On the Accumulated Sojourn Time in Finite–State Markov Processes

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Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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Summary

The subject of this thesis is the joint probability density of the accumulated sojourn time in each state of a Markov process when the initial state is known. The accumulated sojourn time for a Markov process is the total amount of time spent in each state of the process during an interval [0,t).

Sojourn–time problems are notoriously difficult and relatively few explicit results are available. In contrast to equilibrium distributions, which have been determined for a wide range of both finite– and infinite–state Markov processes, exact formulæ for sojourn times have not been available to date even for the general three–state Markov process, though both implicit and computationally–implementable expressions are, of course, available for some processes with a larger number of states. Furthermore, the existing results have mostly been known for some time.

The most comprehensive treatment to date is still that given over thirty years ago by Good [12]. As will be seen, Good's results are implicit rather than explicit, and rendering them explicit (and usable) involves something of a peregrination. Further, his derivation, while very elegant, may be regarded as informal by present–day standards.

In the analysis presented in Chapter 2, the three–state problem is formulated as a system of Kolmogorov equations. Analysis techniques for hyperbolic systems of first–order partial differential equations are used to set up the Kolmogorov equations rigorously. The solution is expedited using transform methods but is presented in terms of direct quantities, not simply through the curtain of their transforms. Our results match those of Good. They also extend them somewhat and provide a 'visual'

methodology which aids manipulation.

In Chapters 3 and 4 we refine the expression obtained for the three–state problem and subsequently generalise it, first to four states and finally to n states. In the process we use techniques from combinatorics and graph theory and present some intrinsically interesting mathematics.

The joint probability density of the total sojourn time has immediate application to reliability problems and also to questions involving transfer times for data in situations where transmission rates vary according to an underlying Markov process. In particular, we can compute the probability distribution of the total amount of data transmitted up to time t when the bit rate available on a link is modulated according to an underlying Markov chain. This is the subject of Chapter 5.

Chapter 1

Introduction

This thesis was originally motivated by real–world problems arising in the design of communications protocols for multi–media networks and HF radio networks. Some of these problems are described in the final chapter. The traffic streams in the relevant systems are best modelled as Markov modulated fluid flows. In the early stages of the research, it was discovered that many of these problems could be represented by the corresponding random walk on a plane with the direction of displacement determined by the state of the underlying Markov process. The accumulated sojourn time in each state gives us the total displacement in that direction.

For a Markov process, $\{S(t)\}$, $(0 \leq t < \infty)$ on a discrete state space, *the accumulated sojourn time up to time t* is the total time spent in each state of the process during an interval [0,t). We define the total sojourn time, $x_k(t)$, in state k by the random variable

$$x_k(t) = \int_0^t \chi_k(u)du, \quad \text{where} \quad \chi_k(u) = \begin{cases} 1 & \text{if } S(u) = k \\ 0 & \text{otherwise} \end{cases}.$$

The main result of this thesis is the rigorous formulation and subsequent solution of a system of Kolmogorov equations which describe the evolution of probability of the accumulated sojourn time for the n–state case. Our solution is in the form of an explicit expression and includes the solution on the boundary. This constitutes a significant extension to a result in Good [12]. Good in fact derives an expression which, as he writes, can "in principle" be used to calculate the joint density function for the total sojourn times in n states when all the states have been visited at least once, that is, the interior solution. In the case of a two–state Markov chain there are some simplifying features and Good provides an explicit evaluation of the joint density function of the sojourn times in the two states for that case.

Bendesson [2] derives an expression for the Laplace transform, with respect to t, of the Laplace–Stieltjes transform of the joint distribution of the times spent in an n–state quasi Markov process. Takács [20] derives an explicit formula for the cumulative distribution of the total sojourn time in a two–state Markov process. Rossiter [18] builds on this and develops a theory of sojourn times for alternating renewal processes. Expressions for the joint probability density function for a three–state stochastic process appear in [15] and [10]. The former derives a more general result but the resulting expression is not as explicit as the one we present here; the latter present an explicit solution for the three–state Markov process with the following restriction: only transitions to A_1 are allowed from states A_2 and A_3.

The structure of the thesis reflects the chronology of the work. In Chapter 2 the three–state problem is formulated as a system of Kolmogorov equations. Analysis techniques for hyperbolic systems of first–order partial differential equations are used to set up the Kolmogorov equations rigorously. The solution arises naturally from the inversion of the solution of the transformed equations. The partial derivatives of these functions are shown to satisfy the original partial differential equations. This work was presented in [8].

The solution obtained originally was in terms of infinite sums of convolutions of modified Bessel functions. Although it could be described as a closed–form solution, it was hoped that it would be possible to transform it into a simpler expression involving familiar special functions. This was considered possible in view of the connection between closed–form solutions to problems in stochastic processes and special functions, see [19]. In Chapter 3 we present a number of identities which are mathematically interesting and which enable us to redefine the three–state solution in terms of functions which, though not easily recognised as special functions, are more elegant than the original expressions and can be interpreted probabilistically. This part of the work entails a great deal of intuitive argument which was developed while writing the code for the numerical approximation of the three-state solution.

In an attempt to reconcile the interior part of our solution with the general result obtained by Good in [12], we manage to extract an explicit expression from his formula (10) for the probability density of the sojourn time in the three-state Markov process. This expression agrees with our refined interior solution, and its numerical approximation is surprisingly close to the approximation derived for the original solution.

The form of the refined three–state solution and an increased understanding of the problem make it possible to deduce the solution for the case $n = 4$, including the boundary terms. Once again the interior part of the solution is shown to be consistent with the explicit expression derived from the implicit form obtained by Good. A similar, but more general, probabilistic argument enables us to derive the general form for the case with n states. In order to extend the argument we make use of graph theory and the matrix-tree theorem [14] and explore some interesting connections between the two. Good did not mention this explicitly in [12], but the connection was clearly known to him as he obtained the solution by applying a generalisation of Lagrange's expansion to the probability distribution of the frequency count of a Markov chain, while in a later publication, [13], he applies the same technique to the enumeration of trees. In the proof of Theorem 3 in Chapter 4, we verify that the intuitive arguments are well–founded and that the deduced solution satisfies the coupled system of partial differential Kolmogorov equations.

In the final chapter, we point out some potential applications of the processes and methodology examined in the previous chapters to the problems which originally motivated us. It is perhaps worth adding *sotto voce* that we are not suggesting that our approach is the only one or the best one.

Chapter 2

Solving the Three-State Problem

2.1 Introduction

In this chapter we use analysis techniques for hyperbolic systems of first–order partial differential equations to set up the Kolmogorov equations for the probability density of the accumulated sojourn time in a three–state Markov process.

In Section 2.2 we formulate the problem rigorously. In Section 2.3 the Kolmogorov equations are expressed in terms of Laplace transforms and the transform equations are solved as Theorem 1. Section 2.4 addresses the inversion of the transformed solution and presents the outcome of the inversion as Theorem 2. A demonstration that the quantities derived actually do satisfy the original Kolmogorov equations forms the content of Section 2.5. We conclude in Section 2.6 by showing that for the special case of a two–state process our solution matches Good's explicit expression [12].

2.2 Problem formulation

2.2.1 Preliminaries

Without further comment we shall adopt the following standard notation – $\boldsymbol{N}$: the set of natural numbers; $\boldsymbol{R}$: the set of reals; $\boldsymbol{R}_+$: $[0, \infty)$; C^∞: the class of all infinitely often differentiable real valued functions on $\boldsymbol{R}^n$.

In this section we set up partial differential equations to describe the evolution with time of the joint distribution of the accumulated sojourn times in the three states. The initial conditions pertaining to these equations involve both Dirac and Kronecker deltas, so that we are in fact dealing with generalised functions (or distributions). In discussing the solution of a differential equation, one must first decide within what class of functions one is seeking a solution. A differential equation (with its attendant side conditions) may well fail to have a solution within some class of functions but admit one if the class is suitably expanded. The legitimacy of operations that are performed on the equation will, of course, depend on the smoothness properties we are anticipating.

Accordingly we introduce a space $\mathcal{D} = C_0^\infty$ of test functions defined by

$$\mathcal{D} = \{\phi \in C^\infty : \text{the support of } \phi \text{ is compact}\}.$$

We define convergence in $\mathcal{D}$ by the following: let $\phi_k, k \in \boldsymbol{N}$, and ϕ be elements of $\mathcal{D}$, then ϕ_k converges to ϕ in $\mathcal{D}$ as $k \to \infty$, if all the ϕ_k vanish outside some compact subset K of $\boldsymbol{R}^n$ and ϕ_k and derivatives of ϕ_k of arbitrary order converge uniformly in K to those of ϕ. The continuous linear functionals on $\mathcal{D}$ are called *distributions* or *generalised functions*. Let $\mathcal{D}'$ denote the space of distributions on $\mathcal{D}$ and (f, ϕ) denote the value assigned to the test function $\phi \in \mathcal{D}$, by the distribution $f \in \mathcal{D}'$.

2.2.2 The Problem

Consider a three–state Markov process with state space $\boldsymbol{X} = \{0, 1, 2\}$ and transition rate matrix $\Lambda = (\lambda_{ij}, i, j \in \boldsymbol{X})$, where $\lambda_{ij} \in \boldsymbol{R}_+$ for $i \neq j$ and $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$.

Let $\boldsymbol{x}(t) = (x_1(t), x_2(t)) \in \boldsymbol{R}_+^2$, where $x_k(t)$ is a random variable representing the time spent in state k up to time $t \in \boldsymbol{R}_+$ and let $x_0(t) = t - x_1(t) - x_2(t)$.

For $i \in \boldsymbol{X}$ let $\rho_i(t)$ denote the probability density of $(x_1(t), x_2(t))$ at time t and final state i. We then assume that ρ_i is a continuous function of $t \in \boldsymbol{R}_+$ valued in $\mathcal{D}'(\boldsymbol{R}^2)$, the space of distributions with respect to $\boldsymbol{x}$ and the support of ρ_i is included in $\boldsymbol{R}_+^2$.

It can be shown (see [22], Chapter II, Section 15) that the partial derivative, $\frac{\partial \rho_i}{\partial t}(t)$, is also a distribution on $\boldsymbol{R}^2$, continuous with respect to t.

The first–order partial derivatives with respect to x_k, $k \in \{1, 2\}$, of $\rho_i(t)$ are given by

$$\left(\frac{\partial \rho_i}{\partial x_k}(t), \phi\right) = \left(\rho_i(t), -\frac{\partial \phi}{\partial x_k}\right)$$

where $\frac{\partial \rho_i}{\partial x_k}(t) \in \mathcal{D}'(\boldsymbol{R}^2)$.

The evolution of probability of the sojourn time in this Markov process is given by the following Kolmogorov equations:

$$\begin{aligned} \frac{\partial \rho_0}{\partial t}(\boldsymbol{x}, t) &= \sum_{i=0}^{2} \lambda_{i0} \rho_i(\boldsymbol{x}, t) && (2.2.1) \\ \frac{\partial \rho_k}{\partial t}(\boldsymbol{x}, t) &= \sum_{i=0}^{2} \lambda_{ik} \rho_i(\boldsymbol{x}, t) - \frac{\partial \rho_k}{\partial x_k}(\boldsymbol{x}, t) && k \in \{1, 2\}. \end{aligned}$$

Since the process is assumed to start in state 0 at $x_1 = x_2 = 0$, the initial probability density is given by $\rho_k(\boldsymbol{x}, 0) = \delta_{k0}\delta(\boldsymbol{x})$, where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and δ is the Dirac delta distribution at $\boldsymbol{x} = (0, 0)$, defined by $(\delta, \phi) = \phi(0)$.

Remark: In the sequel $\delta(x)$ is used to simplify the notation and indicates that δ is operating on test functions which themselves depend on x. Let δ_i denote $\partial\delta/\partial x_i$, that is, $\delta_i(\phi) = -\partial\phi/\partial x_i(0)$, and $H(x)$ denote the Heaviside unit step function. $H(\boldsymbol{x})$ denotes $H(x_1)H(x_2)$ which is equal to 0 when either $x_1 < 0$ or $x_2 < 0$ and 1 when $x_1 > 0$ and $x_2 > 0$.

2.3 Solving the transformed equations

For $k \in \boldsymbol{X}$ let $\epsilon_k : \boldsymbol{R}^3 \mapsto \boldsymbol{R}$ be defined by:

$$\epsilon(x_1, x_2, t) = -\sum_{k=0}^{2} \lambda_{kk} x_k.$$

We can make use of these quantities to simplify our equations. If we substitute $\rho_i = \alpha_i e^{-\epsilon(x_1,x_2,t)}$, $i \in \boldsymbol{X}$, in equations (2.2.1), the diagonal terms λ_{kk} disappear from the right hand side to give

$$\begin{aligned} \frac{\partial\alpha_0}{\partial t}(\boldsymbol{x}, t) &= \lambda_{10}\alpha_1(\boldsymbol{x}, t) + \lambda_{20}\alpha_2(\boldsymbol{x}, t) \\ \frac{\partial\alpha_1}{\partial t}(\boldsymbol{x}, t) + \frac{\partial\alpha_1}{\partial x_1}(\boldsymbol{x}, t) &= \lambda_{01}\alpha_0(\boldsymbol{x}, t) + \lambda_{21}\alpha_2(\boldsymbol{x}, t) \\ \frac{\partial\alpha_2}{\partial t}(\boldsymbol{x}, t) + \frac{\partial\alpha_2}{\partial x_2}(\boldsymbol{x}, t) &= \lambda_{02}\alpha_0(\boldsymbol{x}, t) + \lambda_{12}\alpha_1(\boldsymbol{x}, t). \end{aligned} \tag{2.3.2}$$

Note that, since $\exp(-\epsilon(x_1, x_2, t)) \in C^\infty$, then $\alpha_i \in \mathcal{D}'(\boldsymbol{R}^2)$.

Taking Laplace transforms in (2.3.2), with respect to t, we derive the equations

$$\begin{aligned} s\hat{\alpha}_0(\boldsymbol{x},s) &= \alpha_0(\boldsymbol{x},0)+\lambda_{10}\hat{\alpha}_1(\boldsymbol{x},s)+\lambda_{20}\hat{\alpha}_2(\boldsymbol{x},s), \\ \hat{\alpha}_{1,1}(\boldsymbol{x},s) &= -s\hat{\alpha}_1(\boldsymbol{x},s)+\alpha_1(\boldsymbol{x},0)+\lambda_{01}\hat{\alpha}_0(\boldsymbol{x},s)+\lambda_{21}\hat{\alpha}_2(\boldsymbol{x},s), \\ \hat{\alpha}_{2,2}(\boldsymbol{x},s) &= -s\hat{\alpha}_2(\boldsymbol{x},s)+\alpha_2(\boldsymbol{x},0)+\lambda_{02}\hat{\alpha}_0(\boldsymbol{x},s)+\lambda_{12}\hat{\alpha}_1(\boldsymbol{x},s) \end{aligned} \tag{2.3.3}$$

for $s \in \boldsymbol{R}, s \geq 0$, where $\hat{\alpha}_{k,i}$ is the partial derivative of $\hat{\alpha}_k$ with respect to x_i.

We can now incorporate our known initial conditions by substituting for $\alpha_0(\boldsymbol{x},0)$ and $\alpha_i(\boldsymbol{x},0)$, and subsequently $\hat{\alpha}_0$ in the last 2 rows of (2.3.3). We derive

$$\begin{aligned} \hat{\alpha}_{1,1}(\boldsymbol{x},s) &= -s\hat{\alpha}_1(\boldsymbol{x},s)+\frac{\lambda_{01}}{s}\left(\delta+\lambda_{10}\hat{\alpha}_1(\boldsymbol{x},s)+\lambda_{20}\hat{\alpha}_2(\boldsymbol{x},s)\right)+\lambda_{21}\hat{\alpha}_2(\boldsymbol{x},s), \\ \hat{\alpha}_{2,2}(\boldsymbol{x},s) &= -s\hat{\alpha}_2(\boldsymbol{x},s)+\frac{\lambda_{02}}{s}\left(\delta+\lambda_{10}\hat{\alpha}_1(\boldsymbol{x},s)+\lambda_{20}\hat{\alpha}_2(\boldsymbol{x},s)\right)+\lambda_{12}\hat{\alpha}_1(\boldsymbol{x},s). \end{aligned}$$

For notational simplicity we introduce

$$\begin{aligned} k_1(s) &= \left(\frac{\lambda_{01}\lambda_{10}}{s}-s\right), \quad & k_2(s) &= \left(\frac{\lambda_{01}\lambda_{20}}{s}+\lambda_{21}\right), \\ k_3(s) &= \left(\frac{\lambda_{10}\lambda_{02}}{s}+\lambda_{12}\right), \quad & k_4(s) &= \left(\frac{\lambda_{02}\lambda_{20}}{s}-s\right). \end{aligned}$$

In terms of these functions, our equations become

$$\begin{aligned} \hat{\alpha}_{1,1}(\boldsymbol{x},s) &= k_1(s)\hat{\alpha}_1(\boldsymbol{x},s)+k_2(s)\hat{\alpha}_2(\boldsymbol{x},s)+\frac{\lambda_{01}}{s}\delta, \\ \hat{\alpha}_{2,2}(\boldsymbol{x},s) &= k_3(s)\hat{\alpha}_1(\boldsymbol{x},s)+k_4(s)\hat{\alpha}_2(\boldsymbol{x},s)+\frac{\lambda_{02}}{s}\delta. \end{aligned} \tag{2.3.4}$$

A further substitution enables us to further simplify these equations.

Put $\hat{\beta}_i(\boldsymbol{x}, s) = \hat{\alpha}_i(\boldsymbol{x}, s) \exp(-(k_1(s)x_1 + k_4(s)x_2))$ in (2.3.4), ($\hat{\beta}_i \in \mathcal{D}'(\boldsymbol{R}^2)$),

$$\begin{aligned} \hat{\beta}_{1,1}(\boldsymbol{x}, s) &= k_2(s)\hat{\beta}_2(\boldsymbol{x}, s) + \frac{\lambda_{01}}{s}\delta, \\ \hat{\beta}_{2,2}(\boldsymbol{x}, s) &= k_3(s)\hat{\beta}_1(\boldsymbol{x}, s) + \frac{\lambda_{02}}{s}\delta. \end{aligned} \tag{2.3.5}$$

Now we differentiate $\hat{\beta}_1$ with respect to x_2 and $\hat{\beta}_2$ with respect to x_1 so as to get expressions for $\hat{\beta}_{1,1}$ in terms of $\hat{\beta}_2$ and $\hat{\beta}_{2,2}$ in terms of $\hat{\beta}_1$

$$\begin{aligned} \hat{\beta}_{1,12}(\boldsymbol{x}, s) &= k_2(s)\hat{\beta}_{2,2}(\boldsymbol{x}, s) + \frac{\lambda_{01}}{s}\delta_2, \\ \hat{\beta}_{2,21}(\boldsymbol{x}, s) &= k_3(s)\hat{\beta}_{1,1}(\boldsymbol{x}, s) + \frac{\lambda_{02}}{s}\delta_1. \end{aligned} \tag{2.3.6}$$

Finally, substituting (2.3.5) into (2.3.6) we get a decoupled pair of equations

$$\begin{aligned} \hat{\beta}_{1,12}(\boldsymbol{x}, s) &= k_2(s)k_3(s)\hat{\beta}_1(\boldsymbol{x}, s) + k_2(s)\frac{\lambda_{02}}{s}\delta + \frac{\lambda_{01}}{s}\delta_2, \\ \hat{\beta}_{2,21}(\boldsymbol{x}, s) &= k_2(s)k_3(s)\hat{\beta}_2(\boldsymbol{x}, s) + k_3(s)\frac{\lambda_{01}}{s}\delta + \frac{\lambda_{02}}{s}\delta_1. \end{aligned}$$

Consider first the elementary solution $G_i \in \mathcal{D}'(\boldsymbol{R}^2)$ satisfying

$$G_{i,12} - k_2(s)k_3(s)G_i = k_{i+1}(s)\frac{\lambda_{0j}}{s}\delta, \quad i, j \in \{1, 2\}, j \neq i. \tag{2.3.7}$$

For $x_1, x_2 > 0$, G_i satisfies the homogeneous equation

$$G_{i,12} - k_2(s)k_3(s)G_i = 0.$$

If G_i is a function of $y = x_1x_2$, then the equation is of the form

$$u'' + \frac{u'}{y} - \frac{k_2(s)k_3(s)}{y}u = 0,$$

which yields the power series solution

$$u(y) = c_1 I_0\left(2\sqrt{k_2(s)k_3(s)y}\right) + c_2 K_0\left(2\sqrt{k_2(s)k_3(s)y}\right),$$

with c_1, c_2 arbitrary constants and I_0, K_0 modified Bessel functions of the first and second kind respectively.

To find c_1 and c_2, assume that $\hat{\beta}_i(0,0,s) = 0$. So

$$G_{i,12} = k_{i+1}(s)\frac{\lambda_{0j}}{s}\delta$$

which can be integrated to get,

$$G_i = k_{i+1}(s)\frac{\lambda_{0j}}{s}H(\boldsymbol{x}).$$

So let $c_1 = k_{i+1}(s)\frac{\lambda_{0j}}{s}H(\boldsymbol{x})$ and $c_2 = 0$, that is,

$$G_i(\boldsymbol{x}, s) = k_{i+1}(s)\frac{\lambda_{0j}}{s}H(\boldsymbol{x})I_0\left(2\sqrt{k_2(s)k_3(s)x_1x_2}\right).$$

Now find an elementary solution E_i to

$$E_{i,12} - k_2(s)k_3(s)E_i = \frac{\lambda_{0j}}{s}\delta_2.$$

We utilise the solution to (2.3.7) and obtain

$$\begin{aligned} E_1(\boldsymbol{x}, s) &= \frac{\lambda_{01}}{s}H(x_1)\delta(x_2)I_0(\psi(\boldsymbol{x})) + \frac{\lambda_{01}}{s}H(\boldsymbol{x})\frac{\sqrt{k_2(s)k_3(s)x_1}}{\sqrt{x_2}}I_1(\psi(\boldsymbol{x})) \\ &= \frac{\lambda_{01}}{s}H(x_1)\delta(x_2) + \frac{\lambda_{01}}{s}H(\boldsymbol{x})\frac{\sqrt{k_2(s)k_3(s)x_1}}{\sqrt{x_2}}I_1(\psi(\boldsymbol{x})), \end{aligned}$$

where $\psi(\boldsymbol{x}) = 2\sqrt{k_2(s)k_3(s)x_1x_2}$.

The complete solution is

$$\begin{aligned}\hat{\beta}_1(\boldsymbol{x},s) &= E_1(\boldsymbol{x},s)+G_1(\boldsymbol{x},s)\\ &= \frac{\lambda_{01}}{s}[H(x_1)\delta(x_2)+H(\boldsymbol{x})\frac{\sqrt{k_2(s)k_3(s)x_1}}{\sqrt{x_2}}I_1(\psi(\boldsymbol{x}))]+k_2(s)\frac{\lambda_{02}}{s}H(\boldsymbol{x})I_0(\psi(\boldsymbol{x})).\end{aligned}$$

Substituting $\hat{\beta}_1$ and its derivative into (2.3.5) we obtain a symmetric expression for $\hat{\beta}_2$

$$\hat{\beta}_2(\boldsymbol{x},s)=\frac{\lambda_{02}}{s}[\delta(x_1)H(x_2)+H(\boldsymbol{x})\frac{\sqrt{k_2(s)k_3(s)x_2}}{\sqrt{x_1}}I_1(\psi(\boldsymbol{x}))]+k_3(s)\frac{\lambda_{01}}{s}H(\boldsymbol{x})I_0(\psi(\boldsymbol{x})).$$

For $i\in\{1,2\}$, $\hat{\alpha}_i=\exp(k_1(s)x_1+k_4(s)x_2)\hat{\beta}_i$, which gives us the result of Theorem 1.

Theorem 1 *The solutions to Equations (2.3.3) are given by*

$$\begin{aligned}\hat{\alpha}_1(\boldsymbol{x},s) &= e^{k_1(s)x_1+k_4(s)x_2}\{\frac{\lambda_{01}}{s}[H(x_1)\delta(x_2)+H(\boldsymbol{x})\frac{\sqrt{k_2(s)k_3(s)x_1}}{\sqrt{x_2}}I_1(\psi(\boldsymbol{x}))]\\ &\quad + k_2(s)\frac{\lambda_{02}}{s}H(\boldsymbol{x})I_0(\psi(\boldsymbol{x}))\},\\ \hat{\alpha}_2(\boldsymbol{x},s) &= e^{k_1(s)x_1+k_4(s)x_2}\{\frac{\lambda_{02}}{s}[\delta(x_1)H(x_2)+H(\boldsymbol{x})\frac{\sqrt{k_2(s)k_3(s)x_2}}{\sqrt{x_1}}I_1(\psi(\boldsymbol{x}))]\\ &\quad + k_3(s)\frac{\lambda_{01}}{s}H(\boldsymbol{x})I_0(\psi(\boldsymbol{x}))\},\\ \hat{\alpha}_0(\boldsymbol{x},s) &= \frac{1}{s}[\delta+\lambda_{10}\hat{\alpha}_1(\boldsymbol{x},s)+\lambda_{20}\hat{\alpha}_2(\boldsymbol{x},s)], \qquad (2.3.8)\end{aligned}$$

where I_0, I_1 are modified Bessel functions of the first kind with argument, $\psi(\boldsymbol{x})=2\sqrt{k_2(s)k_3(s)x_1x_2}$, and

$$\begin{aligned}k_1(s) &= \left(\frac{\lambda_{01}\lambda_{10}}{s}-s\right), \quad k_2(s) = \left(\frac{\lambda_{01}\lambda_{20}}{s}+\lambda_{21}\right),\\ k_3(s) &= \left(\frac{\lambda_{10}\lambda_{02}}{s}+\lambda_{12}\right) \text{ and } k_4(s) = \left(\frac{\lambda_{02}\lambda_{20}}{s}-s\right).\end{aligned}$$

Theorem 1 can be verified by direct substitution.

