The 6 × 6 *Latin squares*. By R. A. Fisher, Sc.D., Gonville and Caius College, and F. Yates, B.A., St John's College.

[*Received* 9 August, *read* 29 October 1934.]

1. *Introduction.*

The problem of the enumeration of the different arrangements of n letters in an $n \times n$ Latin square, that is, in a square in which each letter appears once in every row and once in every column, was first discussed by Euler (1). A complete algebraic solution has been given by MacMahon (3) in two forms, both of which involve the action of differential operators on an expanded operand. If MacMahon's algebraic apparatus be actually put into operation, it will be found that different terms are written down, corresponding to all the different ways in which each row of the square could conceivably be filled up, that those arrangements which conflict with the conditions of the Latin square are ultimately obliterated, and those which conform to these conditions survive the final operation and each contribute unity to the result. The manipulation of the algebraic expressions, therefore, is considerably more laborious than the direct enumeration of the possible squares by a systematic and exhaustive series of trials. It is probably this circumstance which has introduced inaccuracies into the numbers of 5 × 5 and 6 × 6 Latin squares published in the literature.

The problem of the Latin square has become of practical interest in recent years in connection with the development of an adequate theoretical basis for the design of biological experiments, for as soon as the underlying principles of such design began to be understood, it appeared that the Latin square arrangement was in many respects extremely suitable to a large class of field trials with agricultural crops. The reason for its special suitability lies in its satisfactorily fufilling two distinct requirements: (1) in equalising more thoroughly than can be done in other ways the fertility of the land on which the different treatments are to be tested, and (2) in allowing, subject to the fixed restrictions of the Latin square, of a random choice among the different possible squares which could be laid down on the same area. This element of randomisation is now recognised to be a necessary condition for the validity of the estimate of error by which the results of the experiment are to be judged, and it is the fact that it is not a particular Latin square but a random selection from an aggregate of possible squares which is required for agricultural practice, that has given a renewed interest to Euler's problem of their enumeration.

For the numbers of reduced squares MacMahon had given

n	1	2	3	4	5
Number	1	1	1	4	52

but on constructing the possible 5 × 5 squares for agricultural use it was at once found that there were 56 and not 52 possible reduced squares. The corrected number was communicated by one of the authors in 1924 to Professor MacMahon in time to be incorporated in the copies of *Combinatory Analysis* then unsold. As will be seen below, this number, 56, was actually given "*d'après un dénombrement exact*" by Euler in 1781.

The 6 × 6 squares are too numerous to be enumerated *seriatim* without risk of error. Since this size is eminently suitable for agricultural purposes the method of enumeration given in subsequent sections was developed. In this case also it has been found that the number given by a previous author is incorrect, for Jacob (4), using a systematised method of progressive trial, had arrived at the number 8192, whereas, as will be shown, there are in reality 9408 reduced 6 × 6 squares.

The enumeration of the 6 × 6 squares is particularly apposite to Euler's main concern, which was to solve or to demonstrate the insolubility of the problem of constructing a 6 × 6 Graeco-Latin square (*quarré magique complet*). He uses a method of transformation similar to that which we shall call intramutation, but not possessing the invariant diagonal properties which we shall use, in order to show that the 6 × 6 Latin squares may be treated in sets such that of each set all or none are eligible as a basis for forming Graeco-Latin squares. Without making an exhaustive enumeration of the different sets, which must be at least as numerous as the 12 sets which may be generated by a transformation more general than Euler's, Euler argued as follows (p. 229):

"De la il est clair, que s'il existoit un seul quarré magique complet de 36 cases on en pourroit déduire plusieures autres moyennant ces transformations, qui satisferoient également aux conditions du probléme. Or ayant examiné un grand nombre de tels quarrés, sans avoir rencontré un seul, il est plus que probable, qu'il n'y en ait aucun; Car le nombre des latins ne sçauroit être si énorme, que la quantité de ceux que j'ai examiné n'en devroit avoir fourni un qui admet des directrices, s'il y en avoit; vû que le cas $n=2$ et $n=3$ ne fournit qu'un seul, le cas de $n=4$ quatre, le cas de $n=5$ cinquante six, d'après un dénombrement exact, d'où l'on voit que le nombre des variations pour le cas de $n=6$ ne sçauroit être si prodigieux, que le nombre de 50 ou 60, que je pourrois avoir examiné n'en fut qu'une petite partie."

