The University of Adelaide
Department of Applied Mathematics
Faculty of Mathematical and Computer Sciences

Jadranka Šunde

# Pólya–type inequalities

Thesis submitted for the degree of
Doctor of Philosophy
at The University of Adelaide

Adelaide, October, 1997.

# SIGNED STATEMENT

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

SIGNED.................................................................DATE....3/3/98.........

# ACKNOWLEDGMENTS

**Acknowledgments**

Special thanks are due to my supervisors,

Assoc. Prof. Charles Pearce who taught me mathematics andante, for his willingness to share his great experience in interpretation, as well as acquainting me with style,

Prof. Josip Pečarić who taught me mathematics prestissimo, for his everflowing stream of ideas, optimism and endless energy.

To my family, Vido and Matea, for encouragement and support.

**Zahvala**

Posebnu zahvalu dugujem mojim mentorima,

Prof. Josipu Pečariću koji je uvijek imao nepresušnu zalihu novih ideja, optimizma i beskrajne energije,

Assoc. Prof. Charlesu Pearceu koji mi je pomogao u primjenama materijala iz ove teze i pokazao da se i u matematici može imati stila.

Mojoj obitelji, Vidi i Matei, zahvaljujem na strpljenju i potpori.

# Contents

# 1. INTRODUCTION

Gauss' inequality, dating from 1821, is one of the most seminal in mathematics, as we shall see with our somewhat encyclopaedic Chapter 2 on history. Two major lines of generalization have come from it, one due to Winckler (but not proven correctly until decades later by Faber), and the other springing from a pair of results of Pólya. Both carry the probabilistic interpretation of the original Gauss result.

In 1990, Alzer discovered a surprising and elegant way to generalize one of the two Pólya results. This has stimulated fresh work by Pečarić, Varošanec and Pearce, who have found a variety of extensions. The generality of these ideas is shown by the fact that there exist also operator versions of at least some of them, as demonstrated by recent work with Mond.

This thesis consists of six chapters.

After an introductory Chapter 1, Chapter 2 presents an historical overview of the subject.

Chapter 3 deals with generalizations of Gauss-Pólya inequalities, and inequalities involving means (weighted, quasiarithmetic and logarithmic).

Chapter 4 concentrates on further generalizations of results given in Chapter 3, involving Stolarsky and Gini means. Integral and summation results are also given, as well as results involving generalized quasiarithmetic means and some further generalizations.

Chapter 5 contains operator versions of a number of classical inequalities with special attention being given to Pólya inequalities for positive linear operators.

Chapter 6 looks at Abel type inequalities with application to the Gauss-Pólya results.

The reader will note that to avoid undue traditionalism and at the same time honour my teachers and heritage, I use Čebyšev as a Romanized form rather than the nineteenth century Tchebychev or Tchebycheff.

As we will repeatedly make use of the Jensen and Jensen-Steffensen inequalities, this is a good place to remind the reader of them.

**Jensen inequality** *If $f$ is a convex function on an interval $I \subset R$, $x = (x_1, \ldots, x_n) \in I^n$, $(n \geq 2)$ and $p$ is a positive $n$-tuple $\left(P_k = \sum_{i=1}^{k} p_i\right)$, then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{1.1}$$

*If $f$ is strictly convex, then inequality (1.1) is strict except when $x_1 = \cdots = x_n$.*

**Jensen–Steffensen inequality** *If $f : I \to R$ is a convex function, $x$ is a real monotone $n$-tuple such that $x_i \in I$, $(i = 1, \ldots, n)$, and $p$ is a real $n$-tuple such that*

$$0 \leq P_k \leq P_n \quad (1 \leq k \leq n), \qquad P_n > 0,$$

*then (1.1) holds. If $f$ is strictly convex, then the inequality (1.1) is strict except when $x_1 = \cdots = x_n$.*

For details on the above inequalities and reverse results, (see [26],p. 6). The thesis contains many new results which complement and generalize established ones. Several papers have been accepted by or submitted to journals for publication (see [9], [10], [30], [31], [32], [36], [38], [39], [40], [59], [60] in the consolidated list of references).

# 2. HISTORY

**2.1 The Gauss inequality.** The Gauss inequality concerns the absolute moments of a probability distribution with nonnegative support. Suppose the distribution function involved is denoted by $Q(\cdot)$. That $Q$ is a distribution function with nonnegative support means that $Q : [0, \infty) \to [0, 1]$ is a nondecreasing function such that $Q(0) = 0$ and $\lim_{x\to\infty} Q(x) = 1$.

The $r$-th absolute moment is defined for $r \geq 0$ by

$$\nu_r = \int_0^\infty x^r \, dQ(x). \tag{2.1.1}$$

In the limiting case $r = 0$ we have $\nu_0 = 1$.

A variety of interesting results exist connecting absolute moments. Thus the fact that the variance of a probability distribution is nonnegative may be expressed as

$$\nu_2 \geq \nu_1^2. \tag{2.1.2}$$

This is in fact the simplest case of the fundamental inequality for power means, which states that

$$\nu_n^{1/n} \leq \nu_r^{1/r} \quad \text{for} \quad n \leq r. \tag{2.1.3}$$

The primitive result (2.1.2) is at heart a manifestation of Jensen's inequality. Suppose $X$ is a random variable, $g$ the convex function given by $g(x) = x^2$ ($x \in R$) and

$E$ expectation. Then (2.1.2) may be expressed as

$$E(g(X)) \geq g(E(X)).$$

In the case when $Q'$ is continuous and nonincreasing on $(0, +\infty)$, that is, the distribution has a nonincreasing density function, Gauss [15] gave without proof the improvement

$$\nu_4 \geq \frac{9}{5}\nu_2^2 \tag{2.1.4}$$

for (2.1.2).

**2.2 Gauss-Winckler Inequality.** A generalization of (2.1.3) was given by Winckler [63].

If $Q'$ is continuous and nonincreasing on $(0, +\infty)$, then

$$((n+1)\nu_n)^{1/n} \leq ((r+1)\nu_r)^{1/r} \quad \text{for} \quad n \leq r. \tag{2.2.1}$$

This subsumes (2.1.4) in the case $n = 2$, $r = 4$. Winckler obtained (2.2.1) by an invalid argument. Faber published the first proof in [11]. Another proof of the Gauss-Winckler inequality was given by M. Fujiwara [14], while S. Narumi [33] gave a generalization of (2.2.1).

Other proofs of (2.1.2) and (2.2.1) were given by F. Bernstein and M. Krafft [5], S. Izumi [17] and M. Krafft [18].

**2.3 Pólya Inequalities.** In the book "Problems and Theorems in Analysis I, II" by G. Pólya and G. Szego [49], Pólya gave two theorems which were to become seminal.

**Theorem 2.3.1.** *Let the function $f : [0,1] \to R$ be nonnegative and increasing. If $a$ and $b$ are nonnegative real numbers, then*

$$\left(\int_0^1 x^{a+b} f(x)dx\right)^2 \geq \left(1 - \left(\frac{a-b}{a+b+1}\right)^2\right) \int_0^1 x^{2a} f(x)dx \int_0^1 x^{2b} f(x)dx. \tag{2.3.1}$$

**Theorem 2.3.2.** *Let the function $f : [0,\infty) \to R$ be nonnegative and decreasing. If $a$ and $b$ are nonnegative real numbers, then*

$$(2.3.2) \quad \left(\int_0^\infty x^{a+b} f(x)dx\right)^2 \leq \left(1 - \left(\frac{a-b}{a+b+1}\right)^2\right) \int_0^\infty x^{2a} f(x)dx \int_0^\infty x^{2b} f(x)dx$$

*whenever the integrals exist.*

It is, of course, implicit in both theorems that $f$ is Lebesgue integrable. With applications in mind, we remark that this will be the case if $f$ is continuous.

Theorem 2.3.2 is a generalization of the Gauss inequality (2.1.4), which arises as the special case $b = 0$, $a = 2$.

Let us pause to consider the significance of these theorems. Theorem 2.3.2 is closer to (2.2.1) so we address it first.

We may divide both sides of (2.3.2) by $\left[\int_0^\infty f(x)dx\right]^2$. Now set

$$\overline{f}(x) = f(x) / \int_0^\infty f(x)dx.$$

Then (2.3.2) can be written as

$$\left(\int_0^\infty x^{a+b} \overline{f}(x)dx\right)^2 \leq \left(1 - \left(\frac{a-b}{a+b+1}\right)^2\right) \int_0^\infty x^{2a} \overline{f}(x)dx \int_0^\infty x^{2b} \overline{f}(x)dx.$$

This is of the same form as (2.3.2) but has $\overline{f}$ in place of $f$. Now $\overline{f}$ is nonegative and satisfies

$$\int_0^\infty \overline{f}(x)dx = 1,$$

so $\overline{f}$ represents a probability density function. Thus without loss of generality the function $f$ in (2.3.2) may be interpreted as a probability density function $Q'$. If we assume continuity of $Q'$ as noted above, then the conditions are just those of

Gauss–Winckler and moreover the result of the theorem may be written compactly as

$$\nu_{a+b}^2 \leq \left(1 - \left(\frac{a-b}{a+b+1}\right)^2\right) \nu_{2a}\nu_{2b}. \tag{2.3.3}$$

Theorem 2.3.2 may thus be viewed as a sort of three–parameter version of the Gauss-Winckler result.

Suppose we set $b = 0$ in this result. Then since $\nu_0 = 1$, (2.3.3) simplifies to

$$[(a+1)\nu_a]^2 \leq (2a+1)\nu_{2a}.$$

Taking $2a$-th roots yields

$$[(a+1)\nu_a]^{1/a} \leq [(2a+1)\nu_{2a}]^{1/(2a)}$$

which is, of course, just (2.2.1) for the case $n = a$, $r = 2a$.

The original Gauss result (2.1.4) is recovered when we further restrict to $a = 2$.

Thus Theorem 2.3.2 is a natural three-parameter offspring from the original Gauss result that reduces to a special case of the Gauss-Winckler result as a two-parameter specialisation.

The other obvious specialisation $a = b$ gives only the tautology

$$\nu_{2a}^2 \leq \nu_{2a}^2.$$

We now turn to Theorem 2.3.1.

Again we may divide both sides in (2.3.1) by $[\int_0^1 f(x)dx]^2$ etc. as argued above to show that without loss of generality we can take

$$\int_0^1 f(x)dx = 1,$$

that is, we may assume $f$ is a probability density function on [0,1]. Here the density is increasing, which represents a breakaway from Gauss. It is because

$$\int_0^\infty f(x)dx = 1$$

cannot occur for an everywhere increasing function $f$ that the theorem for $f$ increasing has to be for $f$ over a finite interval only. A simple change of scale reduces the interval to $[0, 1]$, which is thus a standardised format for the general case.

For a distribution $Q$ with finite support $[0, 1]$, the $r$-th absolute moment as given by (2.1.1) becomes

$$\nu_r = \int_0^1 x^r dQ(x),$$

so that (2.3.1) can now be recast as

$$\nu_{a+b}^2 \geq \left(1 - \left(\frac{a-b}{a+b+1}\right)^2\right) \nu_{2a}\nu_{2b}.$$

This is just the reverse inequality to (2.3.3).

It is now clear that the two Polya inequalities, although couched abstractly, are in fact elegant probabilistic inequalities covering the important cases of distributions with nonnegative support and respectively increasing and decreasing density functions. This probabilistic interpretation has largely been lost sight of in the literature but is not far from the surface. It is tedious to draw it out with every result in this thesis, but we shall occasionally refer to it lest it be overlooked.

**2.4. Volkov's Inequalities.** V.N. Volkov [62] proved a general result and obtained the following special cases.

1) If $a \geq 0, b \geq 0, p > 1, p^{-1} + q^{-1} = 1$, and $g$ is a nonnegative and decreasing function, then

$$\int_0^\infty x^{a+b} g(x)dx \leq c\left(\int_0^\infty x^{ap} g(x)dx\right)^{1/p} \left(\int_0^\infty x^{bq} g(x)dx\right)^{1/q}, \tag{2.4.1}$$

where $c = \dfrac{(ap+1)^{1/p}(bq+1)^{1/q}}{(1+a+b)}$.

2) If $a, b, p, q$ are defined as in 1) and if $g$ is a nonnegative, nonincreasing convex function, then (2.4.1) holds with constant

$$c = \frac{((ap+1)(ap+2))^{1/p}((bq+1)(bq+2))^{1/q}}{(1+a+b)(2+a+b)}.$$

Generalizations of Volkov's general result are obtained by Mitrinović and Pečarić [25]. A further generalization was obtained by Varošanec [54].

Since $p^{-1}+q^{-1}=1$, we may divide through both sides of (2.4.1) to obtain the same result with $g$ replaced by

$$\overline{g}(x) = g(x)/\int_0^\infty g(x)dx.$$

Since

$$\int_0^\infty \overline{g}(x)dx = 1,$$

we again may without loss of generality interpret $g$ as a probability density function and rewrite (2.4.1) as

$$\nu_{a+b} \leq c\nu_{ap}^{1/p}\nu_{pq}^{1/q}.$$

**Theorem 2.4.1.** *Let $f_i : [0,\infty) \to \mathbf{R}$, $i = 1,2,\ldots,2n$, be nonnegative functions and $f : [0,\infty) \to \mathbf{R}$ defined by*

$$f(x) = \int_0^\infty K(x,t)dh(t),$$

*where $K(x,t) \geq 0$ for $x,t \in \mathbf{R}^+$ and $h$ is an increasing function. Let $p_i$, $i = 1,2,\ldots,n$ be positive numbers such that $\sum_{i=1}^n \frac{1}{p_i} = 1$ and $Kf_1\ldots f_n$, $Kf_{n+j} \in \mathcal{L}^1([0,\infty)^2, \mu_h \times \lambda)$ $(j = 1,\ldots,n)$. Then*

$$\int_0^\infty \prod_{j=1}^n f_j(x)f(x)dx \leq C\prod_{j=1}^n \left(\int_0^\infty f_{n+j}(x)f(x)dx\right)^{1/p_j},$$

*where*

$$C = \sup_t \left\{ \frac{\int_0^\infty K(x,t)f_1(x)\ldots f_n(x)dx}{\prod_{j=1}^n \left(\int_0^\infty K(x,t)f_{n+j}(x)dx\right)^{1/p_j}} \right\}.$$

**2.5. Generalizations of the Gauss-Winckler Inequality.** P.R. Beesack [4] proved implicitly the following result.

**Theorem 2.5.1.** *If* $(-1)^{k-1}Q^{(k)}$ *is positive, continuous and decreasing on* $(0,+\infty)$ *for* $k = 1,2,\ldots,n$, *then* $f_k(r) = \log((r+k)\,\nu_r)$ *is a convex function for* $k = 1,\ldots,n$.

This leads to the following generalization of Gauss-Winckler's inequality (Mitrinović and Pečarić [24]).

$$\left(\binom{n+k}{k}\nu_n\right)^{1/n} \le \left(\binom{r+k}{k}\nu_r\right)^{1/r} \quad (n \le r). \tag{2.5.1}$$

It leads also to the following results.

If $m \le n \le r$, then

$$\left(\binom{n+k}{k}\nu_n\right)^{r-m} \le \left(\binom{m+k}{k}\nu_m\right)^{n-r} \cdot \left(\binom{r+k}{k}\nu_r\right)^{n-m}.$$

If $m \le n$ and $r \le s$ then

$$\left(\binom{r+k}{k}\nu_r/\binom{m+k}{k}\nu_m\right)^{1/(r-m)} \le \left(\binom{s+k}{k}\nu_s/\binom{n+k}{k}\nu_n\right)^{1/(s-n)}.$$

Moreover for $Q$ defined and nondecreasing on $[0,\alpha]$ $(0 < \alpha \le +\infty)$ , $Q(0) = 0$, $Q(\alpha) = 1$, and

$$\nu_r = \int_0^\alpha t^r dQ(t), \tag{2.5.2}$$

D.S. Mitrinović and J. Pečarić [24] in the same paper proved the following result.

**Theorem 2.5.2.** *Let* $f : [0,1] \to R$ *be a nondecreasing positive function. If the function* $x \mapsto f(Q(x))/x$ *is nondecreasing, then*

$$\nu_r^{1/r}/\nu_n^{1/n} \le \left(\int_0^1 f(t)^r dt\right)^{1/r} / \left(\int_0^1 f(t)^n dt\right)^{1/n} \quad (n \le r). \tag{2.5.3}$$

*If the function* $x \mapsto f(Q(x))/x$ *is nonincreasing, then the reverse inequality holds.*

**2.6. Alzer's Inequality.** H. Alzer [1] gave the following generalization of Polya's inequality (2.3.1).

*Let $f : [a,b] \to R$ be nonnegative and increasing and let $g : [a,b] \to R$ and $h : [a,b] \to R$ be nonnegative and increasing functions with a continuous first derivative. If $g(a) = h(a)$ and $g(b) = h(b)$, then*

$$(2.6.1) \qquad \left(\int_a^b \left(\sqrt{g(x)h(x)}\right)' f(x)dx\right)^2 \geq \int_a^b g'(x)f(x)dx \int_a^b h'(x)f(x)dx.$$

The introduction of the derivatives is novel. This motif runs through much of this thesis. Where did this idea come from? That it is a natural (and simple) progression from Pólya's inequalities can be seen as follows.

First, rewrite (2.3.1) as

$$\left(\int_0^1 (a+b+1)x^{a+b}f(x)dx\right)^2 \geq \int_0^1 (2a+1)x^{2a}f(x)dx \int_0^1 (2b+1)x^{2b}f(x)dx.$$

This can be expressed as

$$\left(\int_0^1 \left(\frac{d}{dx}x^{a+b+1}\right) f(x)dx\right)^2 \geq \int_0^1 \left(\frac{d}{dx}x^{2a+1}\right) f(x)dx \int_0^1 \left(\frac{d}{dx}x^{2b+1}\right) f(x)dx.$$

Now observe that

$$x^{a+b+1} = \sqrt{x^{2a+1}x^{2b+1}}.$$

This immediately suggests

$$\left(\int_0^1 \left(\frac{d}{dx}\sqrt{gh}\right) f(x)dx\right)^2 \geq \int_0^1 \left(\frac{dg}{dx}\right) f(x)dx \int_0^1 \left(\frac{dh}{dx}\right) f(x)dx,$$

which is simply a standardization of (2.6.1). The relevant properties of $g(x) = x^{2a+1}$, $h(x) = x^{2b+1}$ on $[0,1]$ that need to be carried over turn out to be $g(0) = h(0)$, $g(1) = h(1)$ and $g, h$ increasing and nonnegative with continuous first derivatives.

**2.7. Improvement of Polya's Inequality.** A. M. Fink and M. Jodeit Jr. [12] showed that inequality (2.3.1) holds not only for nonnegative $a$ and $b$, but for $a, b$ greater than $-1/2$.

In fact they proved that (2.3.1) can be written as

$$(2.7.1)\qquad \begin{aligned}&(a+b+1)^2\int_0^1\int_0^1 f(x)f(y)x^{a+b}y^{a+b}dxdy\\ &\geq \frac{(2a+1)(2b+1)}{2}\int_0^1\int_0^1 f(x)f(y)[x^{2a}y^{2b}+x^{2b}y^{2a}]dxdy.\end{aligned}$$

Using the idea of their proof, J. Pečarić ([26], p. 261) noted that the product $f(x)f(y)$ can be replaced by a function $f(x,y)$ whose partial derivatives $f_1$, $f_2$, and $f_{12}$ are nonnegative. See also [46].

A further generalization was obtained by S. Varošanec and J. Pečarić [55].

**Theorem 2.7.1.** *Let $n$ be an even natural number, $a,b>0$ and $f:[0,1]\times[0,1]\to\mathbf{R}$ function with continuous partial derivatives $f_1,f_2,f_{12}$ such that $f_1(x,0)\geq 0, f_2(0,y)\geq 0$ and $f_{12}(x,y)\geq 0$ for all $x,y\in[0,1]$. Then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^1\int_0^1 f(x^{\frac{1}{(n-k)a+kb}},y^{\frac{1}{(n-k)b+ka}})dxdy\leq 0\ .$$

**2.8. Stolarsky's Inequality.** K.B. Stolarsky [52] proved the following result.

*If $g$ is a nonnegative and nonincreasing function on $[0,1]$, then for all positive numbers $a$ and $b$ we have*

$$(2.8.1)\qquad (a+b)g(0)\int_0^1 x^{a+b-1}g(x)dx\geq ab\int_0^1 x^{a-1}g(x)dx\int_0^1 x^{b-1}g(x)dx.$$

Moreover, J. Pečarić [45] proved that if $g$ is a nonnegative nondecreasing function on $[0,1]$, then the inequality in (2.8.1) is reversed. Pečarić [44] also gave a generalization of (2.8.1) including several constants and integrals.

**2.9. Generalizations of Alzer's Inequality.** J. Pečarić and S. Varošanec [47] proved the following two theorems.

**Theorem 2.9.1.** *Let $f : [a,b] \to R$ be nonnegative and increasing, and let $x_i : [a,b] \to R$, $i = 1,\dots,n$ be nonnegative increasing functions with continuous first derivatives. If $p_i, i = 1,\dots,n$ are positive real numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, then*

$$\int_a^b \left(\prod_{i=1}^{n} (x_i(t))^{1/p_i}\right)' f(t)dt \geq \prod_{i=1}^{n} \left(\int_a^b x_i'(t)f(t)dt\right)^{1/p_i}. \tag{2.9.1}$$

*If $x_i(a) = 0$ for all $i = 1,\dots,n$ and if $f$ is a decreasing function, then the reverse inequality holds.*

We are now perhaps sufficiently far from our starting point that a further reference to probabilities may not be unwelcome. We offer a physical interpretation for (2.9.1).

Imagine a collection of $n$ physical quantities $x_i(t)(i = 1,\dots,n)$ varying with time $t$, which runs from $a$ to $b$. A time point is chosen in accordance with the density function

$$f(t)/\int_a^b f(t)dt$$

on $[a,b]$. The numbers $1/p_i$ are regarded as probability weights. Then the expression in large parentheses on the left in (2.9.1) is the weighted geometric mean of the values of our quantities taken at time $t$, while that on the right is the expected value of the derivative of the $i$-th quantity. Inequality (2.9.1) thus states that the average of the derivative of the weighted geometric mean of the quantities exceeds the weighted geometric mean of the average of the derivatives of those quantities. In terms of applicability, we have come a long way from a comparison of two moments of a single random variable, which is where we started.

**Theorem 2.9.2.** *Let $f : [a,b] \to R$ be nonnegative and decreasing, and let $x_i : [a,b] \to R$, $i = 1,\dots,n$ be nonnegative increasing functions with a continuous first derivatives and $x_i(a) = 0$ for all $i = 1,\dots,n$. If $p_i(i = 1,\dots,n)$, are positive real numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, then*

$$\int_a^b \left(\prod_{i=1}^{n} (x_i(t))^{1/p_i}\right)' f(t)dt \leq \prod_{i=1}^{n} \left(\int_a^b x_i'(t)f(t)dt\right)^{1/p_i}. \tag{2.9.2}$$

S. Varošanec [53] also gave a generalization of Alzer's inequality (2.6.1) in which instead of a geometric mean we may have power means of arbitrary orders.

**2.10. Gauss-Winckler and Stolarsky's Inequalities.** J. Pečarić and S. Varošanec [48] have used (2.5.1) and (2.5.3) in proofs of the following results of Stolarsky type.

**Theorem 2.10.1.** *Let $Q$ be a probability distribution function with $Q(x) = 0$ for $x \leq 0$, $\lim_{x\to\infty} Q(x) = 1$ and suppose $(-1)^{k-1}Q^{(k)}$ is positive, continuous and decreasing on $(0,\infty)$ for $k = 1, 2, \ldots, n$. If $r_1, \ldots, r_n > 0$, then*

$$\binom{r_1 + \cdots + r_n + k}{k} \nu_{r_1+\cdots+r_n} \geq \binom{r_1 + k}{k} \cdots \binom{r_n + k}{k} \nu_{r_1} \cdots \nu_{r_n}$$

*for $k = 1, 2, \ldots, n$, where $\nu_r$ is defined by (2.1.1).*

**Theorem 2.10.2.** *Let $f : [0,1] \to R$ be a nondecreasing positive function. If the function $x \mapsto f(Q(x))/x$ is nondecreasing and $r_1, \ldots, r_n > 0$, then*

$$\frac{\nu_{r_1+\cdots+r_n}}{\nu_{r_1} \cdots \nu_{r_n}} \leq \frac{\int_0^1 (f(x))^{r_1+\cdots+r_n}\,dx}{\int_0^1 (f(x))^{r_1}\,dx \cdots \int_0^1 (f(x))^{r_n}\,dx},$$

*where $\nu_r$ is defined by (2.5.2). If the function $x \to f(Q(x))/x$ is nonincreasing, then the reverse inequality applies.*

**2.11. Generalizations of Stolarsky's Inequality.** A generalization of Stolarsky's inequality (2.8.1) which has general weights was given by L. Maligranda, J. Pečarić and L.E. Persson [23].

Let us define a ratio

$$Q(g, w) = \frac{\int_0^1 g(x)w(x)dx}{\int_0^1 w(x)dx},$$

where $w \in \mathcal{L}^1([0,1],\lambda)$ is a nonnegative weight function and $g$ a function of bounded variation such that $gw \in \mathcal{L}^1([0,1],\lambda)$ . If $w_1$, $w_2$, $w_3$ are weight functions, we introduce

$$W_i(x) = \frac{\int_0^x w_i(t)dt}{\int_0^1 w_i(t)dt} \quad , \; i = 1,2,3.$$

**Theorem 2.11.1.** *Let $g$ be a function of bounded variation such that $0 \leq g(1) \leq g(x) \leq g(0)$ for all $x \in [0,1]$ and let*

$$W_1(x) \cdot W_2(x) = W_3(x) \quad \text{for all } x \in [0,1].$$

*Then*

$$g(0) \cdot Q(g,w_3) \geq Q(g,w_1)Q(g,w_2).$$

A modified version of this result and a similar generalization of Pečarić's reverse result was given by S. Varošanec [53].

**Theorem 2.11.2.** *Suppose $f : [a,b] \to$ is a function of bounded variation such that $0 \leq f(b) \leq f(x) \leq f(a)$ for all $x \in [a,b]$. If $g, h[a,b] \to R$ are nonnegative nondecreasing function with continous first derivatives and $g(a) = h(a) = 0$, then*

$$f(a)\int_a^b (g(t)h(t))' f(t)dt \geq \int_a^b g(t)' f(t)dt \int_a^b h(t)' f(t)dt. \tag{2.11.1}$$

*If $0 \leq f(a) \leq f(x) \leq f(b)$, the inequality is reversed.*

**2.12. Inequalities for Concave Functions.** S. Varošanec and J. Pečarić [57] proved the following.

**Theorem 2.12.1.** *If $f$ is a nonnegative differentiable function on $[0,1]$ with nonincreasing first derivative, then the function $r \mapsto \binom{r+2}{2}\int_0^1 x^r f(x)dx$ is log-concave.*

Many inequalities arise as simple consequences of well-known inequalities for concave functions. For example, Jensen's inequality gives the reverse inequality to (2.4.1) with integrals on [0,1], where the constant $c$ is defined as in 2) of 2.4.

In [57] some generalizations of inequalities of Gauss type are obtained involving $(r+1)\int_a^b g(x)^r f(x)dx$.

