CONCERNING t-SPREADS OF $PG((s+1)(t+1)-1,q)$

by

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Statement

The contents of this thesis have not been submitted to any university for the purpose of obtaining any other degree or diploma. Also, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text. I consent to the thesis being made available for photocopying and loan.

Christine M. O'Keefe

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SUMMARY

In this thesis the theory of 1-spreads of $PG(3,q)$ is generalised to a theory of t-spreads of $PG((s+1)(t+1)-1,q)$. There is a well developed theory for t-spreads of $PG(2t+1,q)$, but so far there are limited results in other cases. This thesis extends much of the existing theory to the general case of t-spreads of $PG((s+1)(t+1)-1,q)$.

After a short Introduction containing a literature review, Chapter One of the thesis gives a brief account of the concepts involved.

In Chapter Two the theory of t-spreads of $PG(2t+1,q)$ is revised, setting the scene for the generalisation to come in Chapter Three. Most of the work in this Chapter is well known, but in order to facilitate the later generalisation, some of the presentation is different from the original. For example the concept of regularity is presented in the light of the connection between a regulus of $PG(2t+1,q)$ and the classical Segre Variety which is the product of a line and a t-dimensional space of $PG(2t+1,q)$. In addition, a new and straightforward construction is given for a spread set (originally defined in Bruck and Bose (1964)) corresponding to a t-spread of $PG(2t+1,q)$. This new construction uses the space $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ introduced by Thas (1971).

Chapter Three gives results for t-spreads of $PG((s+1)(t+1)-1,q)$ suggested by the theory studied in Chapter Two. A generalised t-spread set of matrices for certain of these t-spreads is found, using a construction similar to that given by Bruck and Bose (1964). In addition, the new construction of a spread set discussed in Chapter Two generalises naturally to give a new but related entity, to be called a projective t-spread set. This concept is more general because any t-spread of

$PG((s+1)(t+1)-1,q)$ admits a projective t-spread set, but not every t-spread admits a t-spread set. Regularity of a t-spread of $PG((s+1)(t+1)-1,q)$ is explored using the properties of the classical Segre Variety. Different subvarieties produce different reguli of a t-spread, and therefore corresponding different types of regularity. It is shown, however, that all these types are equivalent and coincide with the usual notion of regularity in the few cases where a definition has previously been given, for example in the case of 1-spreads of $PG(2s+1,q)$ in Ebert (1983). The approach developed in this Chapter leads to the construction of an indicator set for a t-spread of $PG((s+1)(t+1)-1,q)$, extending the work of Lunardon (1984). It also yields a representation for regular t-spreads of $PG((s+1)(t+1)-1,q)$, generalising that due to Bruck (1969) for 1-spreads of $PG(3,q)$. A t-spread is called geometric if for any pair X, Y of its elements, the other elements of the t-spread are either contained in or are skew to the space $< X, Y >$ spanned by X and Y. It is shown that a geometric t-spread allows a construction of an affine space $AG(s+1, q^{t+1})$, and the projective t-spread set provides homogeneous coordinates for the corresponding projective space. Examples are given to illustrate the ideas presented.

The next Chapter considers certain partial t-spreads, and in particular those called k-sets of t-dimensional subspaces. These have been studied in the space $PG(3t+2,q)$ by the authors Thas (1971), Declerck et al (1987), Casse and Wild (1983), Casse et al (1985) and Wild (1986). Some new concepts and results are given. The definition of k-sets is then extended to (k,n)-sets of $PG(3t+2,q)$, and connections with work already done by Beutelspacher (1975) and Declerck et al (1987) (in the case of $s=2$) are explored. A maximal (k,n)-set is defined, and its size is determined. A condition guaranteeing that such a set arises from the construction of Thas is found, and applied to maximal $(k,3)$-sets of $PG(5,3^h)$ and maximal (k, n)-sets of $PG(3t+2, 2)$ when $t > 1$. An example of a 4-set ((4,2)-set) of lines of $PG(5, 2)$ is given, which does not arise from the construction due to Thas (1971). This set is contained in a spread which contains no regulus.

A short conclusion and suggestions for further research appear in Chapter Five.

INTRODUCTION

The subject of this thesis is the theory of t-spreads of $PG((s+1)(t+1)-1,q)$. A t-spread of $PG(n,q)$ is a partition of the points of $PG(n,q)$ into t-dimensional subspaces. It is known that $PG(n,q)$ admits a t-spread if and only if $t+1$ divides $n+1$, thus we choose $n+1=(s+1)(t+1)$.

Our aim is to develop a unified algebraic and geometric theory applicable to all t-spreads of $PG((s+1)(t+1)-1,q)$ and which contains the known results on t-spreads of certain projective spaces as special cases of the new theory.

The rather natural idea of partitioning the points of a finite structure has appeared in many contexts in the development of mathematics, particularly in the field of abelian groups. This is especially significant because the abelian groups have been shown to correspond to finite projective geometries, and certain partitions of the abelian groups correspond to t-spreads in the projective geometries (see Carmichael (1937)). In fact the result quoted above on the existence of t-spreads of $PG(n,q)$ appeared in Burnside (1911) as a theorem on partitions of abelian groups.

It was not until later that t-spreads of $PG(n,q)$ were studied in their own right. Burnside's result was rediscovered in geometrical language by André (1954), Bruck and Bose (1964) and Segre (1964). At the same time the connection between t-spreads of $PG(2t+1,q)$ and finite translation planes was discovered by André (1954) and Bruck and Bose (1964) and (1966). It was shown that a t-spread of $PG(2t+1,q)$ could be used to construct a translation plane of order q^{t+1}, and that all finite translation planes arise from such a construction. This connection added great impetus to the study of t-spreads, yielding many interesting results on translation planes (see for example Ostrom (1968), Hughes and Piper (1973), or Lüneburg (1980)).

The research on t-spreads of $PG(2t+1, q)$ was directed by the discovery (Bruck and Bose (1964)) that the translation plane corresponding to a t-spread of $PG(2t+1, q)$ is Desarguesian if and only if the corresponding t-spread is regular. A classification of all t-spreads of $PG(2t+1, q)$ would imply a classification of all finite translation planes. Much effort has been directed to the study of regular, subregular and aregular t-spreads of $PG(2t+1, q)$. The problem of classifying the t-spreads of $PG(2t+1, q)$ has only been effected in a small number of cases, for example every t-spread of $PG(2t+1, 2)$ is regular (see Dembowski (1968), p221), and every 1-spread of $PG(3, q)$ is either regular or is subregular of index 1 (see Bruck (1969) and Denniston (1973b)). Complete results in other cases seem difficult to obtain, see for example Bruck (1969) and (1973c), Bruen (1975), Orr (1976) and Ebert (1983). Examples of aregular t-spreads are given by Denniston (1973a), Bruck (1969) and Bruen (1972a). An interesting result in this area is the construction by Denniston (1976) of a t-spread which contains reguli but is not subregular.

In Bruck and Bose (1964) and (1966), it was also shown that to a t-spread of $PG(2t+1, q)$ there corresponds a set of q^{t+1} $(t+1) \times (t+1)$ matrices, called a *spread set*. This construction of a spread set enabled an easy coordinatisation of the translation plane corresponding to the t-spread. The following characterisation was given: the spread set forms a field under addition and multiplication of matrices if and only if the t-spread is regular. Later Maduram (1975) used this representation of translation planes to describe in terms of matrices the condition that two t-spreads represent isomorphic translation planes, and to exhibit a new characterisation of Desarguesian planes: a translation plane is Desarguesian if and only if all pairs of matrix representations are equivalent.

Apart from purely geometrical arguments conducted entirely in the space $PG(2t+1, q)$, an important method of studying t-spreads of $PG(2t+1, q)$ is by indicator sets. These were introduced for 1-spreads of $PG(3, q)$ in Bruen (1972a), and used for the study of the regularity, subregularity and aregularity of 1-spreads in $PG(3, q)$ by Bruen (1972a) and (1975) and Sherk and Pabst (1977). The generalisation to indicator sets of t-spreads of $PG(2t+1, q)$ was effected by Sherk (1979), with a construction relying on knowledge of a spread set for the spread, as defined by Bruck and Bose (1964). Lunardon (1984) was able to construct indicator sets in a purely geometrical manner, avoiding the use of spread sets. Again, this was employed to study different types of t-spreads and different translation planes arising from the t-spreads.

The current uses of the term regularity appearing in the literature apply only in the cases of t-spreads of $PG(2t+1, q)$ (Dembowski (1968)) and 1-spreads of $PG(2s+1, q)$ (Ebert (1983)). A t-spread $\mathcal{W}$ of $PG(2t+1, q)$ is called *regular* if given any line of $PG(2t+1, q)$, not meeting any element of $\mathcal{W}$ in more than one point, then the elements of $\mathcal{W}$ meeting the line form a regulus in $PG(2t+1, q)$. A 1-spread $\mathcal{W}$ of $PG(2s+1, q)$ is called *regular* if given any line of $PG(2s+1, q)$, not contained in $\mathcal{W}$, then the elements of $\mathcal{W}$ meeting the line form a regulus in a 3-dimensional subspace $PG(3, q)$ of $PG(2s+1, q)$. Thus both notions of regularity rely on the concept of a regulus of t-dimensional spaces in $PG(2t+1, q)$. To be able to extend the definition of regularity to t-spreads of $PG((s+1)(t+1)-1, q)$, a more general concept of regulus is required.

This thesis addresses the problem of defining a more general regulus. We re-examine the known case of t-reguli of $PG(2t+1, q)$, altering our point of view into one which is easily generalised. To be precise, it is shown that a t-regulus of $PG(2t+1, q)$ is just the set of t-dimensional spaces lying on a classical Segre variety $\mathcal{SV}_{t+1,2}$ whose opposite system of subspaces consists of lines. The natural generalisation leads us to say that the set of t-dimensional spaces lying on a Segre variety $\mathcal{SV}_{t+1,r+1}$ in $PG((r+1)(t+1)-1,q)$ is a *t-regulus of rank r*.

The notion of the t-regulus of rank r is used in the definition of regularity of rank r of a t-spread of $PG((s+1)(t+1)-1,q)$ as follows. A t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1,q)$ is said to be *regular of rank r* if whenever S_r is an r-dimensional subspace of $PG((s+1)(t+1)-1,q)$ not meeting any element of $\mathcal{W}$ in more than one point, then the set of elements of $\mathcal{W}$ meeting S_r is a regulus of rank r in some $((r+1)(t+1)-1)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$. This definition agrees with the two definitions mentioned above, which are examples of regularity of rank 1. It is shown that a t-spread of $PG((s+1)(t+1)-1,q)$ is regular of rank r if and only if it is regular of every rank $0,1,\ldots,s$ and in this case it is called *regular*. It is interesting that in proving various results about regular t-spreads of $PG((s+1)(t+1)-1,q)$, it is the fact that it is regular of rank s that seems most useful, in other words, the use of the t-reguli of rank s yields the most information about the t-spread.

Now that we have a notion of regularity for t-spreads of $PG((s+1)(t+1)-1,q)$, we can extend many of the known results on regular t-spreads of $PG(2t+1,q)$ to the general case. For example, in a representation due to Bruck (1969) it is shown that a 1-spread of $PG(3,q)$ is regular if and only if there is a line l of the extension $PG(3,q^2)$ skew to $PG(3,q)$ and meeting every element of the 1-spread, necessarily in a unique point. We generalise this to $PG((s+1)(t+1)-1,q)$, showing that: a t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1,q)$ is regular if and only if there is an s-dimensional subspace of the extension $PG((s+1)(t+1)-1,q^{t+1})$ skew to $PG((s+1)(t+1)-1,q^t)$ and meeting every element of the t-spread in a unique point. This result is used to show that a t-spread of $PG((s+1)(t+1)-1,q)$

is regular if and only if it is geometric.

One of the fundamental results we present is the connection between two seemingly different sets, each constructed from a t-spread of $PG(2t+1,q)$. First there is the spread set of matrices due to Bruck and Bose (1964), which we will call a *t-spread set*. Second there is the set of points of the space $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$ introduced by Thas (1971). This set of points is represented as a set of equivalence classes of pairs of $(t+1)\times(t+1)$ matrices, where two pairs are equivalent if and only if the matrices of one can be obtained from the matrices of the other by multiplication by a non-singular matrix. This set will be called a *projective t-spread set*. It is shown in Chapter Two that in $PG(2t+1,q)$ a t-spread set yields a projective t-spread set, and conversely a projective t-spread set yields a t-spread set. We show that a t-spread of $PG(2t+1,q)$ is regular if and only if its projective t-spread set is isomorphic to $PG(1,q)$. Since a projective t-spread set can be constructed knowing only a basis for each element of the t-spread, this result gives a new and straightforward construction of the t-spread set of any t-spread of $PG(2t+1,q)$.

The above methods are generalised in Chapter Three to construct projective t-spread sets and t-spread sets for t-spreads of $PG((s+1)(t+1)-1,q)$, by using the space $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$. It is shown that a t-spread of $PG((s+1)(t+1)-1,q)$ is regular if and only if its corresponding projective t-spread set is isomorphic to $PG(s,q^{t+1})$. The new definition of projective t-spread set emphasises the projective nature of t-spreads, which was possibly obscured by Bruck and Bose's construction of the spread set as a non-homogeneous entity.

We are able to generalise the construction of an affine plane from a t-spread of $PG(2t+1,q)$ to construct an affine space $AG(s+1,q^{t+1})$ from a t-spread of

$PG((s+1)(t+1)-1,q)$ with $s > 1$ only under the additional assumption that the t-spread is geometric, or equivalently regular. The fact that we need to make this extra assumption is not surprising if we recall that in the case $s = 1$ our affine plane is a translation plane, and is Desarguesian if and only if the t-spread is regular. When $s > 1$ the affine space $AG(s+1, q^{t+1})$ is always Desarguesian, and so its subplanes are always Desarguesian. In fact these arguments are used to give a new proof of the known result: a geometric t-spread induces a regular t-spread on a $(2t+1)$-dimensional subspace which is the space spanned by two t-spread elements.

Now that we have constructed a projective t-spread set for any t-spread of $PG((s+1)(t+1)-1,q)$, we are able to construct an indicator set for any t-spread of $PG((s+1)(t+1)-1,q)$. Thus concepts which have proved fruitful in the study of t-spreads of $PG(2t+1,q)$ can now be applied to the study of t-spreads of any projective space $PG((s+1)(t+1)-1,q)$.

In the literature, attention has also been given to partial t-spreads of $PG(n,q)$, where a partial t-spread is a collection of pairwise skew t-dimensional subspaces. Particular emphasis has been given to the study of *maximal partial t-spreads*, where a partial t-spread is maximal if it is not a t-spread and further is not contained in any t-spread as a proper subset. These are also called *maximal k-spans* (see Hirschfeld (1979)), and in this thesis will be called *complete partial t-spreads*. The main interest has been in answering the questions: how many elements may a partial t-spread have? In particular, how many elements can a complete partial t-spread have? Finally, how many elements must a partial t-spread have to guarantee that it is contained in a unique t-spread? These questions are not treated here, but results appear in Mesner (1967), Glynn (1982), Bruen (1971), (1972b) and (1975), Jungnickel (1984), Beutelspacher (1975), (1976) and

(1980), Bruen and Thas (1976), Ebert (1978) and (1979), and Freeman (1980). These results are collected in Hirschfeld (1985).

The partial t-spreads which are of interest to us in Chapter Four are those which are connected with k-arcs and (k,n)-arcs of projective planes. This connection originates in Thas (1971), but is closely related to the representation of t-spreads of $PG((s+1)(t+1)-1,q)$ in $PG((s+1)(t+1)-1,q^{t+1})$ as the set of t-dimensional spaces meeting an s-dimensional space.

The partial t-spreads connected with k-arcs are called k-sets of $PG(3t+2,q)$, and are sets of k t-dimensional spaces of $PG(3t+2,q)$, any three of which span $PG(3t+2,q)$ (see Casse and Wild (1983)). The connections with the space $\mathcal{S}_2(\mathcal{M}_{t+1}(GF(q)))$ allow the use of the projective t-spread set defined in Chapter Three.

For a k-set of $PG(3t+2,q)$, it is known that $k \leq q^{t+1}+2$ if q is even and $k \leq q^{t+1}+1$ if q is odd. The only known examples of $(q^{t+1}+1)$- and $(q^{t+1}+2)$-sets are constructed in Thas (1971) as follows. Let σ denote the field automorphism $\sigma : x \mapsto x^q$ of $GF(q^{t+1})$. Let Π be a plane of $PG(3t+2,q^{t+1})$ whose $t+1$ conjugates $\Pi, \Pi^{\sigma}, \ldots, \Pi^{\sigma^t}$ span $PG(3t+2,q^{t+1})$. Such a plane Π is called imaginary. Then Π gives rise to a t-spread of $PG(3t+2,q)$ all of whose elements meet it in a unique point. The partial t-spread comprising all the elements of the t-spread meeting Π in the points of a $(q^{t+1}+1)$- or $(q^{t+1}+2)$-arc are a $(q^{t+1}+1)$- or $(q^{t+1}+2)$-set of $PG(3t+2,q)$. The points of the plane Π are a partial indicator set for the partial t-spread whose indicator set is Π. In a similar way, a k-arc of $PG(2,q^{t+1})$ gives rise to a k-set of $PG(3t+2,q)$.

The converse is an interesting question. Given a k-set $\mathcal{K}$ (with possibly some restriction on the size of k) of $PG(3t+2,q)$, is there always an imaginary plane Π

of $PG(3t+2, q^{t+1})$ meeting the extension of every element of $\mathcal{K}$? Or equivalently, is every k-set $\mathcal{K}$ contained in a regular t-spread of $PG(3t+2, q)$? This question has been addressed by, for example, Casse and Wild (1983), Casse et al (1985), Wild (1986) and Declerck et al (1987). They have shown that under certain circumstances, a k-set is contained in a regular t-spread of $PG(3t+2, q)$ or, in other words, arises from the construction described above.

We turn to partial t-spreads which correspond in the same way to (k, n)-arcs of projective planes. The above connection between partial t-spreads and k-arcs suggested a study of (k, n)-arcs of projective planes by the same methods, hopefully leading to examples of maximal (k, n)-arcs, or to demonstrations of the non-existence of these maximal (k, n)-arcs in projective planes of certain orders. The appropriate set of t-dimensional subspaces in $PG(3t+2, q)$ will be called a (k, n)-set. A (k, n)-set of $PG(3t+2, q)$ is defined to be a geometric partial t-spread of $PG(3t+2, q)$ which satisfies the additional property: no $(2t+1)$-dimensional subspace of $PG(3t+2, q)$ contains more than n elements of $\mathcal{K}$, but there is some $(2t+1)$-dimensional subspace containing exactly n elements of $\mathcal{K}$. It is shown that a (k, n)-set has at most $(n-1)q^{t+1}+n$ points, and that sets of this size (called *maximal (k, n)-sets*) can be constructed as above by taking a maximal (k, n)-arc in the plane Π, provided such an arc exists.

The question of existence of maximal (k, n)-sets is therefore closely linked with the question of existence of maximal (k, n)-arcs. It is an open question whether, for $2 \le n \le q^{t+1}-1$, there exist any maximal (k, n)-arcs in $PG(2, q^{t+1})$ with q odd, and hence whether there exist any maximal (k, n)-sets in $PG(3t+2, q)$.

As in the case of k-sets of $PG(3t+2, q)$, the following question is of interest: Do there exist maximal (k, n)-sets of $PG(3t+2, q)$ which do not arise from the above construction using a (k,n)-arc of an imaginary plane Π? This question can be rephrased in another way: Is every maximal (k,n)-set contained in a regular t-spread of $PG(3t+2,q)$? In Chapter Four we obtain some results towards answering this question.

The theory presented in this thesis is an extension of the existing theory of t-spreads. Thus any known use of t-spreads of $PG(2t+1,q)$, especially with regard to regularity, spread sets or indicator sets, can now be examined for possible extensions to the case of t-spreads of $PG((s+1)(t+1)-1,q)$. A few ideas on this theme are collected into Chapter Five.

CHAPTER ONE

PRELIMINARIES AND FUNDAMENTAL CONCEPTS

1.1 PROJECTIVE SPACES AND VECTOR SPACES

In the study of projective geometry, especially from the algebraic point of view, it is often useful to represent an n-dimensional projective space $PG(n,q)$ over $GF(q)$ by an $(n+1)$-dimensional vector space $\mathcal{V}_{n+1}$ over $GF(q)$. This equivalence is explained in Hirschfeld (1979) and briefly is as follows:

Each point P of an n-dimensional projective space $PG(n,q)$ is represented by an equivalence class of $(n+1)$-tuples of elements of $GF(q)$, not all zero, written as column vectors

$$P = \{\rho(x_0, x_1, \ldots, x_n)^T : \ \rho \in GF(q) - \{0\}\}$$

where ^{T} denotes the transpose of a vector or of a matrix. Interpreted as an equivalence class of points of an $(n+1)$-dimensional vector space $\mathcal{V}_{n+1}$ over $GF(q)$, this set of $(n+1)$-tuples is a line of $\mathcal{V}_{n+1}$ through the origin $(0,0,\ldots,0)^T$, excluding the origin.

An m-dimensional subspace of $PG(n,q)$ is a set of points all of whose representing vectors, together with the zero vector, form an $(m+1)$-dimensional subspace of $\mathcal{V}_{n+1}$. We will adopt the following convention with regard to subspaces of $PG(n,q)$ having no common point.

1.1.1 Definition

Two subspaces S_1 and S_2 of $PG(n,q)$ are called *skew* if they have no common point in $PG(n,q)$. The corresponding subspaces S_1' and S_2' of $\mathcal{V}_{n+1}$ have only the zero vector in common, and we shall say that S_1' and S_2' are *skew* also.

In this way two subspaces are skew if and only if as projective subspaces they are disjoint, and as vector subspaces they have only the zero vector in common.

We will also be using the idea of the *direct sum* of two vector spaces, say $\mathcal{V}_n$ and $\mathcal{V}_m$ of dimensions n and m respectively. The direct sum of $\mathcal{V}_n$ and $\mathcal{V}_m$ is

$$\begin{aligned}\mathcal{V}_n \oplus \mathcal{V}_m = \{&(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) : \\ &\qquad (x_1, x_2, \ldots, x_n) \in \mathcal{V}_n \text{ and } (y_1, y_2, \ldots, y_m) \in \mathcal{V}_m\}.\end{aligned}$$

The point $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ is often written as

$$(x_1, x_2, \ldots, x_n) \oplus (y_1, y_2, \ldots, y_m).$$

The direct sum is an $(n+m)$-dimensional vector space over $GF(q)$, and contains the subspaces

$$\begin{aligned}S_1 = \{&(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0) : \\ &\qquad (x_1, x_2, \ldots, x_n) \in \mathcal{V}_n\}\end{aligned}$$

and

$$\begin{aligned}S_2 = \{&(0, 0, \ldots, 0, y_1, y_2, \ldots, y_m) : \\ &\qquad (y_1, y_2, \ldots, y_n) \in \mathcal{V}_m\}\end{aligned}$$

which are isomorphic to $\mathcal{V}_n$ and $\mathcal{V}_m$ respectively, and these isomorphic spaces are sometimes identified. This definition is extended inductively to the direct sum of a finite number of vector spaces.

In the following, the vector e_i of a vector space $\mathcal{V}_n$ will denote the vector with 1 in the ith position and with 0 in every other position.

The next result appears in Bruck and Bose (1964) in the special case of $s = 1$. We will require the extended form in Section 3.2.

1.1.2 Lemma

Let $\mathcal{V}_{(s+1)(t+1)}$ be an $(s+1)(t+1)$-dimensional vector space with a subspace $\mathcal{V}_{s(t+1)}$

of dimension $s(t+1)$ spanned by the $(t+1)$-dimensional subspaces $A_1, A_2, \ldots, A_s$ of $\mathcal{V}_{(s+1)(t+1)}$. Let B be a $(t+1)$-dimensional subspace of $\mathcal{V}_{(s+1)(t+1)}$ skew to $\mathcal{V}_{s(t+1)}$. Write $\mathcal{V}_{(s+1)(t+1)}$ as the direct sum of the spaces $A_1, A_2, \ldots, A_s$ and B, so that any element of $\mathcal{V}_{(s+1)(t+1)}$ can be written as $a_1 \oplus a_2 \oplus \cdots \oplus a_s \oplus b$ where $a_i \in A_i$ for $i = 1, 2, \ldots, s$ and $b \in B$. Conversely any vector of this form is an element of $\mathcal{V}_{(s+1)(t+1)}$. Suppose there exist s non-singular linear transformations from A_1 onto each of $A_1, A_2, \ldots, A_s$ in turn, denoted by

$$(i) : A_1 \to A_i$$
$$a \mapsto a^{(i)}.$$

Let C be any $(t+1)$-dimensional subspace of $\mathcal{V}_{(s+1)(t+1)}$ skew to $\mathcal{V}_{s(t+1)}$ and to B, with the additional property that it is skew to each of the $s(t+1)$-dimensional spaces spanned by B together with $s-1$ of the spaces $A_1, A_2, \ldots, A_s$. (Such a space certainly exists, for example if each A_i has a basis $\{x_0^{(i)}, x_1^{(i)}, \ldots, x_t^{(i)}\}$ and B has a basis $\{y_0, y_1, \ldots, y_t\}$ then a suitable such space C would have basis

$$\{x_0^{(1)} \oplus x_0^{(2)} \oplus \cdots \oplus x_0^{(s)} \oplus y_0, \ldots\ldots, x_t^{(1)} \oplus x_t^{(2)} \oplus \cdots \oplus x_t^{(s)} \oplus y_t\}.)$$

Then there exists a unique non-singular linear transformation

$$' : A_1 \to B$$
$$a \mapsto a'$$

such that the linear transformation

$$a \mapsto a^{(1)} \oplus a^{(2)} \oplus \cdots \oplus a^{(s)} \oplus a'$$

maps A_1 onto C.

Proof: Since a linear transformation is determined by its action on a basis, it is enough to show that there exists a basis

$$\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}, a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}, \ldots\ldots, a_1^{(s)}, a_2^{(s)}, \ldots, a_{t+1}^{(s)}, b_1, b_2, \ldots, b_{t+1}\right\}$$

of $\mathcal{V}_{(s+1)(t+1)}$ such that

$$
\begin{aligned}
A_1 &= \operatorname{lin}\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}\right\}, \\
A_2 &= \operatorname{lin}\left\{a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}\right\}, \\
&\vdots \\
A_s &= \operatorname{lin}\left\{a_1^{(s)}, a_2^{(s)}, \ldots, a_{t+1}^{(s)}\right\}, \\
B &= \operatorname{lin}\left\{b_1, b_2, \ldots, b_{t+1}\right\} \quad \text{and} \\
C &= \operatorname{lin}\left\{a_1^{(1)} \oplus a_1^{(2)} \oplus \cdots \oplus a_1^{(s)} \oplus b_1, a_2^{(1)} \oplus a_2^{(2)} \oplus \cdots \oplus a_2^{(s)} \oplus b_2, \ldots\ldots, \right. \\
&\qquad\qquad \left. a_{t+1}^{(1)} \oplus a_{t+1}^{(2)} \oplus \cdots \oplus a_{t+1}^{(s)} \oplus b_{t+1}\right\}.
\end{aligned}
$$

Then the required linear transformation $'$ is

$$
\begin{aligned}
' &: a_1 \to B \\
&: a_j^{(1)} \mapsto b_j, \quad \text{for } j = 1, 2, \ldots, t+1.
\end{aligned}
$$

We do this by choosing a basis for C, and then since the space $\mathcal{V}_{(s+1)(t+1)}$ is the direct sum of $A_1, A_2, \ldots, A_s$ and B each basis element of C is uniquely expressible as a direct sum of elements of $A_1, A_2, \ldots, A_s$ and B. In this way, suppose that

$$
\begin{aligned}
C &= \operatorname{lin}\left\{c_1, c_2, \ldots, c_{t+1}\right\} \\
&= \operatorname{lin}\left\{a_1^{(1)} \oplus a_1^{(2)} \oplus \cdots \oplus a_1^{(s)} \oplus b_1, a_2^{(1)} \oplus a_2^{(2)} \oplus \cdots \oplus a_2^{(s)} \oplus b_2, \ldots\ldots, \right. \\
&\qquad\qquad \left. a_{t+1}^{(1)} \oplus a_{t+1}^{(2)} \oplus \cdots \oplus a_{t+1}^{(s)} \oplus b_{t+1}\right\}.
\end{aligned}
$$

where $a_j^{(i)} \in A_i$ for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, t+1$, and $b_j \in B$ for each $j = 1, 2, \ldots, t+1$. We must show that the set of vectors $\{a_j^{(i)} : j = 1, 2, \ldots, t+1\}$ is a basis for the space A_i for each i with $i = 1, 2, \ldots, s$, and that the set of vectors $\{b_j : j = 1, 2, \ldots, t+1\}$ is a basis for B. Then since $\mathcal{V}_{(s+1)(t+1)}$ is the direct sum of $A_1, A_2, \ldots, A_s$ and B, the set of all these vectors is a basis for $\mathcal{V}_{(s+1)(t+1)}$. Now, none of the vectors b_j may be zero since C is skew to $\mathcal{V}_{s(t+1)}$, and no vector $a_j^{(i)}$ may be zero for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, t+1$ since C contains no point of the space spanned by B and the set of spaces $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_s$. We now show that $\{b_j : j = 1, 2, \ldots, t+1\}$ is a linearly independent set of vectors. Consider the equation

$$\sum_{i=1}^{t+1} \beta_i b_i = 0, \quad \text{where } \beta_i \in GF(q) \text{ for } i = 1, 2, \ldots, t+1.$$

Suppose that A_1 has basis $\{x_1, x_2, \ldots, x_{t+1}\}$. Then

$$\sum_{i=1}^{t+1} \beta_i (b_i \oplus x_i) = \sum_{i=1}^{t+1} \beta_i x_i \oplus 0.$$

The vector on the right hand side is contained in $A_1 \oplus 0$ and the vector on the left hand side is contained in $(A_1 \oplus B)$. Now $A_1 \oplus 0$ and $(A_1 \oplus B)$ have only the zero vector in common so that

$$\sum_{i=1}^{t+1} \beta_i (b_i \oplus x_i) = \sum_{i=1}^{t+1} \beta_i x_i \oplus 0 = 0 \oplus 0.$$

As $\{x_1, x_2, \ldots, x_{t+1}\}$ is a basis for A_1, we have that $\beta_i = 0$ for all $i = 1, 2, \ldots, t+1$ showing that $\{b_1, b_2, \ldots, b_{t+1}\}$ is a basis for B. The arguments for showing that the set of elements $\{a_1^{(i)}, a_2^{(i)}, \ldots, a_t^{(i)}\}$ is a basis for A_i for each i are analogous. □

Note: The proof follows easily in the case of $s = 1$. The space $\mathcal{V}_{2t+2}$ is spanned by A_1 and B. Any space skew to both of them has basis of the form $\{a_1 \oplus b_1, a_2 \oplus b_2, \ldots, a_{t+1} \oplus b_{t+1}\}$ and none of a_i or b_i may be zero. Then considering linear combinations of the a_i and the b_i establishes linear independence, recalling that A_1, B and C have only the zero vector in common.

We shall sometimes define incidence structures from our projective spaces. An *incidence structure* is a triple $\mathcal{I} = (P, B, I)$, where P and B are disjoint non-empty sets and $I \subseteq (P \times B)$. We normally refer to the elements of P as *points* and the elements of B as *blocks* or *lines*. If $p \in P$, $l \in B$ and $(p, l) \in I$ then we say that p is incident with l, and more commonly that p lies on l or l contains p. If l_1

and l_2 are two blocks of an incidence structure we say that l_1 *meets* l_2 if they are mutually incident with at least one point. We define $l_1 \cap l_2$ to be the set of points mutually incident with both blocks, and we say that l_1 and l_2 *meet* in $l_1 \cap l_2$.

An *isomorphism* from $\mathcal{I} = (P, B, I)$ to $\mathcal{I}' = (P', B', I')$ is a map

$$\phi : P \cup B \to P' \cup B'$$

such that $\phi(P) = P'$, $\phi(B) = B'$ and $(p, l) \in I$ if and only if $(p', l') \in I'$ for all $p \in P$ and $l \in B$.

1.2 t-SPREADS OF $PG(n, q)$

In this Section we will give a brief introduction to those concepts regarding t-spreads of $PG(n, q)$ which we will use in the following chapters. There is a good collection of material on t-spreads contained in Dembowski (1968) and Hirschfeld (1979).

1.2.1 Definition [Segre (1964), p23]
A t-*spread* $\mathcal{W}$ of $PG(n, q)$ is a set of t-dimensional subspaces such that every point of $PG(n, q)$ is contained in exactly one element of $\mathcal{W}$.

1.2.2 Theorem [Segre (1964), p23–25]
The space $PG(n, q)$ contains a t-spread if and only if $t + 1$ divides $n + 1$. If $\mathcal{W}$ is a t-spread of $PG((s + 1)(t + 1) - 1, q)$ then

$$|\mathcal{W}| = \sum_{i=0}^{s} q^{i(t+1)}.$$

Proof: Let $\mathcal{W}$ be a t-spread of $PG(n, q)$. Since the space $PG(n, q)$ is the disjoint union of the subspaces in $\mathcal{W}$, the number $\frac{q^{n+1}-1}{q-1}$ of points of $PG(n, q)$

must be divisible by the number $\frac{q^{t+1}-1}{q-1}$ of points in a t-dimensional subspace. This occurs if and only if $t+1$ divides $n+1$, so that $n = (s+1)(t+1)-1$, and the number of elements in the spread is the quotient $\frac{q^{(s+1)(t+1)}-1}{q^{t+1}-1}$ of these two values.

For the converse, we will construct a t-spread of $PG((s+1)(t+1)-1, q)$ following the presentation in Hirschfeld (1979), Theorem 4.1.1, p72. Let $K = GF(q^{t+1})$ be a field extension of $F = GF(q)$, so that $K = F(\alpha)$. The elements $1, \alpha, \alpha^2, \ldots, \alpha^t$ are linearly independent over $GF(q)$, and any element ζ_i of $GF(q^{t+1})$ may be written uniquely as

$$\zeta_i = x_{i0} + x_{i1}\alpha + \cdots + x_{it}\alpha^t, \text{ where } x_{ij} \in GF(q).$$

In this way, an $(s+1)$-tuple of elements $(\zeta_0, \zeta_1, \ldots, \zeta_s)$ of $GF(q^{t+1})$ gives rise to the following $((s+1)(t+1))$-tuple of elements of $GF(q)$:

$$(x_{00}, x_{01}, \ldots, x_{0t}, x_{10}, x_{11}, \ldots, x_{1t}, \ldots\ldots, x_{s0}, x_{s1}, \ldots, x_{st}).$$

We can interpret this $((s+1)(t+1))$-tuple as homogeneous coordinates of a point in $PG((s+1)(t+1)-1, q)$, and similarly a point of $PG((s+1)(t+1)-1, q)$ is given by an $(s+1)$-tuple of elements of $GF(q^{t+1})$. We choose $s+1$ elements $\tau_0, \tau_1, \ldots, \tau_s$ of $GF(q^{t+1})$, not all zero, and consider the set of equations in $GF(q^{t+1})$ given by

$$\frac{\zeta_0}{\tau_0} = \frac{\zeta_1}{\tau_1} = \cdots = \frac{\zeta_s}{\tau_s}.$$

If we now write each element of $GF(q^{t+1})$ as

$$\zeta_i = x_{i0} + x_{i1}\alpha + \cdots + x_{it}\alpha^t, \; x_{ij} \in GF(q)$$

$$\tau_j = y_{j0} + y_{j1}\alpha + \cdots + y_{jt}\alpha^t, \; y_{jk} \in GF(q),$$

then we obtain $s(t+1)$ linearly independent equations in the variables x_{ij}. This is because there are s linearly independent equations of the form

$$\tau_i\zeta_j = \tau_j\zeta_i, \text{ for } i, j \in \{0, 1, \ldots, s\}$$

involving elements of $GF(q^{t+1})$, and each of these gives rise to $t+1$ equations in elements of $GF(q)$. These $s(t+1)$ equations in the variables x_{ij} determine a subspace of $PG((s+1)(t+1)-1, q)$ of dimension $(s+1)(t+1)-1-s(t+1) = t$. Multiplying each element of the $(s+1)$-tuple $(\tau_0, \tau_1, \ldots, \tau_s)$ by an element of $GF(q^{t+1})$ gives another $(s+1)$-tuple defining the same t-dimensional subspace of $PG((s+1)(t+1)-1, q)$, so that a point $(\tau_0, \ldots, \tau_s)$ of $PG(s, q^{t+1})$ determines a t-dimensional subspace of $PG((s+1)(t+1)-1, q)$. Conversely a t-dimensional subspace of $PG((s+1)(t+1)-1, q)$ determines a point of $PG(s, q^{t+1})$. The set of t-dimensional subspaces of $PG((s+1)(t+1)-1, q)$ determined by all points of $PG(s, q^{t+1})$ is a t-spread of $PG((s+1)(t+1)-1, q)$. To see this, note that $PG(s, q^{t+1})$ has

$$N = \frac{q^{(s+1)(t+1)} - 1}{q^{t+1} - 1}$$

points, so we obtain N t-dimensional subspaces of $PG((s+1)(t+1)-1, q)$. This is the number of elements of a t-spread and it remains to be shown that the t-dimensional spaces we have constructed are disjoint. Each point of the space $PG((s+1)(t+1)-1, q)$ lies in at least one such t-dimensional space, for we can find the corresponding $(s+1)$-tuple $(\zeta_0, \zeta_1, \ldots, \zeta_s)$ of elements of $GF(q^{t+1})$, and produce equations of the above type in ζ_i and τ_i. The number of points of $PG((s+1)(t+1)-1, q)$ covered by the above t-dimensional subspaces is N multiplied by the number of points in such a t-dimensional subspace, or

$$\frac{q^{(s+1)(t+1)} - 1}{q^{t+1} - 1} \times \frac{q^{t+1} - 1}{q - 1}$$

which is exactly the number of points in $PG((s+1)(t+1)-1, q)$. The t-dimensional subspaces must be disjoint and so form a t-spread of $PG((s+1)(t+1)-1, q)$. □

The following idea of a geometric t-spread appeared in both Baer (1963) and Segre (1964), p32, in relation to partitions of abelian groups or as a property of the t-spread constructed in the proof of Theorem 1.2.2.

1.2.3 Definition [Beutelspacher (1975), p212]
A t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1, q)$ is called *geometric* if for every pair of distinct elements X, Y of $\mathcal{W}$ the elements of $\mathcal{W}$ are either contained in or are skew to the join $< X, Y >$ of the spaces X and Y. The elements of $\mathcal{W}$ contained in $< X, Y >$ form a t-spread of $< X, Y >$, called the t-spread *induced* on $< X, Y >$ by $\mathcal{W}$.

For a geometric t-spread $\mathcal{W}$ of $PG(n, q)$ let $\mathcal{I}$ be the following incidence structure:

- the *points* of $\mathcal{I}$ are the elements of $\mathcal{W}$,
- the *blocks* of $\mathcal{I}$ are the subspaces $< V, V' >$ for any two distinct elements V and V' of $\mathcal{W}$, and
- the incidence in $\mathcal{I}$ is set-theoretic inclusion. Then the following holds:

1.2.4 Theorem [Segre (1964)]
(1) If the space $PG(n, q)$ contains a t-spread then it must contain a geometric t-spread. By Theorem 1.2.2, this occurs if and only if $t+1$ divides $n+1$.

(2) If $\mathcal{W}$ is a geometric t-spread of $PG((s+1)(t+1)-1, q)$ then $\mathcal{I}$ is a projective space of order q^{t+1} and dimension s.

Proof: (1) The t-spread constructed in the proof of Theorem 1.2.2 is in fact geometric.
(2) This follows by checking that the incidence structure $\mathcal{I}$ satisfies the axioms for a projective space, for example those appearing in Hirschfeld (1979), p39. □

1.3 THE SEGRE VARIETY $\mathcal{SV}_{s+1,t+1}$ IN $PG((s+1)(t+1)-1,q)$

The Segre variety $\mathcal{SV}_{s+1,t+1}$ appeared first in the work of C. Segre in 1891 (see Segre (1891)), where it was studied in projective spaces over infinite fields. The Segre variety is also referred to as the Segre manifold or the Segre product of two spaces S_t and S_s, of dimensions t and s respectively. For a discussion of the classical Segre variety over an infinite field, see Burau (1961) or Hodge and Pedoe (1952).

The theory is still valid over finite fields, giving the Segre variety in the finite projective space $PG((s+1)(t+1)-1,q)$. There is a close connection with the theory of spreads of $PG((s+1)(t+1)-1,q)$ by t-dimensional or by s-dimensional subspaces. This connection allows us to generalise the known theory of 1-spreads of $PG(3,q)$, in a natural way. In the case of $PG(3,q)$ when $s=t=1$, regularity of a 1-spread is defined using the idea of a regulus of lines, which is related to the Segre variety $\mathcal{SV}_{2,2}$. In the general case, regularity of a t-spread of $PG((s+1)(t+1)-1,q)$ is defined in an analogous way but using the Segre variety $\mathcal{SV}_{s+1,t+1}$.

Here we investigate the behavior of the Segre variety over the finite field $GF(q)$.

Let S_t and S_s be projective spaces of order q and of dimensions t and s respectively, and suppose that they have as systems of homogeneous coordinates respectively $(y_0, y_1, \ldots, y_t)$ and $(z_0, z_1, \ldots, z_s)$. Consider the $((s+1)(t+1)-1)$-dimensional projective space $PG((s+1)(t+1)-1,q)$, with homogeneous coordinates

$$(x_{00}, x_{01}, \ldots, x_{ts}).$$

The set of points of $PG((s+1)(t+1)-1,q)$ with $x_{ij} = y_i z_j$ for all $i = 0, 1, \ldots, t$ and $j = 0, 1, \ldots, s$ is a variety in $PG((s+1)(t+1)-1,q)$, called the *Segre variety*

$\mathcal{SV}_{s+1,t+1}$ in $PG((s+1)(t+1)-1,q)$.

In the following, we will occasionally use semicolons in place of commas to break up the coordinate $(s+1)(t+1)$-tuple

$$(x_{00}, x_{01}, \ldots, x_{ts})$$

into $t+1$ blocks of $s+1$ coordinates each:

$$(x_{00}, x_{01}, \ldots, x_{0s}; x_{10}, x_{11}, \ldots, x_{1s}; \ldots\ldots; x_{t0}, x_{t1}, \ldots, x_{ts}).$$

This has no formal significance, it is just done for ease of notation.

1.3.1 Lemma

(1) The Segre variety $\mathcal{SV}_{s+1,t+1}$ has two systems of linear subspaces of order q lying on it. There are $q^s + q^{s-1} + \cdots + q + 1$ spaces of dimension t, each in projective correspondence with S_t and each determined by one point $(z_0, z_1, \ldots, z_s)$ of S_s. There are $q^t + q^{t-1} + \cdots + q + 1$ spaces of dimension s, each in projective correspondence with S_s and each determined by one point $(y_0, y_1, \ldots, y_t)$ of S_t.

(2) The spaces of each system are skew and there is one space of each system through any given point of $\mathcal{SV}_{s+1,t+1}$. Therefore a space of one system meets each space of the other system in a unique point.

(3) The Segre variety $\mathcal{SV}_{s+1,t+1}$ has exactly

$$(q^t + q^{t-1} + \cdots + q + 1)(q^s + q^{s-1} + \cdots + q + 1)$$

points.

Proof: (1) Fix a point $(z'_0, z'_1, \ldots, z'_s)$ of S_s, and consider the set of points of $\mathcal{SV}_{s+1,t+1}$ given by

$$\{(y_0z'_0, y_0z'_1, \ldots, y_0z'_s; y_1z'_0, y_1z'_1, \ldots, y_1z'_s; \ldots\ldots; y_tz'_0, y_tz'_1, \ldots, y_tz'_s)\}$$

for $y_0, y_1, \ldots, y_t \in GF(q)$, not all zero. This set of points is a t-dimensional subspace of $PG((s+1)(t+1)-1, q)$ since it is spanned by the $t+1$ linearly independent points

$$(z'_0, z'_1, \ldots, z'_s; 0, 0, \ldots, 0; \ldots\ldots; 0, 0, \ldots, 0),$$
$$(0, 0, \ldots, 0; z'_0, z'_1, \ldots, z'_s; 0, 0, \ldots, 0; \ldots\ldots; 0, 0, \ldots, 0),$$
$$\vdots$$
$$(0, 0, \ldots, 0; \ldots\ldots; 0, 0, \ldots, 0; z'_0, z'_1, \ldots, z'_s).$$

It is in projective correspondence with the t-dimensional space S_t with homogeneous coordinates $(y_0, y_1, \ldots, y_t)$. There are $q^s + q^{s-1} + \cdots + q + 1$ choices for the point $(z'_0, z'_1, \ldots, z'_s)$ of S_s, so there are $q^s + q^{s-1} + \cdots + q + 1$ such t-dimensional spaces on $\mathcal{SV}_{s+1,t+1}$. In an analogous way we fix a point $(y'_0, y'_1, \ldots, y'_t)$ of S_t, then the set of points of $\mathcal{SV}_{s+1,t+1}$ given by

$$\{(y'_0 z_0, y'_0 z_1, \ldots, y'_0 z_s; y'_1 z_0, y'_1 z_1, \ldots, y'_1 z_s; \ldots\ldots; y'_t z_0, y'_t z_1, \ldots, y'_t z_s)\}$$

for $z_0, z_1, \ldots, z_s \in GF(q)$, not all zero, forms an s-dimensional subspace of the space $PG((s+1)(t+1)-1, q)$ since it is spanned by the $s+1$ linearly independent points

$$(y'_0, 0, \ldots, 0; y'_1, 0, \ldots, 0; \ldots\ldots; y'_t, 0, \ldots, 0),$$
$$(0, y'_0, 0, \ldots, 0; 0, y'_1, 0, \ldots, 0; \ldots\ldots; 0, y'_t, 0, \ldots, 0),$$
$$\vdots$$
$$(0, \ldots, 0, y'_0; 0, \ldots, 0, y'_1; \ldots\ldots; 0, \ldots, 0, y'_t).$$

Each such s-dimensional space is in projective correspondence with the space S_s with homogeneous coordinates $(z_0, z_1, \ldots, z_s)$.

(2) Given any two distinct points $(y'_0, y'_1, \ldots, y'_t)$ and $(y''_0, y''_1, \ldots, y''_t)$ of S_t, the s-dimensional spaces that they define are skew, and similarly any two t-dimensional spaces on $\mathcal{SV}_{s+1,t+1}$ are skew. The point

$$(y'_0 z'_0, y'_0 z'_1, \ldots, y'_0 z'_s; y'_1 z'_0, y'_1 z'_1, \ldots, y'_1 z'_s; \ldots\ldots; y'_t z'_0, y'_t z'_1, \ldots, y'_t z'_s)$$

lies on the t-dimensional space of $\mathcal{SV}_{s+1,t+1}$ determined by the point $(z'_0, z'_1, \ldots, z'_s)$ of S_s and the s-dimensional space determined by the point $(y'_0, y'_1, \ldots, y'_t)$ of S_t, and these spaces are unique. Conversely the t-dimensional space of $\mathcal{SV}_{s+1,t+1}$ determined by the point $(z'_0, z'_1, \ldots, z'_s)$ of S_s meets the s-dimensional space determined by the point $(y'_0, y'_1, \ldots, y'_t)$ of S_t in the unique point

$$(y'_0z'_0, y'_0z'_1, \ldots, y'_0z'_s; y'_1z'_0, y'_1z'_1, \ldots, y'_1z'_s; \ldots\ldots; y'_tz'_0, y'_tz'_1, \ldots, y'_tz'_s)$$

of $\mathcal{SV}_{s+1,t+1}$.

(3) The number of points of the Segre variety $\mathcal{SV}_{s+1,t+1}$ is the number of t-dimensional spaces on it multiplied by the number of points in such a t-dimensional space. Alternatively, it is the number of s-dimensional spaces multiplied by the number of points in such an s-dimensional space. This is

$$(q^t + q^{t-1} + \cdots + q + 1)(q^s + q^{s-1} + \cdots + q + 1).$$

□

1.3.2 Examples

The Segre variety $\mathcal{SV}_{2,2}$ is a quadric in $PG(3, q)$ and the Segre variety $\mathcal{SV}_{3,2}$ is the rational cubic scroll of planes in $PG(5, q)$. For convenience we include the trivial cases of $\mathcal{SV}_{1,1}$ which is a point, $\mathcal{SV}_{2,1} = \mathcal{SV}_{1,2}$ which is a line, and in general $\mathcal{SV}_{s+1,1}$ which is a projective space of dimension s.

1.3.3 Note

By Lemma 1.3.1 (2) we see that the two systems $\{S_s\}$ and $\{S_t\}$ of projective subspaces lying on $\mathcal{SV}_{s+1,t+1}$ are respectively a partial s-spread and a partial t-spread of $PG((s+1)(t+1)-1, q)$, covering the same points (the points lying on the Segre variety $\mathcal{SV}_{s+1,t+1}$), and such that an element of one spread meets an element of the other spread in a unique point. This property is used in Lemma 1 of Beutelspacher (1978), where he proves that there is a t-spread in the projective space $PG((s+1)(t+1)-1,q)$ containing a partial t-spread which covers the same points as a partial s-spread of the space. In other words Beutelspacher (1978) has constructed a t-spread of $PG((s+1)(t+1)-1,q)$ which contains the set of t-dimensional subspaces of a Segre variety $\mathcal{SV}_{s+1,t+1}$. He uses this to prove that a partial t-spread of $PG(d,q)$ is contained in a t-spread of $PG((d+1)(t+1)-1,q)$.

1.3.4 Lemma

(1) The system $\{S_t\}$ of t-dimensional spaces can be obtained by joining corresponding points of $t+1$ projectively related s-dimensional subspaces of $PG((s+1)(t+1)-1,q)$, no t of which lie in a hyperplane. The system $\{S_s\}$ of s-dimensional spaces is obtained similarly by joining corresponding points of $s+1$ projectively related t-dimensional subspaces no s of which lie in a hyperplane.

(2) There is a unique Segre variety $\mathcal{SV}_{s+1,t+1}$ containing any $t+2$ s-dimensional subspaces of $PG((s+1)(t+1)-1,q)$, no $t+1$ in a hyperplane. Similarly, there is a unique Segre variety $\mathcal{SV}_{s+1,t+1}$ containing $s+2$ t-dimensional subspaces of $PG((s+1)(t+1)-1,q)$, no $s+1$ in a hyperplane.

Proof: (1) We choose a convenient system of homogeneous coordinates for the space $PG((s+1)(t+1)-1,q)$ so that the s-dimensional spaces are:

$$\{(x_0,x_1,\ldots,x_s;0,\ldots,0;\ldots\ldots;0,\ldots,0):\ x_i\in GF(q)\}$$

$$\{(0,\ldots,0;x_0,x_1,\ldots,x_s;0,\ldots,0;\ldots\ldots;0,\ldots,0):\ x_i\in GF(q)\}$$

$$\vdots$$

$$\{(0,\ldots,0;\ldots\ldots;0,\ldots,0;x_0,x_1,\ldots,x_s):\ x_i\in GF(q)\}.$$

For $x'_0,x'_1,\ldots,x'_s\in GF(q)$, not all zero, construct the t-dimensional space spanned by the points

$$(x_0', x_1', \ldots, x_s'; 0, \ldots, 0; \ldots\ldots; 0, \ldots, 0),$$
$$(0, \ldots, 0; x_0', x_1', \ldots, x_s'; 0, \ldots, 0; \ldots\ldots; 0, \ldots, 0),$$
$$\vdots$$
$$(0, \ldots, 0; \ldots\ldots; 0, \ldots, 0; x_0', x_1', \ldots, x_s').$$

The set of t-dimensional spaces constructed is the set of t-dimensional spaces of a Segre variety $\mathcal{SV}_{s+1,t+1}$, and they can be used to give all the s-dimensional spaces as in Lemma 1.3.1 (1). Similarly we can choose homogeneous coordinates for the space $PG((s+1)(t+1)-1, q)$ so that the given t-dimensional spaces are:

$$\{(x_0, 0, \ldots, 0; x_1, 0, \ldots, 0; \ldots\ldots; x_t, 0, \ldots, 0) : \; x_i \in GF(q)\}$$
$$\{(0, x_0, 0, \ldots, 0; 0, x_1, 0, \ldots, 0; \ldots\ldots; 0, x_t, 0, \ldots, 0) : \; x_i \in GF(q)\}$$
$$\vdots$$
$$\{(0, \ldots, 0, x_0; 0, \ldots, 0, x_1; \ldots\ldots; 0, \ldots, 0, x_t) : \; x_i \in GF(q)\}.$$

For $x_0', \ldots, x_t' \in GF(q)$, not all zero, construct the s-dimensional space spanned by the points

$$(x_0', 0, \ldots, 0; x_1', 0, \ldots, 0; \ldots\ldots; x_t', 0, \ldots, 0),$$
$$(0, x_0', 0, \ldots, 0; 0, x_1', 0, \ldots, 0; \ldots\ldots; 0, x_t', 0, \ldots, 0),$$
$$\vdots$$
$$(0, \ldots, 0, x_0'; 0, \ldots, 0, x_1'; \ldots\ldots; 0, \ldots, 0, x_t').$$

The set of s-dimensional spaces constructed is the set of s-dimensional spaces of a Segre variety $\mathcal{SV}_{s+1,t+1}$, and they can be used to give all the t-dimensional spaces as in Lemma 1.3.1 (1).

(2) Through a general point of $PG((s+1)(t+1)-1, q)$ there passes a unique t-dimensional space meeting each of $t+1$ skew s-dimensional spaces, no t in a hyperplane. This space is called a *transversal space* to the s-dimensional spaces,

and meets each of the s-dimensional spaces necessarily in a unique point. So given $t+2$ s-dimensional subspaces of $PG((s+1)(t+1)-1,q)$, no $t+1$ lying in a hyperplane, there are $q^s+q^{s-1}+\cdots+q+1$ t-dimensional spaces meeting all of them, each in a unique point. These t-dimensional spaces are pairwise skew and together with the s-dimensional spaces they define a Segre variety $\mathcal{SV}_{s+1,t+1}$. Similarly, through a general point of $PG((s+1)(t+1)-1,q)$ there passes a unique common transversal s-dimensional space to $s+1$ skew t-dimensional spaces, no s in a hyperplane. So given $s+2$ t-dimensional spaces in $PG((s+1)(t+1)-1,q)$, no $s+1$ in a hyperplane, there are $q^t+q^{t-1}+\cdots+q+1$ s-dimensional subspaces of $PG((s+1)(t+1)-1,q)$ meeting all of them, each in a unique point. These s-dimensional spaces are pairwise skew and together with the t-dimensional spaces they define a Segre variety $\mathcal{SV}_{s+1,t+1}$. □

We now investigate the properties of Segre subvarieties of a Segre variety, an idea which will become important later.

1.3.5 Lemma

The Segre variety $\mathcal{SV}_{s+1,t+1}$ admits Segre subvarieties $\mathcal{SV}_{r+1,t+1}$ for every value of r with $0 \le r \le s$. The t-dimensional spaces of the subvariety are all t-dimensional spaces of $\mathcal{SV}_{s+1,t+1}$, and the r-dimensional spaces of $\mathcal{SV}_{r+1,t+1}$ are subspaces of the s-dimensional spaces of $\mathcal{SV}_{s+1,t+1}$. In particular, $\mathcal{SV}_{r+1,t+1}$ lies in a subspace of $PG((s+1)(t+1)-1,q)$ of dimension $((r+1)(t+1)-1)$.

Proof: Let S_s be one of the spaces of dimension s lying on $\mathcal{SV}_{s+1,t+1}$. For any value of r, with $0 \le r \le s$, let S_r be a subspace of S_s of dimension r. The t-dimensional spaces of $\mathcal{SV}_{s+1,t+1}$ meeting S_s in points of S_r are the t-dimensional spaces of a Segre variety $\mathcal{SV}_{r+1,t+1}$. Each element of the system of r-dimensional subspaces on $\mathcal{SV}_{r+1,t+1}$ is found either by intersecting the t-

dimensional spaces of $\mathcal{SV}_{r+1,t+1}$ with the s-dimensional subspaces of $\mathcal{SV}_{s+1,t+1}$ or alternatively by finding the r-dimensional subspace of each s-dimensional subspace of $\mathcal{SV}_{s+1,t+1}$ which is projectively equivalent to the subspace S_r of S_s under the original projectivity relating the s-dimensional spaces. $\square$

1.3.6 Examples

(0) $t = 1$ and $r = 0$. The Segre subvariety $\mathcal{SV}_{1,2}$ of $\mathcal{SV}_{s+1,2}$ is just a line of the Segre variety.

(1) $t = 1$ and $r = 1$. The Segre subvariety $\mathcal{SV}_{2,2}$ of $\mathcal{SV}_{s+1,2}$, $s \geq 3$ is a quadric surface in a 3-dimensional subspace of $PG(2s+1, q)$. For example if $s = 3$ then we see that the cubic scroll in $PG(5, q)$ has exactly $q^2 + q + 1$ quadrics on it, each lying in a 3-dimensional subspace of $PG(5, q)$, and each pair of them having a line in common, which is necessarily a line of the scroll $\mathcal{SV}_{3,2}$.

