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The Methods of Newton and Leibniz in Their Development of Basic Calculus with Application to the Teaching of Calculus in Schools

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12th December, 1996

Contents

Abstract

A brief history of calculus, leading to Newton and Leibnizs' work in calculus, is given. The work by Newton and Leibniz, in the development of calculus, is investigated with the view of incorporating their methods in a new introductory calculus course for senior Australian school students. It is found that both mathematicians' early manuscripts on calculus are usable in the classroom. A course based mainly on Leibniz's methods, with Newton's study of motion for topics on rates of change, is proposed. Extension material for talented mathematics students is also presented.

Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

D J Woodard-Knight

December 1996

Acknowledgements

I would like to thank my supervisor Dr David Parrott for his comments and guidance throughout the year. Thanks also to Dr Peter Taylor for his comments and for proof reading this thesis. David Beard's ready assistance with LaTeX 2_ε and anything else in the field of computing has made the production of this thesis as a document possible.

Lastly I would like to thank my husband, David Capon, for his comments and patience, and my son Patrick for getting older.

Chapter 1

Introduction

Introductory calculus courses currently being taught in South Australian schools rely on students understanding limit theory in the early stages of their course. Limit theory is required to follow the definition for the derivative function, $f'(x)$ or $\frac{dy}{dx}$. This, stated in functional notation is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This definition arises from the moving of a chord through a curve keeping one point, P, stationary and the other point, Q, moving closer to it. See figure 1.1. If P has the coordinates $(x, f(x))$ and Q is $(x+h, f(x+h))$, where $h > 0$ then the slope of the chord through PQ is

$$\begin{aligned}\frac{ystep}{xstep} &= \frac{f(x+h) - f(x)}{x+h-x} \\ &= \frac{f(x+h) - f(x)}{h}.\end{aligned}$$

As h approaches zero, Q moves closer to P. So, in the limit, we have that the line through PQ is a tangent at P with slope

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

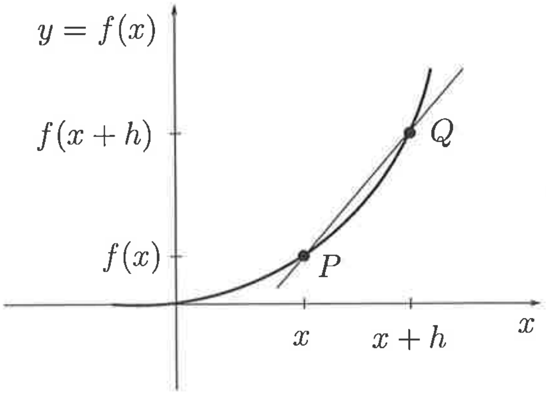

Figure 1.1: The chord PQ becomes a tangent at P as h approaches zero.

This is quite clear to those of us who have studied limit theory and calculus, but to the novice student it requires the introduction of new notation and new terminology, as well as the daunting theory of limits. The theory of limits always seems to present difficulties to students. It is this reason why an alternative course in calculus is proposed which does not rely on limits being taught as a separate topic during the introductory course in calculus, as is currently the case. The development of calculus by mathematicians such as Newton (1642 - 1727) and Leibniz (1646 - 1716) did not involve the use of limits. They did, however, use the concept of "infinitely small" steps or increments in their studies. These ideas are more easily taught and understood than limits. Students would also be removed from the situation of getting caught up in limit problems during the introduction of the concept of the derivative.

To introduce an alternative course in calculus, using Newton and Leibnizs' methods, it is not only necessary to investigate their original work, but also to review the work of earlier mathematicians who influenced the development of calculus. The next chapter presents a brief review of the development of calculus prior to Newton and Leibniz.

Historically, Zeno (c.450 BC) and his paradoxes on the infinite divisibility of magnitude were initially presented. Then the process, referred to now as the Method of Exhaustion, was developed by Eudoxus (c.370 BC). The method of exhaustion resulted from the work initiated by Antiphon (c.430 BC) in his attempt to find the area of a circle. He considered the circle to be composed of a regular polygon with the number of sides doubling until a circle was approximated. Eudoxus refined this method by producing more rigorous proofs. These proofs involved deriving contradictions for inequalities, using the method of exhaustion, such that the equality must hold. Using the same method of proof Archimedes (c.225 BC) showed that the area of a parabolic segment is four thirds of the triangle with the same base and vertex. This example, shown in section 2.4, displays the use of the method of exhaustion within a proof by contradiction.

After Archimedes the next development in calculus did not arise until the fifteenth century in Western Europe. Mathematicians such as Simon Stevin (1548 - 1620) in his study of hydrostatics; Luca Valerio (c.1552 - 1618) in his paper on finding areas of parabolae; Johann Kepler (1571 - 1630) in his study of planetary motion and volumes for capacity calculations of wine barrels, all contributed to methods for finding areas and volumes. This was the initial work in the field of calculus. Bonaventura Cavalieri (1598 - 1647) produced the concept of "indivisibles". That is, the very small parts which make up a line (the points), a surface (the lines) or a solid (the planes). The sum of these indivisibles produces the length, area and volume, respectively.

The need to find maxima and minima resulted in the process of differentiation. Initially, Pierre deFermat (c.1601 - 1665) produced a method for finding a tangent at a point, using what he called a subtangent. Isaac Barrow (1630 - 1677) developed the ratio, known today as $\frac{dy}{dx}$, in a similar style to Fermat. John Wallis (1616 - 1703) studied areas of circles and areas between curves. Using the area of a quadrant of a unit circle he attempted to calculate π.

The work of Newton in the field of differentiation and integration is presented in Chapter 3 and his methods are interpreted using modern mathematical language. Excerpts from his original manuscripts are presented. These excerpts highlight his initial work in finding the ratio of velocities of moving objects, his anti-differentiation technique and his method for finding the area under a curve. These are the basic concepts covered in an introductory calculus course. Similarly, in Chapter 4, the translated works of Leibniz in his development of differences (dx, say), sums ($\int$) and their relationship ($d \int x = x$ and $x = \int dx$), as well as his introduction of the integral sign is presented. Also, his method for finding the area under a curve, using his previously introduced sums and differences, is presented. The modern interpretation of his techniques result in a clear and easy introduction to calculus.

Within Chapter 5 Newton's methods and their possible inclusion in a modern calculus course are discussed in detail. Similarly, the style and methods of Leibniz are studied. Finally, parts of the work of both mathematicians are utilised to produce an alternative course in introductory calculus. Mainly

Leibniz's methods are used to set the course, but the style of Newton in his study of moving bodies and velocity vectors provide an opportunity to discuss rates of change. Their original manuscripts provide useful templates for investigative work for talented mathematics students. Ideas for such extension material are proposed within the final section of Chapter 5.

The incentive behind this work was to present an interesting and innovative introductory calculus course. Using the original works of mathematicians who had a major role in the development of calculus presents the opportunity for students to see for themselves how the pioneers of calculus thought about problems and how they solved them. It also presents an exciting opportunity for students to read Newton and Leibnizs' actual writings on calculus. Teachers of secondary mathematics are often seeking methods for teaching topics which will motivate and stimulate students. The course proposed here, along with ideas for extension material to offer talented students, provides teachers with the opportunity to introduce calculus in a style which will stimulate and interest students.

Chapter 2

The Introduction of Calculus

It is interesting to note that most courses in mathematics at the senior high school level introduce the topic of calculus by first considering differentiation, with integration following. This is in contrast to the historical development of calculus which arose from the need to find areas, volumes and arc lengths, resulting in the creation of integration. Differentiation, on the other hand, arose later as the result of problems requiring tangents to curves and questions concerning maxima and minima. Actually assigning a date and time for the beginning of calculus is impossible, and it is incorrect to say that it is due to Newton (1642 - 1727) and Leibniz (1646 - 1716) alone. Many mathematicians produced work which can be regarded as necessary beginnings to prompt the thinking of later mathematicians. Some historians suggest looking as far back as ancient Greece in the fifth century BC. Below we will briefly consider the contribution to calculus of mathematicians before Newton and Leibniz. The following books were used for information regarding the works of the mathematicians considered: Boyer [1], [2], Smith [10], [11] and Struik [12], [13].

2.1 Zeno (c.450 BC)

Zeno was a philosopher who proposed four paradoxes which were to have a profound effect on mathematics. These particular paradoxes are related to how one thinks of magnitude : magnitude being infinitely divisible, or magnitude being made up of a very large number of small indivisible atomic parts. The following are examples from Eves [6] which illustrate two of the paradoxes :

The Dichotomy

If a straight line segment is infinitely divisible then motion is impossible, for in order to traverse the line segment it is necessary first to reach the midpoint, and to do this one must first reach the one-quarter point, and to do this one must first reach the one-eighth point, and so on, ad infinitum. It follows that the motion can never begin.

The Arrow

If time is made up of indivisible atomic instants, then a moving arrow is always at rest, for at any instant the arrow is in a fixed position. Since this is true of every instant it follows that the arrow never moves.

As a result of these paradoxes there was the development of two schools of thought in mathematics - those following the concept of infinitely divisible magnitudes and those supporting the concept of large numbers of small indivisible atomic parts as the composition of magnitudes.

2.2 Antiphon (c.430 BC)

The first types of problem arising in the calculus area were concerned with finding areas, volumes and arc lengths. Antiphon, who lived at the same time as Socrates, attempted the problem of squaring the circle. That is, finding the area of a circle, by constructing with pencil and compass a square with the same area as a given circle.

Antiphon approached the problem of finding the area of a circle by considering a regular polygon inscribed in a circle and doubling the number of sides of the polygon until a very close approximation of a circle is obtained. The difference in area between the circle and the polygon would eventually be negligible. In Antiphon's time it was possible to construct a square equal in area to any regular polygon. So, he proposed that it is possible to construct a square equal in area to the circle (hence, the use of the term "squaring the circle"). There was much criticism against this argument since it did not hold with the concept of infinitely divisible magnitudes. Supporters of this train of thought held that the whole circle could never be used by the polygon inscribed within it and therefore it is not possible to calculate the area of the circle. This lead to the idea of the Method of Exhaustion which can be used in answering Zeno's paradoxes.

2.3 Eudoxus (c.370 BC)

Although Antiphon did the preliminary work towards the method of exhaustion, it is generally regarded as being due to Eudoxus. This method assumes that a magnitude is infinitely divisible and has the following property:

If from any magnitude there be subtracted a part not less than its half, from the remainder another part not less than its half, and so on, there will at length remain a magnitude less than any preassigned magnitude of the same kind. (Eves [6])

In modern notation this is the same as the following (Boyer [2]):

that if M is a given magnitude, ε is a preassigned magnitude of the same kind, and r is a ratio such that $\frac{1}{2} \leq r < 1$, then we can find a positive integer N such that $M(1-r)^n < \varepsilon$, for all positive integers $n > N$.

That is,

$$\lim_{n \to \infty} M(1-r)^n = 0.$$

According to Archimedes, Eudoxus was the first to use the method of exhaustion by proving that the volume, V, of a tetrahedron is equal to one third the volume, P, of a prism of equal base and height. He assumed that $V > \frac{1}{3}P$ and then $V < \frac{1}{3}P$ and using the method of exhaustion derived contradictions. Hence the equality must hold. This method of proving the equality by producing contradictions for the inequalities was referred to as a *reductio ad absurdum*.

The earlier work of Antiphon, involving the inscribing of regular polygons within a circle and doubling the number of sides indefinitely to find the area of the circle, was made more rigorous by Eudoxus' method of exhaustion. The proof (by the method of exhaustion) that the areas of circles in ratio equal the ratio of their respective squared diameters is given by Euclid in his text *Elements XII.2* (or see Calinger [3], pp136,137) and is probably the work of Eudoxus. Following is a more modern version of the proof (Boyer [2], pp101,102) :

Consider circles c_1 and c_2 with diameters d_1 and d_2, and areas A_1 and A_2. Prove

$$\frac{A_1}{A_2} = \frac{d_1^2}{d_2^2}.$$

Assume

$$\frac{A_1}{A_2} > \frac{d_1^2}{d_2^2}.$$

Then there is a magnitude $A_1' < A_1$ such that

$$\frac{A_1'}{A_2} = \frac{d_1^2}{d_2^2}.$$

Let $A_1 - A_1'$ be a preassigned magnitude $\varepsilon > 0$. Now inscribe, within circles c_1 and c_2, regular polygons of area P_1 and P_2. The polygons have the same number of sides. By doubling the number of sides of these polygons, the area between the polygon and its respective circle would decrease by more than half. The method of exhaustion then says this difference in areas can be reduced by indefinitely doubling the number of sides until $A_1 - P_1 < \varepsilon$.

Since $A_1 - A_1' = \varepsilon$, then $P_1 > A_1'$ using the previous inequality. From Proposition 1 in Book XII of the *Elements*, Euclid says that if two rectilinear figures are similar their areas are in the same ratio as the squares on corresponding sides. In this case, then,

$$\frac{P_1}{P_2} = \frac{d_1^2}{d_2^2}.$$

Now, since $A_1'/A_2 = d_1^2/d_2^2$, then

$$\frac{P_1}{P_2} = \frac{A_1'}{A_2}. \tag{2.1}$$

It has been shown above that $P_1 > A_1'$, so it follows from (2.1) that

$$P_2 > A_2.$$

This is false as the polygon with area P_2 is inscribed <u>within</u> the circle A_2 and therefore cannot be of greater area. Hence it is disproved that

$$\frac{A_1}{A_2} > \frac{d_1^2}{d_2^2}.$$

Similarly it can be shown that

$$\frac{A_1}{A_2} < \frac{d_1^2}{d_2^2}$$

is false. Therefore, the equality holds. That is,

$$\frac{A_1}{A_2} = \frac{d_1^2}{d_2^2}.$$

2.4 Archimedes (c.225 BC)

Archimedes came very close to defining integration when finding the area of a parabolic segment using the method of exhaustion and the *reductio ad absurdum* approach. He showed the area of a parabolic segment is four thirds of the triangle with the same base and vertex.

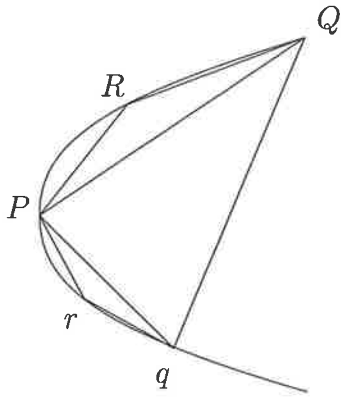

Figure 2.1: The parabolic segment QPq used by Archimedes in Proposition 22 and in his proof on page 15, that a parabolic segment is four thirds of the triangle with the same base and height.

To assist his proof he made the following propositions (Heath [9]):

Proposition 22

If there be a series of areas $A, B, C, D, \ldots$ each of which is four times the next in order, and if the largest, A, be equal to the triangle PQq inscribed in a parabolic segment PQq and having the same base with it and equal height, then $(A + B + C + D + \ldots)$ is less than the area of [parabolic] *segment PQq.*

See figure 2.1.

In an earlier proposition Archimedes proved that

$$\triangle PQq = 4(\triangle PQR + \triangle Pqr).$$

Then, since

$$A = \triangle PQq,$$

we have

$$\triangle PQR + \triangle Pqr = B.$$

The symbol, $\triangle$, refers to the area of the triangle.

In like manner we prove that the triangles similarly inscribed in the remaining segments are together equal to the area C, and so on.

Therefore $A + B + C + D + \ldots$ is equal to the area of a certain inscribed polygon, and is therefore less than the area of the segment.

That is, we have

$$\begin{aligned} B &= \frac{1}{4}A, \\ C &= \frac{1}{4}B = \frac{1}{4^2}A, \\ D &= \frac{1}{4}C = \frac{1}{4^3}A, \quad \text{and so on,} \end{aligned}$$

where A is the area of the triangle QPq. So,

$$A(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots) < \text{the area of the segment.}$$

Proposition 23

Given a series of areas $A, B, C, D, \ldots, Z$, of which A is the greatest, and each is equal to four times the next in order, then

$$A + B + C + D + \ldots + Z + \frac{1}{3}Z = \frac{4}{3}A.$$

Archimedes proves this proposition by partial sums, since the concept of the sum of a geometric progression was not known.

He considered areas a, b, c, d, such that

$$\begin{aligned} b &= \frac{1}{3}B, \\ c &= \frac{1}{3}C, \\ d &= \frac{1}{3}D, \ldots \end{aligned}$$

Then, since

$$\begin{aligned} b &= \frac{1}{3}B \\ \text{and} \quad B &= \frac{1}{4}A \end{aligned}$$

we have

$$B + b = \frac{1}{3}A.$$

Similarly,

$$C + c = \frac{1}{3}B$$

and so on for $D + d$, $E + e$, ...

So,

$$B + C + D + \cdots + Z + b + c + \cdots + z = \frac{1}{3}(A + B + C + \cdots + Y). \tag{2.2}$$

We know that

$$b + c + d + \cdots + y = \frac{1}{3}(B + C + D + \cdots + Y), \tag{2.3}$$

So, subtracting equation (2.3) from equation (2.2) will give

$$B + C + D + \cdots + Z + z = \frac{1}{3}A.$$

Add A to both sides to obtain

$$A + B + C + D + \cdots + Z + \frac{1}{3}Z = \frac{4}{3}A.$$

Now Archimedes has the necessary information to prove the proposition that *Every segment bounded by a parabola and a chord Qq is equal to four thirds of the triangle which has the same base as the segment and equal height.* (Fauvel & Gray [7]).

He begins his proof by letting $K = \frac{4}{3}\triangle PQq$, referring to figure 2.1 on page 12.

