The Pion-Nucleon Sigma Term and the SU(3) Cloudy Bag Model

by

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Declaration

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Abstract

QCD is approximately invariant under chiral $SU(3)_L \times SU(3)_R$ transformations. Experimental evidence (i.e., no parity doublets) tells us that the symmetry must be broken spontaneously. The term in the Hamiltonian which breaks chiral symmetry belongs to the $(3,\overline{3}) + (\overline{3},3)$ representation of $SU(3)_L \times SU(3)_R$. In the quark model, this term is $\overline{q}m_q q$ with the quark fields written as left and right handed fields.

We can construct, from matrix elements which depend only on the symmetry breaking part of the Hamiltonian, the sigma term, and this can be (indirectly) determined from experiment. As such, it is a powerful tool in that it can be used to test symmetry breaking mechanisms. Experimentally, the sigma term is $\Sigma_{\pi N}(t = 2M_\pi^2) = 60 \pm 12$ MeV (t is the square of the momentum transfer). Given this value, it has been found that $\Sigma_{\pi N}(t = 0) = 45 \pm 12$ MeV.

The theoretical value (calculated at $t = 0$ and denoted $\sigma_{\pi N}(0)$), using the $(3,\overline{3})+(\overline{3},3)$ model, is approximately 26 MeV. Using chiral perturbation theory and dispersion relations, this value can be increased to 45 MeV. This value includes a contribution from the leading nonanalytic term and an estimate of higher order corrections to the sigma term (amounting to 10 MeV) and assumes the nucleon has a small strange quark component.

We have made an explicit calculation of the quark and meson contributions to the sigma term within the Cloudy Bag Model. Assuming a current quark mass of 12 ± 3 MeV, at the bag scale of 0.5 GeV, we find that the

valence quarks contribute 17.5 ± 4.5 MeV. Our expression for the meson contribution includes contributions from pion, kaon and eta loops. We find that the kaon and eta contribute less than 1 MeV. The pion contribution is due to πNN and $\pi N\Delta$ loops. If we consider only the first loop, representing contributions from the leading nonanalytic term, then for $0.8 \leq R \leq 1.1$ fm, the pion contributes $12 \leq \sigma^{\pi}_{\pi N}(0) \leq 16$ MeV to the sigma term. Adding contributions from the second loop (representing higher order corrections) increases the pion contribution to $20 \leq \sigma^{\pi}_{\pi N}(0) \leq 26$ MeV. Adding valence quark contributions, we find $37 \leq \sigma_{\pi N}(0) \leq 44$ MeV ± 4.5 MeV (c.f. $\Sigma_{\pi N}(t=0)$ given above). We argue that there is no strange quark component in the nucleon, and that chiral perturbation theory omits contributions from the second loop and, thus, underestimates the contribution from higher order corrections by approximately 7 to 10 MeV.

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Chapter 1

Introduction

This thesis is concerned with two aspects of strong interaction theory - the calculation of the pion-nucleon sigma term, and phenomenological models of hadrons (the Cloudy Bag Model (CBM) in particular).

In this chapter, we introduce the idea of chiral symmetry, and its breaking. We shall also briefly introduce the theoretical models used to calculate the sigma term. They are Quantum Chromodynamics and phenomenological models of low- to medium-energy QCD, namely the CBM and chiral perturbation theory. More detail will be given in chapters 2 and 3.

1.1 The Eight-Fold Way

According to the eight-fold way [1], baryons and mesons can be placed into multiplets, such as the nucleon doublet or pion triplet. The members of a multiplet differ only in mass and charge, the mass difference being due to the electromagnetic interaction within the multiplet. As far as the strong force is concerned, the members of a multiplet are identical.

Each multiplet corresponds to an irreducible representation of the isospin algebra, and will contain $2I+1$ members. The isospin I distinguishes each representation [2,3].

Using the Lie group $U(3)$, Gell-Mann (and, independently, Ne'eman) grouped the baryons and mesons into larger groups consisting of 8 or 10 members, according to charge and strangeness (or isospin) [4].

If we assume that there exists a fundamental triplet, the baryon supermultiplets are found by constructing irreducible representations in the following way [2]

$$\begin{aligned} \underline{3} \times \underline{3} \times \underline{3} &= \underline{6} \times \underline{3} + \underline{3}^* \times \underline{3} \\ &= \underline{10} + \underline{8} + \underline{8} + \underline{1} \end{aligned} \tag{1.1}$$

To see how quarks appear, consider the reduction of $\underline{6} \times \underline{3}$. Define 27 basis vectors $\xi^\alpha(1), \xi^\beta(2)$ and $\xi^\gamma(3)$, with each ξ transforming as $\underline{3}$. If the $\underline{6}$ is generated by ψ^i $(i = 1, \ldots, 6)$, then finding weights [2,5] for the ξ^α and ψ^i will give us weights for the products $\xi^\alpha \psi^i$. Finding the highest weight and applying I-spin, U-spin and V-spin operators will give us the remaining members of the irreducible representation. This method tells us that $\underline{6} \times \underline{3} = \underline{10} + \underline{8}$. We can introduce the idea of quarks by considering, for example, the highest weight of the $\underline{10}$ representation. It is $\xi^1(1)\xi^1(2)\xi^1(3) = uuu \sim \Delta^{++}$.

The members of a supermultiplet may be thought of as a single "particle". The actual members correspond to the eight or ten different orientations of unitary spin. All members of a supermultiplet are related via unitary transformations which turn baryons into baryons, and mesons into mesons. Although the symmetry group of the eightfold way is a unitary group, it will be necessary to reduce this to a special unitary group in order to produce "physical" currents. We will return to this later.

1.2 Chiral Symmetry

At the present time, there are believed to be six quark flavours, u, d, s, c, t and b. Consider the three lightest quarks whose masses are denoted m_u, m_d and m_s. The actual values of these masses are unknown as free quarks do not exist. When we refer to the quark mass, we mean either the constituent quark mass, or the (running) current quark mass. Constituent quark masses are those which sum to the mass of the hadron. Current quark masses appear in the QCD Lagrangian. In future, when we write $m_u, m_d, m_s, \ldots$, we mean current quark masses.

Consider the chiral limit $m_u = m_d = m_s = 0$. Let q be a column vector containing the u, d and s quark fields.

In addition to possessing local gauge invariance (the gauge transformations acting on the colour indices), the QCD Lagrangian also possesses a global symmetry (acting on the flavour indices).

Define helicity projection operators [6]

$$\begin{aligned} q_L &= \frac{1}{2}(1+\gamma_5)q \\ q_R &= \frac{1}{2}(1-\gamma_5)q \end{aligned} \tag{1.2}$$

Then the Lagrangian density ($\not{D}$ is a covariant derivative)

$$\begin{aligned} \mathcal{L} &= i\bar{q}\not{D}q \\ &= i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R \end{aligned} \tag{1.3}$$

is invariant under [7]

$$\begin{aligned} q_L &\rightarrow U_L\, q_L \\ q_R &\rightarrow U_R\, q_R \end{aligned} \tag{1.4}$$

with

$$\begin{aligned} U_L &= exp(i\vec{\alpha}\cdot\vec{\lambda}) \\ U_R &= exp(i\vec{\beta}\cdot\vec{\lambda}) \end{aligned} \tag{1.5}$$

satisfying $UU^\dagger = I$ and $detU = 1$. The transformation group defined here is $SU(3)_L \times SU(3)_R$ (with $U_L \in \frac{1}{2}$ representation of $SU(3)_L$ and $U_R \in \frac{1}{2}$ representation of $SU(3)_R$).

By Noether's theorem there will be 16 conserved currents. The eight vector currents are [8] ($\mu = 0, \ldots, 3; i = 1, \ldots, 8$)

$$V_i^\mu(x) = \bar{q}(x)\gamma^\mu \frac{\lambda_i}{2} q(x) \tag{1.6}$$

and the eight axial vector currents are

$$A_i^\mu(x) = \bar{q}(x)\frac{\lambda_i}{2}\gamma^\mu\gamma_5 q(x) \tag{1.7}$$

The charges associated with these currents are the vector charge

$$Q_i(t) = \int V_i^0(x)\, d^3x \tag{1.8}$$

and the axial charge

$$Q_i^5(t) = \int A_i^0(x)\, d^3x \tag{1.9}$$

In the above, λ_i are the Gell-Mann matrices, γ_5 a Dirac matrix [9]

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\overline{q} = q^\dagger \gamma^0$ with

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

1.2.1 Chiral Algebra

In the chiral limit, the strong interaction Lagrangian is invariant and hence there exists 16 conserved currents and 16 charges associated with these currents.

Gell-Mann assumed that the algebra of the strong interactions can be generated by the hadron vector and axial vector charges given above [4]. The $SU(3)_L \times SU(3)_R$ group is generated by forming the combination [10,11]

$$Q_i^{\pm} = \frac{1}{2}(Q_i \pm Q_i^5) \tag{1.10}$$

with $Q_i^+ \equiv Q_i^L$ and $Q_i^- \equiv Q_i^R$. The charges $Q_i^{\pm}$ obey the following equal-time commutation relations

$$\begin{aligned} [Q_i^L, Q_j^L] &= i f_{ijk} Q_k^L \\ [Q_i^R, Q_j^R] &= i f_{ijk} Q_k^R \\ [Q_i^L, Q_j^R] &= 0 \end{aligned} \tag{1.11}$$

each element separately generating an algebra, and the elements of one algebra commuting with the elements of the other. That is, we have generated the $SU(3)_L \times SU(3)_R$ algebra. The completely antisymmetric f_{ijk} are the $SU(3)$ structure constants [12].

The above commutation relations are easily found from the following

$$\begin{aligned} [Q_i, Q_j] &= i f_{ijk} Q_k \\ [Q_i, Q_j^5] &= i f_{ijk} Q_k^5 \\ [Q_i^5, Q_j^5] &= i f_{ijk} Q_k \end{aligned} \tag{1.12}$$

As $PQ_iP^\dagger = Q_i$ and $PQ_i^5P^\dagger = -Q_i^5$, with P denoting the parity operator ($PP^\dagger = I$), the generators of $SU(3)_L \times SU(3)_R$ are connected in the following way

$$PQ_i^\pm P^\dagger = Q_i^\mp \tag{1.13}$$

and the algebra generated by equations (1.10) and (1.13) defines the chiral $SU(3)_L \times SU(3)_R$ algebra [13].

Symmetry is the important ingredient of this theory. The symmetry of the multiplets is broken by the small mass differences and electromagnetic corrections (we should note that the role of the electromagnetic interaction in the breaking of the isospin symmetry is not well understood). This mechanism cannot be used to explain the large mass differences found within the supermultiplets.

1.2.2 Chiral Symmetry Breaking

In order to study chiral symmetry breaking, we would like to know the representation under which the symmetry breaking part of the Hamiltonian, $\epsilon\mathcal{H}_{SB}$, transforms. One of the simplest models was proposed by Gell-Mann, Oakes and Renner [4] and Glashow and Weinberg [14]. In this model, $\epsilon\mathcal{H}_{SB}$ transforms like $(3,\overline{3}) + (\overline{3},3)$, and there is just one free parameter which needs to be determined. $\epsilon\mathcal{H}_{SB}$ consists of two terms, one term will break the $SU(3)_L \times SU(3)_R$ symmetry, the other will break the $SU(3)_V$ symmetry[1]. These terms are assigned to the $(3,\overline{3}) + (\overline{3},3)$ representation of $SU(3)_L \times SU(3)_R$ (the representation has this form as parity connects the group generators).

Originally, Gell-Mann assumed that chiral symmetry breaking was due to an $SU(3)$ singlet and a term transforming like the eighth component of an octet. That is,

$$\epsilon \mathcal{H}_{SB} = u_0 + c\, u_8 \tag{1.14}$$

The quantity c is a free parameter which can be found from meson mass equations [15]. It is the singlet term which breaks $SU(3)_L \times SU(3)_R$ and the octet piece which breaks $SU(3)_V$.

In the ideal world, we would have an exact $SU(3)_L \times SU(3)_R$ symmetry. This does not mean, however, that there are degenerate $SU(3)_L \times SU(3)_R$ multiplets. The dynamical realizations of this symmetry can be determined by knowing how the invariant part of the Hamiltonian, $\mathcal{H}_0$, is realized on the particle states. The possibilities are determined by the action of the charges on the vacuum [7,13]. When the charge annihilates the vacuum (Wigner-Weyl realization)

$$Q_i|0\rangle = 0 \tag{1.15}$$

the symmetry manifests itself as degenerate multiplets when $\epsilon = 0$. This is certainly true for the baryons where, for example, $M_n \approx M_p$.

For the case

$$Q_i^5|0\rangle \neq 0 \tag{1.16}$$

(Nambu-Goldstone realization) there are massless Goldstone bosons in the symmetry limit [16,17] .

[1] $V = L + R$

Note that we are dealing with the symmetric piece of the Hamiltonian (or Lagrangian). When the Lagrangian is invariant, but the vacuum is not, we say that the symmetry is spontaneously broken (or hidden). This is the situation we have with equation (1.16).

When we have isospin invariance, a group transformation will take, say, the proton to the neutron. Now suppose we have spontaneous symmetry breaking. Applying the corresponding (axial) symmetry transformation to the proton will give us a proton plus a zero energy meson.

If (1.16) was not satisfied and $[Q_i^5, H] = 0$, all isospin multiplets would have at least one mass degenerate partner of opposite parity. As this is not observed, chiral symmetry is broken spontaneously.

Breaking $SU(3)_L \times SU(3)_R$ down to $SU(3)_V$ gives us eight massless Goldstone bosons. As equation (1.15) is believed to be true for the vector symmetry, breaking $SU(3)_V$ gives us degenerate supermultiplets, e.g. $M_N = M_\Sigma$ (this degeneracy is only within a particular representation, e.g. $M_N \neq M_\Delta$).

At this stage, we have chiral $SU(2)_L \times SU(2)_R$ invariance, and massive kaon and eta mesons ($M_\kappa^2, M_\eta^2 \propto \epsilon$). Breaking this group down to the isospin group $SU(2)_V$ breaks the baryon supermultiplet degeneracy. However, we still have multiplet degeneracy, e.g., $M_p = M_n$, and the pion is still massless. The isospin symmetry is broken by quark masses (and electromagnetic corrections). These break the multiplet degeneracy, and gives the pion a mass.

In the quark model, the symmetry breaking part of the Lagrangian density is [18]

$$\begin{aligned}\frac{(m_u+m_d)}{2}(\overline{u}u+\overline{d}d)+m_s\overline{s}s &= \frac{(m_u+m_d+m_s)}{3}(\overline{u}u+\overline{d}d+\overline{s}s)\\ &+(\frac{m_u-m_d}{3})(\overline{u}u-\overline{d}d)\\ &+\frac{1}{3}(\frac{m_u+m_d}{2}-m_s)(\overline{u}u+\overline{d}d-2\overline{s}s)\\ &= c_0u_0+c_3u_3+c_8u_8 \end{aligned} \tag{1.17}$$

The notation used here can be related to that used in equation(1.14) by defining

$$u_i(x)=\overline{q}(x)\lambda_i q(x) \tag{1.18}$$

for $i=0,\ldots,8$. When $i=0$, $\lambda_0=\sqrt{\frac{2}{3}}I$ with I being the 3×3 identity matrix.

We mentioned that the singlet broke $SU(3)_L\times SU(3)_R$ and the octet term broke $SU(3)_V$. From equation(1.17) we see that the chiral group is broken by three quarks acquiring the same mass, and the $SU(3)_V$ symmetry is broken by u, d and s quark mass splitting (but with $m_u=m_d$).

Commuting u_i with the axial current (1.7) and using the identities [8]

$$\begin{aligned}[][AB,CD] &= -AC\{D,B\}+A\{C,B\}D-C\{D,A\}B+\{C,A\}DB\\ [\Gamma_a\lambda_\alpha,\Gamma_b\lambda_\beta] &= \frac{1}{2}\{\Gamma_a,\Gamma_b\}[\lambda_\alpha,\lambda_\beta]+\frac{1}{2}[\Gamma_a,\Gamma_b]\{\lambda_\alpha,\lambda_\beta\}\end{aligned} \tag{1.19}$$

where the Γ's are any of the Dirac matrices, and the equal-time quark commutation relations

$$\begin{aligned}\{q_i^\dagger(x),q_j(x')\} &= \delta_{ij}\delta^3(x-x')\\ \{q_i(x),q_j(x')\} &= \{q_i^\dagger(x),q_j^\dagger(x')\}=0\end{aligned} \tag{1.20}$$

we find(suppressing indices)

$$
\begin{aligned}
[\overline{q}(x)\frac{\lambda_i}{2}\gamma^0\gamma_5 q(x), \overline{q}(y)\lambda_i q(y)] &= q^\dagger(x)[\gamma_5\lambda_i, \gamma^0\lambda_j]\frac{q(y)}{2}\delta^3(x-y) \\
&= -\overline{q}(x)\gamma_5\{\frac{\lambda_i}{2}, \lambda_j\}q(y)\delta^3(x-y) \quad (1.21)
\end{aligned}
$$

Using (1.9), (1.7) and the above, we have

$$[Q_i^5(t), u_j(\vec{y}, t)] = -id_{ijk}\overline{q}(y)\gamma_5\lambda_k q(y) \quad (1.22)$$

$(i = 1\ldots, 8; j, k = 0, \ldots 8)$. In deriving this expression we have made use of the following

$$
\begin{aligned}
[\lambda_i, \lambda_j] &= 2if_{ijk}\lambda_k \\
\{\lambda_i, \lambda_j\} &= 2id_{ijk}\lambda_k \quad (1.23)
\end{aligned}
$$

$i, j, k = 0, \ldots, 8$.

From this expression we can define

$$v_i(y) = \overline{q}(y)\gamma_5\lambda_i q(y) \quad (1.24)$$

At equal-times we then have the following commutation relations

$$
\begin{aligned}
[Q_i(t), u_j(y)] &= i\, f_{ijk}\, u_k(y) \\
[Q_i(t), v_j(y)] &= i\, f_{ijk}\, v_k(y) \\
[Q_i^5(t), u_j(y)] &= -i\, d_{ijk}\, v_k(y) \\
[Q_i^5(t), v_j(y)] &= i\, d_{ijk}\, u_k(y) \quad (1.25)
\end{aligned}
$$

the d_{ijk} being completely symmetric under the interchange of indices.

Using the above commutation relations and equation (1.14) we can determine the extent to which chiral $SU(2)_L \times SU(2)_R$ is broken. The divergence of the axial current is

$$\begin{aligned}\partial_\mu A_i^\mu(x) &= -i\,[Q_i^5(t), \epsilon\mathcal{H}_{SB}(\vec{x},t)] \\ &= -(\,d_{i0k}v_k + \,c\,d_{i8k}v_k) \\ &= -(d_{i0k} + c\,d_{i8k})v_k(x) \end{aligned} \tag{1.26}$$

and as, for $i,j=1,2,3$

$$\begin{aligned} d_{i0k} &= \sqrt{2/3} \quad , i=k \\ d_{i8k} &= \sqrt{1/3} \quad , i=k \end{aligned} \tag{1.27}$$

and zero otherwise, we obtain

$$\begin{aligned} \partial_\mu A_i^\mu &= \frac{-(\sqrt{2}+c)}{\sqrt{3}}v_i \\ &= 0 \end{aligned} \tag{1.28}$$

if $c=-\sqrt{2}$. That is, we have exact $SU(2)_L \times SU(2)_R$ if $c=-\sqrt{2}$.

Using the quark model, we would have

$$\partial_\mu A_i^\mu = \frac{-(c_0\sqrt{2}+c_8)}{\sqrt{3}}v_i \tag{1.29}$$

which would vanish when $c_0\sqrt{2}=-c_8$. Thus, if we define c to be the ratio of the strength of the octet and singlet parts in the symmetry breaking term, i.e., $c=c_8/c_0$, we would have equation(1.28).

How good is the $SU(2)_L \times SU(2)_R$ symmetry? We saw above that the symmetry is exact if $c=-\sqrt{2}$. Therefore the deviation of c from $-\sqrt{2}$ is an indication of the degree of $SU(2)_L \times SU(2)_R$ symmetry breaking (as the pion mass is small, c should be near $-\sqrt{2}$). Experimentally, $c \approx -1.29$ [13]. Thus, $SU(2)_L \times SU(2)_R$ is a good symmetry, and the mass splitting within multiplets will be small. That is, the electromagnetic term in the

Hamiltonian will be small and it is for this reason that we normally choose $c_3 = 0$ in equations(1.14) and (1.17).

What do we mean by saying that symmetries are good or approximate? It means that the terms which break the symmetry must be small compared to the scale of QCD. For example, saying $SU(3)_V$ is approximate means that m_u, m_d and m_s are small compared with the scale of QCD, which is chosen to be M_ρ (alternatively, $m_s - \hat{m}$ must be small compared to M_ρ). Similarly, the symmetry $SU(2)_V$ being approximate means that $m_u - m_d$ must be small compared to M_ρ [19].

We have said that the strong interactions are nearly invariant under the chiral group $SU(3)_L \times SU(3)_R$. There is actually a larger symmetry, and this can be seen by redefining the vector and axial vector currents, (1.6) and (1.7), so that there are now nine, instead of eight (i.e., $i = 0, \ldots, 8$). The commutation relations (1.11) now generate the $U(3)_L \times U(3)_R$ algebra. The $i = 0$ case represents the baryon current.

The unitary transformations can be factored into a transformation corresponding to conservation of baryon number, and transformations generated by the $SU(3)$ group. That is, our symmetry group is the product $SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$. The left and right unitary spin and baryon number are connected by parity (as before). Unfortunately, this symmetry produces major problems. As the $U(3)_L \times U(3)_R$ algebra has nine generators, spontaneous symmetry breaking (in the Nambu-Goldstone mode) is realized by nine Goldstone bosons. There is no experimental evidence to support this realization [13].

In the chiral limit, there are nine conserved vector and axial vector currents. The problem here is that the ninth axial current is gauge dependent.

Its existence is known as the $U(1)$ problem [20].

It is not entirely true to say that the ninth Goldstone boson does not exist. A gauge-dependent current can produce zero mass bosons. Such a current will be a sum of physical and unphysical currents. A gauge-dependent quantity is truly unphysical only if it can be shown, via a gauge transformation, to vanish, i.e., if it can be gauged to zero [21].

The largest flavour symmetry which provides "physical" currents is taken to be $SU(3)_L \times SU(3)_R \times U(1)_V$.

1.3 The Quark Model

In the early 1960's, Gell-Mann [22] and Zweig [23] independently developed a (classification) model of hadrons. They proposed that hadrons consist of elementary particles called quarks (Zweig called them aces). According to Gell-Mann they were mathematical entities. Zweig believed them to be real. The quarks would have half-integral spin, but it was necessary that they have fractional charge.

