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# RAMASWAMI'S DUALITY AND PROBABILISTIC ALGORITHMS FOR DETERMINING THE RATE MATRIX FOR A STRUCTURED $GI/M/1$ MARKOV CHAIN

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**Abstract**

We show that Algorithm H* for the determination of the rate matrix of a block-$GI/M/1$ Markov chain is related by duality to Algorithm H for the determination of the fundamental matrix of a block-$M/G/1$ Markov chain. Duality is used to generate some efficient algorithms for finding the rate matrix in a quasi-birth-and-death process.

## 1. Introduction

In a companion article [8] we constructed a probabilistic algorithm for the determination of the rate matrix $R$ of an irreducible block-$GI/M/1$ Markov chain and showed with benchmark numerical experiments that our procedure, Algorithm H*, compares favourably with existing methods in the literature. Algorithm H* assumes irreducibility of the Markov chain but makes no assumptions about ergodicity.

The notation was chosen to emphasise a duality, indicated by *, with Algorithm H for the determination of the fundamental matrix $G$ for a block-$M/G/1$ Markov chain. The duality was not established in [8]. Algorithm H is derived in [7] and can be shown to reduce to a version of the cyclic reduction algorithm of Bini and Meini for the determination of $G$ when further technical conditions are imposed. For an exposition of the cyclic reduction methodology the reader is referred to [2, 3, 4] and [11].

In this paper we take these ideas further. We manifest explicitly the duality between Algorithms H and H* and show how in the QBD case several other probabilistic algorithms can be constructed for the efficient calculation of $R$. We shall also find relations between Algorithm H*, the logarithmic reduction algorithm of Latouche and

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Ramaswami [10] and the cyclic reduction algorithms of Bini and Meini. To avoid repetition, we assume familiarity with the ideas and notation of [7] and [8].

## 2. The duality $\mathscr{A}_j \longleftrightarrow \mathscr{C}_j$

The duality $^*$ between processes of structured $GI/M/1$ and $M/G/1$ types was introduced by Ramaswami [13]. An alternative derivation based on time reversal was presented subsequently by Asmussen and Ramaswami [1]. Further developments are given in Bright [5].

The duality is between classes of block-$M/G/1$ chains and classes of block-$GI/M/1$ chains. The anomalous leading block row and column in the one-step transition matrices for these two paradigms do not enter into the duality and it is convenient to omit these and relabel the remaining rows and columns as $0, 1, \ldots$. So we replace the $M/G/1$ chain by a chain $\mathscr{A}_0$ on levels $\ell \geq 0$ given by the structured one-step transition matrix

$$P^{(0)} = \begin{bmatrix} A_1 & A_2 & A_3 & \cdots \\ A_0 & A_1 & A_2 & \cdots \\ 0 & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and similarly replace the $GI/M/1$ chain by a chain $\mathscr{C}_0$ on levels $\ell \geq 0$ given by the structured one-step transition matrix

$$P^{*(0)} = \begin{bmatrix} C_1 & C_0 & 0 & 0 & \cdots \\ C_2 & C_1 & C_0 & 0 & \cdots \\ C_3 & C_2 & C_1 & C_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It should be noted that these matrices are substochastic.

The blocks of both matrices are $k \times k$. The matrix $C = \sum_{j=0}^{\infty} C_j$ is taken as stochastic and irreducible and so has an invariant probability measure $c$, that is, $cC = c$. The entries of $c$ are all positive since $C$ is irreducible. Set $\Delta = \operatorname{diag}(c)$. Ramaswami's duality is given by $A_m = \Delta^{-1} C_m^T \Delta$ for all $m \geq 0$. Clearly the matrices $A_m$ have nonnegative entries. For the notion of duality to be meaningful in the context of Markov chains, it is helpful for the matrix $A = \sum_{m=0}^{\infty} A_m$ to be stochastic. This is immediate. We have $A = \Delta^{-1} C^T \Delta$ or $C = \Delta^{-1} A^T \Delta$, so that $cC = c$ can be expressed as $c\Delta^{-1} A^T \Delta = c$ or $c\Delta^{-1} A^T = c\Delta^{-1}$. Since $c\Delta^{-1} = e^T$, where $e$ is a suitable vector of units (here of length $k$), we thus have $Ae = e$ and so $A$ is stochastic.