(2) $t = 1$ and $r = 2$. The Segre subvariety $\mathcal{SV}_{3,2}$ of $\mathcal{SV}_{s+1,2}$, $s \geq 4$ is a rational cubic scroll in a 5-dimensional subspace of $PG(2s+1, q)$.

1.4 THE STRUCTURE OF $PG(n, q^d)$

Lunardon (1984) used the idea of imaginary point and imaginary subspace of $PG(2t-1, q^t)$. This refined the approach of Sherk (1979) who used the concept of linearly independent direction numbers of a line to describe the same phenomenon. In this Section we discuss imaginary subspaces of a general projective space $PG(n, q^d)$.

1.4.1 Definition

Let K be a field extension of the field F. Then $PG(n, F)$ is said to be *embedded in* $PG(n, K)$. If K is not equal to F then the embedding is called *proper*.

$PG(n, F)$ is also said to be a subgeometry of $PG(n, K)$, as in Hirschfeld (1979), p87. For each divisor d_i of d, the projective space $PG(n, q^{d_i})$ is embedded in $PG(n, q^d)$. A point P of $PG(n, q^d)$ is said to lie in $PG(n, q^{d_i})$, if when P is normalised (so that one of its coordinates is 1), then all the coordinates lie in $GF(q^{d_i})$. If q is not prime there are other spaces embedded in $PG(n, q^d)$ but these shall not concern us here.

If S is a subspace of $PG(n, q)$ then we will denote its extension to $PG(n, q^d)$ by $\overline{S}$.

1.4.2 Definition

A point of $PG(n, q)$ will be called a *real* point of $PG(n, q^d)$. A k-dimensional subspace of $PG(n, q^d)$ will be called *real* if it intersects $PG(n, q)$ in a subspace of the same dimension k. A point or subspace which is not real will be called *non-real*.

1.4.3 Example

Let $q = 2$ and $d = 4$. Now $PG(n, 2^4)$ has the two spaces $PG(n, 2)$ and $PG(n, 2^2)$ properly embedded in it. Let ω be a primitive element of $GF(2^4)$, so that $GF(2^4)$ is the set of elements $\{0, 1, \omega, \omega^2, \ldots, \omega^{14}\}$ where $\omega^{15} = 1$ and $\omega^4 + \omega^3 + 1 = 0$. Then $GF(2^2) = \{0, 1, \omega^5, \omega^{10}\}$ with primitive element ω^5 and $GF(2) = \{0, 1\}$. The points of $PG(n, 2)$, or the real points of $PG(n, 2^4)$ are the $(n+1)$-tuples of elements of $GF(2)$ and the points of $PG(n, 2^2)$ are the $(n+1)$-tuples of elements of $GF(2^2)$ where one of the elements is 1. The points of $PG(n, 2^4)$ not lying in $PG(n, 2^2)$ are those points which have at least one coordinate in $GF(2^4)$ no matter which coordinate representation of the point is used.

Let σ denote the field automorphism, called conjugation, of $GF(q^d)$, that is, for $x \in GF(q^d)$,

$$\sigma : x \mapsto x^q.$$

The map σ induces a collineation on $PG(n, q^d)$ which will also be called σ. The image of a point P under σ will be denoted by P^σ and more generally the image of a subspace S of $PG(n, q^d)$ under σ will be denoted by S^σ. More precisely,

$$\sigma : \ P = (x_0, x_1, \ldots, x_n) \mapsto$$
$$P^\sigma = (x_0^q, x_1^q, \ldots, x_n^q)$$

and

$$\sigma : S = \{P_i : P_i \in \ S\} \mapsto S^\sigma = \{P_i^\sigma : P_i \in \ S\}$$

Note that $P^{\sigma^d} = P$ for all P since $x^{q^d} = x$ for all $x \in GF(q^d)$.

1.4.4 Definition

Given a point $P \in PG(n, q^d)$, the d points

$$P, P^\sigma, P^{\sigma^2}, \ldots, P^{\sigma^{d-1}}$$

are called the *conjugates* of P.

1.4.5 Lemma

(1) If $P \in PG(n, q^d)$ then $P^\sigma = P$ if and only if $P \in PG(n, q)$.

(2) If S_m is a subspace of $PG(n, q^d)$ then S_m is fixed by σ, not necessarily pointwise, if and only if S_m intersects $PG(n, q)$ in a space of the same dimension m.

(3) If $P \in PG(n, q^d)$, then $P^{\sigma^{d_i}} = P$ if and only if d_i is a divisor of d and $P \in PG(n, q^{d_i})$.

Proof: (1) If $x \in GF(q^d)$ then $x^q = x$ if and only if $x \in GF(q)$ and (1) follows since $P \in PG(n,q)$ if and only if all the coordinates of P are in $GF(q)$ when one of the coordinates is 1.

(2) In the special case of $n = 2d-1$, this result appears as Lemma 1, p719 of Lunardon (1984). The proof in the general case is analogous. Firstly if S_m is a subspace of $PG(n,q^d)$ meeting $PG(n,q)$ in a space of dimension m, then it is spanned by $m+1$ linearly independent points $P_0, P_1, \ldots, P_m$ of $PG(n,q)$. Since $P_0^\sigma = P_0, P_1^\sigma = P_1, \ldots, P_m^\sigma = P_m$, we see that

$$\begin{aligned} S_m^\sigma &= \text{lin } \{P_0^\sigma, P_1^\sigma, \ldots, P_m^\sigma\} \\ &= \text{lin } \{P_0, P_1, \ldots, P_m\} \end{aligned}$$

which is S_m. The converse is proved by induction. If $m = 0$ then S_m is a point and the property (2) is true by part (1) of this Lemma. Next suppose that any subspace of $PG(n,q^d)$ of dimension $k-1$ which is fixed by σ meets $PG(n,q)$ in a space of the same dimension $k-1$. Let S_k be a k-dimensional subspace of $PG(n,q^d)$ with $S_k^\sigma = S_k$, and suppose that S_k meets $PG(n,q)$ in a space of dimension t, where $t \leq k$. Let H_1 and H_2 be two hyperplanes of $PG(n,q)$, so that their extensions $\overline{H}_1$ and $\overline{H}_2$ are hyperplanes of $PG(n,q^d)$. By the first part of (2) already proved, both $\overline{H}_1$ and $\overline{H}_2$ are fixed by σ. Suppose further that neither of H_1, H_2 contains S_k. Then the spaces $\overline{S}_1 = \overline{H}_1 \cap S_k$ and $\overline{S}_2 = \overline{H}_2 \cap S_k$ are both fixed by σ (as $\overline{H}_1$, $\overline{H}_2$ and S_k are all fixed by σ) and both have dimension $k-1$. By the inductive hypothesis $\overline{S}_1$ and $\overline{S}_2$ meet $PG(n,q)$ in (distinct) spaces both of dimension $k-1$, and so S_k meets $PG(n,q)$ in a space of dimension k.

(3) First suppose that d_i divides d and that $P \in PG(n,q^{d_i}) \subset PG(n,q^d)$. The coordinates of P lie in $GF(q^{d_i})$, and $x \in GF(q^{d_i})$ implies that $x^{q^{d_i}} = x$ so that $P^{\sigma^{d_i}} = P$. Conversely if $P \in PG(n,q)$ and $P^{\sigma^{d_i}} = P$ then $x^{q^{d_i}} = x$ for each coordinate x of P. But $x^{q^{d_i}-1} = 1 = \alpha^{r(q^d-1)}$ for any r and α a primitive element of $GF(q^d)$, so that $x = \alpha^{r(q^d-1)/(q^{d_i}-1)}$. But $GF(q^{d_i})$ has primitive element

$\alpha^{(q^d-1)/(q^{d_i}-1)}$, so $x \in GF(q^{d_i})$ with d_i a divisor of d as required. $\square$

1.4.6 Example

We return to the Example 1.4.3. The automorphic collineation σ of $PG(n, 2^4)$ is

$$\sigma : GF(2^4) \to GF(2^4)$$
$$: x \mapsto x^2.$$

Now $x^2 = x$ if and only if $x = 0$ or 1, that is, if and only if $x \in GF(2)$. The elements which satisfy $x^{2^2} = x$ are the elements $0, 1, \omega^5$ and ω^{10} of $GF(2^2)$. All elements of $GF(2^4)$ satisfy $x^{2^4} = x$.

1.4.7 Definition

(1) A point $P \in PG(n, q^d)$ is called *imaginary* if the subspace $L(P)$ spanned by its d conjugates $P, P^\sigma, \ldots, P^{\sigma^{d-1}}$ has dimension $d-1$ in $PG(n, q^d)$.

(2) An m-dimensional subspace S_m of $PG(n, q^d)$ is called *imaginary* if the subspace $L(S_m)$ spanned by the d conjugates $S_m, S_m^\sigma, \ldots, S_m^{\sigma^{d-1}}$ has dimension $d(m+1)-1$ in $PG(n, q^d)$.

1.4.8 Theorem

(1) If P is an imaginary point of $PG(n, q^d)$ then $L(P)$ meets $PG(n, q)$ in a space of dimension $d-1$.

(2) If S_m is an imaginary subspace of $PG(n, q^d)$ then $L(S_m)$ meets $PG(n, q)$ in a space of dimension $d(m+1)-1$.

(3) If S_m is an imaginary subspace of $PG(n, q^d)$ then all its points are imaginary.

Proof: (1) Now

$$\begin{aligned}(L(P))^\sigma &= \left(\operatorname{lin}\left\{P, P^\sigma, \ldots, P^{\sigma^{d-1}}\right\}\right)^\sigma \\ &= \operatorname{lin}\left\{P^\sigma, \ldots, P^{\sigma^{d-1}}, P\right\} \\ &= L(P)\end{aligned}$$

and so by Lemma 1.4.5 (2), $L(P)$ meets $PG(n,q)$ in a space of dimension $d-1$.

(2) Similarly

$$\begin{aligned}(L(S_m))^\sigma &= \left(\operatorname{lin}\left\{S_m, S_m^\sigma, \ldots, S_m^{\sigma^{d-1}}\right\}\right)^\sigma \\ &= \operatorname{lin}\left\{S_m^\sigma, \ldots, S_m^{\sigma^{d-1}}, S_m\right\} \\ &= L(S_m),\end{aligned}$$

so by Lemma 1.4.5 (2), $L(S_m)$ meets $PG(n,q)$ in a space of dimension $d(m+1)-1$.

(3) Suppose that S_m is an imaginary subspace of $PG(n,q^d)$ containing a point P which is not imaginary. There exists a basis $P, X_1, X_2, \ldots, X_m$ for S_m and

$$\begin{aligned}L(S_m) &= \operatorname{lin}\left\{S_m, S_m^\sigma, \ldots, S_m^{\sigma^{d-1}}\right\} \\ &= \operatorname{lin}\Big\{P, P^\sigma, \ldots, P^{\sigma^{d-1}}, \\ &\qquad X_1, X_1^\sigma, \ldots, X_1^{\sigma^{d-1}}, \\ &\qquad \vdots \\ &\qquad X_m, X_m^\sigma, \ldots, X_m^{\sigma^{d-1}}\Big\} \\ &= \operatorname{lin}\left\{L(P), L(X_1), \ldots, L(X_m)\right\}.\end{aligned}$$

Now $L(P)$ has dimension less than $d-1$, so that $L(S_m)$ has dimension less than $d(m+1)-1$ contradicting Definition 1.4.7 (2). □

1.4.9 Theorem

Let P be a point of $PG(n,q^d)$. Let $\alpha \in GF(q^d)$ be such that $GF(q^d) = GF(q)(\alpha)$, so that P can be written uniquely in the form

$$P = P_0 + P_1\alpha + P_2\alpha^2 + \cdots + P_{d-1}\alpha^{d-1}$$

where $P_0, P_1, \ldots, P_{d-1}$ are real points of $PG(n, q^d)$. Then

$$\begin{aligned} L(P) &= \mathrm{lin}\left\{P, P^{\sigma}, P^{\sigma^2}, \ldots, P^{\sigma^{d-1}}\right\} \\ &= \mathrm{lin}\left\{P_0, P_1, \ldots, P_{d-1}\right\}. \end{aligned}$$

Proof: The d conjugates of P are:

$$\begin{aligned} P &= P_0 + P_1\alpha + P_2\alpha^2 + \cdots + P_{d-1}\alpha^{d-1}, \\ P^{\sigma} &= P_0 + P_1(\alpha^q) + P_2(\alpha^q)^2 + \cdots + P_{d-1}(\alpha^q)^{d-1}, \\ P^{\sigma^2} &= P_0 + P_1(\alpha^{q^2}) + P_2(\alpha^{q^2})^2 + \cdots + P_{d-1}(\alpha^{q^2})^{d-1}, \\ &\vdots \\ P^{\sigma^{d-1}} &= P_0 + P_1(\alpha^{q^{d-1}}) + P_2(\alpha^{q^{d-1}})^2 \cdots + P_{d-1}(\alpha^{q^{d-1}})^{d-1}. \end{aligned}$$

This can be written as

$$\left(P, \ldots, P^{\sigma^{d-1}}\right) = \left(P_0, \ldots, P_{d-1}\right) \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha & \alpha^q & \cdots & \alpha^{q^{d-1}} \\ \alpha^2 & (\alpha^q)^2 & \cdots & (\alpha^{q^{d-1}})^2 \\ \vdots & \vdots & & \vdots \\ \alpha^{d-1} & (\alpha^q)^{d-1} & \cdots & (\alpha^{q^{d-1}})^{d-1} \end{pmatrix}$$

where each of the elements P^{σ^j} and P_j is an $n \times 1$ column vector over $GF(q^d)$ or $GF(q)$ respectively and the matrix is $d \times d$ over $GF(q^d)$. The matrix is a Vandermonde matrix, which is invertible because the elements $\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{d-1}}$ are distinct (see Finkbeiner (1960), p96). Thus, over $GF(q^d)$, the space spanned by the columns of

$$\left(P \quad P^{\sigma} \quad P^{\sigma^2} \quad \ldots \quad P^{\sigma^{d-1}}\right)$$

coincides with the space spanned by the columns of

$$\left(P_0 \quad P_1 \quad P_2 \quad \ldots \quad P_{d-1}\right)$$

and the result is proved. □

1.4.10 Corollary

(1) A point $P = P_0 + P_1\alpha + P_2\alpha^2 + \cdots + P_{d-1}\alpha^{d-1}$ of $PG(n, q^d)$, where P_i is real for $i = 0, 1, \ldots, d-1$, is imaginary if and only if the points $P_0, P_1, \ldots, P_{d-1}$ are linearly independent over $GF(q)$ and hence over $GF(q^d)$.

(2) If P is an imaginary point of $PG(n, q^d)$ then $L(P)$ is the unique $(d-1)$-dimensional subspace of $PG(n, q^d)$ containing P and meeting $PG(n, q^d)$ in a space of dimension $d-1$.

(3) If S_{d-1} is a $(d-1)$-dimensional subspace of $PG(n, q)$, then there exists at least one imaginary point P of $PG(n, q^d)$ such that $L(P) = \overline{S}_{d-1}$.

Proof: (1) By definition, P is imaginary if and only if $L(P)$ has dimension $d-1$ in $PG(n, q^d)$. But by Theorem 1.4.9,

$$L(P) = \text{lin}\,\{P_0, P_1, \ldots, P_{d-1}\}$$

and this has dimension $d-1$ in $PG(n, q^d)$ if and only if the points $P_0, P_1, \ldots, P_{d-1}$ are linearly independent over $GF(q^d)$.

(2) This follows since the points $P_0, P_1, \ldots, P_{d-1}$ are uniquely determined by P.

(3) Suppose that $X_0, X_1, \ldots, X_{d-1}$ are linearly independent points of $PG(n, q)$ such that

$$S_{d-1} = \text{lin}\,\{X_0, X_1, \ldots, X_{d-1}\}\,.$$

Then the point

$$P = X_0 + X_1\alpha + X_2\alpha^2 + \cdots + X_{d-1}\alpha^{d-1}$$

is imaginary (by Theorem 1.4.9) and lies in

$$L(P) = \text{lin}\,\{X_0, X_1, \ldots, X_{d-1}\}\,.$$

□

1.5 THE SEGRE VARIETY $\overline{\mathcal{SV}}_{s+1,t+1}$ IN $PG((s+1)(t+1)-1, q^{t+1})$

In Chapters 2 and 3 (particularly Sections 2.9 and 3.9) we will be interested in the regularity of t-spreads of $PG((s+1)(t+1)-1, q)$, and it turns out that this idea is closely related to the Segre variety, and also to the embedding of $PG((s+1)(t+1)-1, q)$ in $PG((s+1)(t+1)-1, q^{t+1})$. In this Section, therefore, we shall investigate the properties of the Segre variety $\mathcal{SV}_{s+1,t+1}$ when embedded in the space $PG((s+1)(t+1)-1, q^{t+1})$.

In the following, let $GF(q^{t+1})$ be a field extension of $GF(q)$, and denote the corresponding extension of $PG((s+1)(t+1)-1, q)$ by $PG((s+1)(t+1)-1, q^{t+1})$. Recall Definition 1.4.2, so that a point of $PG((s+1)(t+1)-1, q)$ will be called a *real* point of $PG((s+1)(t+1)-1, q^{t+1})$. Similarly, a k-dimensional subspace of $PG((s+1)(t+1)-1, q^{t+1})$ will be called *real* if it intersects $PG((s+1)(t+1)-1, q)$ in a subspace of the same dimension k.

Recall also that σ denotes the field automorphism, called conjugation, of $GF(q^{t+1})$, that is, for $x \in GF(q^{t+1})$,

$$\sigma : x \mapsto x^q.$$

The map σ induces a collineation on $PG((s+1)(t+1)-1, q^{t+1})$ which fixes $PG((s+1)(t+1)-1, q)$ pointwise, and this collineation will also be called σ. The image of a point P under σ will be denoted by P^σ and more generally the image of a subspace S of $PG((s+1)(t+1)-1, q^{t+1})$ under σ will be denoted by S^σ.

Let $\mathcal{SV}_{s+1,t+1}$ be a Segre variety of $PG((s+1)(t+1)-1, q)$. We extend $\mathcal{SV}_{s+1,t+1}$ to $PG((s+1)(t+1)-1, q^{t+1})$ to obtain $\overline{\mathcal{SV}}_{s+1,t+1}$. This could be achieved by first extending $s+2$ of the t-dimensional spaces of $\mathcal{SV}_{s+1,t+1}$, no $s+1$ in a hyperplane, to $PG((s+1)(t+1)-1, q^{t+1})$. As in Lemma 1.3.4 (2), the resulting t-dimensional subspaces of $PG((s+1)(t+1)-1, q^{t+1})$ define a Segre variety $\overline{\mathcal{SV}}_{s+1,t+1}$ in $PG((s+1)(t+1)-1, q^{t+1})$. We could also have done the same thing using s-dimensional subspaces of $\mathcal{SV}_{s+1,t+1}$. Alternatively we could view the equations of the Segre variety $\mathcal{SV}_{s+1,t+1}$ as equations over $GF(q^{t+1})$, and then $\overline{\mathcal{SV}}_{s+1,t+1}$ is the set of points of $PG((s+1)(t+1)-1, q^{t+1})$ which satisfy the equations.

The t-dimensional spaces of $\mathcal{SV}_{s+1,t+1}$ extend to t-dimensional spaces of $\overline{\mathcal{SV}}_{s+1,t+1}$, so that $\overline{\mathcal{SV}}_{s+1,t+1}$ has $q^s + q^{s-1} + \cdots + q + 1$ real t-dimensional spaces. Similarly the s-dimensional spaces of $\mathcal{SV}_{s+1,t+1}$ extend to s-dimensional spaces of $\overline{\mathcal{SV}}_{s+1,t+1}$, so that $\overline{\mathcal{SV}}_{s+1,t+1}$ has $q^t + q^{t-1} + \cdots + q + 1$ real s-dimensional spaces. But $\overline{\mathcal{SV}}_{s+1,t+1}$ has $q^{(t+1)t} + q^{(t+1)(t-1)} + \cdots + q^{t+1} + 1$ s-dimensional spaces and $q^{(t+1)s} + q^{(t+1)(s-1)} + \cdots + q^{t+1} + 1$ t-dimensional spaces. Thus it has

$$q^{(t+1)t} + q^{(t+1)(t-1)} + \cdots + q^{t+1} + 1 - (q^t + q^{t-1} + \cdots + q + 1)$$

non-real s-dimensional spaces and

$$q^{(t+1)s} + q^{(t+1)(s-1)} + \cdots + q^{t+1} + 1 - (q^s + q^{s-1} + \cdots + q + 1)$$

non-real t-dimensional spaces. Since through each real point of $\overline{\mathcal{SV}}_{s+1,t+1}$ there passes a real t-dimensional space and a real s-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$ (namely the extensions to $PG((s+1)(t+1)-1, q^{t+1})$ of the t-dimensional space and s-dimensional space of $\mathcal{SV}_{s+1,t+1}$ through that point), no non-real s-dimensional space or non-real t-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$ may contain any real point (see Lemma 1.3.1 (2)).

A real t-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$ has $q^t + q^{t-1} + \cdots + q + 1$ real points and

$$q^{(t+1)t} + q^{(t+1)(t-1)} + \cdots + q^{t+1} + 1 - (q^t + q^{t-1} + \cdots + q + 1)$$

non-real points. A non-real t-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$ has

$$q^{(t+1)t} + q^{(t+1)(t-1)} + \cdots + q^{t+1} + 1$$

non-real points. Similarly, a real s-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$ has $q^s + q^{s-1} + \cdots + q + 1$ real points and

$$q^{(t+1)s} + q^{(t+1)(s-1)} + \cdots + q^{t+1} + 1 - (q^s + q^{s-1} + \cdots + q + 1)$$

non-real points. A non-real s-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$ has

$$q^{(t+1)s} + q^{(t+1)(s-1)} + \cdots + q^{t+1} + 1$$

non-real points.

A real point of $\overline{\mathcal{SV}}_{s+1,t+1}$ lies on a real t-dimensional space and a real s-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$. A non-real point lies on a t-dimensional space and an s-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$, where the t-dimensional space and the s-dimensional space are not both real. The Segre variety $\overline{\mathcal{SV}}_{s+1,t+1}$ has

$$\left(q^{(t+1)s} + q^{(t+1)(s-1)} + \cdots + q^{t+1} + 1\right)\left(q^{(t+1)t} + q^{(t+1)(t-1)} + \cdots + q^{t+1} + 1\right)$$

points, of which exactly

$$\left(q^s + q^{s-1} + \cdots + q + 1\right)\left(q^t + q^{t-1} + \cdots + q + 1\right)$$

are real.

Let Γ be an s-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$, skew to $PG((s+1)(t+1)-1, q)$. The set $\mathcal{S}$ of the t-dimensional spaces of $\overline{\mathcal{SV}}_{s+1,t+1}$ which are the extensions to $PG((s+1)(t+1)-1, q^{t+1})$ of t-dimensional spaces of $\mathcal{SV}_{s+1,t+1}$ meet Γ in a set B of $q^s + q^{s-1} + \cdots + q + 1$ points. Now B is an s-dimensional subspace of Γ having order q. This is because the extensions of the (real) t-dimensional spaces in S meet Γ in a set of points which is the projective image of the set of points of any of the s-dimensional subspaces of the Segre variety $\mathcal{SV}_{s+1,t+1}$, that is, an s-dimensional projective space of order q.

Conversely, let B be an s-dimensional subspace of order q of the s-dimensional space Γ. Then the set of t-dimensional spaces of $\overline{\mathcal{SV}}_{s+1,t+1}$ which meet it (each in a point) are the extensions of the t-dimensional spaces of a Segre variety $\mathcal{SV}_{s+1,t+1}$ of $PG((s+1)(t+1)-1, q)$ which extends as above to the Segre variety $\overline{\mathcal{SV}}_{s+1,t+1}$ containing the space Γ.

Now a t-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$ meeting Γ in a point P is real and so contains all the conjugates $P, P^{\sigma}, P^{\sigma^2}, \ldots, P^{\sigma^t}$ of P. Thus any t-dimensional space of $\overline{\mathcal{SV}}_{s+1,t+1}$ meets all of $\Gamma, \Gamma^{\sigma}, \ldots, \Gamma^{\sigma^t}$, each in a unique point. As these spaces all have dimension s, they are s-dimensional spaces of $\overline{\mathcal{SV}}_{s+1,t+1}$.

In this way, each s-dimensional subspace B of order q of Γ determines a Segre variety $\overline{\mathcal{SV}}_{s+1,t+1}$ containing Γ and each of the conjugates Γ^{σ}, $\Gamma^{\sigma^2}, \ldots, \Gamma^{\sigma^t}$. This means that we have projective correspondences τ_1, $\tau_2, \ldots, \tau_t$ between Γ and each of $\Gamma^{\sigma}, \Gamma^{\sigma^2}, \ldots, \Gamma^{\sigma^t}$ respectively such that the t-dimensional spaces of $\overline{\mathcal{SV}}_{s+1,t+1}$ are the t-dimensional spaces joining a point P of Γ to the corresponding points $P^{\tau_1}, P^{\tau_2}, \ldots, P^{\tau_t}$ of $\Gamma^{\sigma}, \Gamma^{\sigma^2}, \ldots, \Gamma^{\sigma^t}$ respectively. The real t-dimensional spaces of $\overline{\mathcal{SV}}_{s+1,t+1}$ are exactly those spaces meeting Γ in the points of B, and these are the joins of a point Q in B to the points Q^{σ}, $Q^{\sigma^2}, \ldots, Q^{\sigma^t}$ of $\Gamma^{\sigma}, \Gamma^{\sigma^2}, \ldots, \Gamma^{\sigma^t}$. Thus on the points of B, the projective correspondences τ_1, $\tau_2, \ldots, \tau_t$ determined by $\overline{\mathcal{SV}}_{s+1,t+1}$ coincide with the maps $\sigma, \sigma^2, \ldots, \sigma^t$ from Γ to each of $\Gamma^{\sigma}, \Gamma^{\sigma^2}, \ldots, \Gamma^{\sigma^t}$ in turn.

Conversely, suppose we are given projective correspondences τ_1, $\tau_2, \ldots, \tau_t$ from Γ to each of $\Gamma^{\sigma}, \Gamma^{\sigma^2}, \ldots, \Gamma^{\sigma^t}$ respectively. Then these determine a Segre variety $\overline{\mathcal{SV}}_{s+1,t+1}$ containing each of $\Gamma, \Gamma^{\sigma}, \Gamma^{\sigma^2}, \ldots, \Gamma^{\sigma^t}$. Since $\overline{\mathcal{SV}}_{s+1,t+1}$ contains $q^s + q^{s-1} + \cdots + q + 1$ real t-dimensional spaces, there exists an s-dimensional subspace B of Γ of order q such that the projective correspondences $\tau_1, \tau_2, \ldots, \tau_t$ coincide with the maps $\sigma, \sigma^2, \ldots, \sigma^t$ from Γ to each of $\Gamma^{\sigma}, \Gamma^{\sigma^2}, \ldots, \Gamma^{\sigma^t}$ on exactly the points of B.

1.6 THE SPACE $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$

This space was introduced by Thas (1971), and we will give a short introduction to it in this Section, using the same notation. Let $\mathcal{M}_n(GF(q))$ be the set of all $n \times n$ matrices over $GF(q)$. There are q^{n^2} of them, and the number of $n \times n$ matrices of rank k over GF(q) is

$$\prod_{j=0}^{k-1} \frac{(q^n - q^j)(q^m - q^j)}{q^k - q^j}.$$

Consider the collection of $(m+1)$-tuples of elements of $\mathcal{M}_n(GF(q))$, written as column vectors, say $(\xi_0, \xi_1, \ldots, \xi_m)^T$, such that $\xi_i \in \mathcal{M}_n(GF(q))$ and, over $GF(q)$,

$$\text{rank} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = n.$$

We will usually interpret this as an $(m+1)n \times n$ matrix over $GF(q)$. We introduce an equivalence relation on this set of $(m+1)$-tuples of elements of $M_n(GF(q))$, so that two such $(m+1)$-tuples $(\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_m^{(i)})^T$ and $(\xi_0^{(j)}, \xi_1^{(j)}, \ldots, \xi_m^{(j)})^T$ are equivalent if there exists a non-singular $n \times n$ matrix ρ over GF(q) such that

$$\xi_k^{(i)} = \xi_k^{(j)} \rho \quad \text{for all } k = 0, 1, \ldots, m.$$

1.6.1 Definition [Thas (1971)]

The space $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ is the set of equivalence classes of such $(m+1)$-tuples of elements of $M_n(GF(q))$, and each equivalence class is referred to as a *point* of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$.

The space $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ reflects the "homogeneous" nature of projective spaces, except that in this case the homogeneity is with respect to multiplication by non-singular matrices.

A set of k points $P_1, P_2, \ldots, P_k$ of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$, with $2 \leq k \leq m+1$, is said to be in *clear position* if, given that $P_i = (\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_m^{(i)})^T$,

$$\operatorname{rank}\begin{pmatrix} \xi_0^{(1)} & \xi_0^{(2)} & \cdots & \xi_0^{(k)} \\ \xi_1^{(1)} & \xi_1^{(2)} & \cdots & \xi_1^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_m^{(1)} & \xi_m^{(2)} & \cdots & \xi_m^{(k)} \end{pmatrix} = nk.$$

This matrix will be referred to as the *coordinate matrix* $\mathcal{P}_{1,2,\ldots,k}$ of the points $P_1, P_2, \ldots, P_k$. If $P_1, P_2, \ldots, P_k$ are in clear position, then they define a $(k-1)$-*dimensional subspace*, to be denoted by $\mathcal{S}_{k-1}(\mathcal{M}_n(GF(q)))$, of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ as follows: the points of $\mathcal{S}_{k-1}(\mathcal{M}_n(GF(q)))$ are defined to be precisely those points $(\xi_0, \xi_1, \ldots, \xi_m)^T$ of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ which satisfy

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = \mathcal{P}_{1,2,\ldots,k} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix}$$

where $\alpha_1, \alpha_2 \ldots, \alpha_k$ vary among all the elements of $\mathcal{M}_n(GF(q))$ which satisfy

$$\operatorname{rank}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = n.$$

Now any k points $P_1, P_2, \ldots, P_k$ of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ belong to an l-dimensional subspace $\mathcal{S}_l(\mathcal{M}_n(GF(q)))$ if and only if their coordinate matrix $\mathcal{P}_{1,2,\ldots,k}$ has rank less than or equal to $n(l+1)$. A subspace $\mathcal{S}_{m-1}(\mathcal{M}_n(GF(q)))$ is called a *hyperplane*, a subspace of dimension 1 is called a *line* and a subspace of dimension 0 is a point.

A hyperplane has tangential coordinates $[\pi_0, \pi_1, \ldots, \pi_m]$, written as a row vector, with each matrix $\pi_i \in \mathcal{M}_n(GF(q))$, and the rank of the matrix of tangential coordinates is n. A point $P = (\xi_0, \xi_1, \ldots, \xi_m)^T$ and a hyperplane of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ are said to be in *clear position* if the matrix $[\pi_0\xi_0 + \pi_1\xi_1 + \cdots + \pi_m\xi_m]$ has rank n.

1.6.2 Theorem [Thas (1971)]
There is a bijection f of the space $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ onto the set of all $(n-1)$-dimensional subspaces of a projective space $PG((m+1)n-1, q)$.

Proof: To any point of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ there corresponds an $(n-1)$-dimensional subspace of $PG((m+1)n-1, q)$, defined in the following way. Let $P = (\xi_0, \xi_1, \ldots, \xi_m)^T$ be a point of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$. Then P is mapped under the bijection to $f(P)$ which is the $(n-1)$-dimensional subspace of $PG((m+1)n-1, q)$ spanned by the n columns of P over $GF(q)$, recalling that P has rank n. $f(P)$ is well-defined since if $Q = (\zeta_0, \zeta_1, \ldots, \zeta_m)^T$ is another representation for the point P then by definition there exists a non-singular matrix ρ such that for each $i = 0, 1, \ldots, m$ we have $\zeta_i = \xi_i\rho$ which implies that $Q = P\rho$. The columns of Q are just the coordinate vectors of the columns of P under the change of basis reflected in the matrix ρ. Thus the $(n-1)$-dimensional subspace of $PG((m+1)n-1, q)$ spanned by the columns of Q coincides with the space spanned by the columns of P.

Now let S_{n-1} be an $(n-1)$-dimensional subspace of $PG((m+1)n-1, q)$ and let $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$ be a basis for S_{n-1}. Then $f^{-1}(S_{n-1})$ is defined to be the point P of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ with coordinate matrix $(\xi_0, \xi_1, \ldots, \xi_m)^T$ whose columns are $\mu_0, \mu_1, \ldots, \mu_{n-1}$. This is well defined since if $\nu = (\nu_0, \nu_1, \ldots, \nu_{n-1})$ is another basis for S_{n-1} then the matrix $Q = (\zeta_0, \zeta_1, \ldots, \zeta_m)^T$ whose columns are $\nu_0, \nu_1, \ldots, \nu_{n-1}$ will represent the same point P of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$. This is because if ρ is the (non-singular) transition matrix from the basis μ to the basis ν then $\zeta_i = \xi_i \rho$ for $i = 1, 2, \ldots, n$. □

Thas (1971) showed that for $2 \leq k \leq m+1$, k points of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ are in clear position if and only if the k corresponding $(n-1)$-dimensional subspaces of $PG((m+1)n-1, q)$ span a space of dimension $kn-1$.

The points of a subspace $\mathcal{S}_k(\mathcal{M}_n(GF(q)))$ correspond under the bijection to the set of $(n-1)$-dimensional subspaces of $PG((m+1)n-1, q)$ lying in a subspace $PG((k+1)n-1, q)$. A point and a hyperplane of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ are in clear position if and only if the corresponding subspaces S_{n-1} and S_{mn-1} of $PG((m+1)n-1, q)$ are skew.

In the next theorem, O_i is the $(m+1)n \times n$ matrix which in block form has an $n \times n$ identity matrix in the i^{th} row and zero matrices in every other row. E is the matrix with the identity matrix in every row in its block form. A collineation of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ is a map which involves first applying a field automorphism to the coordinates of each point and then multiplying the coordinate vector by a non-singular matrix of appropriate size.

1.6.3 Theorem [Thas (1971)]
Given a set of $m+2$ points $P_0, P_1, \ldots, P_{m+1}$ of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ such that every $m+1$ of them are in clear position, there exists a collineation Ω of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ such that

$$\Omega(O_i) = P_i, \quad \text{for } i = 0, 1, \ldots, m$$

and

$$\Omega(E) = P_1.$$

□

CHAPTER TWO

t-SPREADS OF $PG(2t+1, q)$

2.1 INTRODUCTION

In this chapter we revise much of the theory of t-spreads of $PG(2t+1, q)$, although sometimes from a different point of view, setting the scene for the generalisation to come in Chapter Three. A very useful reference for the theory of t-spreads is Dembowski (1968) who directs the reader to the original works. More recent texts are Hirschfeld (1979) and (1985). Thus most of the work in this chapter is well known, with the exception of Sections 2.5, 2.8 and 2.9 which I believe to be original.

Let $\mathcal{W}$ be a t-spread of $PG(n, q)$ as defined in Section 1.2. In the following we will set $n = (s+1)(t+1) - 1$, since by Theorem 1.2.2, $t+1$ must divide $n+1$. In this Chapter we make the further restriction that $s = 1$, so that we are considering only t-spreads of $PG(2t+1, q)$. The case of general s will be tackled in Chapter Three.

So for this Chapter, let $\mathcal{W}$ be a t-spread of $PG(2t+1, q)$. Thus $\mathcal{W}$ comprises $q^{t+1}+1$ pairwise skew t-dimensional subspaces covering the points of $PG(2t+1, q)$.

2.2 t-SPREAD SETS

In their papers of (1964) and (1966) Bruck and Bose showed how, given a t-spread of $PG(2t+1, q)$, one could construct a spread set. A spread set is a set of linear transformations of the $(t+1)$-dimensional vector space corresponding to one of the t-spread elements, under the correspondence explained in Section 1.1. The linear transformations can be represented as $(t+1) \times (t+1)$ matrices, and the set of such matrices is also called a spread set. We will in general identify a linear transformation of a $(t+1)$-dimensional vector space with the $(t+1) \times (t+1)$ matrix that it determines. The construction of a spread set will be generalised in Chapter Three so we give the construction of Bruck and Bose (1966) in full here.

Because our coordinate vectors are written as column vectors while Bruck and Bose (1964) and (1969) used row vectors, our linear transformations will act by premultiplication by a $(t+1) \times (t+1)$ matrix instead of postmultiplication by a $(t+1) \times (t+1)$ matrix. The matrices of the spread set constructed here are just the transposes of the matrices constructed in Bruck and Bose (1964).

Let $\mathcal{W}$ be a t-spread of $PG(2t+1,q)$. As in Section 1.1 we represent the space $PG(2t+1,q)$ as a $(2t+2)$-dimensional vector space $\mathcal{V}_{2t+2}$ over the field $GF(q)$. Then $\mathcal{W}$ becomes a collection, still denoted by $\mathcal{W}$, of $(t+1)$-dimensional vector subspaces of $\mathcal{V}_{2t+2}$ pairwise having only the zero vector in common and satisfying the property that each non-zero vector of $\mathcal{V}_{2t+2}$ lies in exactly one element of $\mathcal{W}$.

Let A, B and C be an ordered triple of distinct $(t+1)$-dimensional vector subspaces of $\mathcal{V}_{2t+2}$, pairwise skew (in the usual sense that pairwise they have only the zero vector in common). We will write $\mathcal{V}_{2t+2}$ as the the direct sum of A and B. Then applying Lemma 1.1.2 we see that there exists a unique non-singular linear transformation

$$' : a \mapsto a'$$

of A onto B such that the linear transformation

$$a \mapsto a \oplus a'$$

maps A onto C. To each linear transformation X of A to A over $GF(q)$ there corresponds a unique $(t+1)$-dimensional subspace $J(X)$ of $\mathcal{V}_{2t+2}$ given by

$$J(X) = \{Xa \oplus a' : a \in A\}.$$

In particular (with the following convention for ∞, and denoting the zero linear transformation by 0 and the identity transformation by I),

$$J(\infty) = A = \{a : a \in A\},$$
$$J(0) = B = \{a' : a \in A\}, \quad \text{and}$$
$$J(I) = C = \{a \oplus a' : a \in A\}.$$

Conversely each $(t+1)$-dimensional subspace J of $\mathcal{V}_{2t+2}$ which is skew to $A = J(\infty)$ has the form $J = J(X)$ for a unique linear transformation X of A onto itself.

2.2.1 Lemma

Let X and Y be linear transformations of A into itself. Then

$$\begin{aligned} J(X) \cap J(Y) =& \{Xa \oplus a' : a \in A \text{ and } (X-Y)a = 0\} \\ =& \{Ya \oplus a' : a \in A \text{ and } (X-Y)a = 0\}. \end{aligned}$$

Proof: Suppose $x \in J(X) \cap J(Y)$. Then for unique elements $a, b \in A$,

$$x = Xa \oplus a' = Yb \oplus b'.$$

This can occur if and only if $a' = b'$ and $Xa = Yb$. Now $a' = b'$ implies that $a = b$ and $Xa = Yb$ implies that

$$(X - Y)a = 0$$

and the result follows. □

2.2.2 Corollary

Two spaces $J(X)$ and $J(Y)$ are skew if and only if $X - Y$ is non-singular.

Proof: $J(X)$ and $J(Y)$ have only the zero vector in common if and only if the equation

$$(X - Y)a = 0$$

has only the trivial solution $a = 0$. This occurs if and only if the matrix $X - Y$ is non-singular. □

Now let A, B and C be an ordered triple of three distinct elements of the t-spread $\mathcal{W}$. In terms of the above representation, $\mathcal{W}$ corresponds uniquely to a collection $\mathcal{C} = \mathcal{C}(A, B, C)$ of linear transformations of the vector space A onto itself over $GF(q)$ with the following properties:

(i) $\mathcal{C}$ contains 0 and I,

(ii) If X and Y are distinct elements of $\mathcal{C}$ then $X - Y$ is non-singular, and

(iii) If $b, c \in A$ with $c \neq 0$ there exists a unique X in $\mathcal{C}$ such that $Xc = b$.

To establish these properties, first note that the spaces B and C give rise to the elements 0 and I of $\mathcal{C}$. If X and Y are distinct elements of $\mathcal{C}$, they correspond to distinct elements $J(X)$ and $J(Y)$ of $\mathcal{W}$. These are skew, so by Corollary 2.2.2, $X - Y$ is non-singular. To show (iii), recall that any element of $\mathcal{V}_{2t+2}$ skew to $J(\infty)$ can be written uniquely as $b \oplus c'$ where $b, c \in A$ and $c \neq 0$. Then $b \oplus c'$ is contained in a unique element $J(X)$ of the t-spread $\mathcal{W}$, so that

$$b \oplus c' \in J(X) = \{Xa \oplus a' : a \in A\},$$

and so $c' = a'$ implying that $c = a$ and $b = Xc$. These results suggest the next definition.

2.2.3 Definition [Bruck and Bose (1964)]
A *t-spread set* is a set $\mathcal{C}$ of linear transformations of a $(t + 1)$-dimensional vector space onto itself satisfying the following conditions:

(i) $\mathcal{C}$ has q^{t+1} elements,

(ii) $\mathcal{C}$ contains 0 and I, and

(iii) If X and Y are distinct elements of $\mathcal{C}$ then $X - Y$ is non-singular.

We call the set of linear transformations a t-spread set instead of a spread set to emphasise the fact that it corresponds to a t-spread.

Conditions (ii) and (iii) above ensure that every non-zero element of $\mathcal{C}$ is non-singular.

2.2.4 Theorem [Bruck and Bose (1964)]
Let $\mathcal{C}$ be a t-spread set and let $\{a_1, a_2, \ldots, a_{t+1},\ b_1, b_2, \ldots, b_{t+1}\}$ be a basis of a $(2t+2)$-dimensional vector space $\mathcal{V}_{2t+2}$. Let $J(\infty)$ be the subspace of $\mathcal{V}_{2t+2}$ spanned by the vectors $a_1, a_2, \ldots, a_{t+1}$ and for each $C_i \in \mathcal{C}$ let $J(C_i)$ be the subspace

$$J(C_i) = \mathrm{lin}\left\{C_i a_1 \oplus b_1, C_i a_2 \oplus b_2, \ldots, C_i a_{t+1} \oplus b_{t+1}\right\}.$$

Then the set

$$\mathcal{W} = \{J(\infty)\} \cup \{J(C_i) : C_i \in \mathcal{C}\}$$

is a set of pairwise skew $(t+1)$-dimensional subspaces of $\mathcal{V}_{2t+2}$. It therefore represents the set of elements of a t-spread $\mathcal{W}$ of the corresponding projective space $PG(2t+1, q)$. Conversely every such t-spread may be represented in this manner by a t-spread set.

Proof: The $t+1$ vectors spanning each of the spaces $J(\infty)$ and $J(C_i)$ for $C_i \in \mathcal{C}$ are linearly independent, so each element of $\mathcal{W}$ is a $(t+1)$-dimensional subspace of $\mathcal{V}_{2t+2}$. Now $J(\infty)$ has only the zero vector in common with each space $J(C_i)$, and by Definition 2.2.3 (iii) and Corollary 2.2.2 we see that $J(C_i)$ and $J(C_j)$ are also skew for all $i \neq j$. Thus $\mathcal{W}$ is a set of $q^{t+1}+1$ pairwise skew $(t+1)$-dimensional subspaces of $\mathcal{V}_{2t+2}$, corresponding to a t-spread of $PG(2t+1, q)$. The converse, that every t-spread of $PG(2t+1,q)$ can be represented in this manner is demonstrated in the construction by Bruck and Bose (1964) given above. □

2.2.5 Remark

It is worthwhile to note that in this construction of a t-spread set, a procedure of a "non-homogeneous" nature was used. Suppose that instead of considering the spaces

$$J(X) = \{Xa \oplus a' : a \in A\}$$

we had used spaces of the form

$$J(M,N) = \{Ma \oplus Na' : a \in A\}$$

where N is a linear transformation of A and M is a linear transformation of B then $J(\infty)$ would have arisen as

$$J(\infty) = J(I,0)$$

and any other space would be

$$J(X) = J(X,I).$$

The construction of the t-spread set is reminiscent of the process of restricting a projective line

$$l = \{(x,y) : x,y \in GF(q)\}$$

to an affine line

$$\bar{l} = \{(x,1) : \ x \in GF(q)\} = \{(x) : \ x \in GF(q)\}$$

by deleting the point $(\infty) = (1,0)$.

2.3 CONSTRUCTION OF AN AFFINE PLANE OF ORDER q^{t+1}

Bruck and Bose, again in their papers of (1964) and (1966) developed a very important technique, that of constructing an affine plane of order q^{t+1} from a t-spread of $PG(2t+1,q)$. This affine plane can be completed to a projective plane of order q^{t+1} in the usual way, and is in fact a translation plane. As this construction will be generalised in Chapter Three, it is given in full here.

2.3.1 The construction [Bruck and Bose (1964), p88]

Let $\mathcal{W}$ be a t-spread of $PG(2t+1,q)$, and embed $PG(2t+1,q)$ as a subspace of $PG(2t+2,q)$. We define an incidence structure $\Pi = (P,B,I)$ as follows:

- The *points* of Π are the points of $PG(2t+2,q) - PG(2t+1,q)$.
- The *lines* of Π are the $(t+1)$-dimensional projective subspaces of $PG(2t+2,q)$ which intersect $PG(2t+1,q)$ in a unique element of $\mathcal{W}$, and are not contained in $PG(2t+1,q)$.
- The *incidence* relation of Π is that induced by the incidence relation of $PG(2t+2,q)$.

2.3.2 Theorem [Bruck and Bose (1964), p88]

The incidence structure Π is an affine plane of order q^t.

Proof: The theorem is proved by checking that the incidence structure Π satisfies the axioms of an affine plane of order q^t. The details appear in Bruck and Bose (1964), p88–89. □

The affine plane Π may be completed to a projective plane in the usual manner. Since each element X of the t-spread $\mathcal{W}$ corresponds to a class of parallel lines of Π, namely those containing X, we adjoin each such X to Π as a "point at infinity." The t-spread $\mathcal{W}$ fills the role of the "line at infinity." Hence the corresponding projective plane has a concrete representation in terms of this construction.

2.3.3 Remarks

(1) The affine and projective planes constructed in this Section need not be Desarguesian. In fact it is proved [Bruck and Bose (1966), Theorem 12.1] that the construction yields a Desarguesian plane if and only if the t-spread $\mathcal{W}$ is regular (see Section 2.4 for the definition of regular).

(2) It is interesting to note that the t-spread $\mathcal{W}$ acts like the projective line in this construction, (compare this with Remark 2.2.5). More will be said about this in Section 2.7 and, in the general case, in Chapter Three.

2.4 t-REGULI AND REGULAR t-SPREADS OF $PG(2t+1,q)$

These are very important ideas in the study of t-spreads and, as they too will be generalised in Chapter Three, a brief introduction is given here.

2.4.1 Definition [Dembowski (1968), p220–221]
A *t-regulus* in $PG(2t+1,q)$ is a set $\mathcal{R}$ of t-dimensional subspaces such that

(i) $\mathcal{R}$ has $q+1$ elements,

(ii) the elements of $\mathcal{R}$ are pairwise skew, and

(iii) if a line l meets three distinct elements of $\mathcal{R}$, then it meets them all.

Such a line l is called a *transversal* of $\mathcal{R}$. A transversal meets every element of $\mathcal{R}$ in a unique point, and conversely every point of a transversal belongs to a unique element of $\mathcal{R}$.

There is a unique transversal through each point of an arbitrary element of $\mathcal{R}$, so that in particular all transversals of $\mathcal{R}$ are pairwise skew. The existence of t-reguli is well known; in fact the non-degenerate quadrics of index $t+1$ in $PG(2t+1,q)$ are always covered by reguli.

2.4.2 Lemma [Dembowski (1968), p220–221]
Given any three pairwise skew t-dimensional subspaces A, B and C in $PG(2t+1,q)$ there is a unique t-regulus $\mathcal{R} = \mathcal{R}(A,B,C)$ containing A, B and C.

Proof: Through a fixed point of A, there passes a unique line meeting both B and C, necessarily in a unique point. Thus A, B and C admit

$$\theta = q^t + q^{t-1} + \cdots + q + 1$$

transversal lines $\{l_i : i = 1,2,\ldots,\theta\}$, which are pairwise skew. The t-dimensional spaces A, B and C determine a projective correspondence between l_1 and another transversal line l_j as follows. The image of the point $P_A = A \cap l_1$ of l_1 is the unique point $P_A^{(j)} = A \cap l_j$ of l_j. Similarly, the points $P_B = B \cap l_1$ and $P_C = C \cap l_1$ of l_1 have images $P_B^{(j)} = B \cap l_j$ and $P_C^{(j)} = C \cap l_j$ of l_j, respectively. These three pairs of point and image determine a unique projective correspondence between l_1 and l_j for each $j = 2,3,\ldots,\theta$. The space joining a point $Q \in l_1$ to its images $Q^{(2)}, Q^{(3)}, \ldots, Q^{(\theta)}$ on each of $l_2, l_3, \ldots, l_\theta$ respectively is a t-dimensional subspace of $PG(2t+1,q)$, and two such spaces S_Q and S_R for $Q, R \in l_1$ are skew. We have therefore constructed a set $\mathcal{R}$ of $q+1$ pairwise skew t-dimensional subspaces, each meeting each of $l_1, l_2, l_3, \ldots, l_\theta$ in a unique point. If a line of $PG(2t+1,q)$ meets three elements of $\mathcal{R}$ then it must be one of the lines $l_1, l_2, l_3, \ldots, l_\theta$ since through a point of $PG(2t+1,q)$ there passes a unique transversal line to two skew t-dimensional spaces. By Definition 2.4.1, $\mathcal{R}$ is a t-regulus containing A, B and C. It is unique because of the uniqueness of the projective correspondences

$^{(2)}, {}^{(3)}, \ldots, {}^{(\theta)}$. □

2.4.3 Definition [Dembowski (1968), p220–221]

A t-spread $\mathcal{W}$ of $PG(2t+1, q)$ is *t-regular* (or just *regular*) if whenever $A, B, C \in \mathcal{W}$, then $\mathcal{R}(A, B, C) \subset \mathcal{W}$.

2.4.4 Remark

It is worth noting the connection between the t-reguli of $PG(2t+1, q)$ and the Segre variety $\mathcal{SV}_{2,t+1}$. The set of t-dimensional subspaces in a t-regulus $\mathcal{R}$ together with all the transversal lines is a Segre variety $\mathcal{SV}_{2,t+1}$ in $PG(2t+1, q)$. This is because, as we have already seen, the t-dimensional spaces of the t-regulus $\mathcal{R}$ determine a projective correspondence between each pair of transversal lines, and in a similar way the transversal lines determine a projective correspondence between any pair of t-dimensional spaces. Any three of the t-dimensional spaces determine the Segre variety, and any $t + 2$ of the transversal lines determine the Segre variety. It is this observation which leads to the generalisation of the idea of t-regulus of $PG(2t + 1, q)$ to that of t-regulus of rank r of $PG((s + 1)(t + 1) - 1, q)$ as we shall see in Section 3.4.

2.4.5 Theorem

Let $\mathcal{W}$ be a t-spread of $PG(2t + 1, q)$. Then $\mathcal{W}$ is regular if and only if given any line l of $PG(2t + 1, q)$ not meeting any element of $\mathcal{W}$ in more than one point, the elements of $\mathcal{W}$ meeting l form a t-regulus in $PG(2t + 1, q)$.

Proof: Suppose that $\mathcal{W}$ is regular. Let l be a line of $PG(2t + 1, q)$ not meeting any element of $\mathcal{W}$ in more than one point, and let A, B and C be three distinct elements of $\mathcal{W}$ meeting l. Now by Lemma 2.4.2, A, B and C are in a unique t-regulus $\mathcal{R}(A, B, C)$, and this t-regulus has l as a transversal line. Since

$\mathcal{W}$ is regular, $\mathcal{R}(A,B,C)$ is contained in $\mathcal{W}$ and thus the elements of $\mathcal{W}$ meeting l (which are exactly the lines of $\mathcal{R}(A,B,C)$) form a t-regulus.

Conversely, suppose that given any line l of $PG(2t+1,q)$ not meeting any element of $\mathcal{W}$ in more than one point, the elements of $\mathcal{W}$ meeting l form a t-regulus of $PG(2t+1,q)$. Let A, B and C be three elements of $\mathcal{W}$ and let l be a transversal line of A, B and C. The elements of $\mathcal{W}$ meeting l form a t-regulus $\mathcal{R}(A,B,C)$, the unique t-regulus containing A, B and C. Thus $\mathcal{R}(A,B,C)$ is contained in $\mathcal{W}$ and $\mathcal{W}$ is regular. □

2.4.6 Theorem

Every t-spread of $PG(2t+1,2)$ is regular.

Proof: Let $\mathcal{W}$ be a t-spread of $PG(2t+1,q)$. Now a t-regulus of $PG(2t+1,q)$ has three elements, thus any three elements A, B and C of $\mathcal{W}$ determine a unique t-regulus $\mathcal{R}(A,B,C) = \{A,B,C\}$ which is contained in $\mathcal{W}$. By Definition 2.4.3, $\mathcal{W}$ is regular. □

2.5 CONSTRUCTION OF A PROJECTIVE t-SPREAD SET

In this Section we give a construction for a t-spread set corresponding to any t-spread of $PG(2t+1,q)$ by a method which is easier than finding a set of matrices satisfying the requirements of Definition 2.2.3. For the approach developed in this Chapter, all we need is a basis for each of the subspaces of the t-spread.

The projective t-spread set is constructed by an entirely different method from that used in Section 2.2. In fact we will use the space $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ defined in Thas (1971) and introduced in Section 1.6.

Under the bijection f of Theorem 1.6.2, points of the space $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$

are mapped into $(n-1)$-dimensional subspaces of $PG((m+1)n-1,q)$. Thus to apply these ideas to t-spreads in $PG(2t+1,q)$, we need to put $n=t+1$ and $m=1$.

Thus a t-dimensional subspace of $PG(2t+1,q)$ gives rise to a homogeneous pair of $(t+1)\times(t+1)$ matrices, or a point of $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$. We show that the t-spread sets of Section 2.2 arise naturally from the construction by simply "non-homogenising" the homogeneous pairs of matrices found. The construction given in this section has a "homogeneous" or "projective" flavour and we will call the resulting structure a projective t-spread set.

2.5.1 The construction

A t-spread $\mathcal{W}$ of $PG(2t+1,q)$ maps under the bijection f^{-1} (see Theorem 1.6.2) into a set of $q^{t+1}+1$ points $P_0, P_1, \ldots, P_{q^{t+1}}$ of $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$, each pair of which is in clear position. Note that the idea of clear position can only be applied to pairs of points in $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$ since $m+1=2$. If we choose any three points from among $P_0, P_1, \ldots, P_{q^{t+1}}$, we may use Theorem 1.6.3 to map them under a collineation of $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$ to the points O_0, O_1 and E. Thus without loss of generality suppose that

$$P_0 = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \text{and} \quad P_2 = \begin{pmatrix} I \\ I \end{pmatrix}$$

where each submatrix is $(t+1)\times(t+1)$. Recalling that under the bijection f^{-1}, a t-spread element W_i is the space spanned by the columns of the coordinate matrix

$$P_i = \begin{pmatrix} \xi_0^{(i)} \\ \xi_1^{(i)} \end{pmatrix},$$

we see that this process is equivalent to choosing a basis $(e_1, e_2, \ldots, e_{2t+2})$ for

$PG(2t+1,q)$ so that the t-spread $\mathcal{W}$ contains the t-dimensional spaces

$$W_0 = \text{lin}\{e_{t+2}, e_{t+3}, \ldots, e_{2t+2}\},$$

$$W_1 = \text{lin}\{e_1, e_2, \ldots, e_{t+1}\},$$

$$W_2 = \text{lin}\{e_1 + e_{t+2}, e_2 + e_{t+3}, \ldots, e_{t+1} + e_{2t+2}\}$$

and $\mathcal{V}_{2t+2}$ is the direct sum of W_0 and W_1. Now any point P_i is represented by a column vector of two $(t+1) \times (t+1)$ matrices $(\xi_0^{(i)}, \xi_1^{(i)})^T$, for $i = 0, 1, \ldots, q^{t+1}$. Since every pair of points P_i, P_j for $i, j \in \{0, 1, \ldots, q^{t+1}\}$ is in clear position, the following matrix has rank equal to $2(t+1)$:

$$\begin{pmatrix} \xi_0^{(j)} & \xi_0^{(i)} \\ \xi_1^{(j)} & \xi_1^{(i)} \end{pmatrix}.$$

These results suggest the following definition.