Had Euler realised that the number of 6 × 6 Latin squares of reduced form was as high as 9408, and especially that a number of the transformation sets contain less than a hundredth of the total, he would probably not have judged his conclusion *plus que probable.* On this point a rigorous test is now available by examining for possible systems of *directrices* members of the 12 adjugate sets chosen from the 17 examples given in Section 4. This test has been made, and it was found that none of the 12 sets gave any concordant system of *directrices.* It follows, therefore, as Euler confidently predicted, that no 6 × 6 Graeco-Latin square can exist.

The discussion of the present paper may be contrasted with those of Cayley(2), MacMahon, and Jacob in turning not on the conditions to be satisfied by a permutation by which one line of the square may be transformed into the next, but on the intrinsic symmetry which each solution of the problem of the Latin square presents as a whole. We are therefore concerned with the operations by which a square can be transformed into other squares having the same structural symmetry, whereas the above-mentioned authors have considered the Latin square as a special case of the Latin rectangle.

2. *Definitions.*

1. *Reduced Latin squares.* A square with the first row and first column in alphabetical order *ABCDEF*... has been named by MacMahon a *reduced Latin square.* The diagonal passing through the intersection of the first row and column of a reduced square will be called the *leading diagonal.* A pair of squares is said to be *conjugate* when one is the mirror image of the other in the leading diagonal. *Self-conjugate squares* are symmetrical about the leading diagonal.

2. *Adjugate squares.* Just as the interchange of the rows and columns of a square will give the conjugate square so the interchange of rows and letters in each element (the letters being regarded as possessing, like the rows, a serial order) or of columns and letters will generate a series of squares which may be spoken of as mutually *adjugate.*

3. *Transformations.* Any permutation of the rows, columns, or letters of a square, among themselves, or combination of such permutations, generates another square (possibly identical with the original square). Any rearrangement of this nature will be called a *transformation.*

4. *Intramutations.* In a reduced Latin square any permutation of all the letters other than *A* may be made, and the rows and columns (excluding the first) then rearranged so as to give another reduced

Latin square by arranging the letters of the first row and first column in the standard order. A transformation of this type will be styled an *intramutation*.

Any square of order n can be transformed in $(n!)^3$ ways (including no change). All these transformations do not necessarily give different squares, but all possible squares of order n can be classified in sets of squares which are derivable from one another by some transformation.

It is easily seen that a sequence of transformations is itself a transformation, and that for each transformation there exists a reciprocal transformation which reverses its effect. From these properties it follows that every square of a transformation set is derivable from every other square of the set by the same number of transformations. Thus if there are s and only s transformations which when applied to a square P give square Q, every square in the set is connected to every other square by s transformations, and to itself by $s-1$ transformations (excluding no change); and there are consequently $(n!)^3/s$ squares in the set.

The same property holds good also for intramutation sets of reduced Latin squares: if there are t and only t intramutations connecting any pair of reduced squares then they must be members of an intramutation set containing $(n-1)!/t$ reduced squares.

Each reduced square generates a set of $n!(n-1)!$ squares, all different, by permutation of all the columns, and all the rows except the first. Only the original square is a reduced square. It is therefore sufficient to enumerate all the reduced squares of the size under consideration.