The duality $A_n \longleftrightarrow C_n$ induces a correspondence between $P^{(0)}$ and $P^{*(0)}$. Denote by $(P^{(0)})_{n,m}$, $(P^{*(0)})_{n,m}$ respectively the $(n, m)$ block entries in these two matrices $(n, m \geq 0)$. Then

$$\left(P^{(0)}\right)_{n,m} = \Delta^{-1}\left[\left(P^{*(0)}\right)_{m,n}\right]^T \Delta,$$

which we may express as $P^{(0)} \longleftrightarrow P^{*(0)}$ or

$$\mathscr{A}_0 \longleftrightarrow \mathscr{C}_0. \tag{2.1}$$

This provides a basis for an inductive proof of the following theorem, which extends the duality to one between the sequence $(\mathscr{A}_j)_{j\geq 0}$ of censored processes involved in the construction of Algorithm H and the sequence $(\mathscr{C}_j)_{j\geq 0}$ of censored processes used in the construction of Algorithm H$^*$. This entails an extension of the duality to multistep transitions involving taboo levels.

THEOREM 2.1. *When $A$ is irreducible, we have the duality*

$$\mathscr{A}_j \longleftrightarrow \mathscr{C}_j \quad \textit{for each} \quad j \geq 0. \tag{2.2}$$

PROOF. Suppose (2.2) holds for some $j \geq 0$. We have

$$P^{(j)} = \begin{bmatrix} B_1^{(j)} & B_2^{(j)} & B_3^{(j)} & \cdots \\ A_0^{(j)} & A_1^{(j)} & A_2^{(j)} & \cdots \\ 0 & A_0^{(j)} & A_1^{(j)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad P^{*(j)} = \begin{bmatrix} D_1^{(j)} & C_0^{(j)} & 0 & 0 & \cdots \\ D_2^{(j)} & C_1^{(j)} & C_0^{(j)} & 0 & \cdots \\ D_3^{(j)} & C_2^{(j)} & C_1^{(j)} & C_0^{(j)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

so that $B_n^{(j)} = \Delta^{-1}\left(D_n^{(j)}\right)^T \Delta$, $n > 0$, and $A_n^{(j)} = \Delta^{-1}\left(C_n^{(j)}\right)^T \Delta$, $n \geq 0$.

Therefore

$$\begin{aligned} K_0^{(j+1)} &= \sum_{i=0}^{\infty}\left(A_1^{(j)}\right)^i = \sum_{i=0}^{\infty}\left\{\Delta^{-1}\left(C_1^{(j)}\right)^T \Delta\right\}^i = \Delta^{-1}\sum_{i=0}^{\infty}\left\{\left(C_1^{(j)}\right)^i\right\}^T \Delta \\ &= \Delta^{-1}\left(\left[I - C_1^{(j)}\right]^{-1}\right)^T \Delta = \Delta^{-1}\left(\boldsymbol{K}_0^{(j+1)}\right)^T \Delta, \end{aligned}$$

so that $K_0^{(j+1)} \longleftrightarrow \boldsymbol{K}_0^{(j+1)}$. This provides the basis for an (inner) induction that

$$K_m^{(j+1)} \longleftrightarrow \boldsymbol{K}_m^{(j+1)} \quad \text{for} \quad m \geq 0. \tag{2.3}$$

Suppose that (2.3) holds for $m = 0, 1, \ldots, n-1$ for some $n \geq 1$. Then from the equation

$$K_n^{(j+1)} = \sum_{m=0}^{n-1} K_m^{(j+1)} A_{2(n-m)+1}^{(j)} K_0^{(j+1)}$$

derived in [7, Section 3], we have

$$K_n^{(j+1)} = \sum_{m=0}^{n-1} \left\{\Delta^{-1}\left(\boldsymbol{K}_m^{(j+1)}\right)^T \Delta\right\} \left\{\Delta^{-1}\left(C_{2(n-m)+1}^{(j)}\right)^T \Delta\right\} \left\{\Delta^{-1}\left(\boldsymbol{K}_0^{(j+1)}\right)^T \Delta\right\}$$
$$= \Delta^{-1}\left(\sum_{m=0}^{n-1} \boldsymbol{K}_0^{(j+1)} C_{2(n-m)+1}^{(j)} \boldsymbol{K}_m^{(j+1)}\right)^T \Delta = \Delta^{-1}\left(\boldsymbol{K}_n^{(j+1)}\right)^T \Delta$$

by [8, (3.7)], which gives the inductive step in the inner induction.

