



Generalized Quadrangles and Associated Structures

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Abstract

Our aim in this thesis has been to consider questions concerning the relationship between a Generalized Quadrangle (GQ) and various substructures, with a view to proving characterisation and classification results. We also lay the groundwork for new GQ construction methods, although no new GQs are constructed here.

In Chapter 1 we introduce preliminary concepts and results required for the rest of the thesis, involving graphs, quadrics, geometries, GQs, algebraic topology on a simplicial complex and covers of geometries.

Chapter 2 contains a detailed investigation of the ovoid $K1(\sigma)$ of $Q(4, q)$ constructed by Kantor in [30], including construction of non-elation rosettes of $Q(4, q)$ containing only $K1(\sigma)$ ovoids and rosettes containing both $K1(\sigma)$ ovoids and elliptic quadric ovoids.

In Chapter 3 we show that if \mathcal{S} is a GQ of order (s, s^2) and \mathcal{S}' is a subquadrangle of order s doubly subtended in \mathcal{S} , then the subtended ovoid/rosette structure is a Semi-Partial Geometry (SPG). A new SPG is constructed from a GQ of Kantor ([31]) and a $Q(4, q)$ subquadrangle. For a q -clan GQ \mathcal{S} , q even, Payne constructed a family of subquadrangles \mathcal{S}_α of order q ([45]). We derive the algebraic conditions under which \mathcal{S}_α is doubly subtended in \mathcal{S} , and hence gives an SPG.

In Chapter 4 it is shown that if q is even a non-classical GQ of order (q, q^2) containing a subquadrangle isomorphic to $Q(4, q)$ implies the existence of a new ovoid of $PG(3, q)$. Also, by a homology calculation, it is shown that if \mathcal{S} is a GQ of order (q, q^2) , q odd, such that \mathcal{S} contains a $Q(4, q)$ subquadrangle, with each ovoid of $Q(4, q)$ subtended by \mathcal{S} an elliptic quadric ovoid, then \mathcal{S} is isomorphic to $Q(5, q)$.

In Chapter 5 we show a GQ \mathcal{S} of order s with a regular point (∞) gives rise to a cover of the affine plane constructed from \mathcal{S} and (∞) , as in [49, 1.3.1]. Given an affine plane π of order s and an s -fold cover of π satisfying special conditions we construct a GQ of order s with a regular point. If the cover of π is algebraic the condition on the cover is interpreted in cohomological terms; we investigate these for the remainder of the Chapter 5.