Baryons would consist of three quarks (qqq), and mesons would be quark-antiquark pairs ($\overline{q}q$).

The mathematical basis of this theory is that the quark transforms as a triplet representation under $SU(3)$. The multiplets into which the hadrons are placed are found by constructing irreducible representations of $SU(3)$. This is done by decomposing direct products of triplet representations as mentioned above. For example the $\underline{10}$ will be identified with the $(3/2)^+$ baryon decuplet, the two $\underline{8}$'s with the $(1/2)^+$ baryon octet and the $(0)^-$ meson octet. The baryon singlet has never been observed.

Unfortunately, there is a major flaw in this theory. All spin 1/2 particles must obey Fermi statistics, i.e., the wavefunction must be antisymmetric under the interchange of two quarks. It was found that the Ω^-, with three identical strange quarks, violated this law. This problem was solved by introducing a new quantum number - colour [24]. Quarks now come in three colours, red, green and blue. Hadrons would be colourless combinations. This meant that the Ω^- contained three different strange quarks and its wavefunction was indeed antisymmetric under the interchange of two quarks.

If we now define quarks to transform as a triplet representation under the internal colour group $SU(3)_c$, then hadrons must transform as singlets in order to be colourless. This $SU(3)_c$ symmetry is exact!

The QCD Lagrangian density is $SU(3)_c$ invariant, and as it contains no quark mass term, it is invariant under the chiral flavour group $SU(N)_L \times SU(N)_R \times U(1)_V$ in the fundamental representation (in what is to follow, we will drop the $U(1)_V$ factor and take $N = 3$). The quark fields are written in terms of left and right handed components.

In order to break the $SU(3)_L \times SU(3)_R$ flavour symmetry, we add a mass term which belongs to the representation $(3, \overline{3}) + (\overline{3}, 3)$ but which is an $SU(3)_c$ singlet. The appropriate term is

$$\Delta\mathcal{L} = \overline{q} m_q q \tag{1.30}$$

leaving $\mathcal{L} + \Delta\mathcal{L}$ $SU(2)_L \times SU(2)_R$ invariant, with $m_u = m_d = 0$ and $m_s \neq 0$.

One of the major reasons the quark model was not popular in the 1960's, was the fact that all experiments designed to observe colour triplets failed. That is, no quarks were found. It was concluded that nature only allowed colourless particles and, for reasons unknown, quarks were confined in groups of two or three (or any combination which produced a colourless particle). This is known as the "confinement problem". We will briefly review this in the next section.

Before we can construct the current theory of strong interactions, one more ingredient is required, gauge invariance [25].

Given that a Lagrangian Field theory is globally invariant, it can be made locally invariant by introducing compensating gauge fields [26,27]. A familiar example is electromagnetism. The global symmetry is the $U(1)$ group and making the theory locally invariant introduces the electromagnetic vector potential. In quantum electrodynamics this gauge field is associated with the photon.

In 1954 Yang and Mills (and Shaw) [28] generalized local gauge invariance from the U(1) group to a nonabelian Lie algebra. They showed that for every group generator there is a compensating field. They also found that a property of non-abelian groups is that the gauge fields interact.

Almost twenty years later, Politzer [29] and Gross and Wilczek [30] showed that due to the self interaction of the gauge fields, non-abelian gauge theories exhibit asymptotic freedom, that is, the strength of the interaction mediated by the gauge fields becomes very small at very high energies (or very small distances). Furthermore, only non-abelian gauge theories exhibit this property.

1.4 QCD

In the late 1960's, deep inelastic experiments were carried out at SLAC to test the quark model [31]. If hadrons did not consist of quarks, then the inelastic scattering cross section would decrease rapidly at very high energies. However, it was found that the cross section actually decreased slowly suggesting the proton's electric charge was concentrated in point-like constituents (Feynman's Partons) [32].

A second result from these experiments was that, although the strong interaction of quarks was complicated (strong) at low energies, quarks behaved as if they were free at high energies. This is exactly the asymptotic freedom property of non-abelian gauge theories (discovered a number of years after these experiments were performed).

It was concluded that the theory of strong interactions is a non-abelian gauge theory, and that the interaction between the quarks is mediated by eight massless, spin 1, non-abelian gauge fields called gluons [33].

If QCD is to describe the real world, the theory must be asymptotically free, and must exhibit confinement. It has already been pointed out that QCD is asymptotically free. It is an ongoing problem to prove that QCD is confining.

One way to test this is to work on a 4-dimensional lattice (at low energies) in Euclidean space [34]. Calculating quantities in gauge theories has now been reduced to solving a statistical mechanics problem, usually the Ising model. In lattice theories, the inverse of the lattice spacing acts as an ultraviolet cutoff. The continuum limit (assuming it exists) is obtained by taking the lattice spacing to zero.

It can be shown that a number of theories based on the 4-dimensional Euclidean lattice are asymptotically free, and exhibit confinement [35,36]. As an example, we briefly consider Wilson's confinement mechanism.

A test of confinement in a lattice gauge theory is the existence of a phase transition. Wilson conjectured that, in the weak coupling phase (large $\beta = 1/kT$)[2], there will exist massless gauge fields and free quarks, and in

[2]If we denote the strength of the coupling of the gauge fields to the quarks by g, then $\beta \sim 1/g^2$. Therefore, large β implies small g, and hence, weak coupling.

the strong coupling phase, massive gauge fields and confined quarks. There is a phase transition at some intermediate β_c.

Wilson considered a QCD vacuum consisting of loops representing the virtual creation and annihilation of quark-antiquark pairs. These loops can be large or small. However, if they are sufficiently large, it may be possible to detect individual quarks. Wilson has proposed that, in the strong coupling regime, i.e., in a confining phase, large loops are suppressed.

A useful quantity to work with when one has a theory with phase transitions is an order parameter. The order parameter considered by Wilson was the expectation value of the product of matrices associated with the links of the lattice [37]. This expectation value, known as the Wilson loop (denoted $W(C), C$ denoting the loop), describes the creation of a quark pair at one end of the loop, and destruction at the other end. For small β, the Wilson loop $\sim exp(-kRT)$ for linear interquark energy (k is a constant)[3].

Wilson has found that, for arbitrarily shaped loops, if the weight associated with a given quark path goes like $exp[-kA]$, with A the enclosed area of the loop, then small loops will dominate and individual quarks will not be observed: confinement. Comparing with the above, we see that this area law leds to a linear interquark energy. This is the form of the order parameter for strong coupling. In the weak coupling regime, we have free quarks and a Coulomb-like potential. From this we conclude that a gauge theory satisfying the so-called "Wilson criterion" will exhibit confinement.

It was also pointed out by Wilson that the structure of the strong coupling region is similar to that of hadronic string theories. This point was also raised by Kogut and Susskind [39]. They proposed that hadrons are

[3]It can be shown [38] that two isospin 1/2 charges separated by a distance R, have interquark energy proportional to R.

made up of strings with quarks at the end points (the strings are lines of non-abelian electric flux). Confinement is due to the inability to break a string without producing quark pairs.

The above are by no means the only confinement mechanisms. A third mechanism will be discussed in the next section.

QCD being asymptotically free means that at high energies the quark-gluon coupling constant is small and we can apply perturbative techniques to QCD calculations. Unfortunately, at medium to low energies the coupling constant is large, and so quarks and gluons interact strongly. At these energies we can no longer apply perturbation theory to QCD.

This would not be a problem if we could find another small parameter about which we could expand Green's functions, etc. At low energies there are two such parameters, the current quark mass and the meson momentum. Chiral Perturbation Theory [40] - [46] allows us to make expansions in powers of quark masses and external momentum.

This method, however, is not as straight forward as it sounds. Li and Pagels [42] were able to show that if the Hamiltonian symmetry is realized by Goldstone bosons (Nambu-Goldstone realization), then the S-matrix and matrix elements of currents may no longer analytic in ϵ (or M_π^2) in the chiral limit (the momentum integral becomes infrared divergent due to the strong interaction becoming a long range force in the limit of massless mesons).

In such cases, it becomes necessary to calculate so-called Leading Non-Analytic Contributions (LNAC), and it is not unusual to find contributions of order[4] M_π^3 or $M_\pi^4 ln M_\pi^2$. The LNAC (terms of order (M_π^3)) are assumed to dominate all higher order contributions $(O(M_\pi^4 ln M_\pi^2))$.

[4]In terms of the average running quark mass, $\hat{m}$, these contributions are of order $\hat{m}^{3/2}$ or $\hat{m}^2 ln \hat{m}$

A problem with CPT is that LNAC can be quite large, making quark mass expansions useless. This led to the creation of Improved Chiral Perturbation Theory [19](ICPT) . The LNAC are due to the meson cloud surrounding the baryon (and meson) and are small compared to leading analytic terms (but are still believed to dominate higher order contributions).

The main alternatives to chiral perturbation theory are lattice theories (mentioned above), QCD Sum Rules[47,48] and phenomenological models of hadrons [49,50,51]. A type of phenomenological model, and the one we consider here, is the bag model [52,53,54].

1.5 Bag Models

In bag models, the hadron is fixed in space, and the quarks are confined to the interior of a volume of space - the bag.

One of the more familiar bag models is the MIT bag model [55]. The quark wavefunctions are found by solving the Dirac equation for free spin 1/2 particles

$$(i \not{\partial} - M)\Psi(x) = 0 \tag{1.31}$$

within a bag of radius R. Two solutions are obtained, characterized by the quantum number $\kappa = \mp(j + \frac{1}{2}) = \mp 1$ [56]

$$\Psi_{-1} = q_{1s}(\vec{r}, t) = \frac{N_s}{\sqrt{4\pi}} \begin{bmatrix} \alpha_s^+ j_0(\omega_s r) \\ i\alpha_s^- \vec{\sigma} \cdot \hat{r} j_1(\omega_s r) \end{bmatrix} e^{-i\alpha_s t}\, b\, \theta(R - r)$$

$$\Psi_{+1} = q_{1p}(\vec{r}, t) = \frac{N_p}{\sqrt{4\pi}} \begin{bmatrix} \alpha_p^+ \vec{\sigma} \cdot \hat{r} j_1(\omega_s r) \\ i\alpha_p^- j_0(\omega_p r) \end{bmatrix} e^{-i\alpha_p t}\, b\, \theta(R - r) \tag{1.32}$$

corresponding to $1s_{\frac{1}{2}}$ and $1p_{\frac{1}{2}}$ solutions. In these expressions, N_s and N_p are normalization constants (s and p refer to $s-$ and $p-$wave),

$$N_{s,p} = \frac{1}{R^3 j_0^2(\omega_{s,p} R)} \left[\frac{\alpha_{s,p}(\alpha_{s,p} - \lambda_{s,p})}{2\alpha_{s,p}(\alpha_{s,p} - 1) + \lambda_{s,p}} \right] \tag{1.33}$$

j_0 and j_1 are spherical Bessel functions, and

$$\alpha^{\pm}_{s,p} = [\frac{\alpha_{s,p} \pm \lambda_i}{\alpha_{s,p}}]^{\frac{1}{2}} \tag{1.34}$$

with

$$\alpha_{s,p} = [(\omega_{s,p}R)^2 + \lambda_i^2]^{\frac{1}{2}} \tag{1.35}$$

and

$$\lambda_i = M_i R \tag{1.36}$$

The suffix i denotes the quark flavour. Quark energy is denoted by $\omega_{s,p}$.

The quark energy is quantized by requiring that no current flows across the bag surface. This requirement leads to

$$\alpha_s^+ j_0(\omega_s R) = \alpha_s^- j_1(\omega_s R) \tag{1.37}$$

For massless quarks,

$$j_0(\omega_s R) = j_1(\omega_s R) \tag{1.38}$$

is satisfied by

$$\omega_s R = 2.04\,,\, 5.40\,, \ldots \tag{1.39}$$

Note the parameterization (i.e. $\omega_s R$) we have used here. An alternative notation uses $\omega r/R$ as the argument of the Bessel functions. As with QCD, we require our bag model to be confining. We ensure this in the following way.

As no isolated quarks have been observed, we confine quarks in the bag by requiring the quark energy to become infinite for large bag radius, and hence the quark mass will become infinite and unobservable. An important feature of the MIT bag model is the introduction of a term which produces just this feature.

Confinement means that there is no flow of colour through the bag surface. If the quark current is $\overline{q}\gamma^\mu q$, then this restriction implies that $n_\mu \overline{q}\gamma^\mu q = 0$ on the bag surface ($n^\mu(\vec{x}, t)$ is a unit four vector normal to the surface). It can be shown that this is equivalent to the linear boundary condition $in_\mu \gamma^\mu q = q$ on the bag surface. If we now consider $in_\mu j^\mu$, it is easy to show that, on the bag surface, $\overline{q}q = 0$ (rather than the quark current).

Now consider the energy-momentum tensor inside the bag

$$T^{\mu\nu} = -\frac{i}{2}\overline{q}\gamma^\mu \overleftrightarrow{\partial}^\nu q \tag{1.40}$$

Inside the bag, $\partial_\nu T^{\mu\nu} = 0$. As we do not want any energy-momentum flux to leave the bag, we have $n_\mu T^{\mu\nu} = 0$ on the bag surface. Unfortunately, at the surface, energy-momentum is not conserved

$$\partial_\mu T^{\mu\nu} = \frac{1}{2} n^\nu n \cdot \partial(\overline{q}q)\delta_s \tag{1.41}$$

The solution is to define a new energy-momentum tensor

$$T^{\mu\nu}_{MIT} = (T^{\mu\nu} + Bg^{\mu\nu})\theta_v \tag{1.42}$$

(θ_v is a step function - it is zero outside the bag, and unity inside) where

$$B = -\frac{1}{2} n \cdot \partial(\overline{q}q) \tag{1.43}$$

is the MIT non-linear boundary condition. Note that we now have conservation of energy-momentum at the bag surface, i.e., $\partial_\mu T^{\mu\nu}_{MIT} = 0$.

The total quark energy is thus

$$\sum_i \omega_i + \frac{4\pi}{3} R^3 B \tag{1.44}$$

with ω_i the ground state quark energy (of quark flavour i). The introduced bag "pressure" $B^{\frac{1}{4}} \approx 120$ MeV ensures confinement.

Asymptotic freedom in bag models is ensured by considering only free quarks confined within the bag.

A problem with the MIT model is the lack of a pion (or meson) field. As these fields ensure chiral symmetry (via the sigma model [57]), this is a serious omission.

By introducing a scalar particle and pion field into the model, Chodos and Thorn [58] made the MIT model chiral invariant. By a suitable redefinition of the scalar and meson fields, the scalar field can be removed, leaving the meson field only coupled to the quarks. This redefinition has the effect of making the theory non-linear (details are given in chapter 3). In this model, the pion is allowed into the bag - as opposed to, say, the Brown and Rho model [59] or Hybrid Chiral Bag (HCB) models [60] which exclude the pion from the interior of the bag. These models assume that pions (mesons) are created and annihilated only outside the bag.

An alternative formulation is the Cloudy Bag Model (CBM).

1.5.1 The CBM

The CBM [52,61,62] is similar to the model of Chodos and Thorn in a number of ways. Both require the bag pressure to ensure confinement, and both treat the meson field as a plane wave. The quark wave functions used are the MIT wave functions.

The CBM also allows the meson field inside the bag. As it is possible for mesons ($q\overline{q}$ pairs) to be created inside the bag, we allow the mesons to enter the bag volume. Also, since the meson field is treated as a plane wave, propagating through all of space, it should not be excluded from inside the bag. The CBM is also a non-linear theory. As above, the sigma field is removed by redefining the meson field. We describe this in more detail in chapter 3.

Certain assumptions are made in the formulation of the CBM. It is assumed that the bag is surrounded by few mesons (as we shall see, calculating bare bag probabilities and using this assumption allows us to determine the radius of the bag). It was shown by Dodd, et.al. [63], that the average number of pions surrounding a bag with radius 0.82 fm is $\leq 0.9 \pm 1.0$. Because of their large masses, we do not expect a large number of kaon or eta mesons in the meson cloud.

We can write the CBM Lagrangian density in the form

$$\mathcal{L} = \mathcal{L}_{MIT} + \mathcal{L}_{\pi} + \mathcal{L}_{I} \tag{1.45}$$

for massless quarks and mesons. The first term on the right is the MIT Lagrangian. The second term contains $(\partial_\mu \pi)^2$, and the third term describes the surface interaction. This final term is the major difference between the CBM and the model of Chodos and Thorn.

1.6 The Sigma Term

In all of the above models, it has been assumed that the Hamiltonian symmetry is broken by a term belonging to the $(3, \overline{3}) + (\overline{3}, 3)$ representation of $SU(3)_L \times SU(3)_R$. That is, by adding the mass term $\overline{q} m_q q$.

How can this be tested? We can construct a matrix element which depends only on the symmetry breaking part of the Hamiltonian and which can be found (indirectly) from experiment. From this matrix element we can construct the pion-nucleon sigma term. We shall denote the experimentally determined sigma term by $\Sigma_{\pi N}(t = 2M_\pi^2)$ [13,15] (t is the square of the momentum transfer).

We cannot actually calculate the sigma term directly from experiment. Instead, we find an expression relating the sigma term to a scattering amplitude. This can be done using current algebra. Unfortunately, the amplitude will be off mass shell. We must therefore find some way to relate these amplitudes to on mass shell amplitudes. This was done by Cheng and Dashen by making use of the Adler consistency relations. This solution raises another problem. The Cheng-Dashen amplitude is calculated at an unphysical point. The experimental input (e.g. phase shifts) used to calculate these amplitudes is measured at physical points. The extrapolation from physical to unphysical points is done via forward scattering dispersion relations. With this, $\Sigma_{\pi N}(t = 2M_\pi^2)$ can be found.

Theoretical calculations of the sigma term are made at the point $t = 0$ and are denoted $\sigma_{\pi N}(0)$. Details are given in the next chapter.

In chapter 2, we review a number of sigma term calculations. We shall see that the theoretical value is considerably smaller than the experimental value. We review possible solutions to this problem. These involve chiral perturbation theory and the HCB model. Finally, we review the most recent experimental derivation of the sigma term.

In chapter 3 we discuss the volume coupling $SU(3)$ CBM. We derive bare coupling constants, expressions for the baryon self-energy, bare bag probabilities and renormalized coupling constants. With this we are able to calculate the sigma term.

We have also included a number of appendices dealing with kinematics and dispersion relations.

Chapter 2

The Pion-Nucleon Sigma Term

We define the sigma term to be the nucleon expectation value of the commutator

$$\Sigma_{ij} = [Q_i^5, [Q_j^5, H]] \tag{2.1}$$

The sigma commutator, Σ_{ij}, is symmetric in the SU(3) indices i and j, and vanishes in the chiral limit.

Using the commutation relations given in the previous chapter, the sigma commutator can be shown to be (using the Hamiltonian given by equation(1.14) and $i, j = 1, 2, 3$)

$$\begin{aligned}
\Sigma_{ij} &= d_{j0k}d_{ikl}u_l + cd_{j8k}d_{ikl}u_l \\
&= [d_{j01}d_{i1l} + d_{j02}d_{i2l} + d_{j03}d_{i3l}]u_l \\
&\quad + c\,[d_{j81}d_{i1l} + d_{j82}d_{i2l} + d_{j83}d_{i3l}]u_l \\
&= (\sqrt{2} + c)(\sqrt{2}u_0 + u_8)
\end{aligned} \tag{2.2}$$

summing over repeated indices and $c \approx -1.29$ the ratio of the strength of the octet and singlet parts in the symmetry breaking term. In the limit $M_\pi \to 0$, $c = -\sqrt{2}$ and so Σ_{ij} vanishes in this, the $SU(2) \times SU(2)$, limit.

The sigma term (at zero momentum transfer) is

$$\sigma_{\pi N}(0) = \frac{1}{3}\sum_{i=1}^{3}\langle N(p)|\Sigma_{ii}|N(p)\rangle \tag{2.3}$$

Substituting equation(2.2) we find that

$$\sigma_{\pi N}(0) = \frac{(\sqrt{2}+c)}{3}\langle N|\sqrt{2}u_0 + u_8|N\rangle \tag{2.4}$$

The matrix element $u_8^N = \langle N|u_8|N\rangle$ is known and has a value of 166 ± 10 MeV[1] [13]. Unfortunately, $u_0^N = \langle N|u_0|N\rangle$ is not known. To get around this problem we rearrange the above expression to give

$$\sigma_{\pi N}(0) = \frac{(\sqrt{2}+c)}{3}u_8^N(\frac{\langle N|\sqrt{2}u_0|N\rangle}{u_8^N} + 1) \tag{2.5}$$

It is believed that the magnitude of the $SU(3)_V$ breaking is of the order as the spontaneous breaking of $SU(3)_L \times SU(3)_R$ (so u_0^N will be of the same order of magnitude as u_8^N). Hence $|u_0^N/u_8^N| = 1 - 2$ and

$$\sigma_{\pi N}(0) = 17 - 27\,\text{MeV} \tag{2.6}$$

with $c \neq -\sqrt{2}$.

We now compare this estimate with experiment.

---

[1]Note that this figure is 15 years old. We use it because, at this stage, we are only interested in estimating the sigma term. A detailed calculation will come later.

2.1 Experimental Estimates of the Sigma Term

Before we can consider the actual calculation of the sigma term, it will be necessary to define a transition amplitude for a two body scattering process.

Denote the amplitude of the process (Appendix A,B) [64]

$$N(p) + M_i(q) \to N(p') + M_j(q') \tag{2.7}$$

by $T_{ji}(\nu, t, q^2, q'^2)$ (i and j are $SU(3)$ indices) where

$$\nu = \frac{-(p+p')\cdot q}{2M_N} \tag{2.8}$$

$t = -(p-p')^2$ is (minus) the square of the momentum transfer, and $N(p)$ and $M_i(q)$ represent incoming particles (p and q are (incoming) nucleon and meson four-momenta respectively).

An expression for the T-matrix can be found by "contracting" (or "reducing") the matrix element $\langle q', j; p', s'\, out | q, i; p, s\, in\rangle$ (s denotes nucleon spin/charge states, and i denotes meson charge states).