Suppose the area of the segment is greater than K. Inscribe, in the segments cut off by PQ and Pq, triangles with the same base and height (e.g. $\triangle PRQ$ and $\triangle Prq$). Continue inscribing triangles in the remaining segments and eventually the sum of the segments remaining is less than the area by which segment PQq exceeds K. So, the polygon formed must be greater than K which is impossible, since from Proposition 23

$$A + B + C + \cdots + Z < \frac{4}{3}A,$$

where $A = \triangle PQq$. Thus the supposition that K is less than the area of the segment is false.

Now suppose the area of the segment is less than K.

If

$$\begin{aligned} \triangle PQq &= A, \\ B &= \frac{1}{4}A, \\ C &= \frac{1}{4}B, \end{aligned}$$

and so on, until there is an area X such that

$$X < K - \text{area of the segment} \tag{2.4}$$

then,

$$\begin{aligned} A + B + C + \cdots + X + \frac{1}{3}X &= \frac{4}{3}A \quad \text{from Proposition 23} \\ &= K \end{aligned}$$

Now,

$$K - (A + B + C + \cdots + X) = \frac{1}{3}X \tag{2.5}$$

and from (2.4), there exists $n > 1$, $n \in R$ such that

$$K - \text{area of the segment} \;\; = \;\; nX. \tag{2.6}$$

So, (2.5) gives

$$K - \frac{1}{3}X = A + B + C + \cdots + X \tag{2.7}$$

and from (2.6)

$$K - nX = \text{area of segment}, \quad n > 1, n \in R$$

$$\Rightarrow A + B + C + \cdots + X > \text{area of the segment}$$

which is impossible from Proposition 22. Hence the segment is not less than K.

Thus,

$$K = \frac{4}{3}PQq.$$

It was not until around 1450 that Western Europe became aware of Archimedes' works, and any further development did not take place until the early seventeenth century. Mathematicians such as Simon Stevin, Luca Valerio, Johann Kepler, Bonaventura Cavalieri and Pierre deFermat in the first half of the seventeenth century, and John Wallis and Isaac Barrow working later in that century, contributed to the development of calculus prior to Newton and Leibniz. A brief account of some of their works follows.

2.5 Simon Stevin (1548 - 1620)

Stevin, an engineer from Belgium, worked in the area of hydrostatics. He found the force against a dam wall, due to the pressure of fluid, by dividing the dam into horizontal strips. This method is similar to the modern approach.

2.6 Luca Valerio (c.1552 - 1618)

Valerio was an Italian mathematician who published a paper in 1606 titled *De quadratura parabolae*. In this paper he uses similar methods to Archimedes to find the area underneath parabolae.

2.7 Johann Kepler (1571 - 1630)

Kepler was involved in the study of planetary motion and required a method to find areas related to this work. He also required a method for finding volumes for his work on capacities of wine barrels. To calculate the area of a circle he considered the circumference as the infinite number of sides of a regular polygon. See figure 2.2. These sides represented the base of a triangle with altitude equal to the radius of the circle. The area of the circle then is the sum of all these triangles. That is, the area of the circle is equal to half the product of its circumference and radius.

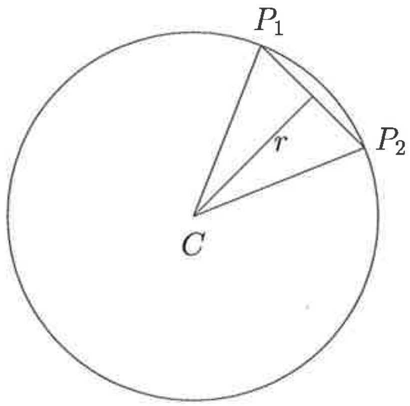

Figure 2.2: Kepler's diagram for finding the area of a circle. Length P_1P_2 is infinitely small, such that the altitude, r, is the radius of the circle with centre C.

To calculate the volume of a sphere Kepler considered an infinite number of cones, with the base of the cones on the sphere surface and their vertices at the centre of the sphere.

2.8 Bonaventura Cavalieri (1598 - 1647)

Cavalieri, an Italian mathematician, established the concept of *indivisibles* to produce a simple form of calculus. Parts which make up an object are the *indivisibles.* For instance, solids are made up of infinitely many planes, surfaces are made up of infinitely many lines and lines are composed of infinitely many points. The sum of the *indivisibles* then produce volumes, areas and lengths, respectively. Cavalieri stated the theorem:

If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, then the volumes of the solids are also in this ratio.

[Smith [10]]

An example of the use of this theorem can be seen in the following problem to find the volume of a sphere:

Consider an hemisphere of radius r and a cylinder with radius r and height r. Inscribe within the cylinder a cone, such that the base of the cone is the upper surface of the cylinder and the vertex of the cone is the centre of the lower base of the cylinder. See figure 2.3. Consider this cone as being taken out of the cylinder. Now place the cylinder with cone removed and the hemisphere on the same plane and cut the solids by a line parallel to the base plane, at a distance h up from it.

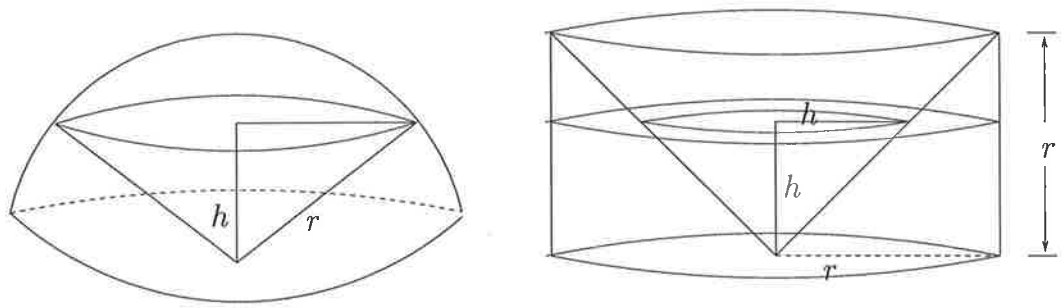

Figure 2.3: Cavalieri used the hemisphere and cylinder with cone removed, cut by a plane at height h, to show that they have equal volumes.

The plane cuts the hemisphere to produce a cross-sectional shape of a circle and the cylinder to produce an annulus as the cross-section. The areas of these resultant cross-sections are $\pi(r^2 - h^2)$. Cavalieri's theorem then implies the hemisphere, and cylinder with cone removed, have equal volumes. From this information the cylinder with cone removed may be used to find an expression for the volume of a sphere. That is

$$\begin{aligned} V &= 2(\text{volume of cylinder} \quad - \quad \text{volume of cone}) \\ &= 2(\pi r^2.r - \frac{\pi r^2.r}{3}) \\ &= \frac{4\pi r^3}{3}. \end{aligned}$$

Differentiation developed due to the need to find maximum and minimum values in problems and also to allow the construction of tangents.

2.9 Pierre deFermat (c.1601 - 1665)

Pierre deFermat was the first to establish ideas in this direction in 1629. In 1638 he communicated a method to Descartes (1596 - 1650) regarding finding the maximum and minimum. According to modern notation he equated $f'(x)$ to zero to find maxima and minima. He also established a method for finding the tangent at a point of a curve, using the subtangent of the point. See figure 2.4. The subtangent is the segment on the x-axis, labelled a, between the foot of the perpendicular drawn down from the point to the x-axis and the intersection of the tangent line with the x-axis.

Let the curve be $f(x, y) = 0$. Through a point (x, y) on the curve draw a tangent line meeting the x-axis. Let e be a very small distance in the x direction and place a point on the tangent line with x coordinate $x+e$. Then the length of p in the diagram is $y(1 + \frac{e}{a})$.

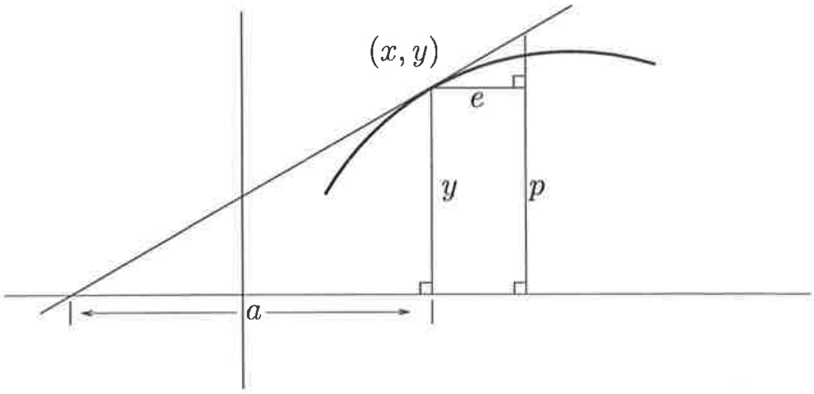

Figure 2.4: deFermat used the subtangent, labelled a, to find the tangent. The tangent slope can be found, once a has been found. Then the equation of the tangent may be found using the point (x, y).

Example

Find the subtangent of

$$x^3 + y^3 = xy \quad \text{i.e. } x^3 + y^3 - xy = 0$$

Substitute $((x+e), y(1+\frac{e}{a}))$ for a point on the tangent very close to the point on the tangent and curve to obtain

$$(x+e)^3 + y^3(1+\frac{e}{a})^3 - y(x+e)(1+\frac{e}{a}) = 0.$$

Now use $f(x, y) = 0$, divide by e, and then let $e = 0$ to obtain the following expression for the subtangent:

$$a = \frac{3y^3 - xy}{y - 3x^2}.$$

The method of dividing by a number which later is used as zero resulted in much criticism against calculus. For instance George Berkeley (1685 - 1753), an Anglican Minister and Philosopher, wrote *The Analyst* in 1734 in which he criticised the use of increments that vanish after a former supposition that they were something. He argued that this *is a false way of reasoning.* [Fauvel & Gray, [7]]

It was not until after Newton that mathematicians became more rigorous in their proofs in calculus. Colin MacLaurin (1698 - 1746) responded to Berkeley's criticism in a paper written in 1742 and provided the basis for a more rigorous approach to the calculus as set up by Newton, Leibniz and their predecessors.

2.10 John Wallis (1616 - 1703)

In 1656 Wallis produced a book called *Arithmetica Infinitorum.* Here he developed a method for using infinite series to assist in solving the problem of finding the area between curves.

In modern notation we write

$$\int_0^1 x^m dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}.$$

Wallis showed this held for m being a positive or negative integer (except for $m = -1$) and for m fractional. He also attempted to calculate π by finding an expression for the area of a quadrant of the unit circle. He did not have knowledge of the binomial theorem, so to calculate the area of the quadrant as $\int_0^1 (1-x^2)^{\frac{1}{2}} dx$ was not possible.

2.11 Isaac Barrow (1630 - 1677)

In 1669 Barrow produced his most important work in the area of calculus in *Lectiones opticae et geometricae.* He contributed to the development of differentiation by providing a method to find a ratio which today we refer to as $\frac{dy}{dx}$. A summary of his method follows.

Figure 2.5 illustrates the constructions required. Given a curve find the slope of the tangent at point P. Let Q be a point close to P on the curve and construct triangle PQR.

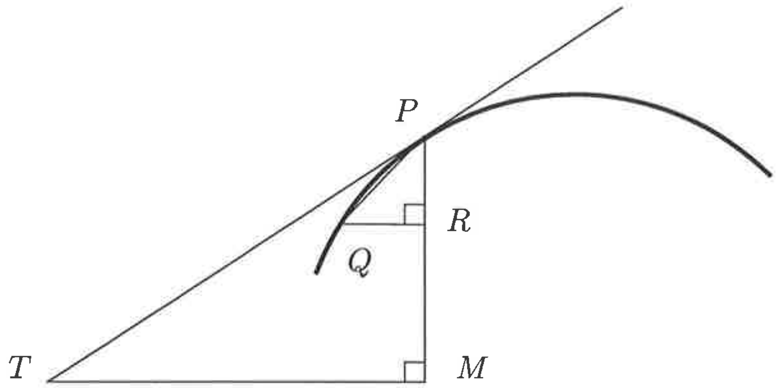

Figure 2.5: Barrow finds the ratio $\frac{a}{e}$ (the modern $\frac{dy}{dx}$), which in this diagram is the ratio of sides PR and QR.

He says triangle PTM is nearly similar to triangle PQR, and that the closer Q is to P then

$$\frac{RP}{QR} = \frac{MP}{TM}.$$

Let $QR = e$ and $RP = a$, then if P is labelled (x, y), Q is $(x - e, y - a)$. Substitute the coordinates for Q into the equation of the curve, neglect squares and higher powers of a and e and find the ratio $\frac{a}{e}$. With the application of limit theory, unknown to Barrow, this method can be made more rigorous. It is interesting to note the similarity between the methods of Fermat and Barrow.

Example

$$x^3 + y^3 = r^3$$

where r is constant. Let x become $x - e$ and y become $y - a$ and substitute into the above equation. Then

$$(x - e)^3 + (y - a)^3 = r^3.$$

So,

$$x^3 - 3x^2e + 3e^2x - e^3 + y^3 - 3y^2a + 3a^2y - a^3 = r^3.$$

Neglecting powers of e^2 and a^2 and higher, and using $x^3 + y^3 = r^3$ results in the following equation

$$3x^2e + 3y^2a = 0.$$

The required ratio is then,

$$\frac{a}{e} = \frac{-x^2}{y^2}.$$

A recognised symbolism and defined sets of rules was needed to tidy up all the preliminary work in calculus. Leibniz and Newton were to provide this for calculus. Their methods are observed in the next chapters. The fundamental basis of calculus required more rigorous proofs and this was provided by the work of Colin MacLaurin, Augustin-Louis Cauchy (1789 - 1857) and mathematicians of the nineteenth century.

Chapter 3

Isaac Newton (1642 - 1727)

Throughout this chapter there are sections cited from Newton's translated papers found in Whiteside's volumes: *The Mathematical Works of Isaac Newton* [14] and *The Mathematical Papers of Isaac Newton* [15]. Newton's short hand requires some explanation before reading further. Words such as "which", "the" and "that" are abbreviated to "w^{ch}", "y^{e}" and "y^{t}", respectively. Spelling is different in some cases, for instance, "uniformely" (uniformly), "bee" (be), "onely" (only), "terme" (term). He sometimes begins sentences in lower case, or with "And", and some of the grammar he uses is not practised in modern English. This all makes reading his manuscripts quite difficult.

Newton referred to variable quantities, such as x and y, as *fluents* and denoted their respective *fluxions* as $\dot{x}$ and $\dot{y}$. In modern notation, the fluxion of x is $\frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$. Newton used the letters p, q and r for fluxions until 1691. That is,

$$p = \frac{dx}{dt},$$
$$q = \frac{dy}{dt},$$

and

$$r = \frac{dz}{dt}.$$

Some of the quotations from Whiteside's [14], [15] volumes in this section use the $\dot{x}$ notation so that arguments are more readily followed. Newton used an arbitrary increase in time, denoted $\circ$ and called it *little zero*. Consequently, $\circ p$, $\circ q$ and $\circ r$ represent increments of the variables x, y and z. In modern notation we call $\circ$, dt giving

$$\begin{aligned} \circ p &= dt\frac{dx}{dt} \\ &= dx \end{aligned}$$

and similarly

$$\begin{aligned} \circ q &= dt\frac{dy}{dt} \\ &= dy \end{aligned}$$

and

$$\begin{aligned} \circ r &= dt\frac{dz}{dt} \\ &= dz. \end{aligned}$$

As the increment vanishes we have,

$$\frac{q}{p} = \frac{dy}{dx}$$

and

$$\frac{r}{p} = \frac{dz}{dx}.$$

For further simplification Newton often chose x to be the independent time - variable with $p = 1$. Then the increments $\circ q$ and $\circ r$ of y and z, with $\circ$ as the increment of x, give

$$q = \frac{dy}{dx}$$

and

$$r = \frac{dz}{dx}.$$

In October 1666 Newton organised his research in calculus from the previous two years (autumn 1664 - May 1666). Whiteside [15] called this work *The October 1666 Tract on Fluxions.* It was first printed in A R Hall and Marie Boas Halls' *Unpublished Scientific Papers of Sir Isaac Newton* [8]. It is this paper that supplies most of the information on Newton's development of calculus.

3.1 Newton's Calculus

Newton's approach to the problem of finding tangents to curves was to consider the x and y coordinates in motion and therefore as functions of time. So, in fact, the curve $f(x, y) = 0$ is the locus of the intersection of a moving vertical line and a moving horizontal line. The movement of a point on the curve can be described by horizontal motion with particular velocity $\dot{x}$ and vertical motion with velocity $\dot{y}$. From figure 3.1 it can be seen that the tangent vector is produced by the vector sum of $\dot{x}$ and $\dot{y}$ and the slope of the tangent vector is $\frac{\dot{y}}{\dot{x}}$.

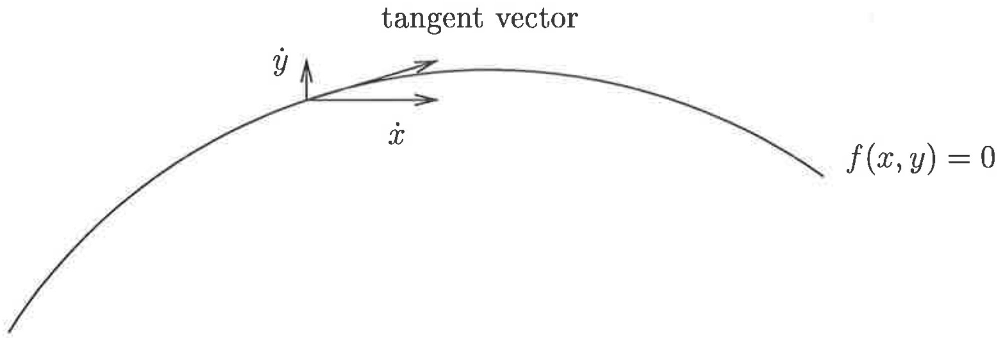

Figure 3.1: The tangent vector, horizontal velocity and vertical velocity components of a curve.