**2.13. Inequalities Involving Derivatives of Higher Order.**

S. Varošanec and J. Pečarić [58] have proved the following results.

**Theorem 2.13.1.** *Let $f, x_i : [a,b] \to R, i = 1,\ldots,m$, be nonnegative functions with continuous derivatives of the n-th order, $n \geq 2$, which satisfy the conditions:*

$1^\circ$ $(-1)^n f^{(n)}(t) \geq 0$ *and* $x_i^{(n)}(t) \geq 0$ *for all* $t \in [a,b], i = 1,\ldots,m$;

$2^\circ$ $(-1)^k f^{(k)}(b) \geq 0$ *for* $k = 0,1,\ldots,n-1$;

$3^\circ$ $x_i^{(k)}(a) = 0$ *and* $x_i^{(k)}(b) \geq 0$ *for* $k = 0,1,\ldots,n-1$ *and* $i = 1,\ldots,m$.

*If* $p_i, i = 1,\ldots,m$, *are positive numbers such that* $\sum_{i=1}^m 1/p_i = 1$, *then*

$$\int_a^b \left(\prod_{i=1}^m x_i^{1/p_i}(t)\right)^{(n)} f(t)dt \leq \prod_{i=1}^m \left(\int_a^b x_i^{(n)}(t) f(t)dt\right)^{1/p_i} + \Delta, \tag{2.13.1}$$

*where*

$$\Delta = \sum_{k=0}^{n-2} (-1)^k f^{(k)}(t) \left( \left(\prod_{i=1}^m x_i^{1/p_i}(t)\right)^{(n-k-1)} - \prod_{i=1}^m \left(x_i^{(n-k-1)}(t)\right)^{1/p_i} \right)\Bigg|_{t=b}.$$

**Theorem 2.13.2.** *Let $f, x_i : [a,b] \to R, i = 1,\ldots,m$, be nonnegative functions with a continuous derivative of the n-th order, $n \geq 2$, which satisfy the conditions:*

$1^\circ$ $(-1)^n f^{(n)}(t) \leq 0, x_i^{(n)}(t) \geq 0, f(b) > 0$ *for all* $t \in [a,b], i = 1,\ldots,m$;

$2^\circ$ $(-1)^k f^{(k)}(b) \leq 0$ *for every* $k = 1,\ldots,n-1$;

$3^\circ$ $x_i^{(k)}(b) \geq 0$ *and* $x_i^{(k)}(a) = 0$ *for* $i = 1,\ldots,m$ *and* $k = 0,1,\ldots,n-1$.

*Then the inequality (2.13.1) is reversed.*

**Theorem 2.13.3.** *Let $f, x_i : [a,b] \to R, i = 1, \dots, m$, be nonnegative functions with continuous derivatives of the n-th order such that $(-1)^{n-1} f^{(n)}$, $\left(\prod_{i=1}^m x_i^{1/p_i}\right)^{(n)}$ and $x_i^{(n)}, i = 1, \dots, m$, are nonnegative continuous functions. Then*

$$\int_a^b \left(\prod_{i=1}^m x_i^{1/p_i}(t)\right)^{(n)} \geq \prod_{i=1}^m \left(\int_a^b x_i^{(n)}(t) f(t) dt\right)^{1/p_i} + \Delta_1,$$

*where*

$$\Delta_1 = \sum_{k=0}^{n-1} (-1)^{n-k-1} f^{(n-k-1)}(t) \left(\sum_{i=1}^m \frac{1}{p_i} x_i^{(k)}(t) - \left(\prod_{i=1}^m x_i^{1/p_i}(t)\right)^{k}\right)\Bigg|_a^b .$$

**2.14. Inequalities of Minkowski Type.** S. Varošanec [56] has proved the following result.

**Theorem 2.14.1.** *Let $f : [a,b] \to R$ be a nonnegative and nondecreasing function, and $x_i : [a,b] \to R$, $(i = 1, \dots, n)$, nonnegative and nondecreasing functions with continuous first derivative. If $p > 1$, then*

$$\left(\int_a^b \left(\left(\sum_{i=1}^n x_i(t)\right)^p\right)' f(t) dt\right)^{1/p} \geq \sum_{i=1}^n \left(\int_a^b \left(x_i^p(t)\right)' f(t) dt\right)^{1/p}. \tag{2.14.1}$$

*If $f$ is a nonincreasing function and $x_i(a) = 0$ for all $i = 1, \dots, n$, then the reverse inequality applies.*

Results involving derivatives of higher order have also been given.

**2.15. Pearce, Pečarić and Varošanec inequalities.** The following results are given in [37].

**Theorem 2.15.1.** *Let $f, g, \varphi : [a,b] \to \mathbf{R}$ be nonnegative functions with $\varphi$ nondecreasing and possessing a continuous first derivative. Further let $p, q$ be real numbers satisfying $p + q = 1$.*

*(a) If $\varphi$ is nondecreasing and $p, q > 0$, then*

$$\int_a^b [(f^p(x) \pm g^p(x))] [(f^q(x) \pm g^q(x))]' \varphi(x) dx$$
$$\gtreqless \left[ \left( \int_a^b f'(x)\varphi(x)dx \right)^p \pm \left( \int_a^b g'(x)\varphi(x)dx \right)^p \right]$$
$$\times \left[ \left( \int_a^b f'(x)\varphi(x)dx \right)^q \pm \left( \int_a^b g'(x)\varphi(x)dx \right)^q \right].$$

*Here the convention is that the greater than or equal possibility is associated with taking the plus throughout and the less than or equal with the minus.*

*(b) If $\varphi$ is a nonincreasing function, $p, q > 0$ and $f(a) = g(a) = 0$, the inequality is reversed.*

*(c) If $\varphi$ is nondecreasing and $pq < 0$, the inequality is reversed.*

**Theorem 2.15.2.** *Let $p$ and $q$ be real numbers such that $p + q = 1$ and let $\varphi, f, g : [a, b] \to \mathbf{R}$ be nonnegative functions with continuous derivatives of $n$-th order and properties (1)–(4) below.*

1. $(-1)^n \varphi^{(n)} < 0$, $f^{(n)} > 0$, $g^{(n)} > 0$, $(f^p \pm g^p)(f^q \pm g^q)^{(n)} > 0$;
2. $(-1)^k \varphi^{(k)}(b) < 0$ *for* $k = 1, 2, \ldots, n-1$, $\varphi(b) > 0$;
3. $f^{(k)}(a) = g^{(k)}(a) = 0$ *for* $k = 0, 1, \ldots, n-1$ *and for* $n \geq 2$;
4. $f^{(k)}(b) > 0$, $g^{(k)}(b) > 0$ *for* $k = 0, 1, \ldots n-1$.

*We have the following.*

*(a) If $p$ and $q$ are positive numbers, then*

$$\int_a^b ((f^p(x) \pm g^p(x))(f^q(x) \pm g^q(x)))^{(n)} \varphi(x) dx$$

$$\gtreqless \Delta + \left[ \left( \int_a^b f^{(n)}(x)\varphi(x)dx \right)^p \pm \left( \int_a^b g^{(n)}(x)\varphi(x)dx \right)^p \right] \tag{2.15.1}$$

$$\times \left[ \left( \int_a^b f^{(n)}(x)\varphi(x)dx \right)^q \pm \left( \int_a^b f^{(n)}(x)\varphi(x)dx \right)^q \right],$$

*where*

$$\Delta = \begin{cases} 0 & \text{for } n = 1 \\ \sum_{k=0}^{n-1} (-1)^k \varphi^{(k)}(b) \Big[ ((f^p \pm g^p)(f^q \pm g^q))^{(n-k-1)}(b) \\ \quad - \Big( (f^{(n-k-1)}(b))^p \pm (g^{(n-k-1)}(b))^p \Big) \\ \quad \times \Big( (f^{(n-k-1)}(b))^q \pm (g^{(n-k-1)}(b))^q \Big) \Big] & \text{for } n \geq 2. \end{cases}$$

*(b) If $f^{(k)}(b) = g^{(k)}(b)$ for all $k = 0, 1, \ldots, n-1$ then $\Delta = 0$.*

*(c) If $pq < 0$, then the sign in (2.15.1) is reversed.*

**Theorem 2.15.3.** *Let $p$ and $q$ be real numbers such that $p + q = 1$ and let $\varphi, f, g : [a, b] \to \mathbf{R}$ be nonnegative functions with continuous derivatives of $n$-th order possessing properties (1)–(4) below.*

1. $(-1)^n \varphi^{(n)} > 0$, $f^{(n)} > 0$, $g^{(n)} > 0$, $(f^p \pm g^p)(f^q \pm g^q)^{(n)} > 0$;
2. $(-1)^k \varphi^{(k)}(b) \geq 0$ *for* $k = 0, 1, 2, \ldots, n-1$;
3. $f^{(k)}(a) = g^{(k)}(a) = 0$ *for* $k = 0, 1, \ldots, n-1$;
4. $f^{(k)}(b) > 0$, $g^{(k)}(b) > 0$ *for* $k = 0, 1, \ldots n-1$.

*We have the following.*

*(a) If $p$ and $q$ are positive numbers, then (2.15.1) holds with the inequality reversed;*

*(b) if $pq < 0$, then (2.15.1) holds.*

**2.16. Overview.** We have now completed our preliminary overview and are ready for some new results of our own. We shall begin the next chapter with some generalizations of Theorem 2.9.2.

# 3. SOME NEW GAUSS–PÓLYA INEQUALITIES

## 3.0. Overview

Like Caesar's Gaul, this chapter is divided into three parts. The first involves integral results and is being prepared for publication under the banner of generalized quasiarithmetic means [38]. The second concerns discrete inequalities and has already been published [59]. The third achieves some special results via the use of the Hölder inequality, and it is again in preparation for publication [10].

## 3.1. Results for weighted means

In this section we provide generalizations of Theorem 2.9.2 in a number of directions. In Subsection 1 we first derive an inequality for weighted means. We note that, as is suggested by the notation for means, our result extends to the case when the ordered pair of weights $(p_1, p_2)$ is replaced by an n-tuple. We derive also a version of our theorem for higher derivatives.

Subsection 2 treats some corresponding results when the mean $M$ is replaced by a quasiarithmetic mean. This can be done when the function involved enjoys appropriate convexity properties. A second theorem in Subsection 2 allows one weight $p_1$ to be positive and the others negative.

Subsection 3 addresses the logarithmic mean.

### 3.1.1. Results connected with weighted means

$M_p^{[s]}(a)$ denotes the weighted mean of order r and weights $p = (p_1, \ldots, p_n)$ of a positive sequence $a = (a_1, \ldots, a_n)$. The n-tuple $p$ is of positive numbers $p_i$ with $\sum_{1=i}^{n} p_i = 1$, that is we deal with probability weights. The mean is defined by

$$M_p^{[r]}(a) = \begin{cases} \left(\sum_{i=1}^{n} p_i a_i^r\right)^{1/r} & \text{for } r \neq 0 \\ \prod_{i=1}^{n} a_i^{p_i} & \text{for } r = 0. \end{cases}$$

In the special cases $r = -1, 0, 1$ we obtain respectively the familiar harmonic, geometric and arithmetic means.
The following theorem, which is a simple consequence of Jensen's inequality for convex functions, is one of the most important inequalities between means.

**Theorem 3.1.1.** *If $a$ and $p$ are positive n-tuples and $s < t$, $s, t \in \mathbf{R}$, then*

$$M_p^{[s]}(a) \leq M_p^{[t]}(a) \quad for \quad s < t \,, \tag{3.1.1}$$

*with equality if and only if $a_1 = \ldots = a_n$.*

A well-known consequence of the above statement is the inequality between arithmetic and geometric means. Previous results and refinements can both be found in [27] and [7].

The following theorem is a generalization of Theorem 2.9.2.

**Theorem 3.1.2.** *Let $g,\ h : [a, b] \to \mathbf{R}$ be nonnegative nondecreasing functions such* *that $g$ and $h$ have a continuous first derivative and $g(a) = h(a)$, $g(b) = h(b)$. Let $p = (p_1, p_2)$ be a pair of positive real numbers $p_1$, $p_2$ such that $p_1 + p_2 = 1$.*

*a) If $f : [a,b] \to \mathbf{R}$ is a nonnegative nondecreasing function, then for $r, s < 1$*

$$M_p^{[r]}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \leq \int_a^b \left(M_p^{[s]}(g(t), h(t))\right)' f(t)dt \tag{3.1.2}$$

*holds, and for $r, s > 1$ the inequality is reversed.*

*b) If $f : [a,b] \to \mathbf{R}$ is a nonnegative nonincreasing function then for $r < 1 < s$ (3.1.2) holds and for $r > 1 > s$ the inequality is reversed.*

**Proof.** Let suppose that $r, s < 1$ and $f$ is nondecreasing. Using inequality (3.1.1) we obtain

$$\begin{aligned}
&M_p^{[r]}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \\
&\leq M_p^{[1]}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \\
&= \int_a^b (p_1 g'(t) + p_2 h'(t))f(t)dt \\
&= f(b)M_p^{[1]}(g(b), h(b)) - f(a)M_p^{[1]}(g(a), h(a)) - \int_a^b M_p^{[1]}(g(t), h(t))df(t) \\
&\leq f(b)M_p^{[1]}(g(b), h(b)) - f(a)M_p^{[1]}(g(a), h(a)) - \int_a^b M_p^{[s]}(g(t), h(t))df(t) \\
&= f(b)M_p^{[1]}(g(b), h(b)) - f(a)M_p^{[1]}(g(a), h(a)) - \\
&\quad - \left(f(b)M_p^{[s]}(g(b), h(b)) - f(a)M_p^{[s]}(g(a), h(a)) - \int_a^b \left(M_p^{[s]}(g(t), h(t))\right)' f(t)dt\right) \\
&= f(b)\left(M_p^{[1]}(g(b), h(b)) - M_p^{[s]}(g(b), h(b))\right) - \\
&\quad - f(a)\left(M_p^{[1]}(g(a), h(a)) - M_p^{[s]}(g(a), h(a))\right) + \int_a^b \left(M_p^{[s]}(g(t), h(t))\right)' f(t)dt \\
&= \int_a^b \left(M_p^{[s]}(g(t), h(t))\right)' f(t)dt.
\end{aligned}$$

A similar proof applies in each of the other cases. □

**Remark 3.1.3.** In Theorem 3.1.2. we deal with two functions $g$ and $h$. Obviously a similar result holds for $n$ functions $x_1, \dots, x_n$ which satisfy the same conditions as $g$ and $h$.

**Remark 3.1.4.** It is obvious that on substituting $r = s = 0$ into (3.1.2), we have inequality (2.9.2) for $n = 2$. The result for $r = s = 0$ is given in [47].

In the following theorem we consider an inequality involving higher derivatives.

**Theorem 3.1.5.** *Let $f : [a,b] \to \mathbf{R}$ , $x_i : [a,b] \to \mathbf{R}$ $(i = 1, \dots, m)$ be nonnegative functions with continuous n-th derivatives such that $x_i^{(n)}$ , $(i = 1, \dots, m)$ are nonnegative functions and $p_i$, $(i = 1, \dots, m)$ be positive real numbers such that $\sum_{i=1}^m p_i = 1$.*

*a) If $(-1)^{n-1} f^{(n)}$ is a nonnegative function, then for $r, s < 1$*

$$M_p^{[r]}\left(\int_a^b x_1^{(n)}(t) f(t)dt, \dots, \int_a^b x_m^{(n)}(t) f(t)dt\right)$$

$$\leq \Delta + \int_a^b \left(M_p^{[s]}\left(x_1(t), \dots, x_m(t)\right)\right)^{(n)} f(t)dt \tag{3.1.3}$$

*holds, where*

$$\Delta = \sum_{k=0}^{n-1} (-1)^{n-k-1} f^{(n-k-1)}(t) \left(\sum_{i=1}^m p_i x_i^{(k)}(t) - \left(M_p^{[s]}(x_1(t), \dots, x_m(t))\right)^{(k)}\right)\Bigg|_a^b .$$

*If*

$$x_i^{(k)}(a) = x_j^{(k)}(a) \text{ and } x_i^{(k)}(b) = x_j^{(k)}(b) \text{ for } i, j \in \{1, \dots, m\} \tag{3.1.4}$$

*and $k = 0, \dots, n-1$, then*

$$\begin{aligned} &M_p^{[r]}\left(\int_a^b x_1^{(n)}(t) f(t)dt, \dots, \int_a^b x_m^{(n)}(t) f(t)dt\right) \\ &\leq \int_a^b \left(M_p^{[s]}\left(x_1(t), \dots, x_m(t)\right)\right)^{(n)} f(t)dt. \end{aligned} \tag{3.1.5}$$

*If $r, s > 1$, then the inequalities (3.1.3) and (3.1.5) are reversed.*

*b) If $(-1)^n f^{(n)}$ is a nonnegative function, then for $r < 1 < s$ the inequalities (3.1.3) and (3.1.5) hold and for $r > 1 > s$ they are reversed.*

**Proof.** a) Let $r$ and $s$ be less than 1. Integrating by parts $n$–times and using (3.1.1), we obtain

$$M_p^{[r]}\left(\int_a^b x_1^{(n)}(t)f(t)dt, \dots, \int_a^b x_m^{(n)}(t)f(t)dt\right)$$

$$\le M_p^{[1]}\left(\int_a^b x_1^{(n)}(t)f(t)dt, \dots, \int_a^b x_m^{(n)}(t)f(t)dt\right)$$

$$= \left(\sum_{k=0}^{n-1}(-1)^{n-k1} f^{(n-k-1)}(t)\sum_{i=1}^{m} p_i x_i^{(k)}(t)\right)\Bigg|_a^b$$

$$- \int_a^b M_p^{[1]}(x_1(t), \dots x_m(t))(-1)^{(n-1)} f^{(n)}(t)dt$$

$$\le \left(\sum_{k=0}^{n-1}(-1)^{n-k1} f^{(n-k-1)}(t)\sum_{i=1}^{m} p_i x_i^{(k)}(t)\right)\Bigg|_a^b$$

$$- \int_a^b M_p^{[s]}(x_1(t), \dots x_m(t))(-1)^{(n-1)} f^{(n)}(t)dt$$

$$= \Delta + \int_a^b \left(M_p^{[s]}(x_1(t), \dots, x_m(t))\right)^{(n)} f(t)dt.$$

We shall prove that $\Delta = 0$ if $x_i$, $i = 1, \dots, m$, satisfy (3.1.4).

Let us use the notation $A_k = x_i^{(k)}(a)$ for $k = 0, 1, \dots, n-1$. Then $\sum_{i=1}^m p_i x_i^{(k)}(a) = A_k$. Consider the $k$-th order derivative of the function $y^p$ where $y$ is an arbitrary function with $k$-th order derivative. First, there exists function $\phi_k^{[p]}$ such that

$$(y^p)^{(k)} = \phi_k^{[p]}(y, y', \dots, y^{(k)}).$$

This follows by induction on $k$ . For $k = 1$ we have $(y^p)' = py^{p-1}y' = \phi_1^{[p]}(y, y')$.

Suppose that proposition is valid for all $j < k+1$. Then using Leibniz's rule we get

$$
\begin{aligned}
(y^p)^{(k+1)} &= (py^{p-1}\cdot y')^{(k)} \\
&= p\sum_{j=0}^{k}\binom{k}{j}(y^{p-1})^{(j)}(y')^{(k-j)} \\
&= p\sum_{j=0}^{k}\binom{k}{j}\phi_j^{[p-1]}(y,y',\dots,y^{(j)})y^{(k-j+1)} \\
&= \phi_{k+1}^{[p]}(y,y',\dots,y^{(k+1)}).
\end{aligned}
\tag{3.1.6}
$$

Suppose that $s \neq 0$ and use the abbreviated notation $M(t)$ for the mean $M_p^{[s]}(x_1(t),\dots,x_m(t))$. Then $M^s(t) = \sum_{i=1}^m p_i x_i^s(t)$. The statement "$M^{(k)}(a) = A_k$" will be proved by induction on $k$. It is easy to check for $k = 0$ and $k = 1$.

Suppose it holds for all $j < k+1$. Then

$$
\begin{aligned}
\left.\left(\sum_{i=1}^m p_i x_i^s(t)\right)^{(k+1)}\right|_{t=a} &= \left.\sum_{i=1}^m p_i\phi_{(k+1)}^{[s]}\left(x_i(t),x_i'(t),\dots,x_i^{(k+1)}(t)\right)\right|_{t=a} \\
&= \phi_{(k+1)}^{[s]}(A_0,A_1,\dots,A_{k+1}) \\
&= s\sum_{j=0}^{k}\binom{k}{j}\phi_j^{[s-1]}(A_0,A_1,\dots,A_j)A_{k-j+1} \\
&\quad +\phi_k^{[s-1]}(A_0,A_1,\dots,A_k)A_{k+1}.
\end{aligned}
$$

On the other hand, using (3.1.6) we get

$$
\begin{aligned}
&(M^s(t))^{(k+1)}|_{t=a} \\
&= s\sum_{j=0}^{k}\binom{k}{j}\phi_j^{[s-1]}(M(a),M'(a),\dots,M^{(j)}(a))M^{(k-j+1)}(a) \\
&\quad +\phi_k^{[s-1]}(M(a),M'(a),\dots,M^{(k)}(a))M^{(k+1)}(a) \\
&= s\sum_{j=0}^{k}\binom{k}{j}\phi_j^{[s-1]}(A_0,A_1,\dots,A_j)A_{k-j+1} + \phi_k^{[s-1]}(A_0,A_1,\dots,A_k)M^{(k+1)}(a).
\end{aligned}
$$

Comparing these two results we obtain that $M^{(k+1)}(a) = A_{k+1}$, which is enough to conclude that $\Delta = 0$.

In the other cases the proof is similar, except in the case $s = 0$, when $\prod_{i=1}^{n} a_i^{p_i}$ should be used instead of $(\sum_{i=1}^{n} p_i a_i^r)^{1/r}$. □

**Applications.** Now we will restrict our attention to the case when $r = 0$ and the $x_i$ are power functions.

**The case when $n = 1$.**
Set: $r = 0$, $n = 1$, $a = 0$, $b = 1$, $x_i(t) = t^{a_i p_i + 1}$ in (3.1.3), where $a_i > -\frac{1}{p_i}$ for $i = 1, \dots, m$, $p_i > 0$ and $\sum_{i=1}^{m} \frac{1}{p_i} = 1$. We obtain that $\Delta = 0$ and

$$(3.1.7) \qquad \int_0^1 t^{a_1+\dots+a_m} f(t)dt \geq \frac{\prod_{i=1}^{m}(a_i p_i + 1)^{1/p_i}}{1 + \sum_{i=1}^{m} a_i} \prod_{i=1}^{m} \left( \int_0^1 t^{a_i p_i} f(t)dt \right)^{1/p_i}$$

if $f$ is a nondecreasing function. This is an improvement of Pólya's inequality (2.3.1). Some other results related to this inequality can be found in [48] and [61].
For example, combining (3.1.7) and the inequality

$$\sum_{i=1}^{m} a_i + 2 \geq \prod_{i=1}^{m} (a_i p_i + 2)^{1/p_i},$$

which follows from the inequality between arithmetic and geometric means, we obtain

$$(3.1.8) \quad \int_0^1 t^{a_1+\dots+a_m} f(t)dt \geq \frac{\prod_{i=1}^{m}((a_i p_i + 1)(a_i p_i + 2))^{1/p_i}}{\left(1 + \sum_{i=1}^{m} a_i\right)\left(2 + \sum_{i=1}^{m} a_i\right)} \prod_{i=1}^{m} \left( \int_0^1 t^{a_i p_i} f(t)dt \right)^{1/p_i}.$$

**The case when $n = 2$.**
Set: $r = 0$, $n = 2$, $a = 0$, $b = 1$, $x_i(t) = t^{a_i p_i + 2}$ in (3.1.3), where $a_i > -\frac{1}{p_i}$ for

$i = 1, \dots, m$, $p_i > 0$ and $\sum_{i=1}^{m} \frac{1}{p_i} = 1$. After some simple calculation, we obtain that $\Delta = 0$ and inequality (3.1.8) holds if $f$ is a concave function. So inequality (3.1.8) applies not only for $f$ nondecreasing, but also for $f$ concave.

### 3.1.2. Results for quasiarithmetic means

**Definition 3.1.6.** Let $f$ be a monotone real function with inverse $f^{-1}$, $p = (p_1, \cdots, p_n) = (p_i)_i$, $a = (a_1, \cdots, a_n) = (a_i)_i$ be real $n$-tuples. The quasiarithmetic mean of the $n$-tuple $a$ is defined by

$$M_f(a;p) = f^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i f(a_i)\right),$$

where $P_n = \sum_{i=1}^{n} p_i$.

For $p_i \geq 0, P_n = 1, f(x) = x^r (r \neq 0)$ and $f(x) = \ln x$ $(r = 0)$ the quasiarithmetic mean $M_f(a;p)$ is the weighted mean $M_p^{[r]}(a)$ of order $r$.

**Theorem 3.1.7.** *Let $p$ be a positive $n$-tuple, $x_i : [a,b] \to \mathbf{R}$ $(i = 1, \cdots, n)$ be nonnegative functions with continuous first derivative such that $x_i(a) = x_j(a)$, $x_i(b) = x_j(b)$, $i, j = 1, \dots, n$.*

*a) If $\varphi$ is a nonnegative nondecreasing function on $[a,b]$ and if $f$ and $g$ are convex increasing or concave decreasing functions , then*

$$M_f\left(\left(\int_a^b x_i'(t)\varphi(t)dt\right)_i ;p\right) \geq \int_a^b M_g'(x_i(t)_i;p)\,\varphi(t)dt. \tag{3.1.9}$$

*If $f$ and $g$ are concave increasing or convex decreasing functions, the inequality is reversed.*

*b) If $\varphi$ is a nonnegative nonincreasing function on $[a,b]$, $f$ convex increasing or concave decreasing and $g$ concave increasing or convex decreasing, then (3.1.9) holds.*

*If $f$ is concave increasing or convex decreasing function and $g$ convex increasing or concave decreasing, then (3.1.9) is reversed.*

**Proof.** Suppose that $\varphi$ is nondecreasing and $f$ and $g$ are convex functions. We shall use integration by parts and the well–known Jensen inequality for convex functions. The latter states that if $(p_i)$ is a positive n-tuple and $a_i \in I$, then for every convex function $f : I \to R$ we have

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(a_i). \tag{3.1.10}$$

We have

$$\begin{aligned}
&M_f\left(\left(\int_a^b x_i'(t)\varphi(t)dt\right)_i ; p\right)\\
&= f^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(\int_a^b x_i(t)\varphi(t)dt\right)\right)\\
&\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \int_a^b x_i'(t)\varphi(t)dt\\
&= \int_a^b \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_i'(t)\right)\varphi(t)dt\\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i(t)\varphi(t)|_a^b - \int_a^b \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_i(t)\right)d\varphi(t)\\
&\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i(t)\varphi(t)|_a^b - \int_a^b g^{-1}\left(\frac{1}{P_n}\left(\sum_{i=1}^{n} p_i g(x_i(t))\right)\right)d\varphi(t)\\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i(t)\varphi(t)|_a^b - \int_a^b M_g\left(x_i(t))_i ; p\right)d\varphi(t)\\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i(t)\varphi(t)|_a^b - M_g\left((x_i(t))_i ; p\right)\varphi(t)|_a^b + \int_a^b M_g'\left((x_i(t))_i ; p\right)\varphi(t)dt\\
&= \int_a^b M_g'\left((x_i(t))_i ; p\right)\varphi(t)dt.
\end{aligned}$$

□

**Theorem 3.1.8.** *Let $x_i, i = 1, \cdots, n$, satisfy the assumptions of Theorem 3.1.7 and let $p$ be a real n-tuple such that*

$$p_1 > 0, \quad p_i \leq 0 \quad (i = 2, \cdots, n), \quad P_n > 0. \tag{3.1.11}$$

*a) If $\varphi$ is a nonnegative nonincreasing function on $[a, b]$ and if $f$ and $g$ are concave increasing or convex decreasing functions, then (3.1.9) holds, while if $f$ and $g$ are convex increasing or concave decreasing then (3.1.9) is reversed.*

*b) If $\varphi$ is a nonnegative nondecreasing function on $[a, b]$, $f$ convex increasing or concave decreasing and $g$ concave increasing or convex decreasing, then (3.1.9) holds.*

*If $f$ is concave increasing or convex decreasing and $g$ convex increasing or concave decreasing, then (3.1.9) is reversed.*

The proof is similar to that of Theorem 3.1.7. Instead of Jensen's inequality, a reverse Jensen's inequality [27, p. 6] is used: that is, if $p_i$ is a real $n$-tuple such that (3.1.11) holds, $a_i \in I, i = 1, \dots, n$, and $(1/P_n)\sum_{i=1}^{n} p_i a_i \in I$, then for every convex function $f : I \to R$ (3.1.10) is reversed.