2.5.2 Definition

A *projective t-spread set* is a set $\mathcal{PC} = \{(\xi_0^{(i)}, \xi_1^{(i)}) : i = 0, 1, \ldots, q^{t+1})\}$ of pairs of $(t+1) \times (t+1)$ matrices such that

(i) $\mathcal{PC}$ has $q^{t+1} + 1$ elements,

(ii) For each i,

$$\text{rank}\begin{pmatrix} \xi_0^{(i)} \\ \xi_1^{(i)} \end{pmatrix} = t+1,$$

(iii) If $(\xi_0^{(i)}, \xi_1^{(i)})$ and $(\xi_0^{(j)}, \xi_1^{(j)})$ are distinct elements of $\mathcal{PC}$ then

$$\text{rank}\begin{pmatrix} \xi_0^{(i)} & \xi_0^{(j)} \\ \xi_1^{(i)} & \xi_1^{(j)} \end{pmatrix} = 2(t+1).$$

A projective t-spread set is said to be *normalised* if it satisfies the additional property,

(iv) $\mathcal{PC}$ contains the elements $(0, I)$, $(I, 0)$, and (I, I).

We have shown that every projective t-spread set can be normalised, which just corresponds to choosing a convenient coordinatisation for $PG(2t+1,q)$. Thus we shall usually assume that a projective t-spread set has been normalised, except where we explicitly mention a non-normalised set. In a normalised projective t-spread set, condition (ii) is implied by (iii) and (iv).

2.5.3 Remark

Referring again to Remark 2.2.5, the projective t-spread set of matrices arises naturally as a set of homogeneous pairs of matrices. In this case it is easy to "non-homogenise" to get the t-spread set of matrices, (see Theorem 2.5.5 below). Only one t-spread element $A = J(\infty)$ is 'lost' in the sense that it does not have an element of the t-spread set corresponding to it. In the standard treatment of Section 2.2, $J(\infty)$ is recovered as the points of $PG(2t+1,q)$ not lying in any other t-spread element. In contrast, the projective t-spread set has a specific element describing $J(\infty)$. The advantages of using a projective t-spread set are more obvious in the general case of $s > 1$ given in Chapter Three. In this situation, not every t-spread has a t-spread set, and any eventual t-spread set comprises elements of different natures corresponding to different t-spread elements. The homogeneous form is preferable as there is only one type of element of the projective t-spread set corresponding to each of the t-spread elements. For the details see Chapter Three.

2.5.4 Theorem

Let $\mathcal{W}$ be a t-spread of $PG(2t+1,q)$. Then there exists a projective t-spread set, and conversely every projective t-spread set gives rise to a t-spread.

Proof: If $\mathcal{W}$ is a t-spread of $PG(2t+1,q)$ then writing the elements as points of the space $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$ yields a projective t-spread set by the arguments preceding Definition 2.5.2. Conversely if $\mathcal{PC}$ is a projective t-spread set then we may interpret its elements as points of the space $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$. Since these points satisfy Definition 2.5.2, they in turn correspond to elements of a t-spread of $PG(2t+1,q)$. □

2.5.5 Theorem

Given a projective t-spread set we may construct a t-spread set, and the t-spreads defined by each of them are isomorphic.

Proof: Let $\mathcal{PC} = \{P_i = (\xi_0^{(i)}, \xi_1^{(i)}) : i = 0, 1, \ldots, q^{t+1}+1\}$ be a projective t-spread set of matrices. Without loss of generality we assume that it is normalised so that $P_0 = (I, 0)$, $P_1 = (0, I)$ and $P_2 = (I, I)$. Since for $i \neq 0$, the point P_i is in clear position with the point P_0,

$$\text{rank}\begin{pmatrix} I & \xi_0^{(i)} \\ 0 & \xi_1^{(i)} \end{pmatrix} = 2(t+1).$$

Therefore it follows that the matrix $\xi_1^{(i)}$ has non-zero determinant and the point P_i may be written in the form $P_i = (C_i, I)$ where $C_i = \xi_0^{(i)}\left(\xi_1^{(i)}\right)^{-1}$ is a $(t+1)\times(t+1)$ matrix, for $i = 1, 2, \ldots, q^{t+1}$. Note that $C_1 = 0$ and $C_2 = I$.

The set $\mathcal{C}$ of matrices $\{C_i : i = 1, 2, \ldots, q^{t+1}\}$ forms a t-spread set, and in fact the t-spread that it defines (in the sense of Section 2.2) is exactly the t-spread $\mathcal{W}$ above.

To see this, first note that $\mathcal{C}$ has q^{t+1} elements and that it contains the matrices 0 and I. Second, choose two elements C_i and C_j of $\mathcal{C}$. Since the corresponding elements of the t-spread are skew, the corresponding points P_i, P_j are in clear position. Thus the matrix

$$\begin{pmatrix} C_i & C_j \\ I & I \end{pmatrix}$$

has rank $2(t+1)$ and so has non-zero determinant, and this implies that the matrix $(C_i - C_j)$ has non-zero determinant.

To show that $\mathcal{C}$ is a t-spread set for $\mathcal{W}$, we need to remember how the bijection f of Theorem 1.6.2 maps a point of the space $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$ to a t-dimensional subspace of $PG(2t+1, q)$. The point $P_i = (C_i, I)$ is mapped to that subspace S_i spanned by the columns of the matrix

$$\begin{pmatrix} C_i \\ I \end{pmatrix}.$$

The columns of the matrix have $2(t+1)$ entries and are points of the $2(t+1)$-dimensional vector space say $\mathcal{V}_{2t+2}$ which corresponds to the projective space $PG(2t+1, q)$, and since the matrix has rank $t+1$, the $t+1$ points corresponding to the columns are linearly independent and so span a $(t+1)$-dimensional subspace of $\mathcal{V}_{2t+2}$. We show that this space is $J(C_i)$, then the two t-spreads have exactly the same elements.

Let the subspace of $\mathcal{V}_{2t+2}$ corresponding to the point P_0 be

$$J(\infty) = \{(x_1, x_2, \ldots, x_{t+1}, 0, \ldots, 0)^T : x_i \in GF(q), \quad \text{not all zero}\}$$

and let the subspace corresponding to the point P_1 be

$$J(0) = \{(0, \ldots, 0, x_{t+2}, x_{t+3}, \ldots, x_{2t+2})^T : x_i \in GF(q), \ \text{not all zero}\}.$$

We now write $\mathcal{V}_{2t+2}$ as the direct sum $J(\infty) \oplus J(0)$. As a $(t+1)$-dimensional vector space, consider $J(\infty)$ to have the basis

$$\begin{aligned} a_1 &= (1, 0, \ldots, 0)^T, \\ a_2 &= (0, 1, 0, \ldots, 0)^T, \\ &\vdots \\ a_{t+1} &= (0, 0, \ldots, 0, 1)^T \end{aligned}$$

and as a $(t+1)$-dimensional vector space, consider $J(0)$ to have the basis

$$\begin{aligned} b_1 =&(1,0,\ldots,0)^T, \\ b_2 =&(0,1,0,\ldots,0)^T, \\ &\vdots \\ b_{t+1} =&(0,0,\ldots,0,1)^T. \end{aligned}$$

Let $'$ denote the non-singular linear transformation

$$\begin{aligned} ' : J(\infty) &\rightarrow J(0) \\ : a_i &\mapsto b_i, \end{aligned}$$

so that for example

$$\begin{aligned} J(I) &= \{a_i \oplus a_i' : a_i \in J(\infty)\} \\ &= \{(x_1, x_2, \ldots, x_{t+1}, x_1, x_2,, \ldots, x_{t+1})^T : x_i \in GF(q), \quad \text{not all zero}\}. \end{aligned}$$

Now the columns of the matrix

$$\begin{pmatrix} C_i \\ I \end{pmatrix}$$

are $c_1 \oplus b_1, c_2 \oplus b_2, \ldots, c_{t+1} \oplus b_{t+1}$ where $c_j \in J(\infty)$ is the jth column of C_i. As a_j is a $t \times 1$ column vector with 1 in the jth position, $c_j = C_i a_j$. Thus the space spanned by these columns is

$$\begin{aligned} &\text{lin}\,\{c_1 \oplus b_1, c_2 \oplus b_2, \ldots, c_{t+1} \oplus b_{t+1}\} \\ &\quad = \text{lin}\,\{C_i a_1 \oplus b_1, C_i a_2 \oplus b_2, \ldots, C_i a_{t+1} \oplus b_{t+1}\} \\ &\quad = \text{lin}\,\{C_i a_1 \oplus a_1', C_i a_2 \oplus a_2', \ldots, C_i a_{t+1} \oplus a_{t+1}'\} \end{aligned}$$

which is $J(C_i)$ (see Theorem 2.2.4). □

2.5.6 Theorem

Given a t-spread set we may construct a projective t-spread set and the t-spreads of $PG(2t+1, q)$ defined by each of these are isomorphic.

Proof: Let $\mathcal{C} = \{C_i : i = 1, 2, \ldots, q^{t+1}\}$ be a t-spread set of matrices, with $C_1 = 0$ and $C_2 = I$. Consider the set $\mathcal{P} = \{P_i : i = 0, 1, \ldots, q^{t+1}\}$ of points of $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$ defined as follows:

$$P_0 = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad \text{and} \quad P_i = \begin{pmatrix} C_i \\ I \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, q^{t+1}.$$

Since

$$\operatorname{rank} \begin{pmatrix} I & C_i \\ 0 & I \end{pmatrix} = 2(t+1)$$

and by the definition of t-spread set

$$\operatorname{rank} \begin{pmatrix} C_i & C_j \\ I & I \end{pmatrix} = 2(t+1),$$

any pair of points of $\mathcal{P}$ are in clear position and so map under the bijection f of Theorem 1.6.2 to $q^{t+1} + 1$ pairwise skew t-dimensional subspaces of $PG(2t+1, q)$. This is a t-spread $\mathcal{W}$ of $PG(2t+1, q)$, and the set

$$\mathcal{P} = \left\{(C_i, I) : i = 1, 2, \ldots, q^{t+1}\right\} \cup \{(I, 0)\}$$

is a (normalised) projective t-spread set. Theorems 2.5.4 and 2.5.5 show that the t-spreads defined by the projective t-spread set and by the t-spread set are isomorphic. □

2.6 COORDINATES FOR THE AFFINE PLANE Π

We use the notation of Section 2.2. Let $\mathcal{W}$ be a t-spread of $PG(2t+1, q)$, and embed the space $PG(2t+1, q)$ as a hyperplane in the projective space $PG(2t+2, q)$. As in Section 1.1 we represent $PG(2t+1, q)$ as a $(2t+2)$-dimensional vector space $\mathcal{V}_{2t+2}$ over the field $GF(q)$, embedded as a hyperplane in the $(2t+3)$-dimensional vector space $\mathcal{V}_{2t+3}$. Then $\mathcal{W}$ becomes a collection, still to be denoted by $\mathcal{W}$, of pairwise skew $(t+1)$-dimensional vector subspaces of $\mathcal{V}_{2t+2}$ over $GF(q)$ which satisfies the property that each non-zero vector of $\mathcal{V}_{2t+2}$ lies in exactly one element of $\mathcal{W}$.

Without loss of generality we may give a special role to some (arbitrarily chosen) ordered triple $J(\infty)$, $J(0)$ and $J(I)$ of distinct elements of the t-spread $\mathcal{W}$. Then we may assume that, in the notation of Section 2.2,

$$\mathcal{W} = \{J(\infty)\} \cup \{J(C_i) : C_i \in \mathcal{C}\}$$

where $\mathcal{C}$ is a t-spread set. Then $\mathcal{V}_{2t+2} = J(\infty) \oplus J(0)$ has a basis of $2t+2$ elements

$$\{a_1, a_2, \ldots, a_{t+1}, b_1, b_2, \ldots, b_{t+1}\}$$

and we need only add one single element e^* say of $\mathcal{V}_{2t+3}$ which is not in $\mathcal{V}_{2t+2}$ in order to obtain a basis for $\mathcal{V}_{2t+3}$.

Each point of the affine plane Π constructed in Section 2.3 is a 1-dimensional vector subspace of $\mathcal{V}_{2t+3}$ not contained in $\mathcal{V}_{2t+2}$ and so has a unique basis element of the form

$$x \oplus y' \oplus e^*$$

where $y' \in J(0)$ so that x and y are in $J(\infty)$. Thus we define the coordinates of that point of Π to be (x, y). Every ordered pair (x, y) of elements of $J(\infty)$ represents a unique point of Π corresponding to the subspace $x \oplus y' \oplus e^*$ of $\mathcal{V}_{2t+3}$. By definition, a line of Π is a $(t+2)$-dimensional subspace of $\mathcal{V}_{2t+3}$ meeting $\mathcal{V}_{2t+2}$ in an element J of $\mathcal{W}$, and so has the form

$$< J, (x, y) > = < J, x \oplus y' \oplus e^* >$$

provided (x, y) is one of its points, or correspondingly, the $(t+2)$-dimensional space contains the 1-dimensional space $x \oplus y' \oplus e^*$. These lines may be divided into two types:

(1) *Lines* $y = t$. If $s, t \in J(\infty)$ the point (x, y) of Π lies on the line $< J(\infty), (s, t) >$ if and only if

$$x \oplus y' \oplus e^* \in < J(\infty), (s, t) > = < \{a : \ a \in J(\infty)\}, s \oplus t' \oplus e^* >$$

which occurs if and only if $y' = t'$ thus if and only if $y = t$.

(2) *Lines* $(x - s) = C_i(y - t)$. If $s, t \in J(\infty)$ and $J(C_i) \in \mathcal{W}$, the point (x, y) lies on the line $< J(C_i), (s,t) >$ if and only if

$$x \oplus y' \oplus e^* \in < J(C_i), (s,t) > = < \{C_i a + a' : \ a \in J(\infty)\}, s \oplus t' \oplus e^* >$$

if and only if $y = a+t$ and $x = C_i a+s$, that is, if and only if $(x-s) = C_i(y-t)$.

We have specified all the points and all the lines of Π by coordinates and linear equations, respectively. We can actually introduce a coordinate ring $(\mathcal{R}, +, \cdot)$ by taking $\mathcal{R}$ to be $J(\infty)$ and defining addition in $\mathcal{R}$ to be the addition in $J(\infty)$ as a vector space. To specify multiplication in $\mathcal{R}$ we choose a non-zero element of $\mathcal{R}$ or, equivalently, we must pick a *unit point* of Π. We pick the point

$$I = (1,1) = 1 \oplus 1' \oplus e^*$$

where 1 is any fixed non-zero element of $\mathcal{R}$. To each $x \in \mathcal{R}$ there corresponds a unique matrix $X \in \mathcal{C}$ such that $x = X1$, by property (iii) immediately following Corollary 2.2.2. Then for $x, y \in \mathcal{R}$ we define

$$xy = (X1)y = Xy.$$

2.6.1 Theorem [Bruck and Bose (1966)]
The system $(\mathcal{R}, +, \cdot)$ is a coordinate ring for Π, and (to within isomorphism) every coordinate ring of the affine plane Π (though not of the corresponding projective plane) may be obtained in this manner.

Proof: See Bruck and Bose (1966), p158–159. □

2.6.2 Theorem [Bruck and Bose (1966)]
(1) The system $(\mathcal{R}, +, \cdot)$ is a division ring precisely when $\mathcal{C}$ is closed under addition, and then $(\mathcal{C}, +)$ is an abelian group isomorphic to $(\mathcal{R}, +)$.

(2) The system $(\mathcal{R}, +, \cdot)$ is a nearfield precisely when $\mathcal{C}$ is closed under multiplication, and then $(\mathcal{C} - \{0\}, \cdot)$ is a group isomorphic to $(\mathcal{R} - \{0\}, \cdot)$.

(3) $(\mathcal{R}, +, \cdot)$ is a field precisely when $(\mathcal{C}, +, \cdot)$ is a ring, and then $(\mathcal{C}, +, \cdot)$ is a field isomorphic to $(\mathcal{R}, +, \cdot)$.

Proof: See Bruck and Bose (1966), p159. □

2.6.3 Corollary [Bruck and Bose (1966)]

Let $\mathcal{W}$ be a t-spread of $PG(2t+1, q)$ with t-spread set $\mathcal{C}$.

(1) The t-spread $\mathcal{W}$ is Desarguesian, that is, the affine plane Π defined by $\mathcal{W}$ is Desarguesian, if and only if the system $(\mathcal{C}, +, \cdot)$ is a field isomorphic to $GF(q^{t+1})$.

(2) Further, $\mathcal{C}$ contains the set of matrices $\{kI : \ k \in GF(q)\}$ which is isomorphic to $GF(q)$.

Proof: (1) See Bruck and Bose (1966), Theorem 11.3, p161.

(2) See Bruck and Bose (1966), p163. □

2.7 REGULAR t-SPREADS OF $PG(2t+1, q)$

The affine plane, and hence also the projective plane, constructed from a t-spread of $PG(2t+1, q)$ are Desarguesian if and only if the system $(\mathcal{R}, +, \cdot)$ is a field, that is, if and only if the t-spread set $\mathcal{C}$ is a field. In this case, as mentioned in Remark (2) of 2.3.3, the t-spread $\mathcal{W}$ is acting like a projective Galois line, the line at infinity of the translation plane constructed from the t-spread. We see that the set of matrices $\mathcal{C}$ is a field of order q^{t+1} and is a non-homogeneous coordinatisation of this line. By Theorem 1.2.4 (2), if $\mathcal{W}$ is geometric then it is isomorphic to the projective line, and in fact the t-spread elements provide an affine coordinate system for this projective line. What we have done in this Chapter is to provide projective coordinates for such a line by using the projective t-spread set. Note that there is a slight difficulty when we try to use the projective t-spread set as projective coordinates for the line, for the pair (X, I) is projectively equivalent to the pair $(\rho X, \rho)$ where ρ may be any non-singular $(t+1) \times (t+1)$ matrix. We are no longer assured that the projective "coordinates" $(\rho X, \rho)$ are elements of the field $\mathcal{C}$. To get a projective coordinatisation for the projective line we just allow a pair (X, I) or $(I, 0)$ to be multiplied only by elements of $\mathcal{C}$.

2.8 t-SPREAD SETS AND INDICATOR SETS

Sherk (1979) used a t-spread set of a t-spread $\mathcal{W}$ of $PG(2t+1, q)$ to construct a set of q^{t+1} points in an affine space $AG(t+1, q^{t+1})$ called an indicator set for $\mathcal{W}$. Lunardon (1984) produced a geometric definition of indicator set, and showed that Sherk's indicator set can be regarded as an example of the geometric construction for a particular choice of indicator space. Lunardon's main criticism of the construction used in Sherk (1979) was that he needed to know a t-spread set for the t-spread, or equivalently, a quasifield coordinatising the translation plane constructed from the t-spread. In the light of Section 2.5, this criticism is no longer valid since given any t-spread, we just need to identify points spanning each of its elements and this immediately gives a t-spread set.

In this Section we re-examine the construction of an indicator set as in Sherk (1979). We modify the construction slightly in view of the ideas presented in this Chapter, giving in some sense a more natural definition of indicator set without losing the spirit of the theory as introduced by Bruen (1972a), Sherk (1979) and Lunardon (1984). The modification involves using a projective t-spread set to define the indicator set, plus another minor alteration. The main advantage of the modification is that the new construction can be generalised in a natural way to t-spreads of $PG((s+1)(t+1)-1,q)$.

This new definition is more natural in another sense because it gives the indicator set in a geometric setting similar to that in Bruen (1972a) and Lunardon (1984), where the indicator set is found by intersecting the t-spread elements with a certain affine space. Thus the algebraic and the geometric approaches are unified.

2.8.1 Construction of an Indicator set [Sherk (1979)]

Let $\mathcal{W}$ be a t-spread of $PG(2t+1,q)$ and let $\mathcal{C} = \{C_i : i = 1,2,\ldots,q^{t+1}\}$ be a t-spread set for $\mathcal{W}$, where

$$C_i = \begin{pmatrix} c_{11}^{(i)} & c_{12}^{(i)} & \cdots & c_{1t+1}^{(i)} \\ c_{21}^{(i)} & c_{22}^{(i)} & \cdots & c_{2t+1}^{(i)} \\ \vdots & & \vdots & \\ c_{t+11}^{(i)} & c_{t+12}^{(i)} & \cdots & c_{t+1t+1}^{(i)} \end{pmatrix}$$

and the t-spread is $\mathcal{W} = \{J(C_i) : i = 1,2,\ldots,q^{t+1}\} \cup \{J(\infty)\}$ as before. Let $GF(q^{t+1})$ be a field extension of $GF(q)$ and let $GF(q^{t+1}) = GF(q)(\alpha)$. Define a set $\mathcal{I}$ of points $\mathcal{I} = \{P_i : i = 1,2,\ldots,q^{t+1}\}$ of a $(t+1)$-dimensional affine space $AG(t+1,q^{t+1})$ as follows:

$$\begin{aligned} P_i &= (1,\alpha,\alpha^2,\cdots,\alpha^t)C_i \\ &= \Big(c_{11}^{(i)} + c_{21}^{(i)}\alpha + \cdots + c_{t+11}^{(i)}\alpha^t, \\ &\qquad c_{12}^{(i)} + c_{22}^{(i)}\alpha + \cdots + c_{t+12}^{(i)}\alpha^t, \\ &\qquad\qquad \ddots \\ &\qquad\qquad\qquad c_{1t+1}^{(i)} + c_{2t+1}^{(i)}\alpha + \cdots + c_{t+1t+1}^{(i)}\alpha^t\Big). \end{aligned}$$

Then $\mathcal{I}$ is called the *indicator set* of the t-spread $\mathcal{W}$ and $AG(t+1,q^{t+1})$ is called the *indicator space*. The subspace $J(\infty)$ is not represented by a point of the indicator space, and this difficulty is overcome by adjoining to the indicator space a single ideal point, denoted by the symbol ∞, with the property that it lies on every line of $AG(t+1, q^{t+1})$. The set $\mathcal{I}^* = \mathcal{I} \cup \{\infty\}$ is called the *augmented indicator set* of the t-spread $\mathcal{W}$.

The direction numbers of the line joining the points $P_i = (p_1^{(i)}, p_2^{(i)}, \ldots, p_{t+1}^{(i)})$ and $P_j = (p_1^{(j)}, p_2^{(j)}, \ldots, p_{t+1}^{(j)})$ are the $t+1$ elements

$$(p_1^{(j)} - p_1^{(i)}),\ (p_2^{(j)} - p_2^{(i)}), \ldots, (p_{t+1}^{(j)} - p_{t+1}^{(i)}).$$

The indicator set $\mathcal{I}$ has the characteristic property that the line joining any two of its points P_i and P_j has direction numbers which are linearly independent over $GF(q)$. To see this, note that the line joining P_i to P_j has direction numbers $(AV_1, AV_2, \ldots, AV_{t+1})$ where V_k is the kth column of the matrix $C_j - C_i$ and A is the row vector $(1, \alpha, \alpha^2, \ldots, \alpha^t)$. These numbers are linearly independent over $GF(q)$ if and only if the vectors $V_1, \ldots, V_{t+1}$ are linearly independent over $GF(q)$, which occurs if and only if the matrix $C_j - C_i$ is non-singular. This follows since C_i and C_j are elements of the t-spread set of $\mathcal{W}$ (see Definition 2.2.3).

2.8.2 Definition of the indicator set from the projective t-spread set

In order to generalise the construction of an indicator set to indicator set of a t-spread in $PG((s+1)(t+1)-1, q)$, we prefer to use the projective t-spread set to define the indicator set. This enables us to give an indicator set for a wider class of t-spreads than just those possessing a t-spread set. The first step is to use the projective t-spread set of a t-spread in $PG(2t+1, q)$ to define an indicator set.

We are going to modify Sherk's definition of indicator set, and the reason for this modification is illustrated with the following example.

2.8.3 Example The case $t = 1$..

Let $\mathcal{W}$ be a 1-spread of $PG(3, q)$ and let $\mathcal{PC}$ be a projective 1-spread set for $\mathcal{W}$.

Then $\mathcal{PC} = \{(I,0)\} \cup \{(C_i, I) : i = 1,2,\ldots,q^2\}$ as in Theorem 2.5.6, where the matrices $\{C_i\}$ form a 1-spread set. Now each matrix C_i is a 2×2 matrix over $GF(q)$, say

$$C_i = \begin{pmatrix} c_{11}^{(i)} & c_{12}^{(i)} \\ c_{21}^{(i)} & c_{22}^{(i)} \end{pmatrix}.$$

Let $GF(q^2)$ be a field extension of $GF(q)$, and let $GF(q^2) = GF(q)(\alpha)$. The points of the indicator set in an affine space $AG(2,q^2)$ in the definition of Sherk given above are

$$\begin{aligned} P_i &= (1,\alpha)C_i \\ &= \left(c_{11}^{(i)} + c_{21}^{(i)}\alpha, \quad c_{12}^{(i)} + c_{22}^{(i)}\alpha \right) \end{aligned}$$

satisfying the condition that the direction numbers of the line joining any two points of the indicator set in $AG(2,q^2)$ are linearly independent, or in other words that the determinant of every matrix $C_j - C_i$ with $j \neq i$, is non-zero.

Now we wish to use the projective 1-spread set to define the indicator set. It would seem natural to use the following definition: The indicator set of the 1-spread $\mathcal{W}$ is the set of points

$$\begin{aligned} Q_i &= (1,\alpha)\,(\,C_i \quad I\,) \\ &= \left(c_{11}^{(i)} + c_{21}^{(i)}\alpha, \quad c_{12}^{(i)} + c_{22}^{(i)}\alpha, \quad 1, \quad \alpha \right). \end{aligned}$$

This is a set of q^2 points lying on the space spanned by the vectors

$$e_1, e_2, \; e_3 + \alpha e_4$$

which is a 2-dimensional projective subspace $PG^*(2,q^2)$ of $PG(3,q^2)$ which meets $PG(3,q)$ in the 1-dimensional subspace spanned by e_1 and e_2. This coincides with the definition of indicator space given by Bruen (1972a), but not the definition of indicator set given there since a point of this new indicator set need not automatically lie on the 1-spread element that it indicates. In the notation of Sherk (1979) the indicator space is the affine space $PG^*(2,q^2) - PG(3,q)$ and the points of the indicator set are exactly the points with affine coordinates $(c_{11}^{(i)}+c_{21}^{(i)}\alpha, c_{12}^{(i)}+c_{22}^{(i)}\alpha)$. What we have done is bring the algebraic construction due to Sherk (1979) closer to the natural geometric setting used by Bruen (1972a).

Now as we have seen in Section 2.5, if we consider the element (C_i, I) of the projective indicator set as a point of the space $\mathcal{S}_1(\mathcal{M}_2(GF(q)))$, then it corresponds under the bijection f of Theorem 1.6.2 to a unique line l_i of the space $PG(3,q)$. This line is spanned by the columns of the matrix

$$\begin{pmatrix} C_i \\ I \end{pmatrix}$$

considered as points of $PG(3,q)$, and so can be written as

$$\begin{aligned} l_i &= \left\{ \begin{pmatrix} C_i \\ I \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} : \ \lambda \in GF(q) \cup \{\infty\} \right\} \\ &= \left\{ \left(c_{11}^{(i)} + c_{12}^{(i)}\lambda, c_{21}^{(i)} + c_{22}^{(i)}\lambda, 1, \lambda \right)^T : \ \lambda \in GF(q) \cup \{\infty\} \right\}. \end{aligned}$$

This consideration suggests the following definition of indicator set $\mathcal{I}'$ of a 1-spread of $PG(3,q)$. Let $\mathcal{I}' = \{Q_i : i = 1, 2, \ldots, q^2\}$ where

$$\begin{aligned} Q_i &= \begin{pmatrix} C_i \\ I \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \\ &= \left(c_{11}^{(i)} + c_{12}^{(i)}\alpha, \quad c_{21}^{(i)} + c_{22}^{(i)}\alpha, \quad 1, \quad \alpha \right)^T. \end{aligned}$$

The set $\mathcal{I}'$ is different from the indicator set $\mathcal{I}$ given by Sherk (1979), but it does satisfy the characteristic property that if Q_i and Q_j are points of $\mathcal{I}'$ then the non-zero direction numbers of the line Q_iQ_j are linearly independent over $GF(q)$. This follows since the direction numbers are actually

$$\left(c_{11}^{(i)} - c_{11}^{(j)}, \quad c_{12}^{(i)} - c_{12}^{(j)} \right) \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \text{ and } \left(c_{21}^{(i)} - c_{21}^{(j)}, \quad c_{22}^{(i)} - c_{22}^{(j)} \right) \begin{pmatrix} 1 \\ \alpha \end{pmatrix}.$$

These are linearly independent over $GF(q)$ because the rows of the matrix $C_i - C_j$ are linearly independent over $GF(q)$.

It is interesting to note that in fact $\mathcal{I}'$ is the indicator set under the construction due to Sherk (1979) of the 1-spread with projective 1-spread set

$$\{(0, I)\} \cup \{(C_i^T, I) : i = 1, 2, \ldots, q^2\}.$$

The points Q_i of $\mathcal{I}'$ all lie in the same indicator space, but now we have the added property that the point Q_i of the indicator set which indicates the line l_i of the 1-spread actually lies on the extension of that line. This is the description of indicator set exactly as in Bruen (1972a).

It is natural to augment the indicator set with the point

$$\begin{aligned} Q_\infty &= \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \\ &= (1, \alpha, 0, 0)^T. \end{aligned}$$

This is no longer an ideal point of the affine indicator space as in Sherk (1979), but a specific point of the line at infinity spanned by the elements e_1 and e_2, giving a more satisfactory augmented indicator set.

We now present the new construction of an indicator set in the general case of a t-spread in $PG(2t+1, q)$, and show connections with the work of Sherk (1979) and Lunardon (1984).

2.8.4 The construction

We return to the notation of Section 2.2, and suppose that A, B, C is an ordered triple of distinct $(t+1)$-dimensional vector subspaces of $\mathcal{V}_{2t+2}$, pairwise having only the zero vector in common. Write $\mathcal{V}_{2t+2} = A \oplus B$ and let $'$ denote the unique non-singular linear transformation of A to B such that the linear transformation defined by $a \mapsto a \oplus a'$ maps A onto C. Then given any linear transformation C_i of A to A over $GF(q)$ there corresponds a unique $(t+1)$-dimensional vector subspace

$J(C_i)$ of $\mathcal{V}_{2t+2}$ where

$$J(C_i) = \{C_i a \oplus a' : a \in A\}.$$

Conversely any $(t+1)$-dimensional vector subspace of $\mathcal{V}_{2t+2}$ having only the zero vector in common with A may be written in this form. We know that $J(C_i)$ and $J(C_j)$ have only the zero vector in common if and only if the matrix $C_j - C_i$ has non-zero determinant.

Let $GF(q^{t+1})$ be a field extension of $GF(q)$ and let $GF(q^{t+1}) = GF(q)(\alpha)$. Let $PG(2t+1, q^{t+1})$ denote the corresponding extension of $PG(2t+1, q)$. If A is a subspace of $PG(2t+1, q)$, we will denote its corresponding extension to $PG(2t+1, q^{t+1})$ by $\overline{A}$. To each subspace $J(C_i)$ we associate a point

$$\begin{aligned} Q_i &= \begin{pmatrix} C_i \\ I \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix} \\ &= (q_1^{(i)}, q_2^{(i)}, \ldots, q_{2t+2}^{(i)})^T \end{aligned}$$

of $PG(2t+1, q^{t+1}) - PG(2t+1, q)$. Now each point Q_i is

$$Q_i = V_1 + V_2\alpha + \ldots + V_{t+1}\alpha^t$$

where V_i is the ith column of the matrix

$$\begin{pmatrix} C_i \\ I \end{pmatrix}.$$

Since the columns $V_1, V_2, \ldots, V_{t+1}$ are linearly independent, by Corollary 1.4.10 Q_i is imaginary.

Let σ denote the automorphism $\sigma : x \mapsto x^q$ of $GF(q^{t+1})$, an also the automorphic collineation of $PG(2t+1, q)$ induced by σ.

2.8.5 Lemma

(1) $\overline{J(C_i)} = \mathrm{lin}\left\{Q_i, Q_i^{\sigma}, \ldots, Q_i^{\sigma^t}\right\}.$

(2) Two distinct spaces $J(C_i)$ and $J(C_j)$ have only the zero vector in common if and only if the line joining the points Q_i and Q_j is imaginary.

Proof: (1) Now $\overline{J(C_i)}$ is a t-dimensional subspace of $PG(2t+1, q^{t+1})$ meeting $PG(2t+1, q)$ in a t-dimensional subspace $J(C_i)$. The imaginary point Q_i lies in $\overline{J(C_i)}$, and by Corollary 1.4.10 (2) $L(Q_i)$ is the unique such subspace. Thus $\overline{J(C_i)} = L(Q_i)$.

(2) Now $J(C_i)$ and $J(C_j)$ are skew if and only if $\overline{J(C_i)}$ and $\overline{J(C_j)}$ are skew. This occurs if and only if the points $Q_i, Q_i^{\sigma}, \ldots, Q_i^{\sigma^t} Q_j, Q_j^{\sigma}, \ldots, Q_j^{\sigma^t}$ span a space of dimension $2t+1$, and this occurs if and only if the lines $Q_iQ_j, (Q_iQ_j)^{\sigma}, \ldots, (Q_iQ_j)^{\sigma^t}$ span a space of dimension $2t+1$, that is, if and only if the line Q_iQ_j is imaginary (see Definition 1.4.7 (2)). $\square$

2.8.6 Remark

The points Q_i all lie in the projective subspace $PG^*(t+1, q^{t+1})$ of $PG(2t+1, q^{t+1})$ which is spanned by the vectors

$$e_1, e_2, \ldots, e_{t+1}, e_{t+2} + \alpha e_{t+3} + \cdots + \alpha^t e_{2t+2}$$

and this space meets the space $PG(2t+1, q)$ in the t-dimensional subspace spanned by the vectors $e_1, e_2, \ldots, e_{t+1}$. This is the subspace $J(\infty) = J(I, 0)$ of $PG(2t+1, q)$. In fact each point Q_i lies in the affine space

$$AG^*(t+1, q^{t+1}) = PG^*(t+1, q^{t+1}) - \overline{J(\infty)}.$$

Now let $\mathcal{W}$ be a t-spread of $PG(2t+1, q)$ containing the subspace $J(\infty)$. Suppose it has a projective t-spread set

$$\mathcal{PC} = \{(I, 0)\} \cup \{(C_i, I) : i = 1, 2, \ldots, q^{t+1}\}$$

as in Theorem 2.5.6. We will denote by $J(C_i)$ either the t-spread element corresponding to the element (C_i, I) itself or the corresponding $(t+1)$-dimensional subspace of $\mathcal{V}_{2t+2}$. Let the set of points of $AG^*(t+1, q^{t+1})$ corresponding to these t-spread elements be

$$\mathcal{I} = \{Q_i : i = 1, 2, \ldots, q^{t+1}\} \quad \text{where } Q_i = \begin{pmatrix} C_i \\ I \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix}.$$

2.8.7 Theorem

In the definition of Sherk (1979), $AG^*(t+1, q^{t+1})$ is an indicator t-space for the set of t-dimensional subspaces of $PG(2t+1, q)$ skew to $J(\infty)$. Further, $\mathcal{I}$ is an indicator set for the t-spread $\mathcal{W}'$ where $\mathcal{W}'$ has projective t-spread set

$$\mathcal{PC}' = \{(I, 0\} \cup \{(C_i^T, I) : \ i = 1, 2, \ldots, q^{t+1}\}.$$

Proof: The construction of an indicator t-space appeared in Sherk (1979), p212–213. He showed that there is a one to one correspondence between the points of $AG^*(t+1, q^{t+1})$ and the t-dimensional subspaces of $PG(2t+1, q)$ skew to $J(\infty)$, as described in 2.8.4. Since

$$Q_i = \begin{pmatrix} C_i \\ I \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix} = ((1, \alpha, \cdots, \alpha^t)(I \quad C_i^T))^T$$

the direction numbers of the line joining any two points of $\mathcal{I}$ are linearly independent over $GF(q)$. Thus the set of points $\mathcal{I}$ in $AG^*(t+1, q^{t+1})$ is an indicator set for the t-spread $\mathcal{W}'$, see Sherk (1979), p213–215. □

2.8.8 Theorem

The space $PG^*(t+1, q^{t+1})$ is an indicator $(t+1)$-space and the set $\mathcal{I}$ is an indicator set on $PG^*(t+1, q^{t+1})$, in the sense of Lunardon (1984).

Proof: The space

$$PG^*(t+1, q^{t+1}) = \text{lin } \{e_1, e_2, \ldots, e_{t+1}, e_{t+2} + \alpha e_{t+3} + \cdots + \alpha^t e_{2t+2}\}$$

is a $(t+1)$-dimensional subspace of $PG(2t+1, q^{t+1})$ which meets $PG(2t+1, q)$ in a space of dimension t, namely the space S_t spanned by $e_1, e_2, \ldots, e_{t+1}$. Further, $PG^*(t+1, q^{t+1})$ contains the imaginary point

$$P = e_{t+2} + \alpha e_{t+3} + \cdots + \alpha^t e_{2t+2}$$

which is not contained in $\overline{S_t}$. The conjugates $P, P^\sigma, \ldots, P^{\sigma^t}$ of P span a t-dimensional space which is skew to $\overline{S_t}$ since $S_t = \text{lin}\{e_1, e_2, \ldots, e_{t+1}\}$. Thus we have shown that $PG^*(t+1, q^{t+1})$ is an indicator $(t+1)$-space under Definition 3 of Lunardon (1984), p721. The set $\mathcal{I}$ of points Q_i comprises q^{t+1} points of $PG^*(t+1, q^{t+1}) - \overline{S_t}$ and it is an indicator set in the indicator $(t+1)$-space $PG^*(t+1, q^{t+1})$ if it satisfies the property (c) of Definition 4, p721 of Lunardon (1984), that the line joining any two of the points meets $\overline{S_t}$ in an imaginary point. Let Q_i and Q_j be two points of $\mathcal{I}$. These are imaginary points (see Construction 2.8.4) and since the line Q_iQ_j is imaginary by Lemma 2.8.5 (2), all its points must be imaginary by Theorem 1.4.8 (3). In particular the point of intersection of Q_iQ_j with $\overline{S_t}$ is imaginary. □

2.8.9 Remark

The element $J(\infty) = J(0, I)$ of the t-spread $\mathcal{W}$ has no point of the indicator set associated to it. Sherk (1979) adjoins an ideal point ∞ to the affine indicator space $AG^*(t+1, q^{t+1})$ and lets this point represent $J(\infty)$. Under our new construction of indicator sets, however, it seems natural to let the point

$$Q_\infty = \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix}$$

represent the t-spread element $J(\infty)$. Then Q_∞ lies on the hyperplane at infinity of $AG^*(t+1, q^{t+1})$ and so is still in some sense an ideal point of the affine space, but it is a particular point of the corresponding projective space. Further, Q_∞ has the added advantage that it is imaginary and $J(\infty) = L(Q_\infty) \cap PG(2t+1, q)$, a property that holds for every other point Q_i of the indicator set $\mathcal{I}$.

2.8.10 Theorem

The set of points $\mathcal{PI} = \{Q_\infty, Q_1, Q_2, \ldots, Q_{q^{t+1}}\}$ constructed as above is a projective indicator set in the sense of Lunardon (1984).

Proof: We check that $\mathcal{PI}$ satisfies the properties listed in Definition 2, p720 of Lunardon (1984), with t replaced by $t+1$. Firstly, $\mathcal{PI}$ comprises $q^{t+1}+1$ imaginary points of $PG(2t+1, q^{t+1})$, and by construction the line joining any two of the points is imaginary (Lemma 2.8.5 (2)). □

2.8.11 Theorem [Lunardon (1984)]

If $\mathcal{PI}$ is a projective indicator set of $PG(2t+1, q^{t+1})$, then the set

$$\mathcal{W} = \{L(Q_i) \cap PG(2t+1, q) : \quad Q_i \in \mathcal{PI}\}$$

is a t-spread of $PG(2t+1, q)$.

Proof: The proof appears in Lunardon (1984), Lemma 2, p720. □

It is also noted in Lunardon (1984) that a t-spread of $PG(2t+1, q)$ may have many projective indicator sets. It is enough to choose an imaginary point in the extension to $PG(2t+1, q^{t+1})$ of each element of the t-spread. In this Chapter, we have used the normalised projective t-spread set to construct a projective indicator set which contains an indicator set in the indicator space $AG^*(t+1, q^{t+1})$. We shall show that conversely an indicator set gives rise naturally to a normalised projective t-spread set (Theorem 2.8.13), but that in general a projective indicator not containing an indicator set corresponds to a projective t-spread set which is not normalised (see Theorem 2.8.12).

Let $\mathcal{PI} = \{Q_\infty, Q_1, Q_2, \ldots, Q_{q^{t+1}}\}$ be a projective indicator set in the space $PG(2t+1, q^{t+1})$, corresponding to the t-spread

$$\mathcal{W} = \{L(Q_i) : \; i = \infty, 1, 2, \ldots, q^{t+1}\}.$$

Let $\alpha \in GF(q^{t+1})$ be such that $GF(q^{t+1}) = GF(q)(\alpha)$. Each point Q_i may be written as

$$Q_i = \begin{pmatrix} \zeta_0^{(i)} \\ \zeta_1^{(i)} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix}$$

where for $k = 1, 2$, $\zeta_k^{(i)}$ is a $(t+1) \times (t+1)$ matrix over $GF(q)$.

2.8.12 Theorem

The set

$$\mathcal{PC} = \{(\zeta_0^{(i)}, \zeta_1^{(i)}) : \; i = \infty, 1, 2, \ldots, q^{t+1}\}$$

is a projective t-spread set, corresponding as in Section 2.5 to the t-spread $\mathcal{W}$. It is not in general normalised. Conversely, any projective t-spread set for $\mathcal{W}$ gives rise to a projective indicator set for $\mathcal{W}$.

Proof: We check that $\mathcal{PC}$ satisfies the conditions of Definition 2.5.2, then it is a projective t-spread set. Firstly, it comprises $q^{t+1} + 1$ pairs of $(t+1) \times (t+1)$ matrices. Since each point $Q_i = (\zeta_0^{(i)}, \zeta_1^{(i)})$ is imaginary, by Corollary 1.4.10 (1)

$$\operatorname{rank} \begin{pmatrix} \zeta_0^{(i)} \\ \zeta_1^{(i)} \end{pmatrix} = t + 1.$$

Two elements $Q_i = (\zeta_0^{(i)}, \zeta_1^{(i)})$ and $Q_j = (\zeta_0^{(j)}, \zeta_1^{(j)})$ can be considered as points of the space $\mathcal{S}_1(\mathcal{M}_{t+1}(GF(q)))$ with the corresponding t-dimensional subspaces of $PG(2t+1,q)$ being $f(Q_i) = L(Q_i)$ and $f(Q_j) = L(Q_j)$. The line Q_iQ_j is imaginary so the spaces $L(Q_i)$ and $L(Q_j)$ are skew, so that the points Q_i and Q_j are in clear position (see Section 1.6) and therefore

$$\operatorname{rank}\begin{pmatrix} \zeta_0^{(i)} & \zeta_0^{(j)} \\ \zeta_1^{(i)} & \zeta_1^{(j)} \end{pmatrix} = 2(t+1).$$

To show that $\mathcal{PC}$ is a projective t-spread set for the t-spread $\mathcal{W}$, for each $Q_i \in \mathcal{PC}$, we need to show that the subspace spanned by the columns of

$$\begin{pmatrix} \zeta_0^{(i)} \\ \zeta_1^{(i)} \end{pmatrix}$$

is precisely the space $L(Q_i) \cap PG(2t+1,q)$. This follows since the space spanned by the columns of

$$\begin{pmatrix} \zeta_0^{(i)} \\ \zeta_1^{(i)} \end{pmatrix}$$

is a t-dimensional subspace of $PG(2t+1,q)$ whose extension to $PG(2t+1,q^{t+1})$ contains the point Q_i. However by Corollary 1.4.10 (2), $L(Q_i)$ is the unique such space. The converse follows by reversing the arguments. □

2.8.13 Theorem

Let $\mathcal{PI} = \{Q_\infty, Q_1, Q_2, \ldots, Q_{q^{t+1}}\}$ be a projective indicator set in $PG(2t+1,q^{t+1})$. Suppose the points $\{Q_1, Q_2, \ldots, Q_{q^{t+1}}\}$ of $\mathcal{PI}$ are an indicator set, and so all lie in a $(t+1)$-dimensional affine subspace $AG^*(t+1,q^{t+1})$ of $PG(2t+1,q^{t+1})$. Suppose further that the projective space $PG^*(t+1,q^{t+1})$ obtained by completing $AG^*(t+1,q^{t+1})$ meets $PG(2t+1,q)$ in a t-dimensional space $PG(t,q)$, such that $Q_\infty \in \overline{PG(t,q)}$. Let $\alpha \in GF(q^{t+1})$ be such that $GF(q^{t+1}) = GF(q)(\alpha)$.Then in a certain coordinatisation of $PG(2t+1,q^{t+1})$, the t-spread $\mathcal{W}$ corresponding to $\mathcal{PI}$ has a projective t-spread set of the form

$$\mathcal{PC} = \{(I,0)\} \cup \{(C_i, I) : \; i = 1,2,\ldots,q^{t+1}\}$$

where

$$Q_\infty = \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix}, \quad \text{and } Q_i = \begin{pmatrix} C_i \\ I \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix}.$$

The converse is also true.

Proof: The converse of this result was demonstrated in Construction 2.8.4. For the forward argument, choose coordinates for $PG(2t+1, q^{t+1})$ so that

$$\overline{PG(t,q)} = \text{lin}\{e_1, e_2, \ldots, e_{t+1}\}$$

$$Q_\infty = e_1 + \alpha e_2 + \cdots + \alpha^t e_{t+1}$$

$$PG^*(t+1, q^{t+1}) = \text{lin}\{e_1, e_2, \ldots, e_{t+1}, e_{t+2} + \alpha e_{t+3} + \cdots + \alpha^t e_{2t+2}\}.$$

Then

$$PG^*(t+1, q^{t+1}) \cap PG(2t+1, q) = PG(t,q), \quad \text{and}$$

$$AG^*(t+1, q^{t+1}) = PG^*(t+1, q^{t+1}) - \overline{PG(t,q)}.$$

Now

$$\begin{aligned} Q_\infty &= (1, \alpha, \ldots, \alpha^t, 0, 0, \ldots, 0)^T \\ &= \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix} \end{aligned}$$

and

$$Q_i = (c_1^{(i)}, c_2^{(i)}, \ldots, c_{t+1}^{(i)}, 1, \alpha, \ldots, \alpha^t)^T$$

where for $k = 1, 2, \ldots, q^{t+1}$ we have $c_k^{(i)} = c_{k1}^{(i)} + c_{k2}^{(i)}\alpha + \cdots + c_{kt+1}^{(i)}\alpha^t$.

If we denote by C_i the $(t+1) \times (t+1)$ matrix whose elements are the $c_{kj}^{(i)}$, then

$$Q_i = \begin{pmatrix} C_i \\ I \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix}.$$

The t-spread of $PG(2t+1,q)$ corresponding to $\mathcal{PI}$ is

$$\mathcal{W} = \{L(Q_i) \cap PG(2t+1,q) : \ i = \infty, 1, 2, \ldots, q^{t+1}\}$$

and by Theorem 2.8.12, this is also the t-spread corresponding to the projective t-spread set $\mathcal{PC}$. □

These last two theorems give a geometric interpretation of the procedure of normalising a projective t-spread set. A projective t-spread set corresponds to a projective indicator set for a t-spread $\mathcal{W}$. Then $\mathcal{W}$ has another projective indicator set, but one that contains an indicator set. This is obtained geometrically by intersecting the elements of $\mathcal{W}$ with an indicator space. The projective t-spread set corresponding to this new indicator set is in the normalised form.

2.9 t-SPREADS OF $PG(2t+1,q)$ IN $PG(2t+1,q^{t+1})$

In the study of sets of t-dimensional subspaces of a space $PG(2t+1,q)$, it is often useful to know whether such a set is contained in a regular t-spread. There is a representation of regular t-spreads which can often be applied to answer this question. It was introduced by Bruck (1969) for regular 1-spreads of $PG(3,q)$, but it can easily be extended to regular t-spreads of $PG(2t+1,q)$ as we shall show below. The representation has the advantage that it yields easily many of the properties of the t-reguli in a regular t-spread.

In the following we shall use the notation and the ideas of Section 1.3. Theorem 2.9.2 is a special case of Theorem 2.9.3. The theory developed in this Chapter allow a simple proof of Theorem 2.9.3, and therefore also of 2.9.2. We shall, however, give an indication of the original proof of the first part of 2.9.2 due to Bruck (1969).

2.9.1 Lemma [Bruck (1969)]
Let $\mathcal{R}$ be a regulus of $PG(3,q)$ and let l be a line not meeting the quadric $\mathcal{Q}$ defined by $\mathcal{R}$. There exists exactly one regular 1-spread $\mathcal{W}$ of $PG(3,q)$ containing both $\mathcal{R}$ and l.

Proof: This is proved using the 1-spread set. □

2.9.2 Theorem [Bruck (1969)]
Let $\mathcal{W}$ be a regular 1-spread of $PG(3,q)$. There exists a line l of $PG(3,q^2)$ skew to $PG(3,q)$ such that l^σ is also skew to $PG(3,q)$ and

$$\mathcal{W} = \{PP^\sigma \cap PG(3,q) : \, P \in l\}.$$

The lines l, l^σ are imaginary and are uniquely determined by the 1-spread $\mathcal{W}$. Conversely any line l of $PG(3,q^2)$ skew to $PG(3,q)$ yields a regular 1-spread of $PG(3,q)$ in this manner.

Proof: This result appears as Theorem 5.3 (i) in Bruck (1969). The second statement is proved in an analogous way to the corresponding statement in Theorem 2.9.3. To prove the first result, we note that if $\mathcal{R}$ is a regulus in $\mathcal{W}$ then it is one set of lines of a quadric surface $\mathcal{Q}$ in $PG(3,q)$ with extension $\overline{\mathcal{Q}}$ to $PG(3,q^2)$. A line l of $\mathcal{W}$ not in $\mathcal{R}$ meets $\overline{\mathcal{Q}}$ in two distinct points P and P^σ which do not lie in $PG(3,q)$. The two distinct lines m and m^σ of the opposite system of lines of $\overline{\mathcal{Q}}$ through P and P^σ respectively define a regular 1-spread of $PG(3,q)$ containing $\mathcal{R}$ and l. By Lemma 2.9.1, this regular 1-spread is $\mathcal{W}$. □

We now give the proof of this result generalised to t-spreads of $PG(2t+1,q)$. The proof relies on the projective t-spread set.

2.9.3 Theorem

(1) Let l be an imaginary line of $PG(2t+1, q^{t+1})$. The set

$$\mathcal{W} = \left\{ \text{lin } \left\{ P, P^{\sigma}, \ldots, P^{\sigma^t} \right\} \cap PG(2t+1, q) : \ P \in l \right\}$$

is a regular t-spread of $PG(2t+1, q)$ meeting each of $l, l^{\sigma}, \ldots, l^{\sigma^t}$.

(2) Conversely a regular t-spread of $PG(2t+1, q)$ can be represented in this manner for a unique set of lines $(l, l^{\sigma}, \ldots, l^{\sigma^t})$.

Proof: (1) Let l be an imaginary line of $PG(2t+1, q^{t+1})$ and let P and Q be distinct points of l. By Theorem 1.4.8 (3), P and Q are imaginary and by Definition 1.4.7 (1) each of the spaces

$$L(P) = \text{lin } \left\{ P, P^{\sigma}, \ldots, P^{\sigma^t} \right\}$$

and

$$L(Q) = \text{lin } \left\{ Q, Q^{\sigma}, \ldots, Q^{\sigma^t} \right\}$$

has dimension t in $PG(2t+1, q^{t+1})$ and so by Theorem 1.4.8 (1) meets $PG(2t+1, q)$ in a t-dimensional subspace. By Lemma 2.8.5, $L(P)$ and $L(Q)$ are skew. Thus $\mathcal{W}$ is a set of $q^{t+1}+1$ pairwise skew t-dimensional subspaces of $PG(2t+1, q)$ and so is a t-spread. We must show that it is regular, according to the Definition 2.4.3. Let A, B and C be three distinct elements of $\mathcal{W}$, let $\mathcal{R}$ be the regulus that they define in $PG(2t+1, q)$ and let $\mathcal{R}^*$ be the regulus that their extensions $\overline{A}$, $\overline{B}$ and $\overline{C}$ define in $PG(2t+1, q^{t+1})$. Now l meets each of the three distinct elements $\overline{A}$, $\overline{B}$ and $\overline{C}$ of $\mathcal{R}^*$ and by the remarks immediately following Definition 2.4.1 l is a transversal of $\mathcal{R}^*$. An element of $\mathcal{R}$, extended to $PG(2t+1, q^{t+1})$, is an element of $\mathcal{R}^*$ and therefore meets l in a unique point. So the extension of every element of $\mathcal{R}$ meets l, and by Corollary 1.4.10 (2) this must be the unique real t-dimensional space meeting l in a given point, and so is in $\mathcal{W}$.

(2) Now let $\mathcal{W}$ be a regular t-spread of $PG(2t+1, q)$. To show that $\mathcal{W}$ can be represented in this manner, we show that there exists an imaginary line l which meets the extension to $PG(2t+1, q^{t+1})$ of every element of $\mathcal{W}$. In this case,

$$\mathcal{W} = \left\{\text{lin } \left\{P, P^{\sigma}, \dots, P^{\sigma^t}\right\} \cap PG(2t+1, q) : \; P \in l\right\}$$

and the uniqueness of the set of lines $l, l^{\sigma}, \dots, l^{\sigma^t}$ follows by Definition 1.4.7 (2). We will prove the result for a particular extension $PG(2t+1, q^{t+1})$ of $PG(2t+1, q)$, and the result follows since all extensions of the same degree are isomorphic. As in Section 2.5, $\mathcal{W}$ gives rise to a normalised projective t-spread set

$$\mathcal{PC} = \{(C_i, I) : \; C_i \in \mathcal{C}\} \cup \{(I, 0)\},$$

where the set $\mathcal{C}$ of matrices is a field of order q^{t+1} under addition and multiplication. Let $\xi \in GF(q^{t+1})$ be such that $GF(q^{t+1}) = GF(q)(\xi)$, then

$$GF(q^{t+1}) = \left\{x_0 + x_1\xi + x_2\xi^2 + \cdots + x_t\xi^t : \; x_i \in GF(q)\right\}$$

where the multiplication is the field (matrix) multiplication of $GF(q^{t+1})$. By Corollary 2.6.3 (2) $\mathcal{C}$ contains the subfield $\{kI : \; k \in GF(q)\}$ which we shall denote by $GF(q)$. Let

$$\{e_1, e_2, \dots, e_{2t+2}\}$$

be a basis for $PG(2t+1, q)$ over $GF(q)$ and hence of $PG(2t+1, q^{t+1})$ over $GF(q^{t+1})$. The following set l of points of $PG(2t+1, q^{t+1})$:

$$\begin{aligned}
l &= \left\{e_1C_i + e_2\xi C_i + \cdots + e_{t+1}\xi^t C_i + e_{t+2}I + e_{t+3}\xi I + \cdots + e_{2(t+1)}\xi^t I : \; C_i \in \mathcal{C}\right\} \\
&\qquad \cup \left\{e_1 I + e_2\xi + \cdots + e_{t+1}\xi^t\right\} \\
&= \left\{(C_i, \xi C_i, \dots, \xi^t C_i, I, \xi, \dots, \xi^t)^T : \; C_i \in \mathcal{C}\right\} \\
&\qquad \cup \left\{(I, \xi, \dots, \xi^t, 0, \dots, 0)^T\right\} \\
&= \left\{C_i\,(I, \xi, \dots, \xi^t, 0, \dots, 0)^T + (0, 0, \dots, 0, I, \xi, \dots, \xi^t)^T : C_i \in \mathcal{C}\right\} \\
&\qquad \cup \left\{(I, \xi, \dots, \xi^t, 0, 0, \dots, 0)^T\right\}
\end{aligned}$$

is a line of $PG(2t+1, q^{t+1})$. Since l is

$$\begin{aligned} l &= (I, \xi, \ldots, \xi^t)^T \oplus (I, \xi, \ldots, \xi^t)^T \\ &= P \oplus Q \end{aligned}$$

the space $L(l)$ spanned by its $t+1$ conjugates is the join of the spaces $L(P)$ and $L(Q)$ spanned by the $t+1$ conjugates of P and the $t+1$ conjugates of Q respectively. Now each of P and Q is imaginary by Corollary 1.4.10 (1) and so $L(P)$ and $L(Q)$ have dimension t, and further by construction $L(P)$ and $L(Q)$ are skew so that $L(l)$ has dimension $2t+1$. By Definition 1.4.7 (2), l is imaginary.

Now recall that because $\mathcal{PC}$ is a projective t-spread set for $\mathcal{W}$, we have

$$\mathcal{W} = \{W_i = J(C_i, I) : \ C_i \in \mathcal{C}\} \cup \{W_\infty = J(I, 0)\}.$$

We will show that the point $(I, \xi, \ldots, \xi^t, 0, \ldots, 0)^T$ of l lies on the extension $\overline{W_\infty}$ of W_∞ to $PG(2t+1, q^{t+1})$, and the point

$$(C_i, \xi C_i, \ldots, \xi^t C_i, I, \xi, \ldots, \xi^t)^T$$

of l lies on $\overline{W_i}$ for each $i = 1, 2, \ldots, q^{t+1}$. Then the line l is the line required for proof of the result as it is imaginary and meets the extension of every element of $\mathcal{W}$.