Newton presented a geometric model, shown in figure 3.2, in which two or more points A and B travel distances x and y along different straight lines, in equal periods of time, with speeds $\dot{x}$ and $\dot{y}$ respectively, such that $f(x, y) = 0$.

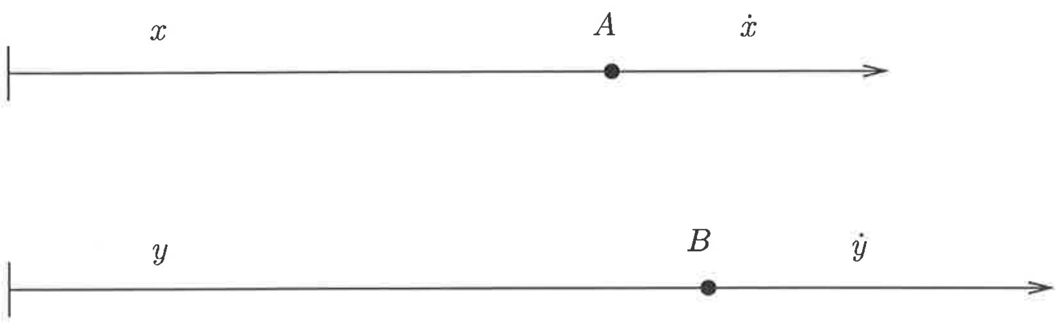

Figure 3.2: Newton's geometric model for the two points A and B moving along different lines in the same space of time.

Newton wanted to find the relationship between $\dot{x}$ and $\dot{y}$, given $f(x, y) = 0$. In *The October 1666 Tract* he writes the following proposition:

Proposition 7

Haveing an Equation expressing y^e relation twixt two or more lines x, y, z &c: described in y^e same time by two or more moveing *bodys A, B, C &c: the relation of their velocities p, q, r &c may bee thus found, viz: Set all y^e termes on one side of y^e Equation that they become equall to nothing. And first multiply each terme by so many times $\frac{p}{x}$ as x hath dimensions in y^t terme. Secondly multiply each terme by so many times $\frac{q}{y}$ as y hath dimensions in it. Thirdly (if there be 3 unknowne quantitys) multiply each terme by so many times $\frac{r}{z}$ as z hath dimensions in y^t terme, (if there bee still more unknowne quantitys doe like to every unknowne quantity). The summe of all these products shall bee equall to nothing. W^{ch} Equation gives y^e relation of y^e velocitys p, q, r &c.*

An example given by Newton of this method is: If $aa + xx - yy = 0$, then

$$2\frac{p}{x}.xx - 2\frac{q}{y}.yy = 0.$$

Here, he has treated a as constant and his notation for x^2 is xx, and similarly for y^2 (although he sometimes does write x^2). Hence the dimensions of x and y are 2 and the multipliers $2\frac{p}{x}$ and $2\frac{q}{y}$ result from his method.

His resultant relationship of velocities is given by

$$x : y = q : p.$$

Example

Given $y = x^3$, find the relationship of the velocities.

Let

$$f(x, y) = y - x^3 = 0.$$

This can be written as

$$x^0 y^1 - x^3 y^0 = 0.$$

So, using Newton's Proposition 7 and using $\dot{x}$ and $\dot{y}$ rather than p and q, we have

$$(0.\frac{\dot{x}}{x} + 1.\frac{\dot{y}}{y})x^0y^1 - (3.\frac{\dot{x}}{x} + 0.\frac{\dot{y}}{y})x^3y^0 = 0$$

or

$$\dot{y} - 3x^2\dot{x} = 0.$$

So

$$\frac{\dot{y}}{\dot{x}} = 3x^2 \quad \text{is the relationship required.}$$

In modern mathematics we would not consider the following demonstration by Newton (Whiteside [15]) a valid proof of Proposition 7. It was mentioned earlier that a major criticism of Newton's work was his lack of rigorous proofs.

Proposition 7 Demonstrated

Lemma. If two bodys A, B move uniformely [see figure 3.3] *y^e $\begin{matrix}\textit{one}\\ \textit{other}\end{matrix}$ from $\begin{matrix}a\\ b\end{matrix}$ to $\begin{matrix}c, d, e, f,\\ g, h, k, l,\end{matrix}$ &c: in y^e same time. Then are y^e lines $\begin{matrix}ac, \&cd, \&de, \&ef\\ bg, \&gh, \&hk, \&kl'\end{matrix}$ &c: as their velocitys $\frac{p}{q}$. And though they move not uniformely yet are y^e infinitely little lines w^{ch} each moment they describe, as their velocitys w^{ch} they have while they describe y^m. As if y^e body A wth y^e velocity p describe y^e infinitely little line $(cd =)p \times o$, in y^t moment y^e body B w^{th} y^e velocity q will describe y^e line $(gh =)q \times o$. For $p : q :: po : qo$. Soe y^t if y^e described lines bee $(ac =)x$, & $(bg =)y$, in one moment, they will bee $(ad =)x + po$, & $(bh =)\ y + qo$ in y^e next.*

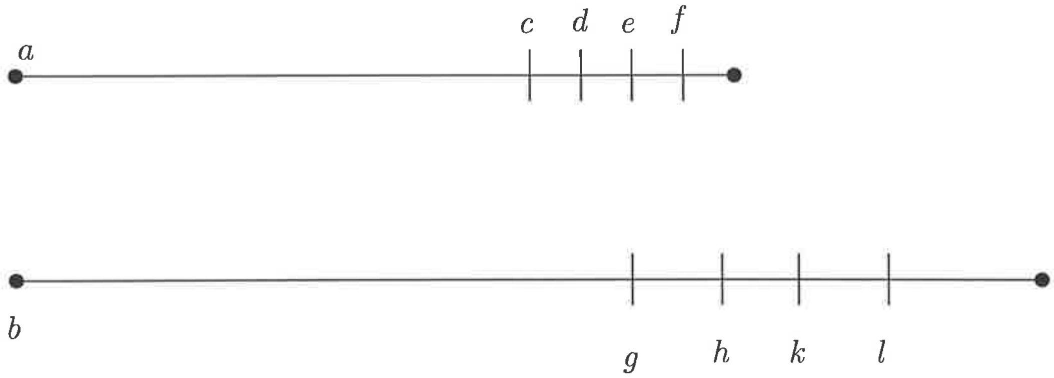

Figure 3.3: To assist the proof of Proposition 7 Newton used this diagram showing two bodies moving uniformly along different lines in the same time.

Newton follows his proof with the following demonstration. It is important to note that the "*d*" in the last term, $-dyy$, in his demonstration is not a derivative. The "*d*" represents a position through which body A moves as shown in figure 3.3. It is also worthy of note that Newton writes cubic powers as x^3, for instance, rather than xxx.

> *Demonstr: Now if* y^e *equation expressing* y^e *relation twixt* y^e *lines* x *&* y *bee* $x^3 - abx + a^3 - dyy = 0$. *I may substitute* $x + p\circ$ *&* $y + q\circ$ *into* y^e *place of* x *&* y*; because (by* y^e *lemma) they as well as* x *&* y*, doe signify* y^e *lines described by* y^e *bodys* A *&* B*. By doeing so there results*
>
> $$x^3 - 3p\circ xx + 3pp\circ\circ x + p^3\circ^3 - abx\; abp\circ + a^3 - dyy - 2dq\circ y - dqq\circ\circ = 0.$$
>
> *But* $x^3 - abx + a^3 - dyy = 0$ *(by supp.). Therefore there remaines onely*
>
> $$3p\circ xx + 3pp\circ\circ x + p^3\circ^3 - 2dq\circ y - dqq\circ\circ - abp\circ = 0.$$

Or dividing by $\circ$ *tis*

$$3px^2 + 3pp \circ x + p^3 \circ \circ - 2dqy - dqq \circ - abp = 0.$$

Also those termes are infinitely little in w^{ch} $\circ$ *is. Therefore omitting them there rests* $3pxx - abp - 2dqy = 0$. *The like may bee done in all other equations.*

Hence I observe. First y^t *those termes ever vanish* w^{ch} *are not multiplyed by* $\circ$, *they being* y^e *propounded equation. Secondly those termes also vanish in* w^{ch} $\circ$ *is of more* y^n *one dimension, because they are infinitely lesse* y^n *those in* w^{ch} $\circ$ *is but of one dimension.*

The expression $x + p\circ$ used in Newton's demonstration is $x + \dot{x}\circ$, or in modern notation $x + \frac{dx}{dt}.dt$. Similarly $y + q\circ$ is $y + \frac{dy}{dt}.dt$. These new terms may be substituted into the original equation in place of x and y since, by the Lemma given on page 30, $x + \frac{dx}{dt}.dt$ and $y + \frac{dy}{dt}.dt$ represent the lines x and y an infinitely small moment later. Upon simplification, and neglecting terms in $(dt)^2$ and higher, the following expression for the relationship between velocities results:

$$\dot{y} : \dot{x} = (3x^2 - ab) : 2dy$$

or

$$\frac{dy}{dt} : \frac{dx}{dt} = (3x^2 - ab) : 2dy$$

or

$$\frac{dy}{dx} = \frac{3x^2 - ab}{2dy}.$$

Again note that the "d" in the term $2dy$ is a position through which the body A moves and is not to be confused with the modern notation for derivative.

To cater for more difficult problems Newton introduced what is now referred to as the chain rule. He describes the process in the following, taken from Whiteside [15]:

> *Note y^t if there happen to bee in any Equation either a fraction or surde quantity ... To find in what proportion the unknowne quantitys increase or decrease doe thus. 1 Take two letters the one (as ξ) to signify y^t quantity, y^e other (a π) its motion of increase or decrease: And making an Equation betwixt y^e letter (ξ) & y^e quantity signified by it, find thereby (by prop 7 ...) y^e valor of y^e other letter (π). 2 Then substituting y^e letter (ξ) signifying y^t quantity, into its place in y^e maine Equation esteeme y^t letter (ξ) as an unknowne quantity & performe y^e worke of [y^e] seavanth proposition; & into y^e resulting Equation instead of those letters ξ & π substitute theire valors. And soe you have y^e Equation required.*
>
> *Example 1. To find p & q y^e motions of x & y whose relation is, $yy = x\sqrt{aa - xx}$. first suppose $\xi = \sqrt{aa - xx}$ or $\xi\xi + xx - aa = 0$. & thereby find π y^e motion of ξ, viz:*
>
> *(by prop 7) $2\pi\xi + 2px = 0$. Or $\frac{-px}{\xi} = \pi = \frac{-px}{\sqrt{aa-xx}}$.*

Secondly in y^e Equation $yy = x\sqrt{aa - xx}$, writing ξ in stead of $\sqrt{aa - xx}$, the result is $yy = x\xi$, whereby find y^e relation of y^e motions $p, q,$ & π: viz (by prop 7) $2qy = p\xi + x\pi$.

In w^{ch} Equation instead of ξ & π writing theire valors, y^e result is, $2qy = p\sqrt{aa - xx} - \frac{pxx}{\sqrt{aa-xx}}$. W^{ch} was required.

Following the method described by Newton above, and using $p = \dot{x}, q = \dot{y}$ and $\pi = \dot{\xi}$, the example used by him can be presented thus:

Let $y^2 = x\sqrt{a^2 - x^2}$. Suppose $\xi = \sqrt{a^2 - x^2}$, then

$$\xi^2 - a^2 + x^2 = 0.$$

Applying Proposition 7 we obtain

$$2\xi\dot{\xi} + 2x\dot{x} = 0,$$

or

$$\begin{aligned} \frac{-x\dot{x}}{\xi} &= \dot{\xi} \\ &= \frac{-\dot{x}x}{\sqrt{a^2 - x^2}}. \end{aligned}$$

Now,

$$y^2 = x\sqrt{a^2 - x^2}$$

so that

$$y^2 = x\xi.$$

So, $2y\dot{y} = \dot{x}\xi + x\dot{\xi}$ using Proposition 7. Substitute for ξ and $\dot{\xi}$ to obtain

$$2y\dot{y} = \dot{x}\sqrt{a^2 - x^2} - \frac{x^2\dot{x}}{\sqrt{a^2 - x^2}}$$

as required.

After further rearranging, the ratio of the velocities can be found to be

$$\frac{\dot{y}}{\dot{x}} = \frac{a^2 - 2x^2}{2x^{1/2}(a^2 - x^2)^{3/4}}.$$

In Proposition 8 Newton presents a method for finding the anti-derivative, y, given the velocities $\dot{x}$ and $\dot{y}$, and the relation between x and $\frac{\dot{y}}{\dot{x}}$. He says that

> *Prop 8th is y^e^ Converse of this 7th Prop. & may bee therefore Analytically demonstrated by it.*
>
> [Proposition] *8 If two Bodys A & B, by their velocitys p & q describe* y^e *lines x & y. & an Equation bee given expressing* y^e *relation twixt one of the* y^e *lines x, &* y^e *ratio* $\frac{q}{p}$ *of their motions q & p; To find* y^e *other line y.*
>
> *Could this bee done all problems whatever might bee resolved. But by* y^e *following rules it may bee very often done. (Note* y^t $\pm m$ *&* $\pm n$ *are logarithmes or numbers signifying* y^e *dimensions of x.)*
>
> *First get* y^e *valor of* $\frac{q}{p}$*. Which if it bee rationall & its Denominator consist of but one terme: Multiply* y^t *valor of x & divide each terme of it by* y^e *logarithme of x in* y^t *terme* y^e *quote shall bee* y^e *valor of y.*

Example

If $\frac{q}{p} = ax^{\frac{m}{n}}$, multiply by x to get $ax^{\frac{m}{n}}x^1$.

Now divide by the power of x to get $\frac{ax^{\frac{m}{n}+1}}{\frac{m}{n}+1}$.

That is,

$$y = \frac{n}{m+n}.ax^{\frac{m+n}{n}}.$$

Newton follows Proposition 8 with many examples for the cases when $\frac{q}{p}$ is rational, irrational, a surd, or combinations of these. One example he gives is:

If

$$\frac{q}{p} = \frac{cx^{n-1}}{a+bx^n},$$

then

$$y = \Box \frac{c}{nab+nbz},$$

where $z = bx^n$ and the symbol $\Box$ is read as "the area of".

To explain how Newton came to derive a solution for y, consider the following:

$$\begin{aligned}\frac{q}{p} &= \frac{\dot{y}}{\dot{x}} \\ &= \frac{cx^{n-1}}{a+bx^n}.\end{aligned}$$

Now,

$$z = bx^n,$$

so that

$$\dot{z} = nbx^{n-1}.\dot{x}.$$

So we can write

$$\begin{aligned}\frac{\dot{y}}{\dot{z}} &= \frac{\dot{y}}{nbx^{n-1}.\dot{x}} \\ &= \frac{cx^{n-1}}{a+bx^n} \times \frac{1}{nbx^{n-1}} \\ &= \frac{c}{nb(a+bx^n)}.\end{aligned}$$

That is,

$$\frac{\dot{y}}{\dot{z}} = \frac{c}{nba+nbz}. \tag{3.1}$$

So we can say

$$y = \Box\frac{c}{nba+nbz}. \tag{3.2}$$

Equation (3.1) can be written as

$$\frac{dy}{dz} = \frac{c}{nba+nbz}.$$

So it can be seen that equation (3.2) is, in modern notation,

$$y = \int \frac{c}{nba+nbz}\,dz.$$

However, to be more accurate the area under a curve is given by a definite integral. So, Newton has really produced a method for the inverse of the chain rule, rather than finding an area.

In the case where the line to be found given the area involved hyperbolic or circular functions Newton used the binomial series expansion and then integrated each term. For example if

$$\frac{\dot{y}}{\dot{x}} = \frac{a}{b+cx},$$

then by using the binomial series expansion we get

$$\frac{\dot{y}}{\dot{x}} = \frac{a}{b} - \frac{acx}{b^2} + \frac{ac^2x^2}{b^3} - \ldots$$

which upon applying Proposition 8 results in

$$y = \frac{ax}{b} - \frac{acx^2}{2b^2} + \frac{ac^2x^3}{3b^3} - \ldots.$$

Problem 5 of *The October 1666 Tract* introduces a method for finding a line given the area. Newton actually uses the Fundamental Theorem of Calculus to solve the problem, that is

$$\int \frac{dy}{dx}\, dx = y,$$

where y is the area.

Prob 5^t. *To find* y^e *nature of* y^e *crooked line whose area is expressed by any given equation. That is;* y^e *nature of* y^e *area being given to find* y^e *nature of* y^e *crooked line whose area it is.*

To assist reading Newton's following solution consider figure 3.4.