**Remark 3.1.9.** In Theorems 3.1.7 and 3.1.8 we deal with first derivatives. We can state an analogous result for higher–order derivatives as in Section 3.1.

**Remark 3.1.10.** The assumption that $p$ is a positive $n$-tuple in Theorem 3.1.7 can be weakened to $p$ being a real $n$-tuple such that

$$0 \le \sum_{i=1}^{k} p_i \le P_n \quad (1 \le k \le n), \quad P_n > 0 \tag{3.1.12}$$

and $\left(\int x_i'(t)\varphi(t)dt\right)_i$ and $(x_i(t))_i, t \in [a, b]$ being monotone $n$-tuples.

In that case, we use Jensen-Steffensen's inequality instead of Jensen's inequality in the proof.

In Theorem 3.1.5, the assumption on the $n$-tuple $p$ can be replaced by $p$ being a real $n$-tuple such that for some $k \in \{1, \cdots, m\}$

$$\sum_{i=1}^{k} p_i \le 0 \ \ (k < m) \quad \text{and} \quad \sum_{i=k}^{n} p_i \le 0 \ \ (k > m) \tag{3.1.13}$$

and $\left(\int x_i'(t)\varphi(t)dt\right)_i$, $(x_i(t))_i, t \in [a, b]$ being monotone $n$-tuples.

We use the reverse Jensen-Steffensen's inequality (see [27, p. 6] and [42]) in the proof. This states the following.

*Let $(p_i)$ be a real $n$-tuple such that (3.1.13) is valid and $(a_i)$ is a monotonic $n$-tuple such $a_i \in I$ and $(1/P_n)\sum_{i=1}^{n} p_i a_i \in I$. Then (3.1.10) is reversed.*

### 3.1.3. Results for logarithmic means

We define the logarithmic mean $L_r(x,y)$ of distinct positive numbers $x, y$ by

$$L_r(x,y) = \begin{cases} \left(\frac{1}{y-x}\frac{y^{r+1}-x^{r+1}}{r+1}\right)^{1/r} & r \neq -1, 0 \\ \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}} & r = 0 \\ \frac{\ln y - \ln x}{y-x} & r = -1 \end{cases}$$

and take $L_r(x,x) = x$. The function $r \mapsto L_r(x,y)$ is nondecreasing.

It is easy to see that $L_1(x,y) = \frac{x+y}{2}$ and using a method similar to that of the previous theorems we obtain the following result.

**Theorem 3.1.11.** *Let $g, h : [a,b] \to \mathbf{R}$ be nonnegative nondecreasing functions with continuous first derivatives and $g(a) = h(a), g(b) = h(b)$.*

*a) If $f$ is a nonnegative increasing function on $[a,b]$, and if $r, s \leq 1$, then*

$$L_r\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \leq \int_a^b L'_s\left(g(t), h(t)\right) f(t)dt. \tag{3.1.14}$$

*If $r, s \geq 1$, then the reverse inequality holds.*

*b) If $f$ is a nonnegative nonincreasing function then for $r < 1 < s$ (3.1.14) holds, and for $r > 1 > s$ the reverse inequality holds.*

**Proof.** Let $f$ be a nonincreasing function and $r < 1 < s$. Using $F = -f$, integration by parts and inequalities between logarithmic means we get

$$\begin{aligned}
&L_r\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right)\\
&\le L_1\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right)\\
&= \frac{1}{2}\int_a^b (g(t)+h(t))'f(t)dt\\
&= \frac{1}{2}(g(t)+h(t))f(t)\Big|_a^b + \int_a^b \frac{1}{2}(g(t)+h(t))dF(t)\\
&\le \frac{1}{2}(g(t)+h(t))f(t)\Big|_a^b + \int_a^b L_s(g(t),h(t))dF(t)\\
&= \frac{1}{2}(g(t)+h(t))f(t)\Big|_a^b - L_s(g(t),h(t))f(t)|_a^b + \int_a^b L_s'(g(t),h(t))f(t)dt\\
&= \int_a^b L_s'(g(t),h(t))f(t)dt.
\end{aligned}$$

## 3.2. Some discrete inequalities

### 3.2.1. Results involving general inequalities

For our next theorem we shall make use of Popoviciu's inequality. For an accessible reference see ([26], p.118). Popoviciu's inequality states the following.

*Let $a_j = (a_{ji})$, $(i = 1, \dots, n; j = 1, \dots, m)$ and $w = (w_i)$ $(i = 1, \dots, n)$ be nonnegative $n$-tuples, such that*

$$w_1 a_{j1} - w_2 a_{j2} - \dots - w_n a_{jn} \ge 0, \quad j = 1, \dots, m.$$

*a) If $p_j > 0, j = 1, \dots, m$, then*

$$\prod_{j=1}^{m}\left(w_1 a_{j1} - \sum_{i=2}^{n} w_i a_{ji}\right)^{p_j} \le w_1 a_{11}^{p_1} a_{21}^{p_2} \dots a_{m1}^{p_m} - \sum_{i=2}^{n} w_i \prod_{k=1}^{m} a_{ki}^{p_k}. \tag{3.2.1}$$

*b) If $p_1 > 0$ and $p_j < 0, j = 2, \ldots, m$, (3.2.1) is reversed.*

**Theorem 3.2.1.** *Let $w = (w_1, \ldots, w_n)$, $a_1 = (a_{11}, \ldots, a_{1n}), \ldots, a_m = (a_{m1}, \ldots, a_{mn})$ be nonnegative n-tuples and let the sums $\sum_{i=1}^{n-1} w_i \Delta a_{ji}$ be nonnegative for all $j = 1, \ldots, m$. Further, let $p_j$ $(j = 1, \ldots, m)$ be real numbers such that $\sum_{i=1}^{m} p_j = 1$.*

**a)** *Suppose $p_j \geq 0$ $(j = 1, \ldots, m)$. If $w$ is nondecreasing, then*

$$\sum_{i=1}^{n-1} w_i \Delta(a_{1i}^{p_1} \ldots a_{mi}^{p_m}) \geq \prod_{j=1}^{m} \left( \sum_{i=1}^{n-1} w_i \Delta a_{ji} \right)^{p_j}, \tag{3.2.2}$$

*where $\Delta a_{ji} = a_{j(i+1)} - a_{ji}$ and*

$\Delta\left(a_{1i}^{p_1} \cdots a_{mi}^{p_m}\right) = a_{1(i+1)}^{p_1} \cdots a_{m(i+1)}^{p_m} - a_{1i}^{p_1} \cdots a_{mi}^{p_m}.$

*If $w$ is a nonincreasing and $a_{j1} = 0$ for $j = 1, \ldots, m$, (3.2.2) is reversed.*

**b)** *Let $p_1 > 0$ and $p_j < 0, j = 2, \ldots, m$. If $w$ is nonincreasing and $a_{j1} = 0$ for $j = 1, \ldots, m$, then (3.2.2) holds.*

**Proof.** To prove assertion a), define $\Delta w_{i-1} = w_i - w_{i-1}$. If $w$ is nondecreasing, then $\Delta w_{i-1} \geq 0$ and we have

$$\begin{aligned}
&\sum_{i=1}^{n-1} w_i \; \Delta(a_{1i}^{p_1} \ldots a_{mi}^{p_m}) \\
&= w_n a_{1n}^{p_1} a_{2n}^{p_n} \ldots a_{mn}^{p_m} - w_1 a_{11}^{p_1} a_{21}^{p_2} \ldots a_{m1}^{p_m} - \sum_{i=2}^{n} a_{i1}^{p_1} a_{i1}^{p_2} \ldots a_{i1}^{p_m} \Delta w_{i-1} \\
&\geq \prod_{j=1}^{m} \left( w_n a_{jn} - w_1 a_{j1} - \sum_{i=2}^{n} a_{ji} \Delta w_{i-1} \right)^{p_j} \\
&= \prod_{j=1}^{m} \left( \sum_{i=1}^{n-1} w_i \Delta a_{ji} \right)^{p_i}.
\end{aligned}$$

Here inequality (3.2.2) is used. If $w$ is a nonincreasing n-tuple then Hölder's inequality is used instead that of Popoviciu's. The proof of assertion b) is similar to the previous one. □

Our next result employs Bellman's inequality [27, p. 118]. This states the following.

*Let $a=(a_i)$ and $b=(b_i)$ be nonnegative $n$- tuples such that*

$$a_1^p - a_2^p - \ldots - a_n^p \geq 0 \text{ and } b_1^p - b_2^p - \ldots - b_n^p \geq 0,$$

*where $p>1$ or $p<0$. Then*

$$\begin{aligned}(3.2.3)\qquad &\left((a_1^p - a_2^p - \ldots - a_n^p)^{1/p} + (b_1^p - b_2^p - \ldots - b_n^p)^{1/p}\right)^p \\ &\leq (a_1+b_1)^p - (a_2+b_2)^p - \ldots - (a_n+b_n)^p.\end{aligned}$$

*If $0<p<1$, the reverse inequality holds.*

**Theorem 3.2.2.** *Let $w=(w_i), a_j=(a_{ji})$ be nonnegative n-tuples such that for some $p \in R$ the sums $\sum_{i=1}^{n-1} w_i \Delta a_{ji}^p$, $j=1,\ldots,m$, are nonnegative.*

*a) Let $w$ be nondecreasing. If $p>1$ or $p<0$, then*

$$(3.2.4)\qquad \sum_{i=1}^{n-1} w_i \Delta (a_{1i} + \ldots + a_{mi})^p \geq \left(\sum_{j=1}^{m}\left(\sum_{i=1}^{n-1} w_i \Delta a_{ji}^p\right)^{1/p}\right)^p.$$

*If $0<p<1$, then the reverse inequality holds.*

*b) Let $w$ be nonincreasing and $a_{j1}=0$, $j=1,\ldots,m$ If $0<p<1$ then (3.2.3) applies. If $p>1$, then the reverse inequality holds.*

**Proof.** To prove assertion a) we use the same idea as in the previous theorem with Bellman's inequality.

An analogous formula applies for $m$ tuples $a_j, j=1,\ldots,m$.

Assertion b) can be proved analogously using the Minkowski inequality. □

**Remark 3.2.3.** An integral version of the previous theorem is given in [56].

**Theorem 3.2.4.** *Let $g=(g_1,\ldots,g_n)$, $h=(h_1,\ldots,h_n)$ be nonnegative and nondecreasing n-tuples such that $g_1=h_1=0$. If $f=(f_1,\ldots,f_n)$ is a nonnegative and nonincreasing n-tuple with $f_1\neq 0$, then*

$$f_1 \sum_{i=1}^{n-1} f_i \Delta(g_i h_i) \geq \left(\sum_{i=1}^{n-1} f_i \Delta g_i\right)\left(\sum_{i=1}^{n-1} f_i \Delta h_i\right).$$

**Proof.** Using Čebyšev's inequality we obtain

$$\begin{aligned}\sum_{i=1}^{n-1} f_i\Delta(g_ih_i) &= f_ng_nh_n - \sum_{i=2}^{n} g_ih_i\Delta f_{i-1}\\ &= f_ng_nh_n + \sum_{i=2}^{n} g_ih_i\Delta\overline{f_{i-1}}\\ &\geq \frac{1}{f_n + \sum_{i=2}^{n}\Delta\overline{f_{i-1}}} \times \left(f_ng_n + \sum_{i=2}^{n} g_i\Delta\overline{f_{i-1}}\right)\left(f_nh_n + \sum_{i=2}^{n} h_i\Delta\overline{f_{i-1}}\right)\\ &= \frac{1}{f_1}\left(\sum_{i=1}^{n-1} f_i\Delta g_i\right)\left(\sum_{i=1}^{n-1} f_i\Delta h_i\right),\end{aligned}$$

where $\overline{f_i} = -f_i$. □

### 3.2.2. Results for weighted, quasiarithmetic and logarithmic means

The preceding results are connected with general inequalities such as Hölder's, Minkowski's and Čebyšev's and their reverse versions. In the following theorem we deal with a weighted mean. Let us recall the definitions of weighted, quasiarithmetic and logarithmic means which are given in Section 3.1.

**Theorem 3.2.5.** *Let $a = (a_i), i = 1, \ldots, n$ and $b = (b_i), i = 1, \ldots, n$ be nonnegative and nondecreasing $n$-tuples such that $a_1 = b_1$ and $a_n = b_n$. Let $p_1$ and $p_2$ be positive real numbers such that $p_1 + p_2 = 1$, and let $r$ and $s$ be arbitrary real numbers. Further, let $f = (f_i), i = 1, \ldots, n$ be a nonnegative $n$-tuple.*

*a) Suppose $f$ is nondecreasing. If $r, s < 1$, then*

$$\sum_{i=1}^{n-1} \Delta M_p^{[r]}(a_i, b_i) f_i \geq M_p^{[s]}\left(\sum_{i=1}^{n-1} f_i\Delta a_i, \sum_{i=1}^{n-1} f_i\Delta b_i\right). \tag{3.2.5}$$

*If $r, s > 1$, (3.2.5) is reversed.*

*b) Suppose $f$ is nonincreasing. If $r < 1 < s$ then (3.2.5) applies, while and if $r > 1 > s$ it is reversed.*

**Proof.** To prove assertion a) let us suppose first that $r, s < 1$. Using the inequality between means, we obtain

$$\begin{aligned}
M_p^{[s]} &\left(\sum_{i=1}^{n-1} f_i \Delta a_i, \sum_{i=1}^{n-1} f_i \Delta b_i\right) \\
&\leq M_p^{[1]} \left(\sum_{i=1}^{n-1} f_i \Delta a_i, \sum_{i=1}^{n-1} f_i \Delta b_i\right) \\
&= \sum_{i=1}^{n-1} (p_1 \Delta a_i + p_2 \Delta b_i) f_i \\
&= f_n M_p^{[1]}(a_n, b_n) - f_1 M_p^{[1]}(a_1, b_1) - \sum_{i=2}^{n} M_p^{[1]}(a_i, b_i) \Delta f_i \\
&\leq f_n M_p^{[1]}(a_n, b_n) - f_1 M_p^{[1]}(a_1, b_1) - \sum_{i=2}^{n} M_p^{[r]}(a_i, b_i) \Delta f_i \\
&= f_n M_p^{[1]}(a_n, b_n) - f_1 M_p^{[1]}(a_1, b_1) - \Big(f_n M_p^{[r]}(a_n, b_n) - f_1 M_p^{[r]}(a_1, b_1) \\
&\quad - \sum_{i=1}^{n-1} \Delta\ M_p^{[r]}(a_i, b_i) f_i\Big) \\
&= \sum_{i=1}^{n-1} \Delta M_p^{[r]}(a_i, b_i) f_i,
\end{aligned}$$

which is the first assertion. The other cases can be proved analogously. □

**Theorem 3.2.6.** *Let $p = (p_i)$ be a positive $n$-tuple, $x_i = (x_{ij})$ $i = 1, \ldots, n$, nonnegative $m$-tuples with $x_{i'1} = x_{i''1}$ and $x_{i'm} = x_{i''m}$ for $1 \leq i', i'' \leq n$, and $w = (w_j), j = 1, \ldots, m$ a nonnegative $m$-tuple. Further, let $f$ and $g$ be real functions and suppose that all the quasiarithmetic means below are well-defined.*

*a) Suppose $w$ is nondecreasing. If $f$ and $g$ are convex increasing or concave decreasing, then*

$$M_f\left(\left(\sum_{k=1}^{m-1} w_k \Delta x_{ik}\right)_i ; p\right) \geq \sum_{k=1}^{m-1} w_k \Delta M_g\left((x_{ik})_i ; p\right). \tag{3.2.6}$$

*If $f$ and $g$ are concave increasing or convex decreasing, then (3.2.6) is reversed.*

*b) Let $w$ be nonincreasing. If $f$ is convex increasing or concave decreasing and $g$ is concave increasing or convex decreasing, then (3.2.6) applies. If $f$ is concave increasing or convex decreasing and $g$ convex increasing or concave decreasing, then (3.2.6) is reversed.*

**Proof.** Let us suppose that $f$ and $g$ are convex increasing. We use Jensen's inequality to obtain

$$\begin{aligned}
&M_f\left(\left(\sum_{k=1}^{m-1} w_k \Delta x_{ik}\right)_i ; p\right) \\
&= f^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(\sum_{k=1}^{m-1} w_k \Delta x_{ik}\right)\right) \\
&\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \sum_{k=1}^{m-1} w_k \Delta x_{ik} \\
&= \sum_{k=1}^{m-1} \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i \Delta x_{ik}\right) w_k \\
&= \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_{im}\right) w_m - \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_{i1}\right) w_1 - \sum_{k=2}^{m} \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_{ik}\right) \Delta w_k \\
&\geq \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_{im}\right) w_m - \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_{i1}\right) w_1 - \sum_{k=2}^{m} g^{-1}\left(\frac{1}{P_n}\left(\sum_{i=1}^{n} p_i g(x_{ik})\right)\right) \Delta w_k \\
&= \sum_{k=1}^{m-1} \Delta g^{-1}\left(\frac{1}{P_n}\left(\sum_{i=1}^{n} p_i g(x_{ik})\right)\right) w_k \\
&= \sum_{k=1}^{m-1} w_k \Delta M_g\left((x_{ik})_i ; p\right),
\end{aligned}$$

which is the first assertion. The other cases can be proved analogously. □

**Remark 3.2.7.** If $p_1 > 0$ and $p_i < 0$ for all $i = 2, \ldots, n$ then using the reverse version of the Jensen inequality we can state similar results. For further weaker conditions on $p$ see [27, p.6].

**Theorem 3.2.8.** *Let $a = (a_i)$ and $b = (b_i)$ be nonnegative and nondecreasing $n$-tuples such that $a_1 = b_1$ and $a_n = b_n$, and $w = (w_i)$ a nonnegative $n$-tuple. Further, let $r$ and $s$ be real numbers.*

*a) Suppose $w$ is nondecreasing. If $r, s \leq 1$, then*

$$L_r\left(\sum_{j=1}^{n-1} w_j \Delta a_j, \sum_{j=1}^{n-1} w_j \Delta b_j\right) \leq \sum_{j=1}^{n-1} w_j \Delta L_s(a_j, b_j). \tag{3.2.7}$$

*If $r, s \geq 1$, then (3.2.7) applies. If $r > 1 > s$, it is reversed.*

*b) Let $w$ be nonincreasing. If $r < 1 < s$ then (3.2.7) holds. If $r > 1 > s$, it is reversed.*

Theorem 3.2.8 can be proved using the inequality $L_r(x, y) \leq L_s(x, y)$ for $r \leq s$ for logarithmic means.

**Remark 3.2.9.** Integral versions of Theorems 3.2.4, 3.2.5 and 3.2.7 are given in the previous section.

## 3.3. A special case of a Gauss–Pólya type inequality

In this section Gauss-Pólya type inequalities are established by the use of Hölder's inequality. We have the following.

**Theorem 3.3.1** *Let $f : [a, b] \to R$ be a nonnegative and increasing function and $x_i : [a, b] \to R$, $(i = 1, \ldots, n)$ be functions with a continuous first derivative. Suppose $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $p_i$ $(i = 1, \ldots\ldots, n)$ are positive real numbers such that*

$\sum_{i=1}^{n} p_i = 1$, *then*

$$\begin{aligned}
&\left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} \left|\int_a^b x_i'(t) f(t) dt\right|^q\right)^{1/q} \\
&\qquad + f(b) \left[\left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i(b)|^q\right)^{1/q} - \sum_{i=1}^{n} p_i x_i(b)\right] \\
&\qquad \geq \left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \int_a^b \left(\left[\sum_{i=1}^{n} |x_i(t)|^q\right]^{1/q}\right)' f(t) dt \\
&\qquad + f(a) \left[\left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i(a)|^q\right)^{1/q} - \sum_{i=1}^{n} p_i x_i(a)\right].
\end{aligned} \tag{3.3.1}$$

**Proof.** First, let observe that, by an integration by parts,

(3.3.2)

$$\begin{aligned}
\sum_{i=1}^{n} p_i \int_a^b x_i'(t) f(t) dt &= \int_a^b \left(\sum_{i=1}^{n} p_i x_i(t)\right)' f(t) dt \\
&= \left(\sum_{i=1}^{n} p_i x_i(t)\right) f(t) dt \Bigg|_a^b - \int_a^b \left(\sum_{i=1}^{n} p_i x_i(t)\right) df(t) \\
&= f(b) A_p(X(b)) - f(a) A_p(X(a)) - \int_a^b \left(\sum_{i=1}^{n} p_i x_i(t)\right) df(t),
\end{aligned}$$

where $X(b) := (x_1(b), \ldots, x_n(b))$ and $X(a) = (x_1(a), \ldots, x_n(a))$.

If we apply Hölder's discrete inequality, we derive

$$\sum_{i=1}^{n} p_i \int_a^b x_i'(t) f(t) dt \leq \left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} \left|\int_a^b x_i'(t) f(t) dt\right|^q\right)^{1/q}$$

and

$$\sum_{i=1}^{n} p_i x_i(t) \leq \left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i(t)|^q\right)^{1/q} \quad \text{for all} \quad t \in [a, b].$$

Thus by (3.3.2) we have that

$$
\begin{aligned}
&\left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} \left|\int_a^b x_i'(t) f(t) dt\right|^q\right)^{1/q} \\
&\quad \geq f(b) A_p(X(b)) - f(a) A_p(X(a)) - \int_a^b \left(\sum_{i=1}^{p} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i(t)|^q\right)^{1/q} df(t) \\
&\quad = f(b) A_p(X(b)) - f(a) A_p(X(a)) - \int_a^b \left(\sum_{i=1}^{p} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i(t)|^q\right)^{1/q} df(t) \\
&\quad\quad - \left(\sum_{i=1}^{p} p_i^p\right)^{1/p} \left[\left(\sum_{i=1}^{n} |x_i(t)|^q\right)^{1/q} f(t)\Bigg|_a^b - \int_a^b \left[\left(\sum_{i=1}^{n} |x_i(t)|^q\right)^{1/q}\right]' f(t) dt\right] \\
&\quad = f(b) A_p(X(b)) - f(a) A_p(X(a)) - \left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i(b)|^q\right)^{1/q} f(b) \\
&\quad\quad + \left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i(a)|^q\right)^{1/q} f(a) + \left(\sum_{i=1}^{n} p_i^p\right)^{1/p} \int_a^b \left[\left(\sum_{i=1}^{n} |x_i(t)|^q\right)^{1/q}\right]' f(t) dt,
\end{aligned}
$$

which is clearly equivalent to (3.3.1).

**Applications.**

**1.** Choose $p = q = 2$ in the above theorem. This gives

$$
\begin{aligned}
&\left(\sum_{i=1}^{n} p_i^2\right)^{1/2} \left(\sum_{i=1}^{n} \left|\int_a^b x_i'(t) f(t) dt\right|^2\right)^{1/2} \\
&\quad + f(b) \left[\left(\sum_{i=1}^{n} p_i^2\right)^{1/2} \left(\sum_{i=1}^{n} |x_i(b)|^2\right)^{1/2} - \sum_{i=1}^{n} p_i x_i(b)\right] \\
&\quad \geq \left(\sum_{i=1}^{n} p_i^2\right)^{1/2} \int_a^b \left(\sqrt{\sum_{i=1}^{n} |x_i(t)|^2}\right)' f(t) dt \\
&\quad + f(a) \left[\left(\sum_{i=1}^{n} p_i^2\right)^{1/2} \left(\sum_{i=1}^{n} |x_i(a)|^2\right)^{1/2} - \sum_{i=1}^{n} p_i x_i(a)\right],
\end{aligned}
$$

which is related to the Cauchy-Buniakowski-Schwarz result.

**2.** If in the above theorem we put $p_1 = \cdots = p_n = \frac{1}{n}$, we deduce that

$$n^{\frac{1}{p}} \left( \sum_{i=1}^{n} \left| \int_a^b x_i'(t) f(t) dt \right| \right)^{1/q} + f(b) \left[ n^{\frac{1}{p}} \left( \sum_{i=1}^{n} |x_i(b)|^q \right)^{1/q} - \sum_{i=1}^{n} x_i(b) \right]$$

$$\geq n^{\frac{1}{p}} \int_a^b \left( \left[ \sum_{i=1}^{n} |x_i(t)|^q \right]^{1/q} \right)' f(t) dt + f(a) \left[ n^{\frac{1}{p}} \left( \sum_{i=1}^{n} |x_i(a)|^q \right)^{1/q} - \sum_{i=1}^{n} x_i(a) \right].$$

**3.** If we assume $x_i(a) = A$, $x_i(b) = B$, $i = 1, \dots, n$ in (3.3.1) we deduce that

$$\left( \sum_{i=1}^{n} p_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} \left| \int_a^b x_i'(t) f(t) dt \right|^q \right)^{1/q} + f(b) \left[ n^{\frac{1}{q}} \left( \sum_{i=1}^{n} p_i^p \right)^{1/p} |B| - B \right]$$

$$\geq \left( \sum_{i=1}^{n} p_i^p \right)^{\frac{1}{p}} \int_a^b \left( \left[ \sum_{i=1}^{n} |x_i(t)|^q \right]^{1/q} \right)' f(t) dt + f(a) \left[ n^{\frac{1}{q}} \left( \sum_{i=1}^{n} p_i^p \right)^{1/p} |A| - A \right].$$

If we further set in this inequality $p_1 = \cdots = p_n = \frac{1}{n}$, we get

$$\frac{1}{n^{\frac{p-1}{p}}} \left( \sum_{i=1}^{n} \left| \int_a^b x_i'(b) f(t) dt \right|^q \right)^{1/q} + f(b) \left[ |B| - B \right]$$

$$\geq \frac{1}{n^{\frac{p-1}{p}}} \int_a^b \left( \left( \sum_{i=1}^{n} |x_i(t)|^q \right)^{1/q} \right)' f(t) dt + f(a) \left[ |A| - A \right].$$

Moreover, if $A > 0$, $B > 0$, this gives

$$\left( \sum_{i=1}^{n} \left| \int_a^b x_i'(t) f(t) dt \right|^q \right)^{1/q} \geq \int_a^b \left( \left( \sum_{i=1}^{n} |x_i(t)|^q \right)^{1/q} \right)' f(t) dt,$$

which holds for all $q > 1$.

