THE SIEVE OF ERATOSTHENES.

By R. A. Fisher, F.R.S.

I am too ignorant of the literature to know what views have been actually expressed by moderns as to the famous sieve of Eratosthenes, yet I think Sir Thomas Heath may be relied on to mention any which seem to throw light upon early Greek mathematics. His description (*Greek Mathematics*, I. 100) from Nicomachus, 100 A.D., whom there is no reason to regard as a mathematician, is as follows:

> The method is this. We set out the series of odd numbers beginning from 3.
>
> 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, ...
>
> Now 3 is a prime number, but multiples of 3 are not; these multiples, 9, 15, ... are got by passing over two numbers at a time beginning from 3; we therefore strike out these numbers as not being prime. Similarly 5 is a prime number, but by passing over four numbers at a time, beginning from 5, we get multiples of 5, namely 15, 25 ...; we accordingly strike out all these multiples of 5. In general if n be a prime number, its multiples appearing in the series are found by passing over $n-1$ terms at a time, beginning from n; and we can strike out all these multiples. When we have gone far enough with this process, the numbers which are still left will be primes. Clearly, however, in order to make sure that the odd number $2n+1$ in the series is prime we should have to try all the prime divisors between 3 and $\sqrt{(2n+1)}$; it is obvious, therefore, that this primitive empirical method would be hopeless as a practical means of obtaining prime numbers of any considerable size.

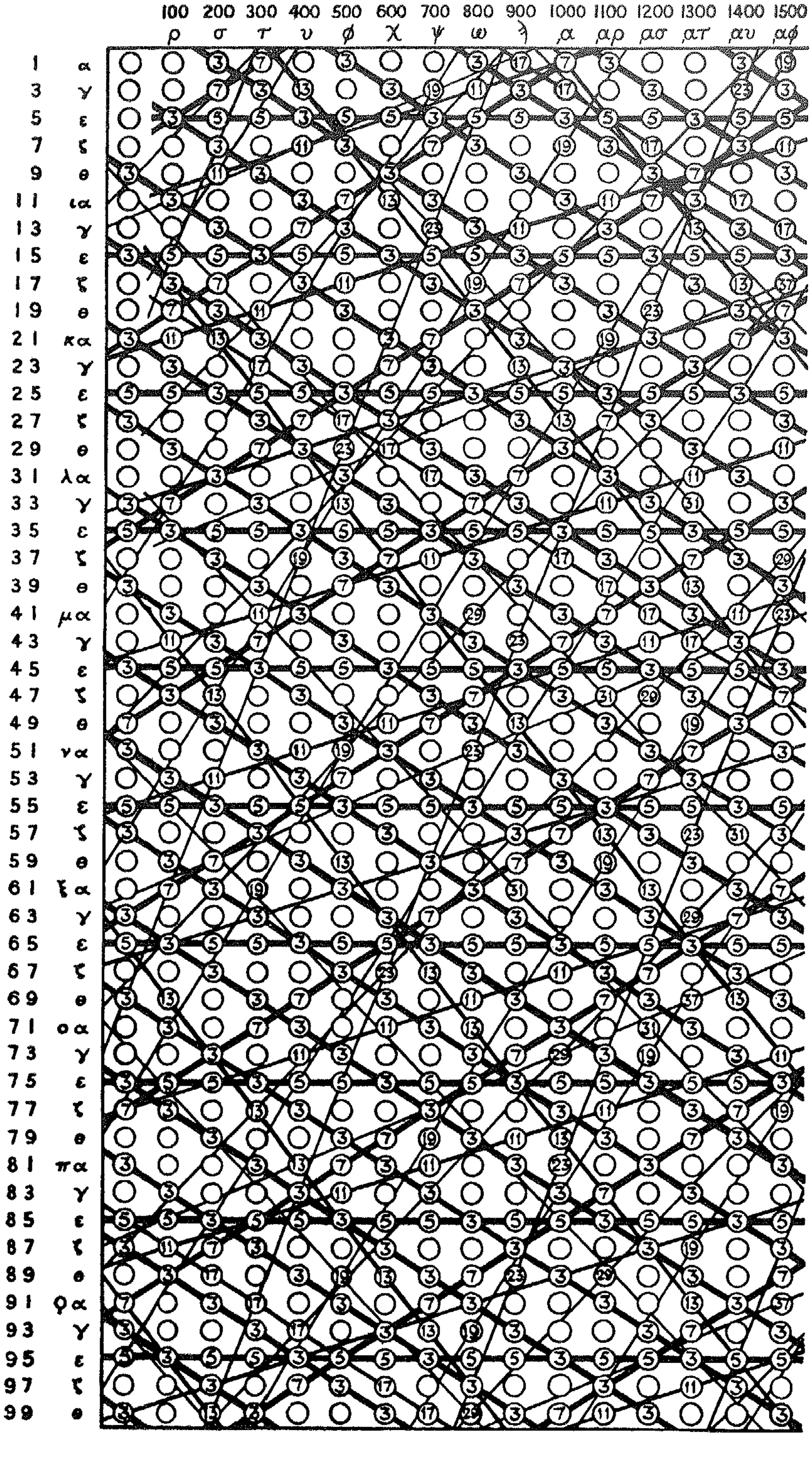
One must feel some surprise that a mathematical contemporary of Archimedes, of great repute in his time—he was nicknamed *pentathlos* from the diversity of his attainments—in Athens and in Alexandria, should have his name associated, with so feeble and useless a device, as though with some considerable or specially ingenious discovery. Heath's words "primitive" and "empirical" are exceedingly moderate in criticism of the method of constructing tables by successive addition, of the multiples of all primes less than $\sqrt{n}$, in order to find out if n is prime.