Resol. If y^e *relation of* $ab = x$, *&* [area]$abc = y$ *bee given &* y^e *relation of* $ab = x$, *&* $bc = q$ *bee required (bc being ordinately applyed at right angles to ab).*

Make $de \parallel ab \perp ad \parallel be = 1$. *&* y^n *is* $\square abed = x$. *Now supposing* y^e *line cbe by parallel motion from ad to describe* y^e *two superficies* $ae = x$, *&* $abc = y$; *The velocity* w^{th} w^{ch} *they increase*

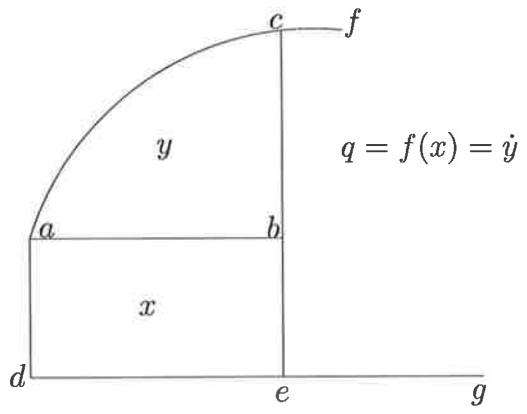

Figure 3.4: Given the area under the curve y, Newton finds the line which determines this area.

will bee, as be to bc: y^t *is,* y^e *motion by* w^{ch} x *increaseth being* $be = p = 1$, y^e *motion by* w^{ch} y *increaseth will bee* $bc = q$. *Which therefore may bee found by prop.*7^{th}.

Example

If

$$\frac{2x}{3}\sqrt{rx} = y,$$

or

$$-4rx^3 + 9y^2 = 0.$$

Then from Proposition 7

$$\begin{aligned} q &= \frac{\dot{y}}{\dot{x}} \\ &= \frac{12rx^2}{18y} \\ &= \sqrt{rx}\,. \end{aligned}$$

So, the curve, whose area is $\frac{2x}{3}\sqrt{rx}$, is $\sqrt{rx}$. In modern notation this means

$$\frac{2x}{3}\sqrt{rx} = \int \sqrt{rx}\,dx.$$

The example supplied by Newton does not clearly show the fundamental theorem of calculus. Consider figure 3.4. The problem states that given the area, y, in terms of x find the curve $q = f(x)$. Also given is $ab = x$, $bc = q$ and $ad = be = 1$.

If the area is regarded as being produced by the movement of the vertical line bc with velocity $\frac{dx}{dt} = \dot{x} = 1$, then, Newton says, the two areas x and y will increase *as be to bc: y^t is, y^e motion by w^{ch} x increaseth being $be = p = (\dot{x}) = 1$, y^e motion by w^{ch} y increaseth will bee $bc = q$.* So, in fact he has said that the rate of change (with respect to time) of the area, y, is q. That is

$$\frac{dy}{dt} = q,$$

or

$$\dot{y} = q.$$

With $\dot{x} = p = 1$ then

$$\frac{\dot{y}}{\dot{x}} = q,$$

or

$$\frac{dy}{dx} = q.$$

So, to find the curve (q) whose area is y, Newton says to use Proposition 7 to find $\frac{dy}{dx}$. In the above example, the area is $y = \frac{2x}{3}\sqrt{rx}$ and following his method results in the curve $\frac{dy}{dx} = \sqrt{rx}$.

In modern notation he has used

$$y = \int \frac{dy}{dx}\, dx,$$

or

$$area = \int (curve)\, dx.$$

Therefore, Newton has used the Fundamental Theorem of Calculus in that $\int \frac{dy}{dx}\, dx = y$ where y is the area and $\frac{dy}{dx}$ is the curve which produces that area.

3.2 Fluents and Fluxions

Newton's first paper introducing the terms fluents and fluxions was written between 1670 and 1671 and is called *De Methodis Serierum et Fluxionum* (Methods of Series and Fluxions). It was published in 1737 and appears in Volume 3 of Whiteside [15]. Whiteside uses the dot notation for speeds rather than letters p, q, r etc. The following excerpts taken from Whiteside highlight the introduction of the terminology used by Newton for quantities and their resultant speeds. A problem very similar to one used earlier to demonstrate Proposition 7 is shown, as well as the proof for this method of solution. Rather than being a repeat of what has been shown earlier the works from his later paper show how his thoughts have developed in the area of calculus.

... to distinguish the quantities which I consider as just perceptibly but indefinitely growing from others which in any equations are to be looked on as known and determined are designated by the initial letters a, b, c and so on, I will hereafter call them fluents and designate them by the final letters $v, x, y,$ and z. And the speeds with which they each flow and are increased by their generating motion (which I might more readily call fluxions or simply speeds) I will designate by the letters $\dot{v}, \dot{x}, \dot{y}$ and $\dot{z}$ ...

Problem I. Given the relation of the flowing quantities to one another, to determine the relation of the fluxions

Arrange the equation by which the given relation is expressed according to the dimensions of some fluent quantity, say x, and multiply its terms by any arithmetical progression and then by $\frac{\dot{x}}{x}$. Carry out this operation separately for each one of the fluent quantities and then put the sum of all the products equal to nothing, and you have the desired equation.

This algorithm is equivalent to that in proposition 7 stated earlier. Here, though, Newton uses his new terminology and in the example following displays an alternative setting out for the solution.

Example. let $x^3 - ax^2 + axy - y^3 = 0$, then considering the x quantity first we get

$$x^3.(3\frac{\dot{x}}{x}) - ax^2.(2\frac{\dot{x}}{x}) + axy.(\frac{\dot{x}}{x}) - y^3.(0) = 0,$$

and then for the y quantity

$$-y^3.(3\frac{\dot{y}}{y}) + axy.(\frac{\dot{y}}{y}) - ax^2.(0) + x^3(0) = 0.$$

The sum of all the products gives

$$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0,$$

which will give the relation between fluxions $\dot{x}$ and $\dot{y}$.

So, with x as the independent variable, that is $\dot{x} = 1$, we can say

$$\frac{dy}{dx} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}.$$

Newton's proof of this method is similar to his proof of Proposition 7, but now he uses the terms "fluents", "fluxions" and "moments" and uses limit increments resulting in a more detailed outcome.

> *The moments of the fluent quantities (that is, their indefinitely small parts, by addition of which they increase during each infinitely small period of time) are as their speeds of flow. Wherefore if the moment of any particular one, say x, be expressed by the product of its speed $\dot{x}$ and an infinitely small quantity $\circ$ (that is, by $\dot{x}\circ$), then the moments of the others, v, y, z, will be expressed by $\dot{v}\circ, \dot{y}\circ, \dot{z}\circ, \ldots$. Now, since the moments (say, $\dot{x}\circ$ and $\dot{y}\circ$) of fluent quantities (x and y, say) are the infinitely small additions by which those quantities increase during each infinitely small interval of time, it follows that those quantities x and y after any infinitely small interval of time will become $x + \dot{x}\circ$ and $y + \dot{y}\circ$. Consequently, an equation which expresses a relationship*

of fluent quantities without variance at all times will express that relationship equally between $x + \dot{x}\circ$ and $y + \dot{y}\circ$ as between x and y; and so $x + \dot{x}\circ$ and $y + \dot{y}\circ$ may be substituted in place of the latter quantities, x and y, in the said equation.

Let there be given, accordingly, any equation

$$x^3 - ax^2 + axy - y^3 = 0$$

and substitute $x + \dot{x}\circ$ in place of x and $y + \dot{y}\circ$ in place of y: there will emerge

$$0 = (x^3 + 3\dot{x}\circ x^2 + 3\dot{x}^2\circ^2 x + \dot{x}^3\circ^3) - (ax^2 + 2a\dot{x}\circ x + a\dot{x}^2\circ^2)$$
$$+ (axy + a\dot{x}\circ y + a\dot{y}\circ x + a\dot{x}\dot{y}\circ^2) - (y^3 + 3\dot{y}\circ y^2 + 3\dot{y}^2\circ^2 y + \dot{y}^3\circ^3).$$

Now by hypothesis $x^3 - ax^2 + axy - y^3 = 0$, and when these terms are erased and the rest divided by $\circ$ there will remain

$$0 = 3\dot{x}x^2 + 3\dot{x}^2\circ x + \dot{x}^3\circ^2 - 2a\dot{x}x - a\dot{x}^2\circ + a\dot{x}y$$
$$+ a\dot{y}x + a\dot{x}\dot{y}\circ - 3\dot{y}y^2 - 3\dot{y}^2\circ y - \dot{y}^3\circ^2 .$$

But further, since $\circ$ is supposed to be infinitely small so that it be able to express moments of quantities, terms which have it as a factor will be equivalent to nothing in respect of the others. I therefore cast them out and there remains

$$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0.$$

As mentioned previously, if t represents time then $\circ = dt$ and $\dot{v}, \dot{x}, \dot{y}$ and $\dot{z}$ are the respective speeds $\frac{dv}{dt}, \frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$.

Although this presentation of Proposition 7 and its"proof" are very similar to his earlier work in *The October 1666 Tract* it is much easier to follow given the new terminology for flowing quantities (fluents) and their respective speeds (fluxions) as well as the introduction of the term "moments". In modern notation moments represent the infinitely small increments dv, dx, dy and dz of the variables v, x, y and z, respectively. By introducing these *infinitely small additions by which those quantities increase during each infinitely small interval of time* Newton has produced a proof using limit increments. Consequently the substitution of $x + \dot{x}\circ$ and $y + \dot{y}\circ$ into his original equation is more plausible.

Chapter 4

Gottfried Leibniz (1646 - 1716)

Leibniz began seriously studying mathematics during 1672 in Paris. He wanted to create a system of notation and terminology to simplify mathematics. His notation made solutions to problems more easily followed and provided an opportunity for more rigorous proofs in calculus. Edwards [5] examines two examples of the simplification of problems due to Leibniz notation, one of which is relevant to the approach taken by Leibniz in the development of calculus.

Examples

1. In Lagrange's functional notation the rule for the derivative of a composite function is :

If

$$h(x) = f(g(x)),$$

then

$$h'(x) = f'(g(x))g'(x).$$

Why this is true is not obvious at first, but using Leibniz notation and setting

$z = f(y)$ and $y = g(x)$ then

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

Now the situation is more readily seen, and there is the opportunity for proof by considering dx, dy and dz as $\triangle x, \triangle y$ and $\triangle z$ and using limits.

2. Consider the problem of finding the area of a surface resulting from rotating $y = f(x)$ about the x axis. An expression for the area can be found to be

$$A = \int 2\pi y \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx.$$

Using Leibniz notation we have the following situation, rather than a lengthy summation of series using a Riemann sum.

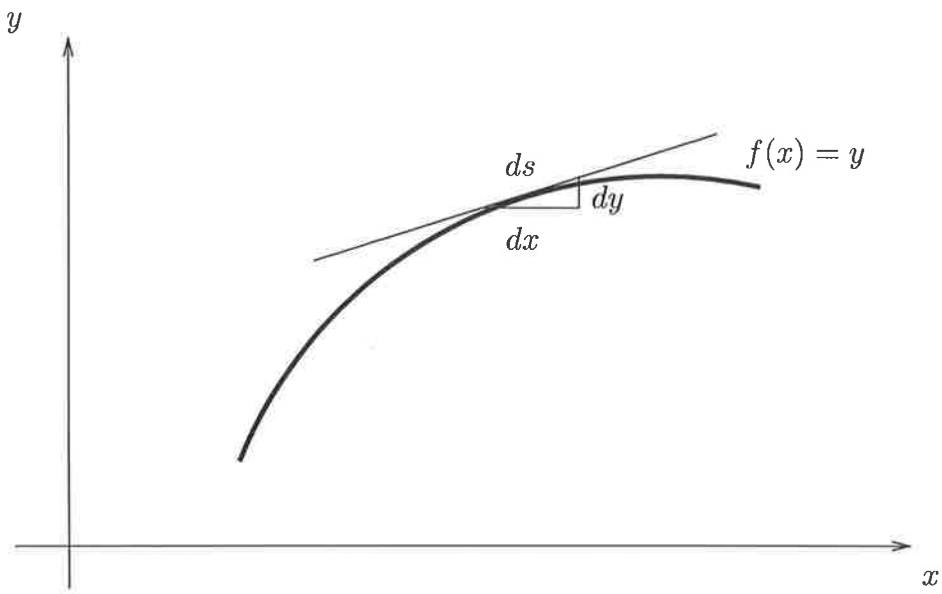

Figure 4.1: ds is an infinitesimal length of the curve $y = f(x)$. dx and dy are infinitesimal lengths representing the horizontal and vertical components of the right angled triangle.

Let ds be an infinitesimal segment of the curve $y = f(x)$ shown in figure 4.1, then

$$\begin{aligned} ds &= \sqrt{(dy)^2 + (dx)^2} \\ &= dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \end{aligned}$$

When the segment, ds, is rotated about the x axis in a circle of radius y, then the expression for an infinitesimal area is

$$\begin{aligned} dA &= 2\pi y.ds \\ &= 2\pi y dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \end{aligned}$$

Therefore the total area is the sum of all infinitesimal areas, dA. That is,

$$\begin{aligned} A &= \int dA \\ &= \int 2\pi y dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \end{aligned}$$

In 1714, two years before his death, Leibniz wrote *Historia et origo calculi differentialis* (History and Origin of the Differential Calculus). In this paper he supplies the history of his own development of calculus. He begins his history by explaining simple number properties which lead him to think of differences and relationships between numbers within sequences. The meaning and the use Leibniz made of the differences and sums of elements of a sequence becomes clear in the following sections.

The English translation of the paper mentioned above is presented in J M Child's [4] *The early mathematical manuscripts of Leibniz.* It is this book to which most of this chapter refers.

4.1 Sequences and Series

In 1672 Leibniz stated that

> *... the sum of the consecutive differences equals the difference of the first and last terms of the original sequence.*

That is, if $a_0, a_1, a_2, \dots, a_n$ is a sequence and denoting the difference between consecutive terms to be

$$d_i = a_i - a_{i-1}$$

then the sum of the differences is

$$d_1 + d_2 + \cdots + d_n = a_n - a_0.$$

Leibniz gives the following example:

$$0, \quad 1, \quad 4, \quad 9, \quad 16, \quad 25$$

has differences

$$1, \quad 3, \quad 5, \quad 7, \quad 9$$

so that the sum of the differences is $1 + 3 + 5 + 7 + 9 = 25 - 0 = 25$.

Leibniz presented his work to Christiaan Huygens (1629 - 1695) who was a well known scientist on the continent at the time. Huygens suggested solving the series

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{1}{n(n+1)/2} + \cdots . \tag{4.1}$$

To solve the problem Leibniz began with Pascal's arithmetic triangle:

$$\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 3 & 6 & 10 & 15 & 21 \\
1 & 4 & 10 & 20 & 35 & 56 \\
1 & 5 & 15 & 35 & 70 & 126
\end{array}$$

In the arithmetic triangle the nth element in each row is the sum of the first n elements in the previous row. Leibniz considered what he called the harmonic triangle:

$$\begin{array}{cccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \frac{1}{30} & \frac{1}{42} & \cdots & \cdots \\
\frac{1}{3} & \frac{1}{12} & \frac{1}{30} & \frac{1}{60} & \frac{1}{105} & \cdots & \cdots & \cdots \\
\frac{1}{4} & \frac{1}{20} & \frac{1}{60} & \frac{1}{140} & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{5} & \frac{1}{30} & \frac{1}{105} & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

where subsequent rows are formed by taking differences, rather than sums in the case of the arithmetic triangle. Using his work from 1672;

for a decreasing sequence $a_1, a_2, \ldots, a_n$ with differences

$$b_i = a_i - a_{i+1},$$

then

$$b_1 + b_2 + \cdots + b_n = a_1 - a_{n+1}$$

which, in the limit, will give the sum of the differences as a_1. So, using the above argument for a decreasing sequence the second row of the harmonic triangle gives

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots = 1. \tag{4.2}$$

That is, the sum of the terms in each row is equal to the first element of the preceding row.

Note that the nth element of the second row of the harmonic triangle is

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

This is half of the nth number in the series posed by Huygens in (4.1) viz; $2/n(n+1)$. So, multiplying (4.2) by 2 gives the solution to Huygen's problem:

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\cdots+\frac{1}{n(n+1)/2}+\cdots=2.$$

Leibniz noticed the inverse relationship between Pascal's arithmetic triangle and his so called harmonic triangle:

Arithmetic Triangle:

each row consists of sums of the terms in the preceding row (and differences of terms in the following row).

Harmonic Triangle:

each row consists of differences of the terms in the preceding row.

The notion of an inverse relationship between the operation of taking differences and that of forming sums of the elements of a sequence, played a major role in Leibniz's development of calculus.

4.2 The Characteristic Triangle

Leibniz was familiar with the work of Pascal. Pascal had proven the theorem stated by Archimedes for measuring the surface of a sphere. He used a method whereby the surface of the solid, produced by rotation about an axis, can be reduced to a plane figure. Leibniz made use of this method and stated that

> *Portions of a straight line normal to a curve, intercepted between the curve and an axis, when taken in order and applied at right*

angles to the axis give rise to a figure equivalent to the moment of the curve about the axis.

This theorem becomes clear when examples of similar triangles are examined with Leibniz's diagram illustrated in figure 4.2. He calls the triangle Y_1DY_2 the *characteristic triangle* and considers three cases of similarity. For each case Leibniz's notation and results will be presented first, followed by an analysis using modern notation.