We recall some notation. The element $J(I,0)$ of $\mathcal{W}$ has basis $\{e_1, e_2, \ldots, e_{t+1}\}$ and the element $J(0, I)$ has basis $\{e_{t+2}, e_{t+3}, \ldots, e_{2t+2}\}$. The vector space $\mathcal{V}_{2t+2}$ corresponding to $PG(2t+1, q)$ has basis $\{e_1, e_2, \ldots, e_{2t+2}\}$ so that

$$\mathcal{V}_{2t+2} = J(I, 0) \oplus J(0, I).$$

Also, $'$ denotes the (non-singular) linear transformation

$$\begin{aligned} ' : J(I,0) &\to J(0, I) \\ : e_k &\mapsto e_{(t+1)+k}. \end{aligned}$$

Then

$$\begin{aligned} W_\infty =& J(I,0) = \{a: \ a \in J(I,0)\} \\ =& \{(a_1, a_2, \dots, a_{t+1}, 0, \dots, 0)^T: \ a_i \in GF(q), \text{ not all zero}\} \end{aligned}$$

so that

$$\begin{aligned} \overline{W}_\infty =& \{(\xi_1, \xi_2, \dots, \xi_{t+1}, 0, \dots, 0)^T: \ \xi_i \in GF(q^{t+1}), \text{ not all zero}\} \\ =& \left\{\xi: \ \xi \in \overline{J(I,0)}\right\}. \end{aligned}$$

We can put

$$\xi = (\xi_1, \dots, \xi_{t+1}, 0, \dots, 0)^T = \left(I, \xi, \dots, \xi^t, 0, \dots, 0\right)^T \in \overline{J(I,0)}$$

and thus the point

$$\left(I, \xi, \dots, \xi^t, 0, \dots, 0\right)^T \in \overline{W}_\infty.$$

Similarly,

$$\begin{aligned} W_i =& J(C_i, I) = \{C_i a \oplus a': \ a \in J(I,0)\} \\ =& \left\{ C_i \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{t+1} \end{pmatrix} \oplus \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{t+1} \end{pmatrix} : \ a_i \in GF(q), \text{ not all zero} \right\} \end{aligned}$$

where the $\oplus$ denotes the direct sum of the first vector which is an element of $J(I,0)$ and the second vector which is an element of $J(0,I)$. Therefore

$$\begin{aligned} \overline{W_i} =& \overline{J(C_i, I)} \\ =& \left\{ C_i \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{t+1} \end{pmatrix} \oplus \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{t+1} \end{pmatrix} : \ \xi_i \in GF(q^{(t+1}), \text{ not all zero} \right\} \\ =& \left\{C_i \xi \oplus \xi': \ \xi \in \overline{J(I,0)}\right\} \end{aligned}$$

Again we can put

$$\xi = (\xi_1, \xi_2, \dots, \xi_{t+1}, 0, \dots, 0)^T = \left(I, \xi, \dots, \xi^t, 0, \dots, 0\right)^T \in \overline{J(I,0)}$$

so that $\xi' = (0,\ldots,0,I,\xi,\ldots,\xi^t)^T \in \overline{J(0,I)}$ and the point

$$(C_i, \xi C_i, \ldots, \xi^t C_i, I, \xi, \ldots, \xi^t)^T = C_i\,(I,\xi,\ldots,\xi^t)^T \oplus I\,(I,\xi,\ldots,\xi^t)^T$$
$$\in \overline{W_i}.$$

Thus we have shown that the extension of every element of $\mathcal{W}$ meets the imaginary line l of $PG(2t+1, q^{t+1})$, which is enough to prove the Theorem. □

2.9.4 Corollary

A regular t-spread of $PG(2t+1,q)$ has an indicator set comprising the $q^{t+1}+1$ points of an imaginary line of $PG(2t+1,q^{t+1})$ and conversely an imaginary line of $PG(2t+1,q^{t+1})$ is an indicator set for a regular t-spread of $PG(2t+1,q)$.

Proof: Let l be an imaginary line of $PG(2t+1,q^{t+1})$. The set

$$\mathcal{W} = \left\{\text{lin}\ \left\{P, P^\sigma, \ldots, P^{\sigma^t}\right\} \cap PG(2t+1,q) :\ P \in l\right\}$$

is a regular t-spread of $PG(2t+1,q)$ and the points of the imaginary line l are an indicator set for $\mathcal{W}$ as they are imaginary points and one point lies on the extension of each t-spread element. Also, the line joining any two such points is l and is therefore imaginary. Conversely, given a regular t-spread of $PG(2t+1,q)$, there exists an imaginary line l of $PG(2t+1,q^{t+1})$ meeting the extension of every t-spread element. Such a line gives an indicator set for the t-spread. □

2.9.5 Corollary

A regular t-spread $\mathcal{W}$ of $PG(2t+1,q)$ is uniquely determined by a t-regulus $\mathcal{R}$ of $\mathcal{W}$ and an element of $\mathcal{W}$ not belonging to $\mathcal{R}$.

Proof: Let $\mathcal{W}$ be a regular t-spread and let l be an imaginary line of the space $PG(2t+1,q^{t+1})$ meeting the extensions of every element of $\mathcal{W}$. The extensions of the elements of a t-regulus $\mathcal{R}$ contained in $\mathcal{W}$ meet l in the points of a projective subline l' of l, and the extension of a further element X_0 of $\mathcal{W}$ meets l in a point P not belonging to l'. Let $\mathcal{W}'$ be a regular t-spread containing the elements of $\mathcal{R}$ and the element X_0 of $\mathcal{W}$. We will show that every element of $\mathcal{W}$ is also an element of $\mathcal{W}'$ and the result follows. Choose elements X_1 and X_2 of $\mathcal{R}$. There is a unique t-regulus $\mathcal{R}'$ of $PG(2t+1,q)$ containing X_0, X_1 and X_2, which is distinct from $\mathcal{R}$. The line l is a transversal to the extension of $\mathcal{R}'$, thus the extensions of the elements of $\mathcal{R}'$ all meet l. But the t-dimensional spaces of $PG(2t+1,q)$ whose extensions meet l are exactly the elements of $\mathcal{W}$. Thus every element of $\mathcal{R}'$ is an element of $\mathcal{W}$. Now since $\mathcal{W}'$ is regular, it contains every element of $\mathcal{R}'$, which are all elements of $\mathcal{W}$. We repeat the argument using different elements of $\mathcal{W}'$ to define t-reguli, all of which are shown to belong to $\mathcal{W}$, continuing until we have shown that every element of $\mathcal{W}'$ is also an element of $\mathcal{W}$. $\square$

Theorem 2.9.3 can be interpreted from the point of view of the Segre variety. Any three distinct elements W_0, W_1 and W_2 of a regular t-spread $\mathcal{W}$ of $PG(2t+1,q)$ are contained in a unique t-regulus $\mathcal{R}$ (see Section 2.4). The $q+1$ elements of $\mathcal{R}$ are all elements of $\mathcal{W}$ and form the set of t-dimensional subspaces of a Segre variety $\mathcal{SV}_{2,t+1}$ in $PG(2t+1,q)$ with lines as its opposite subspaces (see Remark 2.4.4). We now embed $PG(2t+1,q)$ in $PG(2t+1,q^{t+1})$ and extend $\mathcal{SV}_{2,t+1}$ to a Segre variety $\overline{\mathcal{SV}}_{2,t+1}$ in $PG(2t+1,q^{t+1})$ as in Section 1.3. Then $\overline{\mathcal{SV}}_{2,t+1}$ has $q+1$ t-dimensional subspaces which meet $PG(2t+1,q)$, and the remaining q^2-q subspaces are skew to $PG(2t+1,q)$. Since the lines $l, l^{\sigma}, \ldots, l^{\sigma^t}$ meet all $q+1$ elements of $\overline{\mathcal{SV}}_{2,t+1}$ which are extensions of elements of $\mathcal{R}$, they must be lines of $\overline{\mathcal{SV}}_{2,t+1}$. In fact the extensions of the elements of $\mathcal{R}$ meet $l, l^{\sigma}, \ldots, l^{\sigma^t}$ in the points of a projective subline of each of $l, l^{\sigma}, \ldots, l^{\sigma^t}$, respectively. The properties of projective sublines of a projective line of $PG(2t+1,q^{t+1})$ can be used to demonstrate properties of t-reguli and regular t-spreads, as in the following.

2.9.6 Corollary

(1) Two t-reguli in $PG(2t+1,q)$ have either 0, 1, 2 or $q+1$ lines in common.

(2) A regular t-spread of $PG(2t+1,q)$ with $(t+1,2)=1$ is the union of

$$q^t + q^{t-1} + \cdots + q + 1$$

disjoint t-reguli.

(3) A regular t-spread of $PG(2t+1,q)$ has N reguli,where

$$N = \frac{q^t(q^{2t+2}-1)}{q^2-1}.$$

Proof: (1) follows since three points of a projective line l of order q^{t+1} determine a projective subline of order q.

(2) Noting that a such a line l is the union of $q^t+q^{t-1}+\cdots+q+1$ disjoint projective sublines of order q yields (2) (see Hirschfeld (1979), Theorem 4.3.6, Corollary 1, p92).

(3) Recall that since a projective subline of order q is determined by three points of l, the number of such sublines is the number of distinct triples of points of l divided by the number of distinct triples in a subline of order q, giving:

$$\frac{\binom{q^{t+1}+1}{3}}{\binom{q+1}{3}}$$

which is N. □

CHAPTER THREE

t-SPREADS OF $PG((s+1)(t+1)-1,q)$

3.1 INTRODUCTION

This Chapter generalises the work of Chapter Two to the case of t-spreads of $PG((s+1)(t+1)-1,q)$. In Sections 3.2 and 3.5 we investigate the construction of t-spread sets and projective t-spread sets corresponding to t-spreads of the space $PG((s+1)(t+1)-1,q)$, showing the connection between these two ideas. We generalise the construction of an affine plane from a t-spread of $PG(2t+1,q)$ to the construction of an affine space $AG^*(s+1,q^{t+1})$ in Section 3.3, and in Section 3.6 we use the projective t-spread set to provide coordinates for $AG^*(s+1,q^{t+1})$. In Section 3.4 we use the Segre variety to investigate the phenomenon of regularity of t-spreads of $PG((s+1)(t+1)-1,q)$. Section 3.8 generalises the work on indicator sets to define a projective indicator set of a t-spread of $PG((s+1)(t+1)-1,q)$ and Section 3.9 shows a representation of a t-spread of $PG((s+1)(t+1)-1,q)$ when embedded in $PG((s+1)(t+1)-1,q^{t+1})$. The ideas developed in this Chapter are demonstrated by two examples given in Section 3.10.

For this Chapter, let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1,q)$. Then $\mathcal{W}$ comprises $\omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \cdots + q^{(t+1)} + 1$ pairwise skew t-dimensional subspaces covering the points of $PG((s+1)(t+1)-1,q)$.

3.2 t-SPREAD SETS

In this Section, we generalise the construction of Bruck and Bose (1964) and (1966). Where the t-spread set in the case of $s=1$ is a set of matrices, the t-spread set in the case of general s is a set of s-tuples, $(s-1)$-tuples, ... , 2-tuples and single $(t+1)\times(t+1)$ matrices.

Let $\mathcal{V}_{(s+1)(t+1)}$ be an $(s+1)(t+1)$-dimensional vector space over $GF(q)$ and let $\mathcal{V}_{s(t+1)}$ be a fixed $s(t+1)$-dimensional subspace.

Let $A_1, A_2, \ldots, A_s$ be s distinct $(t+1)$-dimensional subspaces of $\mathcal{V}_{(s+1)(t+1)}$ spanning $\mathcal{V}_{s(t+1)}$ and write $\mathcal{V}_{s(t+1)}$ as $A_1 \oplus A_2 \oplus \cdots \oplus A_s$. Define s non-singular linear transformations (i) mapping A_1 to A_i, for $i = 1, 2, \ldots, s$, so that

$$\begin{aligned}(i) : A_1 &\to A_i \\ a &\mapsto a^{(i)}.\end{aligned}$$

Let B and C be an ordered pair of skew $(t+1)$-dimensional vector subspaces of $\mathcal{V}_{(s+1)(t+1)}$, both skew to $\mathcal{V}_{s(t+1)}$, and such that C is skew to each of the $s(t+1)$-dimensional spaces spanned by B together with $s-1$ of the spaces $A_1, A_2, \ldots, A_s$. Consider $\mathcal{V}_{(s+1)(t+1)}$ to be $A_1 \oplus A_2 \oplus \cdots \oplus A_s \oplus B$. The existence of C is shown in Lemma 1.1.2, and we apply this Lemma to show that there exists a unique (non-singular) linear transformation

$$\begin{aligned}' : A_1 &\to B \\ a &\mapsto a'\end{aligned}$$

of A onto B such that the linear transformation

$$a \mapsto a^{(1)} \oplus a^{(2)} \oplus \cdots \oplus a^{(s)} \oplus a'$$

maps A_1 onto C. Now to an s-tuple of $(t+1)\times(t+1)$ matrices $(X_1, X_2, \ldots, X_s)$ over $GF(q)$, there corresponds a unique $(t+1)$-dimensional subspace $J(X_1, X_2, \ldots, X_s)$ of $\mathcal{V}_{(s+1)(t+1)}$ skew to $\mathcal{V}_{s(t+1)}$ given by

$$J(X_1, X_2, \ldots, X_s) = \left\{X_1 a^{(1)} \oplus X_2 a^{(2)} \oplus \cdots \oplus X_s a^{(s)} \oplus a' : a \in A_1\right\}.$$

The s-tuple of $(t+1) \times (t+1)$ matrices can be interpreted as s linear transformations of the spaces $A_1, A_2, \ldots, A_s$ respectively. In particular, (with the following convention for the case ∞, and denoting the $(t+1)\times(t+1)$ zero matrix by 0 and the $(t+1)\times(t+1)$ identity matrix by I),

$$J(\infty) = A_1 = \{a : a \in A_1\}$$

$$J(0,0,\ldots,0) = B = \{a' : a \in A_1\}, \quad \text{and}$$

$$J(I,I,\ldots,I) = C = \{a^{(1)} \oplus a^{(2)} \oplus \ldots \oplus a^{(s)} \oplus a' : a \in A_1\}.$$

Conversely each $(t+1)$-dimensional subspace J of $\mathcal{V}_{(s+1)(t+1)}$ which is skew to $\mathcal{V}_{s(t+1)}$ has the form $J = J(X_1, X_2, \ldots, X_s)$ for a unique s-tuple of $(t+1)\times(t+1)$ matrices $(X_1, X_2, \ldots, X_s)$ over $GF(q)$.

3.2.1 Lemma

Let $(X_1, X_2, \ldots, X_s)$ and $(Y_1, Y_2, \ldots, Y_s)$ be two s-tuples of $(t+1)\times(t+1)$ matrices. Then

$$\begin{aligned} &J(X_1, X_2, \ldots, X_s) \cap J(Y_1, Y_2, \ldots, Y_s) \\ &\quad = \Big\{X_1 a^{(1)} \oplus X_2 a^{(2)} \oplus \cdots \oplus X_s a^{(s)} \oplus a' \colon\ a \in A_1 \text{ and} \\ &\qquad\qquad (X_1 - Y_1)a^{(1)} \oplus (X_2 - Y_2)a^{(2)} \oplus \cdots \oplus (X_s - Y_s)a^{(s)} = 0\Big\} \\ &\quad = \Big\{Y_1 a^{(1)} \oplus Y_2 a^{(2)} \oplus \cdots \oplus Y_s a^{(s)} \oplus a' \colon\ a \in A_1 \text{ and} \\ &\qquad\qquad (X_1 - Y_1)a^{(1)} \oplus (X_2 - Y_2)a^{(2)} \oplus \cdots \oplus (X_s - Y_s)a^{(s)} = 0\Big\}. \end{aligned}$$

Proof: Suppose $x \in J(X_1, X_2, \ldots, X_s) \cap J(Y_1, Y_2, \ldots, Y_s)$. Then for unique elements $a, b \in A_1$,

$$x = X_1 a^{(1)} \oplus X_2 a^{(2)} \oplus \cdots \oplus X_s a^{(s)} \oplus a' = Y_1 b^{(1)} \oplus Y_2 b^{(2)} \oplus \cdots \oplus Y_s b^{(s)} \oplus b'.$$

This implies that

$$X_1 a^{(1)} - Y_1 b^{(1)} \oplus X_2 a^{(2)} - Y_2 b^{(2)} \oplus \cdots \oplus X_s a^{(s)} - Y_s b^{(s)} \oplus a' - b' = 0.$$

Recalling that the spaces $A_1, A_2, \ldots, A_s, B$ pairwise have only the zero vector in common, this can occur if and only if $a' = b'$ so that $a = b$, then we have

$$(X_1 - Y_1)a^{(1)} \oplus (X_2 - Y_2)a^{(2)} \oplus \cdots \oplus (X_s - Y_s)a^{(s)} = 0$$

and the result follows. □

3.2.2 Corollary

Two spaces $J(X_1, X_2, \ldots, X_s)$ and $J(Y_1, Y_2, \ldots, Y_s)$ are skew if and only if $a = (0, 0, \ldots, 0)^T$ is the only common solution to the equations $(X_i - Y_i)a = 0$ for $i = 1, 2, \ldots, s$.

Proof: Since the spaces $A_1, A_2, \ldots, A_s$ have only the zero vector in common, then if

$$(X_1 - Y_1)a^{(1)} \oplus (X_2 - Y_2)a^{(2)} \oplus \cdots \oplus (X_s - Y_s)a^{(s)} = 0$$

with $(X_i - Y_i)a^{(i)} \in A_i$ for all $i = 1, 2, \ldots, s$ it follows that

$$(X_i - Y_i)a^{(i)} = 0 \quad \text{for all } i = 1, 2, \ldots, s.$$

Now $J(X_1, X_2, \ldots, X_s)$ and $J(Y_1, Y_2, \ldots, Y_s)$ have only the zero vector in common if and only if $a = (0, 0, \ldots, 0)^T$ is the only common solution to the equations $(X_i - Y_i)a^{(i)} = 0$ for $i = 1, 2, \ldots, s$. As the linear transformations $(i) : A_1 \to A_i$ are non-singular, this implies that $a = 0$ is the only common solution to the equations $(X_i - Y_i)a = 0$ for $i = 1, 2, \ldots, s$. □

Now let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$, with the property that there exists an $(s(t+1)-1)$-dimensional subspace $\mathcal{S}_{s(t+1)-1} = PG(s(t+1)-1, q)$ such that any element of $\mathcal{W}$ is either contained in $\mathcal{S}_{s(t+1)-1}$ or is skew to it. As in Section 1.1 we represent $PG((s+1)(t+1)-1, q)$ as an $(s+1)(t+1)$-dimensional vector space $\mathcal{V}_{(s+1)(t+1)}$ over the field $GF(q)$. Then $\mathcal{W}$ corresponds to a collection, still denoted $\mathcal{W}$, of $(t+1)$-dimensional vector subspaces of $\mathcal{V}_{(s+1)(t+1)}$ over $GF(q)$ pairwise having only the zero vector in common and satisfying the property that each non-zero vector of $\mathcal{V}_{(s+1)(t+1)}$ lies in exactly one element of $\mathcal{W}$. The space $\mathcal{S}_{s(t+1)-1}$ corresponds to an $s(t+1)$-dimensional subspace $\mathcal{V}_{s(t+1)}$ of $\mathcal{V}_{(s+1)(t+1)}$. Any element of the t-spread $\mathcal{W}$ not lying in the space $\mathcal{V}_{s(t+1)}$ has only the zero vector in common with it.

Suppose there exist elements $A_1, A_2, \ldots, A_s, B, C$ of the t-spread $\mathcal{W}$ such that the elements $A_1, A_2, \ldots, A_s$ span the space $\mathcal{V}_{s(t+1)}$, the elements $A_1, A_2, \ldots, A_s, B$ span the space $\mathcal{V}_{(s+1)(t+1)}$, and further any $s+1$ of the spaces $A_1, A_2, \ldots, A_s, B, C$ span $\mathcal{V}_{(s+1)(t+1)}$. In particular, the last condition implies that C is skew to $\mathcal{V}_{s(t+1)}$ and to B. The elements $A_1, A_2, \ldots, A_s, B$ can always be found, and when $\mathcal{W}$ is geometric (see Definition 1.2.3) an element C satisfying the requirements can be found. To see this, note that there are vectors of $\mathcal{V}_{(s+1)(t+1)}$ skew to each of the $s(t+1)$-dimensional spaces spanned by s of the elements $A_1, A_2, \ldots, A_s, B$. Any one such vector must be contained in an element C of $\mathcal{W}$, and since $\mathcal{W}$ is geometric, C is skew to each of the above $s(t+1)$-dimensional spaces as required.

In terms of the above representation, $\mathcal{W}$ gives rise to a unique collection $\mathcal{C}_s = \mathcal{C}_s(A_1, A_2, \ldots, A_s, B, C)$ of s-tuples of $(t+1) \times (t+1)$ matrices over $GF(q)$ satisfying the following conditions:

(i) $\mathcal{C}_s$ contains $(0, 0, \ldots, 0)$ and $(I, I, \ldots, I)$,

(ii) If $(X_1, X_2, \ldots, X_s)$ and $(Y_1, Y_2, \ldots, Y_s)$ are distinct elements of $\mathcal{C}_s$ then $a = (0, 0, \ldots, 0)^T$ is the only common solution to the equations $(X_i - Y_i)a = 0$ for $i = 1, 2, \ldots, s$, and

(iii) If $x_1, x_2, \ldots, x_s, y \in A_1$ with $y \neq 0$ then there exists a unique s-tuple

$$(X_1, X_2, \ldots, X_s) \in \mathcal{C}_s$$

such that $x_1^{(1)} = X_1 y^{(1)},\ x_2^{(2)} = X_2 y^{(2)},\ \ldots,\ x_s^{(s)} = X_s y^{(s)}$.

To establish these properties, first note that the spaces B and C give rise to the elements $(0, 0, \ldots, 0)$ and $(I, I, \ldots, I)$ of $\mathcal{C}_s$. Two distinct elements

$$(X_1, X_2, \ldots, X_s) \text{ and } (Y_1, Y_2, \ldots, Y_s)$$

of $\mathcal{C}_s$ correspond to distinct elements $J(X_1, X_2, \ldots, X_s)$ and $J(Y_1, Y_2, \ldots, Y_s)$ of $\mathcal{W}$. These are skew, so by Corollary 3.2.2, $a = 0$ is the only common solution to the equations $(X_i - Y_i)a = 0$ for $i = 1, 2, \ldots, s$. To show (iii) recall that since the spaces $A_1, A_2, \ldots, A_s, B$ span $\mathcal{V}_{(s+1)(t+1)}$ and $A_1, A_2, \ldots, A_s$ span $\mathcal{V}_{s(t+1)}$, any vector of $\mathcal{V}_{(s+1)(t+1)} - \mathcal{V}_{s(t+1)}$ can be written uniquely in the form

$$x_1^{(1)} \oplus x_2^{(2)} \oplus \cdots \oplus x_s^{(s)} \oplus y'$$

where $x_1, x_2, \ldots, x_s, y$ are all elements of A_1 and $y \neq 0$. Since $\mathcal{W}$ is a t-spread of $\mathcal{V}_{(s+1)(t+1)}$, this vector is contained in a unique element $J(X_1, X_2, \ldots, X_s)$ of $\mathcal{W}$. Thus

$$x_1^{(1)} \oplus x_2^{(2)} \oplus \cdots \oplus x_s^{(s)} \oplus y' \in \left\{X_1 a^{(1)} \oplus X_2 a^{(2)} \oplus \cdots \oplus X_s a^{(s)} \oplus a' : a \in A_1\right\}$$

and (iii) follows since $y' = a'$ implies that $y = a$.

3.2.3 Definition

An (i, t)-*spread set* is a set $\mathcal{C}_i$ of i-tuples of $(t+1) \times (t+1)$ matrices satisfying the following conditions:

(i) $\mathcal{C}_i$ has $q^{i(t+1)}$ elements,

(ii) $\mathcal{C}_i$ contains $(0,0,\ldots,0)$ and $(I,I,\ldots,I)$, and

(iii) If $(X_1,X_2,\ldots,X_i)$ and $(Y_1,Y_2,\ldots,Y_i)$ are distinct elements of $\mathcal{C}_i$ then $a = (0,0,\ldots,0)^T$ is the only common solution to the equations $(X_j - Y_j)a = 0$ for $j = 1,2,\ldots,i$.

We can also represent the i-tuples of $(t+1)\times(t+1)$ matrices in the set $\mathcal{C}_i$ by i-tuples of linear transformations of i skew $(t+1)$-dimensional vector spaces $A_1, A_2, \ldots, A_i$ respectively. We will say that $\mathcal{C}_i$ is an (i,t)-spread set of linear transformations or an (i,t)-spread set of matrices if we need to distinguish between these two definitions.

3.2.4 Theorem

Let $\mathcal{C}_i$ be an (i,t)-spread set and let

$$\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}; a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}; \ldots\ldots; a_1^{(i)}, a_2^{(i)}, \ldots, a_{t+1}^{(i)}; b_1, b_2, \ldots, b_{t+1}\right\}$$

be a basis of an $((i+1)(t+1))$-dimensional vector space $\mathcal{V}_{(i+1)(t+1)}$. Let $\mathcal{V}_{i(t+1)}$ denote the subspace spanned by the vectors

$$\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}; a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}; \ldots\ldots; a_1^{(i)}, a_2^{(i)}, \ldots, a_{t+1}^{(i)}\right\}.$$

For each $(C_1, C_2, \ldots, C_i) \in \mathcal{C}_i$ let

$$J(C_1, C_2, \ldots, C_i) = \mathrm{lin}\left\{C_1 a_k^{(1)} \oplus C_2 a_k^{(2)} \oplus \cdots \oplus C_i a_k^{(i)} \oplus b_k : \; k = 1, \ldots, t+1\right\}.$$

Then the set

$$\mathcal{W}_i = \left\{J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) : (C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) \in \mathcal{C}_i\right\}$$

is a partition of $\mathcal{V}_{(i+1)(t+1)} - \mathcal{V}_{i(t+1)}$ into pairwise skew $(t+1)$-dimensional subspaces. It is therefore the set of elements belonging to a partial t-spread covering the points $PG((i+1)(t+1)-1,q) - PG(i(t+1)-1,q)$ of the corresponding projective space. Conversely let $\mathcal{W}_i$ be a partial t-spread covering the points of $PG((i+1)(t+1)-1,q) - PG(i(t+1)-1,q)$ and suppose that there exist elements $A_1, A_2, \ldots, A_i, B, C$ of $\mathcal{W}_i$ such that $A_1, A_2, \ldots, A_i$ span $PG(i(t+1)-1,q)$, $A_1, A_2, \ldots, A_i, B$ span $PG((i+1)(t+1)-1,q)$ and any $i+1$ of $A_1, A_2, \ldots, A_i, B, C$ span $PG((i+1)(t+1)-1,q)$. Then $\mathcal{W}_i$ can be represented in this manner.

Proof: For the second statement, such a partial t-spread $\mathcal{W}_i$ covering the points of

$$PG((i+1)(t+1)-1,q) - PG(i(t+1)-1,q).$$

has an (i,t)-spread set as constructed in this Section. This (i,t)-spread set has the required form. To show the first statement, suppose that $\mathcal{C}_i$ is an (i,t)-spread set. We have to show that the set $\mathcal{W}_i$ comprises $q^{i(t+1)}$ pairwise skew $(t+1)$-dimensional subspaces of $\mathcal{V}_{(i+1)(t+1)} - \mathcal{V}_{i(t+1)}$. An element of $\mathcal{W}_i$ is a $(t+1)$-dimensional subspace of $\mathcal{V}_{(i+1)(t+1)}$ since the set of spanning vectors is linearly independent. To see this, let $(C_1, C_2, \ldots, C_i)$ be an element of $\mathcal{C}$, and suppose that for $x_0, x_1, \ldots, x_t \in GF(q)$

$$\sum_{k=0}^{t} x_k \left(C_1 a_k^{(1)} \oplus C_2 a_k^{(2)} \oplus \cdots \oplus C_i a_k^{(i)} \oplus b_k \right) = 0.$$

This implies that

$$\bigoplus_{m=1}^{i} C_m \left(x_0 a_0^{(m)} + x_1 a_1^{(m)} + \cdots + x_t a_t^{(m)} \right) \oplus (x_0 b_0 + x_1 b_1 + \cdots + x_t b_t) = 0.$$

Now the first term on the left hand side is in $\mathcal{V}_{i(t+1)}$, and the second term is in $B = \mathrm{lin}\{b_1, b_2, \ldots, b_{t+1}\}$, and since B has only the zero vector in common with $\mathcal{V}_{i(t+1)}$, both parts of the left hand side of this equation must be zero. But $\{b_1, b_2, \ldots, b_{t+1}\}$ is a linearly independent set of vectors as it is contained in the basis for $\mathcal{V}_{(i+1)(t+1)}$, and this implies that $x_0, x_1, \ldots, x_t$ are all zero. Thus $\mathcal{W}$

certainly comprises $q^{i(t+1)}$ $(t+1)$-dimensional subspaces of $\mathcal{V}_{(i+1)(t+1)} - \mathcal{V}_{i(t+1)}$. Let $J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)})$ and $J(C_1^{(m)}, C_2^{(m)}, \ldots, C_i^{(m)})$ be distinct elements of $\mathcal{W}$. By Corollary 3.2.2, they are skew if and only if $a = (0, 0, \ldots, 0)^T$ is the only common solution to the equations, $(C_k^{(j)} - C_k^{(m)})a = 0$ for $k = 1, 2, \ldots, i$. This is true by Definition 3.2.3 (iii) of (i,t)-spread set. □

3.2.5 Remark

The (i,t)-spread set of $(t+1) \times (t+1)$ matrices as defined here is limited in that it only describes elements of a partial t-spread $\mathcal{W}_i$ covering the points of the subspace $PG((i+1)(t+1)-1,q)$ and skew to a certain subspace $PG(i(t+1)-1,q)$.

Now suppose that we start with a t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1,q)$, where $\mathcal{W}$ contains a partial t-spread $\mathcal{W}_s$ covering the points of

$$PG((s+1)(t+1)-1,q) - PG(s(t+1)-1,q).$$

Suppose further that $\mathcal{W}_s$ has elements $A_1, A_2, \ldots, A_s, B, C$ where $A_1, A_2, \ldots, A_s$ span $PG(s(t+1)-1,q)$, $A_1, A_2, \ldots, A_s, B$ span $PG((s+1)(t+1)-1,q)$ and any $s+1$ of $A_1, A_2, \ldots, A_s, B, C$ span $PG((s+1)(t+1)-1,q)$. Then by Theorem 3.2.4 we can construct an (s,t)-spread set for $\mathcal{W}_s$. However there is no information gained about the elements of $\mathcal{W}$ not contained in $\mathcal{W}_s$.

To try to overcome this difficulty, we suppose further that there exists an $((s-1)(t+1)-1)$-dimensional subspace $PG((s-1)(t+1)-1,q)$ such that the elements of the t-spread $\mathcal{W}$ lying in $PG(s(t+1)-1,q)$ are either contained in $PG((s-1)(t+1)-1,q)$ or are skew to it. Suppose further that $\mathcal{W}$ has elements $A'_1, A'_2, \ldots, A'_{s-1}, B', C'$ such that $A'_1, A'_2, \ldots, A'_{s-1}$ span $PG((s-1)(t+1)-1,q)$, $A'_1, A'_2, \ldots, A'_{s-1}, B'$ span $PG(s(t+1)-1,q)$ and any s of $A'_1, A'_2, \ldots, A'_{s-1}, B', C'$ span $PG(s(t+1)-1,q)$. Then we can construct an $(s-1,t)$-spread set corresponding to the partial t-spread of elements of $\mathcal{W}$ which are (entirely) contained in $PG(s(t+1)-1,q) - PG((s-1)(t+1)-1,q)$.

We could proceed in this manner, provided that the following condition is satisfied. Suppose that we are concerned with the elements of $\mathcal{W}$ contained in the subspace $PG(k(t+1)-1,q)$. This is the kth stage of the process. We require that there exists a $((k-1)(t+1)-1)$-dimensional subspace $PG((k-1)(t+1)-1,q)$ of $PG(k(t+1)-1,q)$ such that the elements of $\mathcal{W}$ in $PG(k(t+1)-1,q)$ are either contained in $PG((k-1)(t+1)-1,q)$ or are skew to it. We require further that $\mathcal{W}$ has elements $A_1'', A_2'', \ldots, A_{k-1}'', B'', C''$ such that $A_1'', A_2'', \ldots, A_{k-1}''$ span $PG((k-1)(t+1)-1,q)$, $A_1'', A_2'', \ldots, A_{k-1}'', B''$ span $PG(k(t+1)-1,q)$ and any k of $A_1'', A_2'', \ldots, A_{k-1}'', B'', C''$ span $PG(k(t+1)-1,q)$. Then we can construct a $(k-1,t)$-spread set corresponding to the elements of $\mathcal{W}$ in $PG(k(t+1)-1,q) - PG((k-1)(t+1)-1,q)$.

Note that in the case of a geometric t-spread $\mathcal{W}$, the condition can always be satisfied at every stage, and in fact the space acting as A_{k-1} in the kth stage can be used as B in the $(k-1)$th stage. The spaces $A_1, A_2, \ldots, A_{k-2}$ of the kth stage are used as $A_1, A_2, \ldots, A_{k-2}$ in the $(k-1)$th stage.

We obtain an (s,t)-spread set, an $(s-1,t)$-spread set, and so on until we get a $(1,t)$-spread set, with a single element $PG(t,q) \in \mathcal{W}$ remaining. The $(1,t)$-spread set is a set of q^{t+1} single matrices corresponding to the elements of $\mathcal{W}$ in a subspace $PG(2t+1,q)$ but skew to $PG(t,q)$. This is the t-spread set constructed by Bruck and Bose (1964), see Section 2.2, and the space $PG(t,q)$ is any t-spread element which is chosen to be $J(\infty)$.

The above remarks could be summarised as follows:

(s): Elements of $\mathcal{W}$ in $PG((s+1)(t+1)-1,q) - PG(s(t+1)-1,q)$ give rise to $q^{s(t+1)}$ s-tuples of $(t+1)\times(t+1)$ matrices

$$\left\{(C_1^{(j)}, C_2^{(j)}, \ldots, C_s^{(j)}) :\ j = 1, 2, \ldots, q^{s(t+1)}\right\},$$

$(s-1)$: Elements of $\mathcal{W}$ in $PG(s(t+1)-1,q) - PG((s-1)(t+1)-1,q)$ give rise to $q^{(s-1)(t+1)}$ $(s-1)$-tuples of $(t+1)\times(t+1)$ matrices

$$\left\{(C_1^{(j)}, C_2^{(j)}, \ldots, C_{s-1}^{(j)}) :\ j = 1, 2, \ldots, q^{(s-1)(t+1)}\right\},$$

$\vdots$

(2): Elements of $\mathcal{W}$ in $PG(3(t+1)-1,q) - PG(2t+1,q)$ give rise to $q^{2(t+1)}$ 2-tuples of $(t+1)\times(t+1)$ matrices

$$\left\{(C_1^{(j)}, C_2^{(j)}) :\ j = 1, 2, \ldots, q^{2(t+1)}\right\},$$

(1): Elements of $\mathcal{W}$ in $PG(2t+1,q) - PG(t,q)$ give rise to q^{t+1} single $(t+1)\times(t+1)$ matrices

$$\left\{C_1^{(j)} :\ j = 1, 2, \ldots, q^{t+1}\right\},$$

(0) The last remaining element of $\mathcal{W}$ is $PG(t,q) = J(\infty)$.

The matrices appearing in the i-tuple at each stage could be taken to be linear transformations on the appropriate $(t+1)$-dimensional vector spaces.

We demonstrate the reverse procedure in the case that $\mathcal{W}$ is geometric. We choose a basis for $PG((s+1)(t+1)-1,q)$ such that the elements

$$A_1, A_2, \ldots, A_s, A_{s+1} = B, A_{s+2} = C$$

of $\mathcal{W}$ are as follows:

$$\begin{aligned}
A_1 &= \text{lin}\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}\right\} \\
&= J(\infty), \\
A_2 &= \text{lin}\left\{a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}\right\} \\
&\in PG(2t+1,q) - J(\infty), \\
A_3 &= \text{lin}\left\{a_1^{(3)}, a_2^{(3)}, \ldots, a_{t+1}^{(3)}\right\} \\
&\in PG(3(t+1)-1,q) - PG(2t+1,q) \\
&\vdots \\
A_s &= \text{lin}\left\{a_1^{(s)}, a_2^{(s)}, \ldots, a_{t+1}^{(s)}\right\} \\
&\in PG(s(t+1)-1,q) - PG((s-1)(t+1)-1,q), \\
A_{s+1} &= \text{lin}\left\{a_1^{(s+1)}, a_2^{(s+1)}, \ldots, a_{t+1}^{(s+1)}\right\} \\
&\in PG((s+1)(t+1)-1,q) - PG(s(t+1)-1,q),
\end{aligned}$$

so that

$$\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}; a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}; \ldots\ldots; \right.$$
$$\left. a_1^{(s)}, a_2^{(s)}, \ldots, a_{t+1}^{(s)}; a_1^{(s+1)}, a_2^{(s+1)}, \ldots, a_{t+1}^{(s+1)}\right\}$$

is then a basis for $PG((s+1)(t+1)-1,q)$.

The elements of $\mathcal{W}$ in $PG((s+1)(t+1)-1,q) - PG(s(t+1)-1,q)$ are

$$\begin{aligned}
&J(C_1^{(j)}, C_2^{(j)}, \ldots, C_s^{(j)}) \\
&\quad = \left\{C_1^{(j)}a^{(1)} \oplus C_2^{(j)}a^{(2)} \oplus \cdots \oplus C_s^{(j)}a^{(s)} \oplus a^{(s+1)} :\ a \in A_1\right\}
\end{aligned}$$

for $j = 1, 2, \ldots, q^{s(t+1)}$.

The elements of $\mathcal{W}$ in $PG(s(t+1)-1,q) - PG((s-1)(t+1)-1,q)$ are

$$\begin{aligned}
&J(C_1^{(j)}, C_2^{(j)}, \ldots, C_{s-1}^{(j)}) \\
&\quad = \left\{C_1^{(j)}a^{(1)} \oplus C_2^{(j)}a^{(2)} \oplus \cdots \oplus C_{s-1}^{(j)}a^{(s-1)} \oplus a^{(s)} :\ a \in A_1\right\}
\end{aligned}$$

for $j = 1, 2, \ldots, q^{(s-1)(t+1)}$, and so on.

The elements of $\mathcal{W}$ in $PG(3(t+1)-1, q) - PG(2t+1, q)$ are

$$J(C_1^{(j)}, C_2^{(j)}) = \left\{ C_1^{(j)} a^{(1)} \oplus C_2^{(j)} a^{(2)} \oplus a^{(3)} : \ a \in A_1 \right\}$$

for $j = 1, 2, \ldots, q^{2(t+1)}$.

The elements of $\mathcal{W}$ in $PG(2t+1, q) - PG(t, q)$ are

$$J(C_1^{(j)}) = \left\{ C_1^{(j)} a^{(1)} + a^{(2)} : \ a \in A_1 \right\}$$

for $j = 1, 2, \ldots, q^{t+1}$. The remaining element of $\mathcal{W}$ is A_1.

These considerations suggest the following definition.

3.2.6 Definition

A *t-spread set* $\mathcal{C}$ is a collection of $q^{s(t+1)}$ s-tuples, $q^{(s-1)(t+1)}$ $(s-1)$-tuples, ..., $q^{2(t+1)}$ 2-tuples and q^{t+1} single $(t+1) \times (t+1)$ matrices such that for each $i = 1, 2, \ldots, s$ the i-tuples of matrices form an (i, t)-spread set.

3.2.7 Definition The Shell Property

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$. Suppose that $PG((s+1)(t+1)-1, q)$ admits subspaces $PG(t, q)$, $PG(2t+1, q)$, $PG(3(t+1)-1, q), \ldots,$ $PG(s(t+1)-1, q)$ such that the sets of points

$$\begin{aligned}
&PG(t, q), \\
&PG(2(t+1)-1, q) - PG(t, q), \\
&\vdots \\
&PG(s(t+1)-1, q) - PG((s-1)(t+1)-1, q), \\
&PG((s+1)(t+1)-1, q) - PG(s(t+1)-1, q)
\end{aligned}$$

are disjoint and so form a partition of $PG((s+1)(t+1)-1,q)$. We shall call a set of points

$$PG((k+1)(t+1),q) - PG(k(t+1)-1,q)$$

a *shell* of $PG((s+1)(t+1)-1,q)$ for $k = 1,2,\ldots,s$. $\mathcal{W}$ is said to have the *Shell property* with respect to this partition of $PG((s+1)(t+1)-1,q)$ into shells if

(i) every element of $\mathcal{W}$ is contained in exactly one shell and has no point in any other shell, and

(ii) for each $k = 1,2,\ldots,s$ there are elements $A_1, A_2, \ldots, A_k, B, C$ of $\mathcal{W}$ such that $A_1, A_2, \ldots, A_k$ span $PG(k(t+1)-1,q)$, the elements $A_1, A_2, \ldots, A_k, B$ span $PG((k+1)(t+1)-1,q)$ and any $k+1$ of $A_1, A_2, \ldots, A_k, B, C$ span $PG((k+1)(t+1)-1,q)$.

When $s = 1$, every t-spread $\mathcal{W}$ has the Shell property with respect to any partition of $PG(2t+1,q)$ into shells, provided that in that partition the shell $PG(t,q)$ is an element of $\mathcal{W}$.

3.2.8 Theorem

A geometric t-spread of $PG((s+1)(t+1)-1,q)$ has the Shell property 3.2.7 with respect to any partition of $PG((s+1)(t+1)-1,q)$ into shells where each shell entirely contains at least one element of $\mathcal{W}$. Conversely if $\mathcal{W}$ is a t-spread of $PG((s+1)(t+1)-1,q)$ with the Shell property 3.2.7 for any partition of $PG((s+1)(t+1)-1,q)$ into shells, where there is at least one element of $\mathcal{W}$ contained in each shell, then $\mathcal{W}$ is geometric.

Proof: Let $\mathcal{W}$ be a geometric t-spread of $PG((s+1)(t+1)-1,q)$. Choose any element W_0 of $\mathcal{W}$, then this is a space $PG(t,q)$ of $PG((s+1)(t+1)-1,q)$. Now choose another element W_1 of $\mathcal{W}$ distinct from W_0. These two spaces span a

$(2t+1)$-dimensional subspace $PG(2t+1,q)$ of $PG((s+1)(t+1)-1,q)$ and any element of $\mathcal{W}$ either lies in $PG(2t+1,q)$ or is skew to it. Further there exists another element C of $\mathcal{W}$ in $PG(2t+1,q)$ skew to $PG(t,q)$ and to W_1 and any two of W_0, W_1, C span $PG(2t+1,q)$. Now choose an element W_2 of $\mathcal{W}$ not lying in $PG(2t+1,q)$. Then W_0, W_1 and W_2 span a $(3(t+1)-1)$-dimensional subspace $PG(3(t+1)-1,q)$ of $PG((s+1)(t+1)-1,q)$, and every element of $\mathcal{W}$ is either contained in $PG(3(t+1)-1,q)$ or is skew to it. Further, there exists another element C' of $\mathcal{W}$ in $PG(3(t+1)-1,q)$ skew to $PG(2t+1,q)$ and to W_2 such that any three of W_0, W_1, W_2, C' span $PG(3(t+1)-1,q)$. We continue in this way until we reach the following: choose an element W_s of $\mathcal{W}$ not lying in the subspace $PG(s(t+1)-1,q)$ of $PG((s+1)(t+1)-1,q)$. Then $W_0, W_1, \ldots, W_s$ span $PG((s+1)(t+1)-1,q)$ and there exists an element C'' of $\mathcal{W}$ such that any $s+1$ of $W_0, W_1, \ldots, W_s, C''$ span $PG((s+1)(t+1)-1,q)$. $PG((s+1)(t+1)-1,q)$ is thus partitioned into shells

$$
\begin{aligned}
&PG(t,q),\\
&PG(2(t+1)-1,q)-PG(t,q),\\
&\vdots\\
&PG(s(t+1)-1,q)-PG((s-1)(t+1)-1,q),\\
&PG((s+1)(t+1)-1,q)-PG(s(t+1)-1,q)
\end{aligned}
$$

and by construction every element of $\mathcal{W}$ is contained in exactly one shell and has no point in any other shell. Given a partition of $PG((s+1)(t+1)-1,q)$ into shells where each shell contains at least one element of $\mathcal{W}$, choose one element from each shell and let these be $W_0, W_1, \ldots, W_s$ as above, and $\mathcal{W}$ has the Shell property 3.2.7 with respect to this partition of $PG((s+1)(t+1)-1,q)$ into shells.

Conversely, suppose that $\mathcal{W}$ has the Shell property 3.2.7 with respect to any partition of $PG((s+1)(t+1)-1,q)$ into shells where each shell contains at least one element of $\mathcal{W}$. Choose $X, Y \in \mathcal{W}$ and choose $X = PG(t,q)$ as the first shell and Y as an element of the second shell $PG(2t+1,q) - PG(t,q)$. By the Shell property 3.2.7, every element of $\mathcal{W} - \{X\}$ is contained in or is skew to $PG(2t+1,q) - PG(t,q)$. Thus every element of $\mathcal{W}$ is contained in or is skew to the $(2t+1)$-dimensional space $< X, Y >$, and $\mathcal{W}$ is geometric. $\square$

It is interesting to ask if there are any t-spreads which are not geometric but which have the Shell property for some division of $PG((s+1)(t+1)-1,q)$ into shells.

3.2.9 Theorem

Let $\mathcal{C}$ be a t-spread set as above, so that

$$\mathcal{C} = \left\{(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) :\ i = 1, 2, \ldots, s,\ j = 1, 2, \ldots, q^{i(t+1)}\right\}.$$

Suppose that $PG((s+1)(t+1)-1,q)$ has a basis

$$\Big\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}; a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}; \ldots\ldots; a_1^{(s)}, a_2^{(s)}, \ldots, a_{t+1}^{(s)};$$
$$a_1^{(s+1)}, a_2^{(s+1)}, \ldots, a_{t+1}^{(s+1)}\Big\}$$

and that

$$J(\infty) = \operatorname{lin}\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}\right\}.$$

For each element $(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)})$ of $\mathcal{C}$, let

$$J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)})$$
$$= \operatorname{lin}\left\{C_1^{(j)} a_k^{(1)} \oplus C_2^{(j)} a_k^{(2)} \oplus \cdots \oplus C_i^{(j)} a_k^{(i)} \oplus a_k^{(i+1)} :\ k = 1, 2, \ldots, t+1\right\}.$$

Then

$$\mathcal{W} = \left\{J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) :\ i = 1, 2, \ldots, s,\ j = 1, 2, \ldots, q^{i(t+1)}\right\} \cup \{J(\infty)\}$$

is a partition of $\mathcal{V}_{(s+1)(t+1)}$ into pairwise skew $(t+1)$-dimensional subspaces. This gives a t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1,q)$ with the Shell property 3.2.7.

Conversely every t-spread of $PG((s+1)(t+1)-1,q)$ which has the Shell property 3.2.7 may be represented in this way by a t-spread set.

Proof: The fact that every t-spread of $PG((s+1)(t+1)-1,q)$ with the Shell property 3.2.7 has a representation as a t-spread set was demonstrated in the construction in Remark 3.2.5. Now let $\mathcal{C}$ be a t-spread set, and let $\mathcal{W}$ be the set constructed as in the statement of the Theorem. By definition, for each $i=1,2,\ldots,s$, the set

$$\mathcal{C}_i = \left\{(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) : \ j=1,2,\ldots,q^{i(t+1)}\right\}$$

is an (i,t)-spread set. This means that, by Theorem 3.2.4, the set

$$\mathcal{W}_i = \left\{J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) : \ j=1,2,\ldots,q^{i(t+1)}\right\}$$

is a partial t-spread of $PG((i+1)(t+1)-1,q) - PG(i(t+1)-1,q)$. Then $\mathcal{W}_1$ is a partial t-spread of $PG(2t+1,q)$ covering the points of $PG(2t+1,q) - J(\infty)$, $\mathcal{W}_2$ is a partial t-spread of $PG(3(t+1)-1,q)$ covering the points lying in the subspace $PG(3(t+1)-1,q) - PG(2(t+1)-1,q)$, and so on until we reach $\mathcal{W}_s$ which is a partial t-spread covering the points of $PG((s+1)(t+1)-1,q) - PG(s(t+1)-1,q)$. Thus,

$$\mathcal{W} = \bigcup_{i=1}^{s} \mathcal{W}_i \cup \{J(\infty)\}$$

is a t-spread of $PG((s+1)(t+1)-1,q)$. To show that $\mathcal{W}$ has the Shell property, first note that $PG((s+1)(t+1)-1,q)$ has been divided into shells and that each element of $\mathcal{W}$ is contained in exactly one shell. To show condition (ii), for

$1 \leq k \leq s$, let

$$A_1 = \text{lin}\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}\right\}$$
$$A_2 = \text{lin}\left\{a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}\right\}$$
$$A_3 = \text{lin}\left\{a_1^{(3)}, a_2^{(3)}, \ldots, a_{t+1}^{(3)}\right\}$$
$$\vdots$$
$$A_k = \text{lin}\left\{a_1^{(k)}, a_2^{(k)}, \ldots, a_{t+1}^{(k)}\right\}.$$

Further, let

$$B = \text{lin}\left\{a_1^{(k+1)}, a_2^{(k+1)}, \ldots, a_{t+1}^{(k+1)}\right\}$$
$$C = J(I, I, \ldots, I)$$

where $(I, I, \ldots, I)$ is in $\mathcal{W}_k$, that is, it is a k-tuple. Then $A_1, A_2, \ldots, A_k$ span $PG(k(t+1)-1, q)$, $A_1, A_2, \ldots, A_k, B$ span $PG((k+1)(t+1)-1, q)$, and any $k+1$ of $A_1, A_2, \ldots, A_k, B, C$ span $PG((k+1)(t+1)-1, q)$. Thus $\mathcal{W}$ has the Shell property. □

3.2.10 Corollary

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$ with the Shell property 3.2.7. Suppose that $PG((s+1)(t+1)-1, q)$ has a basis

$$\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}; a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}; \ldots\ldots; a_1^{(s)}, a_2^{(s)}, \ldots, a_{t+1}^{(s)}; \right.$$
$$\left. a_1^{(s+1)}, a_2^{(s+1)}, \ldots, a_{t+1}^{(s+1)}\right\}$$

and let

$$A_1 = \text{lin}\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}\right\}$$
$$A_2 = \text{lin}\left\{a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}\right\}$$
$$A_3 = \text{lin}\left\{a_1^{(3)}, a_2^{(3)}, \ldots, a_{t+1}^{(3)}\right\}$$
$$\vdots$$
$$A_s = \text{lin}\left\{a_1^{(s)}, a_2^{(s)}, \ldots, a_{t+1}^{(s)}\right\}$$
$$A_{s+1} = \text{lin}\left\{a_1^{(s+1)}, a_2^{(s+1)}, \ldots, a_{t+1}^{(s+1)}\right\}.$$

For each k with $k = 1, 2, \ldots, s+1$, let (k) denote the following (non-singular) linear transformation:

$$(k) : A_1 \to A_k$$
$$a_l^{(1)} \mapsto a_l^{(k)} \qquad \text{for } \ l = 1, 2, \ldots, t+1.$$

For $1 \leq i \leq s$, given any $i+1$ elements

$$a_1 \in A_1, \ a_2 \in A_2, \ldots, \ a_i \in A_i, \ \text{and } y \in A_1,$$

there exists a unique element $(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)})$ of the t-spread set $\mathcal{C}$ of $\mathcal{W}$ such that

$$a_1 = C_1^{(j)} y, \ a_2 = C_2^{(j)} y, \ldots, \ a_i = C_i^{(j)} y.$$

Proof: The vector $a_1 \oplus a_2 \oplus \cdots \oplus a_i \oplus y^{(i+1)}$ represents a point of the $((i+1)(t+1)-1)$-dimensional vector subspace of $\mathcal{V}_{(s+1)(t+1)}$ spanned by the spaces $A_1, A_2, \ldots, A_{i+1}$. Since $\mathcal{W}$ is a t-spread of $\mathcal{V}_{(s+1)(t+1)}$ with the Shell property 3.2.7, this vector is contained in a t-spread element of the form

$$J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)})$$

for some $i \in \{1, 2, \ldots, s\}$ and $j \in \{1, 2, \ldots, q^{i(t+1)}\}$. Thus

$$\begin{aligned} a_1 \oplus a_2 \oplus \cdots \oplus a_i \oplus y^{(i+1)} &\in J\left(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}\right) \\ &= \left\{ C_1^{(j)} a^{(1)} \oplus C_2^{(j)} a^{(2)} \oplus \cdots \oplus C_i^{(j)} a^{(i)} \oplus a^{(i+1)} : \ a \in A_1 \right\} \end{aligned}$$

and we see that $a^{(i+1)} = y^{(i+1)}$ so that $a = y$ and the result follows. □

Apart from the fact that this representation of a t-spread involves different representations for different t-spread elements, only a limited class of t-spreads even have a representation as a t-spread set, namely those with the Shell property 3.2.7. This situation contrasts with Section 3.5 in which we show that every t-spread has a projective t-spread set, and the elements of the projective t-spread set are all the same. In the case of $s = 1$ every t-spread has the Shell property and this difficulty doesn't arise.

3.3 CONSTRUCTION OF AN AFFINE SPACE $AG^*(s+1, q^{t+1})$

This Section generalises the construction of an affine plane which has been given in the case of $s = 1$. The construction yields an affine $(s+1)$-dimensional space of order $q^{(t+1)}$. Since every $(s+1)$-dimensional space is Desarguesian for $s \geq 2$, there are certain implications for the t-spread.

Let $\mathcal{W} = \{W_0, W_1, \ldots, W_\omega\}$, where $\omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \ldots + q^{(t+1)} + 1$, be a t-spread of $PG((s+1)(t+1)-1, q)$. Embed $PG((s+1)(t+1)-1, q)$ as a hyperplane in $PG((s+1)(t+1), q)$, and define an incidence structure $\mathcal{A} = (P, B, I)$ as follows:

- the *points* of $\mathcal{A}$ are the points of $PG((s+1)(t+1), q) - PG((s+1)(t+1)-1, q)$,
- the *blocks* of $\mathcal{A}$ are the $(t+1)$-dimensional subspaces of $PG((s+1)(t+1), q)$ which meet $PG((s+1)(t+1)-1, q)$ in exactly an element of the t-spread $\mathcal{W}$, and
- the *incidence* is that induced by the incidence of $PG((s+1)(t+1), q)$.

Therefore $\mathcal{A}$ has $q^{(s+1)(t+1)}$ points and $q^{s(t+1)}\omega$ blocks. We now show that under the assumption that $\mathcal{W}$ is geometric this is an affine $(s+1)$-dimensional space of order $q^{(t+1)}$, noting first that it has the correct number of points and blocks.

3.3.1 Theorem [Hirschfeld, (1979), p39]
Let $\mathcal{I}$ be an incidence structure with an equivalence relation (parallelism) on its blocks such that

(i) Any two distinct points P_1 and P_2 are incident with exactly one block denoted by $b(P_1P_2)$.

(ii) For every point P and block b, there is a unique block b' parallel to b and containing P.

(iii) If $b(P_1P_2)$ and $b(P_3P_4)$ are parallel blocks and P is a point on $b(P_1P_3)$ distinct from P_1 and P_3, then there is a point P' on $b(PP_2)$ and $b(P_3P_4)$.

(iv) If no block contains more than two points and P_1, P_2, P_3 are distinct points, then the block b_3 through P_3 parallel to $b(P_1P_2)$ and the block b_2 through P_2 parallel to $b(P_1P_3)$ have a point P in common.

(v) Some block contains exactly $q \geq 2$ points.

(vi) There exist two blocks neither parallel nor with a common point.

Then $\mathcal{I}$ is isomorphic to the n-dimensional affine space $AG(n, q)$ of order q, for some $n \geq 3$. □

We now check that $\mathcal{A}$ satisfies these axioms. We say that two blocks of $\mathcal{A}$ are parallel if and only if they are $(t+1)$-dimensional subspaces of $PG((s+1)(t+1), q)$ meeting $PG((s+1)(t+1)-1, q)$ in the same element of the t-spread $\mathcal{W}$. Note that parallel blocks are $(t+1)$-dimensional subspaces of $PG((s+1)(t+1), q)$ which are either coincident and pass through an element of $\mathcal{W}$ or meet only in the points of an element of $\mathcal{W}$. Parallelism so defined is indeed an equivalence relation. Now we check the conditions (i)–(vi) above.

(i) Let P_1 and P_2 be distinct points of $\mathcal{A}$. They are points of

$$PG((s+1)(t+1), q) - PG((s+1)(t+1)-1, q),$$

and the line P_1P_2 of $PG((s+1)(t+1), q) - PG((s+1)(t+1)-1, q)$ meets $PG((s+1)(t+1)-1, q)$ in a point which lies on a unique element W_i of the t-spread $\mathcal{W}$. The $(t+1)$-dimensional subspace $< W_i, P_1P_2 >$ is the unique block of $\mathcal{A}$ containing both P_1 and P_2.