The S-matrix for baryon-meson scattering is (Appendix B)

$$\begin{aligned}
&\langle q', j; p', s'\, in | S | q, i; p, s\, in\rangle \\
&\quad = I - (2\pi)^4 \delta^4(p + q - p' - q')(q^2 - M_\pi^2)(q'^2 - M_\pi^2) \int d^4 z e^{-iq'\cdot z} \langle p' | T(\phi_j(z)\phi_i(0)) | p\rangle \\
&\quad = I + i(2\pi)^4 \delta^4(p + q - p' - q') T_{ji}(\nu, t, q^2, q'^2)
\end{aligned} \tag{2.9}$$

with the off-shell transition amplitude given by

$$T_{ji}(\nu, t, q^2, q'^2) = i\frac{(q^2 - M_\pi^2)}{f_\pi M_\pi^2}\frac{(q'^2 - M_\pi^2)}{f_\pi M_\pi^2}\int d^4 z e^{-iq'\cdot z}\langle p' | T(\partial_\mu A_j^\mu(z)\partial_\nu A_i^\nu(0)) | p\rangle \tag{2.10}$$

where we have made use of PCAC, i.e., $\partial_\mu A_i^\mu(z) = f_\pi M_\pi^2 \phi_i(z)$. Pulling the derivatives through the time-ordering operator we get [11,18]

$$
\begin{aligned}
T_{ji}(\nu,t,q^2,q'^2) &= i\frac{(q^2-M_\pi^2)}{f_\pi M_\pi^2}\frac{(q'^2-M_\pi^2)}{f_\pi M_\pi^2}\int d^4z e^{-iq'\cdot z} \\
&\times\langle p',s'|\big(q'_\mu q_\nu T\{A_j^\mu(z)A_i^\nu(0)\} \\
&+ i\, q'_\mu\delta(z_0)[A_j^\mu(z),A_i^0(0)] \\
&- \delta(z_0)[A_j^0(z),\partial_\nu A_i^\nu(0)]\big)|p,s\rangle \qquad (2.11)
\end{aligned}
$$

From equations(1.9), (1.26) and (2.1), we see that the last term in this equation is the sigma term. In the soft-meson limit, $q \to 0$ and $q' \to 0$, this reduces to

$$
\begin{aligned}
T_{ji}(0,0,0,0) &= -\frac{1}{f_\pi^2}\langle p|[Q_i^5,[Q_j^5,H(0)]]|p\rangle \\
&= -\frac{\Sigma_{\pi N}(0)}{f_\pi^2} \qquad (2.12)
\end{aligned}
$$

and gives an off-shell expression for the sigma term (see figure 2.1). The $SU(3)$ indices i and j are hidden in $\Sigma_{\pi N}(0)$ (we are only interested in the case of the (identical) mesons being pions).

We now relate this to the on shell amplitude. First note (from equation(2.1) or (2.2)) that $\Sigma_{ij} = \Sigma_{ji}$. It is therefore more appropriate to use isospin even scattering amplitudes.

If we let T_+ denote the amplitude for $\pi^+ + p \to \pi^+ + p$ and T_- the amplitude for $\pi^- + p \to \pi^- + p$, we can define an isospin even amplitude (Appendix C)

$$
T^+ = \frac{1}{2}(T_- + T_+) \qquad (2.13)
$$

Writing the amplitude in the form $T^+(\nu,t) = A^+(\nu,t)+\nu B^+(\nu,t)$, Cheng and Dashen [65] extrapolated equation(2.12) on to the mass shell using the following expansion (it has been shown that there are no $O(M_\pi^3)$ terms in the following [66]. The leading nonanalytic term is of order $M_\pi^4 ln M_\pi^2$)

$$T^+(0,2M_\pi^2,M_\pi^2,M_\pi^2) = T^+(0,0,0,0) + M_\pi^2 \frac{\partial}{\partial q^2} T^+(0,0,0,0) + M_\pi^2 \frac{\partial}{\partial q'^2} T^+(0,0,0,0) \quad (2.14)$$

Substituting the Adler consistency relations (which can be found by contracting $\langle p'q'|p\rangle$ and $\langle p'|pq\rangle$) [67]

$$T^+(0,2M_\pi^2,M_\pi^2,0) = T^+(0,0,0,0) + M_\pi^2 \frac{\partial}{\partial q^2} T^+(0,0,0,0) = 0$$
$$T^+(0,2M_\pi^2,0,M_\pi^2) = T^+(0,0,0,0) + M_\pi^2 \frac{\partial}{\partial q'^2} T^+(0,0,0,0) = 0$$

into equation(2.14), we get

$$T^+(0,2M_\pi^2,M_\pi^2,M_\pi^2) = -T^+(0,0,0,0) + O(M_\pi^4) = \frac{\Sigma_{\pi N}(2M_\pi^2)}{f_\pi^2} \quad (2.15)$$

This gives the sigma term on-mass shell, but at an unphysical point ($t > 0$). The point $\nu = 0$ and $t = 2M_\pi^2$ is known as the Cheng-Dashen point.

Using a broad-area subtracted dispersion relation [68], Cheng and Dashen were able to extrapolate to the on-shell unphysical point $\nu = 0, t = 2M_\pi^2$ and

$$\nu_B = -\frac{q \cdot q'}{2M_N} = 0 \quad (2.16)$$

(ν_B is made zero, then ν, to remove the nucleon pole).

A broad area subtraction method is used because of discrepancy's which arise in the calculation of the subtraction constant at threshold [69,70]. In this method, the subtraction is smeared (integrated) over a finite region of the real axis, so reducing the weight of threshold behaviour.

Using the broad-area subtraction method, the amplitude was calculated to be

$$T^+(0,2M_\pi^2,M_\pi^2,M_\pi^2) = 1.7M_\pi^{-1} \quad (2.17)$$

with $f_\pi = 93$ MeV and $M_\pi = 139.56$ MeV [12], this gives

$$\Sigma_{\pi N}(2M_\pi^2) \approx 105 \text{ MeV} \tag{2.18}$$

compared with equation(2.6).

However, this method seems to over emphasize low energy data points, and incompatible low-energy and high-energy phase shifts were used, making the value obtained too large by about 30 MeV. A more reliable estimate was found to be [71]

$$T^+(0, 2M_\pi^2, M_\pi^2, M_\pi^2) = 1.1M_\pi^{-1} - 1.3M_\pi^{-1} \tag{2.19}$$

so that

$$\Sigma_{\pi N}(2M_\pi^2) = 68 - 81 \text{ MeV} \tag{2.20}$$

which is still considerably larger than equation(2.6). The error in equations (2.18) and (2.20) is expected to be around 30%.

The following diagram shows where the above amplitudes are calculated (for $\nu = 0$)[72].

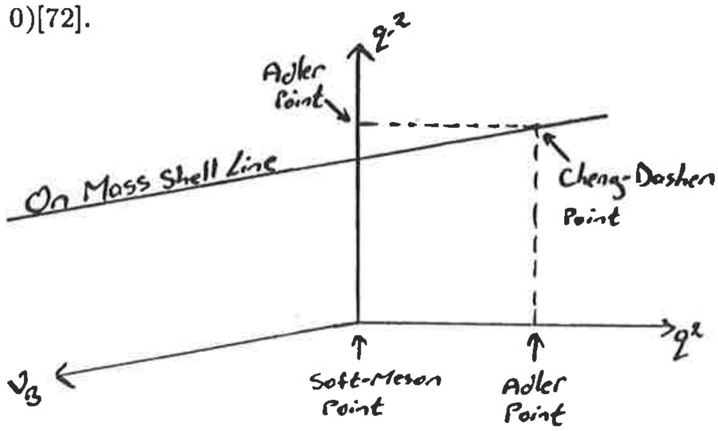


Figure 2.1: Cheng-Dashen point, soft-meson point and Adler points.

Instead of using the transition amplitude given by Cheng and Dashen, we can construct the amplitude [73,74]

$$C^{+}(\nu,t) = A^{+}(\nu,t) + \frac{\nu}{1 - t/4M_N^2} B^{+}(\nu,t) \tag{2.21}$$

A^+ and B^+ being the invariant amplitudes, $C(\nu,t) = C(\nu,t,M_\pi^2,M_\pi^2)$ and

$$\begin{aligned} A^{+}(\nu,t) &= a_1^+ + a_2^+ t + a_3^+ \nu^2 + a_4^+ \nu^2 t + \ldots \\ \nu^{-1} B^{+}(\nu,t) &= b_1^+ + b_2^+ t + b_3^+ \nu^2 + b_4^+ \nu^2 t + \ldots + \text{Born terms} \end{aligned} \tag{2.22}$$

where the a_i and b_i are real numbers. This form of the amplitude is related, in a simple way, to the experimental cross-section. As such, the high energy behaviour of the amplitude can be easily found from the data.

This construction has produced an amplitude in which Born terms are confined to the B^+ amplitude, has a smother threshold behavior, and has a simpler t-channel partial wave expansion.

The analysis of Hohler, Jakob and Strauss [75] involved the use of the polynomial expansion (2.22) of the isospin-even πN amplitude $A^+(\nu,t)$. This amplitude corresponds to the $T^+(\nu,t)$ amplitude of Cheng and Dashen at the point $\nu = 0, t = 2M_\pi^2$ (up to a constant g^2/M_N).

The polynomial must be even in ν^2 because of the crossing symmetry of $A^+(\nu,t)$

$$A^{+}(-\nu,t) = A^{+}(\nu,t) \tag{2.23}$$

That is, A^+ is an even function for $\nu \to -\nu$.

Using an expression obtained by Osypowski [76], they found (ignoring Δ exchange contributions)

$$\begin{aligned} A^+(0, 2M_\pi^2) &= a_1^+ - \frac{g^2}{M_N} + 2M_\pi^2 a_2^+ \\ &= \frac{\Sigma_{\pi N}(2M_\pi^2)}{f_\pi^2} \end{aligned} \tag{2.24}$$

with $g^2/4\pi^2 \approx 14.3$ the pion-nucleon coupling constant [77]. The once subtracted fixed-t dispersion relation for $C^+(\nu, t)$ is [78]

$$\begin{aligned} ReC^+(\nu,t) &= -\frac{g^2}{M_N}\frac{\nu^2}{(\nu^2-\nu_B^2)(1-t/4M_N^2)} + ReC^+(0,t) \\ &+ \frac{2\nu^2}{\pi}\int_{\nu_1}^{\infty}\frac{d\nu'}{\nu'}\frac{ImC^+(\nu',t)}{\nu'^2-\nu^2} \end{aligned} \tag{2.25}$$

with the subtraction constant

$$ReC^+(0,t) = \frac{2}{\pi}\int_{\nu_1}^{\infty}\frac{d\nu'}{\nu'} ImC^+(\nu',t) \tag{2.26}$$

and

$$\nu_1 = M_\pi + t/4M_N \tag{2.27}$$

the branch point in the s-channel.

To obtain values for the expansion parameters, they considered

$$\begin{aligned} ReC^+(\nu,0) &= ReC^+(0,0) - \frac{g^2}{M_N}\frac{\nu^2}{(\nu^2-\nu_B^2)} \\ &+ \frac{2\nu^2}{\pi}\int_{\nu_1}^{\infty}\frac{d\nu'}{\nu'}\frac{ImC^+(\nu',0)}{\nu'^2-\nu^2} \end{aligned} \tag{2.28}$$

with $ReC^+(0,0) = A^+(0,0) = a_1^+$. Rearranging the above we get

$$\begin{aligned} a_1^+ - \frac{g^2}{M_N} &= ReC^+(\nu,0) + \frac{g^2}{M_N}\frac{\nu_B^2}{(\nu^2-\nu_B^2)} \\ &- \frac{2\nu^2}{\pi}\int_{\nu_1}^{\infty}\frac{d\nu'}{\nu'}\frac{ImC^+(\nu',0)}{\nu'^2-\nu^2} \end{aligned} \tag{2.29}$$

An advantage of using forward scattering is that we can now use the optical theorem to relate ImC^+ to the total cross-section

$$ImC^+(\nu', 0) = k_L \sigma^+ \tag{2.30}$$

with σ^+ the total isospin-even $\pi^\pm p$ cross-section

$$\sigma^+ = \frac{1}{2}(\sigma_{\pi^+ p} + \sigma_{\pi^- p}) \tag{2.31}$$

and k_L the momentum of the incident meson in the laboratory system.

Now, consider the case when $\nu = \omega$, the total pion laboratory energy. We have

$$\omega^2 = M_\pi^2 + k_L^2 \tag{2.32}$$

so that

$$\omega d\omega = k_L dk_L \tag{2.33}$$

and hence

$$\begin{aligned} a_1^+ - \frac{g^2}{M_N} &= ReC^+(\omega, 0) + \frac{g^2}{M_N} \frac{\nu_B^2}{(\nu^2 - \nu_B^2)} \\ &- \frac{2\omega^2}{\pi} \int_{\nu_1}^{\infty} \frac{k_L'^2 dk_L'}{\omega'} \frac{\sigma^+(k_L')}{k_L'^2 - k_L^2} \end{aligned} \tag{2.34}$$

Hohler, Jacob and Strauss calculated $ReC^+(\omega, 0)$ from phase shift data and were able to calculate the integral from their table of forward πN amplitudes.

Using (2.16) we see that

$$\nu_B^2 = \frac{M_\pi^4}{4M_N^2} \tag{2.35}$$

and the Born term in equation(2.34) becomes

$$\frac{g^2}{M_N} \frac{\nu_B^2}{(\nu^2 - \nu_B^2)} = \frac{g^2}{4M_N^3} \frac{M_\pi^4}{(\nu^2 - M_\pi^4/4M_N^2)} \tag{2.36}$$

The parameter a_2^+ is found by considering [79]

$$S(\omega) = \frac{\partial}{\partial t} ReC^+(\omega, t)|_{t=0} \tag{2.37}$$

at $\omega = 0$. That is,

$$a_2^+ = S(0) \tag{2.38}$$

They were able to show that

$$\frac{\partial}{\partial t} ImC^+(\omega, t) = \frac{1}{2} b^+(\omega) k_L \sigma^+ \tag{2.39}$$

where b^+ is the slope of the diffraction peak [80]. Hohler, Jacob and Strauss then went on to find an expression which relates a_2^+ and b^+ to a quantity which can be found from phase shifts and total cross-sections in the region $M_\pi < \omega < \overline{\omega} \approx 2$ GeV. Both a_2^+ and b^+ are found from a straight line fit.

The best average result gives [15]

$$\begin{aligned} a_1^+ &= (-1.53 \pm 0.2) M_\pi^{-1} + \frac{g^2}{M_N} \\ a_2^+ &= (1.11 \pm 0.02) M_\pi^{-3} \end{aligned} \tag{2.40}$$

so that, from (2.24)

$$T^+(0, 2M_\pi^2, M_\pi^2, M_\pi^2) = (0.69 \pm 0.24) M_\pi^{-1} \tag{2.41}$$

and

$$\Sigma_{\pi N}(2M_\pi^2) = 43 \pm 15\,\mathrm{MeV} \tag{2.42}$$

compared to equations (2.6) and (2.20).

Nielsen and Oades [80] considered the expansion (2.22) including terms in ν^4 and t^2 (the following expansion is identical to equation(2.22). The only difference is in notation)

$$\begin{aligned}\tilde{C}^+ &= c_1^+ + c_2^+ t + (c_3^+ + c_4^+ t)\nu^2 \\ &\quad + (c_5^+ + c_6^+)\nu^4 + c_7^+ t^2 \end{aligned} \tag{2.43}$$

valid in the region $|t| < 4M_\pi^2$ and $|\nu| < M_\pi + t/4M_N$. They use the notation $\tilde{C}^+$ to denote the even-isospin transition amplitude excluding Born terms, i.e., $\tilde{C}^+ = C^+ - C^+_{Born}$.

In exactly the same way as above, they found that

$$\begin{aligned} c_1^+ - \frac{g^2}{M_N} &= (-1.45 \pm 0.10) M_\pi^{-1} \\ c_2^+ &= (1.18 \pm 0.05) M_\pi^{-3} \end{aligned}$$

and

$$c_7^+ = (0.035 \pm 0.007) M_\pi^{-5} \tag{2.44}$$

They found

$$\begin{aligned} f_\pi^{-2}\Sigma_{\pi N}(2M_\pi^2) &= c_1^+ - \frac{g^2}{M_N} + 2M_\pi^2 c_2^+ + 4M_\pi^4 c_7^+ \\ &= (1.05 \pm 0.14) M_\pi^{-1} \end{aligned} \tag{2.45}$$

giving

$$\Sigma_{\pi N}(2M_\pi^2) = 65 \pm 9 \text{ MeV}. \tag{2.46}$$

In the above examples, the sigma term was obtained by extrapolating the appropriate isospin-even amplitude to the Cheng-Dashen point via a fixed-t dispersion relation. The path of integration being along the (straight line) s-channel cut.

The final experimental estimate we consider, due to Koch [81], extrapolates along a series of hyperbola.

Koch considered the following amplitude

$$D^+(\nu^2,t) = A^+(\nu^2,t) + \nu^2 B^+(\nu^2,t) \tag{2.47}$$

The once subtracted fixed-t dispersion relation for D^+ is

$$\begin{aligned} ReD^+(\nu^2,t) &= A^+(0,t) - \frac{g^2}{M_N}\frac{\nu^2}{\nu^2-\nu_B^2} \\ &+ \frac{\nu^2}{\pi}\int_{\nu_1^2}^{\infty}\frac{d\nu'^2}{\nu}\frac{ImD^+(\nu'^2,t)}{\nu'^2-\nu^2} \end{aligned} \tag{2.48}$$

from which

$$\begin{aligned} A^+(0,t) - \frac{g^2}{M_N} &= ReD^+(\nu^2,t) + \frac{g^2}{M_N}\frac{\nu_B^2}{\nu^2-\nu_B^2} \\ &- \frac{\nu^2}{\pi}\int_{\nu_1^2}^{\infty}\frac{d\nu'^2}{\nu'}\frac{ImD^+(\nu'^2,t)}{\nu'^2-\nu^2} \end{aligned} \tag{2.49}$$

agreeing with equation(2.29) (which is hardly surprising as the amplitudes T^+ (or D^+) and C^+ are identical at the point $\nu = 0$).

By deriving a fixed-ν dispersion relation for $A^+(0,t)$, subtracted at $t = t_1$ (equation (D.82)), and equating it with the above, an expression can be obtained for $A^+(0,t) - g^2/M_N$. It is usually from such an expression that the sigma term is calculated. Instead we consider the following set of hyperbola

$$(\nu^2 - \nu_0^2)(t - t_0) = (t_0 - t_1)\nu_0^2 \tag{2.50}$$

passing through the point $\nu = 0$ and $t = t_1 = 2M_\pi^2$. Each hyperbola will give a value for the sigma term, and the variation is given as the error.

The parameters ν_0^2 and t are chosen so that the curves remain near the physical region of the s-channel and, for $t \geq 4M_\pi^2$, the t-channel partial wave expansion converges.

The t integrations are cut-off at some t_{min} and t_{max} which are close to the branch point. The remainder of the integral, that part that is far away from the branch point, is represented by a disrepancy function [82] which is expected to vary slowly but is unknown.

Koch, using hyperbolic dispersion relations, obtained

$$\begin{aligned}\Sigma_{\pi N}(2M_\pi^2) &= \frac{f_\pi^2}{2}[ReD^+(\nu^2,a) - \frac{\nu^2}{\pi}P\int_{\nu_1^2}^{\infty}\frac{d\nu'^2}{\nu'^2}\frac{ImD^+(\nu'^2,a)}{(\nu'^2-\nu^2)} \\ &\quad -\frac{t-2M_\pi^2}{\pi}\int_{4M_\pi^2}^{t_{max}} dt'\frac{ImD^+(t',a)}{(t'-2M_\pi^2)(t'-t)}] \\ &\quad -(t-2M_\pi^2)\Delta(t,a) \qquad (2.51)\end{aligned}$$

The last term is the disrepancy (consisting of complex poles and distant cuts) and $a^2/2 = \nu_0^2(t_0 - t_1)$. Koch found $\Delta(t,a)$ by solving the above for fixed $\Sigma_{\pi N}(2M_\pi^2)$. He found that $\Delta(t,a)$ varied the least at around 70 MeV.

His analysis gives

$$\Sigma_{\pi N}(2M_\pi^2) = 64 \pm 8 \text{ MeV} \qquad (2.52)$$

in agreement with Nielsen and Oades. Until recently, this was considered to be the most accurate value of the pion-nucleon sigma term. For what is to follow, it suffices to use this value. We will return to this at the end of the chapter.

We see from the above that the sigma term at $t = 2M_\pi^2$ does not agree with the value found at $t = 0$ (equation(2.6)). In the following section we review early attempts to explain this variation. What we believe to be the correct explanation is given at the end of the chapter.

2.2 The Sigma Term in QCD

The formal definition of the pion-nucleon sigma term is

$$\sigma_{\pi N}(t) = \frac{1}{3}\sum_{i=1}^{3}\langle N(p')|[Q_i^5,[Q_i^5,H(0)]]|N(p)\rangle \tag{2.53}$$

($t = -(p-p')^2$) with $H(0)$ the strong interaction Hamiltonian

$$\begin{aligned} H &= H_0 + \epsilon H_{SB} \\ \epsilon H_{SB} &= c_0 u_0 + c_8 u_8 \end{aligned} \tag{2.54}$$

Writing

$$Q_i^5 = \int d^3x\, q_\alpha^\dagger(x)(\gamma_5 \frac{\tau_i}{2})_{\alpha\beta} q_\beta(x) \tag{2.55}$$

and [83]

$$\begin{aligned} H_{SB} &= \int d^3y\, q_\mu^\dagger(y)(\gamma^0 m_q)_{\mu\nu} q_\nu(y) \\ &= \int d^3y\, [m_u \overline{u}u + m_d \overline{d}d + m_s \overline{s}s] \end{aligned} \tag{2.56}$$

it can be shown, with the help of equation(1.19), that

$$[Q_i^5,[Q_i^5,H_{SB}]] = -\int d^3x d^3y\, q^\dagger(x)\frac{m_q}{2}[\gamma_5,\gamma^0\gamma_5]\{\frac{\lambda_i}{2},\lambda_i\}q(y)\delta^3(x-y) \tag{2.57}$$

Using equation (1.23) with $i = 1, 2, 3$, we find that

$$[Q_i^5,[Q_i^5,H_{SB}]] = 3\int d^3x\, [m_u \overline{u}u + m_d \overline{d}d] \tag{2.58}$$

and hence,

$$\sigma_{\pi N}(0) = \int d^3x\, (\hat{m}\langle N|\overline{u}u + \overline{d}d|N\rangle + \frac{(m_u - m_d)}{2}\langle N|\overline{u}u - \overline{d}d|N\rangle) \tag{2.59}$$

having defined

$$\hat{m} = \frac{m_u + m_d}{2} \tag{2.60}$$

to be the average of the renormalized running current quark mass.

The term proportional to $m_u - m_d$, the isospin violating piece, is small and can be dropped. We are then left with

$$\sigma_{\pi N} = \int d^3x \, \hat{m} \langle N | \overline{u}u + \overline{d}d | N \rangle \tag{2.61}$$

We should point out that, although $\hat{m}$ and $\overline{q}q$ vary with the mass scale, the product $\hat{m}\overline{q}q$ is scale independent.