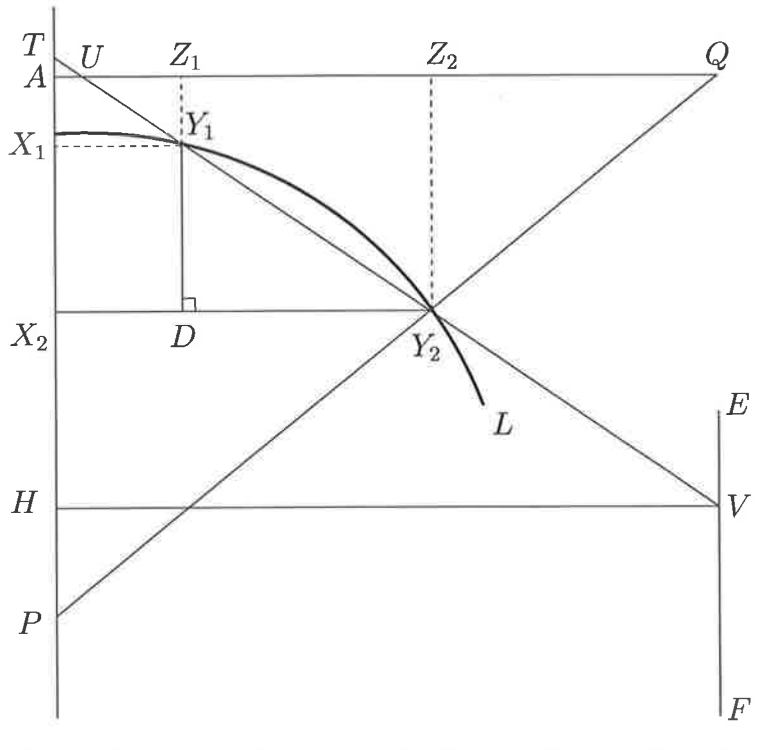

Figure 4.2: Leibniz's diagram showing the characteristic triangle, Y_1DY_2, required for cases 1,2 and 3 following this diagram. Note that the x direction is vertical and the y direction is horizontal.

From figure 4.2 notice that $DY_1 = X_1X_2$, $DY_2 = Z_1Z_2$ and Y_1Y_2 is Y_1Y_2 is part of the tangent TV. The tangent line, TV, should only touch the curve once, but is shown in the diagram cutting the curve at Y_1 and

Y_2. Leibniz is indicating here that the area between the curve and TV is meant to be negligible and that triangle Y_1DY_2 is infinitely small. The axes are referred to as AX (the x axis) and AZ (the y axis) by Leibniz. Then X_1, X_2 or X, and Z_1, Z_2, or Z are positions along the AX and AZ axes, respectively. Leibniz considered the following three sets of similarity using the characteristic triangle in each case. Most important to all the following arguments is that the characteristic triangle Y_1DY_2 is infinitely small.

Case 1. In triangle Y_2X_2P, Y_2P is the perpendicular, or normal, to the curve and X_2P is the subnormal to the curve.

Y_1DY_2 is similar to Y_2X_2P, so

$$PY_2 \times Y_1D = Y_2X_2 \times Y_2Y_1.$$

That is,

$$\text{rectangle area}\, PY_2.Y_1D \quad = \quad \text{rectangle area}\, Y_2X_2.Y_2Y_1. \qquad (4.3)$$

He calls $Y_2X_2.Y_2Y_1$ the moment of the element of the curve about the axis, and that the moment of the curve about the x axis is equal to the area under a second curve (the *quadratrix*) whose y coordinate is the normal, PY_2, to the original curve.

> *Hence the whole moment of the curve is obtained by forming the sum of these perpendiculars to the axis.*

Leibniz uses the phrase *moment of the curve* to represent a portion of the area of the surface formed by the rotation of the curve Y_1Y_2 about the axis AX. If $ds = Y_1Y_2$, $dx = Y_1D$ and $dy = Y_2D$ represent the sides of the infinitely small characteristic triangle, and $y = X_2Y_2$ and $n = Y_2P$ then similar triangles Y_1DY_2 and Y_2X_2P give

$$\frac{ds}{n} = \frac{dx}{y}$$

or

$$y\,ds = n\,dx.$$

So that the whole moment, that is the sum of the infinitesimals, is

$$\int y\,ds = \int n\,dx.$$

The integral sign was not introduced by Leibniz until 1675, so he expressed himself in words (as in equation (4.3)). The area of the surface resulting from the rotation of the original curve about the x axis is found by multiplying the moment by 2π, giving

$$A = \int 2\pi y\,ds.$$

Case 2. The characteristic triangle Y_1DY_2 and triangle THV are similar, giving

$$Y_1Y_2 : Y_2D = TV : VH$$

or

$$VH \times Y_1Y_2 = TV \times Y_2D \tag{4.4}$$

That is,

the rectangle contained by the constant length VH and the element of the curve Y_1Y_2, is equal to the rectangle contained by TV and Y_2D, or the element of the coabscissa, Z_1Z_2. Hence the plane figure produced by applying the lines TV in order at right angles to AZ is equal to the rectangle contained by the curve when straightened out and the constant length HV.

In this case Leibniz has developed a method for rectification of curves. That is, finding a straight line segment equal in length to a given curve. If we let $t = TV$ and a be the constant length VH, then the arclength, s, of the curve can be found by the summation of the elements above in equation (4.4). That is,

$$\int VH\, d(Y_1Y_2) = \int TV.d(Y_2D)$$

or

$$\int a\, ds = \int t\, dy.$$

So, finding the length of a line segment (or arclength, s) is equivalent to finding the area between the y axis and another curve whose x axis is the tangent to the given curve.

Case 3. Triangles Y_1DY_2 and Y_2X_2P are similar, so that

$$Y_1D : DY_2 = Y_2X_2 : X_2P,$$

or

$$X_2P \times Y_1D = Y_2X_2 \times DY_2. \tag{4.5}$$

That is,

the sum of the subnormals X_2P taken in order and applied to the axis, either to Y_1D or to X_1X_2, will be equal to the sum of the products of the ordinates Y_2X_2 and their elements, Y_2D, taken in order.

Using the modern notation, and with $v = X_2P$ equation (4.5) becomes

$$v\,dx = y\,dy.$$

So that we have

$$\int v\,dx = \int y\,dy \tag{4.6}$$

and noting that

$$v = y\left(\frac{dy}{dx}\right) \tag{4.7}$$

we have the result

$$\int y\left(\frac{dy}{dx}\right)\,dx = \int y\,dy.$$

Leibniz referred to this method as [reducing] *these quadratures* [areas] *of figures to an inverse problem of tangents.*

He noted that if $AZ = ZL$, then AZL is a right angled triangle with area $\frac{1}{2}(AZ)^2$.

> *... straight lines that continually increase from zero, when each is multiplied by its element of increase, form altogether a triangle. ...and thus the figure that is produced by taking the subnormals in order and applying them perpendicular to the axis will always be equal to half the square on the ordinate.*

So in equation (4.6) for a curve on an interval of $[0, b]$, passing through $(0,0)$ the area of the triangle, with base and height b, is $\frac{1}{2}b^2$.

The method of reducing an area problem to an inverse tangent problem requires finding another curve for which the subnormal, v, is the given curve. To illustrate the method consider an example posed by Edwards [5]. It is required to find the area under the curve $z = x^n$ on the interval $0 \leqslant x \leqslant a$. So, by Leibniz's method it is required to find a curve, y, with subnormal $v = x^n$ so that

$$\begin{aligned}\int_0^a x^n\,dx &= \int y\,dy \\ &= \frac{1}{2}\Big[y^2\Big]_0^a,\end{aligned} \tag{4.8}$$

where y is a function of x.

If we try $y = bx^k$ then, with the use of $v = y.\frac{dy}{dx}$ from equation (4.7),

$$\begin{aligned}v &= bx^k \times bkx^{k-1} \\ &= b^2kx^{2k-1}.\end{aligned}$$

But

$$v = x^n,$$

so

$$x^n = b^2kx^{2k-1}$$

when

$$k = \frac{1}{2}(n+1)$$

and

$$b = \left[\frac{1}{2}(n+1)\right]^{-1/2}.$$

Substituting $y = bx^k$ into equation (4.8) results in

$$\int_0^a x^n \, dx = \frac{1}{2}\left[\{((n+1)/2)^{-1/2} x^{(n+1)/2}\}^2\right]_0^a$$
$$= \frac{a^{n+1}}{n+1}.$$

In the *Historia et origo*, Leibniz summarises the three cases above: *Thus, to find the area of a given figure, another figure is sought such that its subnormals are equal to the ordinates of the given figure, and then this second figure is the quadratrix of the given one; and thus from this extremely elegant consideration we obtain the reduction of areas of surfaces described by rotation to plane quadratures, as well as the rectification of curves; at the same time we can reduce these quadratures of figures to an inverse problem of tangents.*

The term "quadrature" refers to the method of finding the area of a figure. The method involves constructing a second plane figure, of equal area to the original, and subsequently finding the area of the simpler figure. This second figure is called the quadratrix and is often a square or rectangle.

In a letter to l'Hôpital written twenty years after his initial work in this area, Leibniz summarises that

> ... *use of what I call the* characteristic triangle, *formed from the elements of the coordinates and the curve, I thus found as it*

were in the twinkling of an eyelid nearly all the theorems that I afterward found in the works of Barrow and Gregory.

He continued to write that he did not know the algebra of Descartes, but given encouragement from Huygens he continued his work and came upon *my differential calculus.*

This was as follows. I had for some time previously taken a pleasure in finding the sums of series of numbers, and for this I had made use of the well-known theorem, that, in a series decreasing to infinity, the first term is equal to the sum of all the differences. From this I had obtained what I call the "harmonic triangle", as opposed to the "arithmetic triangle" of Pascal ... Recognising from this the great utility of differences and seeing by the [algebra] of M. Descartes the ordinates of the curve could be expressed numerically, I saw that to find quadratures or the sums of the ordinates was the same thing as to find an ordinate (that of the quadratrix), of which the difference is proportional to the given ordinate. I also recognised almost immediately that to find tangents is nothing else but to find differences, and that to find quadratures is nothing else but to find sums, provided that one supposes that the differences are incomparably small.

4.3 The Integral Sign

In a manuscript dated 29 October 1675 Leibniz introduces the integral sign as we know it in modern mathematics. The manuscript is called *Analy-*

seos Tetragonisticae pars secunda (Second part of analytical quadrature), Child [4]. He begins with a similar diagram to that in the last section but with different labelling as seen in figure 4.3. Initially, he uses *omn* to represent *the sum of ...*, and later introduces the new symbol, $\int$. His working sometimes shows an overline which indicates that the section of text involved should be in brackets. See equation (4.9) for instance.

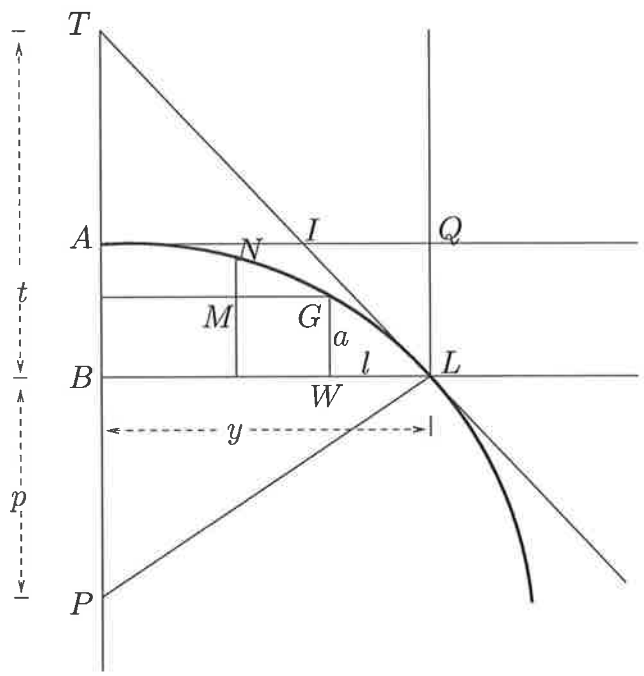

Figure 4.3: Leibniz used this diagram to show his use of the notation *omn*. He replaced this notation with the modern integral sign in his later works.

He states that with $WL = l$, $TB = t$, $GW = a$, $BP = p$, then

$$y = omn.l\,.$$

In other words the total length, y, can be written as the sum of infinitesimally small lengths l.

From similar triangles GWL and LBP,

$$\frac{l}{a} = \frac{p}{omn.l}\,.$$

That is,

$$p = \frac{\overline{omn.l}}{a}.l\,.$$

So,

$$omn.p = omn.\frac{\overline{omn.l}}{a}.l\,. \tag{4.9}$$

If $AQ = QL$, then AQL is a right angled triangle, and from the previous section it was shown that it will have area $\frac{1}{2}(AQ)^2$. So applying Leibniz's Case 3 to this situation with $p = v$, $l = dy$, $a = dx$ and $y = omn.l$ results in

$$omn.p = \frac{1}{2}y^2 = \frac{\overline{omn.l}^2}{2}\,.$$

Therefore, substituting for $omn.p$ in equation (4.9)

$$\frac{\overline{omn.l}^2}{2} = omn.\overline{\overline{omn.l}.\frac{l}{a}}. \tag{4.10}$$

In modern notation $l = dy$ and $a = dx = 1$ so that equation (4.10) becomes

$$\frac{1}{2}\left(\int dy\right)^2 = \int\left(\int dy\right) dy,$$

or

$$\frac{1}{2}y^2 = \int y\,dy.$$

In this manuscript Leibniz also states that

$$omn.xl = x.omn.l - omn.omn.l,$$

where l is taken to be a term of a progression [of differences], *and x is the number which expresses the position or order of the l corresponding to it.*

So, in other words, he is referring to a sequence of differences of ordinates. Using integral notation the above equation becomes

$$\begin{aligned} \int x\,dy &= x\int dy - \iint dy \\ &= xy - \int y\,dx. \end{aligned}$$

He introduces the integral sign later in the manuscript:

> *It will be useful to write $\int$ for omn, so that*
>
> $$\int l \quad = \quad omn.l, \qquad \textit{or the sum of the } l\textit{'s}.$$
>
> *Thus*
>
> $$\int \overline{\frac{l^2}{2}} = \int \overline{\int \bar{l}.\frac{l}{a}}, \tag{4.11}$$
>
> *and*
>
> $$\int \overline{xl} = x\int \bar{l} - \iint l. \tag{4.12}$$

The use of these two equations may be seen when appropriate substitutions are made. For instance, when $l = dx$ in equation (4.11) the result is

$$\frac{1}{2}\int (dx)^2 = \int (\int dx)\,dx$$

that is,

$$\frac{x^2}{2} = \int x\,dx.$$

With $l = x\,dx$ in equation (4.12) Leibniz would obtain

$$\int x.x\,dx = x\int x\,dx - \int\int x\,dx$$

which becomes

$$\int x^2\,dx = x.\frac{x^2}{2} - \int \frac{x^2}{2}\,dx\,.$$

So, upon rearrangement

$$\frac{3}{2}\int x^2\,dx = \frac{x^3}{2}$$

and therefore

$$\int x^2\,dx = \frac{1}{3}x^3.$$

4.4 Later Manuscripts

There is some contention as to the correct date on a manuscript in which Leibniz introduces the notation dx. The manuscript, dated 11 November, 1675 (or 1673) uses dx, but he still considers it to be a constant equal to one. In a manuscript dated 1 November 1675 he introduced the difference notation $\frac{y}{d}$ to represent the modern dy, and in the manuscript mentioned previously he attempts to find an expression for $d(\frac{\nu}{\psi})$ and $d(\nu\psi)$. It was not until 11 July 1677 that he obtained

$$d(xy) = x\,dy + y\,dx.$$

In an undated manuscript, thought to be a revised version of the 11 July 1677 manuscript, Leibniz states *The fundamental principle of the calculus* and provides proofs of the fundamental rules of differentiation. He also considers the integral as the sum of rectangles, by setting up the following as shown in figure 4.4.

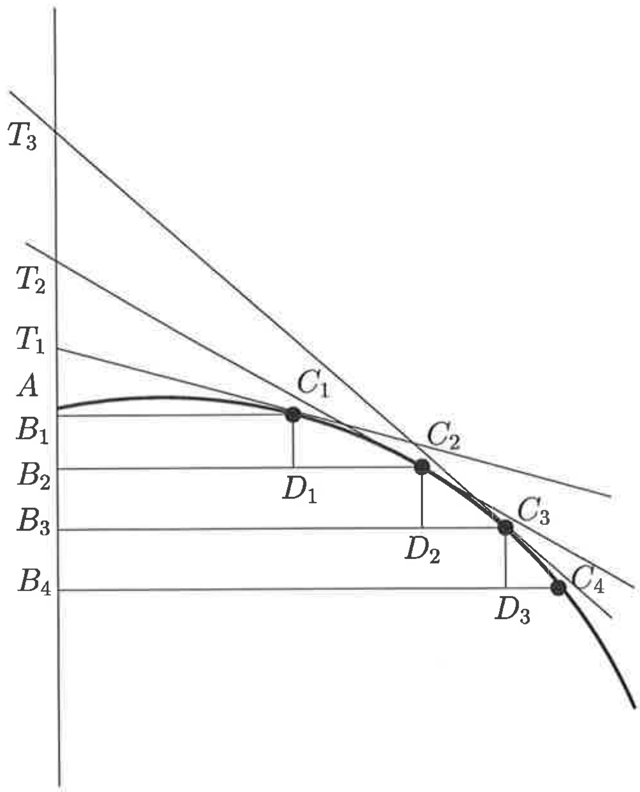

Figure 4.4: Leibniz's diagram to assist his argument for stating that $\int dy = y$ and for finding the area under a curve.