Using this result, we have similarly from the equation

$$L_n^{(j+1)} = \sum_{m=0}^{n} K_m^{(j+1)} A_{2(n-m)}^{(j)}$$

(derived in [7]) and [8, (3.4)] that $L_n^{(j+1)} \longleftrightarrow \boldsymbol{L}_n^{(j+1)}$ for $n \geq 0$.

Finally we have from the relation

$$B_n^{(j+1)} = B_{2n-1}^{(j)} + \sum_{m=1}^{n} B_{2m}^{(j)} L_{n-m}^{(j+1)}$$

derived in [7] for $n \geq 1$ that

$$B_n^{(j+1)} = \Delta^{-1}\left(D_{2n-1}^{(j)}\right)^T \Delta + \sum_{m=1}^{n} \left\{\Delta^{-1}\left(D_{2m}^{(j)}\right)^T \Delta\right\} \left\{\Delta^{-1}\left(\boldsymbol{L}_{n-m}^{(j+1)}\right)^T \Delta\right\}$$
$$= \Delta^{-1}\left[D_{2m}^{(j)} + \sum_{m=1}^{n} \boldsymbol{L}_{n-m}^{(j+1)} D_{2m}^{(j)}\right]^T \Delta = \Delta^{-1}\left[D_n^{(j+1)}\right]^T \Delta$$

by [8, (3.3)], so that $B_n^{(j+1)} \longleftrightarrow D_n^{(j+1)}$ for $n \geq 1$.

A similar argument yields $A_n^{(j+1)} \longleftrightarrow C_n^{(j+1)}$ for $n \geq 0$, completing the outer induction.

COROLLARY 2.2. *It follows from Theorem 2.1 that*

$$\left[I - B_1^{(N)}\right]^{-1} A_0 \longleftrightarrow C_0 \left[I - D_1^{(N)}\right]^{-1}.$$

*The successive approximants to $G$, $R$ derived respectively by Algorithms* H *and* H* *are of the forms of the left- and right-hand sides. Hence the $N$-th approximants $T_N$, $T_N^*$ to $G$ and $R$ given by these algorithms satisfy*

$$T_N \longleftrightarrow T_N^*. \tag{2.4}$$

Relation (2.4) is a path-restricted version of the standard duality result $G \longleftrightarrow R$ between block-$M/G/1$ chains and block-$GI/M/1$ chains. It illustrates that when $A$ is irreducible, duality can be applied directly to Algorithm H for $G$ to produce Algorithm H* for $R$.

## 3. Quasi-birth-and-death-chains

**3.1. Algorithm H*** In the case of a QBD, substantial simplifications occur in the equations prescribing Algorithm H$^*$, as presented in [8, Section 4]. Since $C_n^{(j)} = 0$ for $n > 2$, we have $\boldsymbol{K}_n^{(j)} = 0$ for $n > 0$ and so

$$\boldsymbol{L}_n^{(j+1)} = \begin{cases} C_0^{(j)} \boldsymbol{K}_0^{(j+1)}, & n = 0; \\ C_2^{(j)} \boldsymbol{K}_0^{(j+1)}, & n = 1; \\ 0, & n > 1. \end{cases}$$

Also $D_n^{(j)} = 0$ for $n > 2$.

The relations linking $\mathscr{C}_{j+1}$ and $\mathscr{C}_j$ are thus

$$\begin{aligned} C_n^{(j+1)} &= C_n^{(j)} \boldsymbol{K}_0^{(j+1)} C_n^{(j)} \quad (n = 0, 2), \\ C_1^{(j+1)} &= C_1^{(j)} + C_2^{(j)} \boldsymbol{K}_0^{(j+1)} C_0^{(j)} + C_0^{(j)} \boldsymbol{K}_0^{(j+1)} C_2^{(j)}, \\ D_1^{(j+1)} &= D_1^{(j)} + C_0^{(j)} \boldsymbol{K}_0^{(j+1)} D_2^{(j)}, \\ D_2^{(j+1)} &= C_2^{(j)} \boldsymbol{K}_0^{(j+1)} D_2^{(j)}. \end{aligned}$$

The initialisation is $C_n^{(0)} = C_n$ $(n = 0, 1, 2)$ and $D_n^{(0)} = C_n$ $(n = 1, 2)$.