2.4 The probability density of the sojourn time

For term by term inverse transformation, put $\hat{\alpha}_1(\boldsymbol{x}, s) = \hat{f}_1(\boldsymbol{x}, s) + \hat{g}_1(\boldsymbol{x}, s) + \hat{h}_1(\boldsymbol{x}, s)$, where

$$\begin{aligned} \hat{f}_1(\boldsymbol{x}, s) &= e^{k_1(s)x_1 + k_4(s)x_2} \frac{\lambda_{01}}{s} H(x_1)\delta(x_2), \\ \hat{g}_1(\boldsymbol{x}, s) &= e^{k_1(s)x_1 + k_4(s)x_2} \frac{\lambda_{01}}{s} H(\boldsymbol{x}) \frac{\sqrt{k_2(s)k_3(s)x_1}}{\sqrt{x_2}} I_1(\psi(\boldsymbol{x})), \\ \hat{h}_1(\boldsymbol{x}, s) &= e^{k_1(s)x_1 + k_4(s)x_2} k_2(s) \frac{\lambda_{02}}{s} H(\boldsymbol{x}) I_0(\psi(\boldsymbol{x})). \end{aligned}$$

Let $k_2(s)k_3(s) = \mu_1 s^{-2} + \mu_2 s^{-1} + \mu_3$, with

$$\mu_1 = \lambda_{01}\lambda_{10}\lambda_{02}\lambda_{20}, \ \mu_2 = \lambda_{02}\lambda_{21}\lambda_{10} + \lambda_{01}\lambda_{12}\lambda_{20}, \ \mu_3 = \lambda_{12}\lambda_{21},$$

and $k_1(s)x_1 + k_4(s)x_2 = \gamma(\boldsymbol{x})s^{-1} - \theta(\boldsymbol{x})s$, where

$$\gamma(\boldsymbol{x}) = \lambda_{01}\lambda_{10}x_1 + \lambda_{02}\lambda_{20}x_2 \text{ and } \theta(\boldsymbol{x}) = x_1 + x_2.$$

Noting that $\sqrt{y}\, I_1\left(2\sqrt{y}\right) = y \sum_{k=0}^{\infty} \frac{y^k}{k!(k+1)!}$, we substitute for all occurrences of Bessel functions in $\hat{g}_1$, obtaining

$$\begin{aligned} \hat{g}_1(\boldsymbol{x}, s) &= H(\boldsymbol{x}) e^{\gamma(\boldsymbol{x})/s - \theta(\boldsymbol{x})s} \frac{(\mu_1 s^{-2} + \mu_2 s^{-1} + \mu_3)x_1\lambda_{01}}{s} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(\mu_1 s^{-2} + \mu_2 s^{-1} + \mu_3)^k (x_1 x_2)^k}{k!(k+1)!} \\ &= H(\boldsymbol{x})\lambda_{01}x_1(\mu_1 s^{-2} + \mu_2 s^{-1} + \mu_3) \sum_{k=0}^{\infty} \frac{(x_1x_2)^k}{(k+1)!} \\ &\quad \times \sum_{r_1+r_2+r_3=k} \frac{\mu_1^{r_1}\mu_2^{r_2}\mu_3^{r_3}}{r_1!r_2!r_3!} s^{-2r_1-r_2-1} e^{\gamma(\boldsymbol{x})/s - \theta(\boldsymbol{x})s} \\ &= H(\boldsymbol{x})\lambda_{01}x_1(\mu_1 \hat{F}_{1,3}(\boldsymbol{x}, s) + \mu_2 \hat{F}_{1,2}(\boldsymbol{x}, s) + \mu_3 \hat{F}_{1,1}(\boldsymbol{x}, s)). \end{aligned}$$

For $m, n \in \boldsymbol{N}$, $\hat{F}_{m,n}$ is given by

$$\begin{aligned}
\hat{F}_{m,n}(\boldsymbol{x}, s) &= \sum_{k=0}^{\infty} \frac{(x_1x_2)^k}{(k+m)!} \sum_{r_1+r_2+r_3=k} \frac{\mu_1^{r_1}\mu_2^{r_2}\mu_3^{r_3}}{r_1!r_2!r_3!} s^{-2r_1-r_2-n} e^{\gamma(\boldsymbol{x})/s-\theta(\boldsymbol{x})s} \\
&= e^{-\theta(\boldsymbol{x})s} \sum_{k=0}^{\infty} \frac{(x_1x_2)^k}{(k+m)!} \sum_{r_1+r_2+r_3=k} \frac{\mu_1^{r_1}\mu_2^{r_2}\mu_3^{r_3}}{r_1!r_2!r_3!} \sum_{l=0}^{\infty} \frac{\gamma(\boldsymbol{x})^l}{l!} s^{-2r_1-r_2-l-n} \\
&= e^{-\theta(\boldsymbol{x})s} \sum_{p=0}^{\infty} \sum_{i=0}^{[\frac{p}{2}]} \sum_{j=0}^{p-2i} \sum_{k=j+i}^{\infty} s^{-p-n} \frac{\mu_1^i}{i!} \frac{\mu_2^j}{j!} \frac{\mu_3^{k-j-i}}{(k-j-i)!} \frac{\gamma(\boldsymbol{x})^{p-j-2i}}{(p-j-2i)!} \frac{(x_1x_2)^k}{(k+m)!}.
\end{aligned}$$

From [7] (30.2) we can calculate the inverse Laplace Transform of the series

$$\sum_{\nu=0}^{\infty} a_\nu s^{-(\nu+1)} \xrightarrow{\mathcal{L}^{-1}} \sum_{\nu=0}^{\infty} a_\nu \frac{t^\nu}{\nu!}$$

when the LHS series converges absolutely for $|\, s \,| \geq 0$.

Using this result with $\nu = p + n - 1$, for $n \geq 1$, the translation operator $e^{-\theta(\boldsymbol{x})s}$ and the substitution $x_0 = t - x_1 - x_2 = t - \theta(\boldsymbol{x})$, we obtain the inverse transform

$$\begin{aligned}
F_{m,n}(\boldsymbol{x}, t) &= H(x_0) \sum_{p=0}^{\infty} \frac{x_0^{p+n-1}}{(p+n-1)!} \sum_{i=0}^{[\frac{p}{2}]} \sum_{j=0}^{p-2i} \frac{\gamma(\boldsymbol{x})^{p-j-2i}}{(p-j-2i)!} \frac{\mu_1^i}{i!} \frac{\mu_2^j}{j!} \\
&\quad \times \sum_{k=j+i}^{\infty} \frac{\mu_3^{k-j-i}}{(k-j-i)!} \frac{(x_1x_2)^k}{(k+m)!} \\
&= H(x_0) \sum_{p=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\gamma(\boldsymbol{x})^p}{p!} \frac{x_0^{p+2i+j+n-1}}{(p+2i+j+n-1)!} \frac{(x_1x_2)^{i+j+k}}{(i+j+k+m)!} \frac{\mu_1^i\mu_2^j\mu_3^k}{i!j!k!}.
\end{aligned}$$

We thus obtain this expression for the inverse transform of $\hat{g}_1$

$$g_1(\boldsymbol{x}, t) = H(\boldsymbol{x})\lambda_{01}x_1(\mu_1 F_{1,3(}\boldsymbol{x}, t) + \mu_2 F_{1,2}(\boldsymbol{x}, t) + \mu_3 F_{1,1}(\boldsymbol{x}, t)).$$

By the same technique it can be shown that the inverse transform of $\hat{h}_1$ is

$$h_1(\boldsymbol{x}, t) = H(\boldsymbol{x})(\lambda_{02}\lambda_{20}\lambda_{01}F_{0,2}(\boldsymbol{x}, t) + \lambda_{02}\lambda_{21}F_{0,1}(\boldsymbol{x}, t)).$$

Similarly $f_1(\boldsymbol{x},t) = \lambda_{01}H(t-x_1)H(x_1)\delta(x_2)I_0(2\sqrt{\lambda_{01}\lambda_{10}x_1(t-x_1)})$. For the sake of uniformity we write this Bessel function in terms of $F_{0,n}$

$$f_1(\boldsymbol{x},t) = \lambda_{01}H(t-x_1)H(x_1)\delta(x_2)F_{0,1}(x_1,0,t).$$

Using symmetry arguments we can write similar expressions for f_2, g_2 and h_2

$$\begin{aligned} f_2(\boldsymbol{x},t) &= \lambda_{02}\delta(x_1)H(t-x_2)H(x_2)F_{0,1}(0,x_2,t), \\ g_2(\boldsymbol{x},t) &= H(\boldsymbol{x})\lambda_{02}x_2\left[\mu_1 F_{1,3}(\boldsymbol{x},t) + \mu_2 F_{1,2}(\boldsymbol{x},t) + \mu_3 F_{1,1}(\boldsymbol{x},t)\right], \\ h_2(\boldsymbol{x},t) &= H(\boldsymbol{x})\left[\lambda_{01}\lambda_{10}\lambda_{02}F_{0,2}(\boldsymbol{x},t) + \lambda_{01}\lambda_{12}F_{0,1}(\boldsymbol{x},t)\right]. \end{aligned}$$

In order to derive f_0, g_0 and h_0, we make use of Theorem 1 which gives us the relation

$$\hat{\alpha}_0(\boldsymbol{x},s) = \frac{1}{s}[\delta(\boldsymbol{x}) + \lambda_{10}\hat{\alpha}_1(\boldsymbol{x},s) + \lambda_{20}\hat{\alpha}_2(\boldsymbol{x},s)].$$

From the definition of $\hat{F}_{m,n}$, we observe that $\hat{F}_{m,n+1} = \frac{1}{s}\hat{F}_{m,n}$. Combining this with Theorem 1 and then using the inverse transform technique detailed above, we get

$$\begin{aligned} f_0(\boldsymbol{x},t) = {} & \lambda_{01}\lambda_{10}H(t-x_1)H(x_1)\delta(x_2)F_{0,2}(x_1,0,t) \\ & + \lambda_{02}\lambda_{20}\delta(x_1)H(t-x_2)H(x_2)F_{0,2}(0,x_2,t), \end{aligned}$$

$$\begin{aligned} g_0(\boldsymbol{x},t) = {} & H(\boldsymbol{x})(\lambda_{01}\lambda_{10}x_1 + \lambda_{02}\lambda_{20}x_2) \\ & \times \left[\mu_1 F_{1,4}(\boldsymbol{x},t) + \mu_2 F_{1,3}(\boldsymbol{x},t) + \mu_3 F_{1,2}(\boldsymbol{x},t)\right], \end{aligned}$$

$$h_0(\boldsymbol{x},t) = H(\boldsymbol{x})\left[2\lambda_{01}\lambda_{10}\lambda_{02}\lambda_{20}F_{0,3}(\boldsymbol{x},t) + (\lambda_{10}\lambda_{02}\lambda_{21} + \lambda_{20}\lambda_{01}\lambda_{12})F_{0,2}(\boldsymbol{x},t)\right].$$

We are now able to express α_0, α_1 and α_2 in terms of the component functions $F_{1,n}$ and $F_{0,n}$:

$$\begin{aligned}\alpha_0(\boldsymbol{x},t) &= \delta(\boldsymbol{x}) + \lambda_{01}\lambda_{10}H(x_1)\delta(x_2)F_{0,2}\,(x_1,0,t)\\ &+\ \lambda_{02}\lambda_{20}\delta(x_1)H(x_2)F_{0,2}\,(0,x_2,t)\\ &+\ H(\boldsymbol{x})\gamma(\boldsymbol{x})\left[\mu_1F_{1,4}(\boldsymbol{x},t)+\mu_2F_{1,3}(\boldsymbol{x},t)+\mu_3F_{1,2}(\boldsymbol{x},t)\right]\\ &+\ H(\boldsymbol{x})\left[2\mu_1F_{0,3}(\boldsymbol{x},t)+\mu_2F_{0,2}(\boldsymbol{x},t)\right],\end{aligned}$$

$$\begin{aligned}\alpha_1(\boldsymbol{x},t) &= \lambda_{01}H(x_1)\delta(x_2)F_{0,1}\,(x_1,0,t)\\ &+\ H(\boldsymbol{x})\lambda_{01}x_1\left[\mu_1F_{1,3}(\boldsymbol{x},t)+\mu_2F_{1,2}(\boldsymbol{x},t)+\mu_3F_{1,1}(\boldsymbol{x},t)\right]\\ &+\ H(\boldsymbol{x})\left[\lambda_{02}\lambda_{20}\lambda_{01}F_{0,2}(\boldsymbol{x},t)+\lambda_{02}\lambda_{21}F_{0,1}(\boldsymbol{x},t)\right],\end{aligned}$$

$$\begin{aligned}\alpha_2(\boldsymbol{x},t) &= \lambda_{02}\delta(x_1)H(x_2)F_{0,1}\,(0,x_2,t)\\ &+\ H(\boldsymbol{x})\lambda_{02}x_2\left[\mu_1F_{1,3}(\boldsymbol{x},t)+\mu_2F_{1,2}(\boldsymbol{x},t)+\mu_3F_{1,1}(\boldsymbol{x},t)\right]\\ &+\ H(\boldsymbol{x})\left[\lambda_{01}\lambda_{10}\lambda_{02}F_{0,2}(\boldsymbol{x},t)+\lambda_{01}\lambda_{12}F_{0,1}(\boldsymbol{x},t)\right].\end{aligned}$$

We may now state the main result of this chapter.

Theorem 2 *The solutions to the Kolmogorov equations (2.2.1) of the evolution of probability of the accumulated sojourn time in a three-state Markov process, are given by*

$$\begin{aligned}\rho(\boldsymbol{x},t) &= \rho_0(\boldsymbol{x},t)+\rho_1(\boldsymbol{x},t)+\rho_2(\boldsymbol{x},t)\\ &= e^{-\epsilon(\boldsymbol{x},t)}[\alpha_0(\boldsymbol{x},t)+\alpha_1(\boldsymbol{x},t)+\alpha_2(\boldsymbol{x},t)],\end{aligned}$$

where x_i is the time spent in state i, x_0 is defined to be $t-x_1-x_2$ and $\epsilon(\boldsymbol{x},t) = -\sum_k \lambda_{kk}x_k$.

The generalised functions α_i in Theorem 2 were derived via transform methods. In the next section we show directly that they satisfy Equations (2.3.2), thus proving the theorem. Accordingly we calculate the partial derivatives of $F_{m,n}$ for $m \in \{0, 1\}$.

2.5 The Partial Derivatives of $F_{0,n}$ and $F_{1,n}$

As we saw in the previous section, our solutions are a linear combination of $F_{1,n}$ and $F_{0,n}$ and therefore we need to calculate the partial derivatives of these functions in order to verify that the expressions derived satisfy our system of Kolmogorov equations. Three key technical lemmata serve as a springboard.

In this section, we let $\tilde{F}_{m,n}(x_0, x_1, x_2) = F_{m,n}(\boldsymbol{x}, t)$. Let D_i denote partial differentiation with respect to the ith variable, keeping the other two variables fixed. Since $x_0 = t - x_1 - x_2$, then $\frac{\partial}{\partial t} F_{m,n}(\boldsymbol{x}, t) = D_1 \tilde{F}_{m,n}(x_0, x_1, x_2)$ and $\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial t}\right) F_{m,n}(\boldsymbol{x}, t) = D_2 \tilde{F}_{m,n}(x_0, x_1, x_2)$.

Remark: Since the initial state of the process is state 0, $x_0 > 0$ for all $t > 0$, so that if $x_0 = 0$ then $t = x_1 = x_2 = 0$. Hence all $\delta(x_0)$ terms that arise from the partial differentiation of $H(x_0)$ are deemed to be equal to zero.

Lemma 1

$$\frac{\partial}{\partial t} F_{m,n}(\boldsymbol{x}, t) = F_{m,n-1}(\boldsymbol{x}, t), \qquad n > 1.$$

Proof

$$\begin{aligned}
\frac{\partial}{\partial t} F_{m,n}(\boldsymbol{x}, t) &= D_1 \tilde{F}_{m,n}(x_0, x_1, x_2) \\
&= H(x_0) \sum_{p=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\gamma(\boldsymbol{x})^p x_0^{p+2i+j+n-2} (x_1 x_2)^{i+j+k} \mu_1^i \mu_2^j \mu_3^k}{p!(p+2i+j+n-2)!(i+j+k+m)!i!j!k!} \\
&= F_{m,n-1}(\boldsymbol{x}, t).
\end{aligned}$$

Using the sorts of techniques which are used below to prove Lemma 3, we can define

$$F_{m,0} = \frac{\gamma(\boldsymbol{x})}{x_0} F_{m,2} + \frac{2\mu_1}{x_0} F_{m+1,3} + \frac{\mu_2}{x_0} F_{m+1,2},$$

which is well defined for $t > 0$.

□

Lemma 2

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial t}\right)(x_1 F_{1,n}(\boldsymbol{x}, t)) = \lambda_{01}\lambda_{10} x_1 F_{1,n+1}(\boldsymbol{x}, t) + F_{0,n}(\boldsymbol{x}, t) \qquad (2.5.9)$$

$$\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial t}\right)(x_2 F_{1,n}(\boldsymbol{x}, t)) = \lambda_{02}\lambda_{20} x_2 F_{1,n+1}(\boldsymbol{x}, t) + F_{0,n}(\boldsymbol{x}, t). \qquad (2.5.10)$$

Proof

$$\begin{aligned}
\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial t}\right)(x_1 F_{1,n}(\boldsymbol{x}, t)) &= D_2 \tilde{F}_{m,n}(x_0, x_1, x_2) \\
&= H(x_0)\lambda_{01}\lambda_{10} x_1 \sum_{p=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\gamma(\boldsymbol{x})^{p-1}}{(p-1)!} \frac{x_0^{p+2i+j+n-1}}{(p+2i+j+n-1)!} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(x_1 x_2)^{i+j+k}}{(i+j+k+1)!} \frac{\mu_1^i \mu_2^j \mu_3^k}{i!j!k!} \\
&+ H(x_0) \sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\gamma(\boldsymbol{x})^p}{p!} \frac{x_0^{p+2i+j+n-1}}{(p+2i+j+n-1)!} \frac{(x_1 x_2)^{i+j+k}}{(i+j+k)!} \frac{\mu_1^i \mu_2^j \mu_3^k}{i!j!k!} \\
&= H(x_0)\lambda_{01}\lambda_{10} x_1 \sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\gamma(\boldsymbol{x})^p (x_1 x_2)^{i+j+k} x_0^{p+2i+j+n} \mu_1^i \mu_2^j \mu_3^k}{p!(p+2i+j+n)!(i+j+k+1)!i!j!k!} \\
&\quad + F_{0,n}(\boldsymbol{x}, t) \\
&= \lambda_{01}\lambda_{10} x_1 F_{1,n+1}(\boldsymbol{x}, t) + F_{0,n}(\boldsymbol{x}, t).
\end{aligned}$$

The relation (2.5.10) is proved by symmetry.

□

Lemma 3

$$\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial t}\right)(F_{0,n}(\boldsymbol{x},t)) = \lambda_{01}\lambda_{10}F_{0,n+1}(\boldsymbol{x},t) \; + \\ x_2\left[\mu_1 F_{1,n+2}(\boldsymbol{x},t)+\mu_2 F_{1,n+1}(\boldsymbol{x},t)+\mu_3 F_{1,n}(\boldsymbol{x},t)\right], \qquad (2.5.11)$$

$$\left(\frac{\partial}{\partial x_2}+\frac{\partial}{\partial t}\right)(F_{0,n}(\boldsymbol{x},t)) = \lambda_{02}\lambda_{20}F_{0,n+1}(\boldsymbol{x},t) \; + \qquad (2.5.12) \\ x_1\left[\mu_1 F_{1,n+2}(\boldsymbol{x},t)+\mu_2 F_{1,n+1}(\boldsymbol{x},t)+\mu_3 F_{1,n}(\boldsymbol{x},t)\right].$$

Proof Proceeding as in Lemma 1, we find that,

$$\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial t}\right)(F_{0,n}(\boldsymbol{x},t)) = \lambda_{01}\lambda_{10}F_{0,n+1}(\boldsymbol{x},t) \; + \qquad (2.5.13) \\ H(x_0)x_2 \sum_{p=0}^{\infty}\underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{i+j+k\neq 0} \frac{\gamma(\boldsymbol{x})^p}{p!}\frac{x_0^{p+2i+j+n-1}}{(p+2i+j+n-1)!}\frac{(x_1x_2)^{i+j+k-1}}{(i+j+k-1)!}\frac{\mu_1^i\mu_2^j\mu_3^k}{i!j!k!}.$$

Let the second term on the right be denoted by $\sum_{p=0}^{\infty}\underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{i+j+k\neq 0} A_{p,i,j,k}$.

Clearly

$$\sum_{p=0}^{\infty}\underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{i+j+k\neq 0} A_{p,i,j,k}\frac{i+j+k}{i+j+k} \qquad (2.5.14)$$

$$= \sum_{p=0}^{\infty}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A_{p,i,j,k}\frac{i}{i+j+k} + \sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{j+k\neq 0} A_{m,i,j,k}\frac{j+k}{i+j+k} \qquad (2.5.15)$$

$$= \sum_{p=0}^{\infty}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A_{p,i,j,k}\frac{i}{i+j+k} + \sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} A_{p,i,j,k}\frac{j}{i+j+k} \\ + \sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} A_{p,i,j,k}\frac{k}{i+j+k}. \qquad (2.5.16)$$

Now the following is clearly true

$$\begin{aligned}
&\sum_{p=0}^{\infty}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A_{p,i,j,k}\frac{i}{i+j+k} \\
&= H(x_0)x_2\sum_{p=0}^{\infty}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\gamma(\boldsymbol{x})^p}{p!}\frac{x_0^{p+2i+j+n-1}}{(p+2i+j+n-1)!}(x_1x_2)^{i+j+k-1}\mu_1^i\mu_2^j\mu_3^k \\
&\times \left(\frac{1}{(i+j+k-1)!i!j!k!}-\frac{j+k}{(i+j+k)!i!j!k!}\right) \\
&= H(x_0)\sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\gamma(\boldsymbol{x})^p}{p!}\frac{x_0^{p+2i+j+n+1}}{(p+2i+j+n+1)!}(x_1x_2)^{i+j+k}\mu_1^{i+1}\mu_2^j\mu_3^k \\
&\times \left(\frac{1}{(i+j+k)!(i+1)!j!k!}-\frac{j+k}{(i+j+k+1)!(i+1)!j!k!}\right) \\
&= \mu_1x_2F_{1,n+2}(\boldsymbol{x},t).
\end{aligned}$$

Similar operations on the second two terms of (2.5.16) lead to the following identity:

$$\sum_{p=0}^{\infty}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A_{p,i,j,k} = \mu_1x_2F_{1,n+2}(\boldsymbol{x},t)+\mu_2x_2F_{1,n+1}(\boldsymbol{x},t)+\mu_3x_2F_{1,n}(\boldsymbol{x},t)$$

as required to prove (2.5.11).

□

We can now express the partial derivatives of α_i, for $i \in \boldsymbol{X}$, in terms of $F_{m,n}$ using the results given above. Substitution of the partial derivatives of $F_{m,n}$ verifies (2.3.2).

2.6 The sojourn time for the two–state process

The accumulated sojourn time for a two–state Markov process is, of course, a special case of the one we have been studying. If we let all transition rates to and from state 2 equal zero, we obtain the density, $\omega(x,t)$, of the total time, x, spent in state 1 of the two–state process in $[0,t)$, given it is initially in the zero state.

Let $\lambda_{02} = \lambda_{20} = \lambda_{12} = \lambda_{21} = 0$ in the expression for ρ in Theorem 2, then $\mu_1 = \mu_2 = \mu_3 = 0$ and $\gamma(\boldsymbol{x}) = \lambda_{01}\lambda_{10}x$. Hence

$$\begin{aligned}\omega(x,t) &= e^{-\lambda_{01}t}\delta(x) \\ &+ \lambda_{01}e^{-\lambda_{01}(t-x)-\lambda_{10}x}F_{0,1}(x,0,t) + \lambda_{01}\lambda_{10}e^{-\lambda_{01}(t-x)-\lambda_{10}x}F_{0,2}(x,0,t)\,.\end{aligned} \tag{2.6.17}$$

This expression, with the substitutions $\lambda_{ij} = \alpha_{ij}$ and

$$F_{0,n}(x,0,t) = \left(\sqrt{\frac{(t-x)}{(\lambda_{01}\lambda_{10}x)}}\right)^{n-1} I_{n-1}\left(2\sqrt{(\lambda_{01}\lambda_{10}x(t-x))}\right),$$

appears in Good [12] as the conditional probability density of the total sojourn time in a two–state Markov process up to time t, given that the process is in state 0 at $t = 0$.

Chapter 3

Refining the Solution

3.1 Introduction

In the last chapter we obtained a solution for the three-state Kolmogorov equations in terms of component functions $F_{m,n}$, which arose naturally when the Laplace-transformed solution was inverted. We were able to find partial derivatives of these component functions which, in turn, were a linear composition of the functions $F_{m,n}$.