> *Let CC be a line, of which the axis is AB, and let BC be ordinates perpendicular to this axis, these being called y, and let AB be the abscissae cut off along the axis, these being called x.*

Leibniz calls CD the *differences of the abscissae* and labels C_1D_1, C_2D_2, C_3D_3, etc. as dx. Similarly, the lines D_1C_2, D_2C_3, D_3C_4 etc. are the *differences of ordinates* which Leibniz calls dy. These distances, dx and dy, are taken to be infinitely small and are two sides of his characteristic triangle. He considers the straight lines C_1C_2, C_2C_3, C_3C_4 etc. to be elements *of the curve or a side of the infinite-angled polygon that stands for the curve.* These lines are extended to the AB axis to T_1, T_2, T_3 etc., respectively, to produce tangents. Then, he says

$$T_1B_1 : B_1C_1 = C_1D_1 : D_1C_2,$$

or using that in general T_1B_1, T_2B_2, or T_3B_3 are called t,

$$t : y = dx : dy.$$

In modern notation we would write

$$\frac{dy}{dx} = \frac{y}{t}.$$

Considering the triangle $T_1B_1C_1$ in figure 4.4 it can be seen that $\frac{dy}{dx} = \frac{y}{t}$ results in finding the ratio of the sides $y = B_1C_1$ and $t = T_1B_1$, hence the tangent slope may be obtained. That is, $\frac{dy}{dx}$ is calculated. Similarly for triangles $T_2B_2C_2$ etc. Leibniz summarises:

> *Thus to find the differences of series* [dx and dy] *is to find the tangents.*

Next he considers the sums of differences:

> *Moreover, differences are the opposite of sums; thus B_4C_4 is the sum of all the differences such as D_3C_4, D_2C_3, etc. as far as A,*

even if they are infinite in number. This fact I represent thus, $\int dy = y$.

He illustrates here very clearly how $\int dy = y$ as well as the relationship between sums and differences.

For the entire curve, CC, he says:

Also I represent the area of a figure by the sum of all the rectangles contained by the ordinates and the differences of the abscissae, i.e., by the sum $B_1D_1 + B_2D_2 + B_3D_3 +$ *etc. For the narrow triangles* $C_1D_1C_2, C_2D_2C_3$, *etc., since they are infinitely small compared with the said rectangles, may be omitted without risk; and thus I represent in my calculus the area of the figure by* $\int y\,dx$, *or the sum of the rectangles contained by each* y *and the* dx *that corresponds to it.*

The diagonals B_1D_1, B_2D_2, B_3D_3 etc., referred to in the above text represent the notation for the areas of rectangles B_1C_1 by C_1D_1, B_2C_2 by C_2D_2 etc. So it can be seen that the sum ($\int$) of the rectangles ($y \times dx$) can be represented by $\int y\,dx$. Provided that the triangles $C_1D_1C_2, C_2D_2C_3$, etc. are infinitely small, this is the area under the curve.

Also in this undated manuscript Leibniz supplies a statement on the fundamental principle of his calculus:

The fundamental principle of the calculus

Differences and sums are the inverses of one another, that is to say, the sum of the differences of a series is a term of the series, *and the difference of the sums of a series is a term of the series; and I enunciate the former thus, $\int dx = x$, and the latter thus, $d \int x = x$. Thus, let the differences of a series, the series itself, and the sums of the series, be, let us say,*

Diffs.	1	2	3	4	5	...	dx	
Series	0	1	3	6	10	15	...	x
Sums	0	1	4	10	20	25	...	$\int x$

Then the terms of the series are the sums of the differences, or $x = \int dx$; thus, $3 = 1 + 2, 6 = 1 + 2 + 3$, etc; on the other hand, the differences of the sums of the series are the terms of the series, or $d \int x = x$; thus, 3 is the difference between 1 and 4, 6 between 4 and 10. Also, $da = 0$, if it is given that a is a constant quantity, since $a - a = 0$.

He then supplies information on rules for addition, subtraction, multiplication and division type problems. For instance, he says

$$x + y - v = \int \overline{dx + dy - dv}$$

and

$$\int \overline{x + y - v} = \int x + \int y - \int v,$$

and states that *This is evident at sight, if you take three series, set out their sums and their differences, and take them together correspondingly as above.*

For multiplication he states the modern product rule

$$d(xy) = x\,dy + y\,dx$$

and supplies the following proof:

$d\,xy$ is the same thing as the difference between two successive xy's; let one of these be xy, and the other $x + dx$ into $y + dy$; then we have

$$\begin{aligned} d\,xy &= \overline{\overline{x + dx}.\overline{y + dy}} - xy \qquad (4.13) \\ &= x\,dy + y\,dx + dx\,dy; \end{aligned}$$

the omission of the quantity $dx\,dy$, which is infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small, will leave $x\,dy + y\,dx$.

In equation (4.13) he has just written the difference of successive terms $(x + dx)(y + dy)$ and xy with the overlines representing brackets.

Similarly, he found the quotient rule by considering the difference between successive terms. He states the rule

$$d\,\frac{y}{x} = \frac{x\,dy - y\,dx}{xx}$$

and for the proof he writes

$$\begin{aligned} d\,\frac{y}{x} &= \frac{y + dy}{x + dx} - \frac{y}{x} \\ &= \frac{x\,dy - y\,dx}{xx + x\,dx} \\ &= \frac{x\,dy - y\,dx}{xx}. \end{aligned}$$

The term $x\,dx$ on the denominator may be omitted due to it *being infinitely small in comparison with xx.*

Leibniz shows in this manuscript that, in general, for any positive integer, e,

$$d\,x^e = e.x^{\underline{e-1}}\,dx,$$

by using

$$d\,xvy = xy\,dv + xv\,dy + vy\,dx. \tag{4.14}$$

For instance, let

$$d\,x^3 = d\,xvy$$

where

$$x = v = y \quad \text{and} \quad dx = dv = dy,$$

then, from equation (4.14)

$$d\,x^3 = 3x^2\,dx\,.$$

Hence also

$$d\,\frac{1}{x^h} = -\frac{h\,dx}{\overline{x^{h+1}}}\,.$$

For, if $\frac{1}{x^h} = x^e$, *then* $e = -h$, *and* $x^{e-1} = \frac{1}{\overline{x^{h+1}}}$.

He continues by considering fractions (*The same thing will do for fractions*) and irrationals:

> $d\,\sqrt[r]{x^h} = d\,x^{h:r}$, *(where by* $h : r$ *I mean* $\frac{h}{r}$, *or* h *divided by* r*), or* $d\,x^e$ *(taking* e *equal to* $\frac{h}{r}$*), or* $e.\underline{x^{e-1}}\,dx$, *by what has been said above, or (by substituting once more* $h : r$ *for* e, *and* $\overline{h-r} : r$ *for* $e-1$*)* $\frac{h}{r}.x^{\overline{h-r}:r}.dx$; *and thus finally we get the value of* $d\,\sqrt[r]{x^h}$.

He finally states the converse for the last three derivatives:

Moreover, conversely, we have

$$\int x^e\,dx = \frac{x^{e+1}}{e+1},$$
$$\int \frac{1}{x^e}\,dx = -\frac{1}{\overline{e-1}.x^{e-1}},$$
$$\int \sqrt[r]{x^h}\,dx = \frac{r}{r+h}\sqrt[r]{x^{\overline{h+r:r}}}.$$

These are the elementary principles of the differential and summatory [integral] *calculus, by means of which highly complicated formulas can be dealt with, not only for a fraction or an irrational quantity, or anything else; but also an indefinite quantity, such as x or y, or any other thing expressing generally the terms of any series, may enter into it.*

Chapter 5

Alternative Approaches in the Teaching of Calculus

The conventional way of introducing calculus in the senior years of Australian high schools is to begin with limits in conjunction with investigating slopes of chords through curves. One of the points the chord goes through is considered to move closer to the other fixed point until a tangent line is produced. The slopes of the chords may be calculated by using the gradient formula learnt and practiced in earlier years. Using this approach the slope of the tangent is approximated. Terms such as "tends towards", "approaches" and "in the limit" are introduced as the chord gradually becomes the tangent line at the fixed point. It is advisable to illustrate the above geometrically.

The derivative is defined eventually as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{5.1}$$

This requires knowledge of limit notation and at least a good intuitive idea of what a limit is. Many students are not comfortable with the concept of the limit being an operator, and are confused by the use of the word "of" in the term "the limit of". The word "of" has, for many of their school years, been synonymous with "product".

Before the first principles definition of the derivative is introduced a large amount of work on limits must be covered. All the limit rules must be encountered and lengthy algebraic manipulation within limit problems must be mastered if a complete formal treatment is given. So, initially, the idea of a chord becoming a tangent in the limit is introduced, then limits and applicable rules are introduced and practiced, and finally the two are put together to introduce the derivative function, $f'(x)$, as in equation (5.1). Many problems are set to find the derivative of a function as a function of a variable, x, say. As well, problems are set which enable the slope of the tangent to be found by calculating the derivative at a point on a curve. Then the rules of differentiation are explained, sometimes using the limit definition in equation (5.1) for the proofs. Alternative notation such as $\frac{dy}{dx}$ is also introduced and problems can now be considered using a general formula such as:

$$\text{if } f(x) = x^n \quad \text{then } f'(x) = nx^{n-1} \quad \text{for } n \in R.$$

Later, the product and quotient rules are introduced, usually by the first principles method.

To enhance the understanding of the derivative, there are software packages available. One of these is *ANUgraph*. This may be used to display the changing chord approaching the tangent line with more accuracy than on a whiteboard or blackboard. Obviously the diagrams are not static either. Another approach to assist with the understanding is to consider displacement-time graphs with their corresponding velocity-time graphs. Here, discussions can take place regarding rates of change. For instance, from a curved distance-time graph questions may be posed about the speed: when was it greatest, least, constant? A velocity-time graph may then be drawn by finding approximations for the slopes (of the tangents) at set points along the original curve, and plotting the resultant slopes against time. Discussions from this type of activity will enhance the understanding of the concept of slopes of tangents as well as the ability to interpret graphs.

This technique for introducing calculus is usually followed by considering, graphically, other functions and their resultant gradient or derivative functions. Here, again, the use of computers can enhance the graphical picture and lead into $\frac{f(x+h)-f(x)}{h}$ for the slope of the tangent for small h. Investigations then lead the student to discover a relationship between the original function and the derivative or derived function.

After studying the development of calculus by Newton and Leibniz it becomes apparent that there are alternative methods available for teaching calculus. Newton and Leibniz were criticised for not being rigorous in their proofs: Newton for dismissing the *little zero*, or o, as being negligible and Leibniz for neglecting multiples of differences such as $dxdy$. However, it is possible to produce a course in introductory calculus using mostly Leibniz's methods as a basis with Newton's study of the motion of objects to highlight rates of change. In the following sections, the work of Newton and Leibniz is discussed along with its suitability for inclusion in an introductory calculus course. In the final section a course structure is discussed as well as some ideas for investigative work for talented students. Since the majority of the course suggested uses the methods of Leibniz, his work is discussed initially in the next four sections.

5.1 Leibniz's Differences

Leibniz's approach to calculus is quite different to the moving bodies in Newton's work. Leibniz, in his own history of calculus, says that it was his experimenting with number series which labelled the beginnings of his work in calculus. His terminology and notation are very easy to follow and the use of a simple number series in his explanation for the *fundamental principle of the calculus* in section 4.4 on page 66 highlights the simplicity of his initial approach. His methods are also well within the grasp of students new to calculus. He says, for a series 0 1 3 6 10 15... the differences are 1 2 3 4 5... and are called dx. These are obviously obtained by the differences between consecutive terms of the original series. The sums are 0 1 4 10 20 25... and are labelled $\int x$, indicating that the sum of the first n elements of the series gives the nth term of the sums row. For instance, $0+1+3+6 = 10$, the 4th number of the sums row. Experimenting with series would be the first step to introducing calculus by Leibniz's methods. Students may be given the series row and from that calculate the sums and differences row. Using Leibniz's set up (as on page 66) could be a way for students to find the relationship between the series, differences and sums as Leibniz himself did. That is, the inverse relationship between differences and sums. Consider the following:

Example

Given the series 0 1 4 9 16 25... set the problem out in the style of Leibniz:

Diffs.	1	3	5	7	9	...	dx		
Series	0	1	4	9	16	25	...	x	
Sums	0	1	5	14	30	55	...	$\int x$	

Suggest that students try to obtain the original series by considering only the differences, and also try to obtain the series by considering only the sums. Hopefully they will find, as did Leibniz, that the sum of the differences is a term of the series. Hence, the notation $\int dx$ for the "sum of the differences" equals the series, x. That is, $\int dx = x$. Also, "the difference of the sums", $d \int x$, equals the series, x. That is, $d \int x = x$.

This simple technique is a way of introducing the notation for calculus. It is far less cumbersome than delving straight into limit theory and derivatives. Beyond high school mathematics, the more correct notation, $\sum$, for the sum should be explained.

5.2 Leibniz's Tangents and $\frac{dy}{dx}$

In the *Historia et Origo* Leibniz explains his use of the characteristic triangle in finding the areas of solids of revolutions (in section 4.2 described on pages 51 - 56). For simplicity, consider the characteristic triangle as a right angled triangle with the hypotenuse as the length of an element *of the curve or a side of the infinite-angled polygon that stands for the curve.* In Leibniz's diagram the vertical length is the difference in the x coordinates, labelled dx and the horizontal length is the difference in y coordinates, labelled dy. For the characteristic triangle all these lengths must be infinitely small. To assist the understanding of what "infinitely small" means one could use a phrase by Leibniz: *the two points on the curve are understood to be a distance apart* *that is less than any given length.* To develop the concept of tangent and the relationship to $\frac{dy}{dx}$ consider the following diagram:

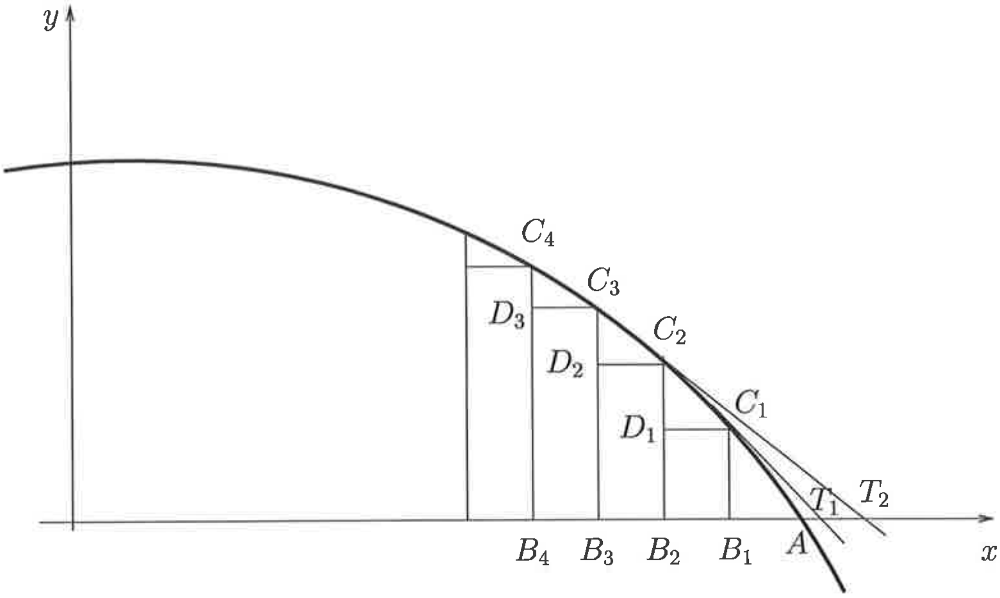

Figure 5.1: This diagram is essentially figure 4.4 shown on page 64. Here, the axes are represented as we would use them today.

Here, the characteristic triangle is $C_1D_1C_2$ and has sides $C_1D_1 = dx$, $D_1C_2 = dy$ and $C_1C_2 = ds$. The line through C_1C_2 meets the x axis at T_1 and since C_1C_2 is an infinitely small length, consider T_1C_1 as the tangent to the curve at C_1. T_1B_1 is a length along the x direction, labelled t_1, and B_1C_1 is a length along the y direction, labelled y_1. Using similar triangles $C_1C_2D_1$ and $T_1C_1B_1$ students should have no problem showing that

$$\frac{dy}{dx} = \frac{y_1}{t_1} \quad \text{at } C_1.$$

Now that this ratio is found the slope of the tangent has consequently been found, since it is the ratio of the y step and x step which gives the slope. That is, the slope of the tangent is really the slope of the hypotenuse of an infinitesimal triangle, which is $\frac{\triangle y}{\triangle x}$. Now, to match modern day usage and notation, denote $\frac{dy}{dx}$ as the slope of the tangent or alternatively call $\frac{dy}{dx}$ the derivative (of the curve).

It is important to illustrate the derivative as an indication of how the curve is changing. One method is to compare other similar triangles to the characteristic triangle. For instance, if $T_1B_1 = t_1$, $T_2B_2 = t_2$, $B_1C_1 = y_1$ and $B_2C_2 = y_2$, then at C_2

$$\frac{dy}{dx} = \frac{B_2C_2}{B_2T_2} = \frac{y_2}{t_2}.$$

Comparing the derivatives at C_1 and C_2 will show that the curvature must be changing.

For the moment, before derivatives are calculated from a curve, observation will need to be used to stress the rate of change. The use of software such as *ANUgraph* would be useful in lessons to display that with a greater derivative, the greater the rate of change and hence the steeper the curve, for instance. Negative cases also need to be considered. Starting with simple parabolae, for example $y = x^2$, one can illustrate the rate of change of the curve by drawing tangents (for $x \geqslant 0$ at first) and assigning values for $\frac{dy}{dx}$ at various points along the curve. This way, the students can see the value of $\frac{dy}{dx}$ changing and the resulting change in the curve - becoming more steep as the derivative becomes greater. The same may be done for $x < 0$. Here the discussion is about the derivative becoming more negative (or the absolute value becoming greater) at values of x becoming more negative, and the consequent change to the slope of the curve and the tangent. Next, discussions regarding a zero slope, or a flat tangent, and where this takes place on curves take place.