In the quark model,

$$u_i = \overline{q}\lambda_i q \tag{2.62}$$

$i = 0, \ldots, 8$. For $i = 1, \ldots, 8$, the λ_i are the usual Gell-Mann matrices. When $i = 0$,

$$\lambda_0 = \sqrt{\frac{2}{3}} I \tag{2.63}$$

Hence [18],

$$u_0 = \sqrt{\frac{2}{3}} (\overline{u}u + \overline{d}d + \overline{s}s) \tag{2.64}$$

and

$$u_8 = \frac{1}{\sqrt{3}} (\overline{u}u + \overline{d}d - 2\overline{s}s) \tag{2.65}$$

The coefficients are

$$\begin{aligned} c_0 &= \frac{1}{\sqrt{6}} (m_u + m_d + m_s) \\ c_8 &= \frac{1}{\sqrt{3}} \left(\frac{m_u + m_d}{2} - m_s\right) \end{aligned} \tag{2.66}$$

so that

$$c_0 u_0 + c_8 u_8 = \left(\frac{m_u + m_d}{2}\right)(\overline{u}u + \overline{d}d) + m_s \overline{s}s \tag{2.67}$$

which is the expression given in equation(1.17).

Now, the baryon, according to QCD [84], is a bound three quark and gluon system surrounded by Goldstone bosons and quark-antiquark pairs. In the case of the nucleon, the bound valence quarks are predominantly the u and d quarks. The $\overline{q}q$ pairs which make up the quark sea are $\overline{u}u, \overline{d}d$ and $\overline{s}s$. We now make the usual assumption (which is confirmed by experiment) that the nucleon wave function is dominated by u and d quarks. This means that only a small percentage of the quark sea consists of $\overline{s}s$ pairs.

Hence, we assume that $\langle N|\overline{s}s|N\rangle \approx 0$ (or $\langle N|\overline{u}u + \overline{d}d|N\rangle >> \langle N|\overline{s}s|N\rangle$), and the sigma term becomes

$$\begin{aligned}
\sigma_{\pi N}(0) &= (\frac{m_u + m_d}{2})\langle N|\overline{u}u + \overline{d}d|N\rangle \\
&\approx (\frac{m_u + m_d}{2})\langle N|\overline{u}u + \overline{d}d - 2\overline{s}s|N\rangle \\
&= \frac{3(m_u + m_d)}{m_u + m_d - 2m_s}\langle N|c_8 u_8|N\rangle
\end{aligned} \tag{2.68}$$

thus relating the sigma term to the $SU(3)$ breaking piece of the Hamiltonian.

We can calculate the matrix element by first considering the decomposition [2]

$$8 \times 8 = 27 + 10^* + 10 + 8_1 + 8_2 + 1 \tag{2.69}$$

We see in this decomposition that there are two 8's. This means that in $8 \times 8 \times 8$ there will be two SU(3) scalars.

To see this, consider the meson octet M transforming as $M \rightarrow UMU^\dagger$ and the baryon octet transforming as $B \rightarrow UBU^\dagger$. We can therefore construct the following two invariants, $Tr(\overline{B}MB)$ and $Tr(\overline{B}BM)$ for the baryon-meson couplings. It is more common to consider linear combinations of these couplings, and for this reason we write [8]

$$\langle B|c_8 u_8|B\rangle = \alpha Tr(\overline{B}u_8 B) + \beta Tr(\overline{B}Bu_8) \tag{2.70}$$

with u_8 written as a 3×3 diagonal matrix and [85]

$$B = \begin{bmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda^0}{\sqrt{6}} \end{bmatrix}$$

Expanding equation(2.70) and noting that $\langle B|c_8u_8|B\rangle$ is the baryon octet mass shift due to SU(3) breaking, we find

$$\begin{aligned} M_N &= M_0 + \alpha - 2\beta \\ M_\Sigma &= M_0 + \alpha + \beta \\ M_\Xi &= M_0 - 2\alpha + \beta \\ M_\Lambda &= M_0 - \alpha - \beta \end{aligned} \tag{2.71}$$

having absorbed all common constants into α and β. From these equations we find

$$\langle N|c_8u_8|N\rangle = \alpha - 2\beta = M_\Lambda - M_\Xi \tag{2.72}$$

As a check of equation(2.71) we see that

$$M_\Lambda - M_\Xi = \frac{2M_N - M_\Sigma - M_\Lambda}{2} \tag{2.73}$$

which is satisfied experimentally. Note that these relations are derived from a Hamiltonian with matrix elements linear in $\hat{m}$ and m_s. This point will be raised later.

Thus

$$\begin{aligned} \sigma_{\pi N}(0) &= \frac{3(m_u + m_d)}{m_u + m_d - 2m_s}(M_\Lambda - M_\Xi) \\ &= 3\frac{(M_\Xi - M_\Lambda)}{m_s/\hat{m} - 1} \end{aligned} \tag{2.74}$$

which measures the amount by which the nucleon mass shifts when m_u and m_d are given non-zero masses (assuming that this is the only cause of symmetry breaking).

The quark masses we use are those found by Leutwyler [19,86,87] at a scale of 1 GeV. Relating matrix elements of vector currents to the pseudoscalar density via the $SU(6)$ symmetry, he found

$$\begin{aligned} m_u + m_d &\approx 11\,\mathrm{MeV} \\ m_s &\approx 130\,\mathrm{MeV} \end{aligned} \tag{2.75}$$

so that

$$\frac{m_s}{\hat{m}} \approx 24 \tag{2.76}$$

as required. Taking [12]

$$\begin{aligned} M_\Lambda &= 1115.6\,\mathrm{MeV} \\ M_\Xi &= 1314.9\,\mathrm{MeV} \end{aligned} \tag{2.77}$$

we have

$$\sigma_{\pi N}^{QCD}(0) \approx 26.5\,\mathrm{MeV} \tag{2.78}$$

which is considerably smaller than the Koch value, but is in agreement with the original estimate of equation(2.6). We therefore interpret equations(2.6) and (2.78) to be the valence quark contribution to the sigma term.

In deriving equation(2.68), we assumed that $\langle N|\overline{s}s|N\rangle \approx 0$ because we do not know this matrix element (some authors invoke the Zweig rule to justify this assumption [70,88]). However, an alternative method of determining the sigma term exists in which we consider the parameter

$$y = \frac{\langle N|\overline{s}s|N\rangle}{\langle N|\overline{u}u + \overline{d}d - 2\overline{s}s|N\rangle} \tag{2.79}$$

which is used as a measure of the strange quark content of the nucleon. Instead of equation(2.68) we write

$$\begin{aligned}\sigma_{\pi N}(0) &= \frac{m_u+m_d}{2}\langle N|\overline{u}u+\overline{d}d-2\overline{s}s|N\rangle\big(1+\frac{2\langle N|\overline{s}s|N\rangle}{\langle N|\overline{u}u+\overline{d}d-2\overline{s}s|N\rangle}\big) \\ &= \frac{3}{m_s/\hat{m}-1}(M_\Xi-M_\Lambda)\big(1+\frac{2\langle N|\overline{s}s|N\rangle}{\langle N|\overline{u}u+\overline{d}d-2\overline{s}s|N\rangle}\big) \qquad (2.80)\end{aligned}$$

using equations (2.71) and (2.72). Dominguez and Langacker [89] have calculated y using the method of Li and Pagels [42,90]. They found that $y \approx 0.36$ which implies that $\sigma_{\pi N}^{QCD}(0) \approx 36$ MeV, compared to equation(2.78) where $y = 0$.

Note that equation(2.80) can also be written

$$\sigma_{\pi N}(0) = \frac{\hat{\sigma}_{\pi N}(0)}{1-y_0} \qquad (2.81)$$

where $\hat{\sigma}_{\pi N}(0) \approx 26.5$ MeV is given by equation(2.74) and

$$y_0 = 2\frac{\langle N|\overline{s}s|N\rangle}{\langle N|\overline{u}u+\overline{d}d|N\rangle} \qquad (2.82)$$

is also interpreted as a measure of the strange quark component of the nucleon.

The parameter y_0 can be related to the baryon bare mass (in the chiral limit) by writing [91]

$$\begin{aligned}2M_N\sigma_{\pi N}(0) &= \hat{m}\langle p|\overline{u}u+\overline{d}d|p\rangle \qquad (2.83) \\ &= \frac{2\hat{m}}{m_s+2\hat{m}}\{M_N^2+\frac{3}{2}\frac{m_s}{m_s-\hat{m}}(M_\Xi^2-M_\Lambda^2)-M_0^2\} \qquad (2.84)\end{aligned}$$

This expression is derived by using the first order mass formula

$$\begin{aligned}M_N^2 &= M_0^2+\hat{m}(B^u+B^d)+m_sB^s \\ M_\Xi^2 &= M_0^2+\hat{m}(B^d+B^s)+m_sB^u \\ M_\Lambda^2 &= M_0^2+\frac{\hat{m}}{3}(B^u+4B^d+B^s)+\frac{m_s}{3}(2B^u-B^d+2B^s) \qquad (2.85)\end{aligned}$$

with $B^q = \langle p|\bar{q}q|p\rangle$.

We use the normalization

$$\langle p'|p\rangle = (2\pi)^3 2p^0 \delta^3(p'-p) \tag{2.86}$$

Proof of (2.83) = (2.84):

The simplest way to prove this is to work backwards and insert equation (2.85) into (2.84), and show equations (2.83) and (2.84) are equivalent.

$$\begin{aligned} M_\Xi^2 - M_\Lambda^2 &= \hat{m}(B^d + B^s) + m_s B^u - \frac{\hat{m}}{3}(B^u + 4B^d + B^s) \\ &\quad -\frac{m_s}{3}(2B^u - B^d + 2B^s) \\ &= \frac{m_s - \hat{m}}{3}(B^u + B^d - 2B^s) \end{aligned} \tag{2.87}$$

so

$$\frac{3}{2}\frac{m_s}{m_s - \hat{m}}(M_\Xi^2 - M_\Lambda^2) = \frac{m_s}{2}(B^u + B^d - 2B^s) \tag{2.88}$$

hence

$$\begin{aligned} & M_N^2 + \frac{3}{2}\frac{m_s}{m_s - \hat{m}}(M_\Xi^2 - M_\Lambda^2) - M_0^2 \\ &= \hat{m}(B^u + B^d) + m_s B^s + \frac{m_s}{2}(B^u + B^d - 2B^s) \\ &= \big(\hat{m} + \frac{m_s}{2}\big)(B^u + B^d) \\ &= \big(\frac{2\hat{m} + m_s}{2}\big)(B^u + B^d) \end{aligned} \tag{2.89}$$

Thus, upon multiplication by $2\hat{m}/(m_s + 2\hat{m})$, we get the desired result.

End of proof.

We should point out that a linear mass formula could have been used in equation(2.85). However, the formula for M_B^2 allows one to calculate the order of magnitude of corrections due to higher terms in the quark mass expansion.

What are these higher terms? One might expect the next term in the expansion for M_B^2 to be of order m_q^2. However, Langacker and Pagels [44] have shown that the next term is actually of order $m_q^{3/2}$. This is the leading nonanalytic term in the quark mass expansion.[2] They are usually due to a finite number of diagrams, and may therefore be calculated exactly - without knowledge of H_0 (which is not true of the analytic terms). The nonanalytic leading terms are believed to dominate higher order terms.

To one loop in chiral perturbation theory, Gasser calculated the Leading NonAnalytic Contributions (LNAC) and higher order (kinematic) corrections to equation(2.81) at $y_0 = 0$, i.e., he included all terms of order $m_q^{3/2}$ and estimated the contribution from terms of order m_q^2. The details can be found in reference [91]. Gasser found that these contributions increase the sigma term by approximately 10 MeV (taking $m_s/\hat{m} \approx 25$ and $m_d/m_u \approx 1.8$). Hence, to one loop

$$\sigma_{\pi N} = \frac{\hat{\sigma}_{\pi N}}{1 - y_0} \tag{2.90}$$

where, now, $\hat{\sigma}_{\pi N} \approx 35$ MeV.

We are now in a position to estimate y_0 (or, through equation(2.84), M_0). If we are to obtain the value $\sigma_{\pi N} \approx 60$ MeV, then $y_0 \approx 0.42$ or $M_0 \approx 300$ MeV (we have used $m_s \approx 140$ MeV). That is, it would appear that there is a large strange quark component of the nucleon, and it contributes more than 50% to the nucleon mass!

The LNAC to equation (2.84) was calculated by Gasser. He found that LNAC ≈ -35 MeV, i.e., large and in the wrong direction. This problem is quite common when calculating LNAC for baryons in an analogous way to those found for mesons (we have not previously mentioned LNAC for

[2]The momentum integral is infrared divergent [19]

mesons. Calculations may be found in references [45] and [91]).

The solution to the problem is to work with models of hadrons described by effective Lagrangians [92]. In such models, LNAC are due to the cloud of virtual mesons surrounding the baryon (this method can also be used to calculate LNAC for mesons. In this case, LNAC are due to the virtual meson cloud surrounding the meson). The LNAC are one loop integrals, essential the self energy of the baryon, but with the physical baryon and meson masses in the propagators.

If we write

$$\begin{aligned} B^u &= \overline{B}^u + \delta B^u \\ B^d &= \overline{B}^d + \delta B^d \end{aligned} \tag{2.91}$$

then

$$\begin{aligned} \delta M_N^2 &= \hat{m}(\delta B^u + \delta B^d) \\ \delta M_\Xi^2 &= \hat{m}\delta B^d + m_s \delta B^u \\ \delta M_\Lambda^2 &= \frac{\hat{m}}{3}(\delta B^u + 4\delta B^d) + \frac{m_s}{3}(2\delta B^u - \delta B^d) \end{aligned} \tag{2.92}$$

so that[3]

$$\begin{aligned} &M_N^2 = M_0^2 + \hat{m}(\overline{B}^u + \overline{B}^d) + m_s B^s + \delta M_N^2 \\ &M_\Xi^2 = M_0^2 + \hat{m}(\overline{B}^d + B^s) + m_s \overline{B}^u + \delta M_\Xi^2 \\ &M_\Lambda^2 = M_0^2 + \frac{\hat{m}}{3}(\overline{B}^u + 4\overline{B}^d + B^s) + \frac{m_s}{3}(2\overline{B}^u - \overline{B}^d + 2B^s) + \delta M_\Lambda^2 \end{aligned} \tag{2.93}$$

where δM_N^2 is a one loop integral given by equation(7.2) of reference [91]. By subtracting the self energy term from these equations one obtains expansions for M_B^2 in the quark mass analytic up to order $m_q^2 ln m_q$.

[3] $\overline{B}^u$, $\overline{B}^d$ and B^s obey $SU(3)$ relations in the absence of loop corrections.

We are now in a position to find LNAC to the sigma term. Using the expansion for M_N^2 given by equation(2.85), we can write the sigma term as

$$\begin{aligned}\sigma_{\pi N} &= \frac{\hat{m}}{2M_N}\langle p|\overline{u}u+\overline{d}d|p\rangle \\ &= \frac{\hat{m}}{2M_N}\frac{\partial M_N^2}{\partial \hat{m}}\end{aligned} \tag{2.94}$$

From equation(2.93)

$$\hat{m}\frac{\partial M_N^2}{\partial \hat{m}} = \hat{m}(\overline{B}^u+\overline{B}^d)+\hat{m}\frac{\partial}{\partial \hat{m}}\delta M_N^2 \tag{2.95}$$

so that, combining equations (2.84) and (2.92) - (2.95) we have

$$\begin{aligned}2M_N\sigma_{\pi N}(0) &= \frac{2\hat{m}}{2\hat{m}+m_s}[M_N^2+\frac{3}{2}\frac{m_s}{m_s-\hat{m}}(M_\Xi^2-M_\Lambda^2)-M_0^2] \\ &\quad -\frac{2\hat{m}}{2\hat{m}+m_s}[\delta M_N^2+\frac{3}{2}\frac{m_s}{m_s-\hat{m}}(\delta M_\Xi^2-\delta M_\Lambda^2)] \\ &\quad +\hat{m}\frac{\partial}{\partial \hat{m}}\delta M_N^2\end{aligned} \tag{2.96}$$

with the second line removing loop contributions to the first line, and contributions from the last two lines cancelling. That is, the second line removes order $m_q^{3/2}$ terms from the first line (making the first line analytic up to order $m_q^2 ln m_q$). The third line is of order $m_q^{3/2}$ and is therefore LNAC.

If we make the approximation $\hat{m} \propto M_\pi^2$, we can write the derivative in the last line in terms of meson masses[4]

$$\hat{m}\frac{\partial}{\partial \hat{m}}\delta M_N^2 = M_\pi^2[\frac{\partial}{\partial M_\pi^2}+\frac{1}{2}\frac{\partial}{\partial M_K^2}+\frac{1}{3}\frac{\partial}{\partial M_\eta^2}]\delta M_N^2 \tag{2.97}$$

so that finally, the pion-nucleon sigma term is

[4] The coefficients in the RHS of this expression are the probabilities of finding a non-strange quark in the meson

$$
\begin{aligned}
2M_N\sigma_{\pi N}(0) \;=\; & \frac{2\hat{m}}{2\hat{m}+m_s}[M_N^2+\frac{3}{2}\frac{m_s}{m_s-\hat{m}}(M_\Xi^2-M_\Lambda^2)-M_0^2] \\
& -\frac{2\hat{m}}{2\hat{m}+m_s}[\delta M_N^2+\frac{3}{2}\frac{m_s}{m_s-\hat{m}}(\delta M_\Xi^2-\delta M_\Lambda^2)] \\
& +M_\pi^2[\frac{\partial}{\partial M_\pi^2}+\frac{1}{2}\frac{\partial}{\partial M_\kappa^2}+\frac{1}{3}\frac{\partial}{\partial M_\eta^2}]\delta M_N^2 \qquad (2.98)
\end{aligned}
$$

The cutoff in the integral for δM_B^2 is determined by the dipole parameterization of the πNN vertex form factor [93,94,95]. One finds from experiment that the cutoff mass $\Lambda_\pi \approx 1$ GeV (it can be shown, in the case of a dipole form factor, that $\Lambda_\pi \approx \sqrt{20}/R$. We see that $\Lambda_\pi = 1$ GeV gives us a bag radius of approximately 0.9 fm). Gasser actually calculates the above expression for various cutoffs, and finds

1. $\Lambda_\pi = 0.5$ GeV

$$\sigma_{\pi N}(0) \approx 42\,\text{MeV} \qquad (2.99)$$

2. $\Lambda_\pi = 0.7$ GeV

$$\sigma_{\pi N}(0) \approx 44\,\text{MeV} \qquad (2.100)$$

3. $\Lambda_\pi = 1.0$ GeV

$$\sigma_{\pi N}(0) \approx 54\,\text{MeV} \qquad (2.101)$$

From the above, we find that in order to agree with the experimental value of the sigma term either y_0 is large[5] or, equivalently, M_0 is small.[6] The conclusion is that there is a high percentage of strange quarks in the nucleon, and that half the nucleon mass ($M_N = 938.28$ MeV) is due to the strange quark [96].

However, we feel that the above values are too large as we do not expact any strange quarks in the nucleon wavefunction.

[5] We see this by equating the values given by equations (2.99) to (2.101) with $\hat{\sigma}_{\pi N}(0)$ given in equation (2.90). We then find that $\sigma_{\pi N}(0) \approx 60$ MeV requires y_0 to be large.

[6] As mentioned on Page 45, using the first-order expression, equation (2.84), to obtain $\sigma_{\pi N}(0) \approx 60$ MeV requires $M_0 \approx 300$ MeV. This value is increased when we use equation (2.98) to calculate M_0. In the above calculation, Gasser uses $M_0 = 750$ MeV, which is still considerably smaller than the baryon's experimental mass.

We have also made a calculation of the sigma term using equation (2.98). Using expressions for the self-energy derived in chapter 3, we found the solutions to equation (2.98) to be unstable. We conclude that numerical results from (2.98) become unreliable when the chiral loop corrections are large.

In contrast to the above, we now consider a bag calculation of the sigma term.

2.2.1 Hybrid Bag Calculation

Jaffe considered a Hybrid Chiral Bag (HCB) model [97,98,99]. In this model, the interior of the bag and the exterior are separate. Inside are quarks and gluons. Outside are the Goldstone bosons which couple to the quarks at the bag surface in a way that ensures conservation of the axial-vector current. It is also assumed that the $\bar{s}s$ component of the quark sea is small.

He found that

$$\frac{\sigma_{\pi N}(0)}{2(M_{\Xi} - M_{\Lambda})} = \frac{1+\sqrt{2}\lambda}{2(m_s/\hat{m} - 1)} \tag{2.102}$$

which also gave $\sigma_{\pi N} \approx 26$ MeV.

We note that this expression can be derived from equation (2.80).

Proof:

Write the matrix elements of equation(2.80) in the form

$$\begin{aligned}
&3\big(1 + \frac{2\langle N|\bar{s}s|N\rangle}{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle}\big)\frac{\langle N|\bar{u}u + \bar{d}d|N\rangle}{\langle N|\bar{u}u + \bar{d}d|N\rangle} \\
&= 3\big(\frac{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle + 2\langle N|\bar{s}s|N\rangle}{\langle N|\bar{u}u + \bar{d}d|N\rangle}\big)\frac{\langle N|\bar{u}u + \bar{d}d|N\rangle}{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle} \\
&= 3\frac{\langle N|\bar{u}u + \bar{d}d|N\rangle}{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle} \\
&= \frac{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle + 2\langle N|\bar{u}u + \bar{d}d + \bar{s}s|N\rangle}{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\langle N|\overline{u}u+\overline{d}d-2\overline{s}s|N\rangle + 2\langle N|\overline{u}u+\overline{d}d+\overline{s}s|N\rangle}{\langle N|\overline{u}u+\overline{d}d-2\overline{s}s|N\rangle} \\
&= \frac{\langle N|(1/\sqrt{3})\,(\overline{u}u+\overline{d}d-2\overline{s}s)|N\rangle + \sqrt{2}\,\langle N|\sqrt{2/3}\,(\overline{u}u+\overline{d}d+\overline{s}s)|N\rangle}{\langle N|(1/\sqrt{3})\,(\overline{u}u+\overline{d}d-2\overline{s}s)|N\rangle} \\
&= \frac{\langle N|u_8|N\rangle + \sqrt{2}\langle N|u_0|N\rangle}{\langle N|u_8|N\rangle} \\
&= 1+\sqrt{2}\lambda
\end{aligned} \tag{2.103}
$$

with

$$
\lambda = \frac{\langle N|u_0|N\rangle}{\langle N|u_8|N\rangle} \tag{2.104}
$$

End of proof.

When considering a model of symmetry breaking, we require that the Gell-Mann-Okubo (GMO) mass formula remain valid. This relation is derived using matrix elements which are linear in the quark masses.