**Remark 3.3.2.** Many particular inequalities can be obtained if we choose the mappings $x_i$ in an appropriate way.

Suppose that all the functions are defined on $[0, 1]$ and let $n = 2$, $x_1(t) = t^a$, $x_2(t) = t^b$, $a, b > 0$, $p_1, p_2 \geq 0$ with $p_1 + p_2 = 1$ and $p, q$ as above. Then we get

$$
\begin{aligned}
&(p_1^p + p_2^p)^{1/p} \left( \left[ \int_0^1 at^{a-1} f(t)dt \right]^q + \left[ \int_0^1 bt^{b-1} f(t)dt \right]^q \right)^{1/q} \\
&\qquad + f(1) \left[ (p_1^p + p_2^p)^{1/p} 2^{1/q} - (p_1 + p_2) \right] \\
&\qquad \geq (p_1^p + p_2^p)^{1/p} \int_a^b \left[ \left( t^{aq} + t^{bq} \right)^{1/q} \right]' f(t)dt,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&(p_1^p + p_2^p)^{1/p} \left( a^q \left[ \int_0^1 t^{a-1} f(t)dt \right]^q + b^q \left[ \int_0^1 t^{b-1} f(t)dt \right]^q \right)^{1/q} \\
&\qquad + f(1) \left[ 2^{1/q} (p_1^p + p_2^p)^{1/p} - 1 \right] \qquad\qquad (3.3.3) \\
&\qquad \geq (p_1^p + p_2^p)^{1/p} \int_0^1 \left( t^{aq} + t^{bq} \right)^{-1/p} \left( at^{aq-1} + bt^{bq-1} \right) f(t)dt,
\end{aligned}
$$

since

$$
\begin{aligned}
\left[ \left( t^{aq} + t^{bq} \right)^{1/q} \right]' &= \frac{1}{q} \left( t^{aq} + t^{bq} \right)^{\frac{1}{q}-1} \times \left( aqt^{aq-1} + bqt^{bq-1} \right) \\
&= \left( t^{aq} + t^{bq} \right)^{-1/p} \left( at^{aq-1} + bt^{bq-1} \right).
\end{aligned}
$$

If we choose $p_1 = p_2 = \frac{1}{2}$ in (3.3.3) we get

$$
\left[ a^q \left( \int_0^1 t^{a-1} f(t)dt \right)^q + b^q \left( \int_0^1 t^{b-1} f(t)dt \right)^q \right]^{1/q} \geq \int_0^1 \frac{at^{aq-1} + bt^{bq-1}}{\left( t^{aq} + t^{bq} \right)^{1/p}} f(t)dt
$$

assuming that the last integral does exist.

# 4. FURTHER GENERALIZATION OF THE FIRST PÓLYA INEQUALITY

## 4.0. Overview

We now return to Pólya's results to embark on generalizations in terms of more general means than in the previous chapter. In the first section we shall address Stolarsky means and in the second Gini means. The third section considers quasiarithmetic means. As in the previous chapter we conclude with some special results. In Section 4 we look at some further generalizations involving functions instead of means.

*If $f : [0,1] \to I\!R$ is a nonnegative and nondecreasing function, then*

$$\left(\int_0^1 x^{a+b} f(x)dx\right)^2 \geq \left(1 - \left(\frac{a-b}{a+b+1}\right)^2\right) \int_0^1 x^{2a} f(x)dx \int_0^1 x^{2b} f(x)dx.$$

For nonnegative real numbers $x, y$ define $G(x,y) := (xy)^{1/2}$. The first Pólya result can be expressed in terms of $G$ as

$$\int_0^1 \left[\frac{d}{dx} G\left(x^{2a+1}, x^{2b+1}\right)\right] f(x)dx \geq G\left(\int_0^1 \left(\frac{d}{dx}x^{2a+1}\right) f(x)dx, \int_0^1 \left(\frac{d}{dx}x^{2b+1}\right) f(x)dx\right).$$

Alzer's generalization (see Section 2.6) reads as follows.

*Let $f, g, h : [a, b] \to \mathbb{R}$ be nonnegative, increasing functions such that $g, h$ and $\sqrt{gh}$ are continuously differentiable on $[a, b]$. If $g(a) = h(a)$ and $g(b) = h(b)$, then (2.6.1) is valid, that is,*

$$\int_a^b \left(G\left(g(x), h(x)\right)\right)' f(x)dx \geq G\left(\int_a^b g'(x)f(x)dx, \int_a^b h'(x)f(x)dx\right).$$

In next section we aim at replacing $G$ in this result by more general means.

Material in this chapter is being prepared for publication in three papers [38], [39] and [40].

## 4.1. Results involving Stolarsky means

### 4.1.1. Preliminaries

Suppose $a, b$ are real numbers and $x, y$ positive numbers. The Stolarsky mean $E_{a,b}(x, y)$ is defined by $E_{a,b}(x, y) = x$ for $x = y$, and for $x \neq y$ by

$$E_{a,b}(x,y) = \begin{cases} \left(\dfrac{b(x^a - y^a)}{a(x^b - y^b)}\right)^{\frac{1}{a-b}} & \text{if } ab(a-b) \neq 0, \\ \left(\dfrac{x^a - y^a}{a(\ln x - \ln y)}\right)^{1/a} & \text{if } a \neq 0, b = 0 \\ \left(\dfrac{b(\ln x - \ln y)}{x^b - y^b}\right)^{-1/b} & \text{if } b \neq 0, a = 0 \\ e^{-1/a}\left(\dfrac{x^{x^a}}{y^{y^a}}\right)^{\frac{1}{x^a - y^a}} & \text{if } a = b \neq 0 \\ \sqrt{xy} & \text{if } a = b = 0. \end{cases}$$

We remark that

$$E_{1,2}(x,y) = E_{2,1}(x,y) = \frac{x+y}{2} = A(x,y)$$

and

$$E_{a,-a}(x,y) = E_{-a,a}(x,y) = \sqrt{xy} = G(x,y).$$

A fundamental question is when is it the case that

$$E_{r,s}(x,y) \le E_{u,v}(x,y) \tag{4.1.1}$$

for all positive and distinct $x, y$? This question has been solved by Leach and Sholander [20]. See also Páles [35], who treats a more general question that subsumes this problem. For clarity, we expand and reword their enunciation slightly.

**Lemma 4.1.1.** *Let $r$, $s$, $u$, $v$ be real numbers with $r \neq s$ and $u \neq v$.*

*(a) If either $0 \le \min(r,s,u,v)$ or $\max(r,s,u,v) \le 0$, then (4.1.1) holds for all distinct positive $x, y$ if and only if*

$$r + s \le u + v$$

*and*

$$e(r,s) \le e(u,v),$$

*where*

$$e(\alpha,\beta) = \begin{cases} (\alpha-\beta)/\ln(\alpha/\beta), & \text{for } \alpha\beta > 0, \alpha \neq \beta \\ 0, & \text{if } \alpha\beta = 0. \end{cases} \tag{4.1.2}$$

*(b) If $\min(r,s,u,v) < 0 < \max(r,s,u,v)$, then (4.1.1) holds for all distinct positive $x, y$ if and only if*

$$r + s \le u + v$$

*and*

$$e(r,s) \le e(u,v),$$

*where*

$$e(\alpha,\beta) = (|\alpha| - |\beta|)/(\alpha - \beta) \quad \text{for } \alpha \neq \beta. \tag{4.1.3}$$

We define sets $A, A^*$ by

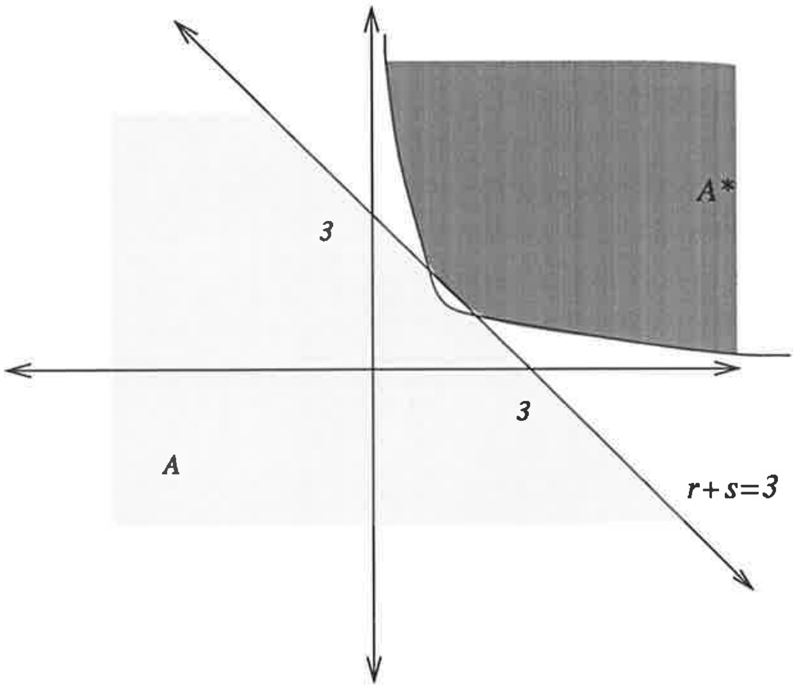
$$A = \{(r,s)|r+s \leq 3 \quad \text{and} \quad e(r,s) \leq e(1,2)\}$$

and

(4.1.4) $$A^{\star} = \{(r,s)|r+s \geq 3 \quad \text{and} \ \ e(r,s) \geq e(1,2)\},$$

(see Figure 1),

**Figure 1**




where $e(x,y)$ is defined by (4.1.2) if $r,s \geq 0$, and by (4.1.3) if $min(r,s) < 0$.

Note that

$$e(1,2) = \begin{cases} 1/\ln 2, & \text{if } \ \min(r,s) \geq 0; \\ 1, & \text{if } \ \min(r,s) < 0. \end{cases}$$

We now establish our basic lemma.

**Lemma 4.1.2.** *Let $r, s$ be real numbers. If $(r,s) \in A$ then*

$$E_{r,s}(x,y) \leq E_{1,2}(x,y), \tag{4.1.5}$$

*while if $(r,s) \in A^\star$, then*

$$E_{r,s}(x,y) \geq E_{1,2}(x,y). \tag{4.1.6}$$

**Proof.** The first fact is immediate from Lemma (4.1.1). We note that $\max(r,s,1,2) \leq 0$ cannot occur and so $e$ is given by (4.1.2) if $r, s \geq 0$ and by (4.1.3) if $\min(r,s) < 0$.

The second part follows similarly.

**Remark 4.1.3.** Lemma 4.1.1 is a generalization of the fact that $E_{r,s}(x,y)$ is a nondecreasing function of $r$ and $s$. So (4.1.1) holds if $r \leq u$ and $s \leq v$, that is, (4.1.5) holds if $r \leq 1$ and $s \leq 2$ and (4.1.6) if $r \geq 1$ and $s \geq 2$.

### 4.1.2. Integral results

**Theorem 4.1.4.** *Suppose $g, h : [a,b] \to I\!R$ are nonnegative nondecreasing functions with continous first derivatives and $g(a) = h(a), g(b) = h(b)$.*

*a) Let $f$ be a nonnegative, nondecreasing, differentiable function on $[a,b]$. If $(r,s) \in A$ and $(u,v) \in A$, then*

$$E_{r,s}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \leq \int_a^b \left(E_{u,v}(g(t),h(t))\right)' f(t)dt. \tag{4.1.7}$$

*If $(r,s) \in A^\star$ and $(u,v) \in A^\star$, the inequality is reversed.*

*b) Let $f$ be a nonnegative, nonincreasing, differentiable function. If $(r,s) \in A$ and $(u,v) \in A^\star$, then (4.1.7) holds, while if $(r,s) \in A^\star$ and $(u,v) \in A$, the inequality is reversed.*

**Proof.** a) Suppose $(r,s)$, $(u,v) \in A$. We have

$$\begin{aligned}
&E_{r,s}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \\
&\leq \frac{1}{2}\left(\int_a^b g'(t)f(t)dt + \int_a^b h'(t)f(t)dt\right) \\
&= \frac{1}{2}(g(t)+h(t))\, f(t)|_a^b - \int_a^b \frac{1}{2}(g(t)+h(t))\, df(t) \\
&\leq \frac{1}{2}(g(t)+h(t))\, f(t)|_a^b - \int_a^b E_{u,v}(g(t),h(t))\, df(t) \\
&= \frac{1}{2}(g(t)+h(t))\, f(t)|_a^b - E_{u,v}(g(t),h(t))\, f(t)|_a^b + \int_a^b (E_{u,v}(g(t),h(t)))'\, f(t)dt \\
&= \int_a^b (E_{u,v}(g(t),h(t)))'\, f(t)dt.
\end{aligned}$$

If $(r,s)$, $(u,v) \in A^\star$, we have trivially that the inequality is reversed.

b) Suppose $(r,s) \in A$ and $(u,v) \in A^{\star}$. Put $F = -f$. Then we have

$$
\begin{aligned}
&E_{r,s}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \\
&\leq \int_a^b \frac{1}{2}(g(t)+h(t))' f(t)dt \\
&= \frac{1}{2}(g(t)+h(t))\, f(t)|_a^b + \int_a^b (g(t)+h(t))\, dF(t) \\
&\leq \frac{1}{2}(g(t)+h(t))\, f(t)|_a^b + \int_a^b E_{u,v}(g(t),h(t))\, dF(t) \\
&= \frac{1}{2}(g(t)+h(t))\, f(t)|_a^b - E_{u,v}(g(t),h(t))\, f(t)|_a^b + \int_a^b (E_{u,v}(g(t),h(t)))' f(t)dt \\
&= \int_a^b (E_{u,v}(g(t),h(t)))' f(t)dt.
\end{aligned}
$$

If $(r,s) \in A^{\star}$ and $(u,v) \in A$, the inequality is clearly reversed.

**Corollary 4.1.5.** *Let $g, h$ be defined as in Theorem 4.1.4.*

*a) Let $f$ be a nonnegative, nondecreasing, differentiable function on $[a,b]$. If $r,u \leq 1$ and $s,v \leq 2$, then (4.1.7) holds. If $r,u \geq 1$ and $s,v \geq 2$, then (4.1.7) is reversed.*

*b) Let $f$ be a nonnegative, nonincreasing, differentiable function on $[a,b]$. If $r \leq 1 \leq u$ and $s \leq 2 \leq v$ then (4.1.7) holds. If $u \leq 1 \leq r$ and $v \leq 2 \leq s$, then (4.1.7) is reversed.*

**Proof.** This follows from Theorem 4.1.4 and Remark 4.1.3.

### 4.1.3. Summation results

We set $\Delta a_i = a_{i+1} - a_i, \quad \Delta a_{ji} = a_{j,i+1} - a_{ji}$.

**Theorem 4.1.6.** *Suppose $a$ and $b$ are nonnegative, nondecreasing $n$-tuples ($n \geq 2$) such that $a_n = b_n$ and $a_1 = b_1$.*

*a) Let $w$ be a nonnegative, nondecreasing $n$-tuple. If $(r,s),(u,v) \in A$ then*

$$E_{r,s}\left(\sum_{j=1}^{n-1} w_j \Delta a_j, \sum_{j=1}^{n-1} w_j \Delta b_j\right) \leq \sum_{j=1}^{n-1} w_j \Delta E_{u,v}(a_j, b_j), \tag{4.1.8}$$

*while if $(r,s),(u,v) \in A^\star$, the inequality is reversed.*

*b) Let $w$ be a nonnegative, nonincreasing $n$-tuple ($n \geq 2$). If $(r,s) \in A$ and $(u,v) \in A^\star$ then (4.1.8) holds. If $(r,s) \in A^\star$ and $(u,v) \in A$ the inequality is reversed.*

**Proof.** a) Let $(r,s),(u,v) \in A$. We have

$$
\begin{aligned}
E_{r,s} & \left(\sum_{j=1}^{n-1} w_j \Delta a_j, \sum_{j=1}^{n-1} w_j \Delta b_j\right) \\
& \leq E_{1,2}\left(\sum_{j=1}^{n-1} w_j \Delta a_i, \sum_{i=1}^{n-1} w_i \Delta b_i\right) \\
& = \frac{1}{2}\left\{\sum_{i=1}^{n-1} w_i \Delta a_i + \sum_{i=1}^{n-1} w_i \Delta b_i\right\} \\
& = \sum_{i=1}^{n-1} w_i \Delta\left(\frac{a_i+b_i}{2}\right) \\
& = w_n \frac{a_n+b_n}{2} - w_1 \frac{a_1+b_1}{2} - \sum_{i=2}^{n} \frac{a_i+b_i}{2} \Delta w_{i-1} \\
& \leq w_n \frac{a_n+b_n}{2} - w_1 \frac{a_1+b_1}{2} - \sum_{i=2}^{n} E_{u,v}(a_i, b_i) \Delta w_{i-1} \\
& = w_n \frac{a_n+b_n}{2} - w_1 \frac{a_1+b_1}{2} \\
& \qquad - \left\{w_n E_{u,v}(a_n, b_n) - w_1 E_{u,v}(a_1, b_1) - \sum_{i=1}^{n-1} \Delta E_{u,v}(a_i, b_i) w_i\right\} \\
& = \sum_{i=1}^{n-1} w_i \Delta E_{u,v}(a_i, b_i).
\end{aligned}
$$

If $(r, s)$, $(u, v) \in A^\star$, the inequality is clearly reversed.

b) Let $(r,s) \in A$ and $(u,v) \in A^\star$. Set $W_i = -w_i \ \ (i-1,\ldots,n-1)$. We have

$$E_{r,s}\left(\sum_{i=1}^{n-1} w_i\Delta a_i, \sum_{i=1}^{n-1} w_i\Delta b_i\right)$$

$$\leq \sum_{i=1}^{n-1} w_i\Delta\left(\frac{a_i+b_i}{2}\right)$$

$$= w_n\frac{a_n+b_n}{2} - w_1\frac{a_1+b_1}{2} + \sum_{i=2}^{n} w_i\frac{a_i+b_i}{2}\Delta W_{i-1}$$

$$\leq w_n\frac{a_n+b_n}{2} - w_1\frac{a_1+b_1}{2} + \sum_{i=2}^{n} E_{u,v}(a_i,b_i)\Delta W_{i-1}$$

$$= w_n\frac{a_n+b_n}{2} - w_1\frac{a_1+b_1}{2}$$

$$-w_nE_{u,v} - \left(w_nE_{u,v}(a_n+b_n) - w_1E_{u,v}(a_1,b_1) - \sum_{i=1}^{n-1}\Delta E_{u,v}(a_i,b_i)w_i\right)$$

$$= \sum_{i=1}^{n-1} w_i\Delta E_{u,v}(a_i,b_i).$$

If $(r,s) \in A^\star$ and $(u,v) \in A$ , the inequality is clearly reversed.

As before, we can make the following deduction.

**Corollary 4.1.7.** *Suppose n-tuples a and b are as in Theorem 4.1.6.*

*a) Let w be a nonnegative, nondecreasing n-tuple.*

*If* $r,u \leq 1$ *and* $s,v \leq 2$, *then (4.1.8) holds. If* $r,u \geq 1$ *and* $s,v \geq 2$, *then (4.1.8) is reversed.*

*b) Let w be a nonnegative, nondecreasing n-tuple.*

*If* $r \leq 1 \leq u$ *and* $s \leq 2 \leq v$, *then (4.1.8) holds. If* $r \geq 1 \geq u$ *and* $s \geq 2 \geq v$, *then (4.1.8) is reversed.*

# 4.2. Results involving Gini means

## 4.2.1. Notation and preliminary results

Let $a, b$ be real numbers. The Gini mean [16] of an $n$-vector $\mathbf{x} = (x_1, \ldots, x_2)$ with weights $\mathbf{w} = (w_1, \ldots, w_n)$ with coordinates in $\mathbb{R} = (0, \infty)$ is defined by

$$
\begin{aligned}
G_{a,b}(\mathbf{x}; \mathbf{w}) &= G_{a,b}(x_1, \ldots, x_n; \mathbf{w}) \\
&= \begin{cases} \left( \dfrac{w_1 x_1^a + \ldots + w_n x_n^a}{w_1 x_1^b + \ldots + w_n x_n^b} \right) & \text{if } a \neq b, \\[2ex] \exp\left( \dfrac{w_1 x_1^a \ln x_1 + \ldots + w_n x_n^a \ln x_n}{x_1^a + \ldots + x_n^a} \right) & \text{if } a = b. \end{cases}
\end{aligned}
$$

If $\mathbf{w} = (1, \ldots, 1)$ we write $G_{a,b}(\mathbf{x}; \mathbf{w}) = G_{a,b}(\mathbf{x})$. Note that we always have $G_{a,b} = G_{b,a}$.

**Lemma 4.2.1.** *[8] Let $a$, $b$, $c$, $d$ be real numbers. Then in order that*

$$G_{a,b}(\mathbf{x}) \leq G_{c,d}(\mathbf{x}) \tag{4.2.1}$$

*be valid for all $n \in \mathbb{N}$ and $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_1, \ldots, x_n > 0$, it is necessary and sufficient that*

$$\min(a, b) \leq \min(c, d) \quad \text{and} \quad \max(a, b) \leq \max(c, d). \tag{4.2.2}$$

A simple consequence is as follows.

**Lemma 4.2.2.** *Let $a$, $b$, $c$, $d$, be real numbers satisfying (4.2.2) . If the $n$-vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ have all positive coordinates, then*

$$G_{a,b}(\mathbf{x}; \mathbf{w}) \leq G_{c,d}(\mathbf{x}; \mathbf{w}).$$

The case $n = 2$ in (4.2.1) is of special interest.

**Lemma 4.2.3.** *[34] Let $a$, $b$, $c$, $d$, be arbitrary real numbers such that $a \neq b$ and $c \neq d$. Then*

$$a + b \leq c + d \quad \text{and} \quad m(a, b) \leq m(c, d) \tag{4.2.3}$$

*is a necessary and sufficient condition that*

$$G_{a,b}(x, y) \geq G_{c,d}(x, y) \tag{4.2.4}$$

*hold for all positive $x$ and $y$. Here*

$$m(\alpha, \beta) = \begin{cases} \min(\alpha, \beta) & \text{if } 0 \leq \min(a, b, c, d) \\ (|\alpha| - |\beta|)/(\alpha - \beta) & \text{if } \min(a, b, c, d) < 0 < \max(a, b, c, d) \\ \max(\alpha, \beta) & \text{if } \max(a, b, c, d) \leq 0. \end{cases} \tag{4.2.5}$$

We shall consider the two special cases

$$G_{a,b}(x, y) \leq G_{0,1}(x, y) = \frac{x + y}{2} \tag{4.2.6}$$

and

$$G_{a,b}(x, y) \geq G_{0,1}(x, y) = \frac{x + y}{2}. \tag{4.2.7}$$

Suppose (without loss of generality) that $a < b$. For (4.2.6) we should set $c = 0$, $d = 1$ in Lemma 4.2.3. From (4.2.3) we get

$$a + b \leq 1 \quad \text{and} \quad m(a, b) \leq m(0, 1). \tag{4.2.3'}$$

As $\max(a, b, 0, 1)$ cannot be $\leq 0$, we only have the first two cases in the definition (4.2.5) of $m(x, y)$.

As $0 \leq \min(a, b, 0, 1)$ is equivalent to $a \geq 0$, we have

$$m(a, b) = a \quad \text{and} \quad m(0, 1) = 0.$$

Applying this to (4.2.3′) we get $a = 0$ and $b \leq 1$.

Similarly $\min(a,b,0,1) < 0 < \max(a,b,0,1)$ is equivalent to $a < 0$. Then

$$m(a,b) = \frac{|b|-|a|}{b-a} \quad \text{and} \quad m(0,1) = 1.$$

Using this in (4.2.3′) we have

$$|b| - |a| \leq b - a,$$

that is,

$$|b| + a \leq b - a. \tag{4.2.8}$$

If $b > 0$, (4.2.8) becomes

$$b + a \leq b - a,$$

while for $b < 0$, (4.2.8) becomes

$$-b + a \leq b - a,$$

which is obviously true.

We have that (4.2.6) holds in the case $a < b$ if $a \leq 0$ and $a + b \leq 1$. Because of symmetry we have that (4.2.6) holds if $(a,b) \in B$, where

$$B = \{(a,b) | a + b \leq 1 \quad \wedge \quad (a \leq 0 \vee b \leq 0)\} \tag{4.2.9}$$

(see Figure 2). Now let us consider (4.2.8).

For $a < b$, set $a = 0$, $b = 1$, $c = a$, $d = b$ in (4.2.4). Then (4.2.3) becomes

$$a + b \geq 1 \quad \text{and} \quad m(a,b) \geq m(0,1), \tag{4.2.3″}$$

where $m(x,y)$ is now defined as above by

$$m(\alpha,\beta) = \begin{cases} \min(\alpha,\beta) & \text{if} \quad a \geq 0 \\ \dfrac{|\alpha|-|\beta|}{\alpha-\beta}, & \text{if} \quad a < 0. \end{cases}$$

So for $a \geq 0$, (4.2.3″) becomes

$$a \geq 0,$$

while for $a < 0$ we have

$$|b| - |a| \geq b - a,$$

that is,

$$|b| + a \geq b - a. \tag{4.2.10}$$

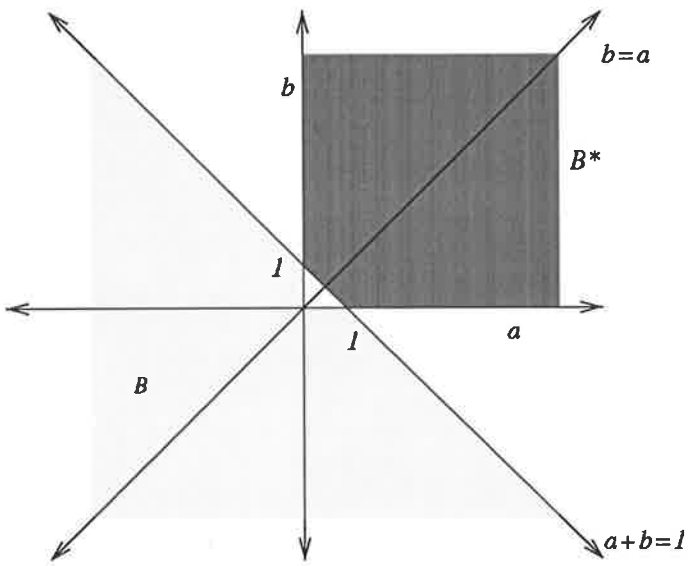
From (4.2.10) there is a contradiction for $b \geq 0$ (which gives $a \geq 0$) and for $b < 0$ (which gives $b \leq a$).

We have that (4.2.7) holds when $a < b$ if $a + b \geq 1$ from (4.2.3″) and $a \geq 0$ applies. Because of symmetry we have that (2.10) holds if $(a, b) \in B^\star$, where

$$B^\star = \{(a, b) | a + b \geq 1, a \geq 0, b \geq 0\} \tag{4.2.11}$$

(see Figure 2).

**Figure 2**




Therefore, we have the following special case of Lemma 4.2.3.