In this chapter we study these functions in an attempt to remove some of the complexity and bring about a probabilistic interpretation. In the process, we obtain as lemmata a number of identities which are mathematically interesting and whose proof requires the use of some elegant combinatorial relations.

In Section 3.4 we study the general result obtained by Good in [12] and extract an explicit expression for the probability density of the sojourn time in the three-state Markov process. This matches our result.

In Section 3.5 we apply the refined solution to two special cases of the three-state problem which arise naturally in applications.

The final section gives a discussion on some numerical aspects of this problem. Once again we reformulate the solution, in two distinct ways, in order to obtain a computationally feasible expression.

3.2 Some Identities

In this section we adopt the convention that when the range of summation is not specified, it is to be understood that the summation variable takes on values over the full range for which the factorial terms are defined. This will save us having to change the limits of summation when the variable is changed by a constant shift.

In the sequel we shall refer to the relations

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k), \tag{3.2.1}$$

the *Vandermonde convolution* formula

$$\binom{n}{m} = \sum_{k} \binom{n-p}{m-k}\binom{p}{k} \tag{3.2.2}$$

and a particular case of (3.2.2), when p = 2,

$$\binom{n}{m} = \binom{n-2}{m} + 2\binom{n-2}{m-1} + \binom{n-2}{m-2}. \tag{3.2.3}$$

The last two identities can be found in [17], Section 1.2.

Let us now introduce some notation which will enable us to better manage unwieldy expressions. By definition

$$F_{m,n}(\boldsymbol{x},t) = H(x_0)\sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\gamma(\boldsymbol{x})^p x_0^{p+2i+j+n-1}(\mu_1 x_1 x_2)^i(\mu_2 x_1 x_2)^j(\mu_3 x_1 x_2)^k}{p!(p+2i+j+n-1)!(i+j+k+m)!i!j!k!}.$$

Substituting $\mu_1 = \lambda_{01}\lambda_{10}\lambda_{02}\lambda_{20}$, $\mu_2 = \lambda_{02}\lambda_{21}\lambda_{10} + \lambda_{01}\lambda_{12}\lambda_{20}$, $\mu_3 = \lambda_{12}\lambda_{21}$ and

$\gamma(\boldsymbol{x}) = \lambda_{01}\lambda_{10}x_1 + \lambda_{02}\lambda_{20}x_2$, and expanding the binomial factors $\gamma(\boldsymbol{x})$ and μ_2, gives

$$F_{m,n}(\boldsymbol{x},t) = H(x_0)\sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} B^{m,n}_{p,i,j,k}(\boldsymbol{x},t),$$

where $B^{m,n}_{p,i,j,k}(\boldsymbol{x},t)$ represents the expression

$$\sum_{r=0}^{p}\sum_{s=0}^{j}\frac{(\lambda_{10}\lambda_{01})^{r+i}(\lambda_{20}\lambda_{02})^{p-r+i}(\lambda_{01}\lambda_{12}\lambda_{20})^{s}(\lambda_{02}\lambda_{21}\lambda_{10})^{j-s}(\lambda_{12}\lambda_{21})^{k}}{(i+j+k+m)!i!k!(p-r)!r!(j-s)!s!} \times \frac{x_0^{p+2i+j+n-1}x_1^{r+i+j+k}x_2^{p-r+i+j+k}}{(p+2i+j+n-1)!}.$$

The numerator can be written more naturally as a product of powers of the factors

$$y_{10} = \lambda_{10}x_0;\quad y_{20} = \lambda_{20}x_0;\quad y_{01} = \lambda_{01}x_1;\quad y_{21} = \lambda_{21}x_1;\quad y_{12} = \lambda_{12}x_2;\quad y_{02} = \lambda_{02}x_2.$$

Substituting these factors into $B^{m,n}_{p,i,j,k}$ and rewriting the finite summations as infinite sums using (3.2.1) gives

$$B^{m,n}_{p,i,j,k}(\boldsymbol{x},t) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{x_0^{n-1}y_{10}^{r+i+j}y_{20}^{p+i+s}y_{01}^{r+i+s}y_{21}^{j+k}y_{12}^{s+k}y_{02}^{p+i+j}}{p!i!j!k!} \times\frac{1}{(p+2i+j+r+s+n-1)!(i+j+k+s+m)!r!s!}. \tag{3.2.4}$$

The substitutions

$$a = r+i+s;\quad b = j+k;\quad c = s+k;\quad d = p+i+j,$$

enable us to eliminate r and s from the numerator, leading to the new form

$$F_{m,n}(\boldsymbol{x},t) = H(x_0)\sum_a\sum_b\sum_c\sum_d C^{m,n}_{a,b,c,d}x_0^{n-1}y_{10}^{a+b-c}y_{20}^{d+c-b}y_{01}^{a}y_{21}^{b}y_{12}^{c}y_{02}^{d} \tag{3.2.5}$$

for $F_{m,n}$, where

$$C^{m,n}_{a,b,c,d} = \frac{1}{(a+d+n-1)!} \times \sum_r\sum_s\frac{1}{(a+b-r+m)!(a-s-r)!(b-c+s)!(c-s)!(d-a-b+c+r)!r!s!}. \tag{3.2.6}$$

Note that since the powers of the y_{ij}'s are non-negative in (3.2.4) we require that $a+b \geq c$ and $d+c \geq b$ in (3.2.5).

Lemma 4

$$F_{0,1}(\boldsymbol{x},t) = H(x_0)\sum_a\sum_b\sum_c\sum_d \frac{y_{10}^{a+b-c}y_{20}^{d+c-b}y_{01}^a y_{21}^b y_{12}^c y_{02}^d}{a!b!c!d!(a+b-c)!(d+c-b)!}.$$

Proof From (3.2.6)

$$\begin{aligned}
C^{0,1}_{a,b,c,d} &= \frac{1}{(a+d)!} \\
\times & \sum_r\sum_s \frac{1}{(a+b-r)!(a-s-r)!(b-c+s)!(c-s)!(d-a-b+c+r)!r!s!} \\
= & \frac{1}{(a+d)!b!}\sum_r \frac{1}{(a+b-r)!r!(d-a-b+c+r)!(a-r)!} \\
\times & \sum_s \frac{(a-r)!b!}{(a-r-s)!s!(b-c+s)!(c-s)!}.
\end{aligned}$$

We use relation (3.2.2) twice in order to eliminate r and s and hence two of the summations.

$$\begin{aligned}
C^{0,1}_{a,b,c,d} &= \\
& \frac{1}{(a+d)!b!}\sum_r \frac{1}{(a+b-r)!r!(d-a-b+c+r)!(a-r)!}\binom{a+b-r}{c} \\
&= \frac{1}{(a+d)!a!b!c!d!}\sum_r \frac{a!d!}{(a-r)!r!(d-a-b+c+r)!(a+b-c-r)!} \\
&= \frac{1}{(a+d)!a!b!c!d!}\binom{a+d}{a+b-c} \\
&= \frac{1}{a!b!c!d!(a+b-c)!(d-b+c)!},
\end{aligned}$$

as required to prove the lemma.

□

Lemma 5

$$\lambda_{01}\lambda_{20}x_1F_{1,2}(\boldsymbol{x},t)+\lambda_{21}x_1F_{1,1}(\boldsymbol{x},t) = H(x_0)\sum_a\sum_b\sum_c\sum_d \frac{y_{10}^{a+b-c-1}y_{20}^{d+c-b+1}y_{01}^a y_{21}^b y_{12}^c y_{02}^d}{a!b!c!d!(a+b-c-1)!(d+c-b+1)!}.$$

Proof From (3.2.5)

$$\lambda_{01}\lambda_{20}x_1F_{1,2}(\boldsymbol{x},t)=H(x_0)\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta C^{1,2}_{\alpha,\beta,\gamma,\delta}y_{10}^{\alpha+\beta-\gamma}y_{20}^{\delta+\gamma-\beta+1}y_{01}^{\alpha+1}y_{21}^\beta y_{12}^\gamma y_{02}^\delta$$

and

$$\lambda_{21}x_1F_{1,1}(\boldsymbol{x},t)=H(x_0)\sum_{\alpha'}\sum_{\beta'}\sum_{\gamma'}\sum_{\delta'} C^{1,1}_{\alpha',\beta',\gamma',\delta'}y_{10}^{\alpha'+\beta'-\gamma'}y_{20}^{\delta'+\gamma'-\beta'}y_{01}^{\alpha'}y_{21}^{\beta'+1}y_{12}^{\gamma'}y_{02}^{\delta'}.$$

In order to sum these two expressions we need to perform term by term matching of the summation variables. By equating the powers of the y_{ij} factors we arrive at the consistency relations

$$\alpha+1=\alpha';\quad \beta=\beta'+1;\quad \gamma=\gamma';\quad \delta=\delta'.$$

Substituting these values into the second summand, we obtain

$$\lambda_{01}\lambda_{20}x_1F_{1,2}(\boldsymbol{x},t)+\lambda_{21}x_1F_{1,1}(\boldsymbol{x},t) = H(x_0)\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta \left[C^{1,2}_{\alpha,\beta,\gamma,\delta}+C^{1,1}_{\alpha+1,\beta-1,\gamma,\delta}\right]y_{10}^{\alpha+\beta-\gamma}y_{20}^{\delta+\gamma-\beta+1}y_{01}^{\alpha+1}y_{21}^\beta y_{12}^\gamma y_{02}^\delta.$$

Using (3.2.2) once on $C^{1,2}_{\alpha,\beta,\gamma,\delta}$ and $C^{1,1}_{\alpha+1,\beta-1,\gamma,\delta}$ gives

$$\begin{aligned}C^{1,2}_{\alpha,\beta,\gamma,\delta} &= \frac{1}{(\alpha+\delta+1)!\beta!}\\ &\times\sum_r\sum_s\frac{1}{(\alpha+\beta-r+1)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha-r)!}\binom{\beta}{\gamma-s}\binom{\alpha-r}{s}\\ &=\frac{1}{(\alpha+\delta+1)!\beta!}\\ &\times\sum_r\frac{(\alpha+\beta-r)!}{(\alpha+\beta-r+1)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha-r)!(\alpha+\beta-r-\gamma)!\gamma!}\end{aligned}$$

and

$$C^{1,1}_{\alpha+1,\beta-1,\gamma,\delta} = \frac{1}{(\alpha+\delta+1)!(\beta-1)!}\sum_s \binom{\beta-1}{\gamma-s}$$
$$\times \sum_r \frac{1}{(\alpha+\beta-r+1)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha-r+1)!}\binom{\alpha-r+1}{s}$$
$$= \frac{1}{(\alpha+\delta+1)!(\beta-1)!}$$
$$\times \sum_r \frac{(\alpha+\beta-r)!}{(\alpha+\beta-r+1)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha-r+1)!(\alpha+\beta-r-\gamma)!\gamma!}.$$

Adding $C^{1,2}_{\alpha,\beta,\gamma,\delta}$ and $C^{1,1}_{\alpha+1,\beta-1,\gamma,\delta}$ and using (3.2.2) once more gives

$$\sum_r \frac{1}{(\alpha+\delta+1)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha+\beta-r-\gamma)!\gamma!}$$
$$\times \left[\frac{1}{\beta!(\alpha-r)!} + \frac{1}{(\beta-1)!(\alpha-r+1)!}\right]\frac{1}{(\alpha+\beta-r+1)}$$
$$= \sum_r \frac{1}{(\alpha+1-r)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha+\beta-\gamma-r)!(\alpha+\delta+1)!\beta!\gamma!}$$
$$= \sum_r \binom{\alpha+1}{r}\binom{\delta}{\alpha+\beta-\gamma-r}\frac{1}{(\alpha+\delta+1)!(\alpha+1)!\beta!\gamma!\delta!}$$
$$= \frac{1}{(\alpha+\beta-\gamma)!(\delta-\beta+\gamma+1)!(\alpha+1)!\beta!\gamma!\delta!}.$$

So

$$\lambda_{01}\lambda_{20}x_1F_{1,2}(\boldsymbol{x},t) + \lambda_{21}x_1F_{1,1}(\boldsymbol{x},t) =$$
$$H(x_0)\sum_a\sum_b\sum_c\sum_d \frac{y_{10}^{\alpha+\beta-\gamma}y_{20}^{\delta+\gamma-\beta+1}y_{01}^{\alpha+1}y_{21}^{\beta}y_{12}^{\gamma}y_{02}^{\delta}}{(\alpha+\beta-\gamma)!(\delta-\beta+\gamma+1)!(\alpha+1)!\beta!\gamma!\delta!},$$

and substituting $a = \alpha+1$, $b = \beta$, $c = \gamma$ and $d = \delta$ into the expression above completes the proof. □

By symmetry we have the following result.

Lemma 6

$$\lambda_{02}\lambda_{10}x_2F_{1,2}(\boldsymbol{x},t) + \lambda_{12}x_2F_{1,1}(\boldsymbol{x},t) =$$
$$H(x_0)\sum_a\sum_b\sum_c\sum_d \frac{y_{10}^{a+b-c+1}y_{20}^{d+c-b-1}y_{01}^{a}y_{21}^{b}y_{12}^{c}y_{02}^{d}}{a!b!c!d!(a+b-c+1)!(d+c-b-1)!}.$$

Lemma 7

$$\begin{aligned}\lambda_{01}\lambda_{10}\lambda_{20}x_1F_{1,3}(\boldsymbol{x},t)+\lambda_{20}F_{0,2}(\boldsymbol{x},t)+\lambda_{21}\lambda_{10}x_1F_{1,2}(\boldsymbol{x},t) &= \\ H(x_0)\sum_a\sum_b\sum_c\sum_d \frac{y_{10}^{a+b-c}y_{20}^{d+c-b+1}y_{01}^a y_{21}^b y_{12}^c y_{02}^d}{a!b!c!d!(a+b-c)!(d+c-b+1)!}.\end{aligned}$$

Proof By (3.2.5)

$$\lambda_{01}\lambda_{10}\lambda_{20}x_1F_{1,3}(\boldsymbol{x},t)=H(x_0)\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta C^{1,3}_{\alpha,\beta,\gamma,\delta}y_{10}^{\alpha+\beta-\gamma+1}y_{20}^{\delta+\gamma-\beta+1}y_{01}^{\alpha+1}y_{21}^{\beta}y_{12}^{\gamma}y_{02}^{\delta},$$

$$\lambda_{20}F_{0,2}(\boldsymbol{x},t)=H(x_0)\sum_{\alpha'}\sum_{\beta'}\sum_{\gamma'}\sum_{\delta'} C^{0,2}_{\alpha',\beta',\gamma',\delta'}y_{10}^{\alpha'+\beta'-\gamma'}y_{20}^{\delta'+\gamma'-\beta'+1}y_{01}^{\alpha'}y_{21}^{\beta'}y_{12}^{\gamma'}y_{02}^{\delta'}$$

and

$$\lambda_{21}\lambda_{10}x_1F_{1,2}(\boldsymbol{x},t)=H(x_0)\sum_{\alpha''}\sum_{\beta''}\sum_{\gamma''}\sum_{\delta''} C^{1,2}_{\alpha'',\beta'',\gamma'',\delta''}y_{10}^{\alpha''+\beta''-\gamma''+1}y_{20}^{\delta''+\gamma''-\beta''}y_{01}^{\alpha''}y_{21}^{\beta''+1}y_{12}^{\gamma''}y_{02}^{\delta''}.$$

Proceeding as in the proof of Lemma 5, we equate the powers of the y_{ij} factors, obtaining the relations

$$\alpha+1=\alpha'=\alpha''; \quad \beta=\beta'=\beta''+1; \quad \gamma=\gamma'=\gamma''; \quad \delta=\delta'=\delta''.$$

Substituting these values into the dummy summation variables in the second and third summands we get

$$\begin{aligned}\lambda_{01}\lambda_{10}\lambda_{20}x_1F_{1,3}(\boldsymbol{x},t)+\lambda_{20}F_{0,2}(\boldsymbol{x},t)+\lambda_{21}\lambda_{10}x_1F_{1,2}(\boldsymbol{x},t) &= \\ H(x_0)\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta \left[C^{1,3}_{\alpha,\beta,\gamma,\delta}+C^{0,2}_{\alpha+1,\beta,\gamma,\delta}+C^{1,2}_{\alpha+1,\beta-1,\gamma,\delta}\right] \\ \times\ y_{10}^{\alpha+\beta-\gamma+1}y_{20}^{\delta+\gamma-\beta+1}y_{01}^{\alpha+1}y_{21}^{\beta}y_{12}^{\gamma}y_{02}^{\delta},\end{aligned}$$

where

$$\begin{aligned}&\left[C^{1,3}_{\alpha,\beta,\gamma,\delta}+C^{0,2}_{\alpha+1,\beta,\gamma,\delta}+C^{1,2}_{\alpha+1,\beta-1,\gamma,\delta}\right] \\ &= \sum_r\sum_s \frac{\delta+1}{(\alpha+\delta+2)!(\delta-\alpha-\beta+\gamma+r)!(\alpha+\beta-r+1)!r!} \\ &\quad\times \frac{1}{(\alpha-r-s+1)!s!(\beta-\gamma+s)(\gamma-s)!}\end{aligned}$$

Rewrite the last factor as a product of binomial coefficients in order to make use of (3.2.2). This gives

$$\begin{aligned}
& \sum_r \frac{\delta+1}{(\alpha+\delta+2)!(\delta-\alpha-\beta+\gamma+r)!(\alpha+\beta-r+1)!r!(\alpha-r+1)!\beta!} \\
\times \quad & \sum_s \binom{\beta}{\gamma-s}\binom{\alpha-r+1}{s} \\
= \quad & \sum_r \frac{\delta+1}{(\alpha-r+1)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha+\beta-\gamma-r+1)!(\alpha+\delta+2)!\beta!\gamma!} \\
= \quad & \sum_r \binom{\alpha+1}{r}\binom{\delta+1}{\alpha+\beta-\gamma+1-r}\frac{1}{(\alpha+\delta+2)!(\alpha+1)!\beta!\gamma!\delta!} \\
= \quad & \frac{1}{(\alpha+\beta-\gamma+1)!(\delta-\beta+\gamma+1)!(\alpha+1)!\beta!\gamma!\delta!}.
\end{aligned}$$

Finally we get

$$\begin{aligned}
\lambda_{01}\lambda_{10}\lambda_{20}x_1F_{1,3}(\boldsymbol{x},t)+\lambda_{20}F_{0,2}(\boldsymbol{x},t)+\lambda_{21}\lambda_{10}x_1F_{1,2}(\boldsymbol{x},t) \quad = \\
H(x_0)\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta \frac{y_{10}^{\alpha+\beta-\gamma+1}y_{20}^{\delta+\gamma-\beta+1}y_{01}^{\alpha+1}y_{21}^{\beta}y_{12}^{\gamma}y_{02}^{\delta}}{(\alpha+\beta-\gamma+1)!(\delta-\beta+\gamma+1)!(\alpha+1)!\beta!\gamma!\delta!}.
\end{aligned}$$

Once again we call α, β, γ and δ, $a-1, b, c$ and d respectively, and the lemma is proved.

□

We invoke symmetry once more to prove the following result.

Lemma 8

$$\begin{aligned}
\lambda_{02}\lambda_{20}\lambda_{10}x_2F_{1,3}(\boldsymbol{x},t)+\lambda_{10}F_{0,2}(\boldsymbol{x},t)+\lambda_{12}\lambda_{20}x_2F_{1,2}(\boldsymbol{x},t) \quad = \\
H(x_0)\sum_a\sum_b\sum_c\sum_d \frac{y_{10}^{a+b-c+1}y_{20}^{d+c-b}y_{01}^{a}y_{21}^{b}y_{12}^{c}y_{02}^{d}}{a!b!c!d!(a+b-c+1)!(d+c-b)!}.
\end{aligned}$$

Lemma 9

$$\begin{aligned}\lambda_{01}\lambda_{20}(\lambda_{10})^2 x_1 F_{1,4}(\boldsymbol{x},t) + \lambda_{02}\lambda_{10}(\lambda_{20})^2 x_2 F_{1,4}(\boldsymbol{x},t) + 2\lambda_{10}\lambda_{20}F_{0,3}(\boldsymbol{x},t)\\ +\lambda_{12}(\lambda_{20})^2 x_2 F_{1,3}(\boldsymbol{x},t) + \lambda_{21}(\lambda_{10})^2 x_1 F_{1,3}(\boldsymbol{x},t) \quad = \\ H(x_0)\sum_a\sum_b\sum_c\sum_d \frac{y_{10}^{a+b-c+1} y_{20}^{d+c-b+1} y_{01}^a y_{21}^b y_{12}^c y_{02}^d}{a!b!c!d!(a+b-c+1)!(d+c-b+1)!}.\end{aligned}$$

Proof By (3.2.5)

$$\begin{aligned}\lambda_{01}\lambda_{20}(\lambda_{10})^2 x_1 F_{1,4}(\boldsymbol{x},t) &= H(x_0)\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta C^{1,4}_{\alpha,\beta,\gamma,\delta} y_{10}^{\alpha+\beta-\gamma+2} y_{20}^{\delta+\gamma-\beta+1} y_{01}^{\alpha+1} y_{21}^\beta y_{12}^\gamma y_{02}^\delta,\\ \lambda_{02}\lambda_{10}(\lambda_{20})^2 x_2 F_{1,4}(\boldsymbol{x},t) &= H(x_0)\sum_{\alpha'}\sum_{\beta'}\sum_{\gamma'}\sum_{\delta'} C^{1,4}_{\alpha',\beta',\gamma',\delta'}\\ &\quad\times\ y_{10}^{\alpha'+\beta'-\gamma'+1} y_{20}^{\delta'+\gamma'-\beta'+2} y_{01}^{\alpha'} y_{21}^{\beta'} y_{12}^{\gamma'} y_{02}^{\delta'+1},\\ \lambda_{12}(\lambda_{20})^2 x_2 F_{1,3}(\boldsymbol{x},t) &= H(x_0)\sum_{\dot\alpha}\sum_{\dot\beta}\sum_{\dot\gamma}\sum_{\dot\delta} C^{1,3}_{\dot\alpha,\dot\beta,\dot\gamma,\dot\delta} y_{10}^{\dot\alpha+\dot\beta-\dot\gamma} y_{20}^{\dot\delta+\dot\gamma-\dot\beta+2} y_{01}^{\dot\alpha} y_{21}^{\dot\beta} y_{12}^{\dot\gamma+1} y_{02}^{\dot\delta},\\ \lambda_{21}(\lambda_{10})^2 x_1 F_{1,3}(\boldsymbol{x},t) &= H(x_0)\sum_{\ddot\alpha}\sum_{\ddot\beta}\sum_{\ddot\gamma}\sum_{\ddot\delta} C^{1,3}_{\ddot\alpha,\ddot\beta,\ddot\gamma,\ddot\delta} y_{10}^{\ddot\alpha+\ddot\beta-\ddot\gamma+2} y_{20}^{\ddot\delta+\ddot\gamma-\ddot\beta} y_{01}^{\ddot\alpha} y_{21}^{\ddot\beta+1} y_{12}^{\ddot\gamma} y_{02}^{\ddot\delta}\end{aligned}$$

and

$$\begin{aligned}2\lambda_{10}\lambda_{20}F_{0,3}(\boldsymbol{x},t) &= 2H(x_0)\sum_{\alpha''}\sum_{\beta''}\sum_{\gamma''}\sum_{\delta''} C^{0,3}_{\alpha'',\beta'',\gamma'',\delta''}\\ &\quad\times\ y_{10}^{\alpha''+\beta''-\gamma''+1} y_{20}^{\delta''+\gamma''-\beta''+1} y_{01}^{\alpha''} y_{21}^{\beta''} y_{12}^{\gamma''} y_{02}^{\delta''}.\end{aligned}$$

Once again we equate the powers of the y_{ij} factors, obtaining the relations

$$\alpha+1=\alpha'=\alpha''=\dot\alpha=\ddot\alpha;\quad \beta=\beta'=\beta''=\dot\beta=\ddot\beta+1;$$

$$\gamma=\gamma'=\gamma''=\dot\gamma+1=\ddot\gamma;\quad \delta=\delta'+1=\delta''=\dot\delta=\ddot\delta.$$

Upon substitution we get

$$\begin{aligned}&\lambda_{01}\lambda_{20}(\lambda_{10})^2 x_1 F_{1,4}(\boldsymbol{x},t) + \lambda_{02}\lambda_{10}(\lambda_{20})^2 x_2 F_{1,4}(\boldsymbol{x},t) + 2\lambda_{10}\lambda_{20}F_{0,3}(\boldsymbol{x},t)\\ &\quad +\ \lambda_{12}(\lambda_{20})^2 x_2 F_{1,3}(\boldsymbol{x},t) + \lambda_{21}(\lambda_{10})^2 x_1 F_{1,3}(\boldsymbol{x},t) \quad =\\ &H(x_0)\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta y_{10}^{\alpha+\beta-\gamma+2} y_{20}^{\delta+\gamma-\beta+1} y_{01}^{\alpha+1} y_{21}^\beta y_{12}^\gamma y_{02}^\delta\\ &\quad\times\ \left[C^{1,4}_{\alpha,\beta,\gamma,\delta} + C^{1,4}_{\alpha+1,\beta,\gamma,\delta-1} + 2C^{0,3}_{\alpha+1,\beta,\gamma,\delta} + C^{1,3}_{\alpha+1,\beta,\gamma-1,\delta} + C^{1,3}_{\alpha+1,\beta-1,\gamma,\delta}\right].\end{aligned} \quad (3.2.7)$$