It is possible that the Hamiltonian has a term which is non-linear in m_s. The GMO relation is violated by an operator transforming as the $\underline{27}$-plet of flavour $SU(3)$. However, the effective interaction beween two valence quarks in a flavour antisymmetric state receives no contribution from this operator. Thus, if this part of the effective interaction dominates over the non-linear interactions, then the GMO relation will be satisfied. This is possible in Chiral Bag models.

Let $H_{SB}^{HCB} = H_{SB}^{quark} + H_{SB}^{meson}$, with H_{SB}^{quark} linear in $\hat{m}$, and H_{SB}^{meson} highly nonlinear in m_s.

Jaffe found the pion contribution to the matrix elements in H_{SB}^{π} to be 27 MeV for $\hat{m} = 15$ MeV and $m_s = 325$ MeV. He was also able to show that the $\underline{27}$ component of H_{SB}^{meson} was negligible, and so the GMO relation would still be valid.

Including contributions to $\sigma_{\pi N}$ from the pion cloud, the sigma term becomes

$$\sigma_{\pi N}(0) \approx 54 \pm 5 \text{ MeV} \tag{2.105}$$

an increase of 27 MeV. That is, pion contributions increase the "effective" sigma term (i.e., the sigma term without pion contributions) by 27 MeV, and hence, a large strange quark content is unnecessary. The small number of strange quark pairs is due to the non-linearity of the HCB model.

An important feature of this calculation is that the pion field is not allowed inside the bag [53]. As such, the pion field will be discontinuous at the bag surface leading to a nonvanishing surface term (in the integral of the axial current). We feel that this may have the effect of increasing the pion contribution. The calculation also assumes that the nucleon and delta are degenerate. This has the effect of overestimating the Δ contribution. We therefore expect the sigma term to be considerably smaller than the value given above.

2.2.2 Latest Experimental Results

What is the value of the sigma term? Experimentally it is given the value $\Sigma_{\pi N}(2M_\pi^2) = 64 \pm 8$ MeV. By calculating the parameter y (or y_0), we saw that, theoretically, the sigma term ranged from 26 MeV to 54 MeV. We now give the latest possible solution to this discrepency [100].

By considering current algebra constraints on on-mass shell amplitudes, Brown, Pardee and Peccei [66] found that

$$f_\pi^2 A^+(0, 2M_\pi^2) = \sigma_{\pi N}(2M_\pi^2) + \text{corrections} \tag{2.106}$$

with corrections expected to be small and of order M_π^4 (i.e., there is no order M_π^3 term in this expression. The leading nonanalytic term is of order

$M_\pi^4 ln M_\pi^2$). Thus, we can write (see equation(2.24))

$$\Sigma_{\pi N}(2M_\pi^2) = \sigma_{\pi N}(2M_\pi^2) + \Delta_R \tag{2.107}$$

with $\Delta_R = 0.35$ MeV to one-loop in chiral perturbation theory [101].

To compare our theoretical results with experiment, we need to know

$$\Delta_\sigma = \sigma_{\pi N}(2M_\pi^2) - \sigma_{\pi N}(0) \tag{2.108}$$

which can be found from dispersion relations.

Write (we are using the notation of reference [100])

$$\Sigma_{\pi N}(2M_\pi^2) = \Sigma_d + \Delta_D \tag{2.109}$$

with (c.f. equation(2.24)) the latest experimental estimate being

$$\Sigma_d = f_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+) = 48 \pm 12 \text{ MeV} \tag{2.110}$$

from experiment. The term Δ_D is determined from the curvature of the isospin-even amplitude (minus Born terms) $\overline{D}^+(0,t)$ [102]. Hence

$$\begin{aligned}\sigma_{\pi N}(0) &= \Sigma_d + \Delta_D - \Delta_\sigma - \Delta_R \\ &= \Sigma_d + \Delta - \Delta_R \end{aligned}\tag{2.111}$$

From dispersion relations, ($\sigma_{\pi N}(0)$ and Σ_d may be thought of as subtractions, with Δ_σ and Δ_D given as an integral) $\Delta_D = 12$ MeV and $\Delta_\sigma = 15$ MeV [104]. Hence, neglecting the contribution from Δ_R, it is found that $\Delta = -3$ MeV.

Thus, with the most recent experimental estimate of $\Sigma_{\pi N}(2M_\pi^2)$, namely

$$\Sigma_{\pi N}(2M_\pi^2) = 60 \pm 12 \text{ MeV} \tag{2.112}$$

the experimental estimate of $\sigma_{\pi N}(0)$ is

$$\sigma_{\pi N}(0) = 45 \pm 12 \text{ MeV} \tag{2.113}$$

These errors are expected to improve with improved experimental results.

Half the difference between the above value of $\sigma_{\pi N}(0)$ and the valence quark value of 26 MeV is believed due to to $SU(3)$ breaking effects in the matrix elements of the scalar currents (as mentioned above, these effects increase $\sigma_{\pi N}(0)$ from 26 MeV to 35 MeV). The remainder is due to the strange quark operator. Using equation(2.90) we find that $y_0 \approx 0.2$. The reason for y_0 becoming smaller is that chiral perturbation theory underestimates Δ_σ by around 10 MeV (if we take $\Delta_\sigma \approx 5$ MeV, then the value obtained would agree with earlier results).

Previous calculations [96,103] found $\Sigma_d \approx 52$ MeV, $\Delta_D \approx 8$ MeV and $\Delta_\sigma \approx 3.5$ MeV, so that $\sigma_{\pi N}(0) \approx 56$ MeV and, hence, $\Sigma_{\pi N}(2M_\pi^2) \approx \sigma_{\pi N}(0)$ at one-loop. The reason these values have altered is that CPT assumes that $\overline{D}^+$, Δ_D and Δ_σ have little t-dependence.

Chiral perturbation theory allows us to expand amplitudes in terms of quark masses and pion momentum. The leading term of such an expansion is the Born term and, at the threshold of the physical region ($t = 4M_\pi^2$), is believed to provide a good approximation to the amplitude.

Now, Δ_D is calculated from the curvature of $\overline{D}^+$, with contributions from the region $t \geq 4M_\pi^2$ believed to be small. Δ_σ was found using CPT. It has recently been found that the curvature of $\overline{D}^+$ receives a large contribution from the region $4M_\pi^2 \leq t \leq (500 \text{ MeV})^2$ [104], and that Δ_σ also receives a large contribution from this region. It has also been found that CPT to one loop underestimates the t-dependence of $\sigma_{\pi N}(t)$. From this it is concluded that, to one-loop, CPT is not a good representation of the scattering amplitude (unless we are only interested in the amplitude at small values of the momentum transfer).

The figures given in equations (2.111) and (2.112) are the latest and most reliable values for the sigma term. The reason for the 15 MeV variation between $\Sigma_{\pi N}(2M_\pi^2)$ and $\sigma_{\pi N}(0)$ is the large t-dependence of $\sigma_{\pi N}(t)$.

Chapter 3

The SU(3) CBM

As we mentioned in chapter 1, the cloudy bag model (CBM) [105,106,107] is a low- to medium-energy model of hadrons. As we require our theory to be invariant under chiral transformations, our starting point will be a Lagrangian which satisfies this property, namely, the Lagrangian of the sigma model. We also require our model to describe particles observed in nature, so we must the remove the scalar particle from the sigma model Lagrangian, while retaining chiral invariance. We do this in the following section.

3.1 The CBM Lagrangian

Scalar fields were introduced by Schwinger [108], and Gell-Mann and Levy [57], in their attempt to produce a chiral symmetric theory of Strong Interactions. The Lagrangian density they proposed was

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - \frac{1}{2f_\pi}\overline{\psi}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)\psi + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 \tag{3.1}$$

plus a potential term. In the above, we define ψ to be the nucleon field (this is therefore an effective theory), σ the scalar field, and $\vec{\pi}$ the pseudoscalar pion triplet field. As usual, $\vec{\tau}$ are the Pauli matrices, and γ_5 was defined in chapter 1. In the above

$$\overleftrightarrow{\not\partial} = (\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\gamma^\mu \tag{3.2}$$

with μ taking the values 0, 1, 2 and 3, and the arrow giving the direction of the operation. Summation over repeated indices is implied.

We can remove the scalar field by making the transformation ($i = 1, 2, 3$)

$$\begin{aligned} \pi_i &= f_\pi \hat{\phi}_i \sin F(x) \\ \sigma &= f_\pi \cos F(x) \end{aligned} \tag{3.3}$$

where $\hat{\phi}_i = \phi_i/\phi$, $x = \phi/f_\pi$ with $\phi = (\vec{\phi}\cdot\vec{\phi})^{\frac{1}{2}}$, and

$$F(x) = x + a_3 x^3 + a_5 x^5 + \ldots. \tag{3.4}$$

Using the identity

$$\cos\theta + i\vec{\tau}\cdot\hat{n}\sin\theta = e^{i\vec{\tau}\cdot\hat{n}\theta} \tag{3.5}$$

we immediately see that

$$\frac{1}{2f_\pi}\overline{\psi}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)\psi = \frac{1}{2}\overline{\psi}e^{\frac{i\vec{\tau}\cdot\vec{\phi}\gamma_5}{f_\pi}}\psi \tag{3.6}$$

In order to redefine the kinetic energy piece, we consider the chiral transformation

$$\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5 \rightarrow e^{\frac{-i\vec{\tau}\cdot\vec{\alpha}\gamma_5}{2}}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)e^{\frac{i\vec{\tau}\cdot\vec{\alpha}\gamma_5}{2}} \tag{3.7}$$

with $\vec{\alpha}$ infinitesimal. The nucleon field is then required to transform as

$$\psi \rightarrow e^{-\frac{i\vec{\tau}\cdot\vec{\alpha}\gamma_5}{2}}\psi \tag{3.8}$$

and the $\vec{\pi}$ and σ fields can be shown to transform like

$$\begin{aligned} \vec{\pi} &\rightarrow \vec{\pi} + \sigma\vec{\alpha} \\ \sigma &\rightarrow \sigma - \vec{\alpha}\cdot\vec{\pi} \end{aligned} \tag{3.9}$$

or

$$\begin{aligned}\delta\pi_i &= \sigma\alpha_i = f_\pi \cos F(x)\alpha_i \\ \delta\sigma &= -\pi_i\alpha_i = -f_\pi \sin F(x)\hat{\phi}_i\alpha_i\end{aligned} \tag{3.10}$$

These equations are found by letting the RHS of equation(3.7) equal

$$\sigma' + i\vec{\tau}\cdot\vec{\pi}'\gamma_5 = (\sigma + \delta\sigma) + i\vec{\tau}\cdot(\vec{\pi} + \delta\vec{\pi})\gamma_5 \tag{3.11}$$

and equating like terms. From (3.3) we also have

$$\delta\sigma = -f_\pi F'(x)\sin F(x)\delta x \tag{3.12}$$

where

$$\begin{aligned}\delta x = \delta(\frac{\phi}{f_\pi}) &= \frac{\phi_i\delta\phi_i}{f_\pi\phi} \\ &= \frac{\hat{\phi}_i\delta\phi_i}{f_\pi}\end{aligned} \tag{3.13}$$

so

$$\delta\sigma = F'(x)\sin F(x)\hat{\phi}_i\delta\phi_i \tag{3.14}$$

$(F'(x) = dF(x)/dx)$. Equating (3.10) with (3.14), we find the transformation property

$$\phi_i\delta\phi_i = \frac{f_\pi}{F'(x)}\phi_i\alpha_i \tag{3.15}$$

We now write the kinetic energy term of equation (3.1) in terms of the new pion field $\vec{\phi}$. Write $\partial_\mu\sigma$ and $\partial_\mu\pi_j$ using the above shorthand notation

$$\begin{aligned}\partial_\mu\sigma &= \frac{\delta\sigma}{\delta\phi_j}\partial_\mu\phi_j \\ &= -F'(x)\sin F(x)\hat{\phi}_j\partial_\mu\phi_j \\ \partial_\mu\pi_i &= \frac{\delta\pi_i}{\delta\phi_j}\partial_\mu\phi_j \\ &= (\partial_\mu\phi_i - \hat{\phi}_i\hat{\phi}_j\partial_\mu\phi_j)\frac{sinF(x)}{x} \\ &\quad + F'(x)\cos F(x)\hat{\phi}_i\hat{\phi}_j\partial_\mu\phi_j\end{aligned} \tag{3.16}$$

Then

$$(\partial_\mu \sigma)^2 = F'(x)^2 \sin^2 F(x)(\hat{\phi}_j \partial_\mu \phi_j)^2 \tag{3.17}$$

and

$$\begin{aligned}(\partial_\mu \pi_i)^2 = [(\partial_\mu \phi_i)^2 \quad &- \quad (\hat{\phi}_j \partial_\mu \phi_j)^2]\frac{\sin^2 F(x)}{x^2} \\ &+ F'(x)^2 \cos^2 F(x)(\hat{\phi}_j \partial_\mu \phi_j)^2\end{aligned} \tag{3.18}$$

Hence, we have

$$\begin{aligned}\mathcal{L}_k &= \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 \\ &= \frac{1}{2}\{[(\partial_\mu \phi_i)^2 - (\hat{\phi}_j \partial_\mu \phi_j)^2]\frac{\sin^2 F(x)}{x^2} \\ &\quad + F'(x)^2(\hat{\phi}_j \partial_\mu \phi_j)^2\}\end{aligned} \tag{3.19}$$

If we let $\sin F(x) = x = \phi/f_\pi$, which is possible by making the choice $F(x) = \arcsin x$, then, for $F'(x) = O(1)$, the second and third terms in equation (3.19) cancel, and we are left with

$$(\partial_\mu \sigma)^2 + (\partial_\mu \pi_i)^2 \approx (\partial_\mu \phi_i)^2 \tag{3.20}$$

Thus the sigma model Lagrangian becomes[1]

$$\mathcal{L} \approx \frac{i}{2}\overline{\psi} \overleftrightarrow{\not{\partial}} \psi - \frac{1}{2}\overline{\psi} e^{\frac{i\vec{\tau}\cdot\vec{\phi}\gamma_5}{f_\pi}} \psi + \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}M_\phi^2 \phi^2 \tag{3.21}$$

which is independent of the scalar field.

As a result of redefining the pion field so that it now transforms non-linearly under the group transformations, those group transformations will transform a one-pion state into a multi-pion state or, an n-pion state into an m-pion state, with $n \neq m$ [50].

The earlier Cloudy Bag Models (a modified version of equation(3.21)) had pions coupling to the quark fields at the bag surface, the invariant

[1]The last term on the RHS of this equation is put in by hand and breaks the $SU(2) \times SU(2)$ symmetry.

Lagrangian density for massless quarks and pions given as

$$\mathcal{L} = [\frac{i}{2}\bar{q}\overleftrightarrow{\not\partial}q - B]\Theta_v - \frac{1}{2}\bar{q}e^{\frac{i\vec{\tau}\cdot\vec{\phi}\gamma_5}{f_\pi}}q\Delta_s + \frac{1}{2}(\partial_\mu\vec{\phi})^2 \tag{3.22}$$

with

$$\begin{aligned}\Theta_v &= 1 \text{ inside the bag volume} \\ &= 0 \text{ outside}\end{aligned} \tag{3.23}$$

and Δ_s the surface delta function. In equation(3.22), q represents a 2-component quark field. The energy density term, $-B\Theta_v$, is required for energy-momentum conservation and confinement. Summation over quark fields is implied.

In what is to follow, we will be interested in the volume coupling model [109,110]. By redefining the quark fields as

$$q_w = e^{\frac{i\vec{\tau}\cdot\vec{\phi}\gamma_5}{2f_\pi}}q \equiv Sq \tag{3.24}$$

the surface term $\bar{q}e^{\frac{i\vec{\tau}\cdot\vec{\phi}\gamma_5}{f_\pi}}q\Delta_s$ becomes

$$\bar{q}_w q_w \Delta_s \tag{3.25}$$

remembering that $\bar{q} = q^\dagger\gamma^0$ and $\{\gamma^0, \gamma_5\} = 0$. By making the transformation

$$\begin{aligned}\frac{i}{2}\bar{q}\overleftrightarrow{\not\partial}q &= \frac{i}{2}\bar{q}_w S\overleftrightarrow{\not\partial}S^\dagger q_w \\ &= \frac{i}{2}\bar{q}_w\overleftrightarrow{\not\partial}q_w + i\bar{q}_w(S\not\partial S^\dagger)q_w\end{aligned} \tag{3.26}$$

the quark kinetic energy term becomes

$$\frac{i}{2}\bar{q}_w\overleftrightarrow{\not\partial}q_w\Theta_v + \frac{\Theta_v}{2f_\pi}\bar{q}_w\gamma^\mu\gamma_5\vec{\tau}\cdot\partial_\mu\vec{\phi}q_w - \frac{\Theta_v}{4f_\pi^2}\bar{q}_w\gamma^\mu\vec{\tau}\cdot(\vec{\phi}\times\partial_\mu\vec{\phi})q_w \tag{3.27}$$

In deriving this equation, we used the identity [52,111]

$$iS\partial_\mu S^\dagger = i\int_0^1 d\lambda S^\lambda(\partial_\mu \ln S^\dagger)S^{\dagger^\lambda} \tag{3.28}$$

(with S given by (3.24)), and equation(3.5).

3.2 The $SU(3)$ Hamiltonian

We write the $SU(3)$ volume coupling CBM Lagrangian density in the form

$$\mathcal{L} = (\frac{i}{2}\bar{q}\overleftrightarrow{\partial\!\!\!/}q - B)\Theta_v - \frac{1}{2}\bar{q}q\Delta_s + \frac{1}{2}(\partial_\mu\vec{\phi})^2 + \frac{\Theta_v}{2f_j}\bar{q}\gamma^\mu\gamma_5\vec{\lambda}q\cdot\partial_\mu\vec{\phi} - \frac{\Theta_v}{4f_j^2}\bar{q}\gamma^\mu\vec{\lambda}q\cdot(\vec{\phi}\times\partial_\mu\vec{\phi}) \tag{3.29}$$

(correct up to $O(\phi^2)$) where $\vec{\phi}$ denotes the meson octet field, $\vec{\lambda}$ are the Gell-Mann matrices, and f_j the octet decay constants, $j \in (\pi, \kappa, \eta)$[2]. In what is to follow, we use [44]

$$\begin{aligned} f_\pi &= 93 \text{ MeV} \\ f_\kappa &= 117 \text{ MeV} \\ f_\eta &= 125 \text{ MeV} \end{aligned}$$

The term linear in $\vec{\phi}$ is responsible for the absorption and emission of mesons, while the $\vec{\phi} \times \partial_\mu\ \vec{\phi}$ term is responsible for s-wave meson-baryon scattering. The quark field q is now a 3-component column vector.

We can decompose the quark-space Hamiltonian into three parts

$$\hat{H} = \hat{H}_0 + \hat{H}_s + \hat{H}_c = \int d^3x\ T^{00}(x) \tag{3.30}$$

with the energy-momentum tensor defined to be

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu q)}(\partial^\nu q) - g^{\mu\nu}\mathcal{L} \tag{3.31}$$

with $g^{\mu\nu} = diag[-1, +1, +1, +1]$. The Hamiltonian $\hat{H}_0$ describes free bags and mesons

$$\hat{H}_0 = \sum_i \omega_i + \frac{4\pi}{3}R^3 B + \int d^3k\ \omega_j a_j^\dagger(\vec{k})a_j(\vec{k}). \tag{3.32}$$

[2] Although we introduced the meson field phenomenologically, and assumed no internal structure, the decay constant reminds us that the mesons do indeed have an internal structure

with ω_i the ground state quark energy, and $\omega_j = \sqrt{k^2 + M_j^2}$ the meson energy. The operator $a_j^\dagger(\vec{k})$ creates a meson with 3-momentum $\vec{k}$ and isospin j.

The coupling of a bare baryon and meson to a baryon state is given by

$$\hat{H}_s = -\int d^3x \, \frac{\Theta_v}{2f_j} \overline{q}\gamma^\mu \gamma_5 \vec{\lambda} q \cdot \partial_\mu \vec{\phi} \tag{3.33}$$

The contact term is

$$\hat{H}_c = \int d^3x \, \frac{\Theta_v}{4f_j^2} \overline{q}\gamma^\mu \vec{\lambda} q \cdot (\vec{\phi} \times \partial_\mu \vec{\phi}) \tag{3.34}$$

The integral $\hat{H}_s$ can be simpilfied upon integration by parts to give

$$\hat{H}_s = \int d^3x \Big[\frac{i}{2f_j} \overline{q}\gamma_5 \vec{\lambda} q \cdot \vec{\phi} \Delta_s - \frac{\Theta_v}{2f_j} \partial_0 (\overline{q}\gamma^0 \gamma_5 \vec{\lambda} q \cdot \vec{\phi}) \Big]. \tag{3.35}$$

by using the massless Dirac equations and the linear boundary condition. In the above expression, $n_\mu(t, \vec{x})$ is the outward normal unit four vector (for a static bag, $n_\mu = (0, \hat{r})$).

The ground state wave function for a massless quark may be written in the form (ω_s is the s-state quark energy)

$$q_{1s}(\vec{r}, t) = \frac{N_s}{\sqrt{4\pi}} \begin{bmatrix} j_0(\omega_s r) \\ i\vec{\sigma} \cdot \hat{r} j_1(\omega_s r) \end{bmatrix} e^{-i\omega_s t} b \theta(R - r) \tag{3.36}$$

with N_s a normalization constant,

$$N_s = \frac{1}{R^3 j_0^2(\omega_s R)} \frac{\omega_s R}{(\omega_s R - 1)} \tag{3.37}$$

and j_0 and j_1 spherical bessel functions. Quark labels on $\vec{\sigma}$ and b have been suppressed. The meson field is written as a plane wave expansion

$$\phi_j(\vec{x}) = \int \frac{d^3k}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} \, [a_j(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + a_j^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}] \tag{3.38}$$

If we consider transitions from 1s states to 1s states, it is straight foward to show that

$$\overline{q}\gamma^0\gamma_5 q = 0$$

and

$$\overline{q}\gamma_5 q = \frac{N_s^2}{4\pi} 2i\vec{\sigma}\cdot\hat{r} j_0(\omega_s r) j_1(\omega_s r) b^\dagger b \theta(R-r) \tag{3.39}$$

Substituting this and equation(3.38) into equation(3.35), we find that $\hat{H}_s$ reduces to

$$\hat{H}_s = -\int d^3k \, [\hat{V}_{0j}(\vec{k}) a_j(\vec{k}) + \hat{V}_{0j}^\dagger(\vec{k}) a_j^\dagger(\vec{k})] \tag{3.40}$$

at $r = R$, where

$$\hat{V}_{0j}(\vec{k}) = \sum_n \sum_a i \frac{g_j^0}{M_j} b_a^\dagger \vec{\sigma}_n \cdot \vec{k} \lambda_j^n \frac{u(kR)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} b_a \tag{3.41}$$

is a bare vertex operator. In this expression, we have defined the bare coupling constant, g_j^0, to be

$$g_j^0 = \frac{M_j}{6f_j}\left(\frac{\omega_s R}{\omega_s R - 1}\right) \tag{3.42}$$

and the form factor

$$u(kR) = 3\frac{j_1(kR)}{kR} \tag{3.43}$$

which rapidly approaches zero for increasing kR (and hence acts as a natural cutoff for the theory). In deriving equations(3.40) and (3.41) we have made use of the following [112,113]

$$e^{i\vec{k}\cdot\vec{R}} = 4\pi \sum_l \sum_m i^l j_l(kR) Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \tag{3.44}$$

and

$$\vec{\sigma}_n \cdot \hat{r} = \sqrt{\frac{4\pi}{3}} \sum_\mu (-1)^\mu \sigma_{n\,-\mu} Y_{1\mu}(\theta', \phi') \tag{3.45}$$

in a spherical basis $\vec{R} = (R, \theta', \phi')$, $\vec{k} = (k, \theta, \phi)$, $\hat{r} = (r, \theta', \phi')$ (note that $\vec{R}$ and $\hat{r}$ are in the same direction).