(ii) Let P be a point of $\mathcal{A}$ and b a block of $\mathcal{A}$, where P is not incident with b. Then P is a point of $PG((s+1)(t+1),q) - PG((s+1)(t+1)-1,q)$ and b is a $(t+1)$-dimensional subspace of $PG((s+1)(t+1),q)$ not containing P and meeting $PG((s+1)(t+1)-1,q)$ in the unique element W_b of $\mathcal{W}$. The $(t+1)$-dimensional space $< W_b, P >$ is the unique block of $\mathcal{A}$ through P and parallel to b.

(iii) Let $b(P_1P_2)$ and $b(P_3P_4)$ be parallel blocks of $\mathcal{A}$, so that they are $(t+1)$-dimensional subspaces of $PG((s+1)(t+1),q)$ meeting $PG((s+1)(t+1)-1,q)$ in exactly the element W_1 say of $\mathcal{W}$. The block $b(P_1P_3)$ meets the space $PG((s+1)(t+1)-1,q)$ in the element W_2 say of $\mathcal{W}$, distinct from W_1, and meets each of $b(P_1P_2)$ and $b(P_3P_4)$ in the unique point P_1 or P_3 respectively. Let P be a point on the block $b(P_1P_3)$ distinct from P_1 and P_3. The block $b(PP_2)$ is a $(t+1)$-dimensional subspace of $PG((s+1)(t+1),q)$ meeting $PG((s+1)(t+1)-1,q)$ in an element W_3 of $\mathcal{W}$. W_3 is distinct from W_1 since $b(PP_2)$ and $b(P_1P_2)$ are not parallel blocks, and W_3 is distinct from W_2 since $b(PP_2)$ and $b(P_1P_3)$ are not parallel blocks. Now $b(PP_2)$ and $b(P_3P_4)$ meet in exactly a point if and only if as $(t+1)$-dimensional subspaces they span a $(2t+2)$-dimensional subspace of $PG((s+1)(t+1),q)$. This happens if and only if the spaces W_1, W_2 and W_3 all lie in a subspace of $PG((s+1)(t+1)-1,q)$ of dimension $2t+1$.

To see this, first suppose that W_1, W_2 and W_3 lie in a $(2t+1)$-dimensional space, then $b(PP_2)$ and $b(P_3P_4)$ lie in a space of dimension one greater as they contain points of $PG((s+1)(t+1),q) - PG((s+1)(t+1)-1,q)$. Conversely if $b(PP_2)$ and $b(P_3P_4)$ span a space of dimension $2t+2$ then they meet the hyperplane $PG((s+1)(t+1)-1,q)$ in a space of dimension $2t+1$. This space contains the t-spread elements W_1 and W_3 (since they are contained in $b(PP_2)$ and $b(P_3P_4)$) and the point of intersection of $PG((s+1)(t+1)-1,q)$ with the line joining the points P and P_3, which lies in W_2. As P may vary over the block $b(P_1P_3)$, and the point of intersection of the line PP_3 must remain in W_2 and in the space spanned by W_1 and W_3 we conclude that W_2 must lie in $< W_1, W_3 >$. The requirement that W_2 must lie in $< W_1, W_3 >$ as W_1, W_3 and W_2 vary over the elements of the t-spread is exactly the condition that $\mathcal{W}$ is geometric.

(iv) This is satisfied vacuously since every block has more than two points.

(v) Every block has exactly $q^{(t+1)}$ points.

(vi) Let W_1 and W_2 be distinct elements of the t-spread $\mathcal{W}$. Let b be a block of $\mathcal{A}$ passing through W_1. There is a block of $\mathcal{A}$ passing through W_2 and skew to b, since b and W_2 span a subspace of $PG((s+1)(t+1),q)$ of dimension $2t+2$, and joining W_2 to any point not in this subspace gives a $(t+1)$-dimensional subspace of $PG((s+1)(t+1),q)$ skew to b.

The above arguments show the following.

3.3.2 Theorem

Let $\mathcal{W}$ be a geometric t-spread of $PG((s+1)(t+1)-1,q)$. Then the incidence structure $\mathcal{I}$ constructed as above is an affine space $AG^*(s+1,q^{t+1})$ of dimension $s+1$ and order q^{t+1} which may be completed to a projective (s+1)-dimensional space $PG^*(s+1,q^{t+1})$.

Proof: Applying Theorem 3.3.1 we see that the incidence structure is indeed an affine space of order q^{t+1} and by comparing the number of points with the number of points of an affine space we see that the space has dimension $s+1$. We complete the affine space to a projective space by adjoining in a special way the space $PG((s+1)(t+1)-1,q)$ as the s-dimensional space at infinity of order q^{t+1} and the elements of the t-spread $\mathcal{W}$ as the points at infinity, one per parallel class. We see then that the s-dimensional space of order q^{t+1} at infinity has the correct number ω of points. The $(i(t+1)-1)$-dimensional spaces joining these are the i-dimensional subspaces of the s-dimensional projective space at infinity, as in Theorem 1.2.4. □

It is known (see Beutelspacher (1980) and Segre (1964)) that a geometric t-spread $\mathcal{W}$ of $PG(d,q)$ for $d > 2t+1$ induces a regular t-spread on the space $< V, V' >$ for any two distinct elements V and V' of $\mathcal{W}$. This result can be recovered as a Corollary of the preceding Theorem 3.3.2.

3.3.3 Corollary

Let $\mathcal{W} = \{W_1, \ldots, W_\omega\}$, $\omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \ldots + q^{t+1} + 1$, be a geometric t-spread of $PG((s+1)(t+1)-1,q)$. Then for any pair W_i, W_j of elements of $\mathcal{W}$, the t-spread induced on the $(2t+1)$-dimensional space $< W_i, W_j >$ is regular.

Proof: Since $\mathcal{W}$ is geometric, it induces a t-spread on any $(2t+1)$-dimensional space $< W_i, W_j >$. Let $AG^*(s+1, q^{t+1})$ be the affine space constructed from $\mathcal{W}$ as above. The hyperplane at infinity of $AG^*(s+1, q^{t+1})$ is the incidence structure $\mathcal{I}$ constructed on the space $PG((s+1)(t+1)-1,q)$, where the points are the elements of $\mathcal{W}$ and the subspaces are the joins of these points. Thus the lines at infinity are the t-spreads induced by $\mathcal{W}$ on the $(2t+1)$-dimensional subspaces of $PG((s+1)(t+1)-1,q)$ of the type $< W_i, W_j >$, $W_i \neq W_j$, and a plane of $AG^*(s+1, q^{t+1})$ meets the hyperplane at infinity in a line at infinity. A plane of $AG^*(s+1, q^{t+1})$ arises from a t-spread induced by $\mathcal{W}$ on a $(2t+1)$-dimensional subspace $< W_i, W_j >$ of $PG((s+1)(t+1)-1, q)$ by the construction described in Section 2.3. Now $AG^*(s+1, q^{t+1})$ is Desarguesian for $s \geq 2$ so any subplane of it is Desarguesian, and thus by Section 2.7, the t-spread induced by $\mathcal{W}$ on $< W_i, W_j >$ is regular. □

3.4 t-REGULI OF RANK r AND REGULAR t-SPREADS

Since a t-regulus of a $(2t+1)$-dimensional projective space is just the set of all t-dimensional subspaces of a Segre variety $\mathcal{SV}_{2,t+1}$ in $PG(2t+1, q)$, it seems natural to generalise this to a "higher dimensional regulus" of $PG((s+1)(t+1)-1, q)$ using the Segre variety $\mathcal{SV}_{s+1,t+1}$ in $PG((s+1)(t+1)-1, q)$.

3.4.1 Definition

For $0 \leq r \leq s$, let $\Gamma^0, \Gamma^1, \ldots, \Gamma^t$ be $t+1$ pairwise skew r-dimensional subspaces spanning a projective space $PG((r+1)(t+1)-1, q)$, and suppose there exist t projective correspondences relating Γ^0 to each of $\Gamma^1, \Gamma^2, \ldots, \Gamma^t$. The set of t-dimensional subspaces of $PG((r+1)(t+1)-1, q)$ joining a point P of Γ^0 to the corresponding points $P^1, P^2, \ldots, P^t$ of $\Gamma^1, \Gamma^2, \ldots, \Gamma^t$ is called a *t-regulus of rank r*, and is denoted by $\mathcal{R}_r$.

The space $PG((r+1)(t+1)-1, q)$ may be a subspace of a projective space $PG(n, q)$. In this case we say that $\mathcal{R}_r$ is a t-regulus of rank r of $PG(n, q)$, but we understand that $\mathcal{R}_r$ lies in a $((r+1)(t+1)-1)$-dimensional subspace of $PG(n, q)$.

3.4.2 Examples

(1) A 1-regulus of rank 0 is just a line in $PG(n, q)$.

(2) A 1-regulus of rank 1 is a regulus of lines of $PG(3, q)$, normally defined as the set of $q+1$ lines of $PG(3, q)$ forming one system of generators of a quadric

surface.

(3) A t-regulus of rank 1 is a t-regulus of $PG(2t+1,q)$ as in Definition 2.4.1.

3.4.3 Theorem

A t-regulus of rank r in $PG(n,q)$ is the set of t-dimensional subspaces of a Segre variety $\mathcal{SV}_{r+1,t+1}$ in some subspace $PG((r+1)(t+1)-1,q)$ of $PG(n,q)$ and conversely.

Proof: Let $\mathcal{R}_r$ be a t-regulus of rank r which is contained in a subspace $PG((r+1)(t+1)-1,q)$ of $PG((s+1)(t+1)-1,q)$. There exist $t+1$ pairwise skew r-dimensional spaces $\Gamma^0,\Gamma^1,\ldots,\Gamma^t$ as in Definition 3.4.1. The elements of $\mathcal{R}_r$ are the t-dimensional subspaces of $PG(n,q)$ joining corresponding points of $\Gamma^0,\Gamma^1,\ldots,\Gamma^t$ under projective correspondences between Γ^0 and each of $\Gamma^1,\Gamma^2,\ldots,\Gamma^t$ respectively. These lie in $PG((r+1)(t+1)-1,q)$. By Lemma 1.3.4 (1) the set $\mathcal{R}_r$ is the set of t-dimensional subspaces of a Segre variety $\mathcal{SV}_{r+1,t+1}$ in $PG((r+1)(t+1)-1,q)$. Conversely, let $\mathcal{R}_r$ be the set of t-dimensional spaces of a Segre variety $\mathcal{SV}_{r+1,t+1}$ in an $((r+1)(t+1)-1)$-dimensional subspace $PG((r+1)(t+1)-1,q)$ of $PG(n,q)$. Then any $t+1$ of the r-dimensional spaces of $\mathcal{SV}_{r+1,t+1}$ spanning $PG((r+1)(t+1)-1,q)$ may be chosen as $\Gamma^0,\Gamma^1,\ldots,\Gamma^t$ by Lemmas 1.3.1 (1) and 1.3.4 (1). □

3.4.4 Corollary

(1) A t-regulus of rank r $\mathcal{R}_r$ has $q^r+q^{r-1}+\cdots+q+1$ elements.

(2) There is a unique t-regulus of rank r through any $r+2$ t-dimensional subspaces in $PG((r+1)(t+1)-1,q)$, no $r+1$ of which lie in a hyperplane.

(3) A t-regulus $\mathcal{R}_r$ of rank r has $q^t+q^{t-1}+\cdots+q+1$ transversal r-dimensional

spaces, that is, r-dimensional spaces which meet every element of $\mathcal{R}_r$ in a unique point.

Proof: (1) By Theorem 3.4.3, $\mathcal{R}_r$ is the set of t-dimensional spaces on a Segre variety $\mathcal{SV}_{r+1,t+1}$ of $PG((r+1)(t+1)-1,q)$, which number $q^r+q^{r-1}+\cdots+q+1$, by Lemma 1.3.1 (1).

(2) By Lemma 1.3.4 (2), there is a unique Segre variety $\mathcal{SV}_{r+1,t+1}$ containing $r+2$ t-dimensional subspaces of $PG((r+1)(t+1)-1,q)$, no $r+1$ in a hyperplane. The t-dimensional spaces of this $\mathcal{SV}_{r+1,t+1}$ form the unique t-regulus of rank r containing these $r+2$ spaces.

(3) By Theorem 3.4.3, $\mathcal{R}_r$ is the set of t-dimensional spaces of a Segre variety $\mathcal{SV}_{r+1,t+1}$ in $PG((r+1)(t+1)-1,q)$. By Lemma 1.3.1 (1) and (2), $\mathcal{SV}_{r+1,t+1}$ has $q^t+q^{t-1}+\cdots+q+1$ r-dimensional spaces, which are transversal r-dimensional spaces of $\mathcal{R}_r$. □

3.4.5 Lemma

A t-regulus of rank r admits t-subreguli of ranks $r-1, r-2, \ldots, 1, 0$. The number of t-subreguli of rank k for $0 \leq k \leq r$ in a t-regulus of rank r is just the number of k-dimensional subspaces of an r-dimensional projective space.

Proof: A t-regulus of rank r is the set of t-dimensional spaces of a Segre variety $\mathcal{SV}_{r+1,t+1}$ in $PG((r+1)(t+1)-1,q)$. As in Lemma 1.3.5, this variety admits Segre subvarieties $\mathcal{SV}_{k+1,t+1}$ for each value of k with $0 \leq k \leq r$. The set of t-dimensional spaces on such a Segre subvariety is then a t-subregulus of rank k of the t-regulus of rank r. This is because the t-dimensional spaces of $\mathcal{SV}_{k+1,t+1}$ are all t-dimensional spaces of $\mathcal{SV}_{r+1,t+1}$, again by Lemma 1.3.5. As in the proof of that result, a Segre subvariety $\mathcal{SV}_{k+1,t+1}$ of the variety $\mathcal{SV}_{r+1,t+1}$ is determined by a k-dimensional subspace of one of the r-dimensional spaces of $\mathcal{SV}_{r+1,t+1}$, so the second statement of the Lemma holds. □

3.4.6 Lemma

Let $\mathcal{R}_r$ be a t-regulus of rank r in $PG((r+1)(t+1)-1,q)$. Two t-subreguli $\mathcal{R}_k$ and $\mathcal{R}_m$ of $\mathcal{R}_r$ (of ranks say k and m respectively) are either disjoint or intersect in a t-subregulus of $\mathcal{R}_r$ (which is also a t-subregulus of $\mathcal{R}_k$ and of $\mathcal{R}_m$) of rank less than or equal to the smaller of the two ranks k and m.

Proof: The t-subreguli of ranks k and m are defined by k- and m-dimensional subspaces of one of the r-dimensional spaces of $\mathcal{R}_r$ as a Segre variety (see Lemma 1.3.5). These meet in a subspace of dimension less than or equal to the smaller of k and m, and this subspace of intersection determines a Segre subvariety which is a t-subregulus of $\mathcal{R}_r$ and of both the t-subreguli of ranks k and m. □

As we now have a definition for a t-regulus of rank r, we can use it to introduce the idea of different sorts of regularity of a t-spread corresponding to the different sorts of t-regulus which it may contain.

3.4.7 Definition

A t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1,q)$ is *t-regular of rank r* for $0 \leq r \leq s$ if whenever S_r is an r-dimensional subspace of $PG((s+1)(t+1)-1,q)$ not meeting any element of $\mathcal{W}$ in more than one point, then the $q^r + q^{r-1} + \cdots + q + 1$ t-dimensional spaces of $\mathcal{W}$ meeting it form a t-regulus of rank r. If there is no confusion then we say that $\mathcal{W}$ is *regular of rank r*. In particular, the $q^r + q^{r-1} + \cdots + q + 1$ lines in the t-regulus of rank r lie in an $((r+1)(t+1)-1)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$.

3.4.8 Examples

(0) Every t-spread of $PG((s+1)(t+1)-1,q)$ is regular of rank 0, since given

any 0-dimensional subspace of $PG((s+1)(t+1)-1,q)$, which is just a point, there is a unique element of the t-spread through it, and this t-dimensional space is a t-regulus of rank 0.

(1) In $PG(2t+1,q)$, a t-spread is regular of rank 1 if and only if it is regular in the usual sense of the word, see Theorem 2.4.5.

(2) In Ebert (1983), a 1-spread $\mathcal{W}$ of $PG(2s+1,q)$ is called regular if for any line l of $PG(2s+1,q)$ not contained in $\mathcal{W}$, the $q+1$ lines of $\mathcal{W}$ meeting l form a regulus. This is precisely the condition that the 1-spread is regular of rank 1. Thus regularity of rank 1 coincides with the usual notion of regularity for 1-spreads of $PG(2s+1,q)$.

The only existing definitions of regularity of t-spreads of $PG((s+1)(t+1)-1,q)$ known to the author are in the cases $t=1$ with general s, and $s=1$ for general t. These are discussed in Section 2.4 and in the Examples above. The new definition of regularity of different ranks given in Definition 3.4.7 does not contradict any of the previous definitions, but refines and generalises the idea of regularity.

3.4.9 Lemma

For some integer r with $1 \leq r \leq s-1$, let S_r be an r-dimensional subspace of $PG((s+1)(t+1)-1,q)$. A set $\mathcal{R}_r$ of $q^r+q^{r-1}+\cdots+q+1$ t-dimensional subspaces of $PG((s+1)(t+1)-1,q)$, each meeting S_r in a unique point, is a t-regulus of rank r if and only if for each line l of S_r, the set of t-dimensional spaces of $\mathcal{R}_r$ meeting l is a t-regulus of rank 1.

Proof: Suppose first that $\mathcal{R}_r$ is a t-regulus of rank r. Then it is the set of t-dimensional spaces of a Segre variety $\mathcal{SV}_{r+1,t+1}$ contained in a given subspace $PG((r+1)(t+1)-1,q)$ of $PG((s+1)(t+1)-1,q)$. Now S_r is one of the r-

dimensional spaces of $\mathcal{SV}_{r+1,t+1}$, by Lemma 1.3.4 (2). Then by Lemma 1.3.5, the set of t-dimensional spaces of $\mathcal{SV}_{r+1,t+1}$ meeting a line l of S_r is a Segre subvariety $\mathcal{SV}_{2,t+1}$ contained in a $(2t+1)$-dimensional subspace of $PG((r+1)(t+1)-1,q)$, that is, a t-regulus of rank 1.

Conversely, suppose the set of $q+1$ t-dimensional spaces of $\mathcal{R}_r$ meeting any line of S_r form a t-regulus of rank 1, so that they are the t-dimensional spaces of a Segre variety $\mathcal{SV}_{2,t+1}$. Let P be a point of S_r and let $l_1, l_2, \ldots, l_r$ be lines of S_r through the point P, such that $l_1, l_2, \ldots, l_r$ span S_r. Let S_t be the unique t-dimensional space of $\mathcal{R}_r$ passing through P. For any i with $1 \leq i \leq r$, consider the t-regulus of rank 1, denoted by $\mathcal{R}_1^i$, comprising the t-dimensional spaces of $\mathcal{R}_r$ meeting l_i. This is the set of t-dimensional spaces of a Segre variety $\mathcal{SV}_{2,t+1}^i$ which contains S_t and has l_i as one of its lines. Any two such varieties, for distinct i and j say, have no common point apart from the points of S_t, as the $\mathcal{R}_1^i$ meet only in the points of S_t. Now two such Segre varieties $\mathcal{SV}_{2,t+1}^i$ and $\mathcal{SV}_{2,t+1}^j$ lie in $(2t+1)$-dimensional subspaces of $PG((s+1)(t+1)-1,q)$ which meet in exactly the t-dimensional space S_t. This is because if they meet in more than just S_t, then they meet in a $(t+1)$-dimensional space through S_t, and such a space through a t-dimensional space of $\mathcal{SV}_{2,t+1}^i$, which is a ruled quadric in $PG(2t+1,q)$, meets it in points outside S_t, and similarly meets $\mathcal{SV}_{2,t+1}^j$ in points outside S_t. So $\mathcal{SV}_{2,t+1}^i$ and $\mathcal{SV}_{2,t+1}^j$ would have common points outside S_t. Through any point Q of S_t there passes a line l_Q^i of the variety $\mathcal{SV}_{2,t+1}^i$, and the set of r such lines through Q span a space of dimension r as each is contained in a $(2t+1)$-dimensional space (as above) and the set of all such $(2t+1)$-dimensional spaces spans a $((r+1)(t+1)-1)$-dimensional space $PG((r+1)(t+1)-1,q)$. As Q varies among the points of S_t, we obtain $q^t+q^{t-1}+\cdots+q+1$ r-dimensional subspaces of $PG((r+1)(t+1)-1,q)$, each meeting $r(q^t+q^{t-1}+\cdots+q)+1$ elements of

$\mathcal{R}_r$. In fact each such r-dimensional space meets every element of $\mathcal{R}_r$ by repeating the above argument choosing another convenient point as P. These r-dimensional spaces define a Segre variety $\mathcal{SV}_{r+1,t+1}$ with $\mathcal{R}_r$ as the set of t-dimensional spaces, and hence $\mathcal{R}_r$ is a regulus of rank r. □

3.4.10 Theorem

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1,q)$ which is regular of rank r, for some r with $1 \leq r \leq s$. Then $\mathcal{W}$ is regular of each rank $r-1, r-2, \ldots, 1, 0$, and it is also regular of each rank $r+1, r+2, \ldots, s$.

Proof: For some value of k with $1 \leq k \leq r-1$, let S_{r-k} be an $(r-k)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$, not meeting any element of $\mathcal{W}$ in more than one point. This lies in an r-dimensional subspace $\mathcal{S}_r$ of the space $PG((s+1)(t+1)-1,q)$ not meeting any element of $\mathcal{W}$ in more than one point. The $q^r + q^{r-1} + \cdots + q + 1$ t-dimensional spaces of $\mathcal{W}$ meeting $\mathcal{S}_r$ are a t-regulus of rank r by assumption, and the $q^{r-k} + q^{r-k-1} + \cdots + q + 1$ t-dimensional spaces of $\mathcal{W}$ meeting S_{r-k} are a subregulus of rank $r-k$ by Lemma 1.3.5. This shows that $\mathcal{W}$ is regular of each rank $r-1, r-2, \ldots, 1$ and it is regular of rank 0 since every t-spread is regular of rank 0 (Examples 3.4.8 (0)).

Now since $\mathcal{W}$ is regular of rank r for some r with $1 \leq r \leq s-1$, then by the first part of the Theorem $\mathcal{W}$ is regular of rank 1. Let S_{r+k}, for some $1 \leq k \leq s-r$, be an $(r+k)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$, not meeting any element of $\mathcal{W}$ in more than one point. There is exactly one element of $\mathcal{W}$ through each point of $\mathcal{S}_{r+k}$, and since $\mathcal{W}$ is regular of rank 1, the set of $q+1$ t-dimensional spaces of $\mathcal{W}$ meeting any line of S_{r+k} is a t-regulus of rank 1. By Lemma 3.4.9, the set of t-dimensional spaces of $\mathcal{W}$ meeting S_{r+k} is a t-regulus of rank $(r+k)$ and so $\mathcal{W}$ is regular of rank $r+k$. This shows that $\mathcal{W}$ is regular of each rank

$r+1, r+2, \ldots, s$. $\square$

3.4.11 Definition

A t-spread $\mathcal{W}$ in $PG((s+1)(t+1)-1,q)$ is called *regular* if it is regular of rank r for some r with $1 \leq r \leq s$, then it is necessarily regular of each rank $0,1,\ldots,s$.

3.4.12 Theorem

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1,q)$. Then $\mathcal{W}$ is geometric if and only if it is regular.

Proof: First, let $\mathcal{W} = \{W_1, W_2, \ldots, W_\omega\}$ be a geometric t-spread of the space $PG((s+1)(t+1)-1,q)$. Then $\mathcal{W}$ induces a regular t-spread on any $(2t+1)$-dimensional subspace $< W_i, W_j >$ for distinct elements W_i and W_j of $\mathcal{W}$ (see Corollary 3.3.3 or Segre (1964)). Let l be a line of $PG((s+1)(t+1)-1,q)$, not contained in any element of $\mathcal{W}$. Without loss of generality, suppose l meets the elements $\{W_1, W_2, \ldots, W_{q+1}\}$ of $\mathcal{W}$. Then l is contained in $< W_1, W_2 >$ and since $W_3, W_4, \ldots, W_{q+1}$ all have a point in common with $< W_1, W_2 >$ (which is their point of intersection with l) then $W_3, W_4, \ldots, W_{q+1}$ are all contained in $< W_1, W_2 >$ as $\mathcal{W}$ is geometric (see Definition 1.2.3). As the t-spread induced on $< W_1, W_2 >$ is regular, the set of spaces $W_1, W_2, \ldots, W_{q+1}$ form a t-regulus which is a t-regulus of rank 1. Thus $\mathcal{W}$ is regular of rank 1 and by Theorem 3.4.10, $\mathcal{W}$ is regular.

Conversely suppose that $\mathcal{W}$ is a regular t-spread, then it is regular of rank 1 by Definition 3.4.11. Choose $W_i, W_j \in \mathcal{W}$, with $W_i \neq W_j$, and consider the $(2t+1)$-dimensional space $< W_i, W_j >$. Any line l of $< W_i, W_j >$ meets $q+1$ elements of the t-spread $\mathcal{W}$, which form a t-regulus of rank 1 in some $(2t+1)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$. Thus if l meets both W_i and W_j then the elements of the t-regulus of rank 1 defined by l all lie in $< W_i, W_j >$, since it has dimension $(2t+1)$. Now let P be any point of $< W_i, W_j >$, and suppose that $P \in W_k$, where $k \neq i, j$. There is a line l of $PG((s+1)(t+1)-1, q)$ through P which meets both W_i and W_j. This is because the space $< W_i, P >$ is contained in $< W_i, W_j >$ and has dimension $t+1$. Thus it meets W_j in a point say Q, and the line $l = PQ$ passes through P and meets both W_i and W_j. In this way we can see that every point $P \in < W_i, W_j >$ lies on some element W_k of the t-spread $\mathcal{W}$, and this element must be contained in $< W_i, W_j >$, and $\mathcal{W}$ is geometric. $\square$

3.5 CONSTRUCTION OF A PROJECTIVE t-SPREAD SET

This construction is different from that given in Section 3.2. A t-spread is now shown to correspond to a set of $(s+1)$-tuples of matrices, or linear transformations. The t-spread set, for t-spreads where it exists, can be obtained from this projective t-spread set by a "non-homogenising" procedure.

Again we use the space $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ introduced by Thas (1971), generalising the results found in Section 2.5.

Under the bijection f given in Theorem 1.6.2, points of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ correspond to $(n-1)$-dimensional subspaces of $PG((m+1)n-1, q)$. Thus to use this space to analyse t-spreads of $PG((s+1)(t+1)-1, q)$, we need to put $n = t+1$ and $m = s$.

3.5.1 The Construction

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$ and let $\omega = |\mathcal{W}|$. Then $\mathcal{W}$ corresponds under the bijection f^{-1}, where f is as in Theorem 1.6.2, to a set $\mathcal{P}$ of ω points of $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$, each pair of which is in clear position. The elements of the t-spread $\mathcal{W}$ are represented by points of $\mathcal{P}$

$$P_i = (\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)})^T,$$

for $i = 0, 1, \ldots, \omega - 1$, where each submatrix $\xi_k^{(i)}$ is $(t+1) \times (t+1)$ and

$$\operatorname{rank}\begin{pmatrix} \xi_0^{(i)} \\ \xi_1^{(i)} \\ \vdots \\ \xi_s^{(i)} \end{pmatrix} = t + 1.$$

The property that every pair of points in $\mathcal{P}$ is in clear position means that if $P_i = (\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)})^T$ and $P_j = (\xi_0^{(j)}, \xi_1^{(j)}, \ldots, \xi_s^{(j)})^T$ are distinct points of $\mathcal{P}$ then

$$\operatorname{rank}\begin{pmatrix} \xi_0^{(i)} & \xi_0^{(j)} \\ \xi_1^{(i)} & \xi_1^{(j)} \\ \vdots & \vdots \\ \xi_s^{(i)} & \xi_s^{(j)} \end{pmatrix} = 2(t+1)$$

If we can choose a set of $s+2$ points of $\mathcal{P}$ such that any $s+1$ of them are in clear position, then Theorem 1.6.3 allows us without loss of generality to suppose that

$$P_0 = (I, 0, \ldots, 0)^T, P_1 = (0, I, 0, \ldots, 0)^T, \ldots, P_{s+1} = (0, \ldots, 0, I)^T$$

and

$$P_{s+2} = (I, I, \ldots, I)^T$$

where each submatrix is $(t+1) \times (t+1)$. Recalling that under the bijection f, a t-spread element W_i is the space spanned by the columns of the coordinate matrix

$$P_i = \begin{pmatrix} \xi_0^{(i)} \\ \xi_1^{(i)} \\ \vdots \\ \xi_s^{(i)} \end{pmatrix},$$

we see that this process is equivalent to choosing a basis $\{e_1, e_2, \ldots, e_{(s+1)(t+1)}\}$ for $PG(((s+1)(t+1)-1, q)$ such that the t-spread $\mathcal{W}$ contains the t-dimensional spaces

$$\begin{aligned}
W_0 &= \operatorname{lin}\{e_1, e_2, \ldots, e_{t+1}\}, \\
W_1 &= \operatorname{lin}\{e_{t+2}, e_{t+3}, \ldots, e_{2(t+1)}\}, \\
&\vdots \\
W_{s+1} &= \operatorname{lin}\{e_{s(t+1)+1}, e_{s(t+1)+2}, \ldots, e_{(s+1)(t+1)}\} \quad \text{and} \\
W_{s+2} &= \operatorname{lin}\{e_1 + e_{t+2} + \cdots + e_{s(t+1)+1}, e_2 + e_{t+3} + \cdots + e_{s(t+1)+2}, \ldots\ldots, \\
&\qquad e_{t+1} + e_{2(t+1)} + \cdots + e_{(s+1)(t+1)}\}.
\end{aligned}$$

This is possible since if we choose the bases for the spaces $W_0, W_1, \ldots, W_s$ as above then Lemma 1.1.2 ensures that there is a suitable basis for W_{s+1} so that W_{s+2} has the required basis.

These considerations prompt the following Definition.

3.5.2 Definition

A *projective t-spread set* is a set $\mathcal{PC}$ of $(s+1)$-tuples of $(t+1) \times (t+1)$ matrices such that

(i) $\mathcal{PC}$ has $\omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \cdots + q^{t+1} + 1$ elements,

(ii) If $P_i = (\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)})$ is an element of $\mathcal{PC}$, then

$$\operatorname{rank} \begin{pmatrix} \xi_0^{(i)} \\ \xi_1^{(i)} \\ \vdots \\ \xi_s^{(i)} \end{pmatrix} = t+1$$

(iii) If $P_i = (\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)})$ and $P_j = (\xi_0^{(j)}, \xi_1^{(j)}, \ldots, \xi_s^{(j)})$ are distinct elements of $\mathcal{PC}$ then

$$\operatorname{rank}\begin{pmatrix} \xi_0^{(i)} & \xi_0^{(j)} \\ \xi_1^{(i)} & \xi_1^{(j)} \\ \vdots & \vdots \\ \xi_s^{(i)} & \xi_s^{(j)} \end{pmatrix} = 2(t+1).$$

A projective t-spread set is said to be *normalised* if it satisfies the additional property,

(iv) $\mathcal{PC}$ contains the $(s+1)$-tuples

$$(I, 0, \ldots, 0),\ (0, I, 0, \ldots, 0), \ldots\ (0, \ldots, 0, I),\ \text{and } (I, I, \ldots, I).$$

3.5.3 Remark

A geometric t-spread has a projective t-spread set which can be normalised, since the extra requirement is satisfied. In a normalised projective t-spread set, condition (ii) is implied by (iii) and (iv).

The above arguments have shown the following:

3.5.4 Theorem

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$. Then there exists a projective t-spread set $\mathcal{PC} = \{P_i : i = 1, 2, \ldots, \omega\}$. If we consider the elements of $\mathcal{PC}$ as points of $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$ then

$$\mathcal{W} = \{f(P_i) : \ i = 1, 2, \ldots, \omega\}$$

where f is the bijection of Theorem 1.6.2.

Proof: Let $\mathcal{W} = \{W_i : \ i = 1, 2, \ldots, \omega\}$. By the Construction 3.5.1, the set $\mathcal{PC} = \{f^{-1}(W_i) : \ i = 1, 2, \ldots, \omega\}$ is a set of ω points of $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$, every pair of which is in clear position. We therefore have constructed a set of ω

$(s+1)$-tuples of $(t+1)\times(t+1)$ matrices satisfying (ii) and (iii) of Definition 3.5.2. $\mathcal{PC}$ is therefore a projective t-spread set. Since

$$\mathcal{PC} = \{f^{-1}(W_i) : \ i = 1,2,\ldots,\omega\}$$

and f is a bijection, we have that

$$\mathcal{W} = \{f(P_i) : \ i = 1,2,\ldots,\omega\}.$$

□

3.5.5 Theorem (The converse)

Let $\mathcal{PC}$ be a projective t-spread set. Then there exists a t-spread $\mathcal{W}$ of the space $PG((s+1)(t+1)-1,q)$ such that

$$\mathcal{PC} = \{f^{-1}(W_i) : \ W_i \in \mathcal{W}\}$$

where f is the bijection of Theorem 1.6.2.

Proof: Let $\mathcal{PC} = \{(\xi_0^{(i)},\xi_1^{(i)},\ldots,\xi_s^{(i)}) : i = 1,2,\ldots,\omega\}$ be a projective t-spread set, that is a set of $(s+1)$-tuples of $(t+1)\times(t+1)$ matrices over $GF(q)$, satisfying the three properties (i), (ii) and (iii) of Definition 3.5.2. The elements of $\mathcal{PC}$ may be regarded as points of the space $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$ every pair of which are in clear position. These correspond under the bijection f of Theorem 1.6.2 to a set $\mathcal{W}$ of ω pairwise skew t-dimensional subspaces of $PG((s+1)(t+1)-1,q)$. Thus $\mathcal{W}$ is a t-spread of $PG((s+1)(t+1)-1,q)$ and

$$\mathcal{PC} = \{f^{-1}(W_i) : \ W_i \in \mathcal{W}\}.$$

□

3.5.6 Theorem

Given a t-spread set, we can construct a (normalised) projective t-spread set and the t-spreads they define are isomorphic.

Proof: Let $\mathcal{C}$ be a t-spread set of matrices, so that it is a set of $q^{s(t+1)}$ s-tuples of matrices, $q^{(s-1)(t+1)}$ $(s-1)$-tuples of matrices, $\ldots$, $q^{2(t+1)}$ 2-tuples of matrices and q^{t+1} single matrices,

$$\mathcal{C} = \left\{(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) : \ i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, q^{i(t+1)}\right\}.$$

All the above matrices are $(t+1) \times (t+1)$ and for each $i = 1, 2, \ldots, s$ the following properties are satisfied:

(ii) the i-tuples $(0, 0, \ldots, 0)$ and $(I, I, \ldots, I)$ are in $\mathcal{C}$, and

(iii) if $(X_1, X_2, \ldots, X_i)$ and $(Y_1, Y_2, \ldots, Y_i)$ are distinct elements of $\mathcal{C}$ then $x = 0$ is the only common solution to the equations $(X_i - Y_i)x = 0$.

For each i-tuple of matrices in the set $\mathcal{C}$, we construct an $(s+1)$-tuple of $(t+1) \times (t+1)$ matrices whose first i entries are the matrices of the i-tuple in the same order. The next entry (the $(i+1)^{th}$ entry) is the identity matrix and the last $(s+1) - (i+1)$ entries are the zero matrix. To the set of such $(s+1)$-tuples, adjoin the $(s+1)$-tuple $(I, 0, \ldots, 0)$, to obtain

$$\mathcal{PC} =$$
$$\left\{P_{ij} = (C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0) : \ i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, q^{i(t+1)}\right\}$$
$$\cup \{P_\infty = (I, 0, \ldots, 0)\}.$$

The set $\mathcal{PC}$ so constructed is a normalised projective t-spread set, and we show this by checking that $\mathcal{PC}$ satisfies Definition 3.5.2. First, $\mathcal{PC}$ is a set of $\omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \cdots + q^{t+1} + 1$ $(s+1)$-tuples of $(t+1) \times (t+1)$ matrices which contains the elements $(I, 0, \ldots, 0), \ldots, (0, \ldots, 0, I)$ and $(I, I, \ldots, I)$. (This follows from the fact that the i-tuple all of whose entries are the zero matrix is in the set for every value of i and performing the above construction on such an i-tuple gives an $(s+1)$-tuple with an identity matrix in position $(i+1)$ and zero matrices elsewhere. The $(s+1)$-tuple all of whose entries are the identity matrix is constructed from the s-tuple all of whose entries are the identity matrix.) Thus (i) and (iv) are satisfied. Since each element of $\mathcal{PC}$ has the identity matrix as one of its elements, property (ii) is satisfied.

Now we must check the condition (iii) of Definition 3.5.2. We shall consider three cases. First, let

$$(X_1, X_2, \ldots, X_i, I, 0, \ldots, 0) \text{ and } (Y_1, Y_2, \ldots, Y_i, I, 0, \ldots, 0)$$

be two $(s+1)$-tuples in $\mathcal{PC}$. Then

$$\operatorname{rank}\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ \vdots & \vdots \\ X_i & Y_i \\ I & I \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} X_1 & Y_1 - X_1 \\ X_2 & Y_2 - X_2 \\ \vdots & \vdots \\ X_i & Y_i - X_i \\ I & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

and we must show that this matrix has rank equal to $2(t+1)$. Since it has $2(t+1)$ columns its rank is at most $2(t+1)$. Because the first $t+1$ columns are linearly independent, and by property (iii) above $x=0$ is the only common solution to the equations $(X_j - Y_j)x = 0$, we see that

$$\operatorname{rank}\begin{pmatrix} Y_1 - X_1 \\ Y_2 - X_2 \\ \vdots \\ Y_i - X_i \end{pmatrix} = t+1$$

and the result follows.

Now let $(X_1, X_2, \ldots, X_i, I, 0, \ldots, 0)$ and $(Y_1, Y_2, \ldots, Y_j, I, 0, \ldots, 0)$ be distinct $(s+1)$-tuples in $\mathcal{PC}$, and suppose without loss of generality that $i < j$. Then the matrix

$$\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ \vdots & \vdots \\ X_i & Y_i \\ I & Y_{i+1} \\ 0 & Y_{i+2} \\ \vdots & \vdots \\ 0 & Y_j \\ 0 & I \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

has rank equal to $2(t+1)$.

Finally let $(X_1, X_2, \ldots, X_i, I, 0, \ldots, 0)$ and $(I, 0, 0, \ldots, 0)$ be two elements of $\mathcal{PC}$. Then the matrix

$$\begin{pmatrix} X_1 & I \\ X_2 & 0 \\ \vdots & \vdots \\ X_i & 0 \\ I & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

has rank equal to $2(t+1)$.

We now show that the t-spreads defined by the t-spread set and the projective t-spread set are isomorphic. Let the set of vectors

$$\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}, a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}, \ldots\ldots, a_1^{(s+1)}, a_2^{(s+1)}, \ldots, a_{t+1}^{(s+1)}\right\}$$

be a basis for $\mathcal{V}_{(s+1)(t+1)}$, the vector space corresponding to $PG((s+1)(t+1)-1, q)$.

Let

$$A_1 = \mathrm{lin}\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}\right\}$$
$$A_2 = \mathrm{lin}\left\{a_1^{(2)}, a_2^{(2)}, \ldots, a_{t+1}^{(2)}\right\}$$
$$\vdots$$
$$A_{s+1} = \mathrm{lin}\left\{a_1^{(s+1)}, a_2^{(s+1)}, \ldots, a_{t+1}^{(s+1)}\right\}.$$

For each value of i with $i = 1, 2, \ldots, s+1$, define a (non-singular) linear transformation (i) from A_1 to A_i as follows:

$$\begin{aligned}(i) &: A_1 \to A_i \\ &: a_k^{(1)} \mapsto a_k^{(i)} \qquad \text{for } k = 1, 2, \ldots, t+1.\end{aligned}$$

The t-spread set

$$\mathcal{C} = \left\{(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) : \ i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, q^{i(t+1)}\right\}$$

defines the t-spread

$$\mathcal{W} = \left\{J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) : \ i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, q^{i(t+1)}\right\} \cup \{J(\infty)\}$$

where

$$J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) = \mathrm{lin}\left\{C_1^{(j)} a_k^{(1)} \oplus C_2^{(j)} a_k^{(2)} \oplus \cdots \oplus C_i^{(j)} a_k^{(i)} \oplus a_k^{(i+1)} : \right.$$
$$\left. k = 1, 2, \ldots, t+1\right\}$$

and

$$J(\infty) = A_1 = \mathrm{lin}\left\{a_1^{(1)}, a_2^{(1)}, \ldots, a_{t+1}^{(1)}\right\}.$$

The projective t-spread set

$$\mathcal{PC} = \left\{P_{ij} = (C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0) : \right.$$
$$\left. i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, q^{i(t+1)}\right\} \cup \{P_\infty\}$$

where $P_\infty = (I, 0, \ldots, 0)$ gives rise to the t-spread

$$\mathcal{W}' = \left\{f(P_{ij}) : i = 1, 2, \ldots, s, \quad j = 1, 2, \ldots, q^{i(t+1)}\right\} \cup \{f(P_\infty)\}$$

where f is the bijection of Theorem 1.6.2. The vectors $e_1, e_2, \ldots, e_{(s+1)(t+1)}$ form a basis for $\mathcal{V}_{(s+1)(t+1)}$, where e_i is the vector with a 1 in its ith position and zeros everywhere else. We shall express $\mathcal{V}_{(s+1)(t+1)}$ as the direct product of $s+1$ $(t+1)$-dimensional vector spaces as follows:

$$\mathcal{V}_{(s+1)(t+1)} = B_1 \oplus B_2 \oplus \cdots \oplus B_{s+1}$$

where

$$\begin{aligned}
B_1 &= \operatorname{lin}\{e_1, e_2, \ldots, e_{t+1}\} \\
B_2 &= \operatorname{lin}\left\{e_{t+2}, e_{t+3}, \ldots, e_{2(t+1)}\right\} \\
&\vdots \\
B_s &= \operatorname{lin}\left\{e_{(s-1)(t+1)+1}, e_{s(t+1)+2}, \ldots, e_{s(t+1)}\right\} \\
B_{s+1} &= \operatorname{lin}\left\{e_{s(t+1)+1}, e_{s(t+1)+2}, \ldots, e_{(s+1)(t+1)}\right\}.
\end{aligned}$$

Now we write each B_j as a $(t+1)$-dimensional vector space, so that for each $j = 1, 2, \ldots, s+1$, B_j has basis

$$\begin{aligned}
b_1^{(j)} &= (1, 0, \ldots, 0)^T \\
b_2^{(j)} &= (0, 1, 0, \ldots, 0)^T \\
&\vdots \\
b_{t+1}^{(j)} &= (0, 0, \ldots, 0, 1)^T.
\end{aligned}$$

For each $j = 1, 2, \ldots, s+1$, we define a (non-singular) linear transformation (j) as follows:

$$\begin{aligned}
(j) &: B_1 \to B_j \\
&: b_k^{(1)} \mapsto b_k^{(j)} \qquad \text{for } k = 1, 2, \ldots, t+1.
\end{aligned}$$

Now $f(P_\infty)$ is the $(t+1)$ dimensional subspace of $\mathcal{V}_{(s+1)(t+1)}$ spanned by the columns of

$$\begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The space spanned by the columns of this matrix is B_1. The space $f(P_{ij})$ is the space spanned by the columns of

$$\begin{pmatrix} C_1^{(j)} \\ C_2^{(j)} \\ \vdots \\ C_i^{(j)} \\ I \\ 0 \\ \vdots \\ 0. \end{pmatrix}$$

These columns are

$$c_{11} \oplus c_{21} \oplus \cdots \oplus c_{i1} \oplus b_1^{(i+1)},$$

$$c_{12} \oplus c_{22} \oplus \cdots \oplus c_{i2} \oplus b_2^{(i+1)},$$

$$\vdots$$

$$c_{1t+1} \oplus c_{2t+1} \oplus \cdots \oplus c_{it+1} \oplus b_{t+1}^{(i+1)}$$

where c_{kl} denotes the lth column of the matrix $C_k^{(j)}$ for $l = 1, 2, \ldots, t+1$ and $k = 1, 2, \ldots, i$. Considered as a point of the space B_k,

$$c_{kl} = C_k^{(j)} b_l^{(k)}$$

and we see that $f(P_{ij})$ is spanned by the vectors

$$C_1^{(j)} b_1^{(1)} \oplus C_2^{(j)} b_1^{(2)} \oplus \cdots \oplus C_i^{(j)} b_1^{(i)} \oplus b_1^{(i+1)},$$

$$C_1^{(j)} b_2^{(1)} \oplus C_2^{(j)} b_2^{(2)} \oplus \cdots \oplus C_i^{(j)} b_2^{(i)} \oplus b_2^{(i+1)},$$

$$\vdots$$

$$C_1^{(j)} b_{t+1}^{(1)} \oplus C_2^{(j)} b_{t+1}^{(2)} \oplus \cdots \oplus C_i^{(j)} b_{t+1}^{(i)} \oplus b_{t+1}^{(i+1)}$$

so that

$$f(P_{ij}) = \left\{ C_1^{(j)} b_k^{(1)} \oplus C_2^{(j)} b_k^{(2)} \oplus \cdots \oplus C_i^{(j)} b_k^{(i)} \oplus b_k^{(i+1)} : \text{ for } k = 1, 2, \ldots, t+1 \right\}.$$

Since the basis $\{a_k^{(j)} : k = 1, 2, \ldots, t+1, \quad j = 1, 2, \ldots, s+1\}$ was arbitrary, we choose

$$a_k^{(j)} = b_k^{(j)} \quad \text{for } j = 1, 2, \ldots, s+1 \text{ and } k = 1, 2, \ldots, s+1$$

so that

$$f(P_\infty) = B_1 = A_1$$

and for each $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, q^{i(t+1)}$

$$\begin{aligned} f(P_{ij}) &= \left\{ C_1^{(j)} b_k^{(1)} \oplus C_2^{(j)} b_k^{(2)} \oplus \cdots \oplus C_i^{(j)} b_k^{(i)} \oplus b_k^{(i+1)} : \text{ for } k = 1, 2, \ldots, t+1 \right\} \\ &= \left\{ C_1^{(j)} a_k^{(1)} \oplus C_2^{(j)} a_k^{(2)} \oplus \cdots \oplus C_i^{(j)} a_k^{(i)} \oplus a_k^{(i+1)} : \text{ for } k = 1, 2, \ldots, t+1 \right\} \\ &= J\left(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)} \right). \end{aligned}$$

Thus the t-spreads $\mathcal{W}$ and $\mathcal{W}'$ are isomorphic, that is, one may be mapped to the other by a homography of $PG((s+1)(t+1)-1, q)$. □

3.5.7 Theorem

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$ with projective t-spread set $\mathcal{PC}$. Suppose that $\mathcal{W}$ has the Shell property 3.2.7. Then we can normalise the projective t-spread set and hence construct a t-spread set which defines a t-spread isomorphic to $\mathcal{W}$.

Proof: Let $\mathcal{PC} = \{P_{ij} = (\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)}) : i = 1, 2, \ldots, \omega\}$ be a projective t-spread set. The elements of $\mathcal{PC}$ may be regarded as points of the space $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$ every pair of which are in clear position. Now $\mathcal{W}$ has the Shell property 3.2.7, so that there exists a partition of $PG((s+1)(t+1)-1, q)$ into "shells",

$$\begin{aligned} &PG(t, q), \\ &PG(2(t+1)-1, q) - PG(t, q), \\ &\vdots \\ &PG(s(t+1)-1, q) - PG((s-1)(t+1)-1, q), \\ &PG((s+1)(t+1)-1, q) - PG(s(t+1)-1, q) \end{aligned}$$

such that every element of $\mathcal{W}$ is contained in exactly one shell and has no point in any other shell. By the Shell property, there exist $s+2$ elements

$$W_\infty, W_1, \ldots, W_s, W_{s+1}$$

of $\mathcal{W}$ as follows:

$$\begin{aligned}
&W_\infty = PG(t,q),\\
&W_1 \in PG(2(t+1)-1,q) - PG(t,q),\\
&\vdots\\
&W_{s-1} \in PG(s(t+1)-1,q) - PG((s-1)(t+1)-1,q),\\
&W_s, W_{s+1} \in PG((s+1)(t+1)-1,q) - PG(s(t+1)-1,q)
\end{aligned}$$

where every $s+1$ of them span $PG((s+1)(t+1)-1,q)$. The elements of the projective t-spread set corresponding to these t-spread elements are the $s+2$ elements $P_\infty, P_1, P_2, \ldots, P_{s+1}$ respectively. When these are considered as points of $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$, the condition that each $s+1$ of the t-spread elements span $PG((s+1)(t+1)-1,q)$ means that every $s+1$ of the points $P_\infty, P_1, P_2, \ldots, P_{s+1}$ are in clear position. By Theorem 1.6.3 there exists a collineation, denoted by Ω, of $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$ such that

$$\begin{aligned}
&\Omega(P_\infty) = (I,0,\ldots,0),\\
&\Omega(P_1) = (0,I,0,\ldots,0),\\
&\vdots\\
&\Omega(P_s) = (0,\ldots,0,I),\\
&\Omega(P_{s+1}) = (I,I,\ldots,I).
\end{aligned}$$

This corresponds to applying a homography to $PG((s+1)(t+1)-1,q)$ such that in the new coordinate system

$$W_\infty = \mathrm{lin}\{e_1, e_2, \dots, e_{t+1}\},$$

$$W_1 = \mathrm{lin}\left\{e_{t+2}, e_{t+3}, \dots, e_{2(t+1)}\right\},$$

$$\vdots$$

$$W_s = \mathrm{lin}\left\{e_{s(t+1)+1}, e_{s(t+1)+2}, \dots, e_{(s+1)(t+1)}\right\} \quad \text{and}$$

$$W_{s+1} = \mathrm{lin}\left\{e_1 + e_{t+2} + \cdots + e_{s(t+1)+1}, e_2 + e_{t+3} + \cdots + e_{s(t+1)+2}, \dots\dots, \right.$$
$$\left. e_{t+1} + e_{2(t+1)} + \cdots + e_{(s+1)(t+1)}\right\}.$$

Now any element $W_1^{(j)}$ of $\mathcal{W}$ lying in $PG(2(t+1)-1, q) - PG(t, q)$, $j = 1, 2, \dots, q^{t+1}$ is in the space spanned by W_∞ and W_1, and so the corresponding element of the projective t-spread set is of the form

$$(\xi_0^{(j)}, \xi_1^{(j)}, 0, \dots, 0)$$

and since this is in clear position with the point $P_\infty = (I, 0, \dots, 0)$,

$$\mathrm{rank}\begin{pmatrix} \xi_0^{(j)} & I \\ \xi_1^{(j)} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} = 2(t+1)$$

and so $\det(\xi_1^{(j)})$ is not zero. This means that we may multiply the coordinates of the point P_{1j} by the non-singular matrix $(\xi_1^{(j)})^{-1}$ to obtain a new representation of the same point of $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$,

$$P_{1j} = (C_1^{(j)}, I, 0, \dots, 0).$$

Similarly, for each i with $i = 2, 3, \dots, s$ an element $W_i^{(j)}$ of $\mathcal{W}$ lying in the space $PG((i+1)(t+1)-1, q) - PG(i(t+1)-1, q)$, has corresponding element of the projective t-spread set

$$P_{ij} = (\xi_0^{(j)}, \xi_1^{(j)}, \dots, \xi_i^{(j)}, 0, \dots, 0).$$

This point of $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$ is in clear position with the space spanned by $P_\infty, P_1, \ldots, P_{i-1}$ and so

$$\operatorname{rank}\begin{pmatrix} \xi_0^{(j)} & I & 0 & 0 & \cdots & 0 \\ \xi_1^{(j)} & 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{i-1}^{(j)} & 0 & 0 & 0 & \cdots & I \\ \xi_i^{(j)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (i+1)(t+1).$$

The matrix $\xi_i^{(j)}$ is non-singular, and we may multiply each coordinate of the point of $\mathcal{S}_s(\mathcal{M}_{t+1}(GF(q)))$ by the inverse of $\xi_i^{(j)}$ to obtain a new representation of the same point:

$$P_{ij} = (C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0).$$

The projective t-spread set is therefore normalised to

$$\mathcal{PC} = \left\{(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0) : \ i = 1, 2, \ldots, s, \ \ j = 1, 2, \ldots, q^{i(t+1)}\right\}$$
$$\cup \{(I, 0, \ldots, 0,)\}$$

which contains the elements $P_\infty, P_1, P_2, \ldots, P_{s+1}$. We may construct the following set

$$\mathcal{C} = \left\{(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}) : \ i = 1, 2, \ldots, s, \ \ j = 1, 2, \ldots, q^{i(t+1)}\right\}.$$

Now $\mathcal{C}$ is a t-spread set, and to see this, we check the conditions of Definition 3.2.6. First note that $\mathcal{C}$ comprises $q^{s(t+1)}$ s-tuples, $q^{(s-1)(t+1)}$ $(s-1)$-tuples, ..., $q^{2(t+1)}$ 2-tuples and q^{t+1} single $(t+1) \times (t+1)$ matrices. We must check that for each $i = 1, 2, \ldots, s$ the i-tuples of matrices form an (i, t)-spread set according to Definition 3.2.3. The set $\mathcal{C}_i$ of i-tuples of matrices has $q^{i(t+1)}$ elements, and it contains the element $(0, 0, \ldots, 0)$ arising from P_{i-1}. If $(X_1, X_2, \ldots, X_i)$ and $(Y_1, Y_2, \ldots, Y_i)$ are distinct elements of $\mathcal{C}_i$ then $(X_1, X_2, \ldots, X_i, I, 0, \ldots, 0)$ and

$(Y_1, Y_2, \ldots, Y_i, I, 0, \ldots, 0)$ are distinct elements of $\mathcal{PC}$ and

$$\operatorname{rank}\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ \vdots & \vdots \\ X_i & Y_i \\ I & I \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} = 2(t+1).$$

This implies that $x = (0, 0, \ldots, 0)^T$ is the only common solution of the equations $(X_k - Y_k)x = 0$. Thus $\mathcal{C}_i$ is an (i,t)-spread set, so that $\mathcal{C}$ is a t-spread set and we still need to show that $\mathcal{PC}$ and $\mathcal{C}$ define isomorphic t-spreads. The projective t-spread $\mathcal{PC}$ is recovered from the t-spread set $\mathcal{C}$ in the manner described in the proof of Theorem 3.5.6, and Theorem 3.5.6 shows that the t-spreads defined by these two sets are isomorphic. □

3.5.8 Corollary

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$ and suppose that $\mathcal{W}$ has the Shell property 3.2.7. Then $\mathcal{W}$ has a projective t-spread set of the form

$$\mathcal{PC} = \left\{(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0) : \ i = 1, 2, \ldots, s, \ \ j = 1, 2, \ldots, q^{i(t+1)}\right\}$$
$$\cup \{(I, 0, \ldots, 0,)\}.$$

Proof: This appears in the proof of Theorem 3.5.7. □

3.6 COORDINATES FOR THE AFFINE SPACE $AG^*(s+1, q^{t+1})$

In this Section we provide coordinates for the affine space $AG^*(s+1, q^{t+1})$ which is constructed (as in Section 3.3) from a geometric t-spread of $PG((s+1)(t+1)-1, q)$. These coordinates are elements of the vector space corresponding to one of the t-spread elements, and are determined by the elements of the normalised t-spread set. In this way, the work of Section 2.6 for the case of $s = 1$ is extended to cover the case of t-spreads of $PG((s+1)(t+1)-1, q)$.

We use the notation of Sections 3.2 and 3.5. Let $\mathcal{W}$ be a geometric t-spread of $PG((s+1)(t+1)-1,q)$, and embed the space $PG((s+1)(t+1)-1,q)$ as a hyperplane in the projective space $PG((s+1)(t+1),q)$. As in Section 1.1 we represent $PG((s+1)(t+1)-1,q)$ as an $((s+1)(t+1))$-dimensional vector space $\mathcal{V}_{(s+1)(t+1)}$ over the field $GF(q)$, embedded as a hyperplane in the $((s+1)(t+1)+1)$-dimensional vector space $\mathcal{V}_{(s+1)(t+1)+1}$. Then $\mathcal{W}$ becomes a collection, still denoted $\mathcal{W}$, of $(t+1)$-dimensional vector subspaces of $\mathcal{V}_{(s+1)(t+1)}$ over $GF(q)$ pairwise having only the zero vector in common and satisfying the property that each non-zero vector of $\mathcal{V}_{(s+1)(t+1)}$ lies in exactly one element of $\mathcal{W}$.