**Lemma 4.2.4.** *If $(a, b) \in B$, where $B$ is defined by (4.2.9), then (4.2.6) holds, while if $(a, b) \in B^\star$, where $B^\star$ is defined by (4.2.11), then (4.2.7) applies.*

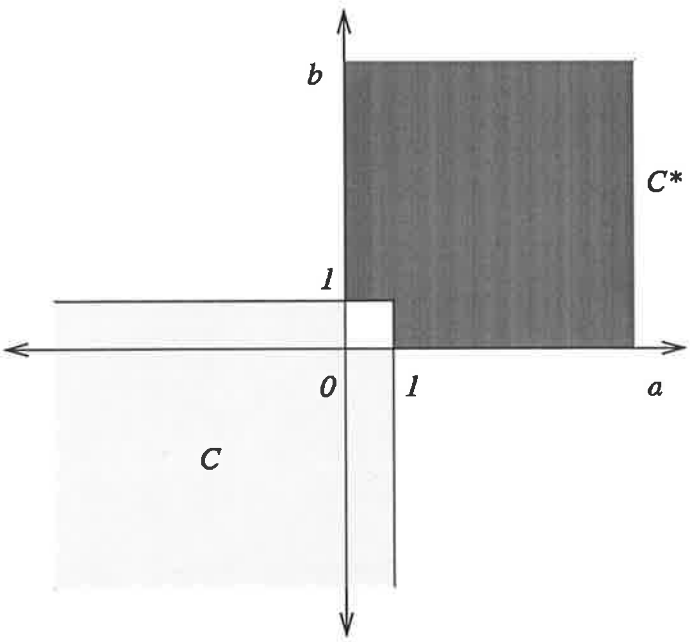
*We shall also use the following special cases of (4.2.1).*

(4.2.13)
$$G_{a,b}(x_1, \ldots, x_n; \mathbf{w}) \leq G_{0,1}(x_1, \ldots, x_n; \mathbf{w}) = \frac{w_1x_1 + \ldots + w_nx_n}{w_1 + \ldots + w_n} \quad (:= A(x_1, \ldots, x_n; \mathbf{w}))$$

*and*

(4.2.14)
$$G_{a,b}(x_1, \ldots, x_n; \mathbf{w}) \geq G_{0,1}(x_1, \ldots, x_n; \mathbf{w}) = \frac{w_1x_1 + \ldots + w_nx_n}{w_1 + \ldots + w_n}.$$

**Figure 3**




The following Lemma is a simple consequence of Lemma 4.2.2.

**Lemma 4.2.5.** *If $(a,b) \in C$, where $C$ is defined by*

$$C = \{(a,b) | ((a \leq 0) \wedge (b \leq 1)) \vee ((a \leq 1) \wedge (b \leq 0))\}$$

*(see Figure 3), then (4.2.13) holds, and if $(a,b) \in C^\star$, where*

$$C^\star = \{(a,b) | ((a \geq 0) \wedge (b \geq 1)) \vee ((a \geq 1) \wedge (b \geq 0))\},$$

*then (4.2.14) applies.*

## 4.2.2. Results

**Theorem 4.2.6.** *Let $g_1, \ldots, g_n : [a,b] \to I\!R$ be nonnegative nondecreasing functions with continuous first derivatives and $g_1(a) = \ldots = g_n(a), g_1(b) = \ldots = g_n(b)$.*

*Suppose* $\mathbf{w}$ *is a positive n-tuple.*

*a) Let f be a nonnegative nondecreasing function on* $[a,b]$*. If* $(r,s)$*,* $(u,v) \in B$*, then*

$$G_{rs}\left(\int_a^b g_1'(t)f(t)dt, \ldots, \int_a^b g_n'(t)f(t)dt; \mathbf{w}\right) \leq \int_a^b \left(G_{uv}(g_1(t), \ldots, g_n(t); \mathbf{w})\right)' f(t)dt. \tag{4.2.15}$$

*If* $(r,s)$*,* $(u,v) \in B^\star$*, then the reverse inequality holds.*

*b) Let f be a nonnegative nonincreasing function. If* $(r,s) \in B$ *and* $(u,v) \in B^\star$*, then inequality (4.2.15) holds, while if* $(r,s) \in B^\star$ *and* $(u,v) \in B$ *then the reverse inequality applies.*

The proof is the same as that of Theorem 4.1.4, except in that we use Lemma 4.2.5 in place of Lemma 4.1.2.

Similarly we can prove the following.

**Theorem 4.2.7.** *Let g and h be nonnegative nondecreasing functions with continuous first derivatives and* $g(a) = h(a), g(b) = h(b)$*.*

*a) Let f be a nonnegative nondecreasing function on* $[a,b]$*. If* $(r,s),(u,v) \in B$*, then*

$$G_{rs}\left(\int_a^b g'(t)f(t)dt, \int_a^b h'(t)f(t)dt\right) \leq \int_a^b G_{uv}\left(g(t),h(t)\right)' f(t)dt. \tag{4.2.16}$$

*If* $(r,s),(u,v) \in B^\star$*, then the reverse inequality holds.*

*b) Let f be a nonnegative nonincreasing function. If* $(r,s) \in B$ *and* $(u,v) \in B^\star$ *then inequality in (4.2.16) holds, while if* $(r,s) \in B^\star$ *and* $(u,v) \in B$ *then the reverse inequality applies.*

Now we shall give discrete analogues to the above results.

**Theorem 4.2.8.** *Let* $\mathbf{a}_1, \ldots, \mathbf{a}_n$ *be nonnegative nondecreasing n-tuples such that* $a_{11} = \ldots = a_{m1}$ *and* $a_{1n} = \ldots = a_{mn}$ *and let* $\mathbf{w}$ *be a positive n-tuple.*

*a) Let* $\mathbf{f}$ *be a nonnegative nondecreasing n-tuple. If* $(r,s), (u,v) \in C$, *then*

$$G_{rs}\left(\sum_{i=1}^{n-1} f_i \Delta a_{1i}, \ldots, \sum_{i=1}^{n-1} f_i \Delta a_{mi}; \mathbf{w}\right) \leq \sum_{i=1}^{n-1} f_i \Delta G_{rs}(a_{1i}, \ldots, a_{mi}; \mathbf{w}). \tag{4.2.17}$$

*If* $(r,s)$, $(u,v) \in C^\star$ *then the reverse inequality holds.*

*b) Let* $\mathbf{f}$ *be a nonnegative nonincreasing n-tuple. If* $(r,s) \in C$ *and* $(u,v) \in C^\star$, *then (4.2.17) applies. If* $(r,s) \in C^\star$ *and* $(u,v) \in C$, *then the inequality is reversed.*

**Theorem 4.2.9.** *Let* $a$ *and* $b$ *be nonnegative nondecreasing n-tuples such that* $a_n = b_n$ *and* $a_1 = b_1$.

*a) Let* $\mathbf{f}$ *be a nonnegative nondecreasing n-tuple. If* $(r,s), (u,v) \in B$, *then*

$$G_{rs}\left(\sum_{i=1}^{n-1} f_i \Delta a_i, \sum_{i=1}^{n-1} f_i \Delta b_i\right) \leq \sum_{i=1}^{n-1} f_i \Delta G_{uv}(a_i, b_i) \tag{4.2.18}$$

*If* $(r,s)$, $(u,v) \in B^\star$, *then the inequality is reversed.*

*b) Let* $\mathbf{f}$ *be a nonnegative nondecreasing n-tuple.*

*If* $(r,s) \in B$ *and* $(u,v) \in B^\star$, *then (4.2.18) applies. If* $(r,s) \in B^\star$ *and* $(u,v) \in B$, *then the reverse inequality holds.*

## 4.3. Inequalities involving generalized quasiarithmetic means

Suppose $I\!R_+ = (0, \infty)$ and $\phi = (\phi_1, \ldots, \phi_n) : (I\!R_+)^n \to (I\!R_+)$. Also suppose $M : I\!R_+ \to I\!R_+$ is strictly monotonic. We define generalized quasiarithmetic means by

$$M_n(\mathbf{a}, \phi) := M_n(a_1, \ldots, a_n; \phi) = M^{-1}\left(\frac{\sum_{i=1}^n \phi_i(a_i) M(a_i)}{\sum_{i=1}^n \phi_i(a_i)}\right).$$

The following results hold (see [21], [6], [7, pp 265-266]).

**Lemma 4.3.1.** *Let $M, K : I\!R_+ \to I\!R_+$ be differentiable strictly monotonic functions and $\chi$, $\phi$ functions from $(I\!R_+)^n$ to $(I\!R_+)^n$. Then*

$$M_n(\mathbf{a}, \chi) \leq K_n(\mathbf{a}, \phi) \tag{4.3.1}$$

*for all $\mathbf{a} \in (I\!R_+)^n$ if for all $u, t \in I\!R_+$*

$$\frac{M(u) - M(t)}{M'(t)} \cdot \frac{\chi_i(u)}{\chi_n(t)} \leq \frac{K(u) - K(t)}{K'(t)} \frac{\phi_i(u)}{\phi_n(t)}, \quad 1 \leq i \leq n. \tag{4.3.2}$$

*If (4.3.2) is reversed, than so is (4.3.1).*

**Definition 4.3.2.** We shall say that a generalized quasiarithmetic mean $M_n(\mathbf{a}; \mathbf{X})$ is subarithmetic if

$$M_n(\mathbf{a}, \chi) \leq A_n(\mathbf{a}, \mathbf{w}) := \frac{w_1 a_1 + \cdots + w_n a_n}{w_1 + \cdots + w_n}, \tag{4.3.3}$$

that is, if for all $u, t \in I\!R_+$

$$\frac{M(u) - M(t)}{M'(t)} \frac{\chi_i(u)}{\chi_n(t)} \leq \frac{w_i}{w_n}(u - t). \tag{4.3.4}$$

If (4.3.4) is reversed, then $M_n(\mathbf{a}, \chi)$ is superarithmetic.

**Theorem 4.3.3.** *Let $g_1, \ldots, g_n : [a, b] \to I\!R$ be nonnegative nondecreasing functions with continuing first derivatives and $g_1(a) = \cdots = g_n(a)$, $g_1(b) = \cdots = g_n(b)$.*

*a) Let $f$ be a nonnegative nondecreasing function on $[a, b]$. If $M_n(\mathbf{a}, \chi)$ and $L_n(\mathbf{a}, \phi)$ are subarithmetic, then*

$$\begin{aligned} &M_n\left(\int_a^b g_1'(t) f(t) dt, \ldots \int_a^b g_n'(t) f(t) dt; \chi\right) \\ &\leq \int_a^b \left(L_n(g_1(t), \ldots, g_n(t); \phi)\right)' f(t) dt. \end{aligned} \tag{4.3.5}$$

*If $M_n(\mathbf{a}, \chi)$ and $L_n(\mathbf{a}, \phi)$ are superarithmetic then (4.3.5) is reversed.*

*b) Let $f$ be a nonnegative nonincreasing function. If $M_n(\mathbf{a}, \chi)$ is subarithmetic and $L_n(\mathbf{a}, \phi)$ superarithmetic, then (4.3.5) holds, while if $M_n(\mathbf{a}, \chi)$ is superarithmetic and $L_n(\mathbf{a}, \phi)$ subarithmetic (4.3.5) is reversed.*

**Proof.** a) Let $M_n$ and $L_n$ be subarithmetic. Then by (4.3.3) we have

$$\begin{aligned}
M_n & \left(\int_a^b g_1'(t)f(t)dt, \ \ldots, \int_a^b g_n'(t)f(t)dt; \chi\right) \\
& \leq \frac{1}{w_1 + \cdots + w_n}\left\{w_1 \int_a^b g_1'(t)f(t)dt + \cdots + w_n \int_a^b g_n'(t)f(t)dt\right\} \\
& = \int_a^b (A(g_1(t), \ldots, g_n(t); \mathbf{w})' f(t)dt \\
& = A(g_1(t), \ldots, g_n(t); \mathbf{w}) f(t)|_a^b - \int_a^b A(g_1(t), \ldots, g_n(t); \mathbf{w}) df(t) \\
& \leq A(g_1(t), \ldots, g_n(t); \mathbf{w}) f(t)|_a^b - \int_a^b L_n(g_1(t), g_2(t); \phi) df(t) \\
& = A(g_1(t), \ldots, g_n(t); \mathbf{w}) f(t)|_a^b - L_n(g_1(t), \ldots, g_n(t); \phi) f(t)|_a^b \\
& \quad + \int_a^b (L_n(g_1(t), \ldots, g_n(t); \phi))' f(t)dt \\
& = \int_a^b (L_n(g_1(t), \ldots, g_n(t); \phi))' f(t)dt.
\end{aligned}$$

If $M_n$ and $L_n$ are superarithmetic, we have reversed inequalities.

b) Let $M_n$ be subarithmetic, $L_n$ superarithmetic and $F = -f$. We have

$$
\begin{aligned}
M_n & \left(\int_a^b g_1'(t)f(t)dt, \quad \dots, \int_a^b g_n'(t)f(t)dt; \chi\right) \\
& \le A_n\left(\int_a^b g_1'(t)f(t)dt, \dots, \int_a^b g_n'(t)f(t)dt; \mathbf{w}\right) \\
& = \int_a^b (A(g_1(t)dt, \dots, g_n(t); \mathbf{w})' f(t)dt \\
& = A\left(g_1(t), \dots, g_n(t); \mathbf{w}\right) f(t)|_a^b + \int_a^b A\left(g_1(t), \dots, g_n(t); \mathbf{w}\right) dF(t) \\
& \le A\left(g_1(t), \dots, g_n(t); \mathbf{w}\right) f(t)|_a^b + \int_a^b L_n\left(g_1(t), \dots, g_n(t); \phi\right) dF(t) \\
& = A\left(g_1(t), \dots, g_n(t); \mathbf{w}\right) f(t)|_a^b - L_n\left(g_1(t), \dots, g_n(t); \phi\right) f(t)|_a^b \\
& \quad + \int_a^b \left(L_n\left(g_1(t), \dots, g_n(t); \phi\right)\right)' f(t)dt \\
& = \int_a^b \left(L_n\left(g_1(t), \dots, g_n(t); \phi\right)\right)' f(t)dt.
\end{aligned}
$$

If $M_n$ is superarithmetic and $L_n$ subarithmetic, we have reversed inequalities.

A discrete analogue of Theorem 4.3.3 is the following.

**Theorem 4.3.4.** *Let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be nonnegative nondecreasing $n$-tuples such that $a_{11} = \cdots = a_{m1}$ and $a_{1n} = \cdots = a_{mn}$.*

*a) Let $\mathbf{f}$ be a nonnegative nondecreasing $n$-tuple. If $M_m(\mathbf{a}, \chi)$ and $L_m(\mathbf{a}, \phi)$ are subarithmetic, then*

$$
M_m\left(\sum_{i=1}^{n-1} f_i \Delta a_{1i}, \dots, \sum_{i=1}^{n-1} f_i \Delta a_{mi}; \chi\right) \le \sum_{i=1}^{n-1} f_i \Delta L_m\left(a_{1i}, \dots, a_{mi}, \phi\right). \tag{4.3.6}
$$

*If $M_m(\mathbf{a}, \chi)$ and $L_m(\mathbf{a}, \phi)$ are superarithmetic, then (4.3.6) is reversed.*

*b) Let $\mathbf{f}$ be a nonnegative nonincreasing $n$-tuple. If $M_m(\mathbf{a}, \chi)$ is subarithmetic* *and $L_m(\mathbf{a},\phi)$ superarithmetic then (4.3.6) holds, while if $M_m(\mathbf{a},\chi)$ is superarithmetic and $L_m(\mathbf{a},\phi)$ subarithmetic, the reverse inequality applies.*

**Proof** . a) Let $M_m(\mathbf{a},\chi)$ and $L_m(\mathbf{a},\phi)$ be subarithmetic. Then we have

$$
\begin{aligned}
M_m&\left(\sum_{i=1}^{n-1} f_i\Delta a_{1i}, \quad \ldots, \sum_{i=1}^{n-1} f_i\Delta a_{mi}; \chi\right)\\
&\le A_m\left(\sum_{i=1}^{n-1} f_i\Delta a_{1i}, \ldots, \sum_{i=1}^{n-1} f_i\Delta a_{mi}; \mathbf{w}\right)\\
&= \sum_{i=1}^{n-1} f_i\Delta A_m(a_{1i},\ldots,a_{mi};\mathbf{w})\\
&= f_nA_m(a_{1n},\ldots,a_{mn};\mathbf{w}) - f_1A_m(a_{11},\ldots,a_{m1};\mathbf{w})\\
&\quad -\sum_{i=2}^{n} A_m(a_{1i},\ldots,a_{mi};\mathbf{w})\Delta f_{i-1}\\
&\le f_nA_m(a_{1n},\ldots,a_{mn};\mathbf{w}) - f_1A_m(a_{11},\ldots,a_{m1};\mathbf{w})\\
&\quad -\sum_{i=2}^{n} L_m(a_{1i},\ldots,a_{mi};\phi)\Delta f_{i-1}\\
&= f_nA_m(a_{1n},\ldots,a_{mn};\ \mathbf{w}) - f_1A_m(a_{11},\ldots,a_{m1};\mathbf{w})\\
&\quad -f_nL_m(a_{1n},\ldots,a_{mn};\phi) - f_1L_m(a_{11},\ldots,a_{m1};\phi)\\
&\quad +\sum_{i=1}^{n-1} f_i\Delta L_m(a_{1i},\ldots,a_{mi};\phi)\\
&= \sum_{i=1}^{n-1} f_i\Delta L_m(a_{1i},\ldots,a_{mi};\phi).
\end{aligned}
$$

If $M_m(\mathbf{a},\chi)$ and $L_m(\mathbf{a},\phi)$ are superarithmetic, then the reverse inequalities apply.

b) Let $M_m(\ \mathbf{a},\chi)$ be subarithmetic and $L_m(\ \mathbf{a},\phi)$ superarithmetic. Write $F_i =$

$-f_i, (i = 1, \dots n)$. Then we have

$$\begin{aligned}
M_m &\left(\sum_{i=1}^{n-1} f_i \Delta a_{1i}, \quad \dots, \sum_{i=1}^{n-1} f_i \Delta a_{mi}; \chi\right) \\
&\leq A_m \left(\sum_{i=1}^{n-1} f_i \Delta a_{1i}, \dots, \sum_{i=1}^{n-1} f_i \Delta a_{mi}; \mathbf{w}\right) \\
&= \sum_{i=1}^{n-1} f_i \Delta A_m \left(a_{1i}, \dots, a_{mi}; \ \mathbf{w}\right) \\
&= f_n A_m \left(a_{1n}, \dots, a_{mn}; \ \mathbf{w}\right) - f_1 A_m \left(a_{11}, \dots, a_{m1}; \ \mathbf{w}\right) \\
&\quad + \sum_{i=2}^{n} A_m \left(a_{1i}, \dots, a_{mi}; \mathbf{w}\right) \Delta F_{i-1} \\
&\leq f_n A_m \left(a_{1n}, \dots, a_{mn}; \ \mathbf{w}\right) - f_1 A_m \left(a_{11}, \dots, a_{m1}; \ \mathbf{w}\right) \\
&\quad + \sum_{i=2}^{n} L_m \left(a_{1i}, \dots, a_{mi}; \phi\right) \Delta F_{i-1} \\
&= f_n A_m \left(a_{1n}, \dots, a_{mn}; \ \mathbf{w}\right) - f_1 A_m \left(a_{11}, \dots, a_{m1}; \ \mathbf{w}\right) \\
&\quad - f_n L_m \left(a_{1n}, \dots, a_{mn}; \phi\right) + f_1 L_m \left(a_{11}, \dots, a_{m1}; \phi\right) \\
&\quad + \sum_{i=1}^{n} f_i \Delta L_m \left(a_{1i}, \dots, a_{mi}; \phi\right) \\
&= \sum_{i=1}^{n} f_i \Delta L_m \left(a_{1i}, \dots, a_{mi}; \phi\right).
\end{aligned}$$

If $M_m(\mathbf{a}, \chi)$ is superarithmetic and $L_m(\ \mathbf{a}, \phi)$ subarithmetic, then the reverse inequality holds.

**Remark 4.3.5** Our results supply generalizations of related results for Gini means given in Section 4.2 as well as of related results for quasiarithmetic means from Subsection 3.1.2.

Of special interest are the following special cases.

**Corollary 4.3.6.** *Let $g_1, \dots, g_n : [a, b] \to I\!R$ be nonnegative nondecreasing functions* *with continuous first derivatives and* $g_1(a) = \cdots = g_n(a)$, $g_1(b) = \cdots = g_n(b)$, *and let* $f$ *be a nonnegative nondecreasing function on* $[a, b]$. *If* $M_n(\mathbf{a}, \chi)$ *is a subarithmetic generalized quasiarithmetic mean, then*

$$\text{(4.3.7)} \qquad \begin{aligned} &M_n\left(\int_a^b g_1'(t)f(t)dt, \ldots, \int_a^b g_n'(t)f(t)dt; \chi\right) \\ &\leq \int_a^b M_n\left(g_1(t), \ldots, g_n(t); \chi\right)' f(t)dt. \end{aligned}$$

*If* $M_n$ *is superarithmetic then (4.3.7) is reversed.*

**Corollary 4.3.7.** *Let* $\mathbf{a}_1, \ldots, \mathbf{a}_m$ *be nonnegative nondecreasing n-tuples such that* $a_{11} = \cdots = a_{m1}$ *and* $a_{1n} = \cdots = a_{mn}$, *and let* $\mathbf{f}$ *be a nonnegative nondecreasing n-tuple. If* $M_m(\mathbf{a}, \chi)$ *is a subarithmetic generalized quasiarithmetic mean, then*

$$\text{(4.3.8)} \qquad M_m\left(\sum_{i=1}^{n-1} f_i \Delta a_{1i}, \ldots, \sum_{i=1}^{n-1} f_i \Delta a_{mi}; \chi\right) \leq \sum_{i=1}^{n-1} f_i \Delta M_m\left(a_{1i}, \ldots, a_{mi}, \chi\right).$$

*If* $M_m(\mathbf{a}, \phi)$ *is superarithmetic, then (4.3.8) is reversed.*

## 4.4. Some further generalizations

We can give further generalizations of results in the previous section.

In place of means our results involve arbitrary functions which satisfy some special properties.

**Definition 4.4.1.** Let $w_1, \ldots, w_m$, be given real numbers such that $\sum_{i=1}^m w_i = 1$. We shall say that a function $F : I^m \to R$ (I is an interval from $R$) belong to the class $W$, if and only if the following conditions are satisfied

(*i*) $F(x, \ldots, x) = x$

and

(*ii*) $F(x_1, \ldots, x_m) \leq \sum_{i=1}^m w_i x_i$.

If in (ii) inequality is reversed, we say that $F$ belongs to the class $W^*$.

**Theorem 4.4.2.** *Let $F, G : I^m \to R$ and $f : [a,b] \to R$ be real functions. Let $g_1, \ldots, g_m : [a,b] \to I$ be continously differentiable functions such that $G(g_1, \ldots, g_m)$ is also continously differentiable and $g_1(a) = \cdots = g_m(a), g_1(b) = \cdots = g_m(b)$, $\int_a^b g_i'(t)f(t)dt \in I$, $i = 1, \ldots, m$.*

*a) If $f$ is a nondecreasing function on $[a,b]$ and if $F, G \in W$, then*

$$F\left(\int_a^b g_1'(t)f(t)dt, \ldots, \int_a^b g_m'(t)f(t)dt\right) \leq \int_a^b (G(g_1(t), \ldots, g_m(t)))' f(t)dt. \tag{4.4.1}$$

*If $F, G \in W^*$, then (4.4.1) is reversed.*

*b) Let $f$ be a nonincreasing function. If $F \in W$ and $G \in W^*$ then (4.4.1) holds. If $F \in W^*$ and $G \in W$, the inequality is reversed.*

**Proof.** The proof is similar to that of Theorem 4.3.4, so we shall give only the proof of part a).

a) Let $F, G \in W$. Then we have

$$
\begin{aligned}
F\left(\int_a^b g_1'(t)f(t)dt, \ \ldots, \int_a^b g_m'(t)f(t)dt\right) & \\
&\leq \sum_{i=1}^m w_i \int_a^b g_i'(t)f(t)dt \\
&= \int_a^b \left(\sum_{i=1}^m w_i g_i'(t)\right) f(t)dt \\
&= \int_a^b \left(\sum_{i=1}^m w_i g_i(t)\right)' f(t)dt \\
&= \left(\sum_{i=1}^m w_i g_i(t)\right) f(t)\Big|_a^b - \int_a^b \left(\sum_{i=1}^m w_i g_i(t)\right) df(t) \\
&= g_1(b)f(b) - g_1(a)f(a) - \int_a^b \left(\sum_{i=1}^m w_i g_i(t)\right) df(t) \\
&\leq g_1(b)f(b) - g_1(a)f(a) - \int_a^b G(g_1(t), \ldots, g_m(t))df(t) \\
&= g_1(b)f(b) - g_1(a)f(a) - G\left(g_1(t), \ldots, g_m(t)\right) f(t)|_a^b \\
&\quad + \int_a^b \left(G(g_1(t), \ldots, g_m(t))\right)' f(t)dt \\
&= \int_a^b \left(G(g_1(t), \ldots, g_m(t))\right)' f(t)dt.
\end{aligned}
$$

If $F, G \in W^*$, we have reversed inequalities.

A discrete analogue of Theorem 4.4.2 is as follows.

**Theorem 4.4.3.** *Let $a_1, \ldots, a_m$ , be real $n$-tuples with components from $I$ such that $a_{11} = \cdots = a_{m1}$ and $a_{1n} = \cdots = a_{mn}$ and let $f$ be a real $n$-tuple such that*

$$\sum_{i=1}^{n-1} f_i \Delta a_{ji} \in I, j = 1, \ldots, m.$$

*a) If $f$ is a nondecreasing $n$-tuple and if $F, G \in W$, then*

$$F\left(\sum_{i=1}^{n-1} f_i \Delta a_{1i}, \ldots, \sum_{i=1}^{n-1} f_i \Delta a_{mi}\right) \leq \sum_{i=1}^{n-1} f_i \Delta G\left(a_{1i}, \ldots, a_{mi}\right). \tag{4.4.2}$$

*If $F, G \in W^*$, then (4.4.2) is reversed.*

*b) Let $f$ be a nonincreasing $n$-tuple. If $F \in W$ and $G \in W^*$, then (4.4.2) holds. If $F \in W^*$ and $G \in W$, then the reversed inequality applies.*

**Proof.** The proof is similar to that of Theorem 4.3.4.

Note that in (2.6.1') we have the same mean on the both sides of the inequality. So the following special cases of Theorem 4.4.2 and 4.4.3 are of special interest.