Hence

$$\begin{aligned} \textstyle\int d\Omega\, \vec{\sigma}_n \cdot \hat{r}[a_j(\vec{k})e^{i\vec{k}\cdot\vec{R}} + a_j^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{R}}] \\ = 4\pi i \vec{\sigma}_n \cdot \hat{k}\,[a_j(\vec{k}) - a_j^\dagger(\vec{k})] j_1(kR) \end{aligned} \tag{3.46}$$

having used the normalization

$$\int d\Omega\, Y_{lm}^*(\theta', \phi') Y_{l'm'}(\theta', \phi') = \delta_{ll'}\delta_{mm'} \tag{3.47}$$

and

$$\begin{aligned} \int d^3r &= \int_0^R r^2 dr \int d\Omega \\ &= \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \end{aligned} \tag{3.48}$$

Note that in equation(3.46) only the $l = 1$ term survives. This means only p-wave mesons couple to the baryon.

In the space of colourless, non-exotic baryons

$$\begin{aligned} H_s &= \sum_{\alpha,\beta} \alpha^\dagger \langle\alpha|\hat{H}_s|\beta\rangle\beta \\ &= -\sum_{\alpha,\beta} \int d^3k \alpha^\dagger [\langle\alpha|\hat{V}_{0j}(\vec{k})|\beta\rangle a_j(\vec{k}) + \langle\alpha|\hat{V}_{0j}^\dagger(\vec{k})|\beta\rangle a_j^\dagger(\vec{k})]\beta \end{aligned} \tag{3.49}$$

As we shall see, $\langle\alpha|$ and $|\beta\rangle$ will be SU(6) spin-flavour wave functions.

If we now define a vertex operator for the absorption of a meson (reading the matrix element from right to left)

$$v_{0j}^{\alpha\beta}(\vec{k}) = \langle\alpha|\hat{V}_{0j}(\vec{k})|\beta\rangle \tag{3.50}$$

then

$$H_s = -\int d^3k\ [V_{0j}(\vec{k})a_j(\vec{k}) + V_{0j}^\dagger(\vec{k})a_j^\dagger(\vec{k})]. \tag{3.51}$$

The following figure shows a 3-point vertex for the absorption of a meson.

We define a general vertex operator to be of the form

$$V_{0j}(k) = \sum_{\alpha,\beta} i\frac{\lambda_{\alpha\beta}}{M_j}\alpha^\dagger \vec{S}^{\alpha\beta}\cdot\vec{k}\,\vec{T}^{\alpha\beta}\frac{u(kR)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}}\frac{\omega_s R}{3(\omega_s R-1)}\beta \tag{3.52}$$

with $\lambda_{\alpha\beta}$ a bare coupling constant. We now identify the above operator with equation (3.41) (and (3.50)). In the above, $\vec{S}$ is a spin s_1 to spin s_2 transition operator, and $\vec{T}$ is an isospin I_1 to isospin I_2 transition operator.

We stated above that only p-wave mesons couple to the baryons. We therefore define irreducible 'spin' tensors of rank 1, and 'isospin' tensors of rank p

$$\begin{aligned}
\vec{S}^{\alpha\beta} &= \sum_{m=-1}^{+1} S_m^{(1)^{\alpha\beta}}\hat{s}_m^* \\
\vec{\sigma}_n &= \sum_m \sigma_{nm}^{(1)}\hat{s}_m^* \\
\vec{T}^{\alpha\beta} &= \sum_{r=-1}^{+1} T_r^{(p)^{\alpha\beta}}\hat{t}_r^* \\
\vec{\lambda}^n &= \sum_r \lambda_r^{n(p)}\hat{t}_r^*
\end{aligned} \tag{3.53}$$

where $\hat{s}^*_m$ and $\hat{t}^*_r$ are spherical basis vectors

$$\begin{aligned} \hat{s}^*_{\pm 1} &= \mp\frac{(e_x \mp ie_y)}{\sqrt{2}} \\ \hat{s}^*_0 &= e_z \end{aligned} \tag{3.54}$$

e_x, e_y and e_z are basis vectors of a polar coordinate system.

It is straight foward to show that, for fixed m and r,

$$\lambda_{\alpha\beta} S^{(1)^{\alpha\beta}}_m T^{(p)^{\alpha\beta}}_r = g_j \sum_{a,n} \langle \alpha | b^\dagger_a \sigma^{(1)}_{nm} \lambda^{n^{(p)}}_r b_a | \beta \rangle \tag{3.55}$$

defining

$$g_j = \frac{M_j}{2f_j} \tag{3.56}$$

and $\lambda_{\alpha\beta}, \vec{S}$ and $\vec{T}$ to be determined. The LHS of equation(3.55) reads

$$S^{(1)^{\alpha\beta}}_m T^{(p)^{\alpha\beta}}_r = \langle j_\alpha, m_\alpha; I_\alpha, I_{3\alpha} | S^{(1)}_m T^{(p)}_r | j_\beta, m_\beta; I_\beta, I_{3\beta} \rangle \tag{3.57}$$

with j_α and m_α the total spin and its z-projection of the state α, and I_α and $I_{3\alpha}$ the total isospin and its z-projection of the state α. As this matrix element involves $m_\alpha, m_\beta, I_{3\alpha}$ and $I_{3\beta}$, it is dependent on the orientation of the coordinate system. We can remove this dependence by using the Wigner-Eckart theorem [112]. We get

$$S^{(1)^{\alpha\beta}}_m T^{(p)^{\alpha\beta}}_r = \frac{\langle j_\alpha I_\alpha || \vec{S}\vec{T} || j_\beta I_\beta \rangle}{[2j_\alpha + 1]^{\frac{1}{2}} [2I_\alpha + 1]^{\frac{1}{2}}} C^{m_\beta\, m\, m_\alpha}_{j_\beta\; 1\; j_\alpha} C^{I_{3\beta}\, r\, I_{3\alpha}}_{I_\beta\; p\; I_\alpha} \tag{3.58}$$

with the help of the relation

$$C^{m_1\, m_2\, m_3}_{j_1\; j_2\; j_3} = (-1)^{j_3 - j_1 - j_2} C^{m_2\, m_1\, m_3}_{j_2\; j_1\; j_3} \tag{3.59}$$

Thus we can make the identification

$$\begin{aligned} S^{(1)^{\alpha\beta}}_m &= C^{m_\beta\, m\, m_\alpha}_{j_\beta\; 1\; j_\alpha} \\ T^{(p)^{\alpha\beta}}_r &= C^{I_{3\beta}\, r\, I_{3\alpha}}_{I_\beta\; p\; I_\alpha} \end{aligned} \tag{3.60}$$

and the reduced matrix element (which is independent of m and I_3 and hence, independent of the orientation of the frame of reference)

$$\langle j_\alpha I_\alpha || \vec{S}\vec{T} || j_\beta I_\beta \rangle = [2j_\alpha + 1]^{\frac{1}{2}}[2I_\alpha + 1]^{\frac{1}{2}} \tag{3.61}$$

Substituting equation(3.60) into equation(3.55) we find the bare coupling constant to be

$$\lambda_{\alpha\beta} = g_j \sum_{a,n} \frac{\langle \alpha | b_a^\dagger \sigma_{nm}^{(1)} \lambda_r^{n(p)} b_a | \beta \rangle}{C_{j_\beta \; 1 \; j_\alpha}^{m_\beta \, m \, m_\alpha} C_{I_\beta \; p \; I_\alpha}^{I_{3\beta} \, r \, I_{3\alpha}}} \tag{3.62}$$

We can simplify equation(3.62) by restricting the baryon states to those within the $\underline{56}$ representation of SU(6). As the wave functions are completely symmetric, we may choose to operate on a particular quark, which we choose to be quark 3, ie, $a = 3$. Thus we have

$$\lambda_{\alpha\beta} = 3g_j \frac{{}_{sf}\langle \alpha | b_3^\dagger \sigma_m^{(1)} \lambda_r^{(p)} b_3 | \beta \rangle_{sf}}{C_{j_\beta \; 1 \; j_\alpha}^{m_\beta \, m \, m_\alpha} C_{I_\beta \; p \; I_\alpha}^{I_{3\beta} \, r \, I_{3\alpha}}} \tag{3.63}$$

with [114]

$$\lambda_r^{(p)} = \begin{cases} \mp\frac{1}{\sqrt{2}}(\lambda_1 \pm i\lambda_2), \; \lambda_3 \\ \mp\frac{1}{\sqrt{2}}(\lambda_4 \pm i\lambda_5), \; \mp\frac{1}{\sqrt{2}}(\lambda_6 \pm i\lambda_7) \\ \lambda_8 \end{cases} \tag{3.64}$$

when $p = 1$ (pions), 1/2 (kaons), and 0 (eta), respectively. Similarly

$$\begin{aligned} \sigma_{\pm 1}^{(1)} &= \pm\frac{1}{\sqrt{2}}(\sigma_1 \mp i\sigma_2) \\ \sigma_0^{(1)} &= \sigma_3 \end{aligned} \tag{3.65}$$

The states ${}_{sf}\langle \alpha|$ and $|\beta\rangle_{sf}$ are now SU(6) spin-flavour states [115,116]. Coupling constants are given in tables 1, 2, 3 and 4 below. The πBB coupling constants are reproduced from references [105] and [107]. Note the correction to the $\lambda_{\pi\Xi^*\Xi^*}$ coupling constant.

Table 1. Unrenormalized πBB coupling constants

$\frac{\lambda_{\alpha\beta}}{g_\pi}$	N	Σ	Λ	Ξ	Δ	Ξ^*	Σ^*
N	5	0	0	0	$4\sqrt{2}$	0	0
Σ	0	$4\sqrt{6}/3$	-2	0	0	0	$-4\sqrt{3}/3$
Λ	0	$2\sqrt{3}$	0	0	0	0	$2\sqrt{6}$
Ξ	0	0	0	-1	0	$-2\sqrt{2}$	0
Δ	$2\sqrt{2}$	0	0	2	5	0	0
Σ^*	0	$2\sqrt{6}/3$	2	0	0	0	$2\sqrt{30}/3$
Ξ^*	0	0	0	0	0	$\sqrt{5}$	0

Table 2. Unrenormalized ηBB coupling constants

$\frac{\lambda_{\alpha\beta}}{g_\eta}$	N	Σ	Λ	Ξ	Δ	Σ^*	Ξ^*	Ω
N	1	0	0	0	0	0	0	0
Σ	0	2	0	0	0	$-2\sqrt{2}$	0	0
Λ	0	0	-2	0	0	0	0	0
Ξ	0	0	0	-3	0	0	$-2\sqrt{2}$	0
Δ	0	0	0	0	$\sqrt{5}$	0	0	0
Σ^*	0	2	0	0	0	0	0	0
Ξ^*	0	0	0	2	0	0	$-\sqrt{5}$	0
Ω	0	0	0	0	0	0	0	$-2\sqrt{5}$

Table 3. Unrenormalized κBB coupling constants

$\frac{\lambda_{\alpha\beta}}{g_\kappa}$	N	Σ	Λ	Ξ	Δ	Ξ^*	Σ^*	Ω
N	0	1	-3	0	0	0	$-2\sqrt{2}$	0
Σ	0	0	0	$-5\sqrt{6}/3$	0	$4/\sqrt{3}$	0	0
Λ	0	0	0	$-\sqrt{2}$	0	-4	0	0
Ξ	0	0	0	0	0	0	0	4
Δ	0	$2\sqrt{2}$	0	0	0	0	$-\sqrt{10}$	0
Σ^*	0	0	0	$2\sqrt{2/3}$	0	$-2\sqrt{10/3}$	0	0
Ξ^*	0	0	0	0	0	0	0	$-\sqrt{10}$
Ω	0	0	0	0	0	0	0	0

Table 4. Unrenormalized $\overline{\kappa}BB$ coupling constants.

$\frac{\lambda_{\alpha\beta}}{g_\kappa}$	N	Σ	Λ	Ξ	Δ	Σ^*	Ξ^*	Ω
N	0	0	0	0	0	0	0	0
Σ	$-\sqrt{2/3}$	0	0	0	$8/\sqrt{3}$	0	0	0
Λ	$3\sqrt{2}$	0	0	0	0	0	0	0
Ξ	0	-5	1	0	0	$-2\sqrt{2}$	0	0
Δ	0	0	0	0	0	0	0	0
Σ^*	$-2\sqrt{2/3}$	0	0	0	$2\sqrt{10/3}$	0	0	0
Ξ^*	0	-2	-2	0	0	$-2\sqrt{5}$	0	0
Ω	0	0	0	-4	0	0	$-2\sqrt{5}$	0

We now turn to the contact Hamiltonian. To simplify matters we deal with s-wave elastic scattering. In this case, the spatial part of the covariant derivative vanishes (as it is p-wave), and we are left with (at the quark level)

$$\hat{H}_c = \int d^3x \, \frac{\Theta_v}{4f_j^2} \overline{q}\gamma^0 \vec{\lambda} q \cdot (\vec{\phi} \times \partial_0 \vec{\phi}) \tag{3.66}$$

with the $SU(3)$ cross product

$$(\vec{\phi} \times \partial_0 \vec{\phi})_i = f_{ijk} \phi_j \partial_0 \phi_k \tag{3.67}$$

Using equation(3.36)

$$\overline{q}\gamma^0 q = \frac{N_s^2}{4\pi}[j_0^2(\omega_s r) + j_1^2(\omega_s r)] b^\dagger b \theta(R - r) \tag{3.68}$$

Expanding the meson field in plane waves and noting that

$$f_{ijk}\phi_j(k)\phi_k(k') = f_{ijk}\phi_j^\dagger(k)\phi_k^\dagger(k') = 0 \tag{3.69}$$

(as f_{ijk} is an odd function and $\phi_i \phi_j$ even under the interchange of two indices) and

$$\begin{aligned} f_{ijk}\phi_j(k)\phi_k^\dagger(k') &= f_{ijk}\phi_k^\dagger(k')\phi_j(k) + f_{ijk}\delta_{jk}\delta^3(k - k') \\ &= f_{ijk}\phi_k^\dagger(k')\phi_j(k) \end{aligned} \tag{3.70}$$

we find that

$$(\vec{\phi} \times \partial_0 \vec{\phi})_i = f_{ijk} \int \frac{d^3k \, d^3k'}{4(2\pi)^3} \frac{[\omega_j(k) + \omega_k(k')]}{[\omega_j(k)\omega_k(k')]^{\frac{1}{2}}} a_j^\dagger(k) a_k(k') e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}} \tag{3.71}$$

Now, applying (3.44) with $l = m = 0$(s-wave),

$$\int d\Omega e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}} = 4\pi j_0(k'x) j_0(kx) \tag{3.72}$$

hence, equation(3.66) becomes

$$\hat{H}_c = \int d^3k d^3k' \hat{V}_{0jk}(k, k') a_j^\dagger(k) a_k(k') \tag{3.73}$$

with the bare vertex operator

$$\hat{V}_{0jk}(k,k') = \frac{-if_{ijk}\lambda_i u_{\alpha\beta}(k,k',R)}{2f_j^2[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}[2\omega_j(k')(2\pi)^3]^{\frac{1}{2}}} \tag{3.74}$$

and defining

$$u_{\alpha\beta}(k,k',R) = N_s^2[\omega_j(k)+\omega_k(k')]\int_0^R r^2 dr\,[j_0^2(\omega_s r)+j_1^2(\omega_s r)]j_0(kr)j_0(k'r) \tag{3.75}$$

Instead of deriving coupling constants from the above vertex operator, consider the following. Define incoming baryon-meson states by

$$i = \langle B_{I_B I_{3B} Y_B} M_{I_M I_{3M} Y_M}| \tag{3.76}$$

and outgoing states by

$$f = |B'_{I_{B'} I_{3B'} Y_{B'}} M'_{I_{M'} I_{3M'} Y_{M'}}\rangle \tag{3.77}$$

Instead of writing the transition matrix in terms of the above states and quantum numbers, we expand the above states in terms of "good" quantum numbers

$$|B_{I_B I_{3B} Y_B} M_{I_M I_{3M} Y_M}\rangle = \sum_{I,Y} C^{I_{3B}\ I_{3M}\ I_3}_{I_B\ \ I_M\ \ I}\,\delta_{Y,Y_B+Y_M}|B_{I I_3 Y} M_{I I_3 Y}\rangle \tag{3.78}$$

defining total isospin

$$I = I_B + I_M, I_B + I_M - 1, \cdots, |I_B - I_M| \tag{3.79}$$

and

$$I_3 = I_{3B} + I_{3M} = I_{3B'} + I_{3M'} \tag{3.80}$$

Therefore

$$\begin{aligned} T^{BB'} &= \langle B_{I_B I_{3B}} M_{I_M I_{3M}}|T|B'_{I_{B'} I_{3B'}} M'_{I_{M'} I_{3M'}}\rangle \\ &= \sum_{I,I_3} C^{I_{3B}\ I_{3M}\ I_3}_{I_B\ \ I_M\ \ I} C^{I_{3B'}\ I_{3M'}\ I_3}_{I_{B'}\ \ I_{M'}\ \ I}\langle B_{I I_3} M_{I I_3}|T^{I,I_3}|B'_{I I_3} M'_{I I_3}\rangle \end{aligned} \tag{3.81}$$

(B denoting a baryon-meson pair) having omitted the hypercharge quantum numbers and delta function. The $\langle B'_{II_3} M'_{II_3}|$ are $SU(3)$ matrix elements.

Using the following representation of the identity

$$I_d = \sum_R |\psi_R^I\rangle\langle\psi_R^I| \tag{3.82}$$

with R the dimension of the irreducible $SU(3)$ representations (for some value of the isospin I), we have

$$T^{\alpha\beta} = \sum_{I,I_3} C^{I_{3B}\ I_{3M}\ I_3}_{I_B\ I_M\ I} C^{I_{3B'}\ I_{3M'}\ I_3}_{I_{B'}\ I_{M'}\ I} \langle B_{II_3} M_{II_3}|\psi_R^I\rangle T_R^{I,I_3} \langle\psi_R^I|B'_{II_3} M'_{II_3}\rangle \tag{3.83}$$

The T_R are real numbers, and the "vectors" $|\psi_R^I\rangle$ can be found in table 3.4 of reference [2]. (Note: In the $\underline{27}$ representation with $Y = 0$ and $I = 1$, the first entry should read $\sqrt{2}(N\overline{K})_1$; In $\underline{8}_1$, $Y = 0$ and $I = 1$, the last entry should read $\sqrt{3}(N\overline{K})_1$; In $\underline{1}$, the $SU(3)$ Clebsch-Gordan coefficients are $1/2, -1/2, \sqrt{6}/4$ and $-\sqrt{2}/4$).

Now, for elastic scattering

$$T = I_{3B} I_{3M} + \frac{3}{4} Y_B Y_M \tag{3.84}$$

allowing matrix elements of the form $\langle BM|T|BM\rangle$ to be calculated and hence, T_R^{I,I_3}. Once these are known, matrix elements like $\langle BM|T|B'M'\rangle$ can be calculated and from these, the coupling constants are found.

Between bare baryon-meson states, the bare vertex operator becomes

$$V_{0ij}(k,k') = \sum_{B_0,B'_0} B_0^\dagger \langle B_0|\hat{V}_{0ij}|B'_0\rangle B'_0 \tag{3.85}$$

with $|B_0^\dagger\rangle$ denoting the incoming baryon-meson pair.

We make the identification

$$\begin{aligned} v_{0ij}^{BB'}(k,k') &= \langle B_0 M_0|\hat{V}_{0ij}|B'_0 M'_0\rangle \\ &= \frac{\langle B_0 M_0| - i f_{ijk}\lambda_i u_{\alpha\beta}|B'_0 M'_0\rangle}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}[2\omega_j(k')(2\pi)^3]^{\frac{1}{2}}} \end{aligned} \tag{3.86}$$

and from equation(3.81)

$$v_{0ij}^{BB'}(k,k') = \sum_{I,I_3} \lambda^I_{\alpha\beta} \frac{-iu_{\alpha\beta}(k,k',R)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}[2\omega_j(k')(2\pi)^3]^{\frac{1}{2}}} C^{I_{3B}\ I_{3M}\ I_3}_{I_B\ \ I_M\ \ I} C^{I_{3B'}\ I_{3M'}\ I_3}_{I_{B'}\ \ I_{M'}\ \ I} \tag{3.87}$$

The coupling constants

$$\lambda^I_{\alpha\beta} = \frac{\langle BM|T^{I,I_3}|B'M'\rangle}{4f_j^2} \tag{3.88}$$

are to be expanded in the form of equation(3.83).

Following through the above with various (at least 4) matrix elements, it can be found that, for $I = Y = 0$

$$\begin{aligned} T^{00}_{27} &= 1 \\ T^{00}_{8_1} &= -3/2 \\ T^{00}_{8_2} &= -3/2 \\ T^{00}_{1} &= -3 \end{aligned} \tag{3.89}$$

and using equation(3.83) $\lambda^0_{\alpha\beta}$ may be found.

In a similar manner, for $I = 1, Y = 0$

$$\begin{aligned} T^{10}_{27} &= 1 \\ T^{10}_{10} = T^{10}_{10^*} &= 0 \\ T^{10}_{8_1} &= -3/2 \\ T^{10}_{8_2} &= -3/2 \end{aligned} \tag{3.90}$$

giving $\lambda^1_{\alpha\beta}$. The same procedure can be used for $I = 1, Y = 1/2$, and so on. The coupling constants can be found in reference [117].

Expressions for p- and d-wave scattering can be found in reference [118].