At this stage, if this method of introducing calculus is used, students do not know how to differentiate functions, but have knowledge of the terminology $dx, dy, \frac{dy}{dx}$, slope of the tangent and derivative. They will also be able to judge what is happening to simple curves given values of the derivative at points on the curve. The problem now, is how to introduce differentiation. Leibniz uses differences between elements of a series to develop some rules of differentiation. To assist the proofs offered by Leibniz for the development of these rules found on pages 67-69 in section 4.4, consider the following explanation:

If a series has a member of the type x then the next member will be $x + dx$. Similarly, if the series has general term xy then the next term will be $(x + dx)(y + dy)$. So, if we considered the differences row and series row we would have

Diffs., (d(xy))	...	$(x + dx)(y + dy) - xy$	...	...
Series, (xy)	...	xy	$(x + dx)(y + dy)$	...

Now $(x + dx)(y + dy) - xy$ is the general term for $d(xy)$. In other words

$$\begin{aligned} d(xy) &= xy + x\,dy + y\,dx + dx\,dy - xy \\ &= x\,dy + y\,dx + dx\,dy. \end{aligned}$$

But the term $dx\,dy$ may be neglected since it is *infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small.*

So we have

$$d(xy) = x\,dy + y\,dx. \tag{5.2}$$

Now if $y = x$, then using equation (5.2)

$$d(x^2) = x\,dx + x\,dx$$
$$= 2x\,dx. \tag{5.3}$$

Leibniz also obtains

$$d(xyv) = xy\,dv + xv\,dy + vy\,dx \tag{5.4}$$

which may be proved in a similar way to equation (5.2). Here if we let $x = y = v$ then as found earlier from equation (5.4),

$$d(x^3) = 3x^2\,dx\,.$$

Students experimenting with these two equations, (5.4) and (5.2), to find simple derivatives of the type x^k, will see a pattern emerge. That is,

$$d(x^k) = kx^{k-1}\,dx \qquad for\ k\epsilon R. \tag{5.5}$$

If we let $y = x^k$ then using equation (5.5)

$$dy = kx^{k-1}\,dx,$$

which may be written in the form

$$\frac{dy}{dx} = kx^{k-1}.$$

Keeping in mind that dy is the difference of y values and dx is the difference of x values, then for a curve $y = x^k$, the ratio $\frac{dy}{dx}$ is an indication of the rate of change of the curve due to the differences of y and x. And since y and x are connected by the relationship $y = x^k$ the rate of change or derivative is determined by that relationship. That is, the slopes of tangents to the curve $y = x^k$ are determined by $\frac{dy}{dx} = kx^{k-1}$ at any point along the curve.

If the cases for k being only positive and an integer have been considered, k negative, as a fraction and an irrational should also be encountered using the general rule in equation (5.5).

The quotient rule as we know it today may be introduced using the same technique as for the product rule (equation (5.2)). That is, consider the general term of a series to be $\frac{u}{v}$ then the next term would be $\frac{u+du}{v+dv}$ so that

$$\begin{aligned} d\left(\frac{u}{v}\right) &= \frac{u + du}{v + dv} - \frac{u}{v} \\ &= \frac{u\,dv - v\,du}{v^2 + v\,dv}. \end{aligned}$$

The term $v\,dv$ may be omitted due to it being infinitely small compared to v^2. For a curve of the type $y = \frac{u}{v}$ the derivative, or slope of the tangent, at any point of the curve will be given by

$$dy = \frac{u\,dv - v\,du}{v^2}.$$

For example, if $u = x^2$ and $v = 3 + 2x$, then $du = 2x\,dx$ and $dv = 2\,dx$, so that

$$dy = \frac{x^2.2\,dx - (3 + 2x).2x\,dx}{(3 + 2x)^2},$$

or

$$\frac{dy}{dx} = \frac{-2x^2 - 6x}{(3 + 2x)^2}.$$

Students who become conversant with this sort of algebraic substitution will be able to cope with composite function derivatives and the chain rule, later in calculus, with relative ease.

5.3 Leibniz's Areas and Integration

To consider Leibniz's approach for finding the area under curves, figure 5.1 is required again. It is repeated here as figure 5.2, without tangent lines, for ease of reference.

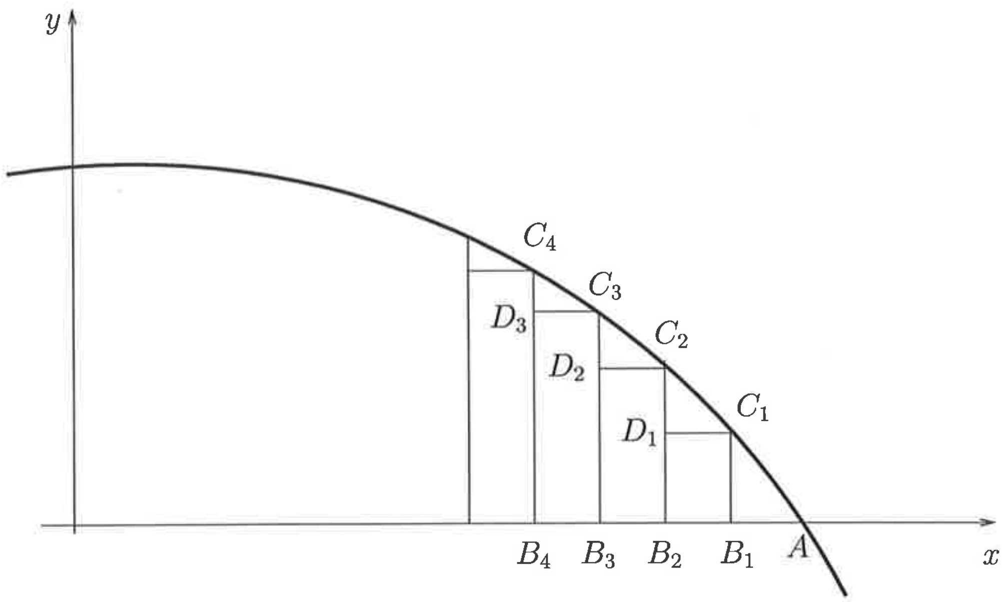

Figure 5.2: This diagram is essentially figure 5.1 shown on page 76. The following text shows how to find the area under the curve, using Leibniz's technique of summing the rectangles under the curve.

Leibniz begins by stating that the sum of the lengths $C_4D_3, C_3D_2, C_2D_1, \ldots$ etc. to A equals the length B_4C_4. The lengths $C_4D_3, C_3D_2, C_2D_1, \ldots$ etc. are differences in y values, dy. So, recalling that he uses $\int$ to represent the sum, he says that $\int dy = y$, the length of C_4B_4. These lengths, dy, are infinitely small. It is easily seen from figure 5.2 that the length C_4B_4 is this sum of dys. He then says that since the triangles $C_4D_3C_3, C_3D_2C_2, C_2D_1C_1$ etc. are infinitely small compared to the rectangles underneath them, the area under the curve may be found by summing all the rectangles $B_1C_1 \times C_1D_1 + B_2C_2 \times C_2D_2 + \ldots$ etc. Each length $B_1C_1, B_2C_2, B_3C_3, B_4C_4$ etc. may be represented by $\int dy = y$. So the total area is the sum of the rectangles, $y \times dx$, where dx is the lengths $C_1D_1, C_2D_2, \ldots$ etc. That is, the area under the curve is $\int y\,dx$.

Having covered Leibniz's use and meaning of sums and differences this method of finding the area is easily explainable and is a way of introducing areas and integration to students. To find the integral of a function, consider the inverse relationship between sums and differences shown earlier by Leibniz. That is, $\int dx = x$ and $d \int x = x$ by using Leibniz's rows of differences (dx), series (x) and sums $(\int x)$. Hence, if we consider the series with general term x^k with difference

$$d(x^k) = kx^{k-1}\,dx,$$

then the converse is

$$\int d(x^k) = x^k,$$

so that

$$\int kx^{k-1}\,dx = x^k.$$

If we let $p = k - 1$, then $kx^{k-1} = (p+1)x^p$ and we have

$$\int x^p\,dx = \frac{1}{p+1}x^{p+1}, \quad \text{for } p \epsilon R,\ p \neq -1.$$

The terms "integral" and "integration" may be introduced here to replace the phrase "finding the sum of" and to match modern terminology.

5.4 Using Leibniz in the Classroom

Since modern notation is that of Leibniz, his ideas are easier to teach. The approach taken by Leibniz in his study of series and consequent development of the inverse relationship between differences and sums is very logical and easy to follow. His method introduces the necessary notation very simply and offers a unique method of introducing calculus. Considering the values on the x and y axes, corresponding to the curve, as a pair of related series leads nicely to his dx and dy of the characteristic triangle, and the consequent ratio $\frac{dy}{dx}$ for the tangent line at any point. The geometric picture becomes clear with this explanation as well as highlighting the concept of related series on the axes. Recognising that the axes represent a pair of related series is not stressed in school mathematics. Using Leibniz's approach students would see the relationship and understand the relevance of series work encountered in their earlier years.

The concept of "infinitely small", when teaching a course based on Leibniz's methods, would need to be covered. Diagrammatically, students can see that the characteristic triangle is small. Consider figure 5.2. Suggest students try to fit in as many rectangles between the line C_4B_4 and A as they can. They will soon see that it is possible to draw more lines simply by reducing the thickness of their pen or pencil.

His explanation for finding the area under a curve, modified in section 5.3, is very clear and suitable for inclusion in an introductory course in calculus.

Leibniz's style of writing is very easy to follow. His ideas and explanations are logical and fluent and his original manuscripts are readable, especially once he begins using notation currently used today.

5.5 Newton's Differentiation.

In section 3.1 it was shown that Newton's ideas are based on the movement of objects as functions of time. He looked at points travelling different distances along straight lines in the same period of time. Proposition 7, shown on page 28, states how to find the ratio of the speeds, $\dot{x}$ and $\dot{y}$, of two objects x and y. In summary, his method for a function $f(x, y) = 0$, where x and y are functions of time, is to multiply the x term by the power of that term as well as $\frac{\dot{x}}{x}$. Similarly for any terms in y. For example, if $f(x, y) = y^2 - x^3 = 0$ then

$$2\frac{\dot{y}}{y}.y^2 - 3\frac{\dot{x}}{x}.x^3 = 0,$$

so that

$$2\dot{y}.y - 3\dot{x}.x^2 = 0$$

and the ratio of velocities is

$$\frac{\dot{y}}{\dot{x}} = \frac{3x^2}{2y}.$$

Introducing calculus using Newton's approach by finding the ratio of speeds of objects is not acceptable. Students need to have studied the formation of curves or loci produced by two moving objects. For instance a curve with functional notation $f(x, y) = 0$, can be thought of as the horizontal motion, x, by an object coinciding with vertical motion, y. Loci problems are usually encountered in the earlier high school years (Years 8 and 9) and to treat the x and y axes as lines along which an object moves, over a certain period of time, should not be an impossible task. It would also be advantageous if students had studied horizontal and vertical velocity vectors in a mathematics or physics course. The reasons become clear by considering the following example. In the example, notation and terminology is introduced for the novice student.

Example

Consider the relationship between two moving bodies to be $y = x^2$ where y and x are both functions of time, t. That is, $y = y(t)$ and $x = x(t)$. The speed of an object is given by $\frac{distance}{time}$, so the speed of one object would be $\dot{y} = \frac{\triangle y}{t}$ and the other $\dot{x} = \frac{\triangle x}{t}$, where $\triangle$ is read to be "change in". Drawing the locus $f(x, y) = y - x^2 = 0$ results in the diagram shown in figure 5.3.

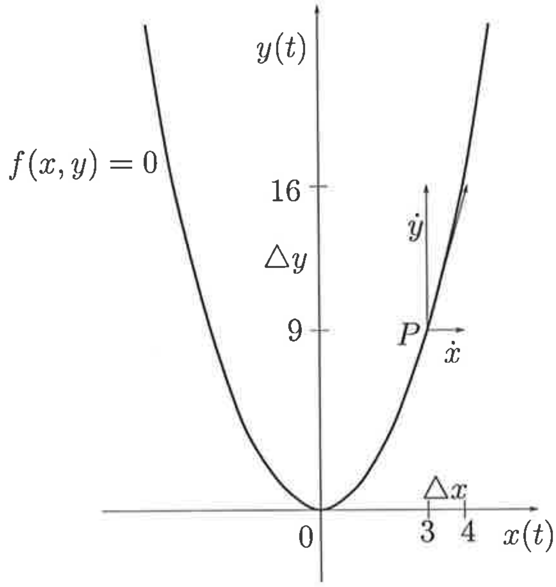

Figure 5.3: The locus of $y - x^2 = 0$, where y and x are functions of time. The tangent line shown at point P is the vector sum of $\dot{y}$ and $\dot{x}$.

At any point, P, on the curve there is a velocity in the horizontal and vertical directions, labelled $\dot{x}$ and $\dot{y}$ respectively. The tangent line, defined as the line which touches the curve once, is the resultant vector sum of $\dot{x}$ and $\dot{y}$ and has a slope or gradient of $\frac{ystep}{xstep}$ which is, $\frac{\dot{y}}{\dot{x}}$. The slope of the tangent may be approximated by:

Consider $\triangle x = 1$ - the size between $x = 3$ and $x = 4$ in figure 5.3. Then the corresponding $\triangle y$ would be given by

$$\begin{aligned} \triangle y &= 4^2 - 3^2 \\ &= 7\,. \end{aligned}$$

So, the slope of the tangent is

$$\frac{\dot{y}}{\dot{x}} = 7\,.$$

Note that if modern methods for finding the derivative at $x = 3$ are used, the slope of the tangent is 6, not 7. The use of a large value for $\triangle x$ results in an obvious inaccuracy.

Now, rather than being faced with this sort of calculation at any point P on the curve, Newton's later version of Proposition 7 in section 3.2 beginning on page 42 (and proof beginning on 43) may be used.

First, introduce the term "moment" to be the very small distance moved by x, say, over a very small period of time. This small increment in time is denoted $\circ$ to use Newton's original notation. By using the original notation of a well known mathematician within his development of calculus, students could well be inspired and interested. Newton called x and y fluents and expressed the moment of x and of y as $\dot{x}\circ$ and $\dot{y}\circ$, respectively. It can be seen that the moments are infinitely small changes in the respective distances of x and y by showing students that

$$\dot{x} = \frac{\textit{change in distance}}{\textit{time}},$$

so

$$\dot{x}\mathrm{o} = \textit{time} \times \frac{\textit{change in distance}}{\textit{time}},$$

that is,

$$\dot{x}\mathrm{o} = \textit{change in distance}.$$

Using Newton's proof of Proposition 7 we can now find a general formula for the ratio $\frac{\dot{y}}{\dot{x}}$ rather than finding values at specific points on the curve. It is also more accurate, since the increments $\dot{x}\mathrm{o}$ and $\dot{y}\mathrm{o}$ are infinitely smaller than the $\triangle x$ and $\triangle y$ used at the beginning of the example.

In figure 5.3, consider a point Q on the curve a very small distance from P. The point Q would have the coordinates $(x + \dot{x}\mathrm{o}, y + \dot{y}\mathrm{o})$ and since Q lies on the curve $f(x, y) = 0$ these coordinates must satisfy this equation. Substitute the coordinates of Q into $y - x^2 = 0$ to obtain

$$y + \dot{y}\,\mathrm{o} - (x + \dot{x}\mathrm{o})^2 = 0\,.$$

That is,

$$y - x^2 - 2x\dot{x}\,\mathrm{o} + \dot{y}\,\mathrm{o} - \dot{x}^2\mathrm{o}^2 = 0\,.$$

This equation simplifies further due to $y - x^2 = 0$:

$$\dot{y}\,\mathrm{o} - 2x\dot{x}\,\mathrm{o} - \dot{x}^2\mathrm{o}^2 = 0\,.$$

Newton says that the period of time, $\circ$, is infinitely small, so when comparing terms with factor $\circ^2$ or higher ordered, these terms may be neglected. So, the above equation is reduced to

$$\dot{y}\circ - 2x\dot{x}\circ = 0\,,$$

or

$$\frac{\dot{y}\circ}{\dot{x}\circ} = 2x$$

leading to the ratio

$$\dot{y} : \dot{x} = 2x : 1$$

or

$$\frac{\dot{y}}{\dot{x}} = 2x\,.$$

After exposure to functions $f(x, y) = 0$, of the same type, and use of the previous method, a pattern should be observed by the students in that, if

$$y = x^n$$

then

$$\frac{\dot{y}}{\dot{x}} = nx^{n-1}\,.$$

The terms "derivative" and "differentiation" need to be introduced. That is, the ratio of velocities, $\frac{\dot{y}}{\dot{x}}$, is the derivative and the process of differentiation is the process in which the derivative of a function at any point on the curve is found. Newton's notation for the derivative shows the ratio of velocities and since this gives the slope of the tangent anywhere on the curve it also gives an indication of how the curve is changing. In this sense, then, the concept of rate of change is introduced to students. This is a valuable inclusion to a calculus course as it is very easy for students to lose sight of the meaning of derivative.