By the definition of $C^{m,n}_{a,b,c,d}$ in (3.2.6) and using (3.2.2) we obtain

$$\begin{aligned}
C^{1,4}_{\alpha,\beta,\gamma,\delta} &= \frac{1}{(\alpha+\delta+3)!\beta!} \\
&\times \sum_r \sum_s \frac{1}{(\alpha+\beta-r+1)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha-r)!} \binom{\beta}{\gamma-s} \binom{\alpha-r}{s} \\
&= \frac{1}{(\alpha+\delta+3)!\beta!\gamma!} \\
&\times \sum_r \frac{(\alpha+\beta-r)!}{(\alpha+\beta-r+1)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha-r)!(\alpha+\beta-r-\gamma)!}.
\end{aligned}$$

Now we repeat this procedure on each of the remaining terms in the last factor of (3.2.7).

$$\begin{aligned}
C^{1,4}_{\alpha+1,\beta,\gamma,\delta-1} &= \frac{1}{(\alpha+\delta+3)!\beta!\gamma!} \\
&\times \sum_r \frac{(\alpha+\beta-r+1)!}{(\alpha+\beta-r+2)!r!(\delta-\alpha-\beta+\gamma+r-2)!(\alpha-r+1)!(\alpha+\beta-r+1-\gamma)!},
\end{aligned}$$

$$\begin{aligned}
C^{1,3}_{\alpha+1,\beta,\gamma-1,\delta} &= \frac{1}{(\alpha+\delta+3)!\beta!(\gamma-1)!} \\
&\times \sum_r \frac{(\alpha+\beta-r+1)!}{(\alpha+\beta-r+2)!r!(\delta-\alpha-\beta+\gamma+r-2)!(\alpha-r+1)!(\alpha+\beta-r+2-\gamma)!},
\end{aligned}$$

$$\begin{aligned}
C^{1,3}_{\alpha+1,\beta-1,\gamma,\delta} &= \frac{1}{(\alpha+\delta+3)!(\beta-1)!\gamma!} \\
&\times \sum_r \frac{(\alpha+\beta-r)!}{(\alpha+\beta-r+1)!r!(\delta-\alpha-\beta+\gamma+r)!(\alpha-r+1)!(\alpha+\beta-r-\gamma)!}
\end{aligned}$$

and

$$2C^{0,3}_{\alpha+1,\beta,\gamma,\delta} = 2\frac{1}{(\alpha+\delta+3)!\beta!\gamma!}$$
$$\times\sum_r \frac{1}{r!(\alpha-r+1)!(\delta-\alpha-\beta+\gamma+r-1)!(\alpha+\beta-r+1-\gamma)!}$$
$$= 2\sum_r \binom{\alpha+1}{r}\binom{\delta}{\alpha+\beta-\gamma+1-r}\frac{1}{(\alpha+1)!\beta!\gamma!\delta!(\alpha+\delta+3)!}.$$

Choosing the most suitable terms for pairwise addition and using (3.2.2) on each resulting sum we obtain

$$C^{1,4}_{\alpha,\beta,\gamma,\delta} + C^{1,3}_{\alpha+1,\beta-1,\gamma,\delta} = \frac{1}{(\alpha+\delta+3)!\beta!\gamma!}$$
$$\times\sum_r \frac{1}{r!(\alpha-r+1)!(\delta-\alpha-\beta+\gamma+r)!(\alpha+\beta-r-\gamma)!}$$
$$= \sum_r \binom{\alpha+1}{r}\binom{\delta}{\alpha+\beta-\gamma-r}\frac{1}{(\alpha+1)!\beta!\gamma!\delta!(\alpha+\delta+3)!}$$

and

$$C^{1,4}_{\alpha+1,\beta,\gamma,\delta-1} + C^{1,3}_{\alpha+1,\beta,\gamma-1,\delta} = \frac{1}{(\alpha+\delta+3)!\beta!\gamma!}$$
$$\times\sum_r \frac{1}{r!(\alpha-r+1)!(\delta-\alpha-\beta+\gamma+r-2)!(\alpha+\beta-r-\gamma+2)!}$$
$$= \sum_r \binom{\alpha+1}{r}\binom{\delta}{\alpha+\beta-\gamma+2-r}\frac{1}{(\alpha+1)!\beta!\gamma!\delta!(\alpha+\delta+3)!}.$$

Finally, adding the three resulting terms and using the relations (3.2.2) and (3.2.3) gives

$$\begin{aligned}
&\left[\left(C^{1,4}_{\alpha,\beta,\gamma,\delta}+C^{1,3}_{\alpha+1,\beta-1,\gamma,\delta}\right)+\left(C^{1,4}_{\alpha+1,\beta,\gamma,\delta-1}+C^{1,3}_{\alpha+1,\beta,\gamma-1,\delta}\right)+2C^{0,3}_{\alpha+1,\beta,\gamma,\delta}\right] \\
&= \sum_r \binom{\alpha+1}{r}\frac{1}{(\alpha+1)!\beta!\gamma!\delta!(\alpha+\delta+3)!} \\
&\quad \times\left[\binom{\delta}{\alpha+\beta-\gamma-r+2}+2\binom{\delta}{\alpha+\beta-\gamma-r+1}+\binom{\delta}{\alpha+\beta-\gamma-r}\right] \\
&= \sum_r \binom{\alpha+1}{r}\binom{\delta+2}{\alpha+\beta-\gamma+2-r}\frac{1}{(\alpha+1)!\beta!\gamma!\delta!(\alpha+\delta+3)!} \\
&= \frac{1}{(\alpha+1)!\beta!\gamma!\delta!(\alpha+\beta-\gamma+2)!(\delta-\beta+\gamma+1)!}.
\end{aligned}$$

The final substitution, $a-1, b, c$ and d for α, β, γ and δ respectively, gives the expression required to prove the lemma. □

3.3 A new expression for $\rho(x,t)$

In this section we derive a new and significantly simpler expression for $\rho(\boldsymbol{x},t)$, of Theorem 2 in Chapter 2. To this end we observe that the expressions on the right-hand side of the identities derived in the last section can be referred to by the functions $L_{m,n}(\boldsymbol{x},t)$, $\quad m,n\in\{-1,0,1\}$, given by

$$L_{m,n}(\boldsymbol{x},t)=H(x_0)\sum_a\sum_b\sum_c\sum_d\frac{y_{10}^{a+b-c+m}y_{20}^{d+c-b+n}y_{01}^a y_{21}^b y_{12}^c y_{02}^d}{a!b!c!d!(a+b-c+m)!(d+c-b+n)!}.$$

Recall that

$$\rho(\boldsymbol{x},t) \;=\; e^{-\epsilon(\boldsymbol{x},t)}[\alpha_0(\boldsymbol{x},t)+\alpha_1(\boldsymbol{x},t)+\alpha_2(\boldsymbol{x},t)],$$

where $\epsilon(\boldsymbol{x},t)=-\sum_k\lambda_{kk}x_k$, and the α_is are defined in Chapter 2.

Using the results from Lemmata 7, 8 and 9, we can express α_0 in terms of the functions $L_{m,n}$ as

$$\begin{aligned}
\alpha_0(\boldsymbol{x},t) \;=\;& H(\boldsymbol{x})\left\{\delta(\boldsymbol{x})+\lambda_{01}\lambda_{10}\delta(x_2)F_{0,2}(x_1,0,t)+\lambda_{02}\lambda_{20}\delta(x_1)F_{0,2}(0,x_2,t)\right. \\
&+\left.\lambda_{01}\lambda_{02}L_{1,1}(\boldsymbol{x},t)+\lambda_{01}\lambda_{12}L_{0,1}(\boldsymbol{x},t)+\lambda_{02}\lambda_{21}L_{1,0}(\boldsymbol{x},t)\right\}.
\end{aligned}$$

Similarly, we make use of Lemmata 7, 5, 4 and 8, 4, 7 to obtain the following expressions for α_1 and α_2, respectively.

$$\begin{aligned}\alpha_1(\boldsymbol{x},t) &= H(\boldsymbol{x})\left\{\lambda_{01}\delta(x_2)F_{0,1}(x_1,0,t)\right.\\ &\quad + \left.\lambda_{01}\lambda_{02}L_{0,1}(\boldsymbol{x},t) + \lambda_{01}\lambda_{12}L_{-1,1}(\boldsymbol{x},t) + \lambda_{02}\lambda_{21}L_{0,0}(\boldsymbol{x},t)\right\}.\end{aligned}$$

$$\begin{aligned}\alpha_2(\boldsymbol{x},t) &= H(\boldsymbol{x})\left\{\lambda_{02}\delta(x_1)F_{0,1}(0,x_2,t)\right.\\ &\quad + \left.\lambda_{01}\lambda_{02}L_{1,0}(\boldsymbol{x},t) + \lambda_{01}\lambda_{12}L_{0,0}(\boldsymbol{x},t) + \lambda_{02}\lambda_{21}L_{1,-1}(\boldsymbol{x},t)\right\}.\end{aligned}$$

We combine these three expressions to get

$$\begin{aligned}\rho(\boldsymbol{x},t) &= e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\left\{\delta(\boldsymbol{x}) + \lambda_{01}\delta(x_2)[\lambda_{10}F_{0,2}(x_1,0,t) + F_{0,1}(x_1,0,t)]\right.\\ &\quad + \lambda_{02}\delta(x_1)[\lambda_{20}F_{0,2}(0,x_2,t) + F_{0,1}(0,x_2,t)]\\ &\quad + \lambda_{01}\lambda_{02}\left[L_{1,1}(\boldsymbol{x},t) + L_{0,1}(\boldsymbol{x},t) + L_{1,0}(\boldsymbol{x},t)\right]\\ &\quad + \lambda_{01}\lambda_{12}\left[L_{0,1}(\boldsymbol{x},t) + L_{-1,1}(\boldsymbol{x},t) + L_{0,0}(\boldsymbol{x},t)\right]\\ &\quad + \left.\lambda_{02}\lambda_{21}\left[L_{1,0}(\boldsymbol{x},t) + L_{0,0}(\boldsymbol{x},t) + L_{1,-1}(\boldsymbol{x},t)\right]\right\}.\end{aligned} \tag{3.3.8}$$

3.4 Comparison with Good's result

In this section we look at formula (10) in Good [12] "which provides, in principle, the joint density function" of the time spent in k states of a continuous–time Markov process. We take the liberty of changing the notation in Good's paper to match ours in order to make comparison possible. Hence we let $\tau_r = x_r$ for $r \in \{0..k\}$ and $\alpha_{r,s} = \lambda_{r,s}$ for $r, s \in \{0..k\}$, such that $r \neq s$, since Good defines $\alpha_{r,r} = 0$, while our definition follows the accepted convention $\lambda_{rr} = -\sum_{s\neq r}\lambda_{rs}$. This should not present a problem since, in our solution, λ_{rr} only appears in $\epsilon(\boldsymbol{x},t)$, which we shall rewrite as

$$\epsilon(\boldsymbol{x},t) = -\sum_{r}\sum_{s\neq r}\lambda_{rs}x_r. \tag{3.4.9}$$

Good first derives a probability generating function for the joint distribution of the frequencies of the k possible letters in a chain, N letters long, generated by a discrete–time Markov process with k states. The letters represent unit length sojourns in each state. He then uses a generalisation to several variables of Lagrange's expansion of an implicit function as a power series to obtain a "pseudo" generating function. By making the transition from discrete to continuous–time, he then obtains from the pseudo P.G.F. an implicit solution to the k-state continuous–time problem.

We shall investigate Good's claim that the probability density, when $\Pi_{r=0}^{k} x_r \neq 0$, is equal to the constant term in

$$\exp\left(-\sum_r \sum_{s\neq r} \lambda_{r,s} x_r\right) \frac{\sum z_r}{\Pi z_r} \sum p_r D_r \exp\left(\sum_r \sum_{s\neq r} \lambda_{s,r} x_r z_s / z_r\right), \qquad (3.4.10)$$

where p_i is the probability that the Markov process is initially in state i and D_i is the cofactor of the ith diagonal element of the matrix

$$\left(\delta_r^s \sum_{u\neq r} \lambda_{ur} z_u - \lambda_{rs} z_s\right) = (a_{r,s})\,.$$

Good gives an explicit expression for the probability density of x_0 and x_1 in the case $k = 2$ and compares it to a previously known result. He points out that his paper is of interest for its methods and the relationship between discrete and continuous time. Formula (3.4.10) is in fact a potentially powerful result if we can extract an explicit form for $k > 2$ which can be reconciled with our solution. To this end we now explore the case $k = 3$.

If the process starts in state 0, $p_0 = 1$ and $p_i = 0$ for $i = 1, 2$. Hence we only need to calculate D_0. For $k = 3$, $(a_{r,s})$ is the matrix

$$\begin{pmatrix} \lambda_{10} z_1 + \lambda_{20} z_2 & -\lambda_{01} z_1 & -\lambda_{02} z_2 \\ -\lambda_{10} z_0 & \lambda_{01} z_0 + \lambda_{21} z_2 & -\lambda_{12} z_2 \\ -\lambda_{20} z_0 & -\lambda_{21} z_1 & \lambda_{02} z_0 + \lambda_{12} z_1 \end{pmatrix},$$

and hence

$$D_0 = \lambda_{01}\lambda_{02}z_0^2 + \lambda_{01}\lambda_{12}z_0z_1 + \lambda_{02}\lambda_{21}z_0z_2.$$

In this case, putting $\epsilon(\boldsymbol{x},t) = \exp\left(-\sum_r \sum_{s\neq r} \lambda_{r,s}x_r\right)$ as defined above, Formula (3.4.10) becomes

$$e^{\epsilon(\boldsymbol{x},t)}\frac{z_0+z_1+z_2}{z_0z_1z_2}D_0 \exp\left(\sum_r^3\sum_{s\neq r}^3 \lambda_{s,r}x_rz_s/z_r\right). \tag{3.4.11}$$

Let $u = z_1/z_0$ and $v = z_2/z_0$. To find the constant term in (3.4.11) we need to calculate the coefficient of u^0v^0 in

$$\begin{aligned} &e^{\epsilon(\boldsymbol{x},t)}\exp\left(\lambda_{10}x_0u + \lambda_{01}x_1u^{-1} + \lambda_{20}x_0v + \lambda_{02}x_2v^{-1} + \lambda_{21}x_1u^{-1}v + \lambda_{12}x_2uv^{-1}\right)\\ &\quad\times\left[\lambda_{01}\lambda_{02}u^{-1}v^{-1} + \lambda_{01}\lambda_{12}v^{-1} + \lambda_{02}\lambda_{21}u^{-1} + \lambda_{01}\lambda_{02}v^{-1} + \lambda_{01}\lambda_{12}uv^{-1}\right.\\ &\qquad\left. + \lambda_{02}\lambda_{21} + \lambda_{01}\lambda_{02}u^{-1} + \lambda_{01}\lambda_{12} + \lambda_{02}\lambda_{21}u^{-1}v\right]. \end{aligned} \tag{3.4.12}$$

We now expand the second exponential term above and write it in terms of the factors y_{ij} defined in Section 3.2 as

$$\begin{aligned} &\exp\left(\lambda_{10}x_0u + \lambda_{01}x_1u^{-1} + \lambda_{20}x_0v + \lambda_{02}x_2v^{-1} + \lambda_{21}x_1u^{-1}v + \lambda_{12}x_2uv^{-1}\right)\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(y_{10}u)^i(y_{20}v)^j(y_{01}u^{-1})^k(y_{21}u^{-1}v)^l(y_{12}uv^{-1})^p(y_{02}v^{-1})^q}{i!j!k!l!p!q!}\\ &= \sum_m\sum_n\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{y_{10}^{m+k+l-p}y_{20}^{n+q-l+p}y_{01}^ky_{21}^ly_{12}^py_{02}^qu^mv^n}{(m+k+l-p)!(n+q-l+p)!k!l!p!q!}\\ &= \sum_m\sum_n L_{m,n}(\boldsymbol{x},t)u^mv^n, \end{aligned} \tag{3.4.13}$$

where $L_{m,n}$ is as defined in Section 3.3. Substituting (3.4.13) into (3.4.12) provides

$$\begin{aligned} &e^{\epsilon(\boldsymbol{x},t)}\sum_m\sum_n L_{m,n}(\boldsymbol{x},t)u^mv^n\\ &\quad\times\left[\lambda_{01}\lambda_{02}u^{-1}v^{-1} + \lambda_{01}\lambda_{12}v^{-1} + \lambda_{02}\lambda_{21}u^{-1} + \lambda_{01}\lambda_{02}v^{-1} + \lambda_{01}\lambda_{12}uv^{-1}\right.\\ &\qquad\left. + \lambda_{02}\lambda_{21} + \lambda_{01}\lambda_{02}u^{-1} + \lambda_{01}\lambda_{12} + \lambda_{02}\lambda_{21}u^{-1}v\right], \end{aligned}$$

which enables us to write an expression for the coefficient of u^0v^0, that is, the probability density of the three–state sojourn time. We obtain

$$\begin{aligned} e^{\epsilon(\boldsymbol{x},t)} & [\lambda_{01}\lambda_{02}\left(L_{1,1}(\boldsymbol{x},t) + L_{0,1}(\boldsymbol{x},t) + L_{1,0}(\boldsymbol{x},t)\right) \\ & + (\lambda_{01}\lambda_{12}\left(L_{0,1}(\boldsymbol{x},t) + L_{-1,1}(\boldsymbol{x},t) + L_{0,0}(\boldsymbol{x},t)\right) \\ & + (\lambda_{02}\lambda_{21}\left(L_{1,0}(\boldsymbol{x},t) + L_{0,0}(\boldsymbol{x},t) + L_{1,-1}(\boldsymbol{x},t)\right)] . \end{aligned} \quad (3.4.14)$$

This is identical to our expression for $\rho(\boldsymbol{x},t)$ in (3.3.8) when the "boundary" terms (when either x_1 or x_2 or both are equal to zero) are subtracted. We compared the solutions arising from each technique for the case $k = 2$ in the previous chapter.

3.5 Special cases of the three–state problem

In this section we apply the solution of the general three–state problem, (3.3.8), to two special cases in which only particular transitions are allowed. We chose these two examples as they often arise in applications.

In the birth–and–death process, transitions are allowed between neighbouring states only. This process is often used to model systems that gradually degrade or improve with time. For example, if the three states in the process represent three possible transmission rates on a high–frequency radio link, poor, fair and good, then we would expect the quality of the link to go from poor to fair then to good and vice–versa rather from poor straight to good.

As its name suggests, the cyclic transition process only allows transitions in a cyclic fashion, that is, from state 0 to 1 to 2 and back to 0. This model is used in certain reliability problems in which the system starts off with all components in working order and they fail one by one until all are not functioning, at which time they are all repaired and the system returns to state 0.

In this section and the next we conveniently make use of the function

$$G_n(y) = \sum_{k=0}^{\infty} \frac{y^k}{(n+k)!k!}.$$

Note that when $x_1 = 0$ and hence $y_{01}, y_{21} = 0$, (3.2.5) becomes

$$\begin{aligned} F_{m,n}(0, x_2, t) &= H(x_0) \sum_d \frac{x_0^{n-1}(y_{20}y_{02})^d}{m!d!(d+n-1)!} \\ &= H(x_0)\frac{x_o^{n-1}}{m!}G_{n-1}(y_{20}y_{02}). \end{aligned}$$

Similarly when $x_2 = 0$,

$$F_{m,n}(x_1, 0, t) = H(x_0)\frac{x_o^{n-1}}{m!}G_{n-1}(y_{10}y_{01}).$$

3.5.1 The birth–and–death process

Since only transitions to neighbouring states are allowed in this case, we let $\lambda_{02} = 0$ and $\lambda_{20} = 0$, and hence $y_{20} = y_{02} = 0$. Substituting for these values, we obtain these expressions for ϵ and $L_{m,n}$.

$$\epsilon^{BD}(\boldsymbol{x}, t) = -\lambda_{01}x_0 - (\lambda_{10} + \lambda_{12})x_1 - \lambda_{21}x_2,$$

and

$$L_{m,n}^{BD}(\boldsymbol{x}, t) = H(x_0)\sum_a \sum_b \frac{y_{10}^{a+n+m}y_{01}^a y_{21}^b y_{12}^{b-n}}{a!b!(a+n+m)!(b-n)!}.$$

We put $b - n = k$ in the latter to get

$$\begin{aligned} L_{m,n}^{BD}(\boldsymbol{x}, t) &= H(x_0)y_{10}^m(y_{10}y_{21})^n \sum_k \frac{(y_{21}y_{12})^k}{k!(k+n)!} \sum_a \frac{(y_{10}y_{01})^a}{a!(a+n+m)!} \\ &= H(x_0)y_{10}^m(y_{10}y_{21})^n G_n(y_{21}y_{12})G_{n+m}(y_{10}y_{01}). \end{aligned}$$

Writing ρ^{BD} in terms of the above functions,

$$\begin{aligned} \rho^{BD}(\boldsymbol{x}, t) &= e^{-\epsilon^{BD}(\boldsymbol{x},t)}H(x_0)H(\boldsymbol{x})\left\{\delta(\boldsymbol{x}) + \lambda_{01}\delta(x_2)[y_{10}G_1(y_{10}y_{01}) + G_0(y_{10}y_{01})]\right. \\ &+ \left.\lambda_{01}\lambda_{12}\left[y_{10}y_{21}G_1(y_{21}y_{12})G_1(y_{10}y_{01}) + y_{21}G_1(y_{21}y_{12})G_0(y_{10}y_{01}) + G_0(y_{21}y_{12})G_0(y_{10}y_{01})\right]\right\} \end{aligned}$$

3.5.2 The cyclic transition process

In this process transitions cycle in the order (0,1,2). We obtain an expression for the density by letting $\lambda_{10}, \lambda_{21}$ and λ_{02} and hence y_{10}, y_{21} and y_{02} all equal zero. We first evaluate

$$L^{CT}_{m,n}(\boldsymbol{x},t) = H(x_0)\sum_a \frac{y_{20}^{a+n+m} y_{01}^a y_{12}^{a+m}}{a!(a+n+m)!(a+m)!}$$

and

$$\epsilon^{CT}(\boldsymbol{x},t) = -\lambda_{01}x_0 - \lambda_{12}x_1 - \lambda_{20}x_2.$$

Defining ρ^{CT} in terms of ϵ^{CT} and $L^{CT}_{m,n}$,

$$\rho^{CT}(\boldsymbol{x},t) = e^{-\epsilon^{CT}(\boldsymbol{x},t)}H(x_0)H(\boldsymbol{x})\left\{\delta(\boldsymbol{x}) + \lambda_{01}\delta(x_2)\right.$$
$$\left. + \quad \lambda_{01}\lambda_{12}\left[y_{20}\sum_a \frac{(y_{20}y_{01}y_{12})^a}{a!a!(a+1)!} + y_{20}y_{01}\sum_a \frac{(y_{20}y_{01}y_{12})}{(a+1)!(a+1)!a!} + \sum_a \frac{(y_{20}y_{01}y_{12})}{a!a!a!}\right]\right\}.$$

3.6 Numerics for the three–state problem

An important aspect of the solution of any mathematical problem is an investigation into its numerical behaviour. In this section we do not attempt to carry out a thorough numerical analysis which would be beyond the scope of this thesis. We are merely concerned with whether the expressions derived for the solution of the general three–state problem are in a feasible form for numerical computation.

We first coded ρ as a linear combination of the functions $F_{m,n}$, as defined in Theorem 2 in Chapter 2. The complexity of these functions needed careful consideration. Initially we attempted to code them as a sum of Bessel functions using a recurrence relation satisfied by the sum. However this proved to be unstable. Next we tried to compute the $F_{m,n}$ as sums of products of Bessel functions, as detailed below, using the in–built Bessel functions in Maple version V. Not only did this solve the instability problem but it converged to an approximation extremely quickly.

We recall the function defined in the last section,

$$G_n(y) = \sum_{k=0}^{\infty} \frac{y^k}{(n+k)!k!},$$

and note that, for $n \in \boldsymbol{N}$, $G_n(y) = y^{-n/2} I_n(2\sqrt{y})$.

We can express $F_{m,n}$ as

$$\begin{aligned}
&F_{m,n}(\boldsymbol{x}, t) \\
&= H(x_0) \sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\gamma(\boldsymbol{x})^p x_0^{p+2i+j+n-1} (\mu_1 x_1 x_2)^i (\mu_2 x_1 x_2)^j (\mu_3 x_1 x_2)^k}{p!(p+2i+j+n-1)!(i+j+k+m)!i!j!k!} \\
&= H(x_0) x_0^{n-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\mu_1 x_0^2 x_1 x_2)^i (\mu_2 x_0 x_1 x_2)^j}{i!j!} \\
&\times \quad \sum_{k=0}^{\infty} \frac{(\mu_3 x_1 x_2)^k}{k!(i+j+m+k)!} \sum_{p=0}^{\infty} \frac{(\gamma(\boldsymbol{x}) x_0)^p}{p!(p+2i+j+n-1)!} \\
&= H(x_0) x_0^{n-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\mu_1 x_0^2 x_1 x_2)^i (\mu_2 x_0 x_1 x_2)^j}{i!j!} G_{i+j+m}(\mu_3 x_1 x_2) G_{2i+j+n-1}(\gamma(\boldsymbol{x}) x_0).
\end{aligned}$$

We implemented this expression by approximating the infinite summations by suitable finite sums. The stopping condition used was that the difference between two consecutive partial sums must be less than the chosen accuracy. In most cases fewer than ten terms of each sum were evaluated before the stopping condition was met. The same results (with 15 decimal digits precision) were obtained when the summations were interchanged. We experimented with higher precision, up to 50 decimal digits, and a constant upper limit on the summation range in place of the stopping condition. We found, for example, that for a precision of 50 decimal digits, the stopping criterion was met after 20 terms and produced an identical result for a constant upper limit of 500.