We once again consider the Hamiltonian given by equation(3.33) but, now, we consider the transition $1s \rightarrow 1p$ [117], with the quark wave functions given by

$$q_{1s}(\vec{r},t) = \frac{N_s}{\sqrt{4\pi}} \begin{bmatrix} j_0(\omega_s r) \\ i\vec{\sigma}\cdot\hat{r} j_1(\omega_s r) \end{bmatrix} e^{-i\omega_s t} b\theta(R-r)$$
$$q_{1p}(\vec{r},t) = \frac{N_p}{\sqrt{4\pi}} \begin{bmatrix} -\vec{\sigma}\cdot\hat{r} j_1(\omega_p r) \\ i j_0(\omega_p r) \end{bmatrix} e^{-i\omega_p t} b\theta(R-r) \qquad (3.91)$$

The normalization factors are

$$N_{s,p}^2 = \frac{1}{2j_0^2(\omega_{s,p}R)R^3} \frac{\omega_{s,p}R}{\omega_{s,p}R \mp 1} \qquad (3.92)$$

and $\omega_p R = 3.81, 7.0, \ldots$, the energy of the first excited (massless) quark state.

As the bessel functions satisfy

$$j_0(\omega_s) = j_1(\omega_s)$$
$$j_0(\omega_p) = -j_1(\omega_p) \qquad (3.93)$$

at the bag surface, we find that

$$\bar{q}_{1p}\gamma_5 q_{1s} = i\frac{N_s N_p}{4\pi}[j_0(\omega_s r)j_0(\omega_p r) - j_1(\omega_s r)j_1(\omega_p r)]e^{i(\omega_p-\omega_s)t}b^\dagger b\Theta(R-r) \qquad (3.94)$$

will be non zero. We also find that

$$\bar{q}\gamma^0\gamma_5 q = -i\frac{N_s N_p}{4\pi}[j_0(\omega_s r)j_0(\omega_p r) + j_1(\omega_s r)j_1(\omega_p r)]e^{i(\omega_p-\omega_s)t}b^\dagger b\Theta(R-r) \qquad (3.95)$$

As

$$\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, e^{\pm i\vec{k}\cdot\vec{r}} = 4\pi j_0(kr) \qquad (3.96)$$

we find the Hamiltonian

$$\hat{H}_s = \frac{N_s N_p}{2f_j} \int \frac{d^3k}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} \{ \big[2R^2 j_0(\omega_s R) j_1(\omega_p R) j_0(kR) - (\omega_s - \omega_p + \omega_j(k))$$
$$\times \int_0^R r^2 dr (j_0(\omega_s r) j_0(\omega_p r) + j_1(\omega_s r) j_1(\omega_p r)) j_0(kr) \big] \lambda_j a_j(k) b^\dagger b$$
$$+ \big[2R^2 j_0(\omega_s R) j_1(\omega_p R) j_0(kR) - (\omega_s - \omega_p - \omega_j(k))$$
$$\times \int_0^R r^2 dr (j_0(\omega_s r) j_0(\omega_p r) + j_1(\omega_s r) j_1(\omega_p r)) j_0(kr) \big] \lambda_j a_j^\dagger(k) b^\dagger b \} \quad (3.97)$$

at $t = 0$.

As before (equation(3.40)), we write

$$\hat{H}_s = - \int d^3k \; [\hat{V}_{0j}(\vec{k}) a_j(\vec{k}) + \hat{V}_{0j}^\dagger(\vec{k}) a_j^\dagger(\vec{k})] \quad (3.98)$$

and if we define form factors

$$u_{\alpha\beta}(kR) = N_s N_p \{ 2R^2 j_0(\omega_s R) j_1(\omega_p R) j_0(kR) - \big[(\omega_s - \omega_p + \omega_j(k))$$
$$\times \int_0^R r^2 dr (j_0(\omega_s r) j_0(\omega_p r) + j_1(\omega_s r) j_1(\omega_p r)) j_0(kr) \big] \} \quad (3.99)$$

and

$$\tilde{u}_{\alpha\beta}(kR) = N_s N_p \{ 2R^2 j_0(\omega_s R) j_1(\omega_p R) j_0(kR) - \big[(\omega_s - \omega_p - \omega_j(k))$$
$$\times \int_0^R r^2 dr (j_0(\omega_s r) j_0(\omega_p r) + j_1(\omega_s r) j_1(\omega_p r)) j_0(kr) \big] \} \quad (3.100)$$

then, at the quark level,

$$\hat{V}_{0j}(\vec{k}) = \frac{1}{2f_j} b^\dagger \lambda_j \frac{u_{\alpha\beta}(kR)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} b \quad (3.101)$$

and

$$\hat{V}_{0j}^\dagger(\vec{k}) = \frac{1}{2f_j} b^\dagger \lambda_j \frac{\tilde{u}_{\alpha\beta}(kR)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} b \quad (3.102)$$

If we define a general vertex operator at the bag level

$$v_{0j}^{\alpha\beta}(\vec{k}) = \lambda_{\alpha\beta} \vec{T}^{\alpha\beta} \frac{u_{\alpha\beta}(kR)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} \quad (3.103)$$

using the Wigner-Eckart theorem we find

$$T_r^{(p)^{\alpha\beta}} = C_{I_\beta \ p \ I_\alpha}^{I_{3\beta} \ r \ I_{3\alpha}} \tag{3.104}$$

and so

$$v_{0j}^{\alpha\beta}(\vec{k}) = \lambda_{\alpha\beta} \frac{u_{\alpha\beta}(kR)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} C_{I_\beta \ p \ I_\alpha}^{I_{3\beta} \ r \ I_{3\alpha}} \tag{3.105}$$

with

$$\lambda_{\alpha\beta} = 3 \frac{{}_{sf}\langle\alpha|b_3^\dagger \lambda_r^{(p)} b_3|\beta\rangle_{sf}}{2f_j C_{I_\beta \ p \ I_\alpha}^{I_{3\beta} \ r \ I_{3\alpha}}} \tag{3.106}$$

again choosing to operate on quark 3 (which will be a $1p$-quark).

These coupling constants are found in the first column of table I in reference [117].

3.3 Mass Renormalization

3.3.1 Baryon self energy and bare bag probabilities

In this section, we follow the method of Chew and Low [119,120], which allows us to deal with low-energy meson-nucleon interactions.

Let $|B\rangle$ denote eigenstates of the Hamiltonian corresponding to the physical baryon with observed mass M_B. That is,

$$H|B\rangle = M_B|B\rangle \tag{3.107}$$

Bare baryon states are denoted $|B_0\rangle$.

As we have seen, we can write the Hamiltonian in the form

$$H = M_0 + H_0 + H_I \tag{3.108}$$

with M_0 the bare mass of the baryon, H_0 describes free quarks and mesons, and H_I is the interaction term. The physical mass is introduced in the following way. Write

$$\begin{aligned} H &= M_0 + H_0 + H_I \\ &= M_0 + \delta M_B + H_0 + H_I - \delta M_B \\ &= \tilde{M}_0 + \tilde{M}_I \end{aligned} \tag{3.109}$$

having defined

$$\tilde{H}_I = H_I - \delta M_B \tag{3.110}$$

and

$$\begin{aligned} \tilde{H}_0 &= M_0 + \delta M_B + H_0 \\ &= M_B + H_0 \end{aligned} \tag{3.111}$$

We now decompose the physical baryon state $|B\rangle$ into a vector along $|B_0\rangle$ and a vector orthogonal to $|B_0\rangle$ which we denote $|\chi\rangle$. If $Z_2^B(E_B)$ is the probability that the physical baryon is a bare baryon, then

$$|B\rangle = Z_2^B(E_B)^{\frac{1}{2}}|B_0\rangle + |\chi\rangle \tag{3.112}$$

Now, as

$$\tilde{H}_0|B_0\rangle = M_B|B_0\rangle \tag{3.113}$$

we have

$$\begin{aligned} (\tilde{H}_0 - M_B)|B\rangle &= (\tilde{H}_0 - M_B)|\chi\rangle \\ &= -\tilde{H}_I|B\rangle \end{aligned} \tag{3.114}$$

Remembering that $|\chi\rangle$ is orthogonal to $|B_0\rangle$, we can solve this equation to give

$$|\chi\rangle = \frac{\Lambda}{M_B - \tilde{H}_0}\tilde{H}_I|B\rangle \tag{3.115}$$

defining a projection operator

$$\Lambda = 1 - \sum_{|B_0\rangle} |B_0\rangle\langle B_0| \tag{3.116}$$

such that $\Lambda|\chi\rangle = |\chi\rangle$, i.e., Λ projects the state $|\chi\rangle$ onto the subspace orthogonal to $|B_0\rangle$. Thus

$$|B\rangle = Z_2^B(E_B)^{\frac{1}{2}}|B_0\rangle + \frac{\Lambda}{M_B - \tilde{H}_0}\tilde{H}_I|B\rangle \tag{3.117}$$

and upon iteration

$$|B\rangle = Z_2^B(E_B)^{\frac{1}{2}}\{1 + \frac{\Lambda}{M_B - \tilde{H}_0}\tilde{H}_I + \frac{\Lambda}{M_B - \tilde{H}_0}\tilde{H}_I\frac{\Lambda}{M_B - \tilde{H}_0}\tilde{H}_I + \ldots\}|B_0\rangle \tag{3.118}$$

which is the perturbation expansion of the physical state in terms of the bare state.

The mass shift introduced by the perturbation is found by considering

$$\begin{aligned}\langle B_0|\tilde{H}_I|B\rangle &= \langle B_0|H - \tilde{H}_0|B\rangle = 0 \\ &= \langle B_0|H_I - \delta M_B|B\rangle \end{aligned}\tag{3.119}$$

and as

$$\langle B_0|B\rangle = Z_2^B(E_B)^{\frac{1}{2}} \tag{3.120}$$

we find that

$$\delta M_B = Z_2^B(E_B)^{\frac{1}{2}}\langle B_0|H_I|B\rangle \tag{3.121}$$

With $|B\rangle$ given by equation(3.118), we can find contributions to the mass shift to any order.

The interaction Hamiltonian is given by equations (3.51) and (3.73). However, as $a_j(\vec{k})|B_0\rangle = 0$, only H_s will contribute to the mass shift and hence (dropping the minus sign)

$$H_I = \int d^3k\ [V_{0j}(\vec{k})a_j(\vec{k}) + V_{0j}^\dagger(\vec{k})a_j^\dagger(\vec{k})] \tag{3.122}$$

with the vertex operator given by equation (3.50).

The interaction Hamiltonian creates or annihilates a meson. Therefore, to first order(or indeed, any odd order), the mass shift is zero.

To second order

$$\delta M_B^{(2)} = \langle B_0 | H_I \frac{\Lambda}{M_B - \tilde{H}_0} H_I | B_0 \rangle \tag{3.123}$$

with $V_{0j}(k)a_j(\vec{k})$ the only surviving term in the Hamiltonian on the left, and $V_{0j}^\dagger(k)a_j^\dagger(\vec{k})$ the only surviving term in the Hamiltonian on the right.

Now, $[\tilde{H}_0, V_{0j}(k)] = 0$, $[V_{0j}(k), a_j(\vec{k})] = 0$ and

$$\begin{aligned} [\tilde{H}_0, a_j^\dagger(\vec{k})] &= [H_0, a_j^\dagger(\vec{k})] \\ &= \omega_j(k) a_j^\dagger(\vec{k}) \end{aligned} \tag{3.124}$$

with $H_0 = \sum_j \int d^3k \, \omega_j(k) a_j^\dagger(\vec{k}) a_j(\vec{k})$.

Thus, using $[a_i(\vec{k}), a_j(\vec{k'})] = \delta_{ij}\delta^3(\vec{k} - \vec{k'})$ and $a_i(\vec{k})|B_0\rangle = 0$ equation(3.123) becomes

$$\delta M_B^{(2)} = \sum_j \int d^3k \langle B_0 | \frac{V_{0j}(k) V_{0j}^\dagger(k)}{M_B - \tilde{H}_0 - \omega_j(k)} | B_0 \rangle \tag{3.125}$$

or, to second order (we denote intermediate states by a)

$$\delta M_B^{(2)} = \sum_j \sum_a \int d^3k \frac{v_{0j}^{Ba}(k) v_{0j}^{Ba*}(k)}{M_B - M_a - \omega_j(k)} \tag{3.126}$$

Using equation(3.52), (3.60) and the results [113]

$$\sum_{m_a} C_{j_B \; 1 \; j_a}^{m_B \, m \, m_a} C_{j_a \; 1 \; j_B}^{m_a \, m' \, m_B} = \delta_{mm'} \tag{3.127}$$

$$\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \, Y_{1\mu}(\theta, \phi) Y_{1\nu}(\theta, \phi) = (-1)^\nu \delta_{-\mu\nu} \tag{3.128}$$

$$\hat{s}^*_{m-\mu}\,\hat{s}_{m-\nu} = (-1)^\mu \hat{s}_{m\mu}\,\hat{s}_{m-\nu} = \delta_{\mu\nu} \tag{3.129}$$

and

$$\sum_{j,r} \hat{t}^*_r \cdot e_j\, \hat{t}_r \cdot e_j = 1 \tag{3.130}$$

with $\hat{s}_m$ and $\hat{t}_m$ spherical basis vectors, we have

$$\sum_j \sum_a \int d^3k v_{0j}^{Ba}(k) v_{0j}^{Ba*}(k) = \frac{4\pi}{3} \sum_a \int dk\, k^4 \left(\frac{\lambda_{Ba}}{M_j}\right)^2 \frac{u^2(kR)}{2\omega_j(k)(2\pi)^3} \tag{3.131}$$

and hence

$$\delta M_B^{(2)} = \frac{1}{12\pi^2} \sum_a \left(\frac{\lambda_{Ba}}{M_j}\right)^2 \int_0^\infty dk \frac{k^4 u^2(kR)}{\omega_j(k)[M_B - M_a - \omega_j(k)]} \tag{3.132}$$

This equation is represented by the following graph. The solid line represents a bare baryon and the dashed line the meson.

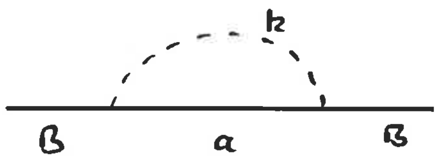


Figure 3.1: Second order self energy graph for emission and absorption of a virtual meson.

The probability that a baryon of energy E_B is a bare baryon is found by considering

$$\begin{aligned}\langle B|B\rangle &= 1 \\ &= Z_2^B(E_B)^{\frac{1}{2}}\langle B_0|1 + \tilde{H}_I\frac{\Lambda}{E_B - \tilde{H}_0} + \tilde{H}_I\frac{\Lambda}{E_B - \tilde{H}_0}\tilde{H}_I\frac{\Lambda}{E_B - \tilde{H}_0} + \ldots|B_0\rangle\end{aligned}$$

As the only surviving terms are those even in $\tilde{H}_I$ we see that

$$\begin{aligned}Z_2^B(E_B)^{-1} &= \langle B_0|1 + \tilde{H}_I\frac{\Lambda}{(E_B - \tilde{H}_0)^2}\tilde{H}_I|B_0\rangle \\ &= 1 - \frac{\partial}{\partial E_B}\delta E_B^{(2)} \qquad (3.133)\end{aligned}$$

to second order. Using the second order expression for the self energy, we find that

$$Z_2^B(E_B)^{-1} = 1 + \frac{1}{12\pi^2}\sum_a\left(\frac{\lambda_{Ba}}{M_j}\right)^2\int_0^\infty dk\frac{k^4u^2(kR)}{\omega_j(k)[M_B - M_a - \omega_j(k)]^2} \qquad (3.134)$$

The probabilities for the $(1/2)^+$ octet are shown on p 90.

3.4 Renormalized Coupling Constants

In this section, we shall "shift" the coupling constants so that the renormalized πNN coupling constant is that given by experiment.

For physical processes, we consider matrix elements of $V_{0j}(\vec{k})$ between dressed states $\langle\alpha|$ and $|\beta\rangle$. Define the renormalized vertex function $v_{rj}^{\alpha\beta}$ by

$$v_{rj}^{\alpha\beta}(\vec{k}) = \langle\alpha|V_{0j}(\vec{k})|\beta\rangle \qquad (3.135)$$

As dressed states and bare states have the same quantum numbers, we can relate the dressed vertex operator to the bare vertex operator

$$v_{rj}^{\alpha\beta}(\vec{k}) = \eta^{\alpha\beta}(E_\alpha, E_\beta)v_{0j}^{\alpha\beta}(\vec{k}) \qquad (3.136)$$

with $\eta^{\alpha\beta}(E_\alpha, E_\beta)$ independent of m_α, m_β, I_{3a} and $I_{3\beta}$.

Using the perturbation expansion given by equation(3.118) we find

$$v_{rj}^{\alpha\beta}(k) = Z_2^\alpha(E_\alpha)^{\frac{1}{2}} Z_2^\beta(E_\beta)^{\frac{1}{2}} \{ v_{0j}^{\alpha\beta}(k) + \langle\alpha_0|\frac{\Lambda\tilde{H}_I}{E_\alpha - \tilde{H}_0} V_{0j}(k) \frac{\Lambda\tilde{H}_I}{E_\beta - \tilde{H}_0}|\beta_0\rangle \} \qquad (3.137)$$

and as

$$\Lambda\tilde{H}_I|\beta_0\rangle = \tilde{H}_I|\beta_0\rangle \qquad (3.138)$$

we find, to second order

$$v_{rj}^{\alpha\beta}(k) = Z_2^\alpha(E_\alpha)^{\frac{1}{2}} Z_2^\beta(E_\beta)^{\frac{1}{2}} \{ v_{0j}^{\alpha\beta} + \langle\alpha_0|\frac{\tilde{H}_I}{E_\alpha - \tilde{H}_0} V_{0j}(k) \frac{\tilde{H}_I}{E_\beta - \tilde{H}_0}|\beta_0\rangle \} \qquad (3.139)$$

By (3.136), the second term must be proportional to $v_{0j}^{\alpha\beta}(k)$

$$\langle\alpha_0|\frac{\tilde{H}_I}{E_\alpha - \tilde{H}_0} V_{0j}(k) \frac{\tilde{H}_I}{E_\beta - \tilde{H}_0}|\beta_0\rangle = \lambda^{\alpha\beta}(E_\alpha, E_\beta) v_{0j}^{\alpha\beta}(k) \qquad (3.140)$$

If we define a vertex function

$$Z_1^{\alpha\beta}(E_\alpha, E_\beta) = [1 + \lambda^{\alpha\beta}(E_\alpha, E_\beta)]^{-1} \qquad (3.141)$$

then

$$\eta^{\alpha\beta}(E_\alpha, E_\beta) = \frac{Z_2^\alpha(E_\alpha)^{\frac{1}{2}} Z_2^\beta(E_\beta)^{\frac{1}{2}}}{Z_1^{\alpha\beta}(E_\alpha, E_\beta)} \tag{3.142}$$

with the vertex function $Z_1^{\alpha\beta}(E_\alpha, E_\beta)$ still to be determined.

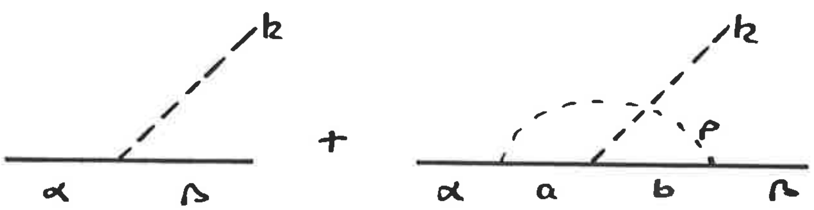
Using equations(3.136), (3.139) and (3.142)

$$Z_1^{\alpha\beta}(E_\alpha, E_\beta)^{-1} v_{0j}^{\alpha\beta}(k) = v_{0j}^{\alpha\beta}(k) + \langle\alpha_0|\frac{\tilde{H}_I}{E_\alpha - \tilde{H}_0} V_{0j}(k) \frac{\tilde{H}_I}{E_\beta - \tilde{H}_0}|\beta_0\rangle \tag{3.143}$$

or, using equation(3.110) and neglecting terms of order $(\delta M)^2$

$$Z_1^{\alpha\beta}(E_\alpha, E_\beta)^{-1} v_{0j}^{\alpha\beta}(k) = v_{0j}^{\alpha\beta}(k) + \langle\alpha_0|\frac{H_I}{E_\alpha - \tilde{H}_0} V_{0j}(k) \frac{H_I}{E_\beta - \tilde{H}_0}|\beta_0\rangle \tag{3.144}$$

Graphically, the RHS looks like



The purpose of the following is to write the matrix element in the above equation in the form $v_{0j}^{\alpha\beta} \times$ some quantity to be determined. From this we easily find $Z_1^{\alpha\beta}(E_\alpha, E_\beta)$.