3. *Enumeration of the* 6×6 *squares.*

There does not appear to be any generating process which when applied to a reduced square will generate a fixed number of other reduced squares, all different. The process of intramutation, however, enables the enumeration to be carried out by sets of varying sizes, of which 120 is the largest and most frequent, members of the same set having certain characteristics of the leading diagonal which are unaltered by intramutation. The actual enumeration consists of three stages.

(*a*) *The exhaustive enumeration of the possible types of leading diagonals.* These are listed for the 6×6 squares in the first and last columns of Table I. For example, diagonals containing two different letters other than A and four A's can all be derived from diagonals containing four A's, one B and one C. All diagonals containing these letters are not, however, derivable from one another; for if, for example, B falls in the column headed by C and C in the column headed by B this property is invariant, and intramutations of the other letters, or B and C, will not change it. It will be seen that any diagonal containing two different letters other than A and four A's can be derived from one of the diagonals $ACBAAA$, $ACAAAB$, or $AAAABC$, in which respectively both, one, or neither letter falls in the column headed by the other letter. These three diagonals are therefore taken as examples of the three diagonal types containing two different letters other than A.

(*b*) *The determination of the number of distinct diagonals which can be generated by intramutation from each typical diagonal.* This presents no difficulty. There are, for example, 10 possible diagonals of the type $ACBAAA$, and 60 of the type $ACAAAB$.

(*c*) *The enumeration by trial of all possible reduced squares having the given typical diagonals.* This task, though it appears considerable, is not really onerous for 6 × 6 squares. In some cases intramutations which leave the diagonal unchanged may be used to shorten the enumeration.

When these operations have been performed the number of squares is determined. The number of squares having the typical diagonal $ACBAAA$, for example, is 8, and since the number of diagonals derivable from the typical diagonal (including the diagonal itself) is 10, this diagonal contributes 80 squares to the total.

Intramutations which leave the diagonal unaltered may give a different square, and consequently all the squares having a given typical diagonal do not necessarily belong to different intramutation sets. Thus by applying a suitable selection of those intramutations which leave the diagonal $ACBAAA$ unchanged to the 8 squares having this diagonal (4 squares and their conjugates) it is found that two squares are connected by intramutation to their conjugates, and the other two each to the conjugate of the other. There are thus four intramutation sets, all of 20 squares; two of these include conjugates, and the other two form a conjugate pair of sets, every square of one set being conjugate to a square of the other. The four sets are thus representable on three cards. The complete classification is set out in Table I. There are in all, as will be seen from the table, 8 self-conjugate sets, 25 sets including conjugates and 39 conjugate pairs of sets.

To obtain a general permutation of letters, intramutation must be supplemented by a change of another type. This consists of bringing any chosen one of the 36 letters to the top left-hand corner by permutation of the rows and columns, at the same time interchanging this letter with A. The rows and columns can be rearranged in the appropriate order to give a reduced square, and with this restriction there will be 36 such changes (which we call *change of corner element*). The 36 changes of corner element, together with the 6! 5! permutations of rows and columns giving non-reduced squares, combine with the 5! possible intramutations to give the whole $(6!)^3$ permutations of the general transformations.

The determination of the connections between the intramutation sets of the 6×6 squares is fairly easily performed, for when once the 36 diagonals generated by change of corner element have been written down for any one square, and identified with typical diagonals, the number of squares with which the given square can possibly be connected is seen to be very limited, and only a few full transformations are necessary. There cannot be any connection between two squares which give a different set of 36 typical diagonals, so that there are few cases where there is any danger of overlooking a connection. The whole process forms an excellent check on the original enumeration.

These connections are also exhibited in Table I, by means of the Roman numerals. The whole of the 111 intramutation sets are comprised by 22 transformation sets, 10 of which form 5 conjugate pairs. Examples from each of these sets are exhibited in the next section. The pair of conjugate sets with diagonal *ACBAAA*, for instance, belongs to the pair of conjugate transformation sets illustrated in Example XI. The only other two pairs of intramutation sets in this pair of transformation sets are the pairs (containing 60 and 40 squares) with diagonals *ACBBCA* and *ACBBBB*. Each of the pair of transformation sets therefore comprises 120 squares.