**3.2. Other QBD methods** In the previous section we showed how Ramaswami's duality can be used to link Algorithm H for the determination of the fundamental matrix in a block-$M/G/1$ Markov chain and Algorithm H$^*$ for the determination of the rate matrix in a block-$GI/M/1$ Markov chain. Operationally, we could have used duality to induce Algorithm H$^*$ from Algorithm H.

We may also dualise the logarithmic reduction technique Algorithm LR for finding $G$ for a QBD to obtain an Algorithm (LR)$^*$ that can be used for calculating $R$ for a QBD. This was observed by Latouche and Ramaswami in their analysis [10].

There is a further possibility available for a QBD. We have the well-known relations

$$U = C_1 + C_0 G \quad \text{and} \quad R = C_0 (I - U)^{-1}$$

(see Hajek [6] and Latouche [9]). From these, $R$ may be calculated via $U$ once $G$ has been determined. In fact, any Algorithm A for finding $G$ in a QBD gives rise to an Algorithm AU for computing $R$.

This provides us with several methods for computing $R$ for a QBD. Apart from the known Neuts method [12, page 13] and the Schur factorisation, both discussed in [8], we have the new Algorithms (LR)$^*$, HU and H$^*$.

**3.3. Relations between the algorithms** We now consider together the probabilistic Algorithms HU, $H^*$, LRU and $(LR)^*$ proposed for the determination of the rate matrix $R$. Of these only $H^*$ has general applicability beyond the QBD context. We examine the path contributions made to the estimates for $R$ in these algorithms in the QBD case.

Recall that the matrix $U$ is envisaged as referring to visits from level 0 to level 0, with level $-1$ taboo. For $\ell > 0$, the matrix $U(\ell)$ denotes the contribution to $U$ arising from trajectories which do not reach level $\ell$ or higher. Similarly $G$ refers to first visits to level $-1$ from level 0 and $G(\ell)$ is the contribution to $G$ made by trajectories not attaining level $\ell$ $(> 0)$ or higher. Finally, $R$ refers to visits to level 0 from level $-1$ with $-1$ as a taboo level and $R(\ell)$ is the contribution to $R$ made by trajectories not reaching level $\ell$ $(> 0)$ or higher.

The estimate $T_N$ of $G$ made by iteration $N$ of Algorithm H is then $G(2^{N+1})$ and is based on the determination of $U(2^{N+1})$. We have

$$U(2^{N+1}+1) = A_1 + A_2 G(2^{N+1}). \tag{3.1}$$

Hence the estimate of $U$ made in iteration $N$ of Algorithm HU is $U(2^{N+1}+1)$. Also

$$R(2^{N+1}+1) = C_0\left[I - U(2^{N+1}+1)\right]^{-1},$$

so iteration $N$ of Algorithm HU provides the estimate $R(2^{N+1}+1)$ for $R$. The above argument shows incidentally that Algorithm HU is enhanced by the use of (3.1) to calculate a value for $U$ rather than merely using the value of $U$ already employed in estimating $G$.

In the same way, iteration $N$ of Algorithm LR provides the estimate $G(2^{N+1}-1)$ for $G$. Since

$$U(2^{N+1}) = A_1 + A_2 G(2^{N+1}-1) \quad \text{and} \quad R(2^{N+1}) = C_0\left[I - U(2^{N+1})\right]^{-1},$$

the contribution to $R$ from iteration $N$ of Algorithm LRU is $R(2^{N+1})$.

Iteration $N$ of Algorithm $H^*$ incorporates the contributions to $R$ of all paths not involving level $2^{N+1}$ or higher, so that the estimate of $R$ provided by that iteration is also $R(2^{N+1})$.

This shows that, in the QBD case, iteration $N$ of Algorithms LRU and $H^*$ yields a common value to machine accuracy. Runs of the two algorithms with a number of examples confirmed this, so giving a useful check of our codes. Also the CPU times for Algorithm LRU and the simplified form of Algorithm $H^*$ for the QBD case were found to be the same, so that Algorithm LRU may be regarded as simply the QBD case of Algorithm $H^*$.

Finally, iteration $N$ of Algorithm $(LR)^*$ is readily seen to give for $R$ the estimate $R(2^{N+1}-1)$.

Thus we have simple relationships between the estimates of $R$ made by the various algorithms in the QBD case for a common iteration count $N$. Algorithm HU incorporates the contribution of trajectories involving one more level than does the general Algorithm H*, while Algorithm (LR)* involves one fewer. Algorithm LRU coincides with the general algorithm in the QBD case.