If the methods of Newton are followed for more complicated differentiation problems, his version of the modern chain rule is shown by example on pages 33 - 35. For the sake of clarity, consider a simpler example:

If $y = (x^2-4)^3$, Newton would let u, say, be x^2-4, so that $y = u^3$. Now, using Proposition 7 we have,

$$\dot{u} = 2x\dot{x}$$

and

$$\dot{y} = 3u^2\dot{u}.$$

So, substituting for u and $\dot{u}$ gives

$$\dot{y} = 3(x^2-4)^2.2x\dot{x}$$

and the ratio of velocities, or derivative, is

$$\frac{\dot{y}}{\dot{x}} = 6x(x^2-4)^2.$$

5.6 Newton's Anti-differentiation

Newton uses the converse of Proposition 7 to find a curve given the ratio of component velocities (see section 3.1 on page 35 for Prop.8). That is, given

$\frac{\dot{y}}{\dot{x}}$ in terms of x, say, find the curve y. He says to multiply by x, and then divide by the power of the resulting x. So that, for instance, if

$$\frac{\dot{y}}{\dot{x}} = nx^{n-1},$$

then

$$\begin{aligned} y &= \frac{nx^{n-1} \times x^1}{n} \\ &= x^n . \end{aligned}$$

Before simplification the denominator, n, is the resulting power of x on the numerator. This method is essentially the same as current methods in the classroom, and can be used to gain familiarity with simple anti-derivative problems.

The relationship between areas and tangents using Newton's version of the fundamental theorem of calculus is described on pages 38 - 41. He considers an area, y, being formed by the movement of a line $q = bc$ (see figure 3.4 on page 39) and shows that for a given area the curve, $q = f(x)$, which produces that area is, $q = \frac{\dot{y}}{\dot{x}}$. His reasoning is that the rate of increase of the area will be in ratio to the movement of the line, bc. From figure 3.4 it can be seen that the rate of increase of the area, y, is determined by the rate of incease of area x, multiplied by the height, q. So that we have

$$q\dot{x} = \dot{y}$$

or, as Newton says,

$$q = \frac{\dot{y}}{\dot{x}} .$$

Notice that Newton uses q to mean the height bc as well as the curve f, or $f(x)$. So, for example, if an area is given as $y = 4x^3$, then the curve, $q = \frac{\dot{y}}{\dot{x}}$, is found by applying Proposition 7 to the area. That is,

$$\begin{aligned} \dot{y} &= 4.3x^3.\frac{\dot{x}}{x} \\ &= 12x^2.\dot{x} \end{aligned}$$

and

$$\frac{\dot{y}}{\dot{x}} = 12x^2.$$

That is, given the area is $y = 4x^3$, the curve which gives that area is $q = 12x^2$.

Newton's approach for finding the curve whose area is given may be used to highlight the reciprocal relationship between integration and differentiation. The Fundamental Theorem of Calculus states that

$$\int \frac{dy}{dx}\, dx = y$$

where y is the area and $\frac{dy}{dx}$ is the curve. Once integral notation is known to students, using Newton's diagram in figure 3.4 would be a useful way to display the reciprocal relationship within the Fundamental Theorem of Calculus. For instance, given a curve with y coordinate $\frac{x^{n+1}}{n+1}$, the slope of the curve will be x^n, by using Proposition 7. A curve with y coordinate x^n has an area beneath it as $\frac{x^{n+1}}{n+1}$, also by Proposition 7. Since, if

$$y = \frac{x^{n+1}}{n+1}$$

is the given area, then

$$\begin{aligned} q &= \frac{dy}{dx} \\ &= x^n\,. \end{aligned}$$

Conversely, if

$$y = \frac{x^{n+1}}{n+1}$$

is the curve, then by Proposition 7 the slope of the curve will be

$$\frac{dy}{dx} = x^n \, .$$

So that, with Leibniz's notation, we have

$$\int \frac{dy}{dx}\, dx = y \, ,$$

or

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad \text{for the above example.}$$

5.7 Using Newton in the Classroom

The main advantage in using Newton's methods in a calculus course is that the concept of rate of change is constantly present. The notation $\frac{\dot{y}}{\dot{x}}$ indicates this. The difficulties arise when we try to convert his methods into modern mathematical language. Since the x and y he uses are functions of time, then $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$ and his increment of time, $\circ$, is dt in modern notation. Showing students that $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}$ early in a calculus course is not advisable. The meaning would be too difficult for students to grasp at the beginning of a course. Newton often used $\dot{x} = 1$ which is another reason why explaining his work is difficult.

Using Newton's method of differentiating, or finding the ratio of velocities, by introducing moments of fluent quantities requires an understanding of the physics of objects in motion. However, his method of substituting $x + \dot{x}o$ and $y + \dot{y}o$ into an equation is useful for students to experience. Not only do they experience an original method by a famous mathematician, but they also see the reasoning behind infinitely small increments of time being negligible when its dimensions are greater than one. This is also an ideal opportunity to consider the meaning of "infinitely small" and why terms containing o with dimensions greater than one may be neglected.

Another problem is how to introduce the modern integral notation under Newton's calculus. The integral sign, $\int$, was introduced by Leibniz to represent the sum, but Newton did not consider area as the sum of rectangles of infinitesimal width, as did Leibniz. He did, however, use the symbol $\Box$ meaning "the area of", and used Proposition 8, the anti-differentiation technique, to find the area under a curve (see pages 35 - 37). His lack of explanation for why the process of anti-differentiation results in the area under a curve would lead to many problems in the classroom.

Newton's method for finding the curve, given the area under the curve, may be used to explain the reciprocal relationship between integration and differentiation as stated in the Fundamental Theorem of Calculus. As mentioned in the previous section Newton's figure 3.4 and his explanation (with modern interpretation) show how the movement of a line q produces an area, y. If Leibniz's integral notation is known to students, Newton's problem could provide an explanation for why

$$\int \frac{dy}{dx}\, dx = y$$

where $\frac{dy}{dx}$ is the curve and y is the (given) area. The example given in the previous section where the area is $y = \frac{x^{n+1}}{n+1}$ and the curve is x^n, compared to a curve $y = \frac{x^{n+1}}{n+1}$ with slope $\frac{dy}{dx} = x^n$ shows the reciprocal relationship between integration and differentiation.

Newton's style of writing is difficult to follow. Often his propositions are statements of the type "Do it this way", and only upon reading his examples and proofs are the ideas understood. His original manuscripts offer interesting material for students very capable in mathematics. Offering these students the opportunity to study Proposition 7 and proof, for example, within an extension programme would be of value. They would discover how Newton thought about problems and his style of solution. Extension work of this type is presented in the final section of this chapter.

5.8 An Alternative Course in Introductory Calculus

In this section an outline for a course in introductory calculus is presented, as well as some ideas for extending talented students. The works of Leibniz are mainly used, with Newton's work supplementing topics on rates of change and as an alternative notation for the derivative. The following is suggested for a course in calculus:

1. As does Leibniz, first consider series to introduce the notation and to instil the concept of differences and sums and their relationship.

2. Apply related series to a set of axes resulting in the use of a characteristic triangle to find the slope of the tangent. Use problems and discussions to enhance the concept of change to the curve in relation to the values of $\frac{dy}{dx}$ at various stages on the curve.

3. Use differences of general terms of series to generate the product rule and $d(xyv) = xy\,dv + xv\,dy + vy\,dx$. From these rules derive the generalisation that $d(x^k) = kx^{k-1}\,dx$.

4. Generate the quotient rule by using general terms $\left(\frac{u}{v}\right)$ and $\left(\frac{u+du}{v+dv}\right)$.

5. Introduce the idea of finding area by finding sums by using Leibniz's sum of rectangles of length y and width dx, with the infinitely small characteristic triangles under the curve being negligible.

6. Introduce the concept of integration as the converse of differentiation by using the inverse relationship between the sums and differences of the terms of a series. That is, use $\int dx = x$ and $d \int x = x$ to develop, for instance,

$$\int x^k\,dx = \frac{1}{k+1}x^{k+1}\,.$$

7. Discuss the fact that if a curve does not pass through the origin a constant of integration needs to be introduced. (Leibniz says in his *fundamental principle of the calculus* on page 66 that $da = 0$ if a is constant, since the difference in that type of series results in $a - a = 0$. That is, $da = 0$.)

8. To enhance the understanding of rate of change, consider the set of axes as functions of time $x(t), y(t)$ and the resulting curve as a locus of $f(x, y) = 0$. Discuss the meaning of $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$ using velocity vectors. (Provided the definition of velocity is known to students and $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are explained in that context). Use Newton's vector sum of $\dot{y}$ and $\dot{x}$ to obtain $\frac{\dot{y}}{\dot{x}}$. Discuss $\frac{\dot{y}}{\dot{x}}$ in relation to rates of change.

9. Discuss more difficult problems involving methods of differentiation, integration, finding areas and applications to rate of change problems.

This course outline uses Leibniz's style and explanations for the concepts of differentiation and integration, but Newton's bodies in motion and rate of change approach is important too. Newton's approach for finding areas and anti-differentiation are unnecessarily complicated compared to Leibniz's area under the curve, using sums of rectangles, and his integration represented as the converse of differences. From the students' points of view, seeing the original works of Leibniz and setting their work out in a similar fashion at some stages, may be inspiring. For instance, it was suggested earlier to set out three rows; Differences, Series, Sums in the style of Leibniz to generate the relationship between differences and sums, and also to the original series. Showing students the actual translated manuscripts to let them attempt to decipher sections would also be an interesting exercise. Care would need to be taken on the choice of excerpts to ensure there were no mistakes and an easily understood section is represented.

Extension work for mathematically talented students can be found in both Leibniz and Newtons' works. This extension work can be written to suit the investigation style of assessment currently required at the senior level in South Australian schools. The following is an outline of extension work possible for students to undertake:

1. From section 4.2 beginning on page 51 is the introduction of Leibniz's characteristic triangle and his use of it to

(a) find the area of a solid by finding another plane figure (Case 1 pp 53-54),

(b) find the length of a curve (Case 2 pp 54-55) and

(c) reduce an area problem to an inverse tangent problem (Case 3 pp

55-56).

Using the work presented on these pages teachers could set the problem for students to obtain the three similar triangle results themselves and then study the interpretation, in modern notation, of the work presented in that section.

2. Using figure 4.3 supply students with the information that $y = omn.l$, and that by using the appropriate similar triangles the relationship $\frac{l}{a} = \frac{p}{omn.l}$. Use the text on pages 61 - 63 to formulate an investigation on Leibniz's use of overlines and the introduction of the integral sign.

3. Using figure 4.4 on page 64 and Leibniz's explanation for the diagram, set an investigation with the text on pages 64 - 66 such that students interpret Leibniz's explanations for finding the area under a curve in modern notation.

4. Set problems to find the converse of a difference with the expectation that this will lead to Leibniz's integrals on page 70.

5. Give students a copy of Newton's Proposition 7 and example using $\dot{x}$ and $\dot{y}$ notation (from page 41). Set various problems to find the ratio $\frac{\dot{y}}{\dot{x}}$. Give students a copy of the demonstration by Newton of Proposition 7 (on pages 43 - 44). Set problems in which students demonstrate the ratios found previously.

6. Provide a simplified version of Newton's chain rule (see pages 33 and 89). Set problems of this style for students to solve.

7. Provide a copy of Proposition 8 (on page 35) by Newton. Students interpret and attempt various examples following his method.

8. The information on pages 38 - 41 shows Newton's use of The Fundamental Theorem of Calculus. Formulate an investigation such that students

analyse Newton's Problem 5 and interpret it in modern notation. This would be very difficult for students unless they had a sound understanding of calculus. It would therefore represent a good test of their understanding of the calculus.

As can be seen from this section, the original works of Leibniz and Newton present possibilities for a new approach to the introduction of calculus, as well as new supplementary material for extension or investigative work.

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ERRATA

page 2 Sentence beginning with “The theory of limits ...”
replace with
“Having taught for thirteen years, it has been my experience that the theory of limits always seems to present difficulties to students.”

page 71 Sentence beginning with “Many students are not ...”
replace with
“It has been my experience that many students are not comfortable with the concept of the limit being an operator, and are confused by the use of the word “of” in the term “the limit of”.”

page 80 Replace the last three equations with the following equations

$$dy = \frac{v\,du - u\,dv}{v^2}.$$

$$dy = \frac{(3+2x).2x\,dx - x^2.2\,dx}{(3+2x)^2},$$

$$\frac{dy}{dx} = \frac{6x+2x^2}{(3+2x)^2}.$$

page 83 Sentence beginning with “Since modern notation ...”
replace with
“Since modern notation is that of Leibniz, his ideas, in my opinion, are easier to teach.”

page 83 Sentence beginning with “Using Leibniz’s approach ...”
replace with
“Using Leibniz’s approach I suggest that students would see the relationship and understand the relevance of series work encountered in their earlier years.”

page 84 Sentence beginning with “Introducing calculus ...”
replace with
“I would advise against introducing calculus using Newton’s approach by finding the ratio of speeds of objects. This method would result in having to explain the chain rule at a very early stage.”

The following is to be included at the end of Chapter 4.

4.5 Newton, Leibniz and Modern Calculus

In this section a comparison of the work of Leibniz and Newton, and their influence on modern teaching techniques, is presented.

From Chapter 3 one can see that Newton's approach in his development of calculus is to compare velocities of objects to produce a ratio $\frac{q}{p}$ or $\frac{\dot{y}}{\dot{x}}$, where $\dot{y} = \frac{dy}{dt}$ and $\dot{x} = \frac{dx}{dt}$ in modern notation. So, the notation $\frac{dy}{dx}$ does not refer to the slope of the tangent line to a curve, but to the ratio, $\frac{dy}{dt} \Big/ \frac{dx}{dt}$, of two moving objects. This ratio, to Newton, indicates the rate of change to the system over a period of time. The concept of rate of change is not usually encountered at the beginning of a calculus course, and the ratio

$$\begin{aligned} \frac{\dot{y}}{\dot{x}} &= \frac{dy}{dt} \Big/ \frac{dx}{dt} \\ &= \frac{dy}{dx} \end{aligned}$$

would be a difficult way of introducing calculus in my opinion.

Newton's propositions outlining the process of differentiation and anti-differentiation are basically algorithms. In modern calculus we introduce the concept of the derivative as the slope of the tangent at a point on the curve, and use limit notation to establish

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Newton and Leibniz used the "idea" of limits but did not have the concept of real number properties to explain their ideas. Their approach was based on geometric arguments as opposed to the numerical and algebraic approach taken in modern calculus. The algorithms presented by Newton as Propositions 7 and 8 are used in the teaching of calculus today once the "First Principles" approach, in the above equation, has been covered.

It is interesting to note that the notation of Leibniz for the derivative, or slope of the tangent ($\frac{dy}{dx}$), and the integral ($\int$) allow modern calculus to be more easily followed. For instance, we avoid using Newton's symbol, $\Box$, to denote the area under the curve, and his method for finding a curve given the area is not a technique used today.

Comparing Newton and Leibniz's approaches for finding the area under a curve we see that Leibniz's approach is more recognisable in the modern style of teaching calculus. We use Leibniz's integral sign and a similar method of summing areas of rectangles under a curve. Newton's method is to reverse the chain rule (that is, undertake anti-differentiation) and use the symbol, $\Box$, to denote the area under the curve.

The usual approach to introducing calculus today does not use Leibniz's series method to develop the dx notation for the differences of consecutive members of the series, nor the integral sign, $\int$, for the sum of terms. Leibniz's characteristic triangle technique introduces the terminology $\frac{dy}{dx}$ for the slope of the tangent which is an obvious part of any calculus course. The characteristic triangle, although new to basic calculus, is not an uncommon method for solving problems involving similarity.

The modern approach to teaching calculus sees the use of limit theory. The work of Newton and Leibniz was made more rigorous by later mathematicians such as Cauchy. Cauchy defined the derivative as

$$\frac{dy}{dx} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x},$$

hence the notation $\frac{dy}{dx}$ is not seen as a ratio, but the interpretation of Leibniz's $\frac{dy}{dx}$ is that it is a fraction. However, I feel there is an opportunity to produce an alternative course in introductory calculus using mainly Leibniz's work. His method of introducing the currently used notation via his series approach, and his development of some simple concepts of calculus may well be able to be taught at the Year 11 level. The following Year 12 course could then introduce limit theory to establish a firm grounding in modern calculus. We undertake this style of teaching in other areas in the curriculum. For instance, initially only the square root of positive numbers is presented to students, with the square root of negative numbers being introduced in Year 12. Prior to the Year 12 course, students see the square root of a negative number as giving no solution. In fact, it should be said that there is no real solution, so that there is room to introduce the concept of complex numbers.

In the next chapter an alternative course for introducing calculus is presented. Within this chapter there are discussions on the works of both Newton and Leibniz in calculus with the view of including both their works in the alternative course. The resulting course would present interesting, new and more motivating material for students.

On page 98, add the following after the last paragraph

The object of this thesis is to present an outline for a new course in introductory calculus. To verify that this proposed alternative method for teaching calculus is an improvement on the current methods used in schools it needs to be trialled within schools along with the appropriate empirical research material in place.[1]

[1]The material in this thesis has been personally presented, in part, to groups of mathematics teachers attending workshops held by the Mathematical Association of South Australia in February 1997. The seminar consisted of an alternative method for introducing calculus, possibly at the Year 11 level. It involved using Leibniz's fundamental principles of calculus statement, as found on page 66, to introduce the notation dx for the difference of consecutive terms in a series, and $\int$ for the sum of terms in a series. It was also shown how to establish $\int dx = x$ and $d \int x = x$. Leibniz's use of the characteristic triangle to develop the tangent to a curve and the consequent $\frac{dy}{dx}$ notation was illustrated, as well as the method for finding the area under a curve, as shown on pages 64-66. The seminars, in my opinion, were well received, and some teachers seemed interested in trialling parts of this thesis with Year 11 students.