The greatest number of reduced squares found in any one set is 1080. If every transformation gave a different square there would be $6^2 . 5! = 4320$ reduced squares in the set. The least number of connections found in sets of 6×6 squares is therefore 4.

The adjugacy of transformation sets, i.e. the connections introduced by the interchange of rows with columns or either with letters, is easily established and will be discussed in the next section.

The grand total of all reduced 6×6 Latin squares is 9408, and therefore the total number of 6×6 squares is $9408 . 6! 5! =$ 812,851,200. Jacob(4) obtains 8192 as the number of reduced 6×6 squares. He based his enumeration on the enumeration by trial of all reduced squares having given typical second rows. Since this grouping cuts across the intramutation grouping on which our enumeration is based it is not possible to locate the discrepancies without considerable labour. It is, however, certain that the true number is not less than that of those which we have enumerated, even if we consider the remote possibility that any set has been omitted. On this latter point the reader will, we think, find little difficulty in verifying that the typical diagonals have been completely listed.

It is interesting to notice that the number 9408 is 3 times 56^2, 56 being the number of 5×5 reduced squares. Similarly 56 is

$3\frac{1}{2}$ times 4^2, 4 being the number of 4×4 reduced squares, and 4 is 4 times 1^2, 1 being the number of 3×3 reduced squares. This consideration, for what it is worth, suggests that the number of 7×7 reduced squares may be expected to be of the order of two hundred and fifty million.

4. *The twenty-two transformation sets.*

We now give 17 examples illustrating the 22 transformation sets which have been found.

	I	II
No. of squares:	1080, 1080	(180s + 450 + 450c)
	A B C D E F	*A B C D E F*
	B C F A D E	*B C F E A D*
	C F B E A D	*C F B A D E*
	D E A B F C	*D E A B F C*
	E A D F C B	*E A D F C B*
	F D E C B A	*F D E C B A*
Serial-numbers:	0001–1080 1081–2160	2161–3240

Example I stands for a conjugate pair of sets of 1080 reduced squares, and Example II for a set adjugate to this pair, contàining 180 self-conjugate pairs, one of which is chosen as our example, together with 450 unsymmetrical squares with their 450 conjugates. This trio of sets comprises 3240 reduced squares, which is the largest number that can be illustrated by a single example.

In respect of the occurrence of self-conjugate squares it is worth noting that the elements of the leading diagonal in any of the self-conjugate members are invariant in the sense of being unchanged by intramutation, or by change of corner element within the elements of the diagonal. The number of self-conjugate squares in any transformation set must therefore be one or more sixths of the number of squares in the set.

III	IV
(540 + 540c)	(540 + 540c)
A B C D E F	*A B C D E F*
B A F E C D	*B A E F C D*
C F B A D E	*C F B A D E*
D C E B F A	*D E A B F C*
E D A F B C	*E D F C B A*
F E D C A B	*F C D E A B*
3241–4320	4321–5400

Examples III and IV each stand for a set of 1080 reduced squares. Both the sets comprise all squares conjugate or adjugate to any of their members, and consequently, since there are no self-conjugate squares in either group, each comprises 540 squares and their conjugates.

Adjugacy can connect only transformation sets of the same number of reduced squares, and might theoretically connect either six, three, two, or one such set. In the case of only two sets these may be a conjugate pair, but not a pair of sets containing conjugates, such as those of Examples III and IV. This can be shown by considering the effects of successive interchanges of rows with columns and rows with letters.

In the case of the 6 × 6 squares, the only possible adjugate sets which include more than one transformation set are those consisting of a conjugate pair of transformation sets and one set including conjugates. Examples I and II, already given, are illustrative of this grouping, and, as will be seen as we proceed, all conjugate pairs of transformation sets form parts of adjugate trios of this nature, though there does not appear to be any reason why this should hold universally for squares of higher orders.