**3.4. Numerics** The preceding discussion indicates that one might expect results of very comparable but slightly decreasing accuracy as we move through use of Algorithm HU to Algorithms H* and (LR)* in turn to evaluate $R$ for a QBD. Table 1 illustrates this for [8, Experiment 2].

TABLE 1. Results for [8, Experiment 2].

| Case | Method | Iterations $I$ | $\|R_I - C(R_I)\|_\infty$ | CPU Time (sec.) |
|---|---|---|---|---|
| 1 | HU | 12 | 9.5665e-13 | 0.010 |
| | H* | 12 | 9.5847e-13 | 0.010 |
| | (LR)* | 12 | 9.6029e-13 | 0.010 |
| | | | | |
| 2 | HU | 12 | 8.8818e-16 | 0.010 |
| | H* | 12 | 8.8818e-16 | 0.010 |
| | (LR)* | 12 | 9.9920e-16 | 0.010 |
| | | | | |
| 3 | HU | 10 | 4.2960e-12 | 0.010 |
| | H* | 10 | 4.3280e-12 | 0.010 |
| | (LR)* | 10 | 4.3605e-12 | 0.010 |
| | | | | |
| 4 | HU | 10 | 4.6130e-14 | 0.010 |
| | H* | 10 | 4.6629e-14 | 0.010 |
| | (LR)* | 10 | 4.7073e-14 | 0.010 |

## 4. Error measures

In [11] Meini noted that, in the absence of an analysis of numerical stability, the common error measure $\|e - G_I e\|_\infty$ for an approximation $G_I$ to a stochastic fundamental matrix $G$ may not be appropriate for the invariant subspace method. She proposed instead the measure $\|G_I - A(G_I)\|_\infty$, which is also appropriate in the case of substochastic $G$. We now consider related isues for the error measure

$$\|R_I - C(R_I)\|$$

for an approximation $R_I$ to $R$.

We make use of the QBD given by

$$C_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1-rp \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & p \\ rp & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1-p & 0 \\ 0 & 0 \end{bmatrix}$$

with $r \geq 1$ and $0 < p < 1/r$.

With these parameter choices, the QBD is irreducible. It is null recurrent for $r = 1$ and positive recurrent for $r > 1$, with rate matrix

$$R = \begin{bmatrix} 0 & 0 \\ (1-pr)/(1-p) & (1-pr)/(1-p) \end{bmatrix}.$$

We readily verify that, for a matrix

$$R_I = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix} \quad \text{with} \quad 0 \leq x, y \leq \frac{1-pr}{1-p}, \tag{4.1}$$

we have

$$C(R_I) = \begin{bmatrix} 0 & 0 \\ rpy + (1-p)xy & 1-pr+px \end{bmatrix}.$$

Take $r = 1$ and $p = 1/2$ and put $R_0 = \left[\begin{smallmatrix} 0 & 0 \\ 0.5 & 0.5 \end{smallmatrix}\right]$ and $R_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0.6 & 0.9 \end{smallmatrix}\right]$. Then

$$\|R - R_1\|_\infty = 0.4 < 0.5 = \|R - R_0\|_\infty.$$

Also $C(R_0) = \left[\begin{smallmatrix} 0 & 0 \\ 0.375 & 0.75 \end{smallmatrix}\right]$, so that $\|R_0 - C(R_0)\|_\infty = 0.25$, and $C(R_1) = \left[\begin{smallmatrix} 0 & 0 \\ 0.57 & 0.95 \end{smallmatrix}\right]$, so that $\|R_1 - C(R_1)\|_\infty = 0.35$.

We thus have an example for which $\|R - R_1\|_\infty < \|R - R_0\|_\infty$, and in fact $0 \leq R_0 \leq R_1 \leq R$, but $\|R_0 - C(R_0)\|_\infty < \|R_1 - C(R_1)\|_\infty$.

Now let $R_I$ be as in (4.1) with $x < (1-pr)/(1-r)$ and $y < 1$. Then

$$x(1-p)(1-y) < (1-pr)(1-y)$$

or

$$x - [rpy + xy(1-p)] < [1-pr+px] - y,$$

so that $\Phi_1 < -\Phi_2$, where $\Phi_i := [R_I - C(R_I)]_{2,i}$ $(i = 1, 2)$. It follows at once that if $\Phi_2 > 0$, then $\Phi_1 < 0$.

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