As $\mathcal{W}$ is geometric, it has the Shell property 3.2.7, and so by Corollary 3.5.8, $\mathcal{W}$ has a projective t-spread set of matrices of the form

$$\begin{aligned}\mathcal{PC} = &\\ &\left\{P_{ij} = (C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0) : \; i = 1,2,\ldots,s, \; j = 1,2,\ldots,q^{i(t+1)}\right\}\\ &\qquad \cup \{P_\infty = (I, 0, \ldots, 0,)\}\end{aligned}$$

where

$$\begin{aligned}P_{11} &= (0, I, 0, \ldots, 0)\\ P_{21} &= (0, 0, I, 0, \ldots, 0)\\ &\vdots\\ P_{s1} &= (0, \ldots, 0, I)\\ P_{sq^{s(t+1)}} &= (I, I, \ldots, I).\end{aligned}$$

This means that

$$\begin{aligned}\mathcal{W} &= \left\{W_{ij} : \; i = 1,2,\ldots,s, \; j = 1,2,\ldots,q^{i(t+1)}\right\} \cup \{W_\infty\}\\ &= \left\{J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0) : \; i = 1,2,\ldots,s, \; j = 1,2,\ldots,q^{i(t+1)}\right\}\\ &\qquad \cup \{W_\infty = J(I, 0, \ldots, 0,)\}\end{aligned}$$

where in particular $\mathcal{W}$ contains the following $(t+1)$-dimensional subspaces of $\mathcal{V}_{(s+1)(t+1)}$:

$$
\begin{aligned}
W_\infty &= \operatorname{lin}(e_1, e_2, \ldots, e_{t+1}), \\
W_{11} &= \operatorname{lin}\left(e_{t+2}, e_{t+3}, \ldots, e_{2(t+1)}\right), \\
&\vdots \\
W_{s1} &= \operatorname{lin}\left(e_{s(t+1)+1}, e_{s(t+1)+2}, \ldots, e_{(s+1)(t+1)}\right) \quad \text{and} \\
W_{sq^s(t+1)} &= \operatorname{lin}\big(e_1 + e_{t+2} + \cdots + e_{s(t+1)+1}, e_2 + e_{t+3} + \cdots + e_{s(t+1)+2}, \ldots\ldots, \\
&\qquad e_{t+1} + e_{2(t+1)} + \cdots + e_{(s+1)(t+1)}\big).
\end{aligned}
$$

To the basis $\{e_1, e_2, \ldots, e_{(s+1)(t+1)}\}$ for $\mathcal{V}_{(s+1)(t+1)}$ we adjoin the element e^* of $\mathcal{V}_{(s+1)(t+1)+1} - \mathcal{V}_{(s+1)(t+1)}$ to obtain a basis for $\mathcal{V}_{(s+1)(t+1)+1}$.

For each $k = 2, 3, \ldots, s+1$, let (k) be the following (non-singular) linear transformation:

$$
\begin{aligned}
(k) &: W_\infty \to W_{(k-1)1} \\
&: e_j \mapsto e_{(k-1)(t+1)+j} \quad \text{for } j = 1, 2, \ldots, t+1.
\end{aligned}
$$

3.6.1 Coordinates for the points

By construction, each point of $AG^*(s+1, q^{t+1})$ is a 1-dimensional subspace of $\mathcal{V}_{(s+1)(t+1)+1}$. Write

$$\mathcal{V}_{(s+1)(t+1)+1} = W_\infty \oplus W_{11} \oplus W_{21} \oplus \cdots \oplus W_{s1} \oplus \{e^*\}$$

so that a point of $AG^*(s+1, q^{t+1})$ has a unique basis element of the form

$$x_1 \oplus x_2^{(2)} \oplus x_3^{(3)} \oplus \cdots \oplus x_{s+1}^{(s+1)} \oplus e^*$$

where

$$x_1 \in W_\infty,\ x_2^{(2)} \in W_{11},\ x_3^{(3)} \in W_{21}, \ldots, x_{s+1}^{(s+1)} \in W_{s1}$$

and therefore

$$x_1, x_2, \ldots, x_{s+1} \in W_\infty.$$

We define the coordinates of this point of $AG^*(s+1, q^{t+1})$ to be $(x_1, x_2, \ldots, x_{s+1})$. Every such ordered $(s+1)$-tuple $(x_1, x_2, \ldots, x_{s+1})$ of elements of W_∞ represents the unique point of $AG^*(s+1, q^{t+1})$ corresponding to the subspace of $\mathcal{V}_{(s+1)(t+1)+1} - \mathcal{V}_{(s+1)(t+1)}$ spanned by the vector $x_1 \oplus x_2^{(2)} \oplus \cdots \oplus x_{s+1}^{(s+1)} \oplus e^*$. In this way the points of $AG^*(s+1, q^{t+1})$ are represented by $(s+1)$-tuples of elements of W_∞.

3.6.2 Equations for the lines

A line of $AG^*(s+1, q^{t+1})$ is a $(t+2)$-dimensional subspace of $\mathcal{V}_{(s+1)(t+1)+1}$ meeting $\mathcal{V}_{(s+1)(t+1)}$ in an element W_{ij} of $\mathcal{W}$. Such a space has the form

$$< W_{ij}, (x_1, x_2, \ldots, x_{s+1}) >=< W_{ij}, x_1 \oplus x_2^{(2)} \oplus \cdots \oplus x_{s+1}^{(s+1)} \oplus e^* >$$

where $(x_1, x_2, \ldots, x_{s+1})$ is one of the points of $AG^*(s+1, q^{t+1})$ belonging to that line.

The lines may be divided into two main types:

(1) *Lines passing through* W_∞. If

$$a_1, a_2, \ldots, a_{s+1} \in W_\infty$$

then the point $(x_1, x_2, \ldots, x_{s+1})$ of $AG^*(s+1, q^{t+1})$ lies on the line

$$< W_\infty, (a_1, a_2, \ldots, a_{s+1}) >=< \{a : a \in W_\infty\}, a_1 \oplus a_2^{(2)} \oplus \cdots \oplus a_{s+1}^{(s+1)} \oplus e^* >$$

if and only if

$$x_1 \oplus x_2^{(2)} \oplus \cdots \oplus x_{s+1}^{(s+1)} \oplus e^* \in< \{a : a \in W_\infty\}, a_1 \oplus a_2^{(2)} \oplus \cdots \oplus a_{s+1}^{(s+1)} \oplus e^* >$$

which occurs if and only if

$$x_2^{(2)} = a_2^{(2)},\ x_3^{(3)} = a_3^{(3)}, \ldots,\ x_{s+1}^{(s+1)} = a_{s+1}^{(s+1)}$$

and therefore if and only if

$$x_2 = a_2,\ x_3 = a_3, \ldots,\ x_{s+1} = a_{s+1}.$$

(2) *Lines not passing through W_∞.* If

$$a_1, a_2, \ldots, a_{s+1} \in W_\infty$$

and

$$W_{ij} = J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots 0) \in \mathcal{W},$$

then the point $(x_1, x_2, \ldots, x_{s+1}) = x_1 \oplus x_2^{(2)} \oplus \cdots \oplus x_{s+1}^{(s+1)} \oplus e^*$ lies on the line

$$\begin{aligned} &< J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0), (a_1, \ldots, a_{s+1}) > \\ &=< J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0), a_1 \oplus a_2^{(2)} \oplus \cdots \oplus a_{s+1}^{(s+1)} \oplus e^* > \end{aligned}$$

if and only if

$$\begin{aligned} &x_1 \oplus x_2^{(2)} \oplus \cdots \oplus x_{s+1}^{(s+1)} \oplus e^* \\ &\qquad \in< \left\{ C_1^{(j)} a \oplus C_2^{(j)} a^{(2)} \oplus \cdots \oplus C_i^{(j)} a^{(i)} \oplus a^{(i+1)} :\ a \in W_\infty \right\}, \\ &\qquad\qquad a_1 \oplus a_2^{(2)} \oplus \cdots \oplus a_{s+1}^{(s+1)} \oplus e^* > \end{aligned}$$

which occurs if and only if

$$x_{s+1} = a_{s+1},\ x_s = a_s, \ldots, x_{i+2} = a_{i+2}$$

and $x_{i+1}^{(i+1)} = a^{(i+1)} + a_{i+1}^{(i+1)}$ so that $a = x_{i+1} - a_{i+1}$ and thus

$$\begin{aligned} &x_i^{(i)} - a_i^{(i)} = C_i^{(j)}(x_{i+1}^{(i)} - a_{i+1}^{(i)}), \\ &x_{i-1}^{(i-1)} - a_{i-1}^{(i-1)} = C_{i-1}^{(j)}(x_{i+1}^{(i-1)} - a_{i+1}^{(i-1)}), \\ &\vdots \\ &x_2^{(2)} - a_2^{(2)} = C_2^{(j)}(x_{i+1}^{(2)} - a_{i+1}^{(2)}) \\ &x_1 - a_1 = C_1^{(j)}(x_{i+1} - a_{i+1}). \end{aligned}$$

As $\mathcal{PC}$ is a projective t-spread set of matrices, and noting the action of each non-singular linear transformation (k), we see that

$$\begin{aligned}(x_k^{(k)} - a_k^{(k)}) &= C_k^{(j)}(x_{i+1}^{(k)} - a_{i+1}^{(k)}) \\ \Rightarrow (x_k - a_k)^{(k)} &= C_k^{(j)}(x_{i+1} - a_{i+1})^{(k)} \\ \Rightarrow (x_k - a_k) &= C_k^{(j)}(x_{i+1} - a_{i+1}).\end{aligned}$$

The equations of the line are:

$$x_{s+1} = a_{s+1},\ x_s = a_s, \ldots, x_{i+2} = a_{i+2}$$

and

$$x_i - a_i = C_i^{(j)}(x_{i+1} - a_{i+1}), \ldots, x_1 - a_1 = C_1^{(j)}(x_{i+1} - a_{i+1}).$$

In this way, every line of $AG^*(s+1, q^{t+1})$ is specified by a set of s linear equations determined by a point of the line and the element of $\mathcal{W}$ through which the line passes.

We now show that, with an appropriate definition of addition and multiplication, the elements of W_∞ form a field, and so they provide coordinates for $AG^*(s+1, q^{t+1})$ in the usual way. Again, these results extend the results discussed in Section 2.6.

We have already found equations describing the lines of $AG^*(s+1, q^{t+1})$, and they are of the type

$$x = a, \text{ or } (x - a) = C_k^{(j)}(y - b)$$

where x, y, a and b are elements of W_∞ and $C_k^{(j)}$ is some matrix appearing in an element of the projective t-spread set of $\mathcal{W}$. We need to rewrite these with addition and multiplication in the field replacing the vector space addition and multiplication by a matrix in the projective t-spread set.

As W_∞ is a vector space, it has vector addition, and we define addition in W_∞ to be this vector addition. In order to define multiplication, we choose a non-zero element of W_∞ and denote it by 1. By Corollary 3.2.10, given any element $x \in W_\infty$ there exists a unique element $(C_1^{(j)}, I, 0, \ldots, 0)$ of the projective t-spread set such that

$$x = C_1^{(j)} 1.$$

In view of this, for x and y in W_∞, we define

$$xy = (C_1^{(j)} 1)y = C_1^{(j)} y,$$

and the equations of a line in $AG^*(s+1, q^{t+1})$ are written as:

$$x = a, \text{ or } (x - a) = z(y - b)$$

where z is the unique element of W_∞ such that $z = C_1^{(j)} 1$.

3.6.3 Theorem

The system $(W_\infty, +, \cdot\,)$ described above is a field.

Proof: We apply the corresponding result in the case $s = 1$ to a subplane of $AG^*(s+1, q^{t+1})$. Consider the $(1, t)$-spread set

$$\mathcal{C}_1 = \left\{ \left(C_1^{(j)}, I, 0, \ldots, 0 \right) : j = 1, 2, \ldots, q^{t+1} \right\}$$

and the corresponding elements of $\mathcal{W}$

$$\mathcal{W}_1 = \left\{ J\left(C_1^{(j)}, I, 0, \ldots, 0 \right) : j = 1, 2, \ldots, q^{t+1} \right\} \cup \left\{ J(I, 0, \ldots, 0) \right\}.$$

Now $\mathcal{W}_1$ contains the two elements

$$W_\infty = J(I, 0, 0, \ldots, 0) = \operatorname{lin}\{e_1, e_2, \ldots, e_{t+1}\}$$

$$W_{11} = J(0, I, 0, \ldots, 0) = \operatorname{lin}\left\{ e_{t+2}, e_{t+3}, \ldots, e_{2(t+1)} \right\}.$$

If we let (2) denote the (non-singular) linear transformation

$$(2) : W_\infty \to W_{11}$$
$$: e_k \mapsto e_{t+1+k}, \qquad \text{for } k = 1, 2, \ldots, t+1$$

then the elements of $\mathcal{W}_1$ are W_∞ and, for $j = 1, 2, \ldots, q^{t+1}$,

$$\begin{aligned} W_{1j} &= J\left(C_1^{(j)}, I, 0, \ldots, 0\right) \\ &= \left\{C_1^{(j)} a \oplus a^{(2)} : \ a \in W_\infty\right\}. \end{aligned}$$

Therefore $\mathcal{W}_1$ is a t-spread of the $(2t+1)$-dimensional space

$$PG(2t+1, q) = < W_\infty, W_{11} > = \mathrm{lin}\{e_1, e_2, \ldots, e_{2(t+1)}\},$$

with projective t-spread set $\{(C_1^{(j)}, I) : \ j = 1, 2, \ldots, q^{t+1}\} \cup \{(I, 0)\}$. This t-spread $\mathcal{W}_1$ defines an affine plane Π of order q^{t+1} as in Section 2.3, which is an affine subplane of $AG^*(s+1, q^{t+1})$. To see this, we follow the construction of Π given in Section 2.3, but using the space $PG(2t+1, q)$ embedded in $PG((s+1)(t+1)-1, q)$ and the t-spread $\mathcal{W}_1$ as a subset of $\mathcal{W}$.

As in Section 1.1, $PG(2t+1, q)$ corresponds to a $(2t+2)$-dimensional subspace $\mathcal{V}_{2t+2}$ of $\mathcal{V}_{(s+1)(t+1)}$, since

$$\mathcal{V}_{2t+2} = \mathrm{lin}\{e_1, e_2, \ldots, e_{2(t+1)}\}$$
$$\mathcal{V}_{(s+1)(t+1)} = \mathrm{lin}\{e_1, e_2, \ldots, e_{(s+1)(t+1)}\}.$$

Also, the space

$$\mathcal{V}_{2t+3} = \mathrm{lin}\{e_1, e_2, \ldots, e_{2(t+1)}, e^*\}$$

is a $(2t+3)$-dimensional subspace of

$$\mathcal{V}_{(s+1)(t+1)+1} = \mathrm{lin}\{e_1, e_2, \ldots, e_{(s+1)(t+1)}, e^*\}.$$

The points of the plane Π are the elements of $\mathcal{V}_{2t+3} - \mathcal{V}_{2t+2}$, which are also points of $AG^*(s+1, q^{t+1})$ as they are elements of $\mathcal{V}_{(s+1)(t+1)+1} - \mathcal{V}_{(s+1)(t+1)}$. Lines of Π are the $(t+2)$- dimensional subspaces of $\mathcal{V}_{2t+3}$ which meet $\mathcal{V}_{2t+1}$ in exactly an element of $\mathcal{W}_1$. These are also lines of $AG^*(s+1, q^{t+1})$ as they are $(t+2)$-dimensional subspaces of $\mathcal{V}_{(s+1)(t+1)+1}$ meeting $\mathcal{V}_{(s+1)(t+1)}$ in exactly an element of $\mathcal{W}$.

We use the construction of Section 2.6 and the construction presented above to give coordinates for Π, both as a plane in its own right and as a subplane of $AG^*(s+1, q^{t+1})$. In both instances the coordinates come from the set W_∞ and we shall see that the coordinates of Π as a plane in its own right occur as the restriction of the coordinates of Π as a subplane of $AG^*(s+1, q^{t+1})$ as is normally the case.

First we give coordinates for Π as a plane in its own right. A point of Π is a point of $\mathcal{V}_{2t+3} - \mathcal{V}_{2t+2}$ and so has a unique basis vector of the form:

$$x_1 \oplus x_2^{(2)} \oplus e^*$$

for some $x_1, x_2 \in W_\infty$. As in Section 2.6, this point has coordinates (x_1, x_2). Now a line of Π is a $(t+2)$-dimensional subspace of $\mathcal{V}_{2t+3}$ which meets $\mathcal{V}_{2t+1}$ in exactly an element of $\mathcal{W}_1$. If the space passes through $W_\infty = (I, 0)$ and contains a point (a_1, a_2) of Π then it is

$$\{(x_1, x_2) : \ x_2 = a_2\}$$

otherwise if it passes through $W_{1j} = J(C_1^{(j)}, I)$ and contains the point (a_1, a_2) then it is

$$\left\{(x_1, x_2) : \ (x_1 - a_1) = C_1^{(j)}(x_2 - a_2)\right\}.$$

Next we give coordinates to the points of Π as a subset of the points of $AG^*(s+1, q^{t+1})$. A point of Π still has basis vector

$$x_1 \oplus x_2^{(2)} \oplus e^*$$

for some $x_1, x_2 \in W_\infty$, so by 3.6.1 the coordinates of this point are

$$(x_1, x_2, 0, \ldots, 0).$$

By 3.6.2, a line of Π passing through $W_\infty = (I, 0, \ldots, 0)$ and containing the point $(a_1, a_2, 0, \ldots, 0)$ of $AG^*(s+1, q^{t+1})$ is

$$\{(x_1, x_2, \ldots, x_{s+1}) : \; x_2 = a_2, x_3 = 0, \ldots, x_{s+1} = 0\}.$$

Similarly, a line through the element $W_{1j} = J(C_1^{(j)}, I, 0, \ldots, 0)$ and containing the point $(a_1, a_2, 0, \ldots, 0)$ of $AG^*(s+1, q^{t+1})$ is

$$\left\{(x_1, x_2, \ldots, x_{s+1}) : \; (x_1 - a_1) = C_1^{(j)}(x_2 - a_2), x_3 = 0, \ldots, x_{s+1} = 0\right\}.$$

Comparing these two coordinatisations we see that Π has the same coordinates whether considered as a subplane of $AG^*(s+1, q^{t+1})$ or as an affine plane in its own right. In both cases the coordinates come from the set W_∞, on which is defined an addition (vector addition) and a multiplication. The definition of multiplication on W_∞ in Section 2.6 coincides with the definition given above, immediately preceding this Theorem.

Now since Π is a subplane of $AG^*(s+1, q^{t+1})$, it is Desarguesian. By Theorems 2.6.2 and 2.6.3, this means that the set $(W_\infty, +, \cdot)$ is a field. □

3.6.4 Corollary

Let $\mathcal{W}$ be a regular t-spread with projective t-spread set $\mathcal{PC}$. Then $\mathcal{PC}$ is isomorphic to an s-dimensional projective space $PG(s, q^{t+1})$ of order $t+1$.

Proof: $\mathcal{PC}$ is the set of all equivalence classes of $(s+1)$-tuples of elements of W_∞ which is a field of order q^{t+1}, where two such $(s+1)$-tuples are equivalent under multiplication by any element of W_∞. □

3.7 CONNECTIONS BETWEEN 3.3 TO 3.6

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1,q)$ with the Shell Property 3.2.7. In particular the smallest shell $PG(t,q)$ is an element of $\mathcal{W}$, to be denoted by W_∞.

By Corollary 3.5.8, $\mathcal{W}$ has a projective t-spread set

$$\mathcal{PC} =$$
$$\left\{P_{ij} = (C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0) : \ i = 1,2,\ldots,s, \quad j = 1,2,\ldots,q^{i(t+1)}\right\}$$
$$\cup \{P_\infty = (I, 0, \ldots, 0,)\}$$

containing the elements

$$P_{11} = (0, I, 0, \ldots, 0)$$
$$P_{21} = (0, 0, I, 0, \ldots, 0)$$
$$\vdots$$
$$P_{s1} = (0, 0, \ldots, 0, I)$$
$$P_{sq^{s(t+1)}} = (I, I, \ldots, I).$$

This means that $\mathcal{W}$ is of the form

$$\mathcal{W} = \left\{W_{ij} : \ i = 1,2,\ldots,s, \quad j = 1,2,\ldots,q^{i(t+1)}\right\} \cup \{W_\infty\}$$
$$= \left\{J(C_1^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0) : \ i = 1,2,\ldots,s, \quad j = 1,2,\ldots,q^{i(t+1)}\right\}$$
$$\cup \{W_\infty = J(I, 0, \ldots, 0,)\}$$

and contains the spaces

$$W_\infty = \operatorname{lin}\{e_1, e_2, \ldots, e_{t+1}\},$$
$$W_{11} = \operatorname{lin}\left\{e_{t+2}, e_{t+3}, \ldots, e_{2(t+1)}\right\},$$
$$\vdots$$
$$W_{s1} = \operatorname{lin}\left\{e_{s(t+1)+1}, e_{s(t+1)+2}, \ldots, e_{(s+1)(t+1)}\right\} \quad \text{and}$$
$$W_{sq^{s(t+1)}} = \operatorname{lin}\left\{e_1 + e_{t+2} + \cdots + e_{s(t+1+1)}, e_2 + e_{t+3} + \cdots + e_{s(t+1)+2}, \ldots\ldots, \right.$$
$$\left. e_{t+1} + e_{2(t+1)} + \cdots + e_{(s+1)(t+1)}\right\}.$$

Consider the set

$$\mathcal{PC}_1 = \left\{P_{1j} = (C_1^{(j)}, I, 0, \ldots, 0) : \quad j = 1, 2, \ldots, q^{i(t+1)}\right\} \cup \{P_\infty = (I, 0, \ldots, 0,)\}.$$

The set

$$\mathcal{C}_1 = \left\{C_1^{(j)} : \left(C_1^{(j)}, I, 0, \ldots, 0\right) \in \mathcal{PC}_1\right\}$$

is a t-spread set for the t-spread $\mathcal{W}_1$ induced by $\mathcal{W}$ on the $(2t+1)$-dimensional subspace $< W_\infty, W_{11} >$.

3.7.1 Theorem

Every matrix appearing in any element of $\mathcal{PC}$ belongs to the set $\mathcal{C}_1$ above, and the set $\mathcal{C}_1$ is a field under addition and multiplication of matrices.

Proof: We have seen that the set of matrices $\mathcal{C}_1$ is a t-spread set for the t-spread $\mathcal{W}_1$ of $PG(2t+1, q)$. This t-spread defines an affine translation plane as in Section 2.3. This plane is an affine subplane of the affine $(s+1)$-dimensional space defined by the t-spread $\mathcal{W}$ as described in Section 3.3. Thus the affine subplane is Desarguesian, and by Theorem 2.6.3, the system $(\mathcal{C}_1, +, \cdot)$ is a field. It remains to show that each matrix $C_k^{(j)}$ for $k \in \{1, 2, \ldots, i\}$, $i = 2, 3, \ldots, s$ and $j = 1, 2, \ldots, q^{i(t+1)}$ is in $\mathcal{C}_1$.

To show that every matrix in each element of $\mathcal{PC}$ occurs as a matrix in $\mathcal{C}_1$, choose such a matrix $C_k^{(j)}$. Let $a = C_k^{(j)}1 \in W_\infty$, then by Corollary 3.2.10, there exists a unique element

$$(C_1^{(j)}, I, 0, \ldots, 0) \in \mathcal{PC}_1$$

such that

$$a = C_1^{(j)}1.$$

It must be that $C_1^{(j)} = C_k^{(j)}$ and the proof is complete. □

3.7.2 Corollary

The field $\mathcal{C}_1$ contains the subfield $\{kI: k \in GF(q)\}$ which is isomorphic to $GF(q)$.

Proof: The set $\{kI: k \in GF(q)\}$ is contained in $\mathcal{C}_1$ by Corollary 2.6.3 (2)□

3.8 t-SPREADS AND INDICATOR SETS

In this Section we extend the work done in Section 2.8 on indicator sets corresponding to t-spreads of $PG(2t+1,q)$. The point of view on indicator sets developed in Section 2.8 allows a natural generalisation, enabling us to define indicator sets for t-spreads of $PG((s+1)(t+1)-1,q)$. These will be called projective indicator sets since they are constructed using the projective t-spread set.

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1,q)$ where $|\mathcal{W}| = \omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \cdots + q^{t+1} + 1$. Suppose that $\mathcal{W}$ has projective t-spread set

$$\mathcal{PC} = \{P_i = (\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)}) : i = 1,2,\ldots,\omega\}$$

where each matrix $\xi_k^{(i)}$ is $(t+1) \times (t+1)$ and has rank $t+1$ over $GF(q)$. The t-spread $\mathcal{W}$ is

$$\mathcal{W} = \{W_i = f((P_i)^T) : i = 1,2,\ldots,\omega\}$$

where f is the bijection of Theorem 1.6.2. Further, if $(\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)})$ and $(\xi_0^{(j)}, \xi_1^{(j)}, \ldots, \xi_s^{(j)})$ are any two elements of $\mathcal{PC}$ then

$$\text{rank}\begin{pmatrix} \xi_0^{(i)} & \xi_0^{(j)} \\ \xi_1^{(i)} & \xi_1^{(j)} \\ \vdots & \vdots \\ \xi_s^{(i)} & \xi_s^{(j)} \end{pmatrix} = 2(t+1).$$

Let $GF(q^{t+1})$ be a field extension of $GF(q)$, and let $\alpha \in GF(q^{t+1})$ be such that $GF(q^{t+1}) = GF(q)(\alpha)$. Let $PG((s+1)(t+1)-1, q^{t+1})$ denote the corresponding extension of $PG((s+1)(t+1)-1,q)$.

Consider the set of points $\mathcal{PI} = \{Q_i : i = 1, 2, \ldots, \omega\}$ where

$$Q_i = \begin{pmatrix} \xi_0^{(i)} \\ \xi_1^{(i)} \\ \vdots \\ \xi_s^{(i)} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix}$$

and the matrix P_i^T is written as an $(s+1)(t+1) \times (t+1)$ matrix over $GF(q)$. Q_i has $(s+1)(t+1)$ coordinates and so is a point of $PG((s+1)(t+1)-1, q^{t+1})$.

3.8.1 Lemma

(1) $\mathcal{PI}$ has ω points, and for $i = 1, 2, \ldots, \omega$, the point Q_i lies on the extension $\overline{W}_i$ of the corresponding t-spread element W_i,

(2) each point $Q_i \in \mathcal{PI}$ is an imaginary point of $PG((s+1)(t+1)-1, q^{t+1})$,

(3) if $Q_i \in \mathcal{PI}$ then the corresponding t-spread element $f((\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)})^T)$ is the t-dimensional space

$$L(Q_i) = \text{lin}\left\{Q_i, Q_i^\sigma, \ldots, Q_i^{\sigma^t}\right\} \cap PG((s+1)(t+1)-1, q),$$

where σ is defined by $\sigma : x \mapsto x^q$, and

(4) if Q_i and Q_j are any two points of $\mathcal{I}$ then the line Q_iQ_j joining them is imaginary.

Proof: (1) First, $\mathcal{PI}$ has one point corresponding to each t-spread element, and the number of these is ω. Each element W_i of the t-spread $\mathcal{W}$ is the subspace of $PG((s+1)t+1)-1, q)$ spanned by the columns of the matrix

$$\begin{pmatrix} \xi_0^{(i)} \\ \xi_1^{(i)} \\ \vdots \\ \xi_s^{(i)} \end{pmatrix}.$$

The point Q_i is a linear combination of the columns of this matrix with coefficients $1, \alpha, \alpha^2, \ldots, \alpha^t$, that is

$$Q_i = V_0 + \alpha V_1 + \alpha^2 V_2 + \cdots + \alpha^t V_t$$

where V_i denotes the $(i+1)$st column of the matrix

$$\begin{pmatrix} \xi_0^{(i)} \\ \xi_1^{(i)} \\ \vdots \\ \xi_s^{(i)} \end{pmatrix}.$$

Since $\overline{W}_i$ is spanned by the vectors $V_0, V_1, \ldots, V_t$ over $GF(q^{t+1})$, we see that Q_i is a point of $\overline{W}_i$.

(2) Since $Q_i = V_0 + \alpha V_1 + \alpha^2 V_2 + \cdots + \alpha^t V_t$, where the points $V_0, V_1, \ldots, V_{d-1}$ are linearly independent over $GF(q)$, by Corollary 1.4.10 (1) Q_i is imaginary.

(3) Now Q_i is an imaginary point of $\overline{W}_i$, which is a t-dimensional space meeting $PG((s+1)(t+1)-1, q)$ in a t-dimensional space. But $L(Q_i)$ is the unique such space by Corollary 1.4.10 (2) so that $L(Q_i) = \overline{W}_i$, and it follows that $W_i = L(Q_i) \cap PG((s+1)(t+1)-1, q)$.

(4) As Q_i and Q_j are imaginary, the spaces $L(Q_i)$ and $L(Q_j)$ both have dimension t in $PG((s+1)(t+1)-1, q^{t+1})$, and each meet $PG((s+1)(t+1)-1, q)$ in a space of dimension t which must be a t-spread element by (2). Thus $L(Q_i)$ and $L(Q_j)$ are skew, so the points

$$Q_i, Q_i^{\sigma}, \ldots, Q_i^{\sigma^t}, Q_j, Q_j^{\sigma}, \ldots, Q_j^{\sigma^t}$$

span a space of dimension $2t+1$. It follows that the lines

$$Q_iQ_j, (Q_iQ_j)^{\sigma}, \ldots, (Q_iQ_j)^{\sigma^t}$$

span a space which is actually $L(Q_iQ_j)$ and is of dimension $2t+1$. From Definition 1.4.7 (2) we see that the line (Q_iQ_j) is imaginary. □

These observations motivate the following definition:

3.8.2 Definition

A *projective indicator set* is a set $\mathcal{PI}$ of $\omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \cdots + q^{t+1} + 1$ imaginary points of $PG((s+1)(t+1)-1, q^{t+1})$ such that the line joining any two points of $\mathcal{PI}$ is imaginary.

3.8.3 Theorem

Let $\mathcal{PI}$ be a projective indicator set. Then the set

$$\mathcal{W}(\mathcal{PI}) = \{L(Q_i) \cap PG((s+1)(t+1)-1, q) : Q_i \in \mathcal{PI}\}$$

is a t-spread of $PG((s+1)(t+1)-1, q)$.

Proof: Firstly, the set $\mathcal{W}(\mathcal{PI})$ contains ω elements, and by Corollary 1.4.10 (2) since each point $Q_i \in PG((s+1)(t+1)-1, q^{t+1})$ is imaginary, each space $L(Q_i) \cap PG((s+1)(t+1)-1, q)$ has dimension t. It remains to show that any pair $W_i = L(Q_i) \cap PG((s+1)(t+1)-1, q)$ and $W_j = L(Q_j) \cap PG((s+1)(t+1)-1, q)$ of elements of $\mathcal{W}(\mathcal{PI})$ are skew in $PG((s+1)(t+1)-1, q)$. The line Q_iQ_j is imaginary, so the lines

$$Q_iQ_j, (Q_iQ_j)^{\sigma}, \ldots, (Q_iQ_j)^{\sigma^t}$$

span a subspace of $PG((s+1)(t+1)-1, q^{t+1})$ of dimension $2t+1$. Thus the points

$$Q_i, Q_i^{\sigma}, \ldots, Q_i^{\sigma^t} \text{ and } Q_j, Q_j^{\sigma}, \ldots, Q_j^{\sigma^t}$$

also span a subspace of dimension $2t+1$, and this is only possible if the subspaces $L(Q_i)$ and $L(Q_j)$ are skew. □

3.8.4 Theorem

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$. Then there exists a projective indicator set $\mathcal{PI}$ in $PG((s+1)(t+1)-1, q^{t+1})$ such that $\mathcal{W} = \mathcal{W}(\mathcal{PI})$ constructed as in Theorem 3.8.3.

Proof: Let $\mathcal{PC}$ be a projective t-spread set corresponding to the t-spread $\mathcal{W}$. This exists by Theorem 3.5.4. Then we can construct a projective indicator set as in the remarks preceding Lemma 3.8.1, then use the Lemma 3.8.1 to verify that it is indeed a projective indicator set and that $\mathcal{W} = \mathcal{W}(\mathcal{PI})$. □

A projective indicator set in $PG((s+1)(t+1)-1, q^{t+1})$ determines a t-spread of $PG((s+1)(t+1)-1, q)$ uniquely, but a t-spread of $PG((s+1)(t+1)-1, q)$ may have more than one projective indicator set in $PG((s+1)(t+1)-1, q^{t+1})$ since it may have more than one projective t-spread set. Lunardon (1984) was able to improve the situation in $PG(2t+1, q)$ by introducing an indicator $(t+1)$-space in $PG(2t+1, q^{t+1})$ and requiring the imaginary points of the t-spread elements to lie in this indicator $(t+1)$-space. This was the approach used by Sherk (1979) and Bruen (1972a). As we saw in Section 2.8, this corresponded to constructing the indicator set using the t-spread set (or equivalently the normalised projective t-spread set) instead of the projective t-spread set, and since every t-spread of $PG(2t+1, q)$ has both a t-spread set and a projective t-spread set, there was no problem. However, the situation is different in $PG((s+1)(t+1)-1, q)$ for $s > 1$ because as we have already noted, only the t-spreads with the Shell property 3.2.7 even have t-spread sets.

In Section 2.8 we found the projective indicator set

$$\{Q_j : j = \infty, 1, 2, \ldots, q^{t+1}\}$$

of a t-spread of $PG(2t+1, q)$. If a basis for $PG(2t+1, q)$ and the extension $PG(2t+1, q^{t+1})$ is

$$\{e_1, e_2, \ldots, e_{2(t+1)}\}$$

then every point Q_i is contained in a subspace $PG^*(t+1, q^{t+1})$ of $PG(2t+1, q^{t+1})$

spanned by the vectors

$$e_1, e_2, \ldots, e_{t+1}, e_{t+2} + \alpha e_{t+3} + \cdots + \alpha^t e_{2t+2}.$$

In fact each point Q_i apart from $Q_\infty = (e_1 + \alpha e_2 + \cdots + \alpha^t e_{t+1})$ is contained in the affine space

$$AG^*(t+1, q^{t+1}) = PG^*(t+1, q^{t+1}) - \overline{J(\infty)}$$

where

$$\overline{J(\infty)} = \operatorname{lin}\{e_1, e_2, \ldots, e_{t+1}\}.$$

In the case of t-spreads of $PG((s+1)(t+1)-1, q)$ the situation is somewhat more complicated, as we shall now see.

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$ and suppose that $\mathcal{W}$ has a projective t-spread set where the last non-zero matrix in each $(s+1)$-tuple of matrices is the identity matrix. This occurs if and only if $\mathcal{W}$ has a t-spread set and so if and only if $\mathcal{W}$ has the Shell property 3.2.7 (see Theorems 3.5.6 and 3.5.7). Let

$$\mathcal{PC} = \left\{P_{ij} : j = 1, 2, \ldots, s, \quad i = 1, 2, \ldots, q^{j(t+1)}\right\}$$

be a projective t-spread set for $\mathcal{W}$, where

$$\begin{aligned}
P_\infty &= (I, 0, \ldots, 0) \\
P_{1j} &= (C_1^{(j)}, I, 0, \ldots, 0), \ j = 1, 2, \ldots, q^{t+1} \\
P_{2j} &= (C_1^{(j)}, C_2^{(j)}, I, 0, \ldots, 0), \ j = 1, 2, \ldots, q^{2(t+1)} \\
&\vdots \\
P_{sj} &= (C_1^{(j)}, C_2^{(j)}, \ldots, C_s^{(j)}, I), \ j = 1, 2, \ldots, q^{s(t+1)}.
\end{aligned}$$

Let $\alpha \in GF(q^{t+1})$ be such that $GF(q^{t+1}) = GF(q)(\alpha)$. Then the projective indicator set $\mathcal{PI}$ comprises the points

$$\mathcal{PI} = \left\{ Q_{ij} : \ i = 1, 2, \ldots, s, \ j = 1, 2, \ldots, q^{i(t+1)} \right\} \\ \cup \{Q_\infty\}$$

where

$$Q_{ij} = \begin{pmatrix} C_1^{(j)} \\ C_2^{(j)} \\ \vdots \\ C_i^{(j)} \\ I \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix}$$

and

$$Q_\infty = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^t \end{pmatrix}.$$

If we suppose that $PG((s+1)(t+1)-1, q)$ has basis

$$\left\{ e_1, e_2, \ldots, e_{(s+1)(t+1)} \right\}$$

then we see that

$$\begin{aligned}
Q_\infty &= \left(e_1 + \alpha e_2 + \cdots + \alpha^t e_{t+1} \right) \\
Q_{1j} &\in \ \mathrm{lin} \left\{ e_1, e_2, \ldots, e_{t+1}, e_{t+2} + \alpha e_{t+3} + \cdots + \alpha^t e_{2(t+1)} \right\}, \\
&\vdots \\
Q_{ij} &\in \ \mathrm{lin} \left\{ e_1, e_2, \ldots, e_{i(t+1)}, \right. \\
&\qquad \left. e_{i(t+1)+1} + \alpha e_{i(t+1)+2} + \cdots + \alpha^t e_{(i+1)(t+1)} \right\}, \\
&\vdots \\
Q_{sj} &\in \ \mathrm{lin} \left\{ e_1, e_2, \ldots, e_{s(t+1)}, \right. \\
&\qquad \left. e_{s(t+1)+1} + \alpha e_{s(t+1)+2} + \cdots + \alpha^t e_{(s+1)(t+1)} \right\}.
\end{aligned}$$

Recall that $\mathcal{W}$ has the Shell property 3.2.7, and that the shells are as follows:

$$PG(t,q) = J(\infty),$$
$$PG(2(t+1)-1,q) - PG(t,q),$$
$$\vdots$$
$$PG(s(t+1)-1,q) - PG((s-1)(t+1)-1,q),$$
$$PG((s+1)(t+1)-1,q) - PG(s(t+1)-1,q)$$

where

$$PG(t,q) = J(\infty) = \operatorname{lin}\{e_1, e_2, \ldots, e_{t+1}\},$$
$$PG(2(t+1)-1,q) = \operatorname{lin}\{e_1, e_2, \ldots, e_{2(t+1)}\},$$
$$\vdots$$
$$PG(s(t+1)-1,q) = \operatorname{lin}\{e_1, e_2, \ldots, e_{s(t+1)}\},$$
$$PG((s+1)(t+1)-1,q) = \operatorname{lin}\{e_1, e_2, \ldots, e_{(s+1)(t+1)}\}.$$

We introduce some notation for certain subspaces of $PG((s+1)(t+1)-1, q^{t+1})$ as follows.

$$\overline{J(\infty)} = (e_1 + \alpha e_2 + \cdots + \alpha^t e_{t+1})$$
$$PG^*(t+1, q^{t+1}) = \operatorname{lin}\{e_1, e_2, \ldots, e_{t+1}, e_{t+2} + \alpha e_{t+3} + \cdots + \alpha^t e_{2(t+1)}\},$$
$$\vdots$$
$$PG^*(i(t+1), q^{t+1}) = \operatorname{lin}\{e_1, e_2, \ldots, e_{i(t+1)},$$
$$e_{i(t+1)+1} + \alpha e_{i(t+1)+2} + \cdots + \alpha^t e_{(i+1)(t+1)}\},$$
$$\vdots$$
$$PG^*(s(t+1), q^{t+1}) = \operatorname{lin}\{e_1, e_2, \ldots, e_{s(t+1)},$$
$$e_{s(t+1)+1} + \alpha e_{s(t+1)+2} + \cdots + \alpha^t e_{(s+1)(t+1)}\}.$$

Note that every point Q_∞ and Q_{ij} of $\mathcal{PI}$ lies in the space $PG^*(s(t+1), q^{t+1})$, and in fact every point Q_{ij} of $\mathcal{PI}$ lies in $PG^*(s(t+1), q^{t+1}) - \overline{J(\infty)}$. We can be more specific, and investigate exactly which subspaces of $PG^*(s(t+1), q^{t+1})$ contain the various points Q_∞ and Q_{ij} of the projective indicator set $\mathcal{PI}$. Using the notation just introduced for certain subspaces of $PG((s+1)(t+1)-1, q^{t+1})$ and checking where the basis vector of each point Q_{ij} lies, we see that:

$$
\begin{aligned}
Q_\infty &\in \overline{J(\infty)}, \\
Q_{1j} &\in AG^*(t+1, q^{t+1}) = PG^*(t+1, q^{t+1}) - \overline{J(\infty)}, \\
Q_{2j} &\in AG^*(2(t+1), q^{t+1}) = PG^*(2(t+1), q^{t+1}) - PG(2(t+1)-1, q^{t+1}), \\
&\vdots \\
Q_{ij} &\in AG^*(i(t+1), q^{t+1}) = PG^*(i(t+1), q^{t+1}) - PG(i(t+1)-1, q^{t+1}), \\
&\vdots \\
Q_{sj} &\in AG^*(s(t+1), q^{t+1}) = PG^*(s(t+1), q^{t+1}) - PG(s(t+1)-1, q^{t+1}).
\end{aligned}
$$

This is similar to the way that a projective space $PG((s+1)(t+1)-1, q)$ divides into shells, see 3.2.7, and the elements of a t-spread sometimes lie entirely within these shells. For this reason we will call this the Shell property for projective indicator sets.

3.8.5 Definition The Shell property for projective indicator sets

Let $\mathcal{PI}$ be a projective indicator set in $PG((s+1)(t+1)-1, q^{t+1})$. Suppose that every point of $\mathcal{PI}$ is contained in a certain subspace $PG^*(s(t+1), q^{t+1})$ which meets $PG((s+1)(t+1)-1, q)$ in a subspace $PG(s(t+1)-1, q)$. Let

$$AG^*(s(t+1), q^{t+1}) = PG^*(s(t+1), q^{t+1}) - PG(s(t+1)-1, q^{t+1})$$

and suppose that the space $AG^*(s(t+1), q^{t+1})$ is divided into "shells"

$$AG^*(t+1, q^{t+1})$$

$$AG^*(2(t+1), q^{t+1}) - AG^*(t+1, q^{t+1})$$

$$\vdots$$

$$AG^*(i(t+1), q^{t+1}) - AG^*((i-1)(t+1), q^{t+1})$$

$$\vdots$$

$$AG^*(s(t+1), q^{t+1}) - AG^*((s-1)(t+1), q^{t+1}).$$

$\mathcal{PI}$ is said to have the *Shell property* with respect to this partition into shells if it has

(1) exactly one point in $AG^*(t+1, q^{t+1})$,

(2) exactly q^{t+1} points in $AG^*(2(t+1), q^{t+1}) - AG^*(t+1, q^{t+1})$,

(3) exactly $q^{2(t+1)}$ points in $AG^*(3(t+1), q^{t+1}) - AG^*(2(t+1), q^{t+1})$,

and so on until

(s+1) exactly $q^{s(t+1)}$ points in $AG^*(s(t+1), q^{t+1}) - AG^*((s-1)(t+1), q^{t+1})$.

3.8.6 Lemma

In the above notation, an element W of a t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1, q)$ lies in the shell

$$PG((i+1)(t+1)-1, q) - PG(i(t+1)-1, q)$$

of $PG((s+1)(t+1)-1, q)$ if and only if the corresponding point Q of a projective indicator set lies in the shell

$$AG^*((i+1)(t+1), q^{t+1}) - AG^*(i(t+1), q^{t+1})$$

of $AG^*(s(t+1), q^{t+1})$.

Proof: An element W of $\mathcal{W}$ lies in the shell

$$PG((i+1)(t+1)-1, q) - PG(i(t+1)-1, q)$$

of $PG((s+1)(t+1)-1, q)$ if and only if (as in the proof of Theorem 3.5.7) it corresponds to a projective t-spread element of the form

$$(C_i^{(j)}, C_2^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0).$$

This occurs if and only if the corresponding point Q of the projective indicator set is

$$\begin{aligned} Q &\in \text{lin } \left\{e_1, e_2, \ldots, e_{i(t+1)}, e_{i(t+1)+1} + \alpha e_{i(t+1)+2} + \cdots + \alpha^t e_{(i+1)(t+1)}\right\} \\ &\in AG^*((i+1)(t+1), q^{t+1}) - AG^*(i(t+1), q^{t+1}). \end{aligned}$$

□

3.8.7 Theorem

Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1, q)$ with projective indicator set $\mathcal{PI}$ in $PG((s+1)(t+1)-1, q^{t+1})$. Then $\mathcal{W}$ has the Shell property 3.2.7 if and only if $\mathcal{PI}$ has the Shell property 3.8.5.

Proof: Let $\mathcal{W}$ be a t-spread set with the Shell property 3.2.7, so that by Theorem 3.2.9 we may suppose that

$$\mathcal{W} = \left\{W_{ij} : \ i = 1, 2, \ldots, s, \ \ j = 1, 2, \ldots, q^{i(t+1)}\right\} \cup \{W_\infty\}$$

where

$$\begin{aligned} W_\infty &= PG(t, q) \\ W_{ij} &\in PG((i+1)(t+1)-1, q) - PG(i(t+1)-1, q). \end{aligned}$$

Let $\mathcal{PI}$ be a projective indicator set for $\mathcal{W}$, so that

$$\mathcal{PI} = \left\{P_{ij} : \ i = 1, 2, \ldots, s, \ \ j = 1, 2, \ldots, q^{i(t+1)}\right\} \cup \{P_\infty\}$$

where P_{ij} corresponds to W_{ij} and P_∞ corresponds to W_∞. By Lemma 3.8.6,

$$P_{ij} \in \ AG^*((i+1)(t+1), q^{t+1}) - AG^*(i(t+1), q^{t+1})$$

and $\mathcal{PI}$ has the Shell property 3.8.5. Conversely, if a projective indicator set $\mathcal{PI}$ has the Shell property 3.8.5, then again by Lemma 3.8.6, the corresponding t-spread $\mathcal{W}$ has the Shell property 3.2.7. □

3.9 t-SPREADS OF $PG((s+1)(t+1)-1, q)$ IN $PG((s+1)(t+1)-1, q^{t+1})$

The following result in the case of $t = 1$ appears in Ebert (1983), however without an indication of the proof. It was used by Ebert (1983) to study the subregular 1-spreads of $PG(2s+1, 2)$.

3.9.1 Theorem

(1) Let S_s be an imaginary s-dimensional subspace of $PG((s+1)(t+1)-1, q^{t+1})$.

The set

$$\mathcal{W} = \left\{ \text{lin} \left\{ P, P^\sigma, \ldots, P^{\sigma^t} \right\} \cap PG((s+1)(t+1)-1, q) \right.$$
$$\left. : P \in S_s \right\}$$

is a regular t-spread of $PG((s+1)(t+1)-1, q)$.

(2) Conversely a regular t-spread of $PG((s+1)(t+1)-1, q)$ can be represented in this manner for a unique set of s-dimensional subspaces $(S_s, S_s^\sigma, \ldots, S_s^{\sigma^t})$.

Proof: (1) Let P and Q be distinct points of S_s. By Theorem 1.4.8 (3), P and Q are imaginary. By Definition 1.4.7 (1) each of the spaces

$$L(P) = \text{lin} \left\{ P, P^\sigma, \ldots, P^{\sigma^t} \right\}$$

and

$$L(Q) = \text{lin} \left\{ Q, Q^\sigma, \ldots, Q^{\sigma^t} \right\}$$

has dimension t in $PG((s+1)(t+1)-1, q^{t+1})$ and by Theorem 1.4.8 (1) meets $PG((s+1)(t+1)-1, q)$ in a t-dimensional subspace. By the same arguments as those used in the proof of Theorem 3.8.3, $L(P)$ and $L(Q)$ are skew. Thus $\mathcal{W}$ is a set of $q^{s(t+1)} + q^{(s-1)(t+1)} + \cdots + q^{t+1} + 1$ pairwise skew t-dimensional subspaces of $PG((s+1)(t+1)-1, q)$ and so is a t-spread. We must show that it is regular, according to the Definition 3.4.11, and in fact we will prove that it is regular of rank s according to Definition 3.4.7. We must show that if S is an s-dimensional subspace of $PG((s+1)(t+1)-1, q)$, then the set of elements of $\mathcal{W}$ meeting S is a t-regulus of rank s. Let S be such an s-dimensional subspace of $PG((s+1)(t+1)-1, q)$. Let $W_1, W_2, \ldots, W_{s+2}$ be elements of $\mathcal{W}$ which meet S in distinct points and are such that no $s+1$ of them lie in a hyperplane. Let $\mathcal{R}_s$ be the t-regulus of rank s defined by $W_1, W_2, \ldots, W_{s+2}$ in $PG((s+1)(t+1)-1, q)$ and let $\mathcal{R}_s{}^*$ be the t-regulus of rank s defined by their extensions $\overline{W}_1, \overline{W}_2, \ldots, \overline{W}_{s+2}$ in $PG((s+1)(t+1)-1, q^{t+1})$. Now S_s meets each of the t-dimensional spaces $\overline{W}_1, \overline{W}_2, \ldots, \overline{W}_{s+2}$ in a point and Theorems 3.4.3 and 1.3.4 (2) imply that S_s is a transversal s-dimensional space of $\mathcal{R}_s{}^*$. A t-dimensional space of $\mathcal{R}_s$, extended to $PG((s+1)(t+1)-1, q^{t+1})$, is a t-dimensional space of $\mathcal{R}_s{}^*$ and therefore meets S_s in a unique point. So the extension of a t-dimensional space X of $\mathcal{R}_s$ meets S_s in a point P. By Corollary 1.4.10 (2) $L(P)$ is the unique such space meeting S_s in the point P, so that $L(P) \cap PG((s+1)(t+1)-1, q)$ must belong to $\mathcal{W}$.

(2) Now let $\mathcal{W}$ be a regular t-spread of $PG((s+1)(t+1)-1, q)$. To show that $\mathcal{W}$ can be represented in this manner, we show that there exists an imaginary s-dimensional subspace S_s which meets the extension to $PG((s+1)(t+1)-1, q^{t+1})$ of every element of $\mathcal{W}$. In this case,

$$\mathcal{W} = \left\{ \text{lin } \left\{ P, P^{\sigma}, \ldots, P^{\sigma^t} \right\} \cap PG((s+1)(t+1)-1, q) : \ P \in S_s \right\}$$

and the uniqueness of the set of s-dimensional subspaces $S_s, S_s^{\sigma}, \ldots, S_s^{\sigma^t}$ follows by Corollary 1.4.10 (2). We will prove that the result holds for a particular extension

$PG((s+1)(t+1)-1,q^{t+1})$ of $PG((s+1)(t+1)-1,q)$, and the result follows since all extensions of the same degree are isomorphic. As in Section 3.5, $\mathcal{W}$ gives rise to a normalised projective t-spread set

$$\mathcal{PC} = \left\{\left(C_1^{(j)},C_2^{(j)},\ldots,C_i^{(j)},I,0,\ldots,0\right) : i=1,2,\ldots,s,\ \ j=1,2,\ldots,q^{i(t+1)}\right\} \cup \{(I,0,\ldots,0)\},$$

where the set

$$\mathcal{C}=\{C_i^{(j)} : i=1,2,\ldots,s,\ j=1,2,\ldots,q^{i(t+1)}\}$$

of matrices is a field of order q^{t+1} under addition and multiplication. Let $\xi \in GF(q^{t+1})$ be a such that $GF(q^{t+1}) = GF(q)(\xi)$. By Corollary 3.7.2 $\mathcal{C}$ contains the subfield $\{kI : \ k \in GF(q)\}$ which we will denote by $GF(q)$, so that

$$GF(q^{t+1}) = \{x_0 + x_1\xi + x_2\xi^2 + \cdots + x_t\xi^t : \ x_i \in GF(q)\}$$

where multiplication is (matrix) multiplication in the field $GF(q^{t+1})$. Let

$$\{e_1, e_2, \ldots, e_{(s+1)(t+1)}\}$$

be a basis for $PG((s+1)(t+1)-1,q)$ over $GF(q)$ and for $PG((s+1)(t+1)-1,q^{t+1})$ over $GF(q^{t+1})$. The set

$$\begin{aligned}
S_s = \Big\{ & e_1C_1^{(j)} + e_2\xi C_1^{(j)} + \cdots + e_{t+1}\xi^t C_1^{(j)} + \\
& e_{t+2}C_2^{(j)} + e_{t+3}\xi C_2^{(j)} + \cdots + e_{2(t+1)}\xi^t C_2^{(j)} + \\
& \cdots + \\
& e_{(i-1)(t+1)+1}C_i^{(j)} + e_{(i-1)(t+1)+2}\xi C_i^{(j)} + \cdots + e_{i(t+1)}\xi^t C_i^{(j)} + \\
& e_{i(t+1)+1} + e_{i(t+1)+2}\xi + \cdots + e_{(i+1)(t+1)}\xi^t \\
& : \ i=1,2,\ldots,s,\ \ j=1,2,\ldots,q^{i(t+1)}\Big\} \cup \{e_1 I + e_2\xi + \cdots + e_{t+1}\xi^t\} \\
= \Big\{ & \Big(C_1^{(j)}, C_1^{(j)}\xi, \ldots, C_1^{(j)}\xi^t; C_2^{(j)}, C_2^{(j)}\xi, \ldots, C_2^{(j)}\xi^t; \ldots\ldots; \\
& C_i^{(j)},, C_i^{(j)}\xi, \ldots, C_i^{(j)}\xi^t; I, \xi, \ldots, \xi^t, 0, \ldots\ldots, 0\Big)^T \\
& : \ C_i \in \mathcal{C}\} \cup \left\{(I,\xi,\ldots,\xi^t,0,\ldots,0)^T\right\}
\end{aligned}$$

is an s-dimensional subspace of $PG((s+1)(t+1)-1, q^{t+1})$ which contains no point of $PG((s+1)(t+1)-1, q)$.

Since S_s can be written as

$$\begin{aligned} S_s &= (I, \xi, \ldots, \xi^t)^T \oplus (I, \xi, \ldots, \xi^t)^T \oplus \cdots \\ &\qquad \oplus (I, \xi, \ldots, \xi^t)^T \\ &= P_0 \oplus P_1 \oplus \cdots \oplus P_s, \end{aligned}$$

the space $L(S_s)$ spanned by its $t+1$ conjugates is the join of the spaces

$$L(P_0),\ L(P_1), \ldots, L(P_s)$$

spanned by the $t+1$ conjugates of each of $P_0, P_1, \ldots, P_s$. Now each of $P_0, P_1, \ldots, P_s$ is imaginary by Corollary 1.4.10 (1) and so $L(P_0), L(P_1), \ldots, L(P_s)$ each have dimension t, and further by construction if $i \neq j$ then $L(P_i)$ and $L(P_j)$ are skew so that $L(S_s)$ has dimension $(s+1)(t+1)-1$. By Definition 1.4.7 (2), S_s is imaginary.

Now as $\mathcal{PC}$ is a projective t-spread set for $\mathcal{W}$ we have

$$\begin{aligned} &\mathcal{W} = \\ &\left\{W_{ij} = J\left(C_1^{(j)}, \ldots, C_i^{(j)}, I, 0, \ldots, 0\right) : \ i = 1, 2, \ldots, s, \ \ j = 1, 2, \ldots, q^{i(t+1)}\right\} \\ &\qquad \cup \{W_\infty = J(I, 0, \ldots, 0)\}. \end{aligned}$$

We will show that the point $(I, \xi, \ldots, \xi^t, 0, \ldots, 0)^T$ of S_s lies on the extension $\overline{W}_\infty$ of W_∞ to $PG((s+1)(t+1)-1, q^{t+1})$, and the point

$$\begin{aligned} &\Big(C_1^{(j)}, C_1^{(j)}\xi, \ldots, C_1^{(j)}\xi^t; C_2^{(j)}, C_2^{(j)}\xi, \ldots, C_2^{(j)}\xi^t; \ldots\ldots; \\ &\qquad\qquad C_i^{(j)}, C_i^{(j)}\xi, \ldots, C_i^{(j)}\xi^t; I, \xi, \ldots\ldots, \xi^t\Big)^T \end{aligned}$$

of S_s lies on $\overline{W}_{ij}$ for each $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, q^{i(t+1)}$. Then the s-dimensional subspace S_s is the s-dimensional subspace required for proof of the result as S_s meets the extension of every element of $\mathcal{W}$.