**Corollary 4.4.4.** *Let $F : I^m \to R$, $g_1, \ldots, g_m : [a,b] \to I$, $f : [a,b] \to R$ be real functions such that $g_1, \ldots, g_n$, $F(g_1, \ldots, g_n)$ are continuously differentiable and $g_1(a) = \cdots = g_m(a)$,$g_1(b) = \cdots = g_m(b)$, $\int_a^b g_i'(t)f(t)df(t) \in I, i = 1, \ldots, m$ and $f$ is nondecreasing.*

*If $F \in W$, then*

$$F\left(\int_a^b g_1'(t)f(t)dt, \ldots, \int_a^b g_m'(t)f(t)dt\right) \leq \int_a^b \left(F(g_1(t), \ldots, g_m(t)\right)' f(t)dt. \tag{4.4.3}$$

*If $F \in W^*$, then the reverse inequality applies.*

**Corollary 4.4.5.** *Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be real $n$-tuples with components in $I$ such that $a_{11}, \cdots = a_{m1}$ and let $a_{1n} \cdots = a_{mn}$ and $\mathbf{f}$ be a nondecreasing real $n$-tuple such that $\sum_{i=1}^{n-1} f_i \Delta a_{ji} \in I$, $j = 1, \ldots, n$. If $F \in W$, then*

$$F\left(\sum_{i=1}^{n-1} f_i \Delta a_{1i}, \ldots, \sum_{i=1}^{n-1} f_i \Delta a_{m1}\right) \leq \sum_{i=1}^{n-1} f_i \Delta F\left(a_{1i}, \ldots, a_{m1}\right). \tag{4.4.4}$$

*If $F \in W^*$, then the reverse inequality applies.*

# 5. OPERATOR VERSIONS OF PÓLYA'S INEQUALITIES

## 5.0. Overview

Inequalities are relatively difficult to establish for operators. The development of a consolidated theory and indeed any theory at all is largely due to the genius of Kubo and Ando [19]. We begin this chapter by noting some of their key concepts. From their foundation we develop a variety of results paralleling those of our earlier chapters. This material is the content of two published papers [30], [31] and a further paper [32] accepted for publication.

## 5.1. Operator versions of some classical inequalities

### 5.1.1. Preliminaries

Let us consider bounded, linear and positive (that is, positive semi-definite) operators on an infinite–dimensional Hilbert space. A scalar multiple of the identity operator is denoted by the scalar itself; in particular, 1 is the identity operator. The order relation $A \leq B$ means that $B - A$ is positive. That $A_1 \geq A_2 \geq \ldots$, and $A_n$ converges strongly to $A$ is denoted by $A_n \downarrow A$.

A binary operator $\sigma$ on the class of positive operators, $(A, B) \to A\sigma B$ is called a **connection** if the following requirements are fulfilled [19]:

(I) $A \leq C$ and $B \leq D \Rightarrow A\sigma B \leq C\sigma D$,

(II) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$,

(III) $A_n \downarrow A$ and $B_n \downarrow B \Rightarrow (A_n\sigma B_n) \downarrow A\sigma B$.

A **mean** is a connection with normalization condition

(IV) $1\sigma 1 = 1$.

The following results are also valid [19]:

Every mean $\sigma$ possesses the property

(IV′) $A\sigma A = A$ for every $A$.

Every connection $\sigma$ possesses the property

(I′) $(A\sigma B) + (C\sigma D) \leq (A + C)\sigma(B + D)$.

The simplest examples of means are

ARITHMETIC MEAN: $A\nabla B = \frac{1}{2}(A + B)$,

HARMONIC MEAN: $A!B = 2(A^{-1} + B^{-1})^{-1}$,

GEOMETRIC MEAN: $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$

for invertible $A$ and $B$.

Moreover, weighted versions of these means can also be defined.

Let $A, B$ be invertible and $\lambda \in (0, 1)$ be a real number. Then the arithmetic, geometric and harmonic means are defined, respectively by

$$A\nabla_\lambda B = \lambda A + (1 - \lambda)B,$$

$$A\#_\lambda B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1-\lambda}A^{1/2},$$

$$A!_\lambda B = (\lambda A^{-1} + (1 - \lambda)B^{-1})^{-1}.$$

We know that

$$A!_\lambda B \leq A\#_\lambda B \leq A\nabla_\lambda B. \tag{5.1.1}$$

Every mean possesses the property [29] $A\sigma A = A$ for every A. A mean is *symmetric* by definition if $A\sigma B = B\sigma A$.

Arithmetic, geometric and harmonic means ($\nabla$,! and #) are symmetric [29].

The arithmetic mean is the maximum of all symmetric means while the harmonic mean is the minimum, that is, the following generalization of (5.1.1) holds [29].

For every symmetric mean $\sigma$, we have

$$A!B \leq A\sigma B \leq A\nabla B. \tag{5.1.1a}$$

### 5.1.2. Operator versions of Cauchy, Hölder and other classical inequalities

Mathematical induction from (I′) gives the following.

**Theorem 5.1.1.** *Let $A_i, B_i, i = 1, \ldots, n$, be bounded linear and positive operators and let $\sigma$ be a connection. Then*

$$\sum_{i=1}^{n} (A_i \sigma B_i) \leq \left(\sum_{i=1}^{n} A_i\right) \sigma \left(\sum_{i=1}^{n} B_i\right). \tag{5.1.2}$$

In the next examples, we assume that $A_i$ and $B_i$ are invertible.

**Examples:** [30]

1°. Cauchy's inequality:

$$\sum_{i=1}^{n} A_i^2 \# B_i^2 \leq \left(\sum_{i=1}^{n} A_i^2\right) \# \left(\sum_{i=1}^{n} B_i^2\right). \tag{5.1.3}$$

2°. Hölder's inequality:

Let $p, q > 0$ with $p^{-1} + q^{-1} = 1$.

$$\sum_{i=1}^{n} A_i^p \#_{1/p} B_i^q \leq \left(\sum_{i=1}^{n} A_i^p\right) \#_{1/p} \left(\sum_{i=1}^{n} B_i^q\right). \tag{5.1.4}$$

3°. Minkowski's inequality:

$$\sum_{i=1}^{n} (A_i + B_i)^{-1} \leq \left[\left(\sum_{i=1}^{n} A_i^{-1}\right)^{-1} + \left(\sum_{i=1}^{n} B_i^{-1}\right)^{-1}\right]^{-1}. \tag{5.1.5}$$

Indeed, this last inequality follows by letting $\sigma$ be the parallel sum, that is,

$$A \sigma B = (A^{-1} + B^{-1})^{-1} \quad \text{and} \quad A_i = A_i^{-1}, B_i = B_i^{-1}.$$

**Theorem 5.1.2.** *Let $A_i, B_i, i = 1, \ldots, n$ be bounded, linear and positive operators such that*

$$A_1 - A_2 - \ldots - A_n \geq 0 \quad \textit{and} \quad B_1 - B_2 - \ldots - B_n \geq 0. \tag{5.1.6}$$

*Then*

$$A_1 \sigma B_1 - \sum_{i=2}^{n} A_i \sigma B_i \geq \left(A_1 - \sum_{i=2}^{n} A_i\right) \sigma \left(B_1 - \sum_{i=2}^{n} B_i\right). \tag{5.1.7}$$

**Proof.** With the substitutions

$$A_1 \to A_1 - A_2 - \ldots - A_n, \; B_1 \to B_1 - B_2 - \ldots - B_n,$$

(5.1.2) becomes

$$(A_1 - A_2 - \ldots - A_n) \sigma (B_1 - B_2 - \ldots - B_n) + \sum_{i=2}^{n} A_i \sigma B_i \leq A_1 \sigma B_1,$$

that is, (5.1.7).

In the following examples $A_i$ and $B_i$ are again invertible.

**Examples.**

4°. Aczél's inequality:

If

$$A_1^2 - A_2^2 - \ldots - A_n^2 > 0 \quad \text{and} \quad B_1^2 - B_2^2 - \ldots - B_n^2 > 0,$$

then

$$A_1^2 \# B_1^2 - \sum_{i=2}^n A_i^2 \# B_i^2 \geq \left(A_1^2 - \sum_{i=2}^n A_i^2\right) \# \left(B_1^2 - \sum_{i=2}^n B_i^2\right).$$

5°. Popoviciu's inequality:

If $p, q > 0, p^{-1} + q^{-1} = 1$ and

$$A_1^p - A_2^p - \ldots - A_n^p > 0,\ B_1^q - B_2^q - \ldots - B_n^q > 0,$$

then

$$A_1^p \#_{1/p} B_1^q - \sum_{i=2}^n A_i^p \#_{1/p} B_i^q \geq \left(A_1^p - \sum_{i=2}^n A_i^p\right) \#_{1/p} \left(B_1^q - \sum_{i=2}^n B_i^q\right).$$

6°. Bellman's inequality:

If

$$A_1^{-1} - A_2^{-1} - \ldots - A_n^{-1} > 0 \quad \text{and} \quad B_1^{-1} - B_2^{-1} - \ldots - B_n^{-1} > 0,$$

then

$$\begin{aligned} (A_1 + B_1)^{-1} - & \sum_{i=2}^n (A_i + B_i)^{-1} \\ & \geq [(A_1^{-1} - \sum_{i=2}^n A_i^{-1})^{-1} + (B_1^{-1} - \sum_{i=2}^n B_1^{-1})^{-1}]^{-1}. \end{aligned} \tag{5.1.8}$$

**Remark 5.1.3.**

1°. Note that (5.1.5) and (5.1.8) can be given in the forms

$$\left(\sum_{i=1}^n (A_i + B_i)^{-1}\right)^{-1} \geq \left(\sum_{i=1}^n A_i^{-1}\right)^{-1} + \left(\sum_{i=1}^n B_i^{-1}\right)^{-1} \tag{5.1.5'}$$

and

$$\begin{aligned} [(A_1 + B_1)^{-1} - & \sum_{i=2}^n (A_i + B_i)^{-1}]^{-1} \\ & \leq \left(A_1^{-1} - \sum_{i=2}^n A_i^{-1}\right)^{-1} + \left(B_1^{-1} - \sum_{i=2}^n B_i^{-1}\right)^{-1}. \end{aligned} \tag{5.1.8'}$$

Note that the following generalization of (5.1.5′) is obtained in [3] for positive invertible operators $A_{ij}, (i = 1, \ldots, n;\ j = 1, \ldots, m)$:

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{-1}\right)^{-1} \leq \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m} A_{ij}\right)^{-1}\right)^{-1} \tag{5.1.9}$$

We can use (5.1.9) in the proof of the following extension of (5.1.8′).

If positive invertible operators $A_{ij}$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ satisfy the conditions

$$A_{1j}^{-1} - A_{2j}^{-1} - \ldots - A_{mj}^{-1} > 0,\ j = 1, \ldots, n,$$

then

$$\left(\sum_{j=1}^{m} A_{1j}\right)^{-1} - \sum_{i=2}^{m}\left(\sum_{j=1}^{n} A_{ij}\right)^{-1} \geq \left[\sum_{j=1}^{n}(A_{1j}^{-1} - \sum_{i=2}^{m} A_{ij}^{-1})^{-1}\right]^{-1}.$$

**Remark 5.1.4.** A simpler form of Hölder's inequality is

$$\sum_{i=1}^{n} A_i \#_\alpha B_i \leq \left(\sum_{i=1}^{n} A_i\right) \#_\alpha \left(\sum_{i=1}^{n} B_i\right), \tag{5.1.4′}$$

where $0 \leq \alpha \leq 1$.

Setting $A_i \to A_i^s$, $B_i \to A_i^r$ in (5.1.4′) we get

$$\begin{aligned}\sum_{i=1}^{n} A_i^{\alpha s+(1-\alpha)r} &\leq \left(\sum_{i=1}^{n} A_i^s\right) \#_\alpha \left(\sum_{i=1}^{n} A_i^r\right)\\ &\leq \alpha\left(\sum_{i=1}^{n} A_i^s\right) + (1-\alpha)\left(\sum_{i=1}^{n} A_i^r\right),\end{aligned}$$

where we have used (5.1.1).

This proves that the function

$$x \mapsto \sum_{i=1}^{n} A_i^x$$

is convex.

### 5.1.3. Inequalities for solidarities

An extension of the Kubo-Ando theory was given by J.I. Fujii, M. Fujii and Y. Seo [13].

A binary operation $s$ on positive operators is an **abstract solidarity** if it satisfies, assuming the existence of $AsB$ as a bounded operator, conditions

(S1) $B \le C$ implies $AsB \le AsC$,

(S2r) $B_n \downarrow B$ implies $AsB_n \downarrow AsB$,

(S2$\ell$) $A_n \to A$ strongly implies $A_n s1 \to As1$ strongly and

(S3) $T^*(AsB)T \le T^*ATsT^*BT$.

The solidarity $s$ is superadditive in that

$$(A+B)s(C+D) \ge AsC + BsD. \tag{5.1.10}$$

Of special interest is the relative operator entropy $S(A|B)$ for invertible $A$, $B$ defined by

$$S(A|B) = A^{1/2}(\log A^{-1/2}BA^{-1/2})A^{1/2}.$$

Using (5.1.10), we can prove the following by mathematical induction.

Let $A_i, B_i, i = 1, \dots, n$ be positive operators. Then

$$\sum_{i=1}^{n}(A_i s B_i) \le \left(\sum_{i=1}^{n} A_i\right) s \left(\sum_{i=1}^{n} B_i\right). \tag{5.1.11}$$

Also, if (5.1.5) holds, then

$$A_1 s B_1 - \sum_{i=2}^{n} A_i s B_i \ge \left(A_1 - \sum_{i=2}^{n} A_i\right) s \left(B_1 - \sum_{i=2}^{n} B_i\right). \tag{5.1.12}$$

For operator entropy, we have

$$\sum_{i=1}^{n} S(A_i|B_i) \le S\left(\sum_{i=1}^{n} A_i \Big| \sum_{i=1}^{n} B_i\right)$$

and, if (5.1.6) holds,

$$S(A_1|B_1) - \sum_{i=2}^{n} S(A_i|B_i) \geq S\left(A_1 - \sum_{i=2}^{n} A_i|B_1 - \sum_{i=2}^{n} B_i\right).$$

## 5.2. Pólya inequalities for positive linear operators

### 5.2.1. Main results

We shall also use the classical notation for a finite difference

$$\triangle A_i = A_{i+1} - A_i \quad (i = 1, \ldots, n-1)$$

and

$$\triangle A_{ji} = A_{j,i+1} - A_{ji}.$$

**Theorem 5.2.1.** *Let $A_1 \leq \ldots \leq A_n$ (not all equal) and $B_1 \leq \ldots \leq B_n$ (not all equal) be bounded linear positive and invertible operators on an infinite dimensional Hilbert space such that $A_1 = B_1$ and $A_n = B_n$, and let $a_1 \leq \ldots \leq a_n$ be positive numbers. If $\sigma$ and $m$ are symmetric means, then*

$$\left(\sum_{i=1}^{n-1} a_i \triangle A_i\right) \sigma \left(\sum_{i=1}^{n-1} a_i \triangle B_i\right) \leq \sum_{i=1}^{n-1} a_i \triangle (A_i m B_i). \tag{5.2.1}$$

**Proof.** We have, using the second inequality of (5.1.1) or (5.1.1a),

$$\begin{aligned}
\sum_{i=1}^{n-1} a_i\Delta(A_i m B_i) &= a_n(A_n m B_n) - a_1(A_1 m B_1) - \sum_{i=2}^{n}(A_i m B_i)\Delta a_{i-1} \\
&\geq a_n(A_n m B_n) - a_1(A_1 m B_1) - \sum_{i=2}^{n}(A_i \nabla B_i) a_{i-1} \\
&= a_n(A_n m B_n) - a_1(A_1 m B_1) - a_n(A_n \nabla B_n) + a_1(A_1 \nabla B_1) \\
&\quad + \sum_{i=1}^{n-1} a_i\Delta(A_i \nabla B_i) \\
&= \sum_{i=1}^{n-1} a_i\Delta(A_i \nabla B_i) \\
&= \left(\sum_{i=1}^{n-1} a_i\Delta A_i\right) \nabla \left(\sum_{i=1}^{n-1} a_i\Delta B_i\right) \\
&\geq \left(\sum_{i=1}^{n-1} a_i\Delta A_i\right) \sigma \left(\sum_{i=1}^{n-1} a_i\Delta B_i\right).
\end{aligned}$$

**Corollary 5.2.2.** Let $\{A_i\}, \{B_i\}$ and $\{a_i\}$ satisfy the conditions of Theorem 5.2.1. Then

$$\left(\sum_{i=1}^{n-1} a_i\Delta A_i\right) \# \left(\sum_{i=1}^{n-1} a_i\Delta B_i\right) \leq \sum_{i=1}^{n-1} a_i\Delta(A_i \# B_i). \tag{5.2.2}$$

**Theorem 5.2.3.** *Let $A_1 \leq \ldots \leq A_n$ (not all equal) and $B_1 \leq \ldots \leq B_n$ (not all equal) be bounded, linear and positive operators and let $a = \{a_1, \ldots, a_n\}$ be a nondecreasing positive n-tuple of real numbers and let $\sigma$ be a connection. Then*

$$\left(\sum_{i=1}^{n-1} a_i\Delta A_i\right) \sigma \left(\sum_{i=1}^{n-1} a_i\Delta B_i\right) \leq \sum_{i=1}^{n} a_i\Delta(A_i \sigma B_i). \tag{5.2.3}$$

*If $a$ is a nonincreasing positive n-tuple of real numbers and $A_1 = B_1 = 0$, then (5.2.3) is reversed.*

**Proof.**

**(i)** We have

$$
\begin{aligned}
\sum_{i=1}^{n-1} a_i\triangle(A_i\sigma B_i) &= a_n(A_n\sigma B_n) - a_1(A_1\sigma B_1) - \sum_{i=2}^{n}(A_i\sigma B_i)\triangle a_{i-1} \\
&= a_n(A_n\sigma B_n) - a_1(A_1\sigma B_1) - \sum_{i=2}^{n}[(A_i\triangle a_{i-1})\sigma(B_i\triangle a_{i-1})] \\
&\geq (a_nA_n)\sigma(a_nB_n) - (a_1A_1)\sigma(a_1B_1) \\
&\quad - \left(\sum_{i=2}^{n} A_i\triangle a_{i-1}\right)\sigma\left(\sum_{i=2}^{n} B_i\triangle a_{i-1}\right) \qquad \text{(by (5.1.2))} \\
&\geq (a_nA_n - a_1A_1 - \sum_{i=2}^{n} A_i\triangle a_{i-1})\sigma(a_nB_n - a_1B_1 - \sum_{i=2}^{n} B_i\triangle a_{i-1}) \\
&\quad \text{(by (5.1.7) for 3 terms)} \\
&= \left(\sum_{i=1}^{n-1} a_i\triangle A_i\right)\sigma\left(\sum_{i=1}^{n-1} a_i\triangle B_i\right).
\end{aligned}
$$

**(ii)** Moreover, suppose $a$ is nonincreasing and $A_1 = B_1 = 0$. Since $-a$ is nondecreasing, we have as a consequence of (5.1.2) that

$$
\begin{aligned}
\sum_{i=1}^{n-1} a_i\triangle(A_i\sigma B_i) &= a_n(A_n\sigma B_n) + \sum_{i=2}^{n}(A_i\sigma B_i)\triangle(-a_{i-1}) \\
&= (a_nA_n)\sigma(a_nB_n) + \sum_{i=2}^{n}[(A_i\triangle(-a_{i-1}))\sigma(B_i\triangle(-a_{i-1}))] \\
&\leq (a_nA_n)\sigma(a_nB_n) + \left(\sum_{i=2}^{n} A_i\triangle(-a_{i-1})\right)\sigma\left(\sum_{i=2}^{n} B_i\triangle(-a_{i-1})\right) \\
&\leq (a_nA_n + \sum_{i=2}^{n} A_i\triangle(-a_{i-1}))\sigma(a_nB_n + \sum_{i=2}^{n} B_i\triangle(-a_{i-1})) \\
&= \left(\sum_{i=1}^{n-1} a_i\triangle A_i\right)\sigma\left(\sum_{i=1}^{n-1} a_i\triangle B_i\right).
\end{aligned}
$$

**Remark 5.2.4.** Theorem 5.2.3 gives the inequality (5.2.1) for $m = \sigma$ for arbitrary connections (not only symmetric means) and also without the conditions $A_1 = B_1$ and $A_n = B_n$. Also, we have a converse result in our theorem.

**Corollary 5.2.5.** *Let $A_1 \leq \ldots \leq A_n$ (not all equal) and $B_1 \leq \ldots \leq B_n$ (not all equal) be bounded, linear, positive and invertible operators and let $\lambda \in [0,1]$. If $a$ is a nondecreasing $n$-tuple of positive numbers, then*

$$\left(\sum_{i=1}^{n-1} a_i \triangle A_i\right) \#_\lambda \left(\sum_{i=1}^{n-1} a_i \triangle B_i\right) \leq \sum_{i=1}^{n-1} a_i (A_i \#_\lambda B_i), \tag{5.2.4}$$

*and*

$$\left(\sum_{i=1}^{n-1} a_i \triangle A_i\right) !_\lambda \left(\sum_{i=1}^{n-1} a_i \triangle B_i\right) \leq \sum_{i=1}^{n-1} a_i (A_i !_\lambda B_i). \tag{5.2.5}$$

*If $a$ is a nonincreasing $n$-tuple of positive numbers and $A_1 = B_1 = 0$, then (5.2.4) and (5.2.5) are reversed.*

As in Theorem 5.2.3, we can also prove the following, as a consequence of (5.1.11) and (5.1.12).

**Theorem 5.2.6.** *Let $A_1 \leq \ldots \leq A_n$ (not all equal) and $B_1 \leq \ldots \leq B_n$ (not all equal) be bounded, linear and positive operators and let $a = \{a_1, \ldots, a_n\}$ be a nondecreasing $n$-tuple of positive numbers and let $s$ be an abstract solidarity. Then*

$$\left(\sum_{i=1}^{n-1} a_i \triangle A_i\right) s \left(\sum_{i=1}^{n} a_i \triangle B_i\right) \leq \sum_{i=1}^{n-1} a_i \triangle (A_i s B_i). \tag{5.2.6}$$

*If $a$ is a nonincreasing $n$-tuple of positive numbers and $A_1 = B_1 = 0$, then (5.2.6) is reversed.*

In the case of operator entropy, (5.2.6) becomes

$$S\left(\sum_{i=1}^{n-1} a_i \triangle A_i | \sum_{i=1}^{n-1} a_i \triangle B_i\right) \leq \sum_{i=1}^{n-1} a_i S(A_i | B_i).$$

## 5.2.2. A generalization of the geometric mean inequality

A weighted generalization of (5.1.1) was recently obtained for matrices in [51]. A related result also holds in the operator case.

Let $w_1, \ldots, w_r$ be positive numbers such that $w_1 + \ldots + w_r = 1$ and let $C_1, \ldots, C_r$ be bounded, linear positive and invertible operators. Consider the arithmetic, geometric and harmonic means of the operators $C_i$ defined by

$$A_r(C_1, \ldots, C_r) = w_1 C_1 + \ldots + w_r C_r,$$

$$G_r(C_1, \ldots, C_r) = C_r^{1/2}(C_r^{-1/2} C_{r-1}^{1/2} \ldots (C_3^{-1/2} C_2^{1/2} (C_2^{-1/2} C_1 C_2^{-1/2})^{u_1}$$

$$C_2^{1/2} C_3^{-1/2})^{u_2} \ldots C_{r-1}^{1/2} C_r^{-1/2})^{u_{r-1}} C_r^{1/2},$$

$$H_r(C_1, \ldots, C_r) = (w_1 C_1^{-1} + \ldots + w_r C_r^{-1})^{-1},$$

where $u_i = 1 - w_{i+1} / \sum_{k=1}^{i+1} w_k$ for $i = 1, \ldots, r-1$. Then

$$H_r(C_1, \ldots, C_r) \le G_r(C_1, \ldots, C_r) \le A_r(C_1, \ldots, C_r). \tag{5.2.8}$$

(see [51]).

Note also that $G_r(C, \ldots, C) = A_r(C, \ldots, C) = H_r(C, \ldots, C) = C$.

**Theorem 5.2.7.** *Let $A_{j1} \le \ldots \le A_{jn}$ (not all equal), $j = 1, \ldots, r$ be bounded, linear, positive and invertible operators on an infinite-dimensional Hilbert space such that $A_{11} = \ldots = A_{r1}$ and $A_{1n} = \ldots = A_{rn}$ and let $a_1 \le \ldots \le a_n$ be positive numbers. Then*

$$G_r\left(\sum_{i=1}^{n-1} a_i \triangle A_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle A_{ri}\right) \le \sum_{i=1}^{n-1} a_i \triangle G_r(A_{1i}, \ldots, A_{ri}). \tag{5.2.9}$$

**Proof.** We proceed as in the proof of Theorem 5.2.1, but in place of (5.1.1), we shall use (5.2.8), that is, the second inequality in (5.2.8). Thus we have

$$\begin{aligned}
\sum_{i=1}^{n-1} a_i \, \triangle G_r(A_{1i}, \ldots, A_{ri}) &= A_n G_r(A_{1n}, \ldots, A_{rn}) - a_1 G_r(A_{11}, \ldots, A_{r1}) \\
&\quad - \sum_{i=2}^{n} G_r(A_{1i}, \ldots, A_{ri}) \triangle a_{i-1} \\
&\geq a_n G_r(A_{1n}, \ldots, A_{rn}) - a_1 G_r(A_{11}, \ldots, A_{r1}) - \sum_{i=2}^{n} A_r(A_{1i}, \ldots, A_{ri}) \triangle a_{i-1} \\
&= a_n G_r(A_{1n}, \ldots, A_{rn}) - a_1 G_r(A_{11}, \ldots, A_{r1}) \\
&\quad - a_n A_r(A_{1n}, \ldots, A_{rn}) + a_1 A_r(A_{11}, \ldots, A_{r1}) + \sum_{i=1}^{n-1} a_i \triangle A_r(A_{1i}, \ldots, A_{ri}) \\
&= \sum_{i=1}^{n-1} a_i \triangle A_r(A_{1i}, \ldots, A_{ri}) \\
&= A_r \left( \sum_{i=1}^{n-1} a_i \triangle A_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle A_{ri} \right) \\
&\geq G_r \left( \sum_{i=1}^{n-1} a_i \triangle A_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle A_{ri} \right).
\end{aligned}$$

**Remark 5.2.8.** Similarly, using (5.2.8), we can prove

$$H_r \left( \sum_{i=1}^{n-1} a_i \triangle A_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle A_{ri} \right) \leq \sum_{i=1}^{n-1} a_i \triangle H_r(A_{1i}, \ldots, A_{ri}); \tag{5.2.10}$$

$$G_r \left( \sum_{i=1}^{n-1} a_i \triangle A_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle A_{ri} \right) \leq \sum_{i=1}^{n-1} a_i \triangle H_r(A_{1i}, \ldots, A_{ri}); \tag{5.2.11}$$

and

$$H_r \left( \sum_{i=1}^{n-1} a_i \triangle A_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle A_{ri} \right) \leq \sum_{i=1}^{n-1} a_i \triangle G_r(A_{1i}, \ldots, A_{ri}). \tag{5.2.12}$$

(see [30]).

## 5.3. Further inequalities of Pólya type for positive linear operators

In this section, we give various generalizations of (5.2.10) by using results from [29]. The results are published in Mond, Pečarić, Šunde and Varošanec [31].

We denote by $S(J)$ the set of all self-adjoint operators on a Hilbert space whose spectra are contained in an interval $J$. If $J = (0, \infty)$, we write $S(0, \infty)$.

Let $X = (X_1, \dots, X_m)$ be an $m$-tuple of operators from $S(0, \infty)$, $A_j (j = 1, \dots, m)$ be contractions such that

$$\sum_{j=1}^{m} A_j^* A_j = I, \tag{5.3.1}$$

where $I$ is the identity operator, and let $A$ denote the $m$-tuple $(A_1, \dots, A_m)$. The power means

$$M_m^{[r]}(X; A) = \left( \sum_{j=1}^{m} A_j^* X_j^r A_j \right)^{1/r}$$

of $X$ with weights $A$ of order $r (\in R \backslash \{0\})$ were considered in [29].