In order to increase confidence in this floating point approximation we also coded ρ as a linear combination of the functions $L_{m,n}$, as in Equation (3.3.8). Once again we used the Maple Bessel functions, obtaining for $L_{m,n}$ the form

$$
\begin{aligned}
&L_{m,n}(\boldsymbol{x}, t) = \\
&\quad H(x_0) \sum_a \sum_b \sum_c \sum_d \frac{y_{10}^{a+b-c+m} y_{20}^{d+c-b+n} y_{01}^a y_{21}^b y_{12}^c y_{02}^d}{a!b!c!d!(a+b-c+m)!(d+c-b+n)!} \\
&= \quad H(x_0) \sum_b \sum_c \frac{y_{10}^{b-c+m} y_{20}^{c-b+n} y_{21}^b y_{12}^c}{b!c!} \sum_a \sum_d \frac{(y_{10}y_{01})^a (y_{20}y_{02})^d}{a!d!(a+b-c+m)!(d+c-b+n)!} \\
&= \quad H(x_0) \sum_b \sum_c \frac{y_{10}^{b-c+m} y_{20}^{c-b+n} y_{21}^b y_{12}^c}{b!c!} G_{m+b-c}(y_{10}y_{01}) G_{n-b+c}(y_{20}y_{02}).
\end{aligned}
$$

Summing was stopped under the same condition as described above. Once again we obtained the same results for ρ, up to a precision of 15 decimal digits. We do not claim that this proves convergence, merely that the solutions obtained, expressed as detailed above, are computationally feasible, though not necessarily the most computationally efficient.

This numerical investigation was instrumental in the construction of the Lemmata of Section 3.2. We were able to use probabilistic intuition to find the identities and decide whether they were satisfied numerically before embarking on a conclusive analytic proof.

Chapter 4

Extending the State Space

4.1 Introduction

In this chapter we use probabilistic interpretation of the three–state solution to deduce the solution for the case $n = 4$. We then repeat the procedure detailed in Section 3.4, this time extending the number of states from three to four, and compare the results. The boundary terms for the four-state case are deduced from the full solutions obtained directly from the partial differential equations in Chapter 2. We extend the probabilistic argument to derive the general form for the case with n states. In the process we delve briefly into graph theory and the matrix-tree theorem [14] and present some interesting connections between the two. Good has applied a generalisation of Lagrange's expansion to the enumeration of trees (see [13]) as well as the probability distribution of the frequency count of a Markov chain in the previously cited [12], so that the foundations for making the connection are well established. We make the observation that the tree-generating determinant for rooted trees, directed away from the root, (see [21]), can be used to deduce the structure for the interior part of the solution for the general case with n states.

4.2 A probabilistic interpretation of $\rho(x,t)$

A close inspection of the explicit solutions obtained thus far for $n = 2$ and $n = 3$ gives an insight into a possible probabilistic structure of the solution for any n.

It will be convenient to express the functions $F_{m,n}(\boldsymbol{x}, t)$, in which one of the x_i's equals 0, in terms of the functions

$$L_m(x_i, t) = H(x_0) \sum_a \frac{y_{i0}^{a+m} y_{0i}^a}{(a+m)!a!},$$

for $i \in \{1, \cdots, n\}$, which have a similar form to $L_{m,n}$. So we have

$$\begin{aligned} F_{0,m+1}(x_1, 0, t) &= \lambda_{10}^{-m} L_m(x_1, t), \\ F_{0,m+1}(0, x_2, t) &= \lambda_{20}^{-m} L_m(x_2, t). \end{aligned}$$

Denote by $\rho^{(n)}(\boldsymbol{x}, t)$ the probability density of the total sojourn time in the case of the n-state Markov process, where $\boldsymbol{x} = (x_1, x_2, \cdots, x_{n-1})$, that is, $\rho^{(n)}$ has n arguments. From (2.6.17), using the substitution defined above, the two-state solution is

$$\rho^{(2)}(\boldsymbol{x}, t) = e^{\epsilon(\boldsymbol{x},t)} H(\boldsymbol{x})[\delta(x_1) + \lambda_{01} L_1(x_1, t) + \lambda_{01} L_0(x_1, t)].$$

Let us express $\rho^{(2)}(\boldsymbol{x}, t)$ in terms of the component densities $\rho_0^{(2)}, \rho_1^{(2)}$. Recall that $\rho = \sum_{i=0}^{n-1} \rho_i$, where ρ_i accounts for the probability density when the final state of the process is i. We assume that the initial state is always 0.

$$\rho_0^{(2)}(\boldsymbol{x},t) = e^{\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})[\delta(x_1) + \lambda_{01}L_1\,(x_1,t)], \tag{4.2.1}$$

$$\rho_1^{(2)}(\boldsymbol{x},t) = e^{\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\lambda_{01}L_0\,(x_1,t)\,. \tag{4.2.2}$$

The first term in (4.2.1), the boundary solution, arises from the cases in which the process stops before it ever leaves state 0, while the second term, $\lambda_{01}L_1$, represents those events for which there are an equal number of transitions from 0 to 1 as from 1 to 0, but with one fewer sojourn in state 1. Similarly $\lambda_{01}L_0$ arises from those events in which there are an equal number of full sojourns in each state and an additional transition into state 1, the final state.

Let us study the three-state solution as derived in Section 3.3. We write $\rho^{(3)}$ in terms of the component densities $\rho_0^{(3)}, \rho_1^{(3)}$ and $\rho_2^{(3)}$ and replace the functions $F_{m,n}$ by the L_m's defined above:

$$\begin{aligned}
\rho_0^{(3)}(\boldsymbol{x},t) &= e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\,\{\delta(\boldsymbol{x}) + \lambda_{01}\delta(x_2)L_1(x_1,t) + \lambda_{02}\delta(x_1)L_1(x_2,t) \\
&\quad + \lambda_{01}\lambda_{02}L_{1,1}(\boldsymbol{x},t) + \lambda_{01}\lambda_{12}L_{0,1}(\boldsymbol{x},t) + \lambda_{02}\lambda_{21}L_{1,0}(\boldsymbol{x},t)\}\,, \\
\rho_1^{(3)}(\boldsymbol{x},t) &= e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\,\{\lambda_{01}\delta(x_2)L_0(x_1,t) \\
&\quad + \lambda_{01}\lambda_{02}L_{0,1}(\boldsymbol{x},t) + \lambda_{01}\lambda_{12}L_{-1,1}(\boldsymbol{x},t) + \lambda_{02}\lambda_{21}L_{0,0}(\boldsymbol{x},t)\}\,, \\
\rho_2^{(3)}(\boldsymbol{x},t) &= e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\,\{\lambda_{02}\delta(x_1)L_0(x_2,t) \\
&\quad + \lambda_{01}\lambda_{02}L_{1,0}(\boldsymbol{x},t) + \lambda_{01}\lambda_{12}L_{0,0}(\boldsymbol{x},t) + \lambda_{02}\lambda_{21}L_{1,-1}(\boldsymbol{x},t)\}\,.
\end{aligned}$$

We first look at the boundary terms. If we gather all the terms with support in the (x_0, x_1) plane, including the $\delta(\boldsymbol{x})$ term, we obtain

$$e^{-\epsilon(\boldsymbol{x},t)}\left\{\delta(\boldsymbol{x}) + \lambda_{01}\delta(x_2)L_1(x_1,t) + \lambda_{01}\delta(x_2)L_0(x_1,t)\right\}.$$

This is just $\delta(x_2)\rho^{(2)}(\bar{\boldsymbol{x}}_{\{2\}},t)$, where $\bar{\boldsymbol{x}}_{\{i\}}$ is the vector $\boldsymbol{x}$ with the ith component removed, that is, $\boldsymbol{x} \in \boldsymbol{R}^2_+$ and $\bar{\boldsymbol{x}}_{\{2\}} = x_1$ and hence $\rho^{(2)}$ has two arguments as required. In addition, the $\rho_0^{(2)}$ terms can be matched to the corresponding $\rho_0^{(3)}$ terms and the same holds for the $\rho_1^{(2)}$ terms. A symmetric argument applies for the terms with values in the (x_0, x_2) plane,

$$e^{-\epsilon(\boldsymbol{x},t)}\left\{\delta(\boldsymbol{x}) + \lambda_{02}\delta(x_1)L_1(x_2,t) + \lambda_{02}\delta(x_1)L_0(x_2,t)\right\} = \delta(x_1)\rho^{(2)}(\bar{\boldsymbol{x}}_{\{1\}},t),$$

with all instances of x_1 replaced by x_2 and all subscripts of λ_{ij} which equal 1 incremented by unity.

Let us attempt to clarify this for the general case. When the argument of $\rho^{(n)}$ is $\bar{\boldsymbol{x}}_{\{i\}}$ and $\boldsymbol{x} \in \boldsymbol{R}^n_+$, all the x_j's with $j > i$ are shifted along by one place and the matrix of transition rates Λ is replaced by $\bar{\Lambda}_{\{i\}}$, an $(n \times n)$ matrix obtained from $\Lambda = (\lambda)_{i,j},\ i,j \in \{0,1,\cdots,n\}$, by removing the $(i+1)$st row and column:

$$\bar{\Lambda}_{\{i\}} = \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} & \cdots & \lambda_{0,i-1} & \lambda_{0,i+1} & \cdots & \lambda_{0,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \lambda_{i-1,0} & \lambda_{i-1,1} & \cdots & \lambda_{i-1,i-1} & \lambda_{i-1,i+1} & \cdots & \lambda_{i-1,n} \\ \lambda_{i+1,0} & \lambda_{i+1,1} & \cdots & \lambda_{i+1,i-1} & \lambda_{i+1,i+1} & \cdots & \lambda_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \lambda_{n,0} & \lambda_{n,1} & \cdots & \lambda_{n,i-1} & \lambda_{n,i+1} & \cdots & \lambda_{n,n} \end{pmatrix}.$$

In the process those subscripts of y_{jk} which are greater than i are incremented by unity.

The value of $\rho^{(3)}$ on the boundary is

$$\delta(x_2)\rho^{(2)}(\bar{\boldsymbol{x}}_{\{2\}},t)+\delta(x_1)\rho^{(2)}(\bar{\boldsymbol{x}}_{\{1\}})-\delta(\boldsymbol{x}). \tag{4.2.3}$$

Consider the "interior" terms of $\rho^{(3)}$. These represent the probability mass of those events in which each of the three states is visited at least once. Recall the definition of $L_{m,n}$:

$$L_{m,n}(\boldsymbol{x},t)=H(x_0)\sum_a\sum_b\sum_c\sum_d\frac{y_{10}^{a+b-c+m}y_{20}^{d+c-b+n}y_{01}^a y_{21}^b y_{12}^c y_{02}^d}{a!b!c!d!(a+b-c+m)!(d+c-b+n)!}.$$

In order to get a better idea of the transition paths these functions describe, we rewrite $L_{m,n}$ as

$$L_{m,n}(\boldsymbol{x},t)=y_{10}^m y_{20}^n H(x_0)\sum_a\sum_b\sum_c\sum_d\frac{(y_{01}y_{10})^a(y_{02}y_{20})^d\left(\frac{y_{12}y_{20}}{y_{10}}\right)^c\left(\frac{y_{21}y_{10}}{y_{20}}\right)^b}{a!b!c!d!(a+b-c+m)!(d+c-b+n)!}.$$

We can intuitively infer that each of the factors in the numerator of the summand represents a two-step transition event. The first two factors correspond to those paths starting at state 0, then visiting one of the other two states before returning to state 0. The third factor corresponds to those transitions from state 1 to state

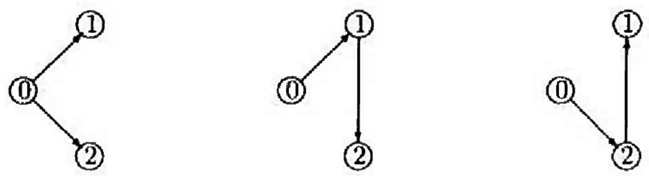

Figure 4.2.1: Spanning trees of a 3–node digraph rooted from node 0

0 with an intermediate visit to state 2. The denominator in the third factor, y_{10}, ensures that these events replace an equal number of direct transitions from 1 to 0. The fourth factor can be similarly interpreted. In other words, $L_{0,0}$ counts all possible transition cycles with state zero as the initial and final state.

The subscripts of L can be thought of as m additional transitions from 1 to 0 and n of them from 2 to 0, as well as $m+n$ full sojourns in state 0.

Now associate with each λ_{kj} a directed line in a three–node spanning tree. So the coefficients of the $L_{m,n}$'s in each of the densities $\rho_i^{(3)}$ correspond to the sequence of edges which make up each directed tree rooted from node 0. There are three such trees as can be seen in Figure 4.2.1. The coefficients then determine the subscripts of L necessary to ensure that the final state is i. For example, in the case of $\rho_0^{(3)}$, we have the first interior term, $\lambda_{01}\lambda_{02}L_{1,1}$, whose coefficient represents the first tree in Figure 4.2.1. In order to make 0 the final state we need additional transitions from 1 to 0 and from 2 to 0, hence we need $m = 1$ and $n = 1$ for this to occur. Similarly, we require that $m = 0, n = 1$, that is a transition from 2 to 0, for the second term and $m = 1, n = 0$, a transition from 1 to 0, for the third term.

To determine the terms in $\rho_1^{(3)}$ and $\rho_2^{(3)}$, we simply substitute m by $m-1$ for the former and n by $n-1$ for the latter, thus eliminating the final transition to state 0.

This connection between spanning trees and the probability densities can be traced to Good's formula (3.4.10) in Section 3.4 and the matrix

$$(a_{r,s}) = \left(\delta_r^s \sum_{u\neq r} \lambda_{ur} z_u - \lambda_{rs} z_s\right).$$

If we replace each z_j, $j \in \{0, 1, 2\}$, by unity in $(a)_{r,s}$, we obtain the matrix $(-\Lambda)$,

$$\left(\delta_r^s \sum_{u\neq r} \lambda_{ur} - \lambda_{rs}\right)_{3\times 3}.$$

The determinant of $(-\Lambda)$ is actually the tree-generating determinant for the spanning trees of a digraph rooted from a given node. A corollary of the "matrix–tree theorem" (see [14], Section 3.3 and [9], Section 11) enumerates the number of out–directed spanning arborescences of a three–node graph rooted at node i by calculating the cofactor, $\mathrm{cof}_{ii}(-\Lambda)$, of the ith diagonal element of $(-\Lambda)$, with each $\lambda_{r,s}$ replaced by the number of edges directed from node r to node s. Temperley [21] goes further and explains in detail why the expansion of cof_{ii} determines the particular lines that make up each of these trees.

Good does not explicitly mention tree enumeration in [12], however he arrived at the solution by using a generalisation to several variables of Lagrange's expansion of an implicit formula as a power series. Later, in [13], he applies Lagrange's expansion to the enumeration of trees. In fact he had already noted that this could be done in [11].

4.3 The four–state solution and beyond

In this section we use the observations made in the last section to extrapolate from the three–state to the four–state problem. For this case $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\bar{\boldsymbol{x}}_{\{i\}}$ is $\boldsymbol{x}$ with the ith component removed, that is $\bar{\boldsymbol{x}}_{\{i\}} \in \boldsymbol{R}_+^2$. Again, it is necessary to consider the solution on the boundary separately.

4.3.1 The boundary solution

Using the same reasoning as in the lead-up to Equation (4.2.3), we arrive at this expression for the boundary terms of $\rho^{(4)}$:

$$\begin{aligned}&\delta(x_3)\rho^{(3)}(\bar{\boldsymbol{x}}_{\{3\}},t)+\delta(x_2)\rho^{(3)}(\bar{\boldsymbol{x}}_{\{2\}},t)+\delta(x_1)\rho^{(3)}(\bar{\boldsymbol{x}}_{\{1\}},t)\\&\quad-2\delta(\boldsymbol{x})-\sum_{i,j,k\in\Omega}\lambda_{0k}\delta(x_i)\delta(x_j)[L_1(x_k,t)+L_0(x_k,t)],\end{aligned}\qquad(4.3.4)$$

where $\Omega=\{i,j,k\in\{1,2,3\}:i\neq k, i<j\}$.

We substitute the three–state solution into the above expression, thus obtaining an explicit solution for the boundary part of $\rho^{(4)}$. Using the probabilistic interpretation of the functions L_m and $L_{m,n}$, with the same argument as in the three–state case we can partition this expression into the boundary terms for the component densities, $\rho_i^{(4)}$, thus arriving at an explicit expression for the boundary part of $\rho^{(4)}$. Once again we start with those cases for which 0 is the final state and at least one of the other three states is never visited:

$$\begin{aligned}&e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\left\{\delta(\boldsymbol{x})+\lambda_{01}\delta(x_2)\delta(x_3)L_1(x_1,t)+\lambda_{02}\delta(x_1)\delta(x_3)L_1(x_2,t)\right.\\&\quad+\lambda_{03}\delta(x_1)\delta(x_2)L_1(x_3,t)+\lambda_{01}\lambda_{02}\delta(x_3)L_{1,1}(\bar{\boldsymbol{x}}_{\{3\}},t)+\lambda_{01}\lambda_{12}\delta(x_3)L_{0,1}(\bar{\boldsymbol{x}}_{\{3\}},t)\\&\quad+\lambda_{02}\lambda_{21}\delta(x_3)L_{1,0}(\bar{\boldsymbol{x}}_{\{3\}},t)+\lambda_{01}\lambda_{03}\delta(x_2)L_{1,1}(\bar{\boldsymbol{x}}_{\{2\}},t)+\lambda_{01}\lambda_{13}\delta(x_2)L_{0,1}(\bar{\boldsymbol{x}}_{\{2\}},t)\\&\quad+\lambda_{03}\lambda_{31}\delta(x_2)L_{1,0}(\bar{\boldsymbol{x}}_{\{2\}},t)+\lambda_{02}\lambda_{03}\delta(x_1)L_{1,1}(\bar{\boldsymbol{x}}_{\{1\}},t)+\lambda_{02}\lambda_{23}\delta(x_1)L_{0,1}(\bar{\boldsymbol{x}}_{\{1\}},t)\\&\quad\left.+\lambda_{03}\lambda_{32}\delta(x_1)L_{1,0}(\bar{\boldsymbol{x}}_{\{1\}},t)\right\}.\end{aligned}$$

Consider the boundary part of $\rho_1^{(4)}$. Since 1 is the final state it must be visited at least once, so we take all those terms above which do not have a $\delta(x_1)$ factor and adjust the subscripts of $L_{m,n}$ accordingly, thus obtaining

$$\begin{aligned} e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\,\{&\lambda_{01}\delta(x_2)\delta(x_3)L_0(x_1,t) \\ &+ \lambda_{01}\lambda_{02}\delta(x_3)L_{0,1}(\bar{\boldsymbol{x}}_{\{3\}},t) + \lambda_{01}\lambda_{12}\delta(x_3)L_{-1,1}(\bar{\boldsymbol{x}}_{\{3\}},t) + \lambda_{02}\lambda_{21}\delta(x_3)L_{0,0}(\bar{\boldsymbol{x}}_{\{3\}},t) \\ &+ \lambda_{01}\lambda_{03}\delta(x_2)L_{0,1}(\bar{\boldsymbol{x}}_{\{2\}},t) + \lambda_{01}\lambda_{13}\delta(x_2)L_{-1,1}(\bar{\boldsymbol{x}}_{\{2\}},t) + \lambda_{03}\lambda_{31}\delta(x_2)L_{0,0}(\bar{\boldsymbol{x}}_{\{2\}},t)\Big\}. \end{aligned}$$

Similarly for $\rho_2^{(4)}$

$$\begin{aligned} e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\,\{&\lambda_{02}\delta(x_1)\delta(x_3)L_0(x_2,t) \\ &+ \lambda_{01}\lambda_{02}\delta(x_3)L_{1,0}(\bar{\boldsymbol{x}}_{\{3\}},t) + \lambda_{01}\lambda_{12}\delta(x_3)L_{0,0}(\bar{\boldsymbol{x}}_{\{3\}},t) + \lambda_{02}\lambda_{21}\delta(x_3)L_{1,-1}(\bar{\boldsymbol{x}}_{\{3\}},t) \\ &+ \lambda_{02}\lambda_{03}\delta(x_1)L_{0,1}(\bar{\boldsymbol{x}}_{\{1\}},t) + \lambda_{02}\lambda_{23}\delta(x_1)L_{-1,1}(\bar{\boldsymbol{x}}_{\{1\}},t) + \lambda_{03}\lambda_{32}\delta(x_1)L_{0,0}(\bar{\boldsymbol{x}}_{\{1\}},t)\Big\}. \end{aligned}$$

In addition, we now have a density component, $\rho_3^{(4)}$, for the cases when the final sojourn is in state 3,

$$\begin{aligned} e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\,\{&\lambda_{03}\delta(x_1)\delta(x_2)L_0(x_3,t) \\ &+ \lambda_{01}\lambda_{03}\delta(x_2)L_{1,0}(\bar{\boldsymbol{x}}_{\{2\}},t) + \lambda_{01}\lambda_{13}\delta(x_2)L_{0,0}(\bar{\boldsymbol{x}}_{\{2\}},t) + \lambda_{03}\lambda_{31}\delta(x_2)L_{1,-1}(\bar{\boldsymbol{x}}_{\{2\}},t) \\ &+ \lambda_{02}\lambda_{03}\delta(x_1)L_{1,0}(\bar{\boldsymbol{x}}_{\{1\}},t) + \lambda_{02}\lambda_{23}\delta(x_1)L_{0,0}(\bar{\boldsymbol{x}}_{\{1\}},t) + \lambda_{03}\lambda_{32}\delta(x_1)L_{1,-1}(\bar{\boldsymbol{x}}_{\{1\}},t)\Big\}. \end{aligned}$$

4.3.2 The arborescence connection

The interior part of the solution requires that we define a function $L_{m,n,o}$, the equivalent four–state version of $L_{m,n}$. This time there are twelve possible y_{ij}'s and nine summations. It is therefore convenient to introduce a notation loosely based on multipartite numbers (see [13]) and the multi–index notation used in the partial differential equations literature.

A *multipartite number* of order n, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, is an n–component vector with suffixes $1, 2, \cdots, n$ or $0, 1, \cdots, n-1$, such that each $\alpha_i \in \boldsymbol{R}_+$ or each $\alpha_i \in \boldsymbol{Z}$, that is, we do not restrict the components to the non-negative integers. Furthermore, we denote by $\boldsymbol{\alpha} = (\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \cdots, \boldsymbol{\alpha_n})$, a *multipartite vector* of order nm, that is, each $\boldsymbol{\alpha_i} = (\alpha_{1i}, \alpha_{2i}, \cdots, \alpha_{mi})$ is a multipartite number of order m. We define $\{\alpha_j\}$ to be that subset of $\boldsymbol{\alpha}$ such that $\{\alpha_j\} = (\alpha_{j1}, \alpha_{j2}, \cdots, \alpha_{jn})$.

The multipartite number, $|\boldsymbol{\alpha}|$, denotes $(|\boldsymbol{\alpha_1}|, |\boldsymbol{\alpha_2}|, \cdots, |\boldsymbol{\alpha_n}|)$, with each $|\boldsymbol{\alpha_i}|$ defined as the scalar $\alpha_{1i} + \alpha_{2i} + \cdots + \alpha_{ni}$; for $\alpha_{ij} \in \boldsymbol{Z}_+$, let $\boldsymbol{\alpha}!$ denote $\boldsymbol{\alpha_1}!\boldsymbol{\alpha_2}! \cdots \boldsymbol{\alpha_n}!$, in which each $\boldsymbol{\alpha_i}! = \alpha_{1i}!\alpha_{2i}! \cdots \alpha_{ni}!$. If α, β and γ are multipartite numbers of order n, and the components of α and β are integers then γ^β is defined as $\gamma_1^{\beta_1}\gamma_2^{\beta_2} \cdots \gamma_n^{\beta_n}$, and $\sum_\beta$ denotes $\sum_{\beta_1}\sum_{\beta_2} \cdots \sum_{\beta_n}$. The sum of two multipartite numbers is again a multipartite number defined as $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n)$. Hence $\gamma^{\beta+\alpha} = \gamma_1^{\beta_1+\alpha_1}\gamma_2^{\beta_2+\alpha_2} \cdots \gamma_n^{\beta_n+\alpha_n}$ and $(\alpha + \beta)! = (\alpha_1 + \beta_1)!(\alpha_2 + \beta_2)! \cdots (\alpha_n + \beta_n)!$.

Let us define a so–called "non–diagonal" multipartite vector, $\boldsymbol{b}$, of order $(nm - \min(n, m))$, which is derived from an nm multipartite vector, $\boldsymbol{\beta}$, by deleting all the "diagonal" components β_{ii}. We can thus classify the variables y_{ij}, defined in the last chapter, as a non–diagonal multipartite vector of order $n(n-1)$, such that $\boldsymbol{y} = (\boldsymbol{y_0}, \boldsymbol{y_1}, \cdots, \boldsymbol{y_{n-1}})$ and each $\boldsymbol{y_i} = (y_{0i}, y_{1i}, \cdots y_{i-1i}, y_{i+1i}, \cdots, y_{n-1i})$, with $y_{ij} \in \boldsymbol{R}_+$, for each $i, j \in \{0, \cdots, n-1\}$ such that $i \neq j$.

