribov copy free, but nonlocal gauge fixing scheme on the lattice.

The second part of this thesis was concerned with gauge fixing in a BRST symmetric theory and began with an explanation of the fundamental concepts of BRST symmetry. The Fadeev-Popov gauge fixed Lagrangian was obtained by making a particular choice of a BRST exact term to be added to the ungauged action. Therefore gauge fixing in the BRST formalism was found to be a more general and natural method, compared to that of the Fadeev-Popov procedure.

We then discussed BRST quantisation where BRST symmetry was elevated to the guiding principle in the quantisation process and thus we found the most general BRST invariant Lagrangian. We found that in a BRST invariant theory the physical observables are those BRST invariant functions of the fields, with ghost number zero, such that they cannot be written as the BRST variation of something else. So physical observables are exactly the BRST cohomology.

Chapter seven consisted of an investigation of the Neuberger problem. Firstly, BRST symmetry was defined in the lattice setting and then the Neuberger problem was proved explicitly as Neuberger had done previously. In order to gain further insight into the problem we considered a simple lattice model, where calculations could be carried out explicitly. This U(1) model consisted of one lattice link, the BRST transformations were derived and found to be of a simple form. A gauge fixing term was added to the action, which would correspond to the generalised covariant gauge. The gauge fixing function was restricted to be periodic when we required that the expectation value of a BRST exact object vanishes, as must be true in any BRST invariant theory. With this constraint the expectation value of some observable was calculated and found to be $\frac{0}{0}$. This is exactly what the Neuberger problem tells us it should be. In this case however we know the mechanism that produces this result. The numerator and denominator both consist of a sum over the Gribov copies weighted by the derivative of the gauge fixing function. Since the gauge fixing function is periodic the sign of the derivative will alternate and there will be an even number overall. Therefore the contributions from the Gribov copies will exactly cancel producing the indeterminate form of the expectation value. The solution to the Gribov problem in this case was found by redefining the delta function which imposed the gauge fixing constraint. When a periodic delta function was used instead of the usual delta function then the number of zeros of the gauge fixing function was odd and the same cancellation between contributions from the Gribov copies was impossible.

In a final remark it was noted that the Neuberger problem depended on the BRST variation of the auxiliary field being zero. Thus within a BRST framework where the BRST transformation of the auxiliary field involved ghost terms, the ghosts and the auxiliary field could not be decoupled and the Neuberger argument broke down. A simple generalisation from the lattice model discussed to such a BRST formalism was found not to be possible. However, Schaden has already put forward a BRST structure with the required property, and he claims that one can avoid the Neuberger problem through his equivariant lattice gauge theory. The obvious continuation of this work would be to examine the work of Schaden with the aim to further understand the nature of the Neuberger problem.

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