**Theorem 5.3.1.** *The inequality*

$$M_m^{[r]}(X; A) \leq M_m^{[s]}(X; A)$$

*holds, if either*

*(a)* $r \leq s$, $r \notin (-1, 1)$, $s \notin (-1, 1)$; *or*

*(b)* $s \geq 1 \geq r \geq 1/2$ *or*

*(c)* $r \leq -1 \leq s \leq -1/2$.

### 5.3.1. Pólya–type inequalities for power means

Let $J_1 = (-\infty, -1] \cup [1/2, 1]$ and $J_2 = [1, \infty)$. We prove the following.

**Theorem 5.3.2.** *Let $C_{j1} \leq \ldots C_{jn}$ (not all equal), $j = 1, \ldots, n$ be operators from $S(0, \infty)$ such that $C_{11} = \ldots = C_{m1}$ and $C_{1n} = \ldots = C_{mn}$, $A_j(j = 1, \ldots, m)$ be contractions such that (5.3.1) holds and let $a_1 \leq \ldots \leq a_n$ ($a_1 \geq \ldots \geq a_n$, resp.) be positive numbers. If $r, s \in J_1$($r \in J_1$ and $s \in J_2$, resp.), then*

$$M_m^{[r]}\left(\sum_{i=1}^{n-1} a_i \triangle C_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle C_{mi}; A\right) \leq \sum_{i=1}^{n-1} a_i \triangle M_m^{[s]}(C_{1i}, \ldots, C_{mi}; A). \tag{5.3.4}$$

*If $r, s \in J_2$($r \in J_2$ and $s \in J_1$, resp.), then the inequality is reversed.*

**Proof.** Let $r, s \in J_1$ and $a_1 \leq \ldots \leq a_n$. We have, by Theorem 5.3.1,

$$\begin{aligned}
&\sum_{i=1}^{n-1} a_i \ \triangle M_m^{[s]}(C_{1i}, \ldots, C_{mi}; A) \\
&= a_n M_m^{[s]}(C_{1n}, \ldots, C_{mn}; A) - a_1 M_m^{[s]}(C_{11}, \ldots, C_{m1}; A) \\
&\quad - \sum_{i=2}^{n} M_m^{[s]}(C_{1i}, \ldots, C_{mi}; A) \triangle a_{i-1} \\
&\geq a_n M_m^{[s]}(C_{1n}, \ldots, C_{mn}; A) - a_1 M_m^{[s]}(C_{11}, \ldots, C_{m1}; A) \\
&\quad - \sum_{i=2}^{n} M_m^{[1]}(C_{1i}, \ldots, C_{mi}; A) \triangle a_{i-1} \\
&= a_n M_m^{[s]}(C_{1n}, \ldots, C_{mn}; A) - a_1 M_m^{[s]}(C_{11}, \ldots, C_{m1}; A) \\
&\quad - a_n M_m^{[1]}(C_{1n}, \ldots, C_{mn}; A) + a_1 M_m^{[1]}(C_{11}, \ldots, C_{m1}; A) \\
&\quad + \sum_{i=1}^{n-1} a_i \triangle M_m^{[1]}(C_{1i}, \ldots, C_{mi}; A) \\
&= \sum_{i=1}^{n-1} a_i \triangle M_m^{[1]}(C_{1i}, \ldots, C_{mi}; A) \\
&= M_m^{[1]}\left(\sum_{i=1}^{n-1} a_i \triangle C_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle C_{mi}; A\right) \\
&\geq M_m^{[r]}\left(\sum_{i=1}^{n-1} a_i \triangle C_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle C_{mi}; A\right).
\end{aligned}$$

If $r, s \in J_2$, we have reverse inequalities in the above argument.

If $a_1 \geq \ldots \geq a_n$ and if $r \in J_1$ and $s \in J_2$, the above argument still holds, since $\triangle a_{i-1} \leq 0$, while if $r \in J_2$ and $s \in J_1$, we have the reverse inequalities.

### 5.3.2. Inequalities involving quasiarithmetic means for operators

The following result also holds [29].

**Theorem 5.3.3.** *Let $f$ be a continuous real-valued function on $J$, an interval of $R$. If $f$ is operator convex, $X_j \in S(J)$ $(j = 1, \ldots, m)$ and $A_j$ $(j = 1, \ldots, m)$ are contractions such that (5.3.1) holds, then*

$$f\left(\sum_{j=1}^{m} A_j^* X_j A_j\right) \leq \sum_{j=1}^{m} A_j^* f(X_j) A_j. \tag{5.3.5}$$

*If $f$ is operator concave, the inequality is reversed.*

Henceforth, we shall use the expression an "operator increasing" function for an "operator monotone" function, while if $-f$ is operator monotone, we shall say that $f$ is an operator decreasing function. The inverse function of $f$, denoted by $f^{-1}$, is assumed to exist with range $J$.

A simple consequence of (5.3.5) is the following.

**Corollary 5.3.4.** *If either*

*(i) $f$ is an operator convex function and $f^{-1}$ is operator increasing, or*

*(ii) $f$ is an operator concave function and $f^{-1}$ is operator decreasing, then*

$$\sum_{j=1}^{m} A_j^* X_j A_j \leq f^{-1}\left(\sum_{j=1}^{m} A_j^* f(X_j) A_j\right). \tag{5.3.6}$$

*Moreover, if either*

*(iii) $f$ is operator convex and $f^{-1}$ is operator decreasing, or*

*(iv) $f$ is operator concave and $f^{-1}$ is operator increasing, then*

*the inequality is reversed.*

Of course the expression on the right hand side of (5.3.6) can be used as the definition of the quasiarithmetic mean

$$M_f(X;A) = f^{-1}\left(\sum_{j=1}^{m} A_j^* f(X_j) A_j\right) \tag{5.3.7}$$

for contractions $A_j$ satisfying (5.3.1).

Corollary 5.3.4 gives only inequalities between the quasiarithmetic mean and the arithmetic mean. However, we can use Theorem 5.3.3 to obtain a related result between two quasiarithmetic means. The following result holds.

**Theorem 5.3.5.** *Let $f$, $g$ continuous real-valued functions on $J$, $X_j \in S(J)$, $j = 1,\dots,m$ and $A_i (i = 1,\dots,m)$ contractions such that (5.3.1) holds. If either $H = f \circ g^{-1}$ is operator convex and $F = f^{-1}$ operator increasing or $H$ is operator concave and $F$ operator decreasing, then*

$$M_g(X;A) \le M_f(X;A). \tag{5.3.8}$$

*Moreover, if either $H$ is operator convex and $F$ operator decreasing or $H$ is operator concave and $F$ operator increasing, then (5.3.8) is reversed.*

**Proof.** Let $H$ be operator convex. Then we have, from (5.3.5), for $f \to H$ and $X_i \to g(X_i)$, that

$$f\left\{g^{-1}\left[\sum_{j=1}^{m} A_j^* g(X_j) A_j\right]\right\} \le \sum_{j=1}^{m} A_j^* f(g^{-1}(g(X_j))) A_j,$$

that is,

$$f\left\{g^{-1}\left[\sum_{j=1}^{m} A_j^* g(X_j) A_j\right]\right\} \le \sum_{j=1}^{m} A_j^* f(X_j) A_j.$$

If $F$ is operator increasing, we have

$$g^{-1}\left[\sum_{j=1}^{m} A_j^* g(X_j) A_j\right] \le f^{-1}\left[\sum_{j=1}^{m} A_j^* f(X_j) A_j\right],$$

that is, (5.3.8). The other cases are proved similarly.

Another generalization of (5.2.9) is the following. See [31].

**Theorem 5.3.6.** *Let $f, g$ be continuous real-valued functions on $J$, $C_{j1} \leq \ldots \leq C_{jn}$ (not all equal) $j = 1, \ldots, m$ be operators from $S(J)$ such that $C_{11} = \ldots = C_{m1}$ and $C_{1n} = \ldots = C_{mn}$, and $A_j (j = 1, \ldots, m)$ contractions such that (5.3.1) holds.*

*(i) Let $a_1 \leq \ldots \leq a_m$ be positive numbers. If either $f$ and $g$ are operator convex and $f^{-1}$ and $g^{-1}$ operator increasing or $f$ and $g$ are operator concave and $f^{-1}$ and $g^{-1}$ operator decreasing, then*

$$M_f\left(\sum_{i=1}^{n-1} a_i \triangle C_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle C_{mi}; A\right) \geq \sum_{i=1}^{n-1} a_i \triangle M_g(C_{1i}, \ldots, C_{mi}; A). \tag{5.3.9}$$

*If either $f$ and $g$ are operator concave and $f^{-1}$ and $g^{-1}$ operator increasing or $f$ and $g$ are operator convex and $f^{-1}$ and $g^{-1}$ operator decreasing, then the reverse inequality applies.*

*(ii) Let $a_1 \geq \ldots \geq a_n$ be positive real numbers.*

*(a) $f$ is operator convex and $f^{-1}$ operator increasing*

*(b) $f$ is operator concave and $f^{-1}$ operator decreasing*

*(c) $g$ is operator concave and $g^{-1}$ operator increasing*

*(d) $g$ is operator convex and $g^{-1}$ operator decreasing*

*If ((a) or (b)) and ((c) or (d)) then (5.3.9) is valid.*

*(e) $f$ is operator concave and $f^{-1}$ operator increasing*

*(f) $f$ is operator convex and $f^{-1}$ operator decreasing*

*(g) $g$ is operator convex and $g^{-1}$ operator increasing*

*(h) $g$ is operator concave and $g^{-1}$ operator decreasing,*

*If ((e) or (f)) and ((g) or (h)) then (5.3.9) is reversed.*

**Proof.** (i) We make use of Corollary 5.3.4.

$$\begin{aligned}
&M_f\left(\sum_{i=1}^{n-1} a_i\triangle C_{1i},\dots,\sum_{i=1}^{n-1} a_i\triangle C_{mi};A\right)\\
&\geq \sum_{j=1}^{m} A_j^*\left(\sum_{i=1}^{n-1} a_i\triangle C_{ji}\right)A_j\\
&= \sum_{i=1}^{n-1} a_i\sum_{j=1}^{m} A_j^*\triangle C_{ji}A_j\\
&= \sum_{i=1}^{n-1} a_i\triangle\left(\sum_{j=1}^{m} A_j^*C_{ji}A_j\right)\\
&= a_n\sum_{j=1}^{m} A_j^*C_{jn}A_j - a_1\sum_{j=1}^{m} A_j^*C_{j1}A_j - \sum_{i=2}^{n}\left(\sum_{j=1}^{m} A_j^*C_{ji}A_j\right)\triangle a_{i-1}\\
&\geq a_n\sum_{j=1}^{m} A_j^*C_{jn}A_j - a_1\sum_{j=1}^{m} A_j^*C_{j1}A_j - \sum_{i=2}^{n} M_g(C_{1i},\dots,C_{mi};A)\\
&= a_n\sum_{j=1}^{m} A_j^*C_{jn}A_j - a_1\sum_{j=1}^{m} A_j^*C_{j1}A_j\\
&\qquad -a_nM_g(C_{1n},\dots,C_{mn};A)+a_1M_g(C_{11},\dots,C_{m1};A)+\sum_{i=1}^{n-1} a_i\triangle M_g(C_{1i},\dots,C_{mi};A)\\
&= \sum_{i=1}^{n-1} a_i\triangle M_g(C_{1i},\dots,C_{mi};A).
\end{aligned}$$

(ii) The proof is similar, but now $\triangle a_{i-1}\leq 0$, that is, $-\triangle a_{i-1}\geq 0$.

In inequality (5.3.5), we can have real numbers as weights instead of contractions. If $w_i$ are positive numbers such that $w_1+\ldots+w_m=1$, then by mathematical induction, we derive

$$f\left(\sum_{j=1}^{m} w_jX_j\right)\leq\sum_{j=1}^{m} w_jf(X_j), \tag{5.3.5'}$$

while the quasiarithmetic means are defined by

$$M_f(X;w)=f^{-1}\left(\sum_{j=1}^{m} w_jf(X_j)\right) \tag{5.3.7'}$$

instead of (5.3.7).

Theorems 5.3.5 and 5.3.6 hold with the same substitutions ($M_g(X;w)$ instead of $M_g(X;A)$, etc.).

Moreover, the following reversal of (5.3.5′) was obtained in [28].

**Theorem 5.3.7.** *Let $w$ be a real $n$-tuple such that*

$$w_1 > 0,\ w_i < 0,\ i = 2, \ldots, m,\ w_1 + \ldots + w_m = 1. \tag{5.3.10}$$

*If $X_j \in S(J)$, $j = 1, \ldots, m, \sum_{j=1}^{m} w_j X_j \in S(J)$, then we have the inequality in (5.3.5), is reversed, that is,*

$$f\left(\sum_{j=1}^{m} w_j X_j\right) \geq \sum_{j=1}^{m} w_j f(X_j)$$

*holds for every operator convex function $f$ on $J$.*

Similar to the proof of Theorem 5.3.5, we can establish the following.

**Theorem 5.3.8** *Let $C_{j1} \leq \ldots \leq C_{jn}$ (not all equal), $j = 1, \ldots, m$ be operators from $S(J)$ such that $C_{11} = \ldots = C_{m1}$ and $C_{1n} = \ldots\ C_{mn}$, $w_j (j = 1, \ldots, m)$ real numbers such that (5.3.10) holds.*

*(i) Let $a_1 \leq \ldots \leq a_n$ be positive numbers. If $f$ and $g$ are operator concave and $f^{-1}$ and $g^{-1}$ operator increasing or $f$ and $g$ are operator convex and $f^{-1}$ and $g^{-1}$ operator decreasing, then*

$$M_f\left(\sum_{i=1}^{n-1} a_i \triangle C_{1i}, \ldots, \sum_{i=1}^{n-1} a_i \triangle C_{mi}; w\right) \geq \sum_{i=1}^{n-1} a_i \triangle M_g(C_{1i}, \ldots, C_{mi}; w). \tag{5.3.12}$$

*(a) If $f$ is operator convex and $f^{-1}$ operator increasing*

*(b) $f$ is operator concave and $f^{-1}$ is operator decreasing*

*(c) $g$ is operator concave and $g^{-1}$ operator increasing*

*(d) $g$ is operator convex and $g^{-1}$ operator decreasing*

*If ((a) or (b)) and ((c) or (d)) then (5.3.12) is reversed.*

*(ii) Let* $a_1 \geq \ldots \geq a_n$ *be positive numbers.*

*(e)* $f$ *and* $g$ *are operator concave and* $f^{-1}$ *and* $g^{-1}$ *operator increasing*

*(f)* $f$ *and* $g$ *are operator convex and* $f^{-1}$ *and* $g^{-1}$ *operator decreasing*

*If (e) or (f) then (5.3.12) holds.*

*(g)* $f$ *and* $g$ *are operator convex and* $f^{-1}$ *and* $g^{-1}$ *operator increasing*

*(h)* $f$ *and* $g$ *are operator concave and* $f^{-1}$ *and* $g^{-1}$ *operator decreasing*

*If (g) or (h) then the inequality is reversed.*

# 6. ABEL, POPOVICIU and ČEBYŠEV INEQUALITIES

## 6.0. Overview

For our concluding chapter we again dig deep into the history of inequalities, and make extensive use of Abel's inequality. In Section 1 we develop the Abel motif and employ the results of our efforts to obtain fresh leverage on inequalities of Gauss–Pólya type. These efforts spilled over into additional insights, which are used to improve the Čebyšev and Popoviciu inequalities in Section 6.2. This seems fitting, as the Čebyšev inequality is one of the most fundamental in probability and statistics, that fertile ground from which the Gauss-Pólya results originally sprung. The substance of Section 6.1 constitutes a paper in preparation [9], while that of 6.2 has already been published as [36].

## 6.1. Inequalities of Abel type and application to Gauss–Pólya type integral inequalities

The following result is well-known in the literature as Abel's inequality (see [27], p. 335).

**Theorem 6.1.1** *Let $p$ be a real $n$-tuple, $a$ be a nonnegative nonincreasing $n$-tuple.*

*Then for* $P_k := \sum_{i=1}^{k} p_i$ *we have*

$$a_1 \min_{1\le k\le n} P_k \le \sum_{i=1}^{n} p_i a_i \le a_1 \max_{1\le k\le n} P_k.$$

The following generalization of Abel's result was proved by Bromwick (see [27], p. 337).

**Theorem 6.1.2.** *For a given real n-tuple p, and given integer v* $(1 \le v \le n)$ *define* $H_1 = h_1 = 0, H_v = \max(P_1, \ldots, P_{v-1}), h_v = \min(P_1, \ldots, P_{v-1}), H_v' = \max(P_v, \ldots, P_n), h_v' = \min(P_v, \ldots, P_n)$*. If a is a positive nonincreasing n-tuple, then we have*

$$h_v(a_1 - a_v) + h_v' a_v \le \sum_{i=1}^{n} p_i a_i \le H_v(a_1 - a_v) + H_v' a_v.$$

These inequalities contain in their proof the following identities due to Abel:

$$\begin{aligned}\sum_{i=1}^{n} p_i a_i &= a_1 \sum_{i=1}^{n} p_i + \sum_{i=2}^{n}\left(\sum_{k=i}^{n} p_k\right)\Delta a_{i-1}\\ &= a_n \sum_{i=1}^{n} p_i - \sum_{i=1}^{n-1}\left(\sum_{k=1}^{i} p_k\right)\Delta a_i,\end{aligned} \tag{6.1.1}$$

where $\Delta a_i = a_{i+1} - a_i$.

In this chapter we will point out some other inequalities of Abel type which hold for nondecreasing n-tuples $a = (a_1, \ldots, a_n)$. Some applications to Gauss–Pólya type inequalities are also given.

## 6.1.1. Inequalities for real numbers

We will start with the following theorem.

**Theorem 6.1.3.** *Let* $a = (a_1, \ldots, a_n)$ *and* $p = (p_1, \ldots, p_n)$ *be* $n$*-tuples of real numbers such that* $a_1 \leq \ldots \leq a_n$ *and* $\sum_{k=i}^n p_k \geq 0$ *for* $i = 2, \ldots, n$. *Then*

$$\sum_{i=1}^n p_i a_i \geq a_1 P_n + \left| \sum_{i=1}^n p_i |a_i| - |a_1| P_n \right|. \tag{6.1.2}$$

**Proof.** As $a$ is nondecreasing, we have that

$$\Delta a_{i-1} = a_i - a_{i-1} = |a_i - a_{i-1}| \geq ||a_i| - |a_{i-1}|| = |\Delta |a_{i-1}|| \geq 0$$

for all $i = 2, \ldots, n$ and

$$0 \leq \sum_{k=i}^n p_k = \left| \sum_{k=i}^n p_k \right|$$

for all $i = 2, \ldots, n$.

Thus, by the first equality in (6.1.1), we have

$$\begin{aligned}
\sum_{i=1}^n p_i a_i - a_1 P_n &= \sum_{i=2}^n \left( \sum_{k=i}^n p_k \right) \Delta a_{i-1} \\
&= \sum_{i=2}^n \left| \sum_{k=i}^n p_k \right| |\Delta a_{i-1}| \\
&\geq \sum_{i=2}^n \left| \sum_{k=i}^n p_k \right| |\Delta |a_{i-1}|| \\
&= \sum_{i=2}^n \left| \left( \sum_{k=i}^n p_k \right) \Delta |a_{i-1}| \right| \\
&\geq \left| \sum_{i=2}^n \left( \sum_{k=i}^n p_k \right) \Delta |a_{i-1}| \right|.
\end{aligned}$$

By Abel's identity for $|a|$ we also have

$$\sum_{i=1}^n p_i |a_i| - |a_1| \sum_{i=1}^n p_i = \sum_{i=2}^n \left( \sum_{k=i}^n p_k \right) \Delta |a_{i-1}|.$$

Thus

$$\sum_{i=1}^{n} p_i a_i - a_1 \sum_{i=1}^{n} p_i \geq \left| \sum_{i=1}^{n} p_i |a_i| - |a_1| P_n \right| \geq 0$$

and the inequality (6.1.2) is proved.

The second result is embodied in the following theorem.

**Theorem 6.1.4.** *Let $a = (a_1, \ldots, a_n)$ and $p = (p_1, \ldots, p_n)$ be n-tuples of real numbers such that $a_1 \leq \ldots \leq a_n$ and $\sum_{k=1}^{i} p_k \geq 0, i = 1, \ldots, n-1$. Then we have the inequality*

$$a_n P_n - \sum_{i=1}^{n} p_i a_i \geq \left| |a_n| P_n - \sum_{i=1}^{n} p_i |a_i| \right| \geq 0. \tag{6.1.3}$$

**Proof.** By the second identity in (6.1.1) we can write

$$a_n P_n - \sum_{i=1}^{n-1} p_i a_i = \sum_{i=1}^{n-1} \left( \sum_{k=1}^{i} p_k \right) \Delta a_i.$$

As

$$\Delta a_i = a_{i+1} - a_i = |a_{i+1} - a_i| \geq ||a_{i+1}| - |a_i|| = |\Delta |a_i||$$

and $\sum_{k=1}^{i} p_k \geq 0$ for $i = 1, \ldots, n-1$, we have that

$$\begin{aligned} \sum_{i=1}^{n-1} \left( \sum_{k=1}^{i} p_k \right) \Delta a_i &= \sum_{i=1}^{n-1} \left| \sum_{n=1}^{i} p_k \right| |\Delta a_i| \\ &\geq \sum_{i=1}^{n-1} \left| \sum_{k=1}^{i} p_k \right| |\Delta |a_i|| \\ &= \sum_{i=1}^{n-1} \left| \sum_{n=1}^{i} p_k \Delta |a_i| \right| \\ &\geq \left| \sum_{i=1}^{n-1} \left( \sum_{k=1}^{i} p_k \right) \Delta |a_i| \right|. \end{aligned}$$

By Abel's identity written for $|a|$ we also have

$$\sum_{i=1}^{n} p_i|a_i| = |a_n|P_n - \sum_{i=1}^{n-1}\left(\sum_{k=1}^{i} p_k\right)\Delta|a_i|.$$

Hence we get

$$a_nP_n - \sum_{i=1}^{n} p_ia_i \geq \left||a_n|P_n - \sum_{i=1}^{n} p_i|a_i|\right| \geq 0$$

and the inequality (6.1.3) is proved.

**Remark 6.1.5.** The condition $\sum_{k=i}^{n} p_k \geq 0, (i = 2, \ldots, n)$ is equivalent to $P_n - P_{i-1} \geq 0, (i = 2, \ldots, n)$ or $P_n \geq P_i$ for $i = 1, \ldots, n-1$. The condition $\sum_{k=1}^{i} p_k \geq 0, (i = 1, \ldots, n-1)$ is equivalent to $P_i \geq 0, (i = 1, \ldots, n-1)$.

The following corollary also holds.

**Corollary 6.1.6.** *Let $a$ be nondecreasing and $p \in \mathbb{R}^n$ with $P_n \geq P_i \geq 0$ for all $i = 1, \ldots, n-1$. Then*

$$a_nP_n - \left||a_n|P_n - \sum_{i=1}^{n} p_i|a_i|\right| \geq \sum_{i=1}^{n} p_ia_i \geq a_1P_n + \left|\sum_{i=1}^{n} p_i|a_i| - |a_1|P_n\right|.$$

**Remark 6.1.7.** Note that the above inequality is similar to Abel's result as it provides an upper and a lower bound for the sum $\sum_{i=1}^{n} p_ia_i$ when the sequence $a$ is nondecreasing and $p$ is such that $0 \leq P_i \leq P_n$ for all $i = 1, \ldots, n-1$.

## 6.1.2. Inequalities for complex numbers

We now point out some similar results valid for complex numbers.

**Theorem 6.1.8.** *Let* $z = (z_1, \dots, z_n), w = (w_1, \dots, w_n) \in C^n$ *and* $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ *such that*

$$|z_i - z_{i-1}| \le a_i - a_{i-1} \tag{6.1.4}$$

*for all* $i = 2, \dots, n$. *Then we have the inequality:*

$$\sum_{i=1}^n |w_i| a_i - a_1 \sum_{i=1}^n |w_i| \ge \max \left\{ \left| \sum_{i=1}^n w_i z_i - z_1 \sum_{i=1}^n w_i \right|, \left| \sum_{i=1}^n w_i |z_i| - |z_1| \sum_{i=1}^n w_i \right|, \right.$$
$$\left. \left| \sum_{i=1}^n w_i |z_i| - |z_1| \sum_{i=1}^n |w_i| \right|, \left| \sum_{i=1}^n |w_i||z_i| - |z_1| \sum_{i=1}^n |w_i| \right| \right\}.$$

**Proof.** By Abel's identity we have

$$\begin{aligned} \sum_{i=1}^n |w_i| a_i - a_1 \sum_{i=1}^n |w_i| &= \sum_{i=2}^n \left( \sum_{k=i}^n |w_k| \right) \Delta a_{i-1} \\ &\ge \sum_{i=2}^n \left( \sum_{k=i}^n |w_k| \right) |\Delta z_{i-1}| =: A \end{aligned}$$

(by (6.1.4)).

Now, by the properties of the modulus mapping, we have

$$\sum_{k=i}^n |w_k| \ge |\sum_{k=i}^n w_k|$$

and so

$$\begin{aligned} A &\ge \sum_{i=2}^n \left| \left( \sum_{k=i}^n w_k \right) \Delta z_{i-1} \right| \\ &\ge \left| \sum_{i=2}^n \left( \sum_{k=i}^n w_k \right) \Delta z_{i-1} \right| \\ &= \left| \sum_{i=1}^n w_i z_i - z_1 \sum_{i=1}^n w_i \right|. \end{aligned}$$

Also, we can write

$$|\Delta z_{i-1}| \ge |\Delta |z_{i-1}||$$

for $i = 2, \ldots, n+1$. Thus

$$
\begin{aligned}
A &\geq \sum_{i=2}^{n} \left| \left( \sum_{k=i}^{n} w_k \right) \Delta z_{i-1} \right| \\
&\geq \left| \sum_{i=2}^{n} \left( \sum_{k=i}^{n} w_k \right) \Delta |z_{i-1}| \right| \\
&= \left| \sum_{i=1}^{n} w_i |z_i| - |z_1| \sum_{i=1}^{n} w_i \right| .
\end{aligned}
$$

In the same way we have

$$
\begin{aligned}
A &= \sum_{i=2}^{n} \left( \sum_{k=i}^{n} |w_k| \right) |\Delta z_{i-1}| \\
&= \sum_{i=2}^{n} \left| \left( \sum_{k=i}^{n} |w_k| \right) \Delta z_{i-1} \right| \\
&\geq \left| \sum_{i=2}^{n} \left( \sum_{k=i}^{n} |w_k| \right) \Delta z_{i-1} \right| \\
&= \left| \sum_{i=1}^{n} |w_i| z_i - z_1 \sum_{i=1}^{n} |w_i| \right|
\end{aligned}
$$

and

$$
\begin{aligned}
A &= \sum_{i=2}^{n} \left( \sum_{k=i}^{n} |w_k| \right) |\Delta z_{i-1}| \\
&\geq \sum_{i=2}^{n} \left( \sum_{k=i}^{n} |w_k| \right) |\Delta |z_{i-1}|| \\
&\geq \left| \sum_{i=2}^{n} \left( \sum_{k=i}^{n} |w_k| \right) \Delta |z_{i-1}| \right| \\
&= \left| \sum_{i=1}^{n} |w_i| |z_i| - |z_1| \sum_{i=1}^{n} |w_i| \right| .
\end{aligned}
$$

and the proof of the theorem is thus finished.