We shall use the three–state example as a demonstration of this notation. Let $\kappa = (\kappa_1, \kappa_2)$ be a multipartite number of order 2, $\boldsymbol{b} = (\boldsymbol{b_1}, \boldsymbol{b_2}) = (b_{01}, b_{21}, b_{02}, b_{12})$ be a non–diagonal multipartite vector of order 4, such that each $\kappa_i \in \boldsymbol{Z}$ and each b_{ij} is a non-negative integer, and let $\bar{\boldsymbol{b}} = (\{b_1\}, \{b_2\}) = (b_{12}, b_{21})$. Of course $\boldsymbol{y} = (\boldsymbol{y_0}, \boldsymbol{y_1}, \boldsymbol{y_2})$ is of order 6. Then we can rewrite $L_{m,n}$ as

$$\begin{aligned}
L_{\kappa}^{(3)}(\boldsymbol{x}, t) &= H(x_0) \sum_{b_{01}} \sum_{b_{21}} \sum_{b_{02}} \sum_{b_{12}} \frac{y_{10}^{\kappa_1 + b_{01} + b_{21} - b_{12}} y_{20}^{\kappa_2 + b_{02} + b_{12} - b_{21}}}{(\kappa_1 + b_{01} + b_{21} - b_{12})!(\kappa_2 + b_{02} + b_{12} - b_{21})!} \\
&\times \frac{y_{01}^{b_{01}} y_{21}^{b_{21}} y_{02}^{b_{02}} y_{12}^{b_{12}}}{b_{01}! b_{21}! b_{02}! b_{12}!} \\
&= H(x_0) \sum_{\boldsymbol{b_1}} \sum_{\boldsymbol{b_2}} \frac{y_{10}^{\kappa_1 + |\boldsymbol{b_1}| - \{b_1\}} y_{20}^{\kappa_2 + |\boldsymbol{b_2}| - \{b_2\}} \boldsymbol{y_1}^{\boldsymbol{b_1}} \boldsymbol{y_2}^{\boldsymbol{b_2}}}{(\kappa_1 + |\boldsymbol{b_1}| - \{b_1\})!(\kappa_2 + |\boldsymbol{b_2}| - \{b_2\})! \boldsymbol{b_1}! \boldsymbol{b_2}!} \\
&= H(x_0) \sum_{\boldsymbol{b}} \frac{\boldsymbol{y_0}^{\kappa + |\boldsymbol{b}| - \bar{\boldsymbol{b}}} \boldsymbol{y_1}^{\boldsymbol{b_1}} \boldsymbol{y_2}^{\boldsymbol{b_2}}}{(\kappa + |\boldsymbol{b}| - \bar{\boldsymbol{b}})! \boldsymbol{b}!}.
\end{aligned}$$

We return to the four-state version of L. Let $\kappa = (\kappa_1, \kappa_2, \kappa_3)$, $\kappa_i \in \boldsymbol{Z}$ be a multipartite number of order 3, $\boldsymbol{b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3})$ be a non-diagonal multipartite vector of order 9, with each $b_{ij} \in \boldsymbol{Z}_+$, such that $0 \le i, j \le 3$, $i \ne j$ and $\bar{\boldsymbol{b}} = (\{b_1\}, \{b_2\}, \{b_3\}) = (b_{12}, b_{13}, b_{21}, b_{23}, b_{31}, b_{32})$. For this case $\boldsymbol{y}$ is a non-diagonal multipartite vector of order 12, and is defined as $(\boldsymbol{y_0}, \boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{y_3})$, with each $y_{ij} = \lambda_{ij} x_j$. Then we define $L_{\kappa}^{(4)}$ as

$$L_{\kappa}^{(4)}(\boldsymbol{x}, t) = H(x_0) \sum_{\boldsymbol{b}} \frac{\boldsymbol{y_0}^{\kappa + |\boldsymbol{b}| - |\bar{\boldsymbol{b}}|} \boldsymbol{y_1}^{\boldsymbol{b_1}} \boldsymbol{y_2}^{\boldsymbol{b_2}} \boldsymbol{y_3}^{\boldsymbol{b_3}}}{(\kappa + |\boldsymbol{b}| - |\bar{\boldsymbol{b}}|)! \boldsymbol{b}!}.$$

Applying the same probabilistic reasoning as in the last section we can write an expression for the interior solution of $\rho_0^{(4)}$. We must first list all four-node spanning trees rooted from node 0. As remarked in the last section, we could use the cofactors of the diagonal elements of the matrix $(-\Lambda)$ to do this, however, in this case, it is easier to simply write them down. All sixteen trees appear in Figure 4.3.2.

Each of these three-edged trees is represented by a product of three distinct λ_{ij}'s, which is a coefficient of some $L_{\kappa}^{(4)}$. Recall that $\kappa = (\kappa_1, \kappa_2, \kappa_3)$ is determined

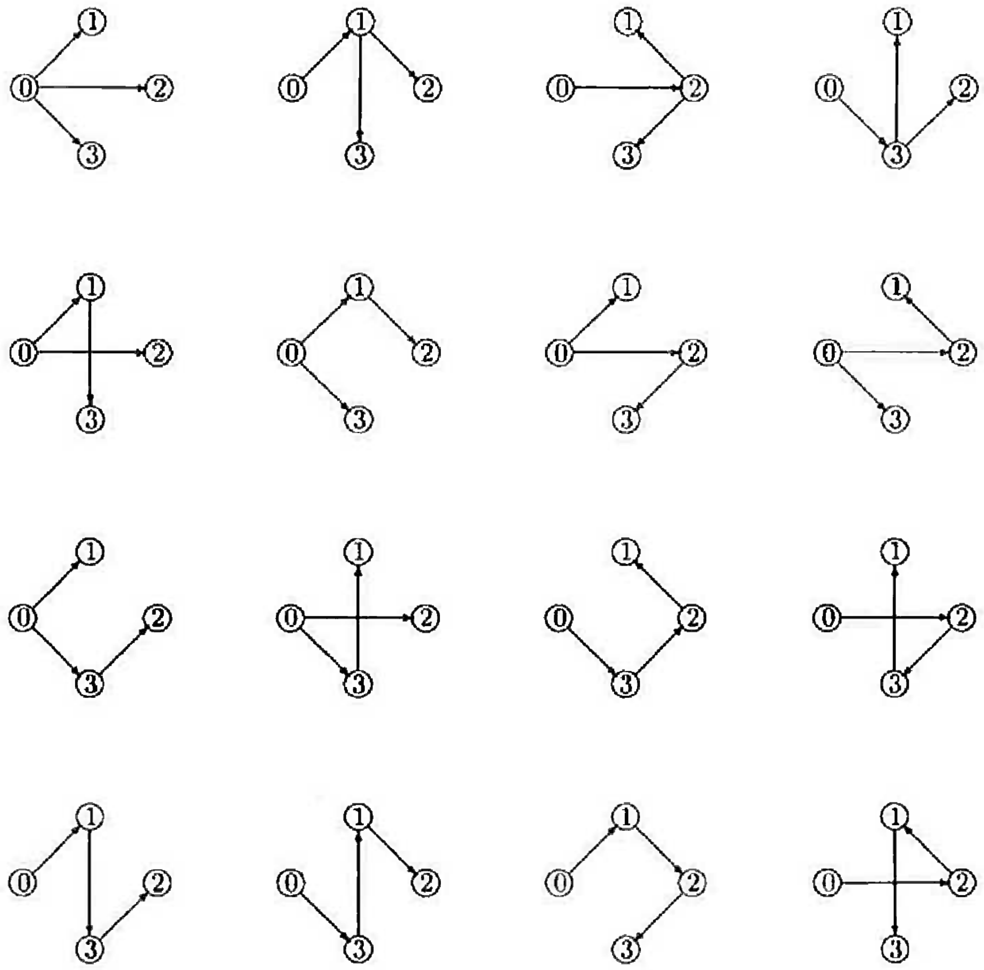

Figure 4.3.2: Spanning trees of a 4–node digraph rooted from node 0

by the coefficient of L_κ and that each κ_i corresponds to a transition into or from state i. Let τ denote the spanning tree represented by the coefficient of $L_{\kappa(\tau)}$, that is, κ is determined by τ. Since we require 0 to be the final state for $\rho_0^{(4)}$, we must set the support of a particular $\kappa(\tau)$ so that it "neutralises" the spanning tree, τ. To achieve this we increment each κ_i by one for each transition into state i (that is, each λ_{ji}, representing the edges from all nodes j to node i in τ) and decrement it by one for each transition out of state i (that is, each λ_{ik} in the coefficient representing τ). Thus, for a node, i, in τ,

$$\kappa_i(\tau) = indegree(\tau, i) - outdegree(\tau, i), \tag{4.3.5}$$

where $indegree(\tau, i)$ is the number of edges directed into node i in $\tau$and $outdegree(\tau, i)$ is the number of edges directed out of node i. However, since each node is only visited once, $indegree(\tau, i) = 1$, for all i and all τ, so $\kappa_i(\tau) = 1 - outdegree(\tau, i)$.

For example, consider the first tree in the second row of Figure 4.3.2, call it τ. This is represented by the sequence of edges, 01, 02 and 13, that is, by the coefficient $\lambda_{01}\lambda_{02}\lambda_{13}$. In the sequel we will, for convenience, refer to such a coefficient by the spanning tree it represents, that is, when we talk about τ we really mean $\lambda_{01}\lambda_{02}\lambda_{13}$. In order to make 0 the final state of the process, we need a further two transitions, one from 2 to 0 and one from 3 to 0, that is, $\kappa(\tau) = (1-1, 1, 1) = (0, 1, 1)$.

Using this technique we obtain as the interior part of $\rho_0^{(4)}$,

$$\begin{aligned}
&e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\,\{\lambda_{01}\lambda_{02}\lambda_{03}L_{1,1,1}(\boldsymbol{x},t) + \lambda_{01}\lambda_{12}\lambda_{13}L_{-1,1,1}(\boldsymbol{x},t)\\
&\quad + \lambda_{02}\lambda_{21}\lambda_{23}L_{1,-1,1}(\boldsymbol{x},t) + \lambda_{03}\lambda_{31}\lambda_{32}L_{1,1,-1}(\boldsymbol{x},t) + \lambda_{02}\lambda_{01}\lambda_{13}L_{0,1,1}(\boldsymbol{x},t)\\
&\quad + \lambda_{03}\lambda_{01}\lambda_{12}L_{0,1,1}(\boldsymbol{x},t) + \lambda_{01}\lambda_{02}\lambda_{23}L_{1,0,1}(\boldsymbol{x},t) + \lambda_{03}\lambda_{02}\lambda_{21}L_{1,0,1}(\boldsymbol{x},t)\\
&\quad + \lambda_{01}\lambda_{03}\lambda_{32}L_{1,1,0}(\boldsymbol{x},t) + \lambda_{02}\lambda_{03}\lambda_{31}L_{1,1,0}(\boldsymbol{x},t) + \lambda_{03}\lambda_{32}\lambda_{21}L_{1,0,0}(\boldsymbol{x},t)\\
&\quad + \lambda_{02}\lambda_{23}\lambda_{31}L_{1,0,0}(\boldsymbol{x},t) + \lambda_{01}\lambda_{13}\lambda_{32}L_{0,1,0}(\boldsymbol{x},t) + \lambda_{03}\lambda_{31}\lambda_{12}L_{0,1,0}(\boldsymbol{x},t)\\
&\quad + \lambda_{01}\lambda_{12}\lambda_{23}L_{0,0,1}(\boldsymbol{x},t) + \lambda_{02}\lambda_{21}\lambda_{13}L_{0,0,1}(\boldsymbol{x},t)\}\,.
\end{aligned}$$

The expressions for $\rho_1^{(4)}$, $\rho_2^{(4)}$ and $\rho_3^{(4)}$ are obtained by deleting a transition from the desired final state to state 0. Thus $\rho_i^{(4)}$ is simply $\rho_0^{(4)}$ with suitably adjusted subscripts to $L_\kappa^{(4)}$, that is each κ_i in $\rho_i^{(4)}$ is decremented by one.

The full interior solution for the four-state case is given by

$$
\begin{aligned}
e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\,\{&\lambda_{01}\lambda_{02}\lambda_{03}[L_{1,1,1}(\boldsymbol{x},t) + L_{0,1,1}(\boldsymbol{x},t) + L_{1,0,1}(\boldsymbol{x},t) + L_{1,1,0}(\boldsymbol{x},t)]\\
+\ &\lambda_{01}\lambda_{12}\lambda_{13}[L_{-1,1,1}(\boldsymbol{x},t) + L_{-2,1,1}(\boldsymbol{x},t) + L_{-1,0,1}(\boldsymbol{x},t) + L_{-1,1,0}(\boldsymbol{x},t)]\\
+\ &\lambda_{02}\lambda_{21}\lambda_{23}[L_{1,-1,1}(\boldsymbol{x},t) + L_{0,-1,1}(\boldsymbol{x},t) + L_{1,-2,1}(\boldsymbol{x},t) + L_{1,-1,0}(\boldsymbol{x},t)]\\
+\ &\lambda_{03}\lambda_{31}\lambda_{32}[L_{1,1,-1}(\boldsymbol{x},t) + L_{0,1,-1}(\boldsymbol{x},t) + L_{1,0,-1}(\boldsymbol{x},t) + L_{1,1,-2}(\boldsymbol{x},t)]\\
+\ &\lambda_{02}\lambda_{01}\lambda_{13}[L_{0,1,1}(\boldsymbol{x},t) + L_{-1,1,1}(\boldsymbol{x},t) + L_{0,0,1}(\boldsymbol{x},t) + L_{0,1,0}(\boldsymbol{x},t)]\\
+\ &\lambda_{03}\lambda_{01}\lambda_{12}[L_{0,1,1}(\boldsymbol{x},t) + L_{-1,1,1}(\boldsymbol{x},t) + L_{0,0,1}(\boldsymbol{x},t) + L_{0,1,0}(\boldsymbol{x},t)]\\
+\ &\lambda_{01}\lambda_{02}\lambda_{23}[L_{1,0,1}(\boldsymbol{x},t) + L_{0,0,1}(\boldsymbol{x},t) + L_{1,-1,1}(\boldsymbol{x},t) + L_{1,0,0}(\boldsymbol{x},t)]\\
+\ &\lambda_{03}\lambda_{02}\lambda_{21}[L_{1,0,1}(\boldsymbol{x},t) + L_{0,0,1}(\boldsymbol{x},t) + L_{1,-1,1}(\boldsymbol{x},t) + L_{1,0,0}(\boldsymbol{x},t)] \qquad (4.3.6)\\
+\ &\lambda_{01}\lambda_{03}\lambda_{32}[L_{1,1,0}(\boldsymbol{x},t) + L_{0,1,0}(\boldsymbol{x},t) + L_{1,0,0}(\boldsymbol{x},t) + L_{1,1,-1}(\boldsymbol{x},t)]\\
+\ &\lambda_{02}\lambda_{03}\lambda_{31}[L_{1,1,0}(\boldsymbol{x},t) + L_{0,1,0}(\boldsymbol{x},t) + L_{1,0,0}(\boldsymbol{x},t) + L_{1,1,-1}(\boldsymbol{x},t)]\\
+\ &\lambda_{03}\lambda_{32}\lambda_{21}[L_{1,0,0}(\boldsymbol{x},t) + L_{0,0,0}(\boldsymbol{x},t) + L_{1,-1,0}(\boldsymbol{x},t) + L_{1,0,-1}(\boldsymbol{x},t)]\\
+\ &\lambda_{02}\lambda_{23}\lambda_{31}[L_{1,0,0}(\boldsymbol{x},t) + L_{0,0,0}(\boldsymbol{x},t) + L_{1,-1,0}(\boldsymbol{x},t) + L_{1,0,-1}(\boldsymbol{x},t)]\\
+\ &\lambda_{01}\lambda_{13}\lambda_{32}[L_{0,1,0}(\boldsymbol{x},t) + L_{-1,1,0}(\boldsymbol{x},t) + L_{0,0,0}(\boldsymbol{x},t) + L_{0,1,-1}(\boldsymbol{x},t)]\\
+\ &\lambda_{03}\lambda_{31}\lambda_{12}[L_{0,1,0}(\boldsymbol{x},t) + L_{-1,1,0}(\boldsymbol{x},t) + L_{0,0,0}(\boldsymbol{x},t) + L_{0,1,-1}(\boldsymbol{x},t)]\\
+\ &\lambda_{01}\lambda_{12}\lambda_{23}[L_{0,0,1}(\boldsymbol{x},t) + L_{-1,0,1}(\boldsymbol{x},t) + L_{0,-1,1}(\boldsymbol{x},t) + L_{0,0,0}(\boldsymbol{x},t)]\\
+\ &\lambda_{02}\lambda_{21}\lambda_{13}[L_{0,0,1}(\boldsymbol{x},t) + L_{-1,0,1}(\boldsymbol{x},t) + L_{0,-1,1}(\boldsymbol{x},t) + L_{0,0,0}(\boldsymbol{x},t)]\}\,.
\end{aligned}
$$

Clearly, we need to devise some way of expressing this solution concisely. In the process we will generalise it to the n-state case.

Let A be a set whose elements are in $\{1, 2, \cdots, n-1\}$, and let $|A|$ denote the order of A. We define the set function $\Upsilon(A)$ to be the set of all out-directed spanning arborescences of a graph with $|A|+1$ nodes, whose labels are the elements of $A \cup \{0\}$, and which are rooted at node 0. If $|A| = n-1$, there are n^{n-2} such trees (see [14], Section 3.3.24). This set can be generated either by calculating the tree-generating determinant, referred to earlier in the section, or by utilising a suitable tree-generating algorithm. Let $\tau \in \Upsilon(A)$ be one of these spanning trees and denote by $o(\tau, i)$, the *outdegree* of node i in τ. Define a function $\kappa : \Upsilon(A) \mapsto \boldsymbol{Z}_+^{|A|}$, such that the components of $\kappa(\tau)$ are given by

$$\kappa_i(\tau) = 1 - o(\tau, i),$$

for $1 \leq i \leq n-1$.

For $N = \{1, 2, \cdots, n-1\}$ and $\tau \in \Upsilon(N)$, $\kappa(\tau)$ gives us the values of the subscripts of $L^{(n)}$ for the component density $\rho_0^{(n)}$. Let $\boldsymbol{e}_k$ be the n-component vector with unity in the kth position and zeros elsewhere, and let $\boldsymbol{e}_0$ denote the zero vector. For the other component densities, $\rho_k^{(n)}, k \in \{1, 2, \cdots, n\}$, we decrement each $\kappa_k(\tau)$ by 1. That is, the required subscripts for $L^{(n)}$ in $\rho_k^{(n)}$ are given by

$$\kappa(\tau) - \boldsymbol{e}_k.$$

Thus if $F = \{1, 2, 3\}$, the interior part of $\rho^{(4)}$ can be rewritten as

$$e^{-\epsilon(\boldsymbol{x},t)} H(\boldsymbol{x}) \sum_{\tau \in \Upsilon(F)} \tau \left[L^{(4)}_{\kappa(\tau)}(\boldsymbol{x}, t) + L^{(4)}_{\kappa(\tau)-\boldsymbol{e}_1}(\boldsymbol{x}, t) + L^{(4)}_{\kappa(\tau)-\boldsymbol{e}_2}(\boldsymbol{x}, t) + L^{(4)}_{\kappa(\tau)-\boldsymbol{e}_3}(\boldsymbol{x}, t) \right].$$

Generalising to n states, we obtain the interior part of $\rho^{(n)}$

$$e^{-\epsilon(\boldsymbol{x},t)} H(\boldsymbol{x}) \sum_{\tau \in \Upsilon(N)} \sum_{i=0}^{n-1} \tau L^{(n)}_{\kappa(\tau)-\boldsymbol{e}_i}(\boldsymbol{x}, t). \tag{4.3.7}$$

4.4 Good's formula revisited

In the last chapter we showed that the interior part of our solution is identical to the constant term in (3.4.10) for $n = 3$, which Good deduced was the probability density function for the sojourn time in an n–state Markov process when $x_i > 0$, for $i \in \{0, 1, , \cdots, n\}$, and when the process starts in state 0. In this section we use Good's formula to derive an explicit expression for the interior solution of the four–state Markov process and compare it with the one we derived via probabilistic interpretation in the last section.

Recall that Good showed that the probability density when $\prod_{r=0}^{n} x_r \neq 0$ is equal to the constant term in

$$\exp\left(-\sum_{r}\sum_{s\neq r}\lambda_{r,s}x_r\right)\frac{\sum z_r}{\Pi z_r}\sum p_r D_r \exp\left(\sum_{r}\sum_{s\neq r}\lambda_{s,r}x_r z_s/z_r\right), \tag{4.4.8}$$

where p_i is the probability that the Markov process is initially in state i and D_i is the cofactor of the ith diagonal element of the matrix

$$\left(\delta_r^s \sum_{u\neq r}\lambda_{ur}z_u - \lambda_{rs}z_s\right) = (a_{r,s}) .$$

Once again, since the process starts in state 0, we only need to calculate D_0. For $n = 4$, $(a_{r,s})$ is the matrix

$$\begin{pmatrix} \sum_{i=1}^{3}\lambda_{i0}z_i & -\lambda_{01}z_1 & -\lambda_{02}z_2 & -\lambda_{03}z_3 \\ -\lambda_{10}z_0 & \sum_{i=0}^{3}\lambda_{i1}z_i - \lambda_{11}z_1 & -\lambda_{12}z_2 & -\lambda_{13}z_3 \\ -\lambda_{20}z_0 & -\lambda_{21}z_1 & \sum_{i=0}^{3}\lambda_{i2}z_i - \lambda_{22}z_2 & -\lambda_{23}z_3 \\ -\lambda_{30}z_0 & -\lambda_{31}z_1 & -\lambda_{32}z_2 & \sum_{i=0}^{3}\lambda_{i3}z_i - \lambda_{33}z_3 \end{pmatrix},$$

and hence

$$D_0 = \lambda_{01}\lambda_{02}\lambda_{03}z_0^3 + \lambda_{01}\lambda_{12}\lambda_{13}z_0z_1^2 + \lambda_{02}\lambda_{21}\lambda_{23}z_0z_2^2 + \lambda_{03}\lambda_{31}\lambda_{32}z_0z_3^2$$
$$+ \lambda_{02}\lambda_{01}\lambda_{13}z_0^2z_1 + \lambda_{03}\lambda_{01}\lambda_{12}z_0^2z_1 + \lambda_{01}\lambda_{02}\lambda_{23}z_0^2z_2 + \lambda_{03}\lambda_{02}\lambda_{21}z_0^2z_2$$
$$+ \lambda_{01}\lambda_{03}\lambda_{32}z_0^2z_3 + \lambda_{02}\lambda_{03}\lambda_{31}z_0^2z_3 + \lambda_{03}\lambda_{32}\lambda_{21}z_0z_2z_3 + \lambda_{02}\lambda_{23}\lambda_{31}z_0z_2z_3$$
$$+ \lambda_{01}\lambda_{13}\lambda_{32}z_0z_1z_3 + \lambda_{03}\lambda_{31}\lambda_{12}z_0z_1z_3 + \lambda_{01}\lambda_{12}\lambda_{23}z_0z_1z_2 + \lambda_{02}\lambda_{21}\lambda_{13}z_0z_1z_2.$$

In this case Formula (3.4.14) becomes

$$e^{\epsilon(\boldsymbol{x},t)}\frac{z_0+z_1+z_2+z_3}{z_0z_1z_2z_3}D_0 \exp\left(\sum_r \sum_{s\neq r} \lambda_{s,r}x_rz_s/z_r\right). \tag{4.4.9}$$

Let $u = z_1/z_0$, $v = z_2/z_0$ and $w = z_3/z_0$. To find the constant term in (4.4.9) we need to calculate the coefficient of $u^0v^0w^0$ in

$$\begin{aligned}
e^{\epsilon(\boldsymbol{x},t)} &\exp\Big(\lambda_{10}x_0u + \lambda_{01}x_1u^{-1} + \lambda_{20}x_0v + \lambda_{02}x_2v^{-1} + \lambda_{03}x_3w^{-1} + \lambda_{30}x_0w \\
+ &\quad \lambda_{21}x_1u^{-1}v + \lambda_{12}x_2uv^{-1} + \lambda_{31}x_1u^{-1}w + \lambda_{13}x_3uw^{-1} + \lambda_{32}x_2v^{-1}w + \lambda_{23}x_3vw^{-1}\Big) \\
\times &\quad \Big[\lambda_{01}\lambda_{02}\lambda_{03}(u^{-1}v^{-1}w^{-1} + v^{-1}w^{-1} + u^{-1}w^{-1} + u^{-1}v^{-1}) \\
&+ \lambda_{01}\lambda_{12}\lambda_{13}(uv^{-1}w^{-1} + u^2v^{-1}w^{-1} + uw^{-1} + uv^{-1}) \\
&+ \lambda_{02}\lambda_{21}\lambda_{23}(u^{-1}vw^{-1} + vw^{-1} + u^{-1}v^2w^{-1} + u^{-1}v) \\
&+ \lambda_{03}\lambda_{31}\lambda_{32}(u^{-1}v^{-1}w + v^{-1}w + u^{-1}w + u^{-1}v^{-1}w^{-1}) \\
&+ (\lambda_{02}\lambda_{01}\lambda_{13} + \lambda_{03}\lambda_{01}\lambda_{12})(v^{-1}w^{-1} + uv^{-1}w^{-1} + w^{-1} + v^{-1}) \\
&+ (\lambda_{01}\lambda_{02}\lambda_{23} + \lambda_{03}\lambda_{02}\lambda_{21})(u^{-1}w^{-1} + w^{-1} + u^{-1}vw^{-1} + u^{-1}) \\
&+ (\lambda_{01}\lambda_{03}\lambda_{32} + \lambda_{02}\lambda_{03})\lambda_{31}(u^{-1}v^{-1} + v^{-1} + u^{-1} + u^{-1}v^{-1}w) \\
&+ (\lambda_{03}\lambda_{32}\lambda_{21} + \lambda_{02}\lambda_{23}\lambda_{31})(u^{-1} + 1 + u^{-1}v + u^{-1}w) \\
&+ (\lambda_{01}\lambda_{13}\lambda_{32} + \lambda_{03}\lambda_{31}\lambda_{12})(v^{-1} + uv^{-1} + 1 + v^{-1}w) \\
&+ (\lambda_{01}\lambda_{12}\lambda_{23} + \lambda_{02}\lambda_{21}\lambda_{13})(w^{-1} + uw^{-1} + vw^{-1} + 1)\Big].
\end{aligned} \tag{4.4.10}$$

We now expand the second exponential term above and write it in terms of the multipartite notation of Section 4.3.2.