V	VI
540, 540	$(90s + 225 + 225c)$
A B C D E F	*A B C D E F*
B A E C F D	*B A F E C D*
C F B A D E	*C F B A D E*
D E F B C A	*D E A B F C*
E D A F B C	*E C D F B A*
F C D E A B	*F D E C A B*
5401–5940	6481–7020
5941–6480	

After the five sets containing 1080 squares each, come four sets each containing 540, these comprising an adjugate trio, and a further single set. Example V illustrates the conjugate pair of sets of 540 squares each, and Example VI the set of 540 which is adjugate to this pair. Of this set one-sixth are self-conjugate (as is the example shown) and the remainder consequently constitute 225 conjugate pairs. This trio therefore comprises 1620 reduced squares.

TABLE I.

Possible diagonals (1)	Number of	
	Self-conjugate sets (2)	Sets including conjugates (3)
AAAAAA	6 (XVI)	30 (IV)
AAAABB	60 (VI)	60 (IX)
AAAABC	—	—
ACAAAB	—	120 (VII)
ACBAAA	—	20 (XII, XVI)
AAABBC	—	120 (IX)
ACAABB	—	120 (IV)
AABABC	—	—
AABACD	—	—
AABCDA	—	—
ACBADA	—	120 (II, VII)
AABBBB	30 (VI)	30 (IV), 120 (III)
AABBCC	120 (II)	120 (IV)
ACBBCA	60 (II, X, XII, XIII)	120 (II)
AABBCD	—	—
AABCBD	—	—
ACABBD	—	—
AADCBB	—	60 (II, VII)
ADACBB	—	—
ACDBAB	—	—
AABCDE	—	—
AABEDC	—	120 (VI, IX)
ACDEBA	—	—
ACBEDA	—	30 (VI, XVI), 60 (VI, IX)
ACBBBB	—	40 (XII, XIV), 120 (II, III)
ADBCBC	—	—
ACBCBD	—	—
AEBCDB	—	—
ADBCBE	—	—
29	8	25

Table I. (*continued*)

squares in **Sets belonging to conjugate pairs** (4)	Cards (5)	Sets (6)	Squares (7)	Impossible diagonals (8)
6 (XV)	3	4	48	
				AAAAAB
—	2	2	120	
60 (V, VIII)	2	4	240	
—	1	1	120	
20 (XI)	3	4	80	
				AAABBB
—	1	1	120	
—	1	1	120	*ACBAAB*
120 (I, IV)	2	4	480	
120 (VII)	1	2	240	
120 (VIII)	1	2	240	*ADBCAA*
—	2	2	240	
				ACABBB
				AABBBC
				ACBABB
30 (V)	4	5	240	
—	2	2	240	
60 (I, X, XI)	8	11	720	
60 (V, VIII)	2	4	240	
120 (VIII)	1	2	240	
120 (I, III)	2	4	480	*ACBABD*
60 (I)	3	4	240	*ACBBAD*
60 (III)	1	2	120	
120 (I, V)	2	4	480	
120 (IV)	1	2	240	
—	2	2	240	*ADBCAE*
30 (V, XV)	2	4	120	
30 (IV)	5	6	240	
40 (XI), 120 (I)	6	8	640	
				ACBCBB
				ADBCBB
				ACBBDB
120 (I, I, V)	3	6	720	*ADBBCC*
120 (I, II, III, IV, VI)	5	10	1200	
120 (V)	1	2	240	*ACBEDB*
120 (I, II, III)	3	6	720	*ACBBDE*
				ACDEFB
				ACBEFD
39	72	111	9408	18

VII

(270 + 270*c*)

A	*B*	*C*	*D*	*E*	*F*
B	*C*	*D*	*E*	*F*	*A*
C	*E*	*A*	*F*	*B*	*D*
D	*F*	*B*	*A*	*C*	*E*
E	*D*	*F*	*B*	*A*	*C*
F	*A*	*E*	*C*	*D*	*B*

7021–7560

There remains one set of 540 squares, illustrated by Example VII. This set comprises all squares conjugate or adjugate to any of its members.