In the same way (using the second part of Abel's identity (6.1.1)) we can prove the following theorem.

**Theorem 6.1.9.** *Let $z, w, a$ be as above. Then we have the inequality*

$$a_n \sum_{i=1}^{n} |w_i| - \sum_{i=1}^{n} |w_i| a_i \geq \max \left\{ \left| z_n \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} w_i z_i \right|, \left| |z_n| \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} |z_i| w_i \right|, \right.$$
$$\left. \left| z_n \sum_{i=1}^{n} |w_i| - \sum_{i=1}^{n} |w_i| z_i \right|, \left| |z_n| \sum_{i=1}^{n} |w_i| - \sum_{i=1}^{n} |w_i| |z_i| \right| \right\}.$$

## 6.1.3. Application to integral inequalities of Gauss–Pólya type

**Theorem 6.1.10.** *Let $f : [a, b] \to I\!R$ be a nonnegative and increasing function and $x_i : [a, b] \to I\!R$ be functions with a continuous first derivative such that*

1) $x_1(t) \leq \cdots \leq x_n(t)$, $t \in [a, b]$,

2) $x_1'(t) \leq \cdots \leq x_n'(t)$, $t \in [a, b]$.

*Suppose also $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

(6.1.5)
$$0 \leq \left| \sum_{i=1}^{n} p_i \left( \left| \int_a^b x_i'(t) f(t) dt \right| - \left| \int_a^b x_1'(t) f(t) dt \right| \right) \right| + \left| \sum_{i=1}^{n} p_i \int_a^b \left( |x_i(t)| - |x_1(t)| \right) df(t) \right|$$
$$\leq f(b) \sum_{i=1}^{n} p_i \left( x_i(b) - x_1(b) \right) - f(a) \sum_{i=1}^{n} p_i \left( x_i(a) - x_1(a) \right).$$

**Proof.** We observe, by an integration by parts, that

$$\sum_{i=1}^{n} p_i \int_a^b x_i'(t)\ f(t) dt = \int_a^b \left( \sum_{i=1}^{n} p_i x_i(t) \right)' f(t) dt$$

(6.1.6)
$$= \left( \sum_{i=1}^{n} p_i x_i(t) \right) f(t) \Big|_a^b - \int_a^b \left( \sum_{i=1}^{n} p_i x_i(t) \right) df(t)$$
$$= f(b) \sum_{i=1}^{n} p_i x_i(b) - f(a) \sum_{i=1}^{n} p_i x_i(a) - \int_a^b \left( \sum_{i=1}^{n} p_i x_i(t) \right) df(t).$$

We can apply the inequality (6.1.2) to obtain

(6.1.7)

$$\sum_{i=1}^{n} p_i \int_a^b x_i'(t) f(t)dt \geq \int_a^b x_1'(t) f(t)dt + \left| \sum_{i=1}^{n} p_i \left| \int_a^b x_i'(t) f(t)dt \right| - \left| \int_a^b x_i'(t) f(t)dt \right| \right|$$

and

$$\sum_{i=1}^{n} p_i x_i(t) \geq x_1(t) + \left| \sum_{i=1}^{n} p_i \left| x_i(t) \right| - \left| x_1(t) \right| \right|$$

for all $t \in [a,b]$.

Intergrating this last inequality, we deduce that

$$\begin{aligned} & \int_a^b \left( \sum_{i=1}^{n} p_i x_i(t) \right) df(t) \\ & \geq \int_a^b x_1(t) df(t) + \int_a^b \left| \sum_{i=1}^{n} p_i \left| x_i(t) \right| - \left| x_1(t) \right| \right| df(t) \\ & \geq \int_a^b x_1(t) df(t) + \left| \sum_{i=1}^{n} p_i \int_a^b \left| x_i(t) \right| df(t) - \int_a^b \left| x_1(t) \right| df(t) \right| \\ & = f(t) x_1(t) |_a^b - \int_a^b x_1'(t) f(t) dt + \left| \sum_{i=1}^{n} p_i \int_a^b \left| x_i(t) \right| df(t) - \int_a^b \left| x_1(t) \right| dt \right| . \end{aligned} \tag{6.1.8}$$

Using the identities (6.1.6), (6.1.7) and (6.1.8) we get

$$\begin{aligned} & \int_a^b x_1' f(t) dt + \left| \sum_{i=1}^{n} p_i \left| \int_a^b x_i' f(t) dt \right| - \left| \int_a^b x_1' f(t) dt \right| \right| \\ & \leq f(b) \sum_{i=1}^{n} p_i x_i(b) - f(a) \sum_{i=1}^{n} p_i x_i(a) - (f(b) x_1(b) - f(a) x_1(a)) \\ & + \int_a^b x_1' f(t) dt - \left| \sum_{i=1}^{n} p_i \int_a^b \left| x_i'(t) \right| df(t) - \int_a^b \left| x_1(t) \right| df(t) \right| , \end{aligned}$$

that is,

$$\left| \sum_{i=1}^{n} p_i \left| \int_a^b x_i'(t) f(t) dt \right| - \left| \int_a^b x_1(t) f(t) dt \right| \right| + \left| \sum_{i=1}^{n} p_i \int_a^b \left| x_i(t) \right| df(t) - \int_a^b \left| x_1(t) \right| df(t) \right|$$

$$\leq f(b)\left(\sum_{i=1}^{n} p_i x_i(b) - x_1(b)\right) - f(a)\left(\sum_{i=1}^{n} p_i x_i(a) - x_1(a)\right),$$

which is clearly equivalent to (6.1.5).

**Remark 6.1.11.** Similar results can be obtained if we use the second Abel– type inequality (6.1.3). We omit the details.

## 6.2. On a refinement of the Čebyšev and Popoviciu inequalities

We establish a refinement of the discrete Čebyšev inequality and an analogous one for the Popoviciu inequality.

The discrete Čebyšev inequality is a fundamental inequality in probability. It states the following.

**Theorem 6.2.1.** *Suppose $a$ and $b$ are $n$–tuples of real numbers, both nondecreasing or both nonincreasing, and $p$ is an $n$–tuple of positive numbers. Then*

(6.2.1) $$T_n(a,b;p) := \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j a_j b_j - \sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j b_j \geq 0.$$

Recently an improvement has been derived by Alzer [2].

**Theorem 6.2.2.** *If $a, b$ and $p$ are defined as above, then*

(6.2.2) $$T_n(a,b;p) \geq \min_{2\leq i,j\leq n} [(a_i - a_{i-1})(b_j - b_{i-1})] \cdot T_n(e,e;p),$$

*where $e = (1,2,\ldots,n)$. Equality holds if and only if*

(6.2.3) $$a_i = a_1 + (i-1)\alpha \quad \text{and} \quad b_i = b_1 + (i-1)\beta \quad (i = 1,\ldots,n),$$

*where $\alpha$ and $\beta$ are positive or negative real numbers according as $a$ and $b$ are both nondecreasing or nonincreasing $n$–tuples.*

In fact it is possible to give a corresponding upper bound for $T_n(a, b; p)$. Set

$$m(a) = \min_{1 \le i < n} (a_{i+1} - a_i), \qquad M(a) = \max_{1 \le i < n} (a_{i+1} - a_i).$$

Lupaş [22] has shown that with the same conditions on $a, b$ and $p$

$$m(a)m(b) \le \frac{T(a, b; p)}{T(e, e; p)} \le M(a)M(b).$$

We note that the first inequality is equivalent to (6.2.2).

The condition that $p$ is a positive $n$–tuple can be weakened to the condition

$$0 \le P_n \le P_k \quad (k = 1, 2, \ldots, n-1), \tag{6.2.4}$$

where $P_k := \sum_{i=1}^{k} p_i$ $(k = 1, 2, \ldots, n)$ (Pečarić, [43]).

The result was established *via* an Abel–type identity. This appears to be of a more general applicability, and we shall employ it to derive two new results: a refinement for the Čebyšev inequality and one for Popoviciu's inequality.

Since the identity is not proved in [43], we present a proof in Subsection 6.2.1. An interesting feature is that although this generalizes Abel's identity, it can be established by repeated use of the basic Abel identity. The latter therefore appears to hold a key role in connection with the cluster of results mentioned above. In Subsection 6.2.2 we prove our new refinements of the Čebyšev and Popoviciu results.

## 6.2.1. An Abel–type identity

Lemma 6.2.3 below is a useful consequence of the repeated use of Abel's identity

$$\sum_{j=1}^{n} p_j c_j = P_n c_n - \sum_{j=1}^{n-1} P_j \Delta c_j,$$

where $\Delta c_j := c_{j+1} - c_j$ and $P_j$ is defined as in the previous subsection.

It will be useful to introduce also a variant. Put $\bar{P}_j = \sum_{i=j}^{n} p_i$ $(j = 1, \ldots, n)$. On substituting for the definitions of $P_j$, $\bar{P}_j$ and interchanging the order of summation, we derive

$$\sum_{j=1}^{n} p_j c_j = c_i P_n - \sum_{j=1}^{i-1} P_j \Delta c_j + \sum_{j=i+1}^{n} \bar{P}_j \Delta c_{j-1} \qquad (1 \leq i \leq n),$$

which is an extension of Abel's identity.

**Lemma 6.2.3.** *[36] Suppose $a = (a_i)_1^n$, $b = (b_i)_1^n$, $p = (p_i)_1^n$ are real $n$-tuples and $T_n(a, b; p)$ is defined by the left-hand relation in (1). Then*

$$T(a, b; p) = \sum_{i=1}^{n-1} \left( \sum_{j=1}^{i-1} \bar{P}_{i+1} P_j \Delta b_j + \sum_{j=i+1}^{n} P_i \bar{P}_j \Delta b_{j-1} \right) \Delta a_i.$$

**Proof.** From its definition, we have

$$T(a, b; p) = \sum_{i=1}^{n} p_i a_i \left( \sum_{j=1}^{n} p_j (b_i - b_j) \right) = \sum_{i=1}^{n} p_i h_i a_i,$$

where

$$h_i := \sum_{j=1}^{n} p_j (b_i - b_j). \tag{6.2.5}$$

Accordingly, by Abel's identity,

$$T(a, b; p) = \left( \sum_{i=1}^{n} p_i h_i \right) a_n - \sum_{i=1}^{n-1} \left( \sum_{j=1}^{i} p_i h_i \right) \Delta a_i,$$

and since

$$\sum_{i=1}^{n} p_i h_i = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j (b_i - b_j) = 0,$$

we thus have

$$T(a, b; p) = - \sum_{i=1}^{n-1} \left( \sum_{j=1}^{i} p_j h_j \right) \Delta a_i. \tag{6.2.6}$$

Again by Abel's identity,

$$\sum_{j=1}^{i} p_j h_j = h_i P_i - \sum_{j=1}^{i-1} P_j \Delta h_j = h_i P_i - \sum_{j=1}^{i-1} P_j P_n \Delta b_j. \tag{6.2.7}$$

Further, from (6.2.5) and our extension of Abel's identity,

$$h_i = \sum_{j=1}^{i-1} P_j \Delta b_j - \sum_{j=i+1}^{n} \bar{P}_j \Delta b_{j-1}, \tag{6.2.8}$$

and so (6.2.6) yields

$$\begin{aligned} T(a,b;p) &= \sum_{i=1}^{n-1} \left( h_i P_i - \sum_{j=1}^{i-1} P_j P_n \Delta b_j \right) \Delta a_i \quad \textit{by } (6.2.7) \\ &= -\sum_{i=1}^{n-1} \Bigg[ P_k \left( \sum_{j=1}^{i-1} P_j \Delta b_j - \sum_{j=i+1}^{n} \bar{P}_j \Delta b_{j-1} \right) \\ &\qquad - \sum_{j=1}^{i-1} P_j P_n \Delta b_j \Bigg] \Delta a_k \quad \textit{by } (6.2.8) \\ &= \sum_{i=1}^{n-1} \left( \bar{P}_{i+1} \sum_{j=1}^{i-1} P_j \Delta b_j + P_i \sum_{j=i+1}^{n} \bar{P}_j \Delta b_{j-1} \right) \Delta a_i, \end{aligned}$$

and we are done.

## 6.2.2. Refinements of Čebyšev and Popoviciu inequalities

We now proceed to an application of Lemma 6.2.3 to give a refinement of Čebyšev's inequality. With the notation

$$|a| = (|a_1|, \dots, |a_n|),$$

we have the following result.

**Theorem 6.2.4.** *[36] Let $a$ and $b$ be $n$-tuples of real numbers, both nondecreasing or both nonincreasing, and $p$ a real $n$–tuple satisfying (6.2.4). Then*

$$T_n(a;b;p) \geq |T_n(|a|,|b|,p)| \geq 0.$$

**Proof.** For a nondecreasing $n$–tuple we have

$$\Delta a_i = a_{i+1} - a_i = |a_{i+1} - a_i| \geq ||a_{i+1}| - |a_i|| = |\Delta |a_i||\,,$$

so that by Lemma 6.2.3

$$\begin{aligned} T(a,b;p) &= \sum_{k=1}^{n-1} \left( \bar{P}_{k+1} \sum_{j=1}^{k-1} P_j \Delta b_j + P_k \sum_{j=k+1}^{n} \bar{P}_j \Delta b_{j-1} \right) \Delta a_k \\ &\geq \sum_{k=1}^{n-1} \left( \bar{P}_{k+1} \sum_{j=1}^{k-1} P_j \, |\Delta |b_j|| + P_k \sum_{j=k+1}^{n} \bar{P}_j \, |\Delta |b_{j-1}|| \right) |\Delta |a_k|| \\ &\geq \left| \sum_{k=1}^{n-1} \left( \bar{P}_{k+1} \sum_{j=1}^{k-1} P_j \Delta |b_j| + P_k \sum_{j=k+1}^{n} \bar{P}_j \Delta |b_{j-1}| \right) \Delta |a_k| \right| \\ &= |T_n(|a|, |b|; p)|\,, \end{aligned}$$

giving the required result.

We conclude by considering Popoviciu's inequality [50], which states the following.

**Theorem 6.2.5.** *Suppose*

$$F(a,b;x) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} a_i b_j,$$

*where all the quantities involved are real numbers. Then*

$$F(a,b;x) \geq 0 \tag{6.2.9}$$

*for all sequences* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_m)$ *which are monotonic in the same sense if and only if*

$$\begin{aligned} &X_{r,s} \geq 0 \quad (r = 2, \ldots, n; \; s = 2, \ldots, m), \\ &X_{r,1} = 0 \;\; (r = 1, \ldots, n), \qquad\qquad (6.2.10) \\ &X_{1,s} = 0 \;\; (s = 2, \ldots, m), \end{aligned}$$

*where* $X_{r,s} = \sum_{i=r}^{n}\sum_{j=s}^{m} x_{i,j}$.

**Remark 6.2.6.** For the case $m = n$, we recover Čebyšev's inequality under condition (6.2.4) with the choice

$$x_{i,j} = \begin{cases} p_i(P_n - p_i) & \text{for } i = j \\ -p_i p_j & \text{for } i \neq j. \end{cases}$$

Relation (6.2.9) is a simple consequence of the identity

$$\begin{aligned} F(a,b;x) &= a_1 b_1 X_{1,1} + a_1 \sum_{s=2}^{m} X_{1,s}\Delta b_{s-1} \\ &\quad + b_1 \sum_{r=2}^{n} X_{r,1}\Delta a_{r-1} + \sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s}\Delta a_{r-1}\Delta b_{s-1} \end{aligned} \tag{6.2.11}$$

(see Pečarić, [41] and also Mitrinović, Pečarić and Fink, [27], p. 341).

Interpolations of (6.2.9) which contain (6.2.2) and (6.2.3) are obtained in [43].

Finally we derive an analogue of Theorem 6.2.4 for $F$.

**Theorem 6.2.7.** *Suppose $x_{i,j}$ ($1 \leq i \leq n$, $1 \leq j \leq m$) are real numbers satisfying (6.2.10). If the sequences $a$ and $b$ are monotone in the same sense, then*

$$F(a,b;x) \geq |F(|a|,|b|;x)| \geq 0.$$

**Proof.** By (6.2.10) $F$ reduces to the last term in (6.2.11), so

$$\begin{aligned}
F(a,b;x) &= \sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s}\Delta a_{r-1}\Delta b_{s-1} \\
&\geq \sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s}\,|\Delta|a_{r-1}||\times|\Delta|b_{s-1}|| \\
&= \sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s}\,|\Delta|a_{r-1}|\times\Delta|b_{s-1}|| \\
&\geq \left|\sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s}\Delta|a_{r-1}|\times\Delta|b_{s-1}|\right| \\
&= |F(|a|,|b|;x)|\,.
\end{aligned}$$

**Remark 6.2.8.** [36] As in Remark 6.2.6 we can obtain Theorem 6.2.4 from Theorem 6.2.5.

# References


[1] H. ALZER, An Extension of an Inequality of G. Pólya, *Buletinul Institutului Politechnic Din Iasi, Tomul* XXXVI (XL), Fasc. 1–4, (1990), 17–18.

[2] H. ALZER, A refinement of Tchebyschef's Inequality, *Nieuw Archief voor Wiskunde*, 10 (1992), 7-9.

[3] W.N. ANDERSON, Jr. and R.J. DUFFIN, Series and parallel addition of matrices, *J. Math. Anal. Appl.* 26 (1969), 576-594.

[4] P. R. BEESACK, Inequalities for Absolute Moments of a Distribution: From Laplace to Von Mises, *J. Math. Anal. Appl.* 98 (1984), 435–457.

[5] F. BERNSTEIN and KRAFFT M., Integralungleichungen konvexer Funktionen., *Nachr. Ges. Wiss. Göttingen* (1914), 299–308.

[6] P.S. BULLEN, On a theorem of L. Losonczi. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser.Mat.Fiz.* No. 412-460 (1973), 105-108.

[7] P.S. BULLEN, D.S.MITRINOVIĆ and P.M.VASIĆ, *Means and their inequalities.* D. Reidel Publ. Company Dordrecht/Boston/Lancaster/Tokyo, 1988.

[8] Z. DARÓCSY and L. LOSONCZY , Über den Vergleich von Mittelwerten *Publ. Math. Debrecen*, 17 (1970), 289-297.

[9] S.S. DRAGOMIR, C.E.M. PEARCE and J. ŠUNDE, On some inequalities of Abel type and application to Gauss–Pólya type integral inequalities, submitted

[10] S.S. DRAGOMIR, C.E.M. PEARCE and J. ŠUNDE, A special case of Gauss–Pólya type inequality, submitted

[11] G. FABER, Bermerkungen zu Sätzen der Gausschen theoria combinationis observationum, Heft 1, *Sitzungber. Bayer Akad. Wiss.* (1922), 7–21.

[12] A.M. FINK and M. JODEIT, Jensen inequalities for functions with higher monotonicities, *Aeq. Math.*, 40 (1990), 26–43.

[13] J.I. FUJII, M. FUJII and Y. SEO, An extension of the Kubo-Ando theory: Solidarities, *Math. Japon.* 35 (1990), 387-396.

[14] M. FUJIWARA, Über Gauss–Fabersche Ungleichung für Integrale , *Japanese Journ. of Math.* (1926), 197–200.

[15] C. F. GAUSS, *"Theoria combinationis observationum"*, 1821; translated into German in *"Abhandlungen zur Methode der kleinsten Quadrate"*, 1887, pp. 9, 12, Neudruck, Würzburg, 1964.

[16] C. GINI, Di una formula comprensiva delle medie, *Metron*, 13 (1938), 3-22.

[17] S. IZUMI, Uber die Gauss–Fabersche Ungleichung für Integrale, , *Japanese Journ. of Math.* 4 (1927), 7–10.

[18] M. KRAFFT, Zwei Sätze aus Gauss' Theoria combinationis observationum , *Deutsche Math.* 2 (1937), 624–630.

[19] F. KUBO and T. ANDO, Means of positive linear operators, *Math. Ann.*, 246 (1980), 205-224.

[20] E. LEACH and M. SHOLANDER, Extended mean values II, *J. Math. Anal. Appl.* 92 (1983), 207-223.

[21] L. LOSONCZI, Uber eine neue Klasse von Mittelwerte. *Acta Sci. Math.* (Szeged), 32 (1971), 71-81.

[22] A. LUPAŞ, On an inequality , *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No. 716-734 (1981), 32-34.

[23] L. MALIGRANDA, J. PEČARIĆ and L.E. PERSSON, Stolarsky's Inequality with General Weights, *Proc. Amer. Math. Soc.*, 123 (1995), 2113-2118.

[24] D.S. MITRINOVIĆ and J. PEČARIĆ, Note on the Gauss–Winckler inequality, *Anz. Österreich Akad. Wiss. Math.-Natur. Kl.*, 6 (1986), 89–92.

[25] D.S. MITRINOVIĆ and J. PEČARIĆ, Two integral inequalities, *South East Asian Bull. Math.*, 15 (1991), 39–42.

[26] D.S. MITRINOVIĆ, J. PEČARIĆ and A.M. FINK, *Inequalities involving functions and their integrals and derivatives*, Kluwer Acad. Publishers, Dordrecht, 1991.

[27] D.S. MITRINOVIĆ, J. PEČARIĆ and A.M. FINK, *Classical and new inequalities in analysis*, Kluwer Acad. Publishers, Dordrecht, 1993.

[28] B. MOND and J. PEČARIĆ, Remarks on Jensen's inequality for operator convex functions, *Ann. Univ. Mariae Curie-Sklodowska*, 47 (1993), 96-103.

[29] B. MOND and J. PEČARIĆ, On Jensen's inequality for operator convex-functions, *Houston J. Math.*, 21 (1995) 739–754.

[30] B. MOND, J. PEČARIĆ, J. ŠUNDE and S. VAROŠANEC, Operator versions of some classical inequalities, *Lin. Alg. Appl.*, 264 (1997), 117-126.

[31] B. MOND, J. PEČARIĆ, J. ŠUNDE and S. VAROŠANEC, Inequalities of Pólya type for positive linear operators, *Houston J. Math.*, 22 (1996) 4, 851-858.

[32] B. MOND, J. PEČARIĆ, J. ŠUNDE and S. VAROŠANEC, Pólya's inequality for positive linear operators, *Rad Hazu*, to appear.

[33] S. NARUMI, On a generalized Gauss–Faber's inequality for integral, *Japanese Journ. of Math.* 4 (1927), 33–39.

[34] Zs. PÁLES, Inequalities for sums of powers *J. Math. Anal. Appl.*, 131 (1988), 265-270.

[35] Zs. PÁLES, Inequalities for differences of powers, *J. Math. Anal. Appl.* 131 (1988), 271-281.

[36] C.E.M. PEARCE, J. PEČARIĆ and J. ŠUNDE, On a refinement of the Čebyšev and Popoviciu inequalities, *Mathematical Communications*, 1 (1996), 121–127.

[37] C.E.M. PEARCE, J. PEČARIĆ and S. VAROŠANEC, Inequalities of Gauss-Minkowski type, *Intern. Ser. Num. Math.*, 123 (1997), 27–37.

[38] C.E.M. PEARCE, J. PEČARIĆ and J. ŠUNDE, A generalization of Pólya's inequality to Stolarsky and Gini means, *Mathematical Inequalities and Appl.*, to appear

[39] C.E.M. PEARCE, J. PEČARIĆ and J. ŠUNDE, Some inequalities for generalized quasiarithmetic means, submitted

[40] C.E.M. PEARCE, J. PEČARIĆ and J. ŠUNDE, Pólya–type inequalities with arbitrary functions replacing means, submitted

[41] J.E. PEČARIĆ, On an inequality of T. Popoviciu, *Bul. Sti. Tech. Inst. Politehn.*, Timisoara 2, 24 38 (1979), 9-15.

[42] J.E. PEČARIĆ, Inverse of Jensen–Steffensen's inequality, *Glas. Mat. Ser. III*, Vol 16(36), No 2, (1981), 229–233.

[43] J.E. PEČARIĆ, On the Ostrowsky generalization of Čebyšev's inequality, *J. Math. Anal. Appl.*, 102 (1984), 479-487.

[44] J. PEČARIĆ, On Stolarsky's quotient, *Prilozi MANU, Skopje*, 14,2 (1993), 55-60.

[45] J. PEČARIĆ, A reverse Stolarsky's inequality, *Amer. Math. Monthly*, 101 (1994), 566–568.

[46] J. PEČARIĆ, On two integral inequalities, *Bull. Inst. Polytehnic Iasy*, 40 (44) 1994.

[47] J. PEČARIĆ and S. VAROŠANEC, A generalization of Pólya's inequalities, *WSSIAA* 3 (1994), 501–504.

[48] J. PEČARIĆ and S. VAROŠANEC, Remarks on Gauss–Winckler's and Stolarsky's inequalities, *Utilitas Mathematica* 48 (1995), 233–241.

[49] G. PÓLYA and G. SZEGÖ, *Aufgaben und Lehrsätze aus der Analysis , vol I and II* , Berlin, Springer Verlag, 1954.

[50] T. POPOVICIU, On an inequality, (Timisoara) *Gaz. Mat. Fiz. Ser.* , A8, 64 (1959), 451-461.

[51] M. SAGAE and K. TANABE, Upper and lower bounds for the arithmetic-geometric-harmonic means of positive definite matrices, *Lin. Multilin. Alg.* 37 (1994), 279-282.

[52] K.B. STOLARSKY, From Wythoff's Nim to Chebyshev's inequality, *Amer. Math. Monthly*, 98 (1991), 889–900.

[53] S. VAROŠANEC, Inequalities of Gauss type, *(Croatian) Doct. Diss. Univ. of Zagreb, Dept. Math.*, 1994.

[54] S. VAROŠANEC, A remark on Volkov's result, *Rad Hazu (Zagreb)* 468 12 (1995).

[55] S. VAROŠANEC and J. PEČARIĆ, On Pólya's inequality, *Conf. Functional Analysis IV, Dubrovnik, November 10–17, 1993*, Aarhus Univ. (1994), 275–278.

[56] S. VAROŠANEC, Inequalities of Minkowski type, *Real Analysis Exchange* 20 (1995), 250–255.

[57] S. VAROŠANEC and J. PEČARIĆ, Gauss' and related inequalities, *Zeit. für Anal. und ihre Anwendungen* 14 (1995), 175–183.

[58] S. VAROŠANEC and J. PEČARIĆ, Integral inequalities involving derivatives of higher order with some application, *J. Austr. Math. Soc.*, Ser B. 38 (1997), 325–335.

[59] S. VAROŠANEC, J. PEČARIĆ and J. ŠUNDE, Some discrete inequalities, *Zeit. für Anal. A.*, 15 (1996), 1033–1044.

[60] S. VAROŠANEC, J. PEČARIĆ and J. ŠUNDE, On Gauss–Pólya's inequality, submitted to *Zeit. für Anal. A.*.

[61] S. VAROŠANEC and J.E. PEČARIĆ, On Gauss type inequalities, *Grazer Math. Brichte*, 328 (1996), 113–118.

[62] V. N. VOLKOV, Utočnenie neravenstva Gel'dera, *Mosk. Gos. Ped. Inst. im V.I. Lenina. Učen. Zap.-Neravenstva* 460 (1972), 110–115.

[63] A. WINCKLER, Allgemeine Sätze zur Theorie der unregelmässigen Beobachtungsfehler, *Sitzungsber. Math.-Natur. Kl. K. Akad. Wiss. Wien, Zweite Abt.* 53 (1866), 6–41.