Let $a = (a_0, a_1, a_2)$, such that $a_i \in \boldsymbol{Z}_+$, for $i \in \{0, 1, 2\}$, $\boldsymbol{b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3})$ is of order 9, and $\bar{\boldsymbol{b}} = (\{b_1\}, \{b_2\}, \{b_3\}) = (b_{12}, b_{13}, b_{21}, b_{23}, b_{31}, b_{32})$, with $b_{ij} \in \boldsymbol{Z}_+$; $\boldsymbol{y}$ is a non–diagonal multipartite vector of order 12, and is defined as $(\boldsymbol{y_0}, \boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{y_3})$, with each $y_{ij} = \lambda_{ij} x_j$. We obtain

$$\begin{aligned}
&\exp\Big(\lambda_{10}x_0u + \lambda_{01}x_1u^{-1} + \lambda_{20}x_0v + \lambda_{02}x_2v^{-1} + \lambda_{03}x_3w^{-1} + \lambda_{30}x_0w + \lambda_{21}x_1u^{-1}v \\
&\qquad + \lambda_{12}x_2uv^{-1} + \lambda_{31}x_1u^{-1}w + \lambda_{13}x_3uw^{-1} + \lambda_{32}x_2v^{-1}w + \lambda_{23}x_3vw^{-1}\Big) \\
&= \sum_{a}\sum_{\boldsymbol{b_1}} \frac{(y_{10}u)^{a_0}(y_{20}v)^{a_1}(y_{30}w)^{a_2}(y_{01}u^{-1})^{b_{01}}(y_{21}u^{-1}v)^{b_{21}}(y_{31}u^{-1}w)^{b_{31}}}{a!\boldsymbol{b_1}!} \\
&\times \sum_{\boldsymbol{b_2}}\sum_{\boldsymbol{b_3}} \frac{(y_{02}v^{-1})^{b_{02}}(y_{12}uv^{-1})^{b_{12}}(y_{32}v^{-1}w)^{b_{32}}(y_{03}w^{-1})^{b_{03}}(y_{13}uw^{-1})^{b_{13}}(y_{23}vw^{-1})^{b_{23}}}{\boldsymbol{b_2}!\boldsymbol{b_3}!}.
\end{aligned}$$

Rewriting the expression above as a power series of u, v and w, gives

$$\begin{aligned}
&\sum_{k}\sum_{\boldsymbol{b_1}}\sum_{\boldsymbol{b_2}}\sum_{\boldsymbol{b_3}} \frac{y_{10}^{k_1+|\boldsymbol{b_1}|-|\{b_1\}|}\, y_{20}^{a+|\boldsymbol{b_2}|-|\{b_2\}|}\, y_{30}^{k_3+|\boldsymbol{b_3}|-|\{b_3\}|}\, \boldsymbol{y_1}^{\boldsymbol{b_1}}\boldsymbol{y_2}^{\boldsymbol{b_2}}\boldsymbol{y_3}^{\boldsymbol{b_3}} u^{a_1}v^{a_2}w^{a_3}}{(k_1+|\boldsymbol{b_1}|-|\{b_1\}|)!(k_2+|\boldsymbol{b_2}|-|\{b_2\}|)!(k_3+|\boldsymbol{b_3}|-|\{b_3\}|)!\boldsymbol{b_1}!\boldsymbol{b_2}!\boldsymbol{b_3}!} \\
= \;& \sum_{k}\sum_{\boldsymbol{b}} \frac{\boldsymbol{y_0}^{k+|\boldsymbol{b}|-|\bar{\boldsymbol{b}}|}\boldsymbol{y_1}^{\boldsymbol{b_1}}\boldsymbol{y_2}^{\boldsymbol{b_2}} u^{a_1}v^{a_2}w^{a_3}}{(k+|\boldsymbol{b}|-|\bar{\boldsymbol{b}}|)!\boldsymbol{b}!} \qquad (4.4.11) \\
= \;& \sum_{a} L_a^4(\boldsymbol{x}, t)u^{a_1}v^{a_2}w^{a_3}.
\end{aligned}$$

The constant term is the coefficient of $u^0v^0w^0$ in the expression obtained by substituting (4.4.11) into (4.4.10),

$$
\begin{aligned}
& e^{\epsilon(\boldsymbol{x},t)} \sum_{a} L_a^4(\boldsymbol{x},t) u^{a_1} v^{a_2} w^{a_3} \\
& \quad \times \Big[\lambda_{01}\lambda_{02}\lambda_{03}(u^{-1}v^{-1}w^{-1} + v^{-1}w^{-1} + u^{-1}w^{-1} + u^{-1}v^{-1}) \\
& \quad + \; \lambda_{01}\lambda_{12}\lambda_{13}(uv^{-1}w^{-1} + u^2v^{-1}w^{-1} + uw^{-1} + uv^{-1}) \\
& \quad + \; \lambda_{02}\lambda_{21}\lambda_{23}(u^{-1}vw^{-1} + vw^{-1} + u^{-1}v^2w^{-1} + u^{-1}v) \\
& \quad + \; \lambda_{03}\lambda_{31}\lambda_{32}(u^{-1}v^{-1}w + v^{-1}w + u^{-1}w + u^{-1}v^{-1}w^{-1}) \\
& \quad + \; (\lambda_{02}\lambda_{01}\lambda_{13} + \lambda_{03}\lambda_{01}\lambda_{12})(v^{-1}w^{-1} + uv^{-1}w^{-1} + w^{-1} + v^{-1}) \\
& \quad + \; (\lambda_{01}\lambda_{02}\lambda_{23} + \lambda_{03}\lambda_{02}\lambda_{21})(u^{-1}w^{-1} + w^{-1} + u^{-1}vw^{-1} + u^{-1}) \\
& \quad + \; (\lambda_{01}\lambda_{03}\lambda_{32} + \lambda_{02}\lambda_{03}\lambda_{31})(u^{-1}v^{-1} + v^{-1} + u^{-1} + u^{-1}v^{-1}w) \\
& \quad + \; (\lambda_{03}\lambda_{32}\lambda_{21} + \lambda_{02}\lambda_{23}\lambda_{31})(u^{-1} + 1 + u^{-1}v + u^{-1}w) \\
& \quad + \; (\lambda_{01}\lambda_{13}\lambda_{32} + \lambda_{03}\lambda_{31}\lambda_{12})(v^{-1} + uv^{-1} + 1 + v^{-1}w) \\
& \quad + \; (\lambda_{01}\lambda_{12}\lambda_{23} + \lambda_{02}\lambda_{21}\lambda_{13})(w^{-1} + uw^{-1} + vw^{-1} + 1) \Big] .
\end{aligned}
$$

It is evident that the coefficient of $u^0v^0w^0$ in the above expression is identical to (4.3.6), the full interior four–state solution deduced in the last section.

4.5 The full n–state solution

In the last section we were able to use graph theory to derive a generalisation to n states of the four–state interior solution and we were able to reconcile the latter with Good's formula. We deduced the boundary solution by extrapolating from three to four states using probabilistic intuition. In this section we use the same reasoning to obtain the full n–state solution. In conclusion we show that the solution satisfies the Kolmogorov equations for the n–state sojourn time problem.

Recall the first expression we derived for the boundary part of $\rho^{(4)}$ in (4.3.4). It seems logical to assume that since we want to calculate the probability density for those events in which at least one of the $n-1$ states is never visited (state 0 is the initial state and therefore x_0 always accrues some density), the solution must be related to the full $(n-1)$–state solution. Therefore the boundary solution is simply the sum of $\rho^{(n-1)}$ calculated over each possible combination of $n-1$ of the n states after the duplicate expressions have been subtracted. An alternative way to think about the problem is that the solution on each boundary (that is, those hyperplanes for which at least one of the x_i's vanishes) is simply the interior solution valued over the number of states which are visited at least once and multiplied by the appropriate delta distributions. Thus we can write the boundary solution by listing each possible subset of $N = \{1, 2, \cdots, n-1\}$, and for each subset A, multiplying the interior part of $\rho^{(|A|)}$ valued over the states in A by $\delta(x_i)$, for each i in the complement of A in N.

Recall that $\Upsilon(N)$ is the set of all spanning trees whose $(n+1)$ nodes are the elements of $N \cup \{0\}$, and which are rooted from node zero. In addition, recall that $\boldsymbol{x} = (x_1, x_2, \cdots, x_{n-1})$ for the n–state case. Let N_k denote the set of all subsets of N, which have order k, N_k^m denote those subsets of N_k which contain m, and $N_K^{\bar{m}}$ denote those subsets of N_k which do not contain m. Let $A = \{a_1, a_2, \cdots, a_k\}$, such that $A \in N_k$ and $a_i < a_j$ for all $i < j$, that is, the elements of A are ordered. We denote by $\boldsymbol{x}_A$ the vector $(x_{a_1}, x_{a_2}, \cdots, x_{a_k})$, so that, when L_κ is valued over x_A, all the subscripts j appearing in the expansion of $L_\kappa^{(k+1)}(\boldsymbol{x}_A, t)$ are replaced by a_j, that is, y_{ij} becomes $y_{a_i a_j}$. Clearly $N_{n-1} = N$, and $\boldsymbol{x}_N = (x_1, x_2, \cdots, x_{n-1})$.

We claim that the boundary part of $\rho^{(n)}$ is given by the expression

$$e^{\epsilon(\boldsymbol{x},t)} H(\boldsymbol{x}) \left[\delta(\boldsymbol{x}) + \sum_{k=1}^{n-2} \sum_{A \in N_k} \left(\prod_{j \in N \setminus A} \delta(x_j) \right) \sum_{\tau \in \Upsilon(A)} \sum_{i=0}^{k} \tau L_{\kappa(\tau)-e_i}^{(k+1)}(\boldsymbol{x}_A, t) \right].$$

Let us use this expression for $n = 4$. In this case $N = \{1, 2, 3\}$, and the power set of N is $\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. If the above claim holds then the boundary solution is given by

$$
\begin{aligned}
e^{\epsilon(\boldsymbol{x},t)} H(\boldsymbol{x}) \Big[& \delta(\boldsymbol{x}) + \delta(x_2)\delta(x_3)\lambda_{01}[L_1^{(2)}(x_1, t) + L_0^{(2)}(x_1, t)] \\
& + \delta(x_1)\delta(x_3)\lambda_{02}[L_1^{(2)}(x_2, t) + L_0^{(2)}(x_2, t)] + \delta(x_1)\delta(x_2)\lambda_{03}[L_1^{(2)}(x_3, t) + L_0^{(2)}(x_3, t)] \\
& + \delta(x_3)\Big(\lambda_{01}\lambda_{02}[L_{1,1}^{(3)}(x_1, x_2, t) + L_{0,1}^{(3)}(x_1, x_2, t) + L_{1,0}^{(3)}(x_1, x_2, t)] \\
& \quad + \quad \lambda_{01}\lambda_{12}[L_{0,1}^{(3)}(x_1, x_2, t) + L_{-1,1}^{(3)}(x_1, x_2, t) + L_{0,0}^{(3)}(x_1, x_2, t)] \\
& \quad + \quad \lambda_{02}\lambda_{21}[L_{1,0}^{(3)}(x_1, x_2, t) + L_{0,0}^{(3)}(x_1, x_2, t) + L_{1,-1}^{(3)}(x_1, x_2, t)]\Big) \\
& + \delta(x_2)\Big(\lambda_{01}\lambda_{03}[L_{1,1}^{(3)}(x_1, x_3, t) + L_{0,1}^{(3)}(x_1, x_3, t) + L_{1,0}^{(3)}(x_1, x_3, t)] \\
& \quad + \quad \lambda_{01}\lambda_{13}[L_{0,1}^{(3)}(x_1, x_3, t) + L_{-1,1}^{(3)}(x_1, x_3, t) + L_{0,0}^{(3)}(x_1, x_3, t)] \\
& \quad + \quad \lambda_{03}\lambda_{31}[L_{1,0}^{(3)}(x_1, x_3, t) + L_{0,0}^{(3)}(x_1, x_3, t) + L_{1,-1}^{(3)}(x_1, x_3, t)]\Big) \\
& + \delta(x_1)\Big(\lambda_{02}\lambda_{03}[L_{1,1}^{(3)}(x_2, x_3, t) + L_{0,1}^{(3)}(x_2, x_3, t) + L_{1,0}^{(3)}(x_2, x_3, t)] \\
& \quad + \quad \lambda_{02}\lambda_{23}[L_{0,1}^{(3)}(x_2, x_3, t) + L_{-1,1}^{(3)}(x_2, x_3, t) + L_{0,0}^{(3)}(x_2, x_3, t)] \\
& \quad + \quad \lambda_{03}\lambda_{32}[L_{1,0}^{(3)}(x_2, x_3, t) + L_{0,0}^{(3)}(x_2, x_3, t) + L_{1,-1}^{(3)}(x_2, x_3, t)]\Big)\Big],
\end{aligned}
$$

and this is the explicit solution given in Section 4.3.1.

Hence the full n-state solution will be given by

$$
\rho^{(n)}(\boldsymbol{x}, t) = e^{-\epsilon(\boldsymbol{x},t)} H(\boldsymbol{x}) \left[\delta(\boldsymbol{x}) + \sum_{k=1}^{n-1} \sum_{A \in N_k} \sum_{\tau \in \Upsilon(A)} \sum_{i=0}^{k} \prod_{j \in N \setminus A} \delta(x_j) \tau L_{\kappa(\tau)-e_i}^{(k+1)}(\boldsymbol{x}_A, t) \right].
$$

Once again it is desirable to separate $\rho^{(n)}$ into its components $\rho_m^{(n)}$,

$$\rho_0^{(n)}(\boldsymbol{x},t) = e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\left[\delta(\boldsymbol{x})+\sum_{k=1}^{n-1}\sum_{A\in N_k}\sum_{\tau\in\Upsilon(A)}\prod_{j\in N\setminus A}\delta(x_j)\tau L_{\kappa(\tau)}^{(k+1)}(\boldsymbol{x}_A,t)\right],$$

and for all $m \geq 1$,

$$\rho_m^{(n)}(\boldsymbol{x},t) = e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\left[\sum_{k=1}^{n-1}\sum_{A\in N_k^m}\sum_{\tau\in\Upsilon(A)}\prod_{j\in N\setminus A}\delta(x_j)\tau L_{\kappa(\tau)-e_m}^{(k+1)}(\boldsymbol{x}_A,t)\right].$$

We now state the main result of this thesis.

Theorem 3 *Let $N = \{1, 2, \cdots, n-1\}$, and for any $B \subseteq N$, we denote by $\bar{B}$, the augmented set $B \cup \{0\}$. Let $\Upsilon(N)$ be the set of all spanning trees whose nodes are the elements of $\bar{N}$, rooted from node zero. Each $\tau \in \Upsilon(N)$ is represented by $\prod \lambda_{ij}$, where each ij is an edge of the tree τ.*

For an n-state process, let the total accumulated sojourn time in state k up to time t be denoted by x_k with $x_0 = t - \sum_{j=1}^{n} x_j$. Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, and let the joint probability density of $\boldsymbol{x}$, when the final state of the process is state m, be denoted by $\rho_m^{(n)}$.

Let $A = \{a_1, a_2, \cdots, a_k\} \in N_k$. Then $\boldsymbol{x}_A$ denotes the vector $(x_{a_1}, x_{a_2}, \cdots, x_{a_k})$. The function $\kappa : \Upsilon(A) \mapsto \boldsymbol{Z}_+^{|A|}$, is defined in terms of its components,

$$\kappa_i(\tau) = 1 - o(\tau, i),$$

for $1 \leq i \leq n-1$, with $\tau \in \Upsilon(A)$ and $o(\tau, i)$ defined to be the outdegree of node i in τ.

The function $L_\kappa^{(i)}, 2 \leq i \leq n$ is defined as

$$L_\kappa^{(k)}(\boldsymbol{x},t) = H(x_0)\sum_{\boldsymbol{b}} \frac{{y_0}^{\kappa+|b|-|\bar{b}|}{y_1}^{b_1}{y_2}^{b_2}\cdots {y_{k-1}}^{b_{k-1}}}{(\kappa+|b|-|\bar{b}|)!b!},$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_{k-1})$, κ *is a multipartite number of order* $k-1$, *with* $\kappa_j \in \boldsymbol{Z}$ *for each* j, $\boldsymbol{b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_{k-1}})$ *is a non-diagonal multipartite vector of order* $(k-1)^2$ *and whose components are defined to be* $\boldsymbol{b_j} = (b_{0j}, b_{1j}, \cdots, b(1-i)j))$, *with each* $b_{ij} \in \boldsymbol{Z}_+$ *and the elements of* $\bar{\boldsymbol{b}} = (\{b_1\}, \{b_2\}, \cdots, \{b_{k-1}\})$ *are contained in* $\boldsymbol{b}$; $\boldsymbol{y}$ *is a non-diagonal multipartite vector of order* $k(k-1)$, *such that each* $y_{ij} = \lambda_{ij} x_j$.

The Kolmogorov equations for the evolution of probability of the total accumulated sojourn time in an n*-state Markov process,*

$$\begin{aligned}
\frac{\partial \rho_0^{(n)}}{\partial t}(\boldsymbol{x}, t) &= \sum_{i=0}^{n-1} \lambda_{i0} \rho_i^{(n)}(\boldsymbol{x}, t), \\
\frac{\partial \rho_m^{(n)}}{\partial t}(\boldsymbol{x}, t) &= \sum_{i=0}^{n-1} \lambda_{ik} \rho_i^{(n)}(\boldsymbol{x}, t) - \frac{\partial \rho_m^{(n)}}{\partial x_k}(\boldsymbol{x}, t) \qquad k \in \{1, 2, \cdots, n\},
\end{aligned}$$

are satisfied by the generalised functions

$$\rho_0^{(n)}(\boldsymbol{x}, t) = e^{-\epsilon(\boldsymbol{x},t)} H(\boldsymbol{x}) \left[\delta(\boldsymbol{x}) + \sum_{k=1}^{n-1} \sum_{A \in N_k} \sum_{\tau \in \Upsilon(A)} \prod_{j \in N \setminus A} \delta(x_j) \tau L_{\kappa(\tau)}^{(k+1)}(\boldsymbol{x}_A, t) \right],$$

and for all $m \geq 1$,

$$\rho_m^{(n)}(\boldsymbol{x}, t) = e^{-\epsilon(\boldsymbol{x},t)} H(\boldsymbol{x}) \left[\sum_{k=1}^{n-1} \sum_{A \in N_k^m} \sum_{\tau \in \Upsilon(A)} \prod_{j \in N \setminus A} \delta(x_j) \tau L_{\kappa(\tau)-e_m}^{(k+1)}(\boldsymbol{x}_A, t) \right].$$

Proof Let $\tilde{L}_\kappa(x_0, \boldsymbol{x}) = L_\kappa(\boldsymbol{x}, t)$, and let D_i denote partial differentiation with respect to the ith variable, keeping the other $n-1$ variables fixed. Since $x_0 = t - \boldsymbol{x}$, then $\frac{\partial}{\partial t} L_\kappa(\boldsymbol{x}, t) = D_1 \tilde{L}_\kappa(x_0, \boldsymbol{x})$, and $\left(\frac{\partial}{\partial x_m} + \frac{\partial}{\partial t}\right) L_\kappa(\boldsymbol{x}, t) = D_{m+1} \tilde{L}_\kappa(x_0, \boldsymbol{x})$, for all $m \in \{1, 2, \cdots, n-1\}$.

In order to differentiate L_κ with respect to x_0 we first of all observe that since the initial state of the process is state 0, then $x_0 > 0$ for all $t > 0$, so that if $x_0 = 0$ then $t = x_1 = x_2 = \cdots = x_{n-1} = 0$. Hence all $\delta(x_0)$ terms that arise from the partial differentiation of $H(x_0)$ are deemed to be equal to zero. The rest of the differentiation becomes immediate if we expand those factors which are dependent on x_0,

$$
\begin{aligned}
D_1 \tilde{L}_\kappa^{(i)}(x_0, \boldsymbol{x}) &= D_1 \left(H(x_0) \sum_{\boldsymbol{b}} \frac{\boldsymbol{y_1}^{b_1} \boldsymbol{y_2}^{b_2} \cdots \boldsymbol{y_{i-1}}^{b_{i-1}}}{\boldsymbol{b}!} \prod_{k=1}^{i-1} \frac{y_{k0}^{(\kappa_k + |b_k| - |\{b_k\}|)}}{(\kappa_k + |b_k| - |\{b_k\}|)!} \right) \\
&= \sum_{j=1}^{i-1} \lambda_{j0} L_{\kappa - e_j}^{(i)}(\boldsymbol{x}, t).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\frac{\partial}{\partial t} L_\kappa(\boldsymbol{x}_A, t) &= D_1 \tilde{L}_\kappa^{(i)}(x_0, \boldsymbol{x}_A) \\
&= \sum_{j \in A} \lambda_{j0} L_{\kappa - e_j}^{(i)}(\boldsymbol{x}_A, t).
\end{aligned}
$$

We repeat the technique in order to obtain the partial derivatives with respect to the rest of the x_m's,

$$
\begin{aligned}
D_{m+1}\tilde{L}_{\kappa}^{(i)}(x_0,\boldsymbol{x}) &= D_{m+1}\left(H(x_0)\sum_{\boldsymbol{b}}\frac{y_0{}^{(\kappa+|\boldsymbol{b}|-|\bar{b}|)}\boldsymbol{y_1}^{\boldsymbol{b_1}}\cdots\prod_{k=1}^{i-1}y_{km}^{b_{km}}\cdots\boldsymbol{y_{i-1}}^{\boldsymbol{b_{i-1}}}}{(\kappa+|\boldsymbol{b}|-|\bar{b}|)!\boldsymbol{b_1}!\cdots\prod_{k=1}^{i-1}b_{km}!\cdots\boldsymbol{b_{i-1}}!}\right)\\
&= \sum_{j=1}^{i-1}\lambda_{jm}L_{\kappa+e_m-e_j}^{(i)}(\boldsymbol{x},t),
\end{aligned}
$$

and, in particular,

$$
D_{m+1}\tilde{L}_{\kappa-e_m}^{(i)}(x_0,\boldsymbol{x}_A) = \sum_{j\in\bar{A}\setminus\{m\}}\lambda_{jm}L_{\kappa-e_j}^{(i)}(\boldsymbol{x}_A,t).
$$

The partial derivative of $\rho_0^{(n)}(\boldsymbol{x},t)$ with respect to x_0:

$$
\begin{aligned}
&\lambda_{00}\rho_0^{(n)}(\boldsymbol{x},t) + e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\left[\sum_{k=1}^{n-1}\sum_{A\in N_k}\sum_{\tau\in\Upsilon(A)}\sum_{i\in A}\prod_{j\in N\setminus A}\delta(x_j)\tau\lambda_{i0}L_{\kappa(\tau)-e_i}^{(k+1)}(\boldsymbol{x}_A,t)\right]\\
&= \lambda_{00}\rho_0^{(n)}(\boldsymbol{x},t) + e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\left[\sum_{i=1}^{n-1}\lambda_{i0}\sum_{k=1}^{n-1}\sum_{A\in N_k^i}\sum_{\tau\in\Upsilon(A)}\prod_{j\in N\setminus A}\delta(x_j)\tau L_{\kappa(\tau)-e_i}^{(k+1)}(\boldsymbol{x}_A,t)\right]\\
&= \lambda_{00}\rho_0^{(n)}(\boldsymbol{x},t) + \sum_{i=1}^{n-1}\lambda_{i0}\rho_i^{(n)}(\boldsymbol{x},t),
\end{aligned}
$$

clearly satisfies the first Kolmogorov equation.

The partial derivative with respect to x_m requires the differentiation of $H(\boldsymbol{x})$ which yields a factor of $\delta(x_m)$. It is therefore necessary to calculate $\rho_m^{(n)}(\boldsymbol{x},t)|_{x_m=0}$. We first of all investigate $L_{\kappa(\tau)-e_m}^{k}(\boldsymbol{x},t)$ with $x_m=0$, which forces $\boldsymbol{b_m}$ and $\{b_m\}$ to equal zero. We remark that by its definition, each component of $\kappa(\tau)\leq 1$, hence all terms for which the mth component of $\kappa(\tau) - \boldsymbol{e}_m$ is negative, vanish, leaving us with the terms for which the mth component is zero. If we eliminate these k–node spanning trees, τ, from our set, $\Upsilon(A)$, we are left with those trees for which there is one edge directed into node m. We can therefore enumerate these trees as $(k-1)$–node trees (node m is removed) each of which has an extra edge, from one of the $k-1$ nodes in turn, to node m. That is, the domain of Υ does not contain m.