VIII

360, 360

A	*B*	*C*	*D*	*E*	*F*
B	*A*	*E*	*F*	*C*	*D*
C	*F*	*A*	*E*	*D*	*B*
D	*C*	*B*	*A*	*F*	*E*
E	*D*	*F*	*C*	*B*	*A*
F	*E*	*D*	*B*	*A*	*C*

7561–7920
7921–8280

IX

(180 + 180*c*)

A	*B*	*C*	*D*	*E*	*F*
B	*A*	*E*	*F*	*C*	*D*
C	*F*	*A*	*B*	*D*	*E*
D	*E*	*B*	*A*	*F*	*C*
E	*D*	*F*	*C*	*B*	*A*
F	*C*	*D*	*E*	*A*	*B*

8281–8640

The nine sets illustrated so far comprise 7560 squares, leaving only 1848 for the remaining 13 smaller sets. Of these three sets have 360 each, and form an adjugate trio accounting for more than half the remainder. Example VIII represents the conjugate pair of this trio and Example IX the single set of 360, which in this case contains no self-conjugates but 180 conjugate pairs.

X

(60*s* + 60 + 60*c*)

(*a*)

A	*B*	*C*	*D*	*E*	*F*
B	*C*	*F*	*A*	*D*	*E*
C	*F*	*B*	*E*	*A*	*D*
D	*A*	*E*	*B*	*F*	*C*
E	*D*	*A*	*F*	*C*	*B*
F	*E*	*D*	*C*	*B*	*A*

(*b*)

A	*B*	*C*	*D*	*E*	*F*
B	*A*	*D*	*C*	*F*	*E*
C	*D*	*F*	*E*	*B*	*A*
D	*C*	*E*	*F*	*A*	*B*
E	*F*	*B*	*A*	*D*	*C*
F	*E*	*A*	*B*	*C*	*D*

8641–8820

Only one of our sets, that illustrated in X (*a*) and X (*b*), comprises 180 reduced squares, every two of which are therefore connected by 24 distinct transformations. One-third of the squares in this set are self-conjugate, there being two invariant sets of elements each of which may constitute a diagonal of symmetry.

A new feature is introduced in this transformation set in that twelve of the arrangements, such as that illustrated in Example X (*b*), consist of nine 2 × 2 squares. These nine 2 × 2 squares must of course consist of three groups of three squares each, arranged in a 3 × 3 Latin square, squares of the same group containing the same pair of letters. The squares of any group may be all oriented alike or two may be oriented alike and the third at right angles to them. In the set of squares under discussion the 2 × 2 squares of the same group are never oriented alike for all three groups. For this reason the squares termed by Euler squares *à double marche* cannot occur in this group.

XI	XII
120, 120	(60*s* + 30 + 30*c*)
A B C D E F	*A B C D E F*
B C A F D E	*B C A E F D*
C A B E F D	*C A B F D E*
D F E B A C	*D E F B A C*
E D F C B A	*E F D A C B*
F E D A C B	*F D E C B A*
8821–8940	9061–9180
8941–9060	

There are three sets of 120, forming an adjugate trio. The two forming the conjugate pair are shown in Example XI, and the third, which includes 60 symmetrical squares, in Example XII. In this set, therefore, there are three sets of six elements each capable of appearing as diagonals of symmetry. These transformation sets (Examples XI and XII) all comprise arrangements consisting of four 3 × 3 Latin squares, as is shown in the examples chosen.