We recall some notation. We can suppose that $\mathcal{W}$ contains the spaces

$$\begin{aligned} W_\infty &= \operatorname{lin}\{e_1, e_2, \ldots, e_{t+1}\}, \\ W_{11} &= \operatorname{lin}\left\{e_{t+2}, e_{t+3}, \ldots, e_{2(t+1)}\right\}, \\ &\vdots \\ W_{s1} &= \operatorname{lin}\left\{e_{s(t+1)+1}, e_{s(t+1)+2}, \ldots, e_{(s+1)(t+1)}\right\}. \end{aligned}$$

The vector space $\mathcal{V}_{(s+1)(t+1)}$ corresponding to $PG((s+1)(t+1)-1, q)$ has basis $\{e_1, e_2, \ldots, e_{(s+1)(t+1)}\}$ so that

$$\mathcal{V}_{(s+1)(t+1)} = W_\infty \oplus W_{11} \oplus W_{21} \oplus \cdots \oplus W_{s1}.$$

Also, for each $k = 2, 3, \ldots, s+1$, let (k) denote the (non-singular) linear transformation

$$\begin{aligned} (k) &: W_\infty \to W_{(k-1)1} \\ &: e_l \mapsto e_{(k-1)(t+1)+l} \quad \text{for } l = 1, 2, \ldots, t+1. \end{aligned}$$

Then

$$\begin{aligned} W_\infty &= \{a : \; a \in W_\infty\} \\ &= \{(a_1, a_2, \ldots, a_{t+1}, 0, \ldots\ldots, 0) : \; a_i \in GF(q) \text{ not all zero}\} \end{aligned}$$

so that

$$\begin{aligned} \overline{W}_\infty &= \{(\xi_1, \xi_2, \ldots, \xi_{t+1}, 0, \ldots\ldots, 0) : \; \xi_i \in GF(q^{t+1}), \quad \text{not all zero}\} \\ &= \left\{\xi : \; \xi \in \overline{W}_\infty\right\}. \end{aligned}$$

We can choose

$$\xi = (\xi_1, \xi_2, \ldots, \xi_{t+1}, 0, \ldots\ldots, 0) = (I, \xi, \ldots, \xi^t, 0, \ldots, 0) \in \overline{W}_\infty$$

which shows that the point

$$(I, \xi, \ldots, \xi^t, 0, \ldots, 0)^T \in \overline{W}_\infty$$

as required. Similarly,

$$W_{ij} = \left\{C_1^{(j)}a \oplus C_2^{(j)}a^{(2)} \oplus \cdots \oplus C_i^{(j)}a^{(i)} \oplus a^{(i+1)} : \ a \in W_\infty\right\}$$

$$= \left\{ C_1^{(j)} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{t+1} \end{pmatrix} \oplus C_2^{(j)} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{t+1} \end{pmatrix} \oplus \cdots \oplus C_i^{(j)} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{t+1} \end{pmatrix} \oplus \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{t+1} \end{pmatrix} \right.$$
$$\left. : \ a_i \in GF(q), \text{ not all zero} \right\}$$

and

$$\overline{W}_{ij} = \left\{ C_1^{(j)} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{t+1} \end{pmatrix} \oplus C_2^{(j)} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{t+1} \end{pmatrix} \oplus \cdots \oplus C_i^{(j)} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{t+1} \end{pmatrix} \oplus \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{t+1} \end{pmatrix} \right.$$
$$\left. : \ \xi_i \in GF(q^{t+1}), \text{ not all zero} \right\}$$
$$= \left\{C_1^{(j)}\xi + C_2^{(j)}\xi^{(2)} + \cdots + C_i^{(j)}\xi^{(i)} + \xi^{(i+1)} : \ \xi \in \overline{W}_\infty\right\}.$$

Again we choose

$$\xi = (\xi_1, \xi_2, \ldots, \xi_{t+1}, 0, \ldots, 0) = (I, \xi, \ldots, \xi^t, 0, \ldots, 0) \in \overline{W}_\infty$$

so that the point

$$\begin{aligned} &\left(C_1^{(j)}, C_1^{(j)}\xi, \ldots, C_1^{(j)}\xi^t; C_2^{(j)}, C_2^{(j)}\xi, \ldots, C_2^{(j)}\xi^t; \ldots\ldots; \right. \\ &\qquad\qquad \left. C_i^{(j)}, C_i^{(j)}\xi, \ldots, C_i^{(j)}\xi^t; I, \xi, \ldots\ldots, \xi^t\right)^T \\ &= C_1^{(j)}(I, \xi, \ldots, \xi^t)^T \oplus C_2^{(j)}(I, \xi, \ldots, \xi^t)^T \oplus \cdots \oplus \\ &\qquad\qquad C_i^{(j)}(I, \xi, \ldots, \xi^t)^T \oplus (I, \xi, \ldots, \xi^t)^T \\ &\in \overline{W}_{ij} \end{aligned}$$

as required. Thus the extension of every element of $\mathcal{W}$ meets the imaginary s-dimensional subspace S_s of $PG((s+1)(t+1)-1, q^{t+1})$ which is enough to prove the Theorem. □

3.9.2 Corollary

A regular t-spread of $PG((s+1)(t+1)-1, q)$ has an indicator set comprising the $q^{s(t+1)} + q^{(s-1)(t+1)} + \cdots + q^{t+1} + 1$ points of an imaginary s-dimensional subspace of $PG((s+1)(t+1)-1, q^{t+1})$ and conversely an imaginary s-dimensional subspace of $PG((s+1)(t+1)-1, q^{t+1})$ is an indicator set for a regular t-spread of $PG((s+1)(t+1)-1, q)$.

Proof: First, let S_s be an imaginary s-dimensional subspace of $PG((s+1)(t+1)-1, q^{t+1})$. The set

$$\mathcal{W} = \left\{\text{lin } \left\{P, P^{\sigma}, \ldots, P^{\sigma^t}\right\} \cap PG((s+1)(t+1)-1, q) : \ P \in S_s\right\}$$

is a regular t-spread of $PG((s+1)(t+1)-1, q)$ and the points of the imaginary s-dimensional subspace S_s are an indicator set for $\mathcal{W}$ as they are imaginary points and one point lies on the extension of each t-spread element. The line PQ joining any two of the points of S_s is imaginary, for if the space $L(PQ)$ had dimension less than $2(t+1)$ then $L(S_s)$ would have dimension less than $(s+1)(t+1)-1$. Conversely, given a regular t-spread of $PG((s+1)(t+1)-1, q)$, there exists an imaginary s-dimensional subspace S_s of $PG((s+1)(t+1)-1, q^{t+1})$ meeting the extension of every t-spread element. The points of such a s-dimensional subspace are an indicator set for the t-spread. □

3.9.3 Corollary

A regular t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1, q)$ is uniquely determined by a t-regulus $\mathcal{R}_s$ of rank s of $\mathcal{W}$ and an element of $\mathcal{W}$ not belonging to $\mathcal{R}_s$.

Proof: Let $\mathcal{W}$ be a regular t-spread and let S_s be an imaginary s-dimensional subspace of the space $PG((s+1)(t+1)-1, q^{t+1})$ meeting the extensions of every element of $\mathcal{W}$.. The extensions of the elements of a t-regulus $\mathcal{R}_s$ of rank s contained in $\mathcal{W}$ meet S_s in the points of an s-dimensional projective subspace S'_s of order

q of S_s, and the extension of a further element X_0 of $\mathcal{W}$ meets S_s in a point P not belonging to S'_s. Let $\mathcal{W}'$ be a regular t-spread containing the elements of $\mathcal{R}_s$ and the element X_0 of $\mathcal{W}$. We will show that every element of $\mathcal{W}$ is also an element of $\mathcal{W}'$ and the result follows. Choose elements $X_1, X_2, \dots, X_{s+1}$ of $\mathcal{R}_s$. There is a unique t-regulus $\mathcal{R}_s{}'$ of rank s of $PG((s+1)(t+1)-1, q)$ containing $X_0, X_1, \dots, X_{s+1}$, which is distinct from $\mathcal{R}_s$. The s-dimensional subspace S_s is a transversal to the extension of $\mathcal{R}'_s$, thus the extensions of the elements of $\mathcal{R}'_s$ all meet S_s. But the t-dimensional spaces of $PG((s+1)(t+1)-1, q)$ whose extensions meet S_s are exactly the elements of $\mathcal{W}$. Thus every element of $\mathcal{R}'_s$ is an element of $\mathcal{W}$. Now since $\mathcal{W}'$ is regular, it contains every element of $\mathcal{R}'_s$, which are all elements of $\mathcal{W}$. We repeat the argument using different elements of $\mathcal{W}'$ to define t-reguli of rank s, all of which are shown to belong to $\mathcal{W}$, continuing until we have shown that every element of $\mathcal{W}'$ is also an element of $\mathcal{W}$. □

We can use the Segre variety to interpret this result. Choose any $s+2$ distinct elements $W_0, W_1, \dots, W_{s+1}$ of the regular t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1, q)$, no $s+1$ in a hyperplane, and let $\mathcal{R}_s$ be the unique t-regulus of rank s containing them (see Section 3.4). The $q^s + q^{s-1} + \cdots + q + 1$ elements of $\mathcal{R}_s$ are all elements of $\mathcal{W}$ and form the set of t-dimensional spaces of a Segre variety $\mathcal{SV}_{s+1,t+1}$ in $PG((s+1)(t+1)-1, q)$ with s-dimensional subspaces as its opposite subspaces (Corollary 3.4.4 (3)).

We now embed $PG((s+1)(t+1)-1, q)$ in $PG((s+1)(t+1)-1, q^{t+1})$ and extend $\mathcal{SV}_{s+1,t+1}$ to a Segre variety $\overline{\mathcal{SV}}_{s+1,t+1}$ of $PG((s+1)(t+1)-1, q^{t+1})$ as in Section 1.5. Then $\overline{\mathcal{SV}}_{s+1,t+1}$ has $q^s + q^{s-1} + \cdots + q + 1$ t-dimensional subspaces which meet $PG((s+1)(t+1)-1, q)$, and the remaining ones are skew to $PG((s+1)(t+1)-1, q)$. Since the spaces $S_s, S_s^{\sigma}, \dots, S_s^{\sigma^t}$ meet all $q^s + q^{s-1} + \cdots + q + 1$ t-dimensional spaces of $\overline{\mathcal{SV}}_{s+1,t+1}$ which are extensions of t-dimensional spaces of $\mathcal{R}_s$, they must belong to the system of s-dimensional spaces of $\overline{\mathcal{SV}}_{s+1,t+1}$. In fact the extensions of the elements of $\mathcal{R}_s$ meet S_s in the points of an s-dimensional projective subspace of order q of S_s. Two t-reguli of the same or different ranks intersect according to how the corresponding subspaces of order q of S_s intersect. The properties of projective subspaces of a projective space in $PG((s+1)(t+1)-1,q)$ can be used to demonstrate properties of the t-reguli of various ranks and regular t-spreads, as in the following.

3.9.4 Corollary

If $(s+1,t+1)=1$ then a regular t-spread of $PG((s+1)(t+1)-1,q)$ can be expressed as a union of disjoint t-reguli of rank s.

Proof: The result follows since an s-dimensional subspace of order q^{t+1} can be expressed as the union of skew s-dimensional subspaces of order q if and only if $(s+1)(t+1)=1$ (see Hirschfeld (1979), p92). $\square$

3.9.5 Corollary

Let $\mathcal{W}$ be a regular t-spread of $PG((s+1)(t+1)-1,q)$. Then $\mathcal{W}$ has a projective t-spread set

$$\mathcal{PC}=\Big\{\Big(C_1^{(j)},C_2^{(j)},\ldots,C_i^{(j)},I,0,\ldots,0\Big):\\ i=1,2,\ldots,s,\;\; j=1,2,\ldots,q^{i(t+1)}\Big\}\\ \cup\{(I,0,\ldots,0)\}$$

isomorphic to $PG(s,q^{t+1})=S_s$ by Corollary 3.6.4. The elements of $\mathcal{PC}$ which correspond to the elements of a t-regulus $\mathcal{R}_r$ of rank r in $\mathcal{W}$ are a projective r-dimensional subspace $PG(r,q)$ of order q of $PG(s,q^{t+1})$.

Proof: We have seen that a t-regulus $\mathcal{R}_s$ of rank s in $\mathcal{W}$ meets S_s in a projective s-dimensional subspace $PG(s,q)$ of order q. As in Lemma 3.4.5, a t-

subregulus $\mathcal{R}_r$ of rank r contained in $\mathcal{R}_s$ meets S_s in an r-dimensional subspace $PG(r,q)$ of $PG(s,q)$. $\square$

3.10 EXAMPLES

In his paper of (1983), Ebert has given examples of 1-spreads of $PG(5,2)$. We use the example of a regular 1-spread appearing in Example (3), Ebert (1983) and the 1-regulus free 1-spread of Section 4, Ebert (1983) to illustrate the ideas in this Chapter. In the rest of this Section, we assume that $PG((s+1)(t+1)-1,q) = PG(5,2)$ and $\mathcal{W}$ is a 1-spread of $PG(5,2)$ so that $t=1$, $s=2$ and $q=2$.

In this Section we will write points of $PG(5,2)$ and $PG(5,4)$ as row vectors, for ease of notation.

3.10.1 Example

We extend $GF(2)$ to $GF(2^2)=GF(4)$ by adjoining a primitive element ω which is a primitive cube root of unity in $GF(4)$, so that $GF(4)=\{0,1,\omega,\omega^2\}$, where $\omega^2=\omega+1$. Let $PG(5,4)$ be the corresponding extension of $PG(5,2)$. As in Example (3) of Ebert (1983), the following 21 points are points of a plane Π of $PG(5,4)$ which has no point in common with the space $PG(5,2)$. The plane is

$$\Pi = \{\alpha(1,\omega,0,0,0,0)+\beta(0,0,1,\omega,0,0)+\gamma(0,0,0,0,1,\omega) : \\ \alpha,\beta,\gamma \in GF(4), \text{ not all zero}\}.$$

The names of the points X_{ij} are consistent with the use of the subscripts ij with $i=1,2$ and $j=1,2,\ldots,2^{2i}$, together with ∞, in this Chapter and will be used later in this example.

$\Pi =$

$$\{ X_{21} = (0,0,0,0,1,\omega),\ X_{22} = (1,\omega,0,0,1,\omega),\ X_{23} = (1,\omega,0,0,\omega^2,1)$$
$$X_{24} = (1,\omega,0,0,\omega,\omega^2),\ X_{25} = (0,0,1,\omega,1,\omega),\ X_{26} = (1,\omega,1,\omega,1,\omega),$$
$$X_{27} = (1,\omega,\omega^2,1,\omega^2,1),\ X_{28} = (1,\omega,\omega,\omega^2,\omega,\omega^2),\ X_{29} = (0,0,1,\omega,\omega^2,1),$$
$$X_{210} = (1,\omega,\omega,\omega^2,1,\omega),\ X_{211} = (1,\omega,1,\omega,\omega^2,1),\ X_{212} = (1,\omega,\omega^2,1,\omega,\omega^2),$$
$$X_{213} = (0,0,1,\omega,\omega,\omega^2),\ X_{214} = (1,\omega,\omega^2,1,1,\omega),\ X_{215} = (1,\omega,\omega,\omega^2,\omega^2,1),$$
$$X_{216} = (1,\omega,1,\omega,\omega,\omega^2),\ X_{11} = (0,0,1,\omega,0,0),\ X_{12} = (1,\omega,1,\omega,0,0),$$
$$X_{13} = (1,\omega,\omega^2,1,0,0),\ X_{14} = (1,\omega,\omega,\omega^2,0,0),\ X_\infty = (1,\omega,0,0,0,0)\}\,.$$

Joining a point X_{ij} of Π to the corresponding point X^*_{ij} of the plane Π^* which is conjugate to Π under the automorphism $\sigma : x \mapsto x^2$ of $GF(4)$ gives a line $\overline{l_{ij}}$ of $PG(5,4)$ which meets $PG(5,2)$ in a line l_{ij}. The set of all such lines

$$\mathcal{W} = \{l_{ij} : i = 1,2,\ j = 1,2,\ldots,2^{2i}\} \cup \{l_\infty\}$$

is a regular 1-spread of $PG(5,2)$.

We wish to construct a projective 1-spread set from this 1-spread $\mathcal{W}$. By Theorem 3.5.4, the appropriate set is

$$\mathcal{PC} = \{f^{-1}(l_{ij}) :\ i = 1,2,\ j = 1,2,\ldots,2^{2i}\} \cup \{f^{-1}(l_\infty)\}$$

where f is the bijection of Theorem 1.6.2 and each element $f^{-1}(l_{ij})$ is a 3-tuple $(\xi_1,\xi_2,\xi_3)^T$ of 2×2 matrices over $GF(2)$. To construct $f^{-1}(l_{ij})$, recall that l_{ij} is the subspace of $PG(5,2)$ spanned by the columns of $f^{-1}(l_{ij})$ considered as points of $PG(5,2)$. To find the element $(f^{-1}(l_{ij}))^T = (\xi_1,\xi_2,\xi_3)^T$ corresponding to a 1-spread element l_{ij}, we reverse this process. For each line l_{ij} of $\mathcal{W}$, we identify a set of two points spanning l_{ij}, then write the two coordinate 6-tuples of these points as columns of a 2×6 matrix over $GF(2)$. We then interpret this matrix as a 2×3 column vector whose entries are 2×2 matrices over $GF(2)$. The transpose of this column vector is the element $f^{-1}(l_{ij})$ of the projective 1-spread set corresponding to l_{ij}.

For example, the point $X_{21} = (0,0,0,0,1,\omega)$ on Π has corresponding point $X^*_{21} = (0,0,0,0,1,\omega^2)$ on Π^*, and the line $\overline{l_{21}}$ of $PG(5,4)$ joining these two points meets $PG(5,2)$ in the line

$$l_{21} = \{(0,0,0,0,1,0),(0,0,0,0,0,1),(0,0,0,0,1,1)\}.$$

Choosing the first two as a basis for the line, we see that the corresponding 2×6 matrix over $GF(2)$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which gives the element $P_{21} = (0,0,I)$ of the projective 1-spread set.

Continuing in this way we find the whole projective 1-spread set which is, written in the same order as the points of Π above

$$\begin{aligned}\mathcal{PC} = \{\, & P_{21} = (0,0,I),\ P_{22} = (I,0,I),\ P_{23} = (I,0,A),\ P_{24} = (I,0,B),\\ & P_{25} = (0,I,I),\ P_{26} = (I,I,I),\ P_{27} = (I,A,A),\ P_{28} = (I,B,B),\\ & P_{29} = (0,I,A),\ P_{210} = (I,B,I),\ P_{211} = (I,I,A),\ P_{212} = (I,A,B),\\ & P_{213} = (0,I,B),\ P_{214} = (I,A,I),\ P_{215} = (I,B,A),\ P_{216} = (I,I,B),\\ & P_{11} = (0,I,0),\ P_{12} = (I,I,0),\ P_{13} = (I,A,0),\ P_{14} = (I,B,0),\\ & P_{\infty} = (I,0,0)\}\end{aligned}$$

where

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},\ A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ \text{and } B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The set of matrices $\{0, I, A, B\}$ forms a field $GF(4)$ (as predicted by Theorem 3.7.1) , since $AB = BA = I$, $A^2 = B$ and $B^2 = A$, and $I+I = A+A = B+B = 0$, $A + B = I$, $A + I = B$ and $B + I = A$. As in Corollary 3.7.2, this field contains the subfield $\{I, 0\}$ isomorphic to $GF(2)$.

If we identify $GF(2)$ and $GF(4)$ in the above construction with the fields $\{I, 0\}$ and $\{I, 0, A, B\}$ of matrices, then for example $\omega = B$ and the projective 1-spread set $\mathcal{PC}$ is a projective plane over the field $GF(4)$ isomorphic to Π. We shall identify Π and $\mathcal{PC}$ as in Section 3.9, and use the ideas explored there.

We can use Theorem 3.9.5 to identify the 1-reguli contained in $\mathcal{W}$. As $s = 2$ $\mathcal{W}$ contains 1-reguli of ranks 0, 1 and 2. A 1-regulus of rank 0 is just a line of $\mathcal{W}$ and corresponds to a 0-dimensional subspace of Π which is a point of Π. A 1-regulus of rank 2 corresponds to a 2-dimensional subspace of Π of order 2, a Baer subplane. The Baer subplane B coordinatised by $GF(2) = (0, I)$, for example, gives a 1-regulus of rank 2 in $\mathcal{W}$. B comprises the points

$$P_\infty,\ P_{11},\ P_{21},\ P_{12},\ P_{22},\ P_{25} \text{ and } P_{26}$$

and the corresponding 1-regulus of rank 2 is

$$\mathcal{R}_2 = \{l_{21},\ l_{17},\ l_1,\ l_{18},\ l_{22},\ l_{25},\ l_{26}\}.$$

Also, $\mathcal{W}$ has many 1-reguli of rank 1, which correspond to projective sublines of order 2 of lines of Π. As $q = 2$, any three collinear points of Π are a projective subline of order 2, so that 1-reguli of rank 1 correspond to triples of collinear points in Π. Some examples of triples of collinear points of Π are

$$(P_{22},\ P_{214},\ P_{11})$$

$$(P_{24},\ P_{29},\ P_{13})$$

$$(P_{25},\ P_{27},\ P_\infty)$$

$$(P_{26},\ P_{211},\ P_{12})$$

corresponding to the following 1-reguli of rank 1:

$$\{l_{22},\ l_{214},\ l_{11}\}$$

$$\{l_{24},\ l_{29},\ l_{13}\}$$

$$\{l_{25},\ l_{27},\ l_{\infty}\}$$

$$\{l_{26},\ l_{211},\ l_{12}\}.$$

The 1-spread $\mathcal{W}$ is regular, and therefore is geometric (by Theorem 3.4.12) and so by Theorem 3.2.8 has the Shell property 3.2.7 for any division of $PG(5,2)$ into shells such that each element of $\mathcal{W}$ lies in a unique shell. We can therefore construct a 1-spread set $\mathcal{C}$ as described in the proof of Theorem 3.5.7. We disregard the element $P_{\infty} = (I,0,0)$ and any other element of $\mathcal{PC}$ is either $P_{2j} = (\xi_1, \xi_2, \xi_3)$ with $\xi_3 \neq 0$ or $P_{1j} = (\xi_1, \xi_2, 0)$ with $\xi_2 \neq 0$. By the property (iii) in the Definition 3.5.2 of projective 1-spread sets, an element P_{2j} has $\det(\xi_3) \neq 0$ and can be written as

$$P_{2j} = (\xi_1\xi_3^{-1}, \xi_2\xi_3^{-1}, I).$$

The corresponding element C_{2j} of the 1-spread set is

$$C_{2j} = (\xi_1\xi_3^{-1}, \xi_2\xi_3^{-1}) \text{ for } j = 1, 2, \ldots, 16.$$

Similarly, an element P_{1j} of $\mathcal{PC}$ has $\det(\xi_2) \neq 0$ and can be written as

$$P_{1j} = (\xi_1\xi_2^{-1}, I, 0)$$

and the corresponding element of the 1-spread set is

$$C_{1j} = (\xi_1\xi_2^{-1}) \text{ for } j = 1, 2, \ldots, 4.$$

The 1-spread set $\mathcal{C}$ is, again written in the same order,

$$
\begin{aligned}
\mathcal{C} = \{ &(0,0),\ (I,0),\ (B,0),\ (A,0),\ (0,I),\ (I,I),\\
&(B,I),\ (A,I),\ (0,B),\ (I,B),\ (B,B),\\
&(A,B),\ (0,A),\ (I,A),\ (B,A),\ (A,A),\\
&(0),\ (I),\ (B),\ (A)\}
\end{aligned}
$$

corresponding to the new projective 1-spread set

$$
\begin{aligned}
\{ &P_{21} = (0,0,I),\ P_{22} = (I,0,I),\ P_{23} = (B,0,I),\ P_{24} = (A,0,I),\\
&P_{25} = (0,I,I),\ P_{26} = (I,I,I),\ P_{27} = (B,I,I),\ P_{28} = (A,I,I),\\
&P_{29} = (0,B,I),\ P_{210} = (I,B,I),\ P_{211} = (B,B,I),\ P_{212} = (A,B,I),\\
&P_{213} = (0,A,I),\ P_{214} = (I,A,I),\ P_{215} = (B,A,I),\ P_{216} = (A,A,I),\\
&P_{11} = (0,I,0),\ P_{12} = (I,I,0),\ P_{13} = (B,I,0),\ P_{14} = (A,I,0)\}\\
&\cup \{ P_\infty = (I,0,0)\}\,.
\end{aligned}
$$

This projective 1-spread set is also a projective 1-spread set for $\mathcal{W}$ as it is obtained from the original one by multiplying each 3-tuple by a non-singular matrix. This operation corresponds to fixing a particular division of $PG(5,2)$ into shells where each line of $\mathcal{W}$ is contained in a unique shell. In this case the shells are chosen to be:

$$
\begin{aligned}
PG(1,2) &= l_\infty = \text{lin } \{e_1, e_2\}\\
PG(3,2) &= < l_\infty, l_{11} >= \text{lin } \{e_1, e_2, e_3, e_4\}\\
PG(5,2) &= < l_\infty, l_{11}, l_{22} >= \text{lin } \{e_1, e_2, e_3, e_4, e_5, e_6\}\,.
\end{aligned}
$$

From now on we shall use this projective 1-spread set for $\mathcal{W}$, denoted by $\mathcal{PC}$. We now demonstrate the ideas of Section 3.9. We proceed to construct a projective indicator set $\mathcal{PI}$ and an indicator set $\mathcal{I}$ for $\mathcal{W}$.

As $\mathcal{PC}$ is a normalised projective 1-spread set for $\mathcal{W}$, a projective indicator set $\mathcal{PI}$ is

$$
\mathcal{PI} = \{Q_{ij} :\ i = 1,2,\ j = 1,2,\ldots,2^{2i}\} \cup \{Q_\infty\}
$$

where

$$Q_{ij} = (P_{ij})^T \begin{pmatrix} 1 \\ \omega \end{pmatrix}.$$

For example, $P_{21} = (0, 0, I)$ so that

$$\begin{aligned} Q_{21} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix} \\ &= (0, 0, 0, 0, 1, \omega), \end{aligned}$$

and $P_{23} = (B, 0, I)$ so that

$$\begin{aligned} Q_{23} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix} \\ &= (\omega, \omega + 1, 0, 0, 1, \omega) \\ &= (\omega, \omega^2, 0, 0, 1, \omega). \end{aligned}$$

The projective indicator set is, therefore,

$$\mathcal{PC} =$$

$$\{\, Q_{21} = (0,0,0,0,1,\omega),\ Q_{22} = (1,\omega,0,0,1,\omega),\ Q_{23} = (\omega,\omega^2,0,0,1,\omega)$$

$$Q_{24} = (\omega^2,1,0,0,1,\omega),\ Q_{25} = (0,0,1,\omega,1,\omega),\ Q_{26} = (1,\omega,1,\omega,1,\omega),$$

$$Q_{27} = (\omega,\omega^2,1,\omega,1,\omega),\ Q_{28} = (\omega^2,1,1,\omega,1,\omega),\ Q_{29} = (0,0,\omega,\omega^2,1,\omega),$$

$$Q_{210} = (1,\omega,\omega,\omega^2,1,\omega),\ Q_{211} = (\omega,\omega^2,\omega,\omega^2,1,\omega),\ Q_{212} = (\omega^2,1,\omega,\omega^2,1,\omega),$$

$$Q_{213} = (0,0,\omega^2,1,1,\omega),\ Q_{214} = (1,\omega,\omega^2,1,1,\omega),\ Q_{215} = (\omega,\omega^2,\omega^2,1,1,\omega),$$

$$Q_{216} = (\omega^2,1,\omega^2,1,1,\omega),\ Q_{11} = (0,0,1,\omega,0,0),\ Q_{12} = (1,\omega,1,\omega,0,0),$$

$$Q_{13} = (\omega,\omega^2,1,\omega,0,0),\ Q_{14} = (\omega^2,1,1,\omega,0,0),\ Q_{\infty} = (1,\omega,0,0,0,0)\}\,.$$

Note that $\mathcal{PC}$ is exactly the plane Π because $Q_{ij} = X_{ij}$ for all $i = 1,2$ and $j = 1,2,\ldots,2^{2i}$ and $Q_\infty = X_\infty$. Now $\mathcal{W}$ has the Shell property 3.2.7 where the shells are

$$PG(1,2) = \text{lin } \{e_1, e_2\}$$

$$PG(3,2) = \text{lin } \{e_1, e_2, e_3, e_4\}$$

$$PG(5,2) = \text{lin } \{e_1, e_2, e_3, e_4, e_5, e_6\}.$$

Let

$$PG^*(1,4) = \text{lin } \{e_1 + \omega e_2\}$$

$$PG^*(2,4) = \text{lin } \{e_1, e_2, e_3 + \omega e_4\}$$

$$PG^*(4,4) = \text{lin } \{e_1, e_2, e_3, e_4, e_5 + \omega e_6\}.$$

Then we see that

$$Q_\infty = (1, \omega, 0, 0, 0, 0) \in PG^*(1,4)$$

$$Q_{1j} = (a_{1j}, b_{1j}, 1, \omega, 0, 0) \in PG^*(2,4) - PG(1,4)$$

$$Q_{2j} = (a_{2j}, b_{2j}, c_{2j}, d_{2j}, 1, \omega) \in PG^*(4,4) - PG(3,4).$$

Thus $\mathcal{PI}$ has the Shell property 3.8.5 for projective indicator sets.

3.10.2 Example

Ebert (1983), Section 4, lists the lines $l_1, l_2, \ldots, l_{21}$ of a 1-regulus free 1-spread $\mathcal{W}$ of $PG(5,2)$. By the same process used in Example 2.8.1, we can construct a projective 1-spread set for this 1-spread $\mathcal{W}$. We obtain the set

$$\begin{aligned}\mathcal{PC} = \{\, & P_1 = (A,I,B),\ P_2 = (I,I,B),\ P_3 = (I,C,B),\ P_4 = (D,E,B),\\ & P_5 = (I,F,B),\ P_6 = (G,H,E),\ P_7 = (E,I,E),\ P_8 = (I,0,J),\\ & P_9 = (I,G,E),\ P_{10} = (I,H,J),\ P_{11} = (K,I,I),\ P_{12} = (L,K,E),\\ & P_{13} = (M,I,K),\ P_{14} = (G,C,I),\ P_{15} = (K,B,J),\ P_{16} = (L,G,J),\\ & P_{17} = (N,I,M),\ P_{18} = (L,I,J),\ P_{19} = (B,E,I),\ P_{20} = (J,P,I),\\ & P_{21} = (0,0,I)\}\end{aligned}$$

where

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ E = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \ F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ K = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \ M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ N = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is impossible to normalise this projective 1-spread set, and impossible to find a 1-spread set. The 1-spread $\mathcal{W}$ does not have the Shell property for any division of $PG(5,2)$ into shells.

We can, however, find a projective indicator set for $\mathcal{W}$, as in Section 3.9 of this Chapter. Let ω be a primitive element for $GF(4)$ as a field extension of $GF(2)$. A projective indicator set is

$$\begin{aligned}
&\mathcal{PI} = \\
&\{Q_1 = (\omega^2,0,\omega,1,1,0),\ Q_2 = (1,\omega,1,\omega,1,0);\ Q_3 = (\omega,1,\omega,\omega^2,1,0), \\
&Q_4 = (0,1,\omega,\omega,1,0),\ Q_5 = (1,\omega,\omega^2,\omega^2,1,0),\ Q_6 = (\omega,0,0,1,\omega,\omega), \\
&Q_7 = (\omega,\omega,1,\omega,\omega,\omega),\ Q_8 = (\omega,1,0,0,0,\omega),\ Q_9 = (1,\omega,\omega,0,\omega,\omega), \\
&Q_{10} = (1,\omega,0,1,0,\omega),\ Q_{11} = (0,\omega^2,1,\omega,1,\omega),\ Q_{12} = (1,1,0,\omega^2,\omega,\omega), \\
&Q_{13} = (\omega,1,1,\omega,0,\omega^2),\ Q_{14} = (\omega,0,\omega,\omega^2,1,\omega),\ Q_{15} = (0,\omega^2,1,0,0,\omega), \\
&Q_{16} = (1,1,\omega,0,0,\omega),\ Q_{17} = (\omega^2,1,\omega,1,1,\omega),\ Q_{18} = (1,1,1,\omega,0,\omega), \\
&Q_{19} = (1,0,\omega,\omega,1,\omega),\ Q_{20} = (0,\omega,\omega^2,\omega,1,\omega),\ Q_{21} = (0,0,0,0,1,\omega)\}.
\end{aligned}$$

To check that the 1-spread $\mathcal{W}$ has no regulus, all we need to do is check that no three points of $\mathcal{PI}$ are collinear in $PG(5,2)$. This is simpler than the approach used by Ebert (1983), who conducted a computer search to show that no three lines in the 1-spread have more than one transversal in $PG(5,2)$.

In fact, the set $\mathcal{PI}$ is a 21-cap in $PG(5,4)$, having no point in common with $PG(5,2)$.

CHAPTER FOUR

PARTIAL t-SPREADS OF $PG((s+1)(t+1)-1,q)$

4.1 INTRODUCTION AND DEFINITIONS

In this Chapter we will consider partial t-spreads of $PG((s+1)(t+1)-1,q)$, which have useful connections with other geometrical objects. There are certain partial t-spreads which can be used to investigate k-arcs and (k,n)-arcs of projective planes.

Partial 1-spreads of $PG(3,q)$ have been studied by many authors, including Mesner (1967), Bruen (1971), Bruen and Thas (1976), Ebert (1979) and Glynn (1982). The main work was devoted to determining the maximum and minimum number of elements that a partial 1-spread could contain while not being embeddable in a larger partial 1-spread of $PG(3,q)$, and classifying these 1-spreads for small values of q. The concept of a partial 1-spread was generalised to cover partial t-spreads of $PG(d,q)$ by Beutelspacher (1975).

We shall concentrate on the connections between the partial t-spreads and k-arcs and (k,n)-arcs in projective planes. First we introduce some definitions and preliminary results on partial t-spreads of $PG(d,q)$.

4.1.1 Definition [Beutelspacher (1975)]

(1) A *partial t-spread* $\mathcal{W}$ of $PG(d,q)$ is a set of pairwise skew t-dimensional subspaces of $PG(d,q)$. In other words, any point of $PG(d,q)$ is contained in at most one element of $\mathcal{W}$.

(2) A partial t-spread which is not a t-spread is called *strictly partial*.

(3) A strictly partial t-spread $\mathcal{W}$ not contained in any partial t-spread $\mathcal{W}'$ as a

proper subset is called a *complete* strictly partial t-spread.

A partial t-spread $\mathcal{W}$ of $PG(d,q)$ is a t-spread of $PG(d,q)$ if each point of $PG(d,q)$ is contained in an element of $\mathcal{W}$, which is necessarily unique. If $\mathcal{W}$ is a strictly partial t-spread of $PG(d,q)$ then there are points of $PG(d,q)$ which lie on no element of $\mathcal{W}$.

In the literature, the idea expressed in (3) of Definition 4.1.1 is often called "maximal". However for the purpose of this work the term "complete" seems more appropriate, as the concept is the same as, for example, that of complete k-arcs (see Hirschfeld (1979), p163). We prefer to reserve the term "maximal" for a concept analogous to that of maximal (k,n)-arcs (see Hirschfeld (1979), p324) which will be introduced for partial t-spreads in Section 4.4.

The next definitions represent a generalisation of the concept of a geometric t-spread of $PG(d,q)$ as in Definition 1.2.3.

4.1.2 Definition [Beutelspacher (1975)]

(1) A partial t-spread $\mathcal{W}$ is said to *induce a partial t-spread* on the subspace $< W_1, W_2 >$ spanned by the distinct elements W_1 and W_2 of $\mathcal{W}$ if any element of $\mathcal{W}$ having a point in common with $< W_1, W_2 >$ is entirely contained in $< W_1, W_2 >$.

(2) A partial t-spread $\mathcal{W}$ is *geometric* if for each pair W_1 and W_2 of distinct elements of $\mathcal{W}$, $\mathcal{W}$ induces a partial t-spread on $< W_1, W_2 >$.

(3) A geometric partial t-spread $\mathcal{W}$ is called *μ-geometric* if for each pair W_1, W_2 of elements of $\mathcal{W}$, the space $< W_1, W_2 >$ contains exactly $(\mu+1)$ elements of $\mathcal{W}$.

(4) A partial t-spread $\mathcal{W}$ is *ν-uniform* if any $(t+1)$-dimensional subspace through an element of $\mathcal{W}$ meets exactly ν elements of $\mathcal{W}$, each in a (necessarily unique) point. (If such a $(t+1)$-dimensional space S_{t+1} through an element $W_1 \in \mathcal{W}$ were to meet an element $W_2 \in \mathcal{W}$ in more than a point, then the intersection $S_{t+1} \cap W_2$ would contain a line which must then meet the first element W_1 of $\mathcal{W}$, giving two elements W_1 and W_2 of $\mathcal{W}$ with a common point).

For a geometric partial t-spread $\mathcal{W}$ of $PG(d,q)$ let $\mathcal{I} = (P, B, I)$ be the following incidence structure:

- the points of $\mathcal{I}$ are the elements of $\mathcal{W}$,
- the blocks of $\mathcal{I}$ are the subspaces $< W_1, W_2 >$ where W_1 and W_2 are distinct elements of $\mathcal{W}$, and
- the incidence is set-theoretic inclusion. Then:

4.1.3 Theorem [Beutelspacher (1975), Theorem 5.1]
Let $\mathcal{W}$ be a geometric complete partial t-spread of $PG(d,q)$, with $d \geq 2t+1$. Then the incidence structure $\mathcal{I}$ consists of the points and lines of a projective space of dimension at least one. □

4.1.4 Corollary [Beutelspacher (1975)]
A geometric complete partial t-spread of $PG(d,q)$ with $d \geq 2t+1$ is μ-geometric where the incidence structure $\mathcal{I}$ is a projective space of order μ. □

The following result characterises the 1-uniform partial t-spreads:

4.1.5 Theorem [Beutelspacher (1975)]
Let $\mathcal{W}$ be a partial t-spread of $PG(d,q)$. Then $\mathcal{W}$ is 1-uniform if and only if the

following conditions hold:

(i) Any three elements of $\mathcal{W}$ span a $(3t+2)$-dimensional subspace of $PG(d,q)$, and

(ii) for any $W_0 \in \mathcal{W}$ the set

$$\{< W_0, W >: \ W \in \mathcal{W} - \{W_0\}\}$$

is a t-spread of the quotient geometry $PG(d,q)/W_0$. Recall (see Dembowski (1968), p25) that the quotient geometry $PG(d,q)/W_0$ consists of all subspaces of $PG(d,q)$ containing W_0. This is a projective space of dimension $d-t-1$ and order q. □

The $(2t+1)$-dimensional subspaces of $PG(d,q)$ are important for finding incidence structures on a partial t-spread. We introduce the following terminology.

4.1.6 Definition

Let $\mathcal{W}$ be a partial t-spread of $PG(d,q)$, with $d \geq 2t+1$.

(1) A $(2t+1)$-dimensional subspace S_{2t+1} of $PG(d,q)$ is called an *i-secant* of $\mathcal{W}$ if it contains exactly i elements of $\mathcal{W}$, and is skew to each of the remaining elements of $\mathcal{W}$.

(2) A *secant* of $\mathcal{W}$ is a $(2t+1)$-dimensional subspace of $PG(d,q)$ which is an i-secant of $\mathcal{W}$ for some value of i.

Not every $(2t+1)$-dimensional subspace of $PG(d,q)$ is a secant of $\mathcal{W}$, as there are $(2t+1)$-dimensional subspaces of $PG(d,q)$ which intersect an element W of $\mathcal{W}$ in at least a point but do not contain the whole of the space W. If $\mathcal{W}$ is geometric then every $(2t+1)$-dimensional subspace which contains two elements of $\mathcal{W}$ is a secant of $\mathcal{W}$. If in addition $\mathcal{W}$ is μ-geometric then a $(2t+1)$-dimensional subspace containing two elements of $\mathcal{W}$ is a $(\mu+1)$-secant.

The methods of projective t-spread sets and projective indicator sets developed in Chapters Two and Three can be used to study partial t-spreads. We therefore make the following definitions 4.1.7 and 4.1.9 with regard to partial t-spreads.

4.1.7 Definition

A *partial projective t-spread set* $\mathcal{PPC}$ is a set of $(s+1)$-tuples of $(t+1)\times(t+1)$ matrices such that

(i) $\mathcal{PPC}$ has k elements, where

$$1 \leq k \leq \omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \cdots + q^{t+1} + 1$$

(ii) If $P_i = (\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)})$ is an element of $\mathcal{PPC}$, then

$$\operatorname{rank}\begin{pmatrix} \xi_0^{(i)} \\ \xi_1^{(i)} \\ \vdots \\ \xi_s^{(i)} \end{pmatrix} = t+1$$

(iii) If $P_i = (\xi_0^{(i)}, \xi_1^{(i)}, \ldots, \xi_s^{(i)})$ and $P_j = (\xi_0^{(j)}, \xi_1^{(j)}, \ldots, \xi_s^{(j)})$ are distinct elements of $\mathcal{PPC}$ then

$$\operatorname{rank}\begin{pmatrix} \xi_0^{(i)} & \xi_0^{(j)} \\ \xi_1^{(i)} & \xi_1^{(j)} \\ \vdots & \vdots \\ \xi_s^{(i)} & \xi_s^{(j)} \end{pmatrix} = 2(t+1).$$

Arguments similar to those given in Theorems 3.5.4 and 3.5.5 can be used to prove the next statement:

4.1.8 Theorem

Let $\mathcal{W}$ be a partial t-spread of $PG((s+1)(t+1)-1,q)$. Then the set

$$\mathcal{PPC} = \{f^{-1}(W_i): \; W_i \in \mathcal{W}\}$$

(where f is the bijection of Theorem 1.6.2) is a partial projective t-spread set. Conversely, let $\mathcal{PPC}$ be a partial projective t-spread set. Then the set

$$\mathcal{W} = \{f(P_i): \; P_i \in \mathcal{PPC}\}$$

is a partial t-spread of $PG((s+1)(t+1)-1,q)$.

Proof: Theorems 3.5.4 and 3.5.5 are proved with $k=\omega$. The same arguments are valid when k is used in place of ω, and the projective t-spread sets and t-spreads are partial. $\square$

4.1.9 Definition

A *partial projective indicator set* is a set $\mathcal{PPI}$ of k imaginary points of the space $PG((s+1)(t+1)-1,q^{t+1})$, where

$$1 \leq k \leq \omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \cdots + q^{t+1} + 1,$$

and with the added property that the line joining any two points of $\mathcal{PPI}$ is imaginary.

4.1.10 Theorem

Let $\mathcal{PPI}$ be a partial projective indicator set, and for each point Q_i of $\mathcal{PPI}$ let

$$L(Q_i) = \operatorname{lin}\left\{Q_i, Q_i^{\sigma}, \ldots, Q_i^{\sigma^t}\right\}.$$

Then the set

$$\mathcal{W}(\mathcal{PPI}) = \{L(Q_i) \cap PG((s+1)(t+1)-1,q) : Q_i \in \mathcal{PPI}\}$$

is a partial t-spread of $PG((s+1)(t+1)-1,q)$. Conversely, let $\mathcal{W}$ be a partial t-spread of $PG((s+1)(t+1)-1,q)$. Then there exists a partial projective indicator set $\mathcal{PPI}$ in $PG((s+1)(t+1)-1,q^{t+1})$ such that

$$\mathcal{W} = \mathcal{W}(\mathcal{PPI})$$

constructed as above.

Proof: Again the proofs of Theorems 3.8.3 and 3.8.4 need only be modified to allow partial t-spreads and partial projective t-spread sets. □

4.2 k-ARCS AND (k,n)-ARCS OF $PG(2,q)$

In this Section we introduce the k-arcs and (k,n)-arcs of the projective plane $PG(2,q)$. These are important objects in the theory of projective geometry, and have been studied by many authors. Hirschfeld (1979) provides a very good introduction to the topic with direction to the original source material.

4.2.1 Definitions [Hirschfeld (1979), p163]

(1) A *k-arc* $\mathcal{C}$ of $PG(2,q)$ is a set of k points, no three of which are collinear.

(2) A k-arc of $PG(2,q)$ is *complete* if it is not contained in a $(k+1)$-arc of $PG(2,q)$.

(3) The maximum number of points that a k-arc of $PG(2,q)$ can have is denoted by $m(2,q)$, and a k-arc with this number of points is called an *oval*.

4.2.2 Theorem [Bose (1947), Qvist (1952), Segre (1955), Cossu (1960)]
If q is odd then $m(2,q)=q+1$ and an oval comprises the points of an irreducible conic. If q is even then $m(2,q)=q+2$, and the ovals of $PG(2,q)$, for q even, have not yet been completely classified. □

The examples of $(q+2)$-arcs in $PG(2,q)$, $q=2^h$, known up to 1979 are given in Hirschfeld (1979). Glynn (1983) gives two new infinite sequences of $(q+2)$-arcs in $PG(2,q)$, $q=2^h$, and gives a complete list of such $(q+2)$-arcs known up to 1982. Recently a new infinite sequence of $(q+2)$-arcs (conjectured by W. Cherowitzo) has been verified by Glynn and Payne (1987).

The (k,n)-arc of $PG(2,q)$ is a natural generalisation of a k-arc of $PG(2,q)$. Again, Hirschfeld (1979) provides excellent introduction and list of references for the topic.

4.2.3 Definition [Barlotti (1955)]

(1) A (k,n)-*arc* $\mathcal{C}$ of $PG(2,q)$ is a set of k points such that some line of $PG(2,q)$ meets $\mathcal{C}$ in exactly n points and such that no line meets $\mathcal{C}$ in more than n points, where $n \geq 2$.

(2) A line of $PG(2,q)$ meeting $\mathcal{C}$ in exactly i points is called an i-*secant* of $\mathcal{C}$.

A $(k,2)$-arc of $PG(2,q)$ is a k-arc of $PG(2,q)$. Irreducible algebraic curves of order n in $PG(2,q)$ give examples of (k,n')-arcs with $n' \leq n$, but very little is known about (k,n)-arcs in general. Even the maximum value for k such that (k,n)-arcs actually exist given q and n is known for only a few values of q and n.

4.2.4 Theorem [Tallini-Scafati (1966)]

If $\mathcal{C}$ is a (k,n)-arc of $PG(2,q)$, then $k \leq (n-1)q+n$. □

We shall concentrate our attention primarily on the (k,n)-arcs which admit the largest possible value of n.

4.2.5 Definition [Hirschfeld (1979), p324]

A (k,n)-arc $\mathcal{C}$ of $PG(2,q)$ with $k=(n-1)q+n$ is called *maximal*.

4.2.6 Theorem [Cossu (1961)]

Let $\mathcal{C}$ be a maximal (k, n)-arc of $PG(2, q)$.

(1) If $n = q + 1$ then $\mathcal{C}$ comprises all the points of $PG(2, q)$,

(2) if $n = q$ then $\mathcal{C} = PG(2, q) - l$ where l is a line of $PG(2, q)$, and

(3) if $2 \le n \le q$ then n divides q and the dual of the complement of $\mathcal{C}$ forms a $(q(q + 1 - n)/n, q/n)$-arc which is also maximal. □

4.2.7 Corollary [Hirschfeld (1979) p324]

A (k, n)-arc is maximal if and only if every line in $PG(2, q)$ is a 0-secant or an n-secant. □

There are few results on the existence of maximal (k, n)-arcs. For $q = 2^h$, there are examples of maximal (k, n)-arcs for every value of n dividing q (see Theorem 4.2.8). When q is odd, there is no example of a maximal (k, n)-arc known, and in fact if $q = 3^h$ and $n = 3$ it is known that no maximal $(k, 3)$-arc exists (see Theorem 4.2.9). First we construct examples of maximal (k, n)-arcs in $PG(2, 2^h)$. This work is due to Denniston (1969), but we follow the presentation in Hirschfeld (1979).

Let $x^2 + bx + 1$ be an irreducible quadratic over $GF(2^h)$ and let $\mathcal{L}$ be the pencil of conics

$$\mathcal{L} = \{Q_\lambda : \ \lambda \in GF(2^h) \cup \infty\}$$

where for $\lambda \in GF(2^h)$ the conic Q_λ has equation

$$x_0^2 + bx_0x_1 + x_1^2 + \lambda x_2^2 = 0$$

and Q_∞ has equation

$$x_2^2 = 0.$$

The additive group of $GF(2^h)$ has subgroups of every order dividing 2^h.

4.2.8 Theorem [Denniston (1969)]
Let H be a subgroup of the additive group of $GF(2^h)$ of order n, where n divides 2^h. Let $\mathcal{C}$ be the set of points of $PG(2, 2^h)$ which lie on some conic Q_λ for $\lambda \in H$, so that

$$\mathcal{C} = \left\{ P \in PG(2, 2^h) : \; P \in \bigcup_{\lambda \in H} Q_\lambda \right\}.$$

Then $\mathcal{C}$ is a maximal (k, n)-arc of $PG(2, 2^h)$. □

Other (maximal) $(2^{3m} - 2^{2m} + 2^m, 2^m)$-arcs of $PG(2, 2^{2m})$ are constructed by Thas (1974). In Cossu (1961) it is shown that there does not exist a (maximal) (21,3)-arc in $PG(2, 9)$. This result is obtained as a corollary of the following.

4.2.9 Theorem [Thas (1975)]
In $PG(2, q)$ where $q = 3^h$ and $h > 1$, there are no (maximal) $(2q + 3, 3)$-arcs. □

As we shall see, the connection mentioned between the partial t-spreads and the k-sets and (k, n)-sets occurs between partial t-spreads of $PG(3t + 2, q)$ and k-sets and (k, n)-sets of $PG(2, q^{t+1})$, for $t \geq 2$. The following Theorem holds in the special case of $t = 1$.

4.2.10 Theorem
Let $\Pi = PG(2, q)$ be a projective subplane of order q of $\overline{\Pi} = PG(2, q^2)$. Suppose that Π contains a maximal (k, n)-arc $\mathcal{C}$. If $\mathcal{C}$ is a proper subset of a $(\overline{k}, n)$-arc $\overline{\mathcal{C}}$ of $\overline{\Pi}$ then $\overline{k} \leq q^2 + n$.

Proof: As $\overline{\mathcal{C}}$ is a $(\overline{k}, n)$-arc, no line of $\overline{\Pi}$ can contain more than n points of $\overline{\mathcal{C}}$. A point of Π is either a point of $\mathcal{C}$ or is a point not on $\mathcal{C}$ through which there pass exactly $q + 1 - q/n$ n-secants of $\mathcal{C}$, so that no point of Π can be a point of $\overline{\mathcal{C}}$.

Thus $\overline{\mathcal{C}}$ has exactly $(n-1)q+n$ points in Π, namely the points of $\mathcal{C}$. Any points of $\overline{\mathcal{C}}$ which lie in $\overline{\Pi}$-Π must lie on lines which are the extensions to $\overline{\Pi}$ of 0-secants of $\mathcal{C}$ in Π. Any such line may contain at most n points of $\overline{\mathcal{C}}$. Thus

$$\overline{k} \leq (n-1)q + n + n\tau_0$$

where τ_0 is the number of 0-secants of $\mathcal{C}$ in Π, which is $(q(q+1-n))/n$. So we have

$$\begin{aligned} \overline{k} &\leq (n-1)q + n + q(q+1-n) \\ &= q^2 + n. \end{aligned}$$

4.2.11 Corollary

If a maximal $(\overline{k}, n)$-arc of $PG(2, q^2)$ contains the points of a maximal (k, n)-arc of a Baer subplane $PG(2, q)$, then $n = 2$.

Proof: Suppose that $\overline{\mathcal{C}}$ is a maximal $(\overline{k}, n)$-arc of $PG(2, q^2)$, then by Definition 4.2.4,

$$\overline{k} = (n-1)q^2 + n.$$

By Theorem 4.2.10, we see that

$$\overline{k} \leq q^2 + n.$$

These can be equal only in the case n=2. □

When $n = 2$, q is a power of 2 and the (k, n)-arcs are just k-arcs. This Corollary shows that an oval of $PG(2, q^2)$ is the only maximal $(\overline{k}, n)$-arc which can admit a subset being a maximal (k, n)-arc in a subplane of order q. Suppose that the oval $\mathcal{C}$ is the set of points of an irreducible conic together with its nucleus in $PG(2, q)$. If we extend $GF(q)$ to $GF(q^2)$ and $PG(2, q)$ to $PG(2, q^2)$ then the conic extends to a conic of $PG(2, q^2)$ with the same nucleus. This is the situation described in the Corollary 4.2.11.

4.3 k-SETS OF $PG(3t+2,q)$

In Thas (1971) a construction is given for k-arcs of the space $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$, where a k-arc of the space $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$ is a set of k points of $\mathcal{S}_m(\mathcal{M}_n(GF(q)))$, every $m+1$ of which are in clear position. Under the bijection f of Theorem 1.6.2, this k-arc corresponds to a set of k pairwise skew $(n-1)$-dimensional subspaces of $PG((m+1)n-1,q)$, every $m+1$ of which span $PG((m+1)n-1,q)$. It was this construction which motivated the definition of k-sets of $(n-1)$-dimensional subspaces in $PG(3n-1,q)$ given in Casse and Wild (1983). The following is a generalisation of that definition. We will adhere to the notation used in the previous Section and Chapters, that is we will put $n=t+1$ and $m=s$.

4.3.1 Definition

A *k-set* of $PG((s+1)(t+1)-1,q)$ is a collection of k t-dimensional subspaces, any $s+1$ of which span $PG((s+1)(t+1)-1,q)$.

In this Section we will be interested particularly in the k-sets of $PG(3t+2,q)$, or collections of k t-dimensional subspaces of $PG(3t+2,q)$, any three of which span $PG(3t+2,q)$.

The following constructions appear in Thas (1971). They show a natural connection between the k-arcs of $PG(2,q^{t+1})$ and the k-sets of $PG(3t+2,q)$.

4.3.2 Constructions [Thas (1971)]

(1) There exist $(q^{t+1}+1)$-sets in $PG(3t+2,q)$.

(2) If $q=2^h$ there exist $(q^{t+1}+2)$-sets in $PG(3t+2,q)$.

Proof: (1) Let $GF(q^{t+1})$ be an extension (of degree $t+1$) of $GF(q)$ and let $PG(3t+2,q^{t+1})$ be the corresponding extension of $PG(3t+2,q)$. Let Π be an imaginary plane of $PG(3t+2, q^{t+1})$. Then the $t+1$ conjugates of Π under the collineation induced by the automorphism $\sigma : x \mapsto x^q$ of $GF(q^{t+1})$ span the whole of $PG(3t+2, q^{t+1})$. Each point P of Π is imaginary (see Theorem 1.4.8 (3)), and by Definition 1.4.7 (1) and Corollary 1.4.10 (2) the $t+1$ conjugates of P span a t-dimensional subspace $L(P)$ of $PG(3t+2, q^{t+1})$ which meets $PG(3t+2, q)$ in a t-dimensional subspace. The points of a $(q^{t+1}+1)$-arc $\mathcal{C}$ in Π determine a set of $q^{t+1}+1$ t-dimensional subspaces $\mathcal{K}$ of $PG(3t+2, q)$. As no three points of $\mathcal{C}$ are collinear, any three of them span Π. The set of all conjugates of any three points of $\mathcal{C}$ span a space of dimension $3t+2$, and so any three elements of $\mathcal{K}$ span a space of dimension $3t+2$. Thus $\mathcal{K}$ is a $(q^{t+1}+1)$-set of $PG(3t+2, q)$.

(2) If $q = 2^h$, we repeat the construction with $\mathcal{C}$ a $(q^{t+1}+2)$-arc of Π. □

4.3.3 Theorem

If $\mathcal{K}$ is a k-set of $PG(3t+2, q)$ arising from Construction 4.3.2 then the corresponding k-arc $\mathcal{C}$ used in the construction is a partial projective indicator set for $\mathcal{K}$.

Proof: Since $\mathcal{C}$ lies on an imaginary plane of $PG(3t+2, q^{t+1})$, it is a set of k imaginary points of $PG(3t+2, q^{t+1})$ such that the line joining any two points of $\mathcal{C}$ is imaginary. By Definition 4.1.9, $\mathcal{C}$ is a partial projective indicator set, and each point of $\mathcal{C}$ lies on the extension to $PG(3t+2, q^{t+1})$ of the corresponding element of $\mathcal{K}$. By Corollary 1.4.10 (2), such a space through an imaginary point Q_i is the unique space $L(Q_i)$, and thus

$$\mathcal{K} = \{L(Q_i) \cap PG(3t+2, q) : \; Q_i \in \mathcal{C}\}.$$

□

The Construction 4.3.3 can be repeated with $\mathcal{C}$ a k-arc of Π, $2 \le k \le m(2, q)$ in which case $\mathcal{K}$ is a k-set of $PG(3t+2, q)$. In fact it can be shown that:

4.3.4 Theorem [Thas (1971)]
Let $\mathcal{K}$ be a k-set of $PG(3t+2,q)$. If q is even then $k \leq (q^{t+1}+2)$, while if q is odd then $k \leq (q^{t+1}+1)$. By Constructions 4.3.2 these bounds are realised. □

We have seen that if there exists a k-arc of $PG(2,q^{t+1})$ then there exists a k-set of $PG(3t+2,q)$. In fact these are the only examples of $(q^{t+1}+1)$-sets and $(q^{t+1}+2)$-sets of $PG(3t+2,q)$ known. However, in Section 4.5 a 4-set of $PG(5,2)$ will be given which does not arise from the Construction 4.3.2. It does not have a partial indicator set which is the set of points of a k-arc of an imaginary plane of $PG(5,2^2)$.

The converse is an interesting question. Given a k-set $\mathcal{K}$ of $PG(3t+2,q)$ (with possibly some restriction on the size of k), is there always an imaginary plane Π of $PG(3t+2,q^{t+1})$ meeting the extension of every element of $\mathcal{K}$? Or equivalently, is every k-set $\mathcal{K}$ contained in a regular t-spread of $PG(3t+2,q)$? This question has been addressed by, for example, Casse and Wild (1983), Casse et al (1985), Wild (1986) and Declerck et al (1987). They have shown that under certain circumstances, a k-set is contained in a regular t-spread of $PG(3t+2,q)$ or, in other words, arises from the Construction 4.3.2. Some results which they found are given in Theorems 4.3.5–4.3.8 below.

4.3.5 Theorem [Casse and Wild (1983), Theorem 3]
Let $\mathcal{K}$ be a $(q^{t+1}+1)$-set of $PG(3t+2,q)$ with q odd. Suppose that the projection of $\mathcal{K}$ from some $X \in \mathcal{K}$ onto a $(2t+1)$-dimensional subspace S_{2t+1} skew to X yields a regular t-spread $\mathcal{W}$. Then $\mathcal{K}$ arises from Construction 4.3.2.