Using the reasoning in the previous paragraph, we obtain

$$\begin{aligned}\delta(x_m)\rho_m^{(n)} &= e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\left[\lambda_{0m}\delta(\boldsymbol{x})\right.\\ &\quad \left.+\sum_{k=1}^{n-2}\sum_{A\in N_k^{\bar{m}}}\sum_{\tau\in\Upsilon(A)}\sum_{i\in\bar{A}}\prod_{j\in N\setminus A}\delta(x_j)\lambda_{im}\tau L_{\kappa(\tau)-\boldsymbol{e}_i}^{(k+1)}(\boldsymbol{x}_A,t)\right].\end{aligned}$$

To conclude, for $1 \le m \le n-1$,

$$\begin{aligned}&\left(\frac{\partial}{\partial x_m}+\frac{\partial}{\partial t}\right)\rho_m^{(n)}(\boldsymbol{x},t) = \lambda_{mm}\rho_m^{(n)}\\ &+e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\left[\lambda_{0m}\delta(\boldsymbol{x})+\sum_{k=1}^{n-2}\sum_{A\in N_k^{\bar{m}}}\sum_{\tau\in\Upsilon(A)}\sum_{i\in\bar{A}}\prod_{j\in N\setminus A}\delta(x_j)\lambda_{im}\tau L_{\kappa(\tau)-\boldsymbol{e}_i}^{(k+1)}(\boldsymbol{x}_A,t)\right.\\ &+e^{-\epsilon(\boldsymbol{x},t)}H(\boldsymbol{x})\left[\sum_{k=1}^{n-1}\sum_{A\in N_k^{m}}\sum_{\tau\in\Upsilon(A)}\sum_{i\in\bar{A}\setminus\{m\}}\prod_{j\in N\setminus A}\delta(x_j)\lambda_{im}\tau L_{\kappa(\tau)-\boldsymbol{e}_i}^{(k+1)}(\boldsymbol{x}_A,t)\right]\\ &=\sum_{i=0}^{n-1}\lambda_{ik}\rho_i^{(n)}(\boldsymbol{x},t),\end{aligned}$$

as required.

□

Chapter 5

Applications in Communications

5.1 Fluid flow models of queueing systems

Multi–state Markov processes arise quite naturally in the study of communications. In particular, continuous–time processes are widely used to model traffic sources as streams of fluid flowing at a constant rate which is modulated by an underlying Markov process. Such a process is known as a Markov modulated rate process (MMRP). This traffic approximation is particularly suited to data traffic which is generated in constant–rate bursts. Typically, messages or data files are transmitted from source to a primary node or server at the available link rate. The data traffic is then queued until transmitted at a constant rate over a common higher bitrate channel. Longer buffering is allowed for data (such as file transfers) which is not as sensitive to delay as interactive traffic. The stochastic characteristics of this system are captured by a model of a superposition of a finite number of independent, identical, continuous on/off sources and a constant service capacity.

An on/off fluid source is accurately modelled as a two–state MMRP. Extending this to three states gives a good approximation of a traffic source which transmits at three different rates, a good example of which is a video source (see [4]). In the communications literature there are several analytical results for variants of the time–dependent two–state problem. As we show in Section 5.3, the joint probability density of the total time spent in each of the n states of a Markov process can be applied to the problem of finding the probability density of the data arriving in a given interval from a source with n transmission rates. The n–state model will also provide the basic structure for multi–dimensional intelligent queueing models. We explore the notion of intelligent queueing in Section 5.2.

The classical one–buffer fluid model of a data–handling switch was first introduced in [1]. In this model a FIFO (First–In First–Out) queue receives messages from N on/off independent sources and buffers incoming traffic which is in excess of the maximum transmission rate of the output link. The authors of [1] exploit the linearity of the system to obtain closed–form expressions of the eigenvalues and eigenvectors of the differential equations for the time–independent equlibrium probabilities.

Bensaou *et al.* [3] study the queue–length distribution for a superposition of on/off fluid sources. They extend a result due to Beneš to fluid models and derive the stationary probability that the queue size exceeds a given value in terms of the probability density of the amount of work arriving in a given interval. Their result can be applied to non–Markovian arrival processes.

A time–dependent result is due to Chen and Samalam [6], who study fluid buffer models with Markovian arrivals and a constant service rate. Their approach is to solve a first–passage time problem described by a Fredholm integral equation of the second kind. They obtain numerical results for the queue–length distribution for those arrival streams which can be modelled as independent Ornstein–Uhlenbeck diffusion processes.

5.2 Intelligent Queueing

Systems in which bandwidth is in high demand could be made more efficient by utilising intelligent queueing. One way of doing this is to buffer the data streams separately on the basis of some traffic classification. For example, one might separate bursts on the basis of length in order to prevent unnecessary delay experienced by (typically) short bursts of interactive traffic which are held up behind long data files. A second classification might be traffic priority, that is, allowing urgent traffic to be sent before routine data transfers and information downloads. We look at such a priority queueing system in Section 5.2.4.

Let us first of all present the mathematical representation of a one–buffer system. We shall later extend this to a two–buffer model as required for intelligent queueing.

5.2.1 Representation of the one–buffer problem

We shall take the model presented in [1] as a typical example: N on/off identical and independent sources with exponentially–distributed on, as well as off, periods. The unit of time is taken to be the average length of an on period. The unit of information is the amount generated by a source in an average on period.

We adopt the following notation.

$\lambda \in (0, \infty)$ = rate at which an off source turns on;

$z(t) \in [0, \infty)$ = buffer occupancy at time t;

$c \in [0, \infty)$ = service rate;

$N \in \boldsymbol{Z}_+$ = total number of sources;

$P_i : \boldsymbol{R}_+^2 \mapsto \boldsymbol{R}_+$, such that for $i \in \{0, .., N\}$,

$$P_i(t, z) = \Pr\{i \text{ sources are on and buffer content} \leq z \text{ at time } t\}.$$

Thus when r sources are on simultaneously, the instantaneous receiving rate at the data switch is r and the instantaneous buffer occupancy drift rate is $r - c$. When all sources are off, the drift rate is $-c$. The buffer length is assumed to be infinite.

We can model the two–state problem, that is, when $N = 1$, as a continuous–time random walk with two directions, positive when the source is on and negative when it is off. The position at time t corresponds to the buffer level at that time. This is analogous to using a simple random walk to model the time-dependent queueing problem with a Poisson arrival stream and negative-exponential service times, see for example [5]. Since we know the drift function for each state and we can calculate the density of the accumulated sojourn times, we can solve the random walk problem to get a solution for the probability density in the interior, that is, when the queue is not empty. On generalising this idea to an n-state problem and by finding the total sojourn time spent in each state of the Markov process, for each time t, with a given initial state, we can get an expression for the density functions of the buffer states between the passage times at which a boundary is first reached (when the buffer first empties or overflows).

5.2.2 Representation of the general two-buffer problem

We can extrapolate from one buffer to two in such a way that the representation of the new system is simply an extension of the state space. We have two classes of traffic arriving at a node which buffers each class of traffic separately. For $k \in \{1, 2\}$ there are $N_k \in \boldsymbol{Z}_+$ identical traffic sources which alternate independently between exponentially distributed periods in the on and off states. A single output link serves each buffer at a different rate depending on a priority weighting and/or the buffer occupancy of each class.

Much as before we write

$\lambda_k \in (0, \infty)$ = rate at which an off class k source turns on;

$\mu_k \in (0, \infty)$ = rate at which an on class k source turns off;

$z_k(t) \in [0, \infty)$ = buffer k occupancy at time t;

$c_k(t, z_1, z_2) \in [0, \infty)$ = class k service rate;

$N_k \in \boldsymbol{Z}_+$ = total number of class k sources;

$P_{ij} : \boldsymbol{R}^3_+ \mapsto \boldsymbol{R}_+$, such that, for $i \in \{0, N_1\}$, $j \in \{0, N_2\}$,

$$P_{ij}(t, z_1, z_2) = \Pr\{(i, j) \text{ sources are on, } z_1(t) \leq z_1 \text{ and } z_2(t) \leq z_2\}.$$

The arrival process for this system has $N_1 \times N_2$ possible states each with its own drift rate vector function, $v_{ij} : \boldsymbol{R}_+^3 \mapsto \boldsymbol{R}_+^2$, such that,

$$v_{ij}(t, z_1, z_2) = [i - c_1(t, z_1, z_2), j - c_2(t, z_1, z_2)].$$

This problem can thus be represented by a continuous–time random walk on a plane, such that the direction of displacement is defined by the drift rate for the current state. Once again, the joint probability density of the total sojourn time in each state gives us the probability density of the position at time t, $(z_1(t), z_2(t))$, between absorption times when $z_1(t) = z_2(t) = 0$.

5.2.3 Some special cases

A queueing system in which the service is shared evenly between two queues (in round–robin fashion) is a particular case of the general two–buffer problem. This is possibly the simplest in this class of problems and yet the performance analysis of such a queue is highly non–trivial. Round–robin servicing is implemented widely in communications systems and hence the analysis has many applications. For two buffers, the classical round robin has a constant service rate equal to half the link capacity while both buffers are occupied. When one of the buffers is empty, the other is served at the full rate.

Stern and Elwalid analyse a two–buffer system representing a constant rate communication channel which serves two classes of traffic modelled as continuous flow processes modulated by an underlying reversible Markov process. Class x traffic is stored if it cannot be transmitted immediately, while class y traffic is discarded if it cannot have instantaneous service. The service strategy modelled is a general type of round robin in which class x traffic is apportioned $1 - \alpha$ of the service, while class y gets the remaining α. The authors use a decomposition technique to get a solution for the equilibrium equations for those cases in which the Markov process is separable.

Yadin [23] analyses a two–queue system with alternating priorities. Each queue has its own independent Poisson stream of arrivals. A shared server switches its service from one queue to the other according to a *switch rule*, which depends on the current queue lengths. The state of the system is represented by an ordered pair, $(x, y) \in \boldsymbol{Z}_+^2$, where x the number of customers in queue I and y, the number in queue II. The process can be modelled as a random walk on the lattice in the positive quadrant of the plane. When the random walk enters a certain *absorbing set* of points determined by the switch rule, the service switches to the other queue. The period between switches is called a *task*. Since only one queue is served during a given task, the random walk is three–directional, the two positive directions corresponding to arrivals at each queue and the negative direction to a service.

Yadin considers three different switch rules and for each, derives the conditional density of the time to complete a task, when the initial state is known.

5.2.4 Representation of an alternating priority system

We can apply the results from our three–state model to a two–buffer fluid system analogous to Yadin's discrete system. The probability density of the time taken to complete each task is a function of the accumulated sojourn time in each of the three states.

Two classes of data traffic are generated from the same source at independent rates, r_1 and r_2, alternating with off periods according to an MMRP. The states of the underlying three–state continuous–time Markov chain, $\{\boldsymbol{X}(t)\}$, correspond either to the traffic class, $i \in \{1, 2\}$, of the current burst, or to an idle period represented by 0. Recall that x_m denotes the total time spent in state m and $x_0+x_1+x_2 = t$. If k is the initial state, then $x_k > 0$. We denote by $\rho_{k,l}^{AP}(x_k, x_{s_1}, x_{s_2})$, the joint probability density of the total sojourn time in each of the three states up to time t, when the initial state is k and the final state is l. When $k \neq 0$, we interchange

0 and k whenever they occur in the subscripts of $\rho_{0,l}^{AP}$ and we set $x_{s_k} = x_0$; when $k = 0$, $x_{s_i} = x_i$, for $i \in \{1, 2\}$. Movement among states is governed by the transition rate matrix, $\Lambda = (\lambda_{rs}, r, s \in \{0, 1, 2\})$, with $\lambda_{12} = \lambda_{21} = 0$.

The shared server switches from one queue to the other according to a switch rule chosen for that queue, which serves each at a constant rate, c. We shall assume that $c \leq r_1, r_2$. Following Yadin's lead we call the period during which queue I is served, task I and similarly for task II. While the Markov chain is in state i, the fluid rate is r_i, with $r_0 = 0$; the drift rates for queue I and queue II are $\delta_i^1 r_1 - c$ and $\delta_i^2 r_2$, respectively, for the duration of task I, and $\delta_i^2 r_2 - c$, $\delta_i^1 r_1$ for queue II and I, respectively, for the duration of task II.

We represent the current level in the two buffers by the point (u, v). We call the high–priority buffer, queue I. It is served according to the *zero switch rule*, determined by the absorbing set,

$$\{(u, v)|u = 0, v \geq 0\}.$$

Since $r_1 - c \geq 0$, we would expect the final state of task I, that is the state of the Markov process when absorption occurs, to be in $\{0, 2\}$ (when the drift rate is negative). Naturally, this state is also the initial state of task II. During task II, queue II is served according to the *constant queue switch rule*, which is defined by the absorbing set

$$\{(u, v)|0 \leq u \leq k, v = 0 \text{ or } u = k, v > 0\}.$$

The final state of task II (and the initial state of task I) is one of $\{0, 1\}$ if the first condition holds and 1 otherwise.

Denote by $\alpha_i(t|u_0, v_0)$ the probability density of the duration of task i, given that the initial buffer state is (u_0, v_0), and let π_m denote the probability that m is the initial state of the chain. Then, letting $l_1(t) = \frac{(r_1-c)t+u_0}{r_1}$, $l_2(t) = \min\left(\frac{k-u_0}{r_1}, \frac{(r_2-c)t+v_0}{r_2}\right)$ and $l_3(t) = \frac{r_1 t-k+u_0}{r_1}$, we obtain

$$
\begin{aligned}
\alpha_1(t|u_0, v_0) &= Pr\left\{[u_0 - ct + x_1 r_1] = 0, X(0) \in \{0,1\}, X(t) \in \{0,2\}\right\} \\
&= \pi_0 \int_0^{l(t)} \left[\rho_{0,0}^{AP}\left(x_0, \frac{ct-u_0}{r_1}, x_2\right) + \rho_{0,2}^{AP}\left(x_0, \frac{ct-u_0}{r_1}, x_2\right)\right] dx_2 \\
&+ \pi_1 \int_0^{l_1(t)} \left[\rho_{1,0}^{AP}\left(\frac{ct-u_0}{r_1}, x_0, x_2\right) + \rho_{1,2}^{AP}\left(\frac{ct-u_0}{r_1}, x_0, x_2,\right)\right] dx_2,
\end{aligned}
$$

and for task II,

$$
\begin{aligned}
\alpha_2(t|u_0, v_0) &= Pr\{[v_0 - ct + x_2 r_2] = 0, [u_0 + x_1 r_1] \le k, X(0) \in \{0,2\}, X(t) \in \{0,1\}\} \\
&+ Pr\{[u_0 + x_1 r_1] = k, X(0) \in \{0,2\}, X(t) = 1\} \\
&= \pi_0 \int_0^{l_2(t)} \left[\rho_{0,0}^{AP}\left(x_0, x_1, \frac{ct-v_0}{r_2}\right) + \rho_{0,1}^{AP}\left(x_0, x_1, \frac{ct-v_0}{r_2}\right)\right] dx_1 \\
&+ \pi_2 \int_0^{l_2(t)} \left[\rho_{2,0}^{AP}\left(\frac{ct-v_0}{r_2}, x_1, x_0\right) + \rho_{2,1}^{AP}\left(\frac{ct-v_0}{r_2}, x_1, x_0\right)\right] dx_1 \\
&+ \int_0^{l_3(t)} \left[\pi_0 \rho_{0,1}^{AP}\left(x_0, \frac{k-u_0}{r_1}, x_2\right) + \pi_2 \rho_{2,1}^{AP}\left(x_2, \frac{k-u_0}{r_1}, x_0\right)\right] dx_2.
\end{aligned}
$$

As remarked above, in those instances for which the initial state is not zero, we need to substitute all occurrences of 0 in the subscripts, with the label of the initial state, k. We denote by $\epsilon_k^{AP}(x_1, x_2, t)$ and ${}_kL_{m,n}^{AP}(x_1, x_2, t)$, the constituent functions of $\rho_{kl}^{AP}(x_k, x_{s_1}, x_{s_2})$. Since $\lambda_{12} = \lambda_{21} = 0$, then $y_{21} = y_{12} = 0$. Substituting for these values, we obtain these expressions,

$$
\epsilon_0^{AP}(x_1, x_2, t) = -(\lambda_{01} + \lambda_{02})x_0 - \lambda_{10}x_1 - \lambda_{20}x_2,
$$

and

$$
\begin{aligned}
{}_0L_{m,n}^{AP}(x_1, x_2, t) &= H(x_0) \sum_a \sum_b \frac{y_{10}^{a+m} y_{20}^{b+n} y_{01}^a y_{02}^b}{a!b!(a+m)!(b+n)!} \\
&= H(x_0) y_{10}^m y_{20}^n G_m(y_{01}y_{10}) G_n(y_{02}y_{20}),
\end{aligned}
$$

where $G_n(y)$ is as defined in Section 3.5.

The above expressions and the results of Chapter 3 give us

$$
\begin{aligned}
\rho_{0,0}^{AP}(x_0, x_1, x_2) &= e^{-\epsilon_0^{AP}(x_1, x_2, t)} H(x_0) H(\boldsymbol{x}) \left[\delta(\boldsymbol{x}) + \lambda_{01}\delta(x_2) y_{10} G_1(y_{10}y_{01})\right. \\
&+ \left. \lambda_{02}\delta(x_1) y_{20} G_1(y_{02}y_{20}) + \lambda_{01}\lambda_{02} y_{10} y_{20} G_1(y_{01}y_{10}) G_1(y_{02}y_{20})\right],
\end{aligned}
$$

$$\rho_{0,1}^{AP}(x_0, x_1, x_2) = e^{-\epsilon_0^{AP}(x_1,x_2,t)} H(x_0) H(\boldsymbol{x}) \left[\lambda_{01}\delta(x_2) G_0(y_{10}y_{01}) + \lambda_{01}\lambda_{02} y_{20} G_0(y_{01}y_{10}) G_1(y_{02}y_{20})\right],$$

$$\rho_{0,2}^{AP}(x_0, x_1, x_2) = e^{-\epsilon_0^{AP}(x_1,x_2,t)} H(x_0) H(\boldsymbol{x}) \left[\lambda_{02}\delta(x_1) G_0(y_{20}y_{02}) + +\lambda_{01}\lambda_{02} y_{10} G_1(y_{01}y_{10}) G_0(y_{02}y_{20})\right].$$

The component functions of $\rho_{k,l}(x_k, x_{s_1}, x_{s_2})$, for $k \in \{1, 2\}$, are obtained from $\epsilon_0^{AP}(x_1, x_2, t)$ and ${}_0L_{m,n}^{AP}(x_k, x_{s_1}, x_{s_2})$ by interchanging 0 and k.

In Section 3.6, we saw that the functions $G_n(y)$ can be coded as Bessel functions, and they are computationally feasible. It follows that the expressions for α_i can be similarly coded to yield a feasible numerical approximation.

5.3 Total work up to time t

Brandt and Brandt (see [4]) derive an expression for the distribution of the number of arriving packets, $N(t)$, approximated as the density of the amount of fluid or work, $A(t)$, in an interval of length t. They present results for a two-state Markov modulated rate process and a superposition of M such sources. Another way to derive the density of $A(t)$ is to calculate the density of the time spent in each state of the underlying two-state Markov process in $[0, t)$. This is equivalent to a two-state continuous-time random walk on a continuum where the time interval between state changes is a random variable which depends only on the state of the random walk. The amount of work up to time t has important applications.

Consider a typical ATM (Asynchronous Transfer Mode) network in which low priority traffic is transmitted at the available bit-rate. The higher priority traffic is typically transmitted at a varying rate which can be modelled as an underlying n–state Markov chain. Thus the residual bit–rate available for low priority traffic is also modulated by the same process. The time–dependent n–state model we have been studying is an ideal candidate for this system, especially over short periods of time during which the parameters of the Markov modulated process are unlikely to change.

One of the quality of service measurements required by communications managers is the time it takes to transfer a quantity of data (for example a file of size w bits) over a link. From the probability density of $A(t)$, the total work up to time t, we can calculate the probability that w bits can be transmitted in less than T seconds given that the usable rate in each state is known up to time $t > T$.

Another application for $A(t)$ is in window control mechanisms for packet–switched networks. In this scenario, packets are accepted for transmission only if if $A(t^*)$ is less than a certain nominated threshold; t^* needs to be chosen to reflect the reaction time of the system.

The third application we will consider arises in HF radio networks. The usable transmission rate on an HF radio link varies in a random fashion according to atmospheric conditions, the time of day and geographic location. At any point in time, it is only possible to predict the transmission rate on a given frequency over a short interval of time. When the errored bit–rate is high the usable transmission rate drops. Observations of the available bit–rate on HF links, linking nodes in the major cities in Australia, show that the usable transmission rate can be modelled with reasonable accuracy as an n–state Markov process in which each state represents a different rate. Extensive measurements are needed to infer accurate state transition rates for this process.

An increase in data throughput is achieved by the latest generation of HF data modems controlled by a purpose–designed Physical Automatic ReQuest protocol (PARQ) (see [16]). A typical modem operates at one of four rates: 300, 600, 1200 and 2400 bits per second. PARQ adapts the modem data rate to suit the prevailing link conditions. A data rate of 1200 bits per second is always selected at start–up. PARQ subsequently makes rate changes upwards, one level at a time, when conditions improve and downwards when they degrade. Rate changes are only implemented after a request by one end of the link is accepted by the other end. If communications cease after a request to change, the requesting end reverts to the suggested rate and awaits acknowledgement. If none arrives the communications link is deemed lost.

A mathematical model of this system is needed to predict the amount of data transmitted up to time t, given the current conditions. We will associate the four rates with states 0, 1, 2 and 3, in this order. State transition rates can be inferred from the protocol specification given such measurements as signal–to–noise ratios, which are relatively constant over a short time span. Since state changes only depend on the current state, the resulting data transmission process corresponds to a four–state Markov process. Furthermore, since only transitions to neighbouring states occur, the process is in fact a birth–and–death–process.

We looked at a three–state birth–and–death process in Section 3.5.1, so it will not be particularly illuminating to reproduce the mathematics for a four–state example. We may write the four–state version of $L^{BD}_{m,n,o}$ as

$$L^{BD}_{m,n,o}(\boldsymbol{x},t) = H(x_0)y_{10}^m(y_{10}y_{21})^n(y_{10}y_{21}y_{32})^o G_o(y_{32}y_{23})G_{n+o}(y_{21}y_{12})G_{m+n+o}(y_{10}y_{01}).$$

Once again the solution we are after is predominantly made up of the numerically friendly functions $G_n(y)$. Furthermore, it is evident from the form of L^{BD}_κ for the three– and four–state models that it is easy to deduce L^{BD}_κ for the n–state process. Let us denote this by $L^{BD}_\kappa(n,\boldsymbol{x},t)$. In this case, $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_{n-1})$.

Let $s(\kappa,i) = \sum_{j=i}^{n} \kappa_j$, then we have

$$L^{BD(n)}_\kappa(n,\boldsymbol{x},t) = H(x_0)\prod_{i=1}^{n-1} y_{i(i-1)}^{s(\kappa,i)} G_{s(\kappa,i)}(y_{i(i-1)}y_{(i-1)i}),$$

which is also computationally feasible.

We conclude with the solution for the n–state birth–and–death process, which we shall denote by $\rho^{BD}(n, \boldsymbol{x}, t)$. This closed–form solution is of a much simpler form than the solution for the general n–state process. For the interior solution, we have just one coefficient which is not zero. All spanning trees with edges other than the ones going from k to $k+1$, for each k such that $0 \le k \le n-1$, feature at least one edge represented by a λ_{ij} which equals zero. The boundary terms are similarly restricted: if state i is never visited in the interval up to time t, then the process never makes a transition to states $i+1, i+2, \cdots, n-1$. Let τ_m denote the spanning tree with edges $01, 12, 23, \cdots, (m-1)m$, and represented by the coefficient $\lambda_{01}\lambda_{12}\cdots\lambda_{(m-1)m}$; $\boldsymbol{e}_m$ denotes the vector with unity in the mth position and zero elsewhere, with $\boldsymbol{e}_0$ defined as the zero vector. As in the last chapter, we will conveniently let τ_m denote either the tree or the coefficient depending on the context. It is easy to verify that for n states ρ^{BD} is given by

$$\rho^{BD}(n, \boldsymbol{x}, t) = \exp(\epsilon^{BD}(n, \boldsymbol{x}, t))H(\boldsymbol{x}) \left[\delta(\boldsymbol{x}) + \sum_{m=1}^{n-1} \sum_{k=0}^{m} \tau_m \prod_{j=m+1}^{n-1} \delta(x_j) L^{BD(m)}_{\boldsymbol{e}_m - \boldsymbol{e}_k} \right],$$

where

$$\epsilon^{BD}(n, \boldsymbol{x}, t) = -\lambda_{01}x_0 - \sum_{i=1}^{n-2} \left(\lambda_{i(i-1)} + \lambda_{i(i+1)}x_i \right) - \lambda_{(n-1)(n-2)}x_{n-1}.$$

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