XIII

60*s*

(*a*)

A	*B*	*C*	*D*	*E*	*F*
B	*C*	*A*	*F*	*D*	*E*
C	*A*	*B*	*E*	*F*	*D*
D	*F*	*E*	*B*	*A*	*C*
E	*D*	*F*	*A*	*C*	*B*
F	*E*	*D*	*C*	*B*	*A*

(*b*)

A	*B*	*C*	*D*	*E*	*F*
B	*C*	*D*	*E*	*F*	*A*
C	*D*	*E*	*F*	*A*	*B*
D	*E*	*F*	*A*	*B*	*C*
E	*F*	*A*	*B*	*C*	*D*
F	*A*	*B*	*C*	*D*	*E*

9181–9240

The remaining sets comprise only 228 squares in all. The set of 60, Example XIII, illustrates the possibilities of symmetry about the diagonal in the highest degree, for every element of the square belongs to one or other of six sets, each of which is capable of appearing as a diagonal of symmetry, and consequently every reduced square of the set is self-conjugate. This set is remarkable in that four members consist of nine 2 × 2 squares (of which one is Euler's square *à double marche*) and six other members of four 3 × 3 squares (including Euler's *triple marche*), but the properties of this peculiarly simple class of Latin squares may be most easily developed by throwing it into the form shown in Example XIII(*b*), in which each line is shifted one place from its position in the line above, and all lines at right angles to the leading diagonal contain only a single letter. This is Euler's arrangement *à simple marche*. The whole transformation set comprises what Jacob has termed "complete cycle" squares (if his definition is taken to include, as he implies, the production of the rows in any order by the "complete cycle" operator). Jacob gives a formula for the number of reduced "complete cycle" squares of side n, namely $(n-1)!/\Phi(n)$, where $\Phi(n)$ is the number of integers, including unity, less than n and prime to it.

XIV

(20 + 20*c*)

A	*B*	*C*	*D*	*E*	*F*
B	*C*	*A*	*E*	*F*	*D*
C	*A*	*B*	*F*	*D*	*E*
D	*F*	*E*	*B*	*A*	*C*
E	*D*	*F*	*C*	*B*	*A*
F	*E*	*D*	*A*	*C*	*B*

9241–9280

There is one set of 40 squares, Example XIV, which comprises 20 squares and their conjugates. Four members of the set consist of four 3 × 3 squares. This set is interesting in that, like the set represented by Example XIII, it and all adjugate squares formed from its members comprise only a single intramutation set.

XV

36, 36

A	*B*	*C*	*D*	*E*	*F*
B	*A*	*F*	*E*	*D*	*C*
C	*D*	*A*	*B*	*F*	*E*
D	*F*	*E*	*A*	*C*	*B*
E	*C*	*B*	*F*	*A*	*D*
F	*E*	*D*	*C*	*B*	*A*

9281–9316
9317–9352

XVI

$(6s + 15 + 15c)$

A	*B*	*C*	*D*	*E*	*F*
B	*A*	*E*	*C*	*F*	*D*
C	*E*	*A*	*F*	*D*	*B*
D	*C*	*F*	*A*	*B*	*E*
E	*F*	*D*	*B*	*A*	*C*
F	*D*	*B*	*E*	*C*	*A*

9353–9388

The three sets of 36 squares, which form the fifth and last adjugate trio, are represented by Examples XV and XVI, the former illustrating the conjugate pair of sets, and the latter the third set of the trio, of which in this case one-sixth are self-conjugate. Although these sets are so small none of them contains squares which can be broken up into 2 × 2 or 3 × 3 squares.

XVII

$(10 + 10c)$

A	*B*	*C*	*D*	*E*	*F*
B	*C*	*A*	*F*	*D*	*E*
C	*A*	*B*	*E*	*F*	*D*
D	*E*	*F*	*A*	*B*	*C*
E	*F*	*D*	*C*	*A*	*B*
F	*D*	*E*	*B*	*C*	*A*

9389–9408

Example XVII represents the smallest set of 20 reduced squares, and concludes our enumeration of the 9408 reduced squares. Like the sets of Examples XIII and XIV it comprises only a single intramutation set, and one-tenth of the members consist of four 3 × 3 squares.