Proof: The proof uses the method of indicator sets, showing that such a set $\mathcal{K}$ has a particular indicator space with an indicator set comprising q^{t+1} points of an irreducible conic in an imaginary plane of $PG(3t+2,q^{t+1})$. □

This result was improved and presented in a different way, using the theory of Generalised Quadrangles, in Casse et al (1985). A $(q^{t+1}+1)$-set gives rise to different types of t-spreads on $(2t+1)$-dimensional spaces S_{2t+1}, classified by the nature of the space S_{2t+1}. When q is odd these types are given as:

(a) Let $\mathcal{K} = \{X_0, X_1, \ldots, X_{q^{t+1}}\}$ be a $(q^{t+1}+1)$-set of $PG(3t+2, q)$, and let Y_i be the tangent space to $\mathcal{K}$ at X_i for $i = 0, 1, \ldots, q^{t+1}$. Let S_{2t+1} be a $(2t+1)$-dimensional subspace of $PG(3t+2, q)$ skew to X_i, for some given i. For $j \neq i$, denote the t-dimensional subspace $< X_i, X_j > \cap S_{2t+1}$ by Δ_j, where $< X_i, X_j >$ is the subspace of $PG(3t+2, q)$ generated by X_i and X_j. Let $\Delta_i = S_{2t+1} \cap Y_i$. Then $W = \{\Delta_0, \Delta_1, \ldots, \Delta_{q^{t+1}}\}$ is a t-spread of S_{2t+1} of type (a).

(b) t-spreads of type (b) occur only when q is even and this possibility is not treated here.

(c) Suppose q is odd and consider a 1-secant space Y_i of $\mathcal{K}$ containing the point X_i for some i. For each $j \neq i$, let $\Delta_j = Y_j \cap Y_i$. Further let $\Delta_i = X_i$. Then $W_i^* = \{\Delta_0, \Delta_1, \ldots, \Delta_{q^{t+1}}\}$ is a t-spread of Y_i of type (c).

4.3.6 Theorem [Casse et al (1985), Theorem 2]

Let $\mathcal{K}$ be a $(q^{t+1}+1)$-set of $PG(3t+2, q)$, where q is odd. At least one of the t-spreads of type (a) is regular if and only if at least one of the t-spreads of type (c) is regular. In such a case all the t-spreads of types (a) and (c) are regular and $\mathcal{K}$ arises from Construction 4.3.2.

Proof: The proof involves the use of a generalised quadrangle arising from the $(q^{t+1}+1)$-set in $PG(3t+2, q)$ as in Casse et al (1985). □

The following result applies in the case q even as well as q odd. When q is odd it is weaker than the results above, however it is useful in the case of q even.

4.3.7 Theorem [Wild (1986), Proposition 1]
Let $\mathcal{K}$ be a $(q^{t+1}+1)$-set of $PG(3t+2,q)$. Suppose that two of the t-spreads arising from projection of $\mathcal{K}$ from elements say X_0 and X_1 onto $(2t+1)$-dimensional spaces $S^{(0)}_{2t+1}$ and $S^{(1)}_{2t+1}$ skew to X_0 and X_1 respectively are regular. By Theorem 2.9.3 there exist lines $l_0 \in \overline{S}^{(0)}_{2t+1}$ and $l_1 \in \overline{S}^{(1)}_{2t+1}$ meeting the extension of every element of the t-spreads of $S^{(0)}_{2t+1}$ and $S^{(1)}_{2t+1}$ respectively, in unique points. If l_0 and l_1 have a common point in $PG(3t+2,q^{t+1})$ then $\mathcal{K}$ arises from Construction 4.3.2.

Proof: The proof is similar to that given for Theorem 4.4.18, so it will be omitted here. □

Casse et al (1985) and Wild (1986) also present representations of k-sets of $PG(3t+2,q)$ as sets of points in translation planes. In each case the k-sets arising from Construction 4.3.2 are characterised in terms of these representations.

4.4 (k,n)-SETS OF $PG(3t+2,q)$

The connection between the k-sets of $PG(3t+2,q)$ and the k-arcs of $PG(2,q^{t+1})$ has been explored in the previous section. It suggested that a study of (k,n)-arcs of projective planes by the same methods would hopefully lead to examples of maximal (k,n)-arcs, or to demonstrations of the non-existence of these maximal (k,n)-arcs in projective planes of certain orders. The appropriate set of t-dimensional subspaces in $PG(3t+2,q)$ will be called a (k,n)-set.

4.4.1 Definition
A (k,n)-set $\mathcal{K}$ of $PG(3t+2,q)$ is a set of k pairwise skew t-dimensional subspaces of $PG(3t+2,q)$ satisfying:

(i) $\mathcal{K}$ is *geometric*, as in Definition 4.1.2,

(ii) No $(2t+1)$-dimensional subspace of $PG(3t+2,q)$ contains more than n elements of $\mathcal{K}$, but there is some $(2t+1)$-dimensional subspace containing exactly n elements of $\mathcal{K}$ and skew to every other element of $\mathcal{K}$. A (k,n)-set $\mathcal{K}$ is called *complete* if there is no $(k+1,n)$-set containing it.

A (k,n)-set of $PG(3t+2,q)$ is a geometric partial t-spread of $PG(3t+2,q)$ which satisfies the additional condition (ii) of Definition 4.4.1. The largest value of i for which $\mathcal{K}$ admits i-secants is $i=n$.

4.4.2 Examples

(1) When $t=0$ then $PG(3t+2,q)$ is a projective plane and a (k,n)-set is just a (k,n)-arc of the plane. The elements of $\mathcal{K}$ are points and the $(2t+1)$-dimensional subspaces are lines. Condition (i) is satisfied automatically.

(2) If $n=2$ then a (k,n)-set is just a k-set of $PG(3t+2,q)$.

(3) Any $(n-1)$-geometric partial t-spread $\mathcal{K}$ is a (k,n)-set of PG(3t+2,q). Such a (k,n)-set admits only 0-secants, 1-secants and n-secants since if a $(2t+1)$-dimensional subspace contains two elements of $\mathcal{K}$ then by definition it must contain exactly n of them. Declerck et al (1987) showed that such a set $\mathcal{K}$ satisfies $n-1$ divides $k-1$, and also that $k \leq (n-1)q^{t+1}+n$, with equality if and only if $\mathcal{K}$ is $(n-1)$-uniform. This is a generalisation of Theorem 4.1.5 which characterises the 1-uniform partial t-spreads of $PG(3t+2,q)$ as k-sets of $PG(3t+2,q)$.

As we will be interested particularly in the application of (k,n)-sets to maximal (k,n)-arcs, we need to know how many points a (k,n)-set may have. This is found in Theorem 4.4.4, but first we need a definition.

4.4.3 Definition

Let $\mathcal{K} = \{X_1, X_2, \ldots, X_k\}$ be a (k,n)-set of $PG(3t+2,q)$. Let S_{2t+1} be a $(2t+1)$-dimensional subspace of $PG(3t+2,q)$ skew to X_i for some given i with $1 \leq i \leq k$. For $j \neq i$, let $\Delta_j =< X_i, X_j > \cap S_{2t+1}$ where $< X_i, X_j >$ is the $(2t+1)$-dimensional subspace of $PG(3t+2,q)$ spanned by X_i and X_j. Since $\mathcal{K}$ is geometric, the spaces Δ_j are skew or coincide. The set of distinct Δ_j form a partial t-spread of S_{2t+1}, called the *partial t-spread arising from the projection of $\mathcal{K}$ onto S_{2t+1} from X_i.*

The elements of $\mathcal{K}$ which lie in a secant $(2t+1)$-dimensional space S_{2t+1} lie in the partial t-spread arising from the projection of $\mathcal{K}$ from any element X of $\mathcal{K}$ skew to S_{2t+1}.

4.4.4 Theorem

Let $\mathcal{K}$ be a (k,n)-set of $PG(3t+2,q)$. Then $k \leq (n-1)q^{t+1} + n$.

Proof: Let X be an element of $\mathcal{K}$ and let S_{2t+1} be a $(2t+1)$-dimensional subspace of $PG(3t+2,q)$ skew to X. The $(2t+1)$-dimensional subspaces joining X to other members of $\mathcal{K}$ meet S_{2t+1} in the elements of a partial t-spread $\mathcal{W}$ of S_{2t+1}, and by Theorem 1.2.2 $\mathcal{W}$ has at most $q^{t+1}+1$ elements. So there are at most $q^{t+1}+1$ such $(2t+1)$-dimensional spaces, and each contains at most n elements of $\mathcal{K}$ including X. Each element of $\mathcal{K}$ lies in one of these spaces by construction, and since $\mathcal{K}$ is geometric each element of $\mathcal{K}$ intersects exactly one of the spaces. Thus

$$\begin{aligned} k &\leq (q^{t+1}+1)(n-1)+1 \\ &= (n-1)q^{t+1}+n. \end{aligned}$$ □

When $t = 0$ we get the classical result, see Theorem 4.2.4.

This result can be slightly improved under further assumptions. For example if we assume that an element X of $\mathcal{K}$ lies on an m_1-secant $(2t+1)$-dimensional subspace, then $k \leq (n-1)q^{t+1} + m_1$.

4.4.5 Definition

A (k,n)-set $\mathcal{K}$ is called *maximal* if $k = (n-1)q^{t+1} + n$.

4.4.6 Theorem

Let $\mathcal{K}$ be a maximal (k,n)-set of $PG(3t+2,q)$. Then $\mathcal{K}$ admits only 0-secant and n-secant $(2t+1)$-dimensional spaces. We can say that $\mathcal{K}$ has type $(0,n)$.

Proof: Let S_{2t+1} be a $(2t+1)$-dimensional subspace of $PG(3t+2,q)$, and let $\mathcal{W}$ be the partial t-spread arising on S_{2t+1} by the projection of $\mathcal{K}$ from an element X of $\mathcal{K}$ skew to S_{2t+1}. The elements of $\mathcal{K} - \{X\}$ lie in the $(2t+1)$-dimensional subspaces which are the joins of X to the elements of $\mathcal{W}$. Since $k = (n-1)q^{t+1} + n$, and no $(2t+1)$-dimensional subspace of $PG(3t+2,q)$ may contain more than n elements of $\mathcal{K}$, it follows that $\mathcal{W}$ is a t-spread of S_{2t+1} and the $(2t+1)$-dimensional subspaces joining X to the elements of $\mathcal{W}$ are all n-secants of $\mathcal{K}$, so that X lies on exactly $q^{t+1}+1$ n-secants and no others. There is a $(2t+1)$-dimensional subspace of $PG(3t+2,q)$ skew to each element of $\mathcal{K}$, so that every element of $\mathcal{K}$ lies on exactly $q^{t+1}+1$ n-secants and no other secants. Because a secant which is not a 0-secant contains an element of $\mathcal{W}$, it is an n-secant and the result follows. □

4.4.7 Corollary

A maximal (k,n)-set of $PG(3t+2,q)$ is an $(n-1)$-geometric and $(n-1)$-uniform partial t-spread, and conversely.

Proof: Firstly let $\mathcal{K}$ be a maximal (k,n)-set of $PG(3t+2,q)$. Then it is a partial t-spread and by definition it is geometric. Since it is of type $(0,n)$ the space joining any two of its elements contains exactly n elements of $\mathcal{K}$, so $\mathcal{K}$ is $(n-1)$-

geometric. Since $k = (n-1)q^{t+1} + n$, by Example 4.4.2 (3), $\mathcal{K}$ is $(n-1)$-uniform.

Now suppose $\mathcal{K}$ is an $(n-1)$-geometric and $(n-1)$-uniform partial t-spread of $PG(3t+2,q)$ with k elements. By Example 4.4.2 (3) we have $k = (n-1)q^{t+1} + n$. Since $\mathcal{K}$ is $(n-1)$-geometric, it is certainly geometric, and in fact since the space joining any two of its elements contains exactly n elements of $\mathcal{K}$, no $(2t+1)$-dimensional space may contain more than n elements of $\mathcal{K}$. Thus $\mathcal{K}$ is a maximal (k,n)-set of $PG(3t+2,q)$. □

The discussion prior to Theorem 4.1.3 showed how to construct an incidence structure from any geometric partial t-spread of $PG(d,q)$. In the case of a (k,n)-set of $PG(3t+2,q)$ more can be shown. As in Section 4.1, we construct an incidence structure $\mathcal{I} = (P, B, I)$ whose points are the elements of $\mathcal{K}$ and whose blocks are the n-secants of $\mathcal{K}$. Incidence is set-theoretic inclusion. Then $\mathcal{I}$ is a $2-((n-1)q^{t+1}+n, n, 1)$-design. (For a discussion of designs, see Hughes and Piper (1985).)

4.4.8 Theorem [Declerck et al (1987)]
Let $\mathcal{K}$ be a maximal (k,n)-set of $PG(3t+2,q)$.

(1) If $n = q^{t+1}+1$ then $\mathcal{I}$ is a projective plane of order q^{t+1}.

(2) If $n = q^{t+1}$ then $\mathcal{I}$ is an affine plane of order q^{t+1}.

(3) If $2 \leq n \leq q^{t+1}-1$ then the number of n-secants of $\mathcal{K}$ through a point of $PG(3t+2,q)$ not contained in any element of $\mathcal{K}$ is equal to

$$q^{t+1} + 1 - \frac{q^{t+1}}{n}.$$

Thus n divides q^{t+1}. □

We now turn to the question of existence of maximal (k,n)-sets. If $n = 2$

then $k = q^{t+1} + 2$ and q must be even by Theorem 4.3.4. Therefore there do not exist maximal $(k, 2)$-sets when q is odd. When q is even, examples of $(q^{t+1} + 2, 2)$-sets have been given in Construction 4.3.2 (2). For $n \geq 3$ the situation is more difficult. However, if there exists a (k, n)-arc in a projective plane $PG(2, q^{t+1})$ for some value of t, then we can construct a (k, n)-set $\mathcal{K}$ in $PG(3t+2, q)$ in a manner analogous to that used in Construction 4.3.2.

4.4.9 Construction

Let $GF(q^{t+1})$ be an extension (of degree $t+1$) of $GF(q)$ and let $PG(3t+2, q^{t+1})$ be the corresponding extension of $PG(3t+2, q)$. Let Π be an imaginary plane of $PG(3t+2, q^{t+1})$. Then the $t+1$ conjugates of Π under the collineation σ induced by the automorphism $\sigma : x \mapsto x^q$ of $GF(q^{t+1})$ span the whole of $PG(3t+2, q^{t+1})$. Each point P of Π is imaginary (see Theorem 1.4.8 (3)), and by Definition 1.4.7 (1) the $t+1$ conjugates of P span a t-dimensional subspace $L(P)$ of $PG(3t+2, q^{t+1})$ which meets $PG(3t+2, q)$ in a t-dimensional subspace (Corollary 1.4.10 (2)). The points of a (k, n)-arc $\mathcal{C}$ in Π determine a set $\mathcal{K}$ of k t-dimensional subspaces of $PG(3t+2, q)$. To show that $\mathcal{K}$ is a (k, n)-set we need to prove that $\mathcal{K}$ is geometric, and that there is no $(2t+1)$-dimensional subspace of $PG(3t+2, q)$ containing more than n elements of $\mathcal{K}$, while some $(2t+1)$-dimensional subspace contains exactly n elements of $\mathcal{K}$. These properties both follow from Corollaries 4.4.11 and 4.4.12.

4.4.10 Theorem

Let l be a line of Π, with points $P_0, P_1, \ldots, P_{q^{t+1}}$. For $i \in \{0, 1, \ldots, q^{t+1}\}$ let $L(P_i) = \mathrm{lin}\{P_i, P_i^{\sigma}, \ldots, P_i^{\sigma^t}\}$. Then

$$\mathcal{W} = \{L(P_i) \cap PG(3t+2, q) : \ i = 0, 1, \ldots, q^{t+1}\}$$

is a regular t-spread of a $(2t+1)$-dimensional subspace of $PG(3t+2, q)$.

Proof: Since Π is imaginary, each point P_i is imaginary, thus by Definition 1.4.7 (1), $L(P_i)$ has dimension t in $PG(3t+2, q^{t+1})$ and meets $PG(3t+2, q)$ in a space of dimension t. By Lemma 2.8.5, the spaces $L(P_i)$ and $L(P_j)$ are skew for $i \neq j$. For each $i = 0, 1, \ldots, q^{t+1}$, $L(P_i)$ is contained in the $(2t+1)$-dimensional space $L(l)$. By Theorem 2.9.3 the t-spread $\mathcal{W}$ of $L(l)$ is regular. □

4.4.11 Corollary

There is a natural isomorphism between the plane Π with its points and lines and certain t-dimensional and $(2t+1)$-dimensional subspaces of $PG(3t+2, q)$. The (k, n)-set $\mathcal{K}$ of $PG(3t+2, q)$ is isomorphic to a (k, n)-arc $\mathcal{C}$ in Π, with the i-secants of $\mathcal{K}$ corresponding to the i-secants of $\mathcal{C}$.

Proof: We define an incidence structure $\mathcal{I}$ on $PG(3t+2, q)$. The points are the t-dimensional subspaces spanned by a point of Π together with all its conjugates, and the blocks are the $(2t+1)$-dimensional subspaces of $PG(3t+2, q)$ spanned by a line of Π together with its t conjugates. The incidence is containment. Then $\mathcal{I}$ is a projective plane of order q^{t+1}, isomorphic to Π. A (k, n)-arc $\mathcal{C}$ of Π determines a (k, n)-arc $\mathcal{C}'$ of $\mathcal{I}$ and the i-secants of the (k, n)-arc in Π determine the i-secants of the (k, n)-arc $\mathcal{C}'$ in $\mathcal{I}$. □

4.4.12 Corollary

The (k, n)-set $\mathcal{K}$ arising from Construction 4.4.9 is contained in a regular t-spread of $PG(3t+2, q)$.

Proof: The elements of the regular t-spread are the subspaces $L(P_i)$ of $PG(3t+2, q)$ corresponding to the points P_i of Π. □

4.4.13 Corollary

Let $\mathcal{K}$ be a (k, n)-set arising from Construction 4.4.9. The t-spread arising from the projection of $\mathcal{K}$ from $X \in \mathcal{K}$ onto a $(2t+1)$-dimensional subspace S_{2t+1} skew to X is a partial t-spread lying in a regular t-spread.

Proof: A partial t-spread arising from the projection of $\mathcal{K}$ from an element X onto a $(2t+1)$-dimensional space skew to X is contained in the regular t-spread in Corollary 4.4.12. □

4.4.14 Examples

(1) If $n = 2$ then $k = q^{t+1} + 2$ and by Theorem 4.3.4, q must be even. There do not exist maximal $(k, 2)$-sets when q is odd.

(2) In $PG(2, 2^h)$, there exist maximal (k, n)-arcs for every integer n dividing 2^h, see Theorem 4.2.8. We can use the Construction 4.4.9 to construct in $PG(3t+2, q)$, with q even, maximal (k, n)-sets for every integer n dividing q^{t+1}.

It is an open question whether, for $3 \leq n \leq q^{t+1} - 1$, there exist any maximal (k, n)-arcs in $PG(2, q^{t+1})$ with q odd, and hence whether there exist any maximal (k, n)-sets in $PG(3t+2, q)$.

As in the case of k-sets of $PG(3t+2, q)$, the following question is of interest: Do there exist maximal (k, n)-sets of $PG(3t+2, q)$ which do not arise from the Construction 4.4.9? In other words, given a maximal (k, n)-set $\mathcal{K}$ of $PG(3t+2, q)$ does there exist an imaginary plane of $PG(3t+2, q)$ meeting the extension of every element of $\mathcal{K}$? Such a plane, then, is a plane $PG(2, q^{t+1})$ containing a maximal (k, n)-arc. This question can be rephrased in another way: Is every maximal (k, n)-set contained in a regular t-spread of $PG(3t+2, q)$? In the following we obtain some results towards answering this question.

4.4.15 Theorem [Thas (1971), (1975) and Denniston (1969)]

(1) If $n = 2$ then there exist maximal (k, n)-sets in $PG(3t + 2, q)$ arising from Construction 4.4.9 only if q is even.

(2) In $PG(3t + 2, q)$, q even, there exist maximal (k, n)-sets arising from Construction 4.4.9 for every integer n dividing q^{t+1}.

(3) In $PG(3t+2, q)$, $q = 3^m$, with $m \geq 1$, there does not exist a maximal $(k, 3)$-set arising from Construction 4.4.9.

Proof: (1) See Example 4.4.14 (1).

(2) When q is even, by Theorem 4.2.8 there exist maximal (k, n)-arcs in $PG(2, q^{t+1})$ for every value of n dividing q^{t+1}. We apply Construction 4.4.9.

(3) Suppose that there exists a maximal $(k, 3)$-set in $PG(3t + 2, q)$, with $q = 3^m$, $m \geq 1$ arising from the Construction 4.4.9. Then there is a maximal $(k, 3)$-arc in a plane $PG(2, q^{m(t+1)})$ where 3 divides $m(t + 1)$ and $m(t + 1) > 1$. This contradicts Theorem 4.2.9. □

We may apply the classical results on (k, n)-arcs of projective planes, (see for example Hirschfeld (1979)), to obtain other results about (k, n)-sets of $PG(3t+2, q)$ arising from Construction 4.4.9, for example: if $\mathcal{K}$ is a $((n - 1)q^{t+1} + n - 1, n)$-set of $PG(3t + 2, q)$ arising from Construction 4.4.9, then it is incomplete and can be completed in a unique manner to a (maximal) $((n-1)q^{t+1} + n, n)$-set by adjoining the t-dimensional subspace which is the intersection of all its $(t - 1)$-secants. Also, if n does not divide q^{t+1}, and $2 \leq n \leq q^{t+1}$ then a (k, n)-set of $PG(3t + 2, q)$ satisfies $k \leq (n - 1)q^{t+1} + n - 2$. A $(k, 3)$-set of $PG(3t + 2, q)$, $q \geq 3$ and $n \geq 2$ satisfies $k \leq 2q^{t+1} + 1$.

The following concept is important in deciding whether a certain (k, n)-set of $PG(3t+2,q)$ arises from the Construction 4.4.9. It appeared implicitly in the work of Casse and Wild (1983).

4.4.16 Definition [Declerck et al (1987)]
Let $\mathcal{W}$ be a partial t-spread of $PG(3t+2,q)$. A secant $(2t+1)$-dimensional projective subspace S_{2t+1} of $PG(3t+2,q)$ is called ***projection stable*** with respect to $\mathcal{W}$ if the partial t-spreads arising from the projection of $\mathcal{W}$ onto S_{2t+1} from any element $X \in \mathcal{W}$ skew to S_{2t+1} belong to a fixed t-spread of S_{2t+1}.

If $\mathcal{W}$ is a maximal (k,n)-set of $PG(3t+2,q)$ then the partial t-spreads arising from the projections of $\mathcal{W}$ onto S_{2t+1} from each element of $\mathcal{W}$ skew to S_{2t+1} are all t-spreads and so must coincide. Declerck et al have shown that if $\mathcal{K}$ is a maximal (k,n)-set of $PG(3t+2,q)$ such that every secant $(2t+1)$-dimensional space is projection stable with respect to $\mathcal{K}$, then $\mathcal{K}$ arises from the Construction 4.4.9. This idea could be useful in spaces $PG(3t+2,q)$ for which there are few t-spreads in $PG(2t+1,q)$.

4.4.17 Lemma
Let $\mathcal{K}$ be a (k,n)-set of $PG(3t+2,q)$. Suppose there exists a secant $(2t+1)$-dimensional subspace S_{2t+1} of $\mathcal{K}$ such that the elements of $\mathcal{K}$ which lie in S_{2t+1} are embeddable in a unique t-spread of S_{2t+1}. Then S_{2t+1} is projection stable with respect to $\mathcal{K}$.

Proof: Let X be an element of $\mathcal{K}$ skew to S_{2t+1}. Then the elements of S_{2t+1} lie in the partial t-spread arising from the projection of $\mathcal{K}$ from X onto S_{2t+1}. Since the elements of $\mathcal{K}$ in S_{2t+1} lie in a unique t-spread say $\mathcal{W}$, then the partial t-spread arising from the projection of $\mathcal{K}$ from X onto S_{2t+1} lies in the fixed t-spread $\mathcal{W}$. By definition, since X is any element of $\mathcal{K}$ skew to S_{2t+1}, we see that

S_{2t+1} is projection stable with respect to $\mathcal{K}$. □

The condition of projection stability can be relaxed a bit, as in the next Theorem. If S is a subspace of $PG(3t+2,q)$, we will denote its extension to $PG(3t+2,q^{t+1})$ by $\overline{S}$.

4.4.18 Theorem

Let $\mathcal{K}$ be a maximal (k,n)-set in $PG(3t+2,q)$. Suppose $\mathcal{K}$ admits a secant $(2t+1)$-dimensional subspace S_0 with the property that there exist two distinct elements X_1, X_2 of $\mathcal{K}$ skew to S_0 such that the two t-spreads on S_0 arising from the projection of $\mathcal{K}$ onto S_0 from each of X_1, X_2 coincide (this would occur, for example, if S_0 were projection stable). Denote this t-spread by $\mathcal{W}_0$. The (2t+1)-dimensional space $S_1 =< X_1, X_2 >$ is a secant of $\mathcal{K}$ meeting S_0 in an element of $\mathcal{W}_0$. Let X_0 be an element of $\mathcal{K}$ in $\mathcal{W}_0$, skew to S_1, and let the t-spread arising from the projection of $\mathcal{K}$ onto S_1 from X_0 be denoted $\mathcal{W}_1$. Note that X_1, X_2 and $S_0 \cap S_1$ are all elements of $\mathcal{W}_1$. Suppose $\mathcal{W}_0$ and $\mathcal{W}_1$ are both regular t-spreads. Then there exists a line l_0 in the space $\overline{S_0}$ which meets the extension of each element of $\mathcal{W}_0$ in a unique point, and a line l_1 in the space $\overline{S_1}$ which meets the extension of each element of $\mathcal{W}_1$ in a unique point. If these lines l_0 and l_1 have a common point (in $PG(3t+2,q^{t+1})$), then $\mathcal{K}$ arises from the Construction 4.4.9.

Proof: Since projecting $\mathcal{K}$ from X_1 onto S_0 yields the t-spread $\mathcal{W}_0$, we denote the points of l_0 as follows:

$$P_j = \overline{< X_1, X_j > \cap S_0} \cap l_0, \quad \text{for } j \neq 1.$$

Then P_j is the point of l_0 belonging to that element of $\mathcal{W}_0$ arising from the projection of X_j onto S_0 from X_1. The points P_j are not all distinct, in fact each of the $q^{t+1}+1$ points of l_0 occurs $n-1$ times among the points P_j. In a similar way we denote the points of l_1 by:

$$Q_i = \overline{< X_0, X_i > \cap S_1} \cap l_1, \quad \text{for } i \neq 0.$$

Now $\Pi =< l_0, l_1 >$ is a plane of $PG(3t+2, q^{t+1})$, and the extension of each element of $\mathcal{K}$ contained in S_0 or S_1 meets Π in a unique point. We show that this is also true of elements of $\mathcal{K}$ skew to S_0 and S_1. Let X_j be an element of $\mathcal{K}$ not lying in S_0 or S_1, and not contained in the space $< X_0, X_1 >$. The space $< \overline{X_0}, \overline{X_j} >$ meets Π in the line $P_0 Q_j$. The space $< \overline{X_j}, \overline{X_1} >$ meets Π in the line $P_j Q_1$. These lines are distinct and so meet in a unique point R_j. Since $\overline{X_j} =< \overline{X_0}, \overline{X_j} > \cap < \overline{X_j}, \overline{X_1} >$, then $R_j \in \overline{X_j}$ and $\overline{X_j}$ meets Π in the unique point R_j. Lastly let X_j be an element of $\mathcal{K}$ skew to S_0 and S_1 but contained in $< X_0, X_1 >$. Then X_j is skew to $< X_0, X_2 >$ so we just repeat the above argument replacing X_1 by X_2 to show that also $\overline{X_j}$ meets Π in a unique point. Thus the extension of every element of $\mathcal{K}$ meets Π in a unique point and $\mathcal{K}$ arises from Construction 4.4.9. □

A maximal (k, n)-set $\mathcal{K}$ arising from Construction 4.4.9 has the property that the projection of $\mathcal{K}$ onto a secant $(2t+1)$-dimensional space S_{2t+1} from any element X skew to S_{2t+1} is a regular t-spread. In some cases the maximal (k, n)-sets can be characterised in terms of this property, as in Theorems 4.4.21 and 4.4.22. We use Theorem 4.4.18 and Lemma 4.4.19.

4.4.19 Lemma

Let $\mathcal{K}$ be a maximal (k, n)-set of $PG(3t+2, q)$ with $n \geq 3$. Suppose that if $q = 2$ then $t \geq 2$, and that if $q \geq 3$ then $t \geq 1$. Let S_{2t+1} be an n-secant $(2t+1)$-dimensional subspace of $PG(3t+2, q)$. Suppose that the t-spreads of S_{2t+1} arising from the projections onto S_{2t+1} from each element of $\mathcal{K}$ skew to S_{2t+1} are all regular. Then S_{2t+1} is projection stable with respect to $\mathcal{K}$.

Proof: We denote the $k-n$ elements of $\mathcal{K}$ skew to S_{2t+1} by $X_1, X_2, \ldots, X_{k-n}$

and denote the t-spread arising from the projection of $\mathcal{K}$ from X_i onto S_{2t+1} by $\mathcal{W}_i$, for $i = 1, \ldots, k-n$. Note that the n elements of $\mathcal{K}$ in S_{2t+1}, denoted by $X_{k-n+1}, X_{k-n+2}, \ldots, X_{k-1}, X_k$, all belong to each of the t-spreads

$$\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{k-n}.$$

Now by Lemma 2.4.2 the three elements X_{k-2}, X_{k-1}, X_k of $\mathcal{K}$ lie in a unique t-regulus of rank 1 $\mathcal{R}$ in S_{2t+1}, and we denote the $q-2$ further t-dimensional subspaces of S_{2t+1} in the t-regulus by $Z_1, Z_2, \ldots, Z_{q-2}$. Some of these elements could be elements of $\mathcal{K}$, since S_{2t+1} is an n-secant of $\mathcal{K}$. Since each of the t-spreads $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{k-n}$ is regular, the subspaces $X_{k-2}, X_{k-1}, X_k, Z_1, \ldots, Z_{q-2}$ belong to each of them.

Let Y_r be an element of the t-spread $\mathcal{W}_1$, not lying in $\mathcal{R}$. The space $< X_1, Y_r >$ is an n-secant of $\mathcal{K}$, and contains either $n-2$ or $n-1$ elements of $\mathcal{K}$ distinct from X_1 and skew to S_{2t+1} according as whether Y_r is in $\mathcal{K}$ or not. In either case $< X_1, Y_r >$ contains at least $n-2$ elements of $\mathcal{K}$ distinct from X_1 and skew to S_{2t+1}. The t-spreads of S_{2t+1} arising from the projections of $\mathcal{K}$ onto S_{2t+1} from X_1 and these further $n-2$ elements of $\mathcal{K}$ all contain the subspaces $X_{k-2}, X_{k-1}, X_k, Z_1, \ldots, Z_{q-2}$ and Y_r. Since by Corollary 2.9.5 a regular t-spread of S_{2t+1} is uniquely determined by one of its t-reguli of rank 1 together with one further t-dimensional space of the t-spread, the $n-1$ t-spreads above are all identical to $\mathcal{W}_1$. For this choice of Y_r there are at least $n-2$ elements of $\mathcal{K}$ whose projections of $\mathcal{K}$ onto S_{2t+1} yield t-spreads identical to $\mathcal{W}_1$. There are $q^{t+1}-q$ such spaces Y_r, so that there are at least $(n-2)(q^{t+1}-q)$ t-spreads $\mathcal{W}_i$ identical to $\mathcal{W}_1$. Without loss of generality suppose the t-spreads $\mathcal{W}_1, \ldots, \mathcal{W}_{(n-2)(q^{t+1}-q)+1}$ are identical.

Consider $X_t \in \mathcal{K}$, skew to S_{2t+1}, and such that $\mathcal{W}_t$ differs from $\mathcal{W}_1$. Then X_t must lie on a secant joining X_1 to an element of $\mathcal{R}$. Similarly, since $\mathcal{W}_t$ differs from $\mathcal{W}_2$, X_t must lie on a secant joining X_2 to an element of $\mathcal{R}$. In this way we see that the joins of X_t to each of $X_1, X_2, \ldots, X_{(n-2)(q^{t+1}-q)+1}$ must meet S_{2t+1} in elements of $\mathcal{R}$ and so $X_1, X_2, \ldots, X_{(n-2)(q^{t+1}-q)+1}$ must all lie on secants joining X_t to an element of $\mathcal{R}$. But the number of elements of $\mathcal{K}$ distinct from X_t and skew to S_{2t+1} lying on secants joining X_t to elements of $\mathcal{R}$ is at most

$$3(n-2)+(q-2)(n-1).$$

This is a contradiction because

$$\begin{aligned}&\left[(n-2)(q^{t+1}-q)+1\right]-[3(n-2)+(q-2)(n-1)]\\&=(n-2)(q^{t+1}-2q-1)-q+3\end{aligned}$$

which is greater than zero for all $q \geq 3$, $n \geq 3$ and $t \geq 1$. If $q=2$ then it is also greater than zero for all $n \geq 3$ and $t \geq 2$. Note particularly that in the case of $q=2$ the argument fails if $t=1$ since n divides q^{t+1} and $n \geq 3$ imply $n \geq 4$. □

4.4.20 Theorem

Let $\mathcal{K}$ be a maximal (k,n)-set of $PG(3t+2,2)$ with $n \geq 3$ and $t \geq 2$. Then every n-secant $(2t+1)$-dimensional subspace is projection stable with respect to $\mathcal{K}$.

Proof: Let S_{2t+1} be a secant $(2t+1)$-dimensional subspace of $PG(3t+2,2)$. Since every t-spread of $PG(2t+1,2)$ is regular (see Theorem 2.4.6), all the t-spreads arising from the projection of $\mathcal{K}$ from any element of $\mathcal{K}$ skew to S_{2t+1} are regular. By Lemma 4.4.19, S_{2t+1} is projection stable with respect to $\mathcal{K}$. □

4.4.21 Theorem

Let $\mathcal{K}$ be a maximal $(k,3)$-set in $PG(5,q)$, where by Theorem 4.4.8, q is a power of 3. Since $t=1$ we see that $\mathcal{K}$ is a $(k,3)$-set of lines. The projection of $\mathcal{K}$ from any one of its elements onto a 3-secant 3-dimensional subspace S_3 yields a 1-spread on that 3-secant. Suppose that all such 1-spreads on each such S_3 are regular. Then $\mathcal{K}$ arises from Construction 4.4.9.

Proof: Choose $X \in \mathcal{K}$ and denote the q^2+1 3-secant 3-spaces through X by $S_3^1, S_3^2, ..., S_3^{q^2+1}$. For $i = 1$ to q^2+1, the projection of $\mathcal{K}$ onto S_3^i from any element of $\mathcal{K}$ skew to it yields a regular 1-spread of S_3^i. By Lemma 4.4.19, the space S_3^i is projection stable and we will denote the 1-spread on it arising from any such projection of $\mathcal{K}$ by $\mathcal{W}_i$.

Let $GF(q^2)$ be a field extension of $GF(q)$, and denote the corresponding extension of $PG(5,q)$ by $PG(5,q^2)$. For each i from 1 to q^2+1 there are two imaginary skew lines in $PG(5,q^2)$ which meet the extension of each element of $\mathcal{W}_i$ in a unique point (see Theorem 2.9.3). These lines are a conjugate pair, that is they are images of one another under the collineation induced by the automorphism $x \to x^2$ of $GF(q^2)$. We thus have $2(q^2+1)$ lines, occurring in conjugate pairs. Since X lies in each of $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{q^2+1}$, it follows that the extension $\overline{X}$ of X to $PG(5,q^2)$ meets each of the imaginary lines, in exactly one point. $\overline{X}$ meets a conjugate pair of lines in conjugate points of $\overline{X}$.

Now since $\overline{X}$ has q^2-q imaginary points, occurring in conjugate pairs, and since $2(q^2+1) > q^2-q$, it follows that there is at least one imaginary point of $\overline{X}$ lying on at least two of the imaginary lines listed above. Theorem 4.4.18 gives the result. □

The proof of Theorem 4.4.21 fails for $t \geq 2$ at the following point: the number of imaginary lines meeting $\overline{X}$ is shown to be $(t+1)(q^{t+1}+1)$ while the number of imaginary points in the extension $\overline{X}$ which is a t-dimensional space is

$$q^{t^2}(q^t-1) + q^{t(t-1)}(q^{t-1}-1) + \cdots + q^t(q-1)$$

and the desired conclusion can only be arrived at in the case $t = 1$.

4.4.22 Theorem

Let $\mathcal{K}$ be a maximal (k,n)-set of $PG(3t+2,2)$ with $n \geq 3$ and $t \geq 2$. Then $\mathcal{K}$ arises from the Construction 4.4.9.

Proof: We show that $\mathcal{K}$ is contained in a regular t-spread of $PG(3t+2,2)$, which is sufficient to prove the result. Let X be an element of $\mathcal{K}$. As in the remarks preceding Theorem 4.4.8, the elements of $\mathcal{K}$ and the n-secant $(2t+1)$-dimensional subspaces form a $2-((n-1)2^{t+1}+n,n,1)$-design. Therefore through X there pass $m = 2^{t+1}+1$ such n-secants, say $S_1, S_2, \ldots, S_m$. For each value of i, $1 \leq i \leq m$, the projection of $\mathcal{K}$ from a given element X_i skew to S_i yields a t-spread $\mathcal{W}_i$ of S_i. Since $q = 2$, by Theorem 2.4.6 each t-spread W_i is regular, and by Theorem 4.4.20 S_i is projection stable. Now by Theorem 2.9.3 there exists a set of $t+1$ conjugate imaginary lines in $PG(3t+2,2^{t+1})$ meeting the extension of every element of the t-spread $\mathcal{W}_i$ in a unique point. Thus there exist $2^{t+1}+1$ sets of $t+1$ conjugate lines in $PG(3t+2,2^{t+1})$, each such line meeting $\overline{X}$ in a unique point. Now $\overline{X}$ has exactly $(2^{t+1}-2^t)/(2-1) = 2^t$ imaginary points, and since

$$(t+1)\left(2^{t+1}-1\right) > 2^t$$

there is an imaginary point of $\overline{X}$ through which there pass at least two such lines. Suppose that these two lines are the lines meeting the extension of every element of the t-spreads $\mathcal{W}_i$ and $\mathcal{W}_j$ in the secants S_i and S_j. By applying Theorem 4.4.18, choosing S_0 and S_1 as the two secants S_i and S_j, we see that $\mathcal{K}$ arises from Construction 4.4.9. □

4.5 MORE ABOUT (k,n)-SETS OF $PG(3t+2,q)$

Let $\mathcal{K}$ be a (k,n)-set of $PG(3t+2,q)$, with $n \geq 2$ so that the remarks in this Section apply to k-sets and (k,n)-sets of $PG(3t+2,q)$. Suppose that $\mathcal{K}$ arises from the Construction 4.3.2, or 4.4.9, so that there exists an imaginary plane Π

of $PG(3t+2, q^{t+1})$ meeting the extension of every element of $\mathcal{K}$, each in a unique point. (In fact the set of these points of Π is a (k,n)-arc of Π).

The regular t-spread $\mathcal{W}$ of $PG(3t+2, q)$ all of whose elements meet Π in a unique point is a regular t-spread of $PG(3t+2, q)$ containing the elements of $\mathcal{K}$. In Corollary 3.9.5 we showed that such a t-spread $\mathcal{W}$ has a projective t-spread set $\mathcal{PC}$ of 3-tuples of $(t+1)\times(t+1)$ matrices isomorphic to Π. The matrices all belong to a field $(\mathcal{C}, +, \cdot)$ of order q^{t+1}, and we can use this field as $GF(q^{t+1})$ in order to coordinatise $PG(3t+2, q^{t+1})$. Under this coordinatisation the isomorphism between Π and $\mathcal{PC}$ is the identity, and

$$\begin{aligned}\mathcal{PC} = \{(I,0,0)\} &\cup \left\{(C_1^{(j)}, I, 0) : \ j = 1, 2, \ldots, q^{t+1}\right\} \\ &\cup \left\{(C_1^{(j)}, C_2^{(j)}, I) : \ j = 1, 2, \ldots, q^{2(t+1)}\right\}.\end{aligned}$$

An element W of $\mathcal{W}$ corresponding to the element $(\xi_0^{(i)}, \xi_1^{(i)}, \xi_2^{(i)})$ of the projective t-spread meets Π in the point with coordinates $(\xi_0^{(i)}, \xi_1^{(i)}, \xi_2^{(i)})^T$. The elements of $\mathcal{W}$ meet Π in a (k,n)-arc of Π and so the elements of the partial projective t-spread set $\mathcal{PPC}$ corresponding to $\mathcal{W}$ are a (k,n)-arc of $\mathcal{PC}$. It seems that trying to construct a (k,n)-set of $PG(3t+2, q)$ is as hard as constructing a (k,n)-arc of $PG(2, q^{t+1})$.

However these ideas can be used to test whether a certain (k,n)-set $\mathcal{K}$ of $PG(3t+2, q)$ arises from Construction 4.4.9. We would simply find the partial projective t-spread set corresponding to $\mathcal{K}$, and see whether each of the elements could be multiplied by a non-singular $(t+1)\times(t+1)$ matrix to obtain an identity matrix in the last non-zero position. If this is not possible then $\mathcal{K}$ does not arise from Construction 4.4.9. If it is possible, we still need to check whether all the matrices appearing in the partial projective t-spread set come from a field of order q^{t+1}. If they do, then that field gives the regular t-spread containing $\mathcal{K}$. Otherwise,

$\mathcal{K}$ is not contained in a regular t-spread and so does not arise from Construction 4.4.9.

4.5.1 Example

We use Examples 3.10.1 and 3.10.2. If we find a (k,n)-set $\mathcal{K}$ of lines among the lines $l_1, l_2, \ldots, l_{21}$ appearing in Example 3.10.1, then the corresponding partial projective t-spread set has matrices belonging to the field $0, I, A, B$ of order 4. Then $\mathcal{K}$ is contained in the regular 1-spread $l_1, l_2, \ldots, l_{21}$ and arises from Construction 4.4.9.

If, on the other hand, we choose a (k,n)-set from the lines $l_1, l_2, \ldots, l_{21}$ of the regulus free 1-spread $\mathcal{W}$ of Example 3.10.2, the situation is different. We try to find a set of lines arising from Construction 4.4.9, as large as possible. As the non-zero matrices appearing in the partial projective t-spread set must be invertible, we cannot choose any line whose corresponding 3-tuple P_i contains any of the matrices: $A, B, D, E, F, G, H, J, K$ and L. There is only one line l_{21} of $\mathcal{W}$ satisfying this condition, so there is no (k,n)-set of $PG(3t+2,q)$ with $k \geq 2$ contained in $\mathcal{W}$ and arising from Construction 4.4.9.

It would be interesting to produce an example of a maximal (k,n)-set of lines contained in the regulus free 1-spread $\mathcal{W}$, answering at least in the space $PG(5,q)$ the question of existence of maximal (k,n)-sets of lines not arising from the Construction 4.4.9. However the best that can be done is to find a 4-set of lines, while a maximal k-set would have six lines. Such a 4-set of lines of $PG(5,2)$, not arising from the construction 4.4.9, is

$$\mathcal{K} = \{l_1, l_2, l_6, l_9\}, \qquad \text{or}$$

$$\mathcal{K} = \{l_1, l_2, l_7, l_{12}\}.$$

CHAPTER FIVE

CONCLUSION

The applications of the ideas of regularity, of t-spread sets and of indicator sets in the case of t-spreads of $PG(2t+1,q)$ existing in the literature suggest many lines of new research in the general case. A few ideas are presented here.

The aim of classifying the 1-spreads of $PG(3,q)$ has led to the definition of *subregular* 1-spreads. A 1-spread is subregular if it can be obtained by reversing a sequence of reguli in turn, starting from a regular 1-spread. The concept of reversing a regulus involves replacing the lines of the regulus (or Segre variety $\mathcal{SV}_{2,2}$) by the lines of the opposite regulus (or lines of the opposite system of lines of $\mathcal{SV}_{2,2}$). Reversing a regulus in a 1-spread yields another 1-spread. Hirschfeld (1985) gives a good survey of results in this area.

In the case of a t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1,q)$, let $\mathcal{R}_r$ be a t-regulus of rank r contained in $\mathcal{W}$. The "opposite" regulus is an r-regulus of rank t, since $\mathcal{R}_r$ is the set of t-dimensional spaces of a Segre variety $\mathcal{SV}_{r+1,t+1}$ whose opposite system comprises r-dimensional spaces of $\mathcal{SV}_{t+1,r+1}$. If $t=r$ then reversing $\mathcal{R}_r$ yields another t-spread of $PG((s+1)(t+1)-1,q)$. Otherwise reversing $\mathcal{R}_r$ gives a partition of $PG((s+1)(t+1)-1,q)$ into pairwise skew t- and r-dimensional subspaces. The results in the case of a 1-spread of $PG(3,q)$ suggest the development of a study of such partitions of $PG((s+1)(t+1)-1,q)$, particularly in the case of $t=s$.

Another important application of 1-spreads of $PG(3,q)$ is to the theory of inversive planes. An *inversive plane of order* q, denoted by $IP(q)$ (see Dembowski (1968), p252) is a $3-(q^2+1,q+1,1)$-design, whose blocks are called *circles*.

Bruck (1969) has shown that in $PG(3,q)$, a regular 1-spread together with its lines and 1-reguli are an inversive plane of order q, where the points are the lines of the 1-spread and the circles are the 1-reguli of the 1-spread. The incidence is containment. Similarly, the indicator set of a regular 1-spread of $PG(3,q)$, which is a line of $PG(3,q^2)$ skew to $PG(3,q)$, is an inversive plane of order q, where the points of the line are the points of the inversive plane and the circles are the projective sublines of order q. These ideas are used in Bruen (1978) to give a new class of translation planes of order q^2.

Bruck (1973a) and (1973b) has defined a d-dimensional circle geometry for each integer $d \geq 2$. The circle geometries of dimension 2 are precisely the inversive planes, and a regular t-spread of $PG(2t+1,q)$ is a $(t+1)$-dimensional circle geometry whose points are the elements of the t-spread and whose circles are the t-reguli of the t-spread.

It would be interesting to define a higher dimensional inversive geometry, which would admit not only circles (as in the case of d-dimensional circle geometries) but also higher dimensional circles. This would be analogous to a projective space with its system of subspaces of different dimensions. The definition of this $(s+1)$-dimensional inversive geometry $IG(s+1)$ of order q^t would admit as an example the regular t-spreads of $PG((s+1)(t+1)-1,q)$ in the following way: the points of $IG(s+1)$ would be the elements of the t-spread, and the r-dimensional circles would be the t-reguli of rank r for $r = 0, 1, \ldots, s$. An inversive plane would be an $IG(2)$ of order q and a d-dimensional circle geometry would be an $IG(2)$ of order q^t.

There are other questions raised by the work presented here: for example the definition of indicator sets suggests a classification of t-spreads using their indicator set. This was begun for 1-spreads of $PG(3,q)$ in Bruen (1972), and Theorem 3.8.7 is also a step in this direction.

On the topic of indicator sets, the work in Chapter Four does not fully address the question of the existence of maximal (k,n)-arcs by studying maximal (k,n)-sets of $PG(3t+2,q)$. It is possible that a maximal (k,n)-set of $PG((s+1)(t+1)-1,q)$ not embeddable in a regular t-spread, if it does in fact exist, could be constructed as a union of t-dimensional spaces belonging to different Segre varieties. The requirement that such a set be geometric seems to be very strong.

The same method of studying certain sets of t-dimensional spaces of the space $PG((s+1)(t+1)-1,q)$ could be employed for studying other sets of points in projective spaces, for example k-caps and (k,n)-caps of $PG(s,q^{t+1})$ correspond to certain sets of t-dimensional spaces in $PG((s+1)(t+1)-1,q)$.

REFERENCES

André, J. (1954). Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe. *Math. Z.* **60**, 156–186.

Baer, R. (1963). Partitionen abelscher Gruppen. *Arch. Math.* **14**, 73–83.

Barlotti, A. (1956). Sui $\{k; n\}$-archi di un piano lineare finito. *Boll. Un. Mat. Ital.* **11**, 553–556.

Beutelspacher, A. (1975). Partial spreads in finite projective spaces and partial designs. *Math. Z.* **145**, 211–229.

Beultelspacher, A. (1976). Correction to "Partial spreads in finite projective spaces and partial designs." *Math. Z.* **147**, 303.

Beutelspacher, A. (1978). Embedding of partial spreads in spreads. *Arch. Math.* **30**, 317–324.

Beutelspacher, A. (1980). Blocking sets and partial spreads in finite projective spaces. *Geom. Dedicata* **9**, 425–449.

Bose, R.C. (1947). Mathematical theory of the symmetrical factorial design. *Sankhya* **8**, 107–166.

Bruck, R.H. (1969). Construction problems of finite projective planes. Conference on Combinatorial Mathematics and its Applications (University of North Carolina, 1967), University of North Carolina Press, 426–514.

Bruck, R.H. (1973a). Circle geometry in higher dimensions. *A survey of Combinatorial Theory.* ed. J.N. Srivastava et al. North Holland. 69–77.

Bruck, R.H. (1973b). Circle geometry in higher dimensions.II. *Geom. Dedicata* **2**, 133–188.

Bruck, R.H. (1973c). Construction problems in finite projective spaces. *Finite Geometric Structures and their applications.* (Bressanone, 1972). Cremonese, 105–188.

Bruck, R.H. and Bose, R.C. (1964). The construction of translation planes from projective spaces. *J. Algebra* **1**, 85–102.

Bruck, R.H. and Bose. R.C. (1966). Linear representations of projective planes in projective spaces. *J. Algebra* **4**, 117–172.

Bruen, A. (1971). Partial spreads and replaceable nets. *Canad. J. Math.* **23**, 381–391.

Bruen, A. (1972a). Spreads and a conjecture of Bruck and Bose. *J. Algebra* **23**, 519–537.

Bruen, A. (1972b). Unimbeddable nets of small deficiency. *Pacific J. Math.* **43**, 51–54.

Bruen, A. (1975). Subregular spreads and indicator sets. *Canad. J. Math.* **27**, 1141–1148.

Bruen, A. (1978). Inversive Geometry and some new translation planes, I. *Geom. Dedicata* **7**, 81–98.

Bruen, A. and Thas, J.A. (1976). Partial spreads, packings and hermitian manifolds in $PG(3,q)$. *Math. Z.* **151**, 207–214.

Burau, W. (1961). *Mehrdimensionale Projektive und Höhere Geometrie.* Mathematische Monographien **5**, ed. W. Blaschke. Berlin.

Burnside, W. (1911). *Theory of groups of finite order.* Second edition, Cambridge, (Dover 1955).

Carmichael, R.D. (1937). *Introduction to the theory of groups of finite order.* Boston (Dover, 1956).

Casse, L.R.A. and Wild, P.R. (1983). k-sets of $(n-1)$-dimensional subspaces of $PG(3n-1,q)$. *Combinatorial Mathematics* X, Lecture Notes in Math. 1036, Springer, Berlin, 410–416.

Casse, L.R.A., Thas J.A. and Wild, P.R. (1985). (q^n+1)-sets of $PG(3n-1,q)$, generalised quadrangles and Laguerre planes. *Simon Stevin* **59**, 21–42.

Cossu, A. (1960). Sulle ovali in un piano proiettivo sopra un corpo finito. *Atti Accad. Naz. Lincei Rend.* **28**, 342–344.

Cossu, A. (1961). Su alcune proprietà dei $\{k,n\}$-archi di un piano proiettivo sopra un corpo finito. *Rend. Mat. e Appl.* **20**, 271–277.

Declerck, F., Gevaert, H. and Thas, J.A. (1987). Translation Partial Geometries. Preprint.

Dembowski, P. (1968). *Finite Geometries.* Springer-Verlag, Berlin–Heidelberg.

Denniston, R.H.F. (1969). Some maximal arcs in finite projective planes. *J. Combinatorial Theory* **6**, 317–319.

Denniston, R.H.F. (1973a). Spreads which are not subregular. *Glasnik Mat.* **8**, 3–5.

Denniston, R.H.F. (1973b). Packings of $PG(3,q)$. Finite Geometric Structures and their Applications. (Bressanone, 1972). Cremonese. 195–199.

Denniston, R.H.F. (1976). Some spreads which contain reguli without being subregular. *Teorie Combinatorie* (Rome, 1973), volume II, Accad. Naz. dei Lincei, 367–371.

Ebert, G.L. (1978). Maximal strictly partial spreads. *Canad. J. Math.* **30**, 483–489.

Ebert, G.L. (1979). A new lower bound for msp spreads. *Proc. Tenth Southeastern Conf. on Combinatorics, Graph Theory and Computing* Vol 1, Congr. Numer. 23, Utilitas Math., Winnipeg, 413–421.

Ebert, G.L. (1983). Subregular 1-spreads of PG(2n+1,2). *Geom. Dedicata* **14**, 343–353.

Finkbeiner, D.T. (1960). *Introduction to matrices and linear transformations.* Freeman. SanFrancisco and London.

Freeman, J.W. (1980). Reguli and pseudo-reguli in $PG(3,s^2)$. *Geom. Dedicata* **9**, 267–280.

Glynn, D.G. (1982). A lower bound for maximal partial spreads in $PG(3,q)$. *Ars. Combin.* **13**, 39-40.

Glynn, D.G. (1983). Two new sequences of ovals in finite Desarguesian planes of even order. *Combinatorial Mathematics* **X**, Lecture Notes in Math. 1036, Springer, Berlin, 217–229.

Glynn, D.G. and Payne, S.E. (1987). The proof of the oval conjectured to exist in $PG(2, q)$ by Cherowitzo. Private communication.

Hirschfeld, J.W.P. (1979). *Projective Geometries over Finite Fields.* Oxford University Press, Oxford.

Hirschfeld, J.W.P. (1985). *Finite Projective Spaces of Three Dimensions.* Oxford University Press, Oxford.

Hodge, W.V.D. and Pedoe, D. (1952). *Methods of Algebraic Geometry.* Volume II. Cambridge University Press.

Hughes, D.R. and Piper, F.C. (1985). *Design Theory.* Cambridge University Press.

Hughes, D.R. and Piper, F.C. (1973). *Projective planes.* Springer, Berlin.

Jungnickel, D. (1984). Maximal partial spreads and translation nets of small deficiency. *J. Algebra* **90**, 119–132.

Lunardon, G. (1984). Insiemi indicatori proiettivi e fibrazioni planari di uno spazio proiettivo finito. *Boll. Un. Mat. Ital.* **6**, 3–B, 717–735.

Lüneburg, H. (1980). *Translation planes.* Springer, Berlin.

Maduram, D.M. (1975). Matrix representation of translation planes. *Geom. Dedicata.* 4, 485–492.

Mesner, D.M. (1967). Sets of disjoint lines in $PG(3, q)$. *Canad. J. Math.* **19**, 273-280.

Orr, W.F. (1976). A characterisation of subregular spreads in finite 3-space. *Geom. Dedicata* **5**, 43–50.

Ostrom, T.G. (1968). Vector spaces and construction of finite projective planes. *Arch. Math* **19**, 1–25.

Qvist, B. (1952). Some remarks concerning curves of second degree in a finite plane. *Ann. Acad. Sci. Fenn. Ser. A,* No. 134.

Segre, B. (1955). Ovals in a finite projective plane. *Canad. J. Math.* **7**, 414–416.

Segre, B. (1964). Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane. *Ann. Mat. Pura Appl.* **64**, 1–76.

Segre, C. (1891). *Rendic. Palermo* **5**, 192–204.

Sherk, F.A. (1979). Indicator sets in an affine space of any dimension. *Canad. J. Math.* **31**, 211–224.

Sherk, F.A. and Pabst, G. (1977). Indicator sets, reguli and a new class of spreads. *Canad. J. Math.* **29**, 132–154.

Tallini-Scafati, M. (1966). $\{k, n\}$-archi di un piano grafico finito, con particolare riguardo a quelli con due caratteri. (Note I; II) *Atti Accad. Naz. Lincei Rend.* **40**, 812–818, 1020–1025.

Thas, J.A. (1971). The m-dimensional projective space $S_m(\mathcal{M}_n(GF(q)))$ over the total matrix algebra $\mathcal{M}_n(GF(q))$ of the $n \times n$ matrices with elements in the Galois field $GF(q)$. *Rend. Mat.* **4**, 459–532.

Thas, J.A. (1974). Construction of maximal arcs and partial geometries. *Geom. Dedicata* **3**, 61–64.

Thas, J.A. (1975). Some results concerning $\{(q+1)(n-1);n\}$-arcs and $\{(q+1)(n-1)+1;n\}$-arcs in finite projective planes of order q. *J. Combinatorial Theory A* **19**, 228–232.

Wild, P.R. (1986). Higher dimensional ovals and dual translation planes. Preprint.