It is perhaps worth while to ignore the description, which reads like a portion of a demonstration, and attend to the name. The sieve is a machine, and therefore a contrivance to save labour. It is not designed to determine the properties of any individual object, for which we might use a measure or a gauge, but to sort out automatically from a bulk of mixed material, that portion which has some particular property. The job is done once for all; and if we may assume that the sieve of Eratosthenes was named by men who understood its purpose, and that the device was thought sufficiently ingenious, or the name sufficiently witty, to be memorable, we may perhaps take it for granted that its purpose was not to determine whether a particular number was a prime, but to sort out the whole of the primes, more rapidly than could otherwise be done, from an extensive series of consecutive numbers. The device would then work like a sieve. If it also looked like a sieve we should the better understand how it was that the name struck the Greek fancy.

The diagram (page 565) shows the 800 odd numbers less than 1600, arranged in 16 columns of 50 each. Since 102 is an even number divisible by 3, we may pass from any one odd multiple of 3 to another by moving to the next column and taking one step of 2 units down, and the continuation of the oblique line joining these two numbers will pass through nothing but numbers divisible by 3, as far as the edge of the diagram. The series of such oblique lines, which automatically strike out the numbers divisible by 3, are equally spaced at intervals of 3 odd numbers from the top of the diagram to the bottom, or

RECONSTRUCTION OF THE

SIEVE OF ERATOSTHENES



perhaps one should say from its N.E. to its S.W. corner, and may be drawn without hesitation without the tedium of counting 3, 530 times or so.

When the threes are complete we can put in the lines for 5, which happen to be horizontal, for 7, which pass obliquely to the N.E., and already build up a tolerable picture of a wicker sieve, and continue the process for the higher primes. For some of these it will be convenient, and later obligatory, to use lines which miss the numbers altogether in some of the columns. Thus for 11 it is more convenient to use the multiple 198, entailing 2 steps to the right and one step of 2 units upwards, than 110, with its 5 steps down for every one to the right.

I should judge from the description that in Eratosthenes' sieve the numbers were actually written down, and the bars of the sieve allowed to delete those which are not primes. In my diagram the numbers are inferred from their position, and the smallest factors, of all but the primes, inserted. In any case I imagine that the sieve, for the first few thousand integers, was a permanent feature on the walls of Eratosthenes' lecture room, and that his pupils were encouraged to construct similar diagrams for appropriate groups of higher numbers. The placing of the lines for these latter would entail some initial calculation, which was by no means beyond the capacity of his time.

It may have added to the humorous comparison with a real sieve that of each set of parallel bars, one is broken before it reaches the first column; indeed it is only through these breaks that the prime numbers manage to escape.

R. A. FISHER.