The classification of 4 × 4 and 5 × 5 squares, on lines similar to those which we have employed with the 6 × 6 squares, may be given here for purposes of comparison. In the case of the 4 × 4 squares the 4 reduced squares, all of which are self-conjugate, form two intramutation sets, one containing three squares (the "complete cycle" squares) and the other a single square. Each of these sets, with the corresponding non-reduced squares, constitutes a transformation set. In the case of the 5 × 5 squares the 56 reduced squares form six intramutation sets, of which five belong to a transformation set containing the 25 unsymmetrical reduced squares and their conjugates, and the other forms a transformation set containing the 6 self-conjugate reduced squares, the set of "complete cycle" 5 × 5 squares: These are tabulated in Table II, an example from each transformation set being given. It is clear that each transformation set of both 4 × 4 and 5 × 5 squares must contain all squares adjugate to any of its members.

TABLE II.

Intramutation sets for 5 × 5 *squares.*

Diagonals	Number of squares in: Self-conjugate sets	Sets including conjugates	Sets belonging to conjugate pairs	Sets	Squares
AAAAA	—	2 (I)	—	1	2
AABBB	—	8 (I)	—	1	8
AAECD	—	—	8 (I)	2	16
ACBBB	—	24 (I)	—	1	24
AEBCD	6 (II)	—	—	1	6
5	1	3	1	6	56

I	II
(25 + 25*c*)	6*s*
A B C D E	*A B C D E*
B A D E C	*B C D E A*
C E A B D	*C D E A B*
D C E A B	*D E A B C*
E D B C A	*E A B C D*

5. *Conclusions.*

1. It has been shown that the number of reduced 6 × 6 Latin squares can be enumerated without extravagant labour. This is done by means of the special type of transformation which we term

intramutation, and is made possible through the existence of properties of the leading diagonal which are invariant under such transformations. The number of reduced squares is thus enumerated for the most part in sets or pairs of sets of 120, i.e. $(n-1)!$; only a minority of the squares belong to smaller sets. If the same method were to be applied to 7 × 7 squares, supposing there to be about 250,000,000 of these, we must anticipate nearly 200,000 sets or pairs of sets, since the greatest number in any such pair will be 2.6! or 1440.

2. On the other hand, using the sets generated by a general transformation involving independent permutations of rows, columns, and letters, the aggregate of 6 × 6 squares has been shown to be easily derivable from only 17 examples, representing only 12 distinct types of squares. It is not easy to suppose that any similar grouping could reduce the number of typical 7 × 7 squares below about 10,000, so that their enumeration would, by any means at present available, be exceedingly tedious.

3. The number of reduced 6 × 6 Latin squares given by Jacob, namely 8192, is too small, for our complete enumeration gives 9408 reduced squares. This happens rather oddly to be 168 times 56, the number of reduced 5 × 5 squares, or three times the square of that number.

4. Euler's conclusion that no Graeco-Latin 6 × 6 square exists is easily verified from the 12 types of 6 × 6 Latin squares exemplified in this paper.

REFERENCES

(1) L. Euler, "Recherches sur une nouvelle espèce de quarrés magiques", *Verh. v. h. Zeeuwsch Genootsch. der Wetensch., Vlissingen*, 9 (1782), 85–239.

(2) A. Cayley, "On Latin squares", *Mess. of Math.* 19 (1890), 135–137.

(3) P. A. MacMahon, *Combinatory Analysis* (Cambridge, 1915, 1916).

(4) S. M. Jacob, "The enumeration of the Latin rectangle of depth three by means of a formula of reduction, with other theorems relating to non-clashing substitutions and Latin squares", *Proc. Lond. Math. Soc.* (2), 31 (1930), 329–354.