With H_I defined by equation(3.122) we find that

$$\begin{aligned}&\langle\alpha_0|\, H_I(E_\alpha - \tilde{H}_0)^{-1} V_{0j}(k)(E_\beta - \tilde{H}_0)^{-1} H_I|\beta_0\rangle \\ &= \sum_l \sum_{a,b} \int d^3p \, \frac{v_{0l}^{\alpha a}(p)}{(E_\alpha - M_a - \omega_j(p))} v_{0j}^{ab}(k) \frac{v_{0l}^{\beta b^*}(p)}{(E_\beta - M_b - \omega_j(p))}\end{aligned} \tag{3.145}$$

with

$$v_{0j}^{ab}(k) = \sum_{m,r} i\frac{\lambda_{ab}}{M_j} C_{j_b\ 1\ j_a}^{m_b\ m\ m_a}\ \hat{s}_m^* \cdot \vec{k}\, C_{I_b\ p\ I_a}^{I_{3b}\ r\ I_{3a}}\ \hat{t}_r^* \cdot e_j \frac{u(kR)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} \tag{3.146}$$

The spin component of the above matrix element produces the following factor

$$\sum_{m_a,m_b} \sum_{m',m,m''} \int p^4 dp \int d\Omega\, C_{j_a\ 1\ j_\alpha}^{m_a\ m'\ m_\alpha} C_{j_b\ 1\ j_a}^{m_b\ m\ m_a} C_{j_b\ 1\ j_\beta}^{m_b\ m''\ m_\beta}\ \hat{s}_{m'}^* \cdot \hat{p}\ \hat{s}_m^* \cdot \vec{k}\ \hat{s}_{m''} \cdot \hat{p} \tag{3.147}$$

reducing to

$$\sum_{m_a,m_b} \sum_{m',m} \frac{4\pi}{3} \int p^4 dp\, C_{j_a\ 1\ j_\alpha}^{m_a\ m'\ m_\alpha} C_{j_b\ 1\ j_a}^{m_b\ m\ m_a} C_{j_b\ 1\ j_\beta}^{m_b\ m'\ m_\beta}\ \hat{s}^* \cdot \vec{k} \tag{3.148}$$

By applying

$$C_{j_1\ j_2\ j_3}^{m_1\ m_2\ m_3} = (-1)^{j_3-j_1-j_2} C_{j_2\ j_1\ j_3}^{m_2\ m_1\ m_3} \tag{3.149}$$

$$C_{j_1\ j_2\ j_3}^{m_1\ m_2\ m_3} = (-1)^{j_1-m_1} \left(\frac{2j_3+1}{2j_2+1}\right)^{\frac{1}{2}} C_{j_3\ j_1\ j_2}^{m_3\ m_1\ m_2} \tag{3.150}$$

and

$$C_{j_1\ j_2\ j_3}^{m_1\ m_2\ m_3} = (-1)^{j_2+m_2} \left(\frac{2j_3+1}{2j_1+1}\right)^{\frac{1}{2}} C_{j_2\ j_3\ j_1}^{m_2\ m_3\ m_1} \tag{3.151}$$

to (3.148) and by defining

$$\begin{array}{llllll} j_\beta = j_1 & j_b = j_2 & j_a = j_3 & j_\alpha = j & 1 = j' & 1 = j'' \\ m_\beta = m_1 & m_b = m_2 & m_a = m_3 & m_\alpha = m & l = m' & m = m'' \end{array}$$

we find that

$$C_{j_a\ 1\ j_\alpha}^{m_a\ m'\ m_\alpha} C_{j_b\ 1\ j_a}^{m_b\ m\ m_a} C_{j_b\ 1\ j_\beta}^{m_b\ m'\ m_\beta}$$
$$= (-1)^{j_\alpha+j_b} \frac{[(2j_\alpha+1)(2j_\beta+1)]^{\frac{1}{2}}}{3} C_{1\ j_a\ j_\alpha}^{m'\ m_a\ m_\alpha} C_{j_b\ j_a\ 1}^{m_b\ m_a\ m} C_{j_\beta\ j_b\ 1}^{m_\beta\ m_b\ m'} \tag{3.152}$$

To this expression we apply the identity

$$\sum_{m',m_2,m_3} C^{m_1\, m_2\, m'}_{j_1\, j_2\, j'} C^{m'\, m_3\, m}_{j'\, j_3\, j} C^{m_2\, m_3\, m''}_{j_2\, j_3\, j''}$$
$$=(-1)^{j_1+j_2+j_3+j}(2j'+1)^{\frac{1}{2}}(2j''+1)^{\frac{1}{2}} C^{m_1\, m''\, m}_{j_1\, j''\, j} \begin{Bmatrix} j_1 & j_2 & j' \\ j_3 & j & j'' \end{Bmatrix} \quad (3.153)$$

where $\left\{ \ldots \right\}$ is a 6-j symbol [121].

Using the above, we find that (3.148) reduces to

$$\frac{4\pi}{3}\sum_m \int p^4 dp (-1)^{j_\beta+j_a}(2j_\beta+1)^{\frac{1}{2}}(2j_a+1)^{\frac{1}{2}} C^{m_\beta\, m\, m_\alpha}_{j_\beta\, 1\, j_\alpha}\, \hat{s}^*_m \cdot \vec{k} \begin{Bmatrix} j_\beta & j_b & 1 \\ j_a & j_\alpha & 1 \end{Bmatrix} \quad (3.154)$$

Equation(3.145) produces the following factor for isospin

$$\sum_l \sum_{r',r,r''} \sum_{I_{3a},I_{3b}} C^{I_{3a}\, r'\, I_{3\alpha}}_{I_a\, p\, I_\alpha} C^{I_{3b}\, r\, I_{3a}}_{I_b\, p\, I_a} C^{I_{3b}\, r''\, I_{3\beta}}_{I_b\, p\, I_\beta}\, \hat{t}^*_{r'} \cdot e_l\, \hat{t}^*_r \cdot e_j\, \hat{t}_{r''} \cdot e_l$$
$$=\sum_{r,r'} \sum_{I_{3a},I_{3b}} C^{I_{3a}\, r'\, I_{3\alpha}}_{I_a\, p\, I_\alpha} C^{I_{3b}\, r\, I_{3a}}_{I_b\, p\, I_a} C^{I_{3b}\, r'\, I_{3\beta}}_{I_b\, p\, I_\beta}\, \hat{t}^*_r \cdot e_j \quad (3.155)$$

which gives an expression similar in form to (3.148), and applying the various symmetry operations, we derive and expression similar to equation(3.154) (the isospin expression will not involve an integration).

Putting the above together, and using equation(3.146), we finally arrive at the following

$$Z_1^{\alpha\beta}(E_\alpha, E_\beta)^{-1} v_{0j}^{\alpha\beta}(k)$$
$$= v_{0j}^{\alpha\beta}(k) + \Big(\sum_{a,b} \frac{\lambda_{\alpha a}\lambda_{ab}\lambda_{\beta b}}{M_j^3} \frac{M_j}{\lambda_{\alpha\beta}} (2\beta+1)^{\frac{1}{2}}(2a+1)^{\frac{1}{2}} \begin{Bmatrix} j_\beta & j_b & 1 \\ j_a & j_\alpha & 1 \end{Bmatrix} \begin{Bmatrix} I_\beta & I_b & p \\ I_a & I_\alpha & p \end{Bmatrix}$$
$$\times \frac{v_{0j}^{\alpha\beta}(k)}{12\pi^2} \int_0^\infty \frac{dp}{\omega_j(p)} \frac{p^4 u^2(pR)}{(M_\alpha - M_a - \omega_j(p))(M_\beta - M_b - \omega_j(p))} \Big)$$

with

$$\begin{aligned} 2\beta+1 &= (-1)^{2j_\beta+2I_\beta}(2j_\beta+1)(2I_\beta+1) \\ 2a+1 &= (-1)^{2j_a+2I_a}(2j_a+1)(2I_a+1) \end{aligned} \quad (3.156)$$

and $p = 1$ for pions, 1/2 for kaons, and 0 for the eta in the 6-j symbol.

Comparing this equation with (3.141) we see that

$$Z_1^{\alpha\beta}(E_\alpha, E_\beta)^{-1} = 1 + \sum_{a,b} \lambda_{ab}^{\alpha\beta}(E_\alpha, E_\beta) \tag{3.157}$$

with

$$\lambda_{ab}^{\alpha\beta}(E_\alpha, E_\beta) = \frac{\lambda_{\alpha a}\lambda_{ab}\lambda_{\beta b}}{\lambda_{\alpha\beta}M_j^2}\frac{1}{12\pi^2}(2\beta+1)^{\frac{1}{2}}(2a+1)^{\frac{1}{2}}\left\{\begin{matrix} j_\beta & j_b & 1 \\ j_a & j_\alpha & 1 \end{matrix}\right\}\left\{\begin{matrix} I_\beta & I_b & p \\ I_a & I_\alpha & p \end{matrix}\right\}$$
$$\times \int_0^\infty \frac{dp}{\omega_j(p)} \frac{p^4 u^2(pR)}{(M_\alpha - M_a - \omega_j(p))(M_\beta - M_b - \omega_j(p))} \tag{3.158}$$

We now have expressions for each function in equation(3.142). We can now use the above expression for the vertex function $Z_1^{\alpha\beta}$, and the expressions for the bare bag probabilities, to calculate the renormalized coupling constants and test their energy dependence.

If we write

$$v_{0j}^{\alpha\beta}(k) = i\frac{\lambda_{\alpha\beta}}{M_j}\langle\alpha_0|\vec{\sigma}\cdot\vec{k}\ \lambda_j|\beta_0\rangle\frac{u^2(kR)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} \tag{3.159}$$

and

$$v_{rj}^{\alpha\beta}(k) = i\frac{\lambda_{\alpha\beta}^r}{M_j}\langle\alpha|\vec{\sigma}\cdot\vec{k}\ \lambda_j|\beta\rangle\frac{u^2(kR)}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} \tag{3.160}$$

with $\lambda_{\alpha\beta}^r$ defined to be the renormalized coupling constant, then, by equations(3.136) and (3.142), we can relate the renormalized coupling constants to the bare coupling constants

$$\lambda_{\alpha\beta}^r = \frac{Z_2^\alpha(E_\alpha)^{\frac{1}{2}} Z_2^\beta(E_\beta)^{\frac{1}{2}}}{Z_1^{\alpha\beta}(E_\alpha, E_\beta)}\lambda_{\alpha\beta} \tag{3.161}$$

The $\lambda_{\alpha\beta}$ are found from tables 1,2,3 and 4 of section 3.2.

Consider the following ratio

$$\begin{aligned} C_r(E) &= \frac{\lambda^r_{\alpha\beta}(M_\alpha, M_\beta - E)}{\lambda^r_{NN}(M_N, M_N - E)} \frac{\lambda_{NN}}{\lambda_{\alpha\beta}} \\ &= \frac{Z_2^\alpha(M_\alpha)^{\frac{1}{2}} Z_2^\beta(M_\beta - E)^{\frac{1}{2}}}{Z_1^{\alpha\beta}(M_\alpha, M_\beta - E)} \frac{Z_1^{NN}(M_N, M_N - E)}{Z_2^N(M_N)^{\frac{1}{2}} Z_2^N(M_N - E)^{\frac{1}{2}}} \end{aligned} \quad (3.162)$$

for varying E. If $C_r(E) \approx 1$ we can rearrange the above and write

$$\lambda^r_{\alpha\beta} \approx \left(\frac{\lambda_{\alpha\beta}}{\lambda_{NN}}\right) \lambda^r_{NN} \quad (3.163)$$

with $\lambda_{\alpha\beta}$ and λ_{NN} known, and λ^r_{NN} given by experiment.

The ratio given by equation(3.163) with $\alpha = N$ and $\beta = \Delta$ is given in reference [106] for pion loops only. We have included kaon and eta loops. The result is given in figure 3.2. It can seen that the ratio decreases with increased bag radius, allowing us to use equation(3.163).

Now consider

$$f_r(E) = \frac{\lambda^r_{NN}(M_N, M_N - E)}{\lambda^r_{NN}(M_N, M_N)} \quad (3.164)$$

This ratio is also shown in reference [106]. The result of including kaon and eta loops is shown in figure 3.3. We see that $f_r(E) \approx 1$, and hence, we can write

$$\lambda^r_{NN}(E_N, E_N) = \lambda^r_{NN}(M_N, M_N) = \lambda^r_{NN} \quad (3.165)$$

We see that the inclusion of kaon and eta loops does not significantly alter these ratios from the $SU(2)$ (pion only) case.

We now relate λ^r_{NN} to the experimental value of the renormalized πNN coupling constant $f^2_{exp} = 0.081$ through [120,122]

$$\lambda^r_{NN} = 6\sqrt{\pi} f_{exp} = 3.027 \quad (3.166)$$

Hence

$$\lambda^r_{\alpha\beta} \approx 3.03 \frac{\lambda_{\alpha\beta}}{\lambda_{NN}} \quad (3.167)$$

It is this value which is to be used in calculating bag probabilities and the sigma term.

We mentioned in the chapter 1 that an assumption is made about the number of mesons surrounding the bag, namely that there would be few. We see from figure 3.4 that this assumption is true for a bag radius in the range $0.8 \leq R \leq 1.1$ fm.

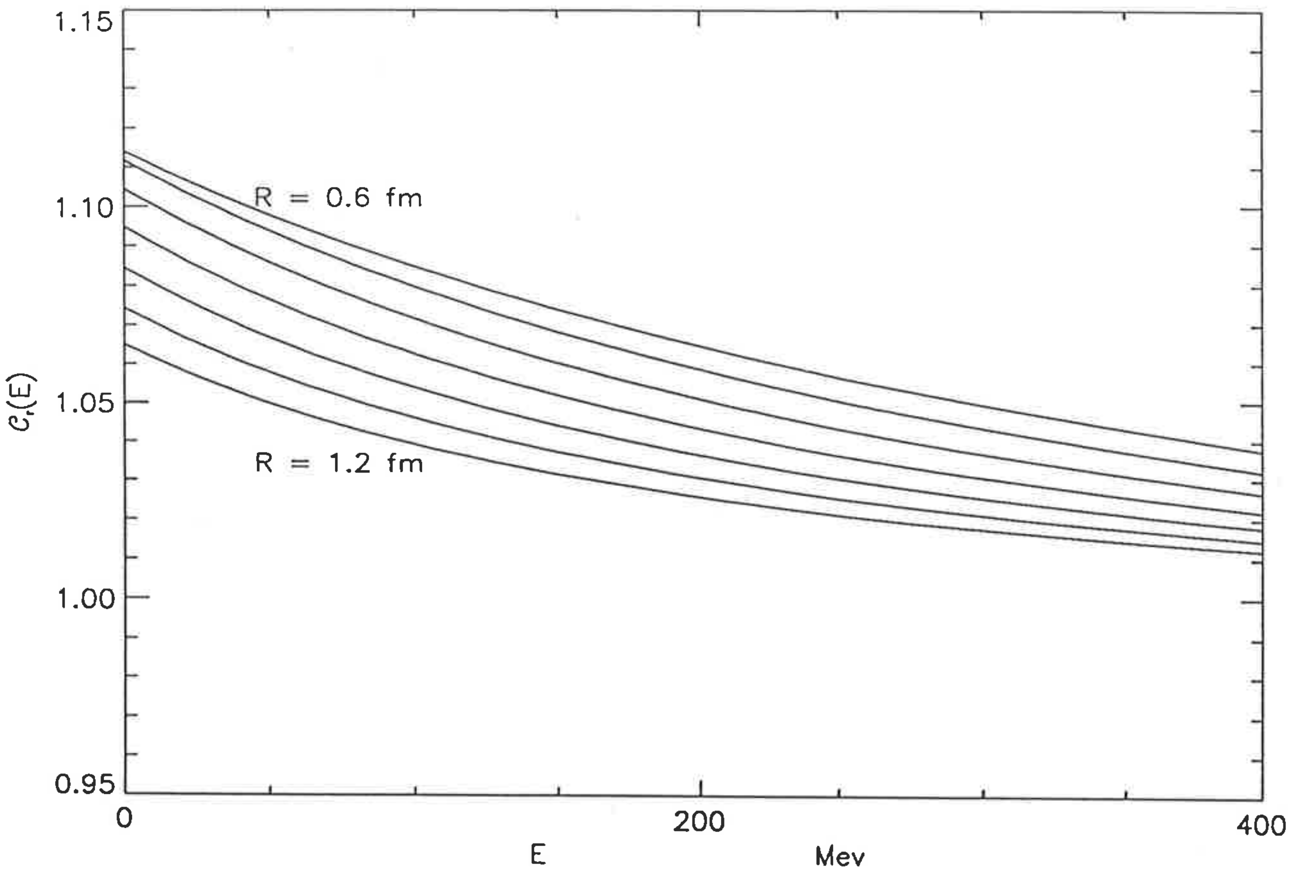


Figure 3.2: Energy dependence of the πNN and $\pi\Delta N$ coupling constants.

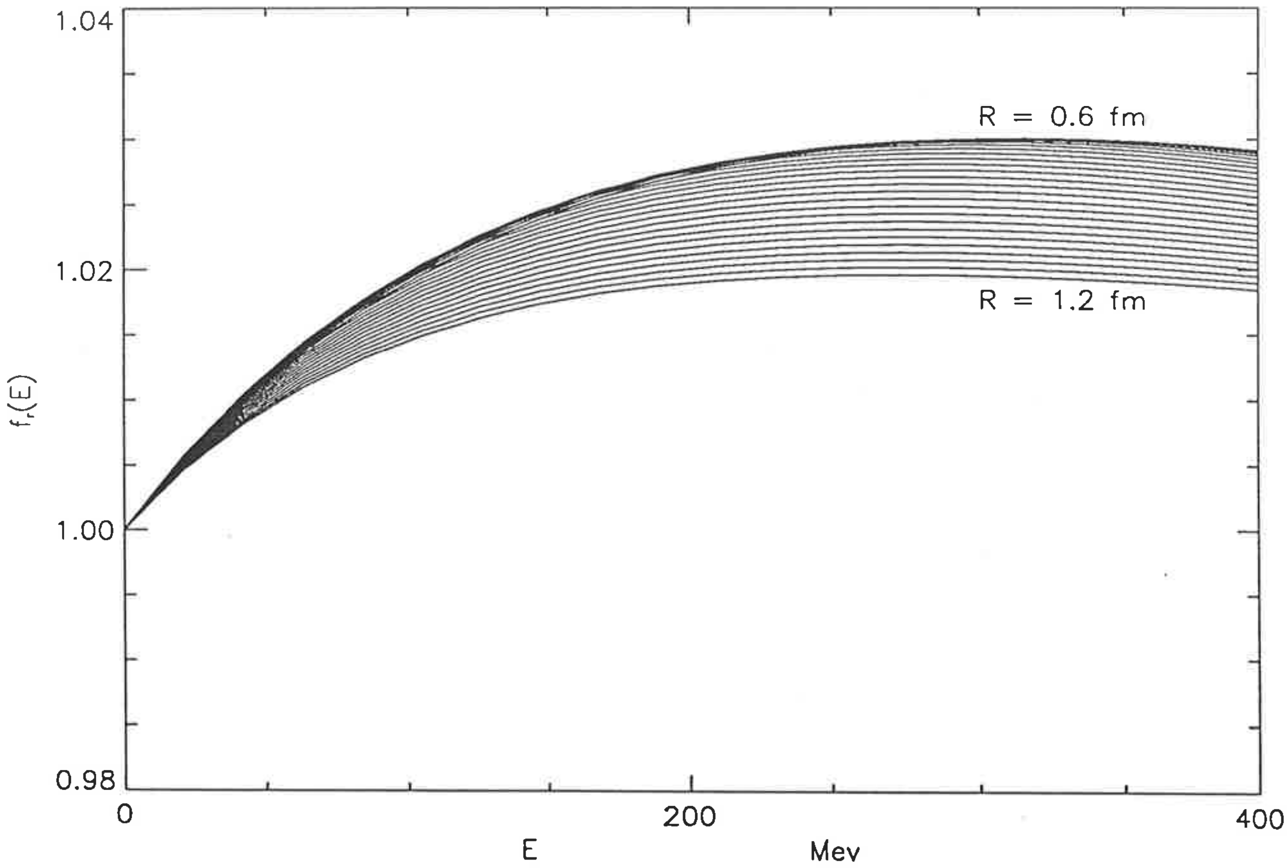


Figure 3.3: Energy dependence of the renormalized coupling constant.

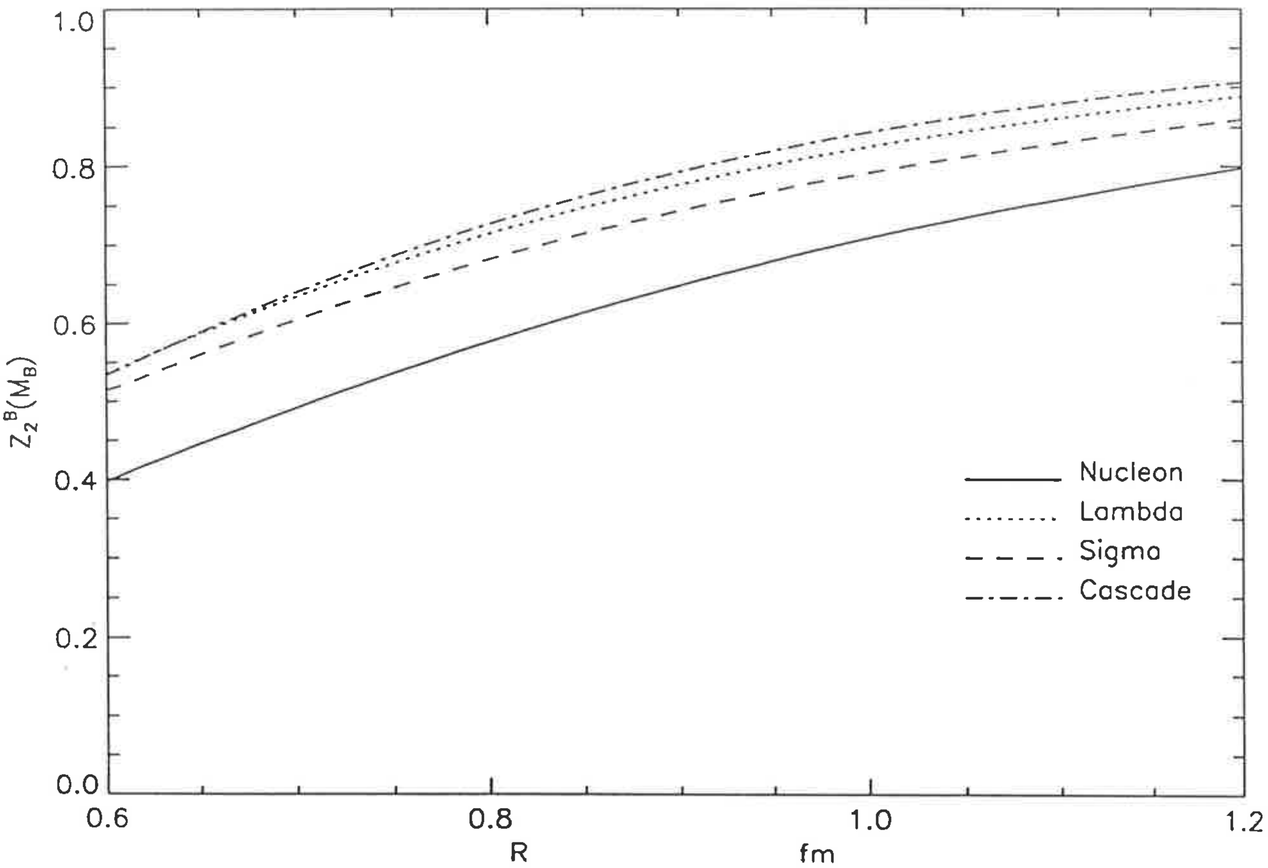


Figure 3.4: SU(3) Bare Bag Probabilities.

3.5 Quark Masses

Quark masses are calculated using sum rules [123,124]. These are found by equating the QCD calculation of a two point function to the asymptotic expression for the dispersion relation of the same function. In particular, one considers [19,125,126]

$$P(q^2) = i \int d^4x e^{iqx} \langle 0|T\partial_\mu A^\mu(x)(\partial_\nu A^\nu(0))^\dagger|0\rangle \tag{3.168}$$

One is able to equate these two representations if the average quark mass found in the dispersion relation expression is identified with the running quark mass defined by renormalization group equations.

More recent calculations have made use of the so-called "Laplace transform technique" [87,127] in which one considers

$$L(M) = \frac{1}{\pi} \int ds\, e^{-s/M} ImP(s) \tag{3.169}$$

with M variable. It is this method which is used by Gasser and Leutwyler in calculating their renormalized current quark masses. Details are found in reference [19]. Using the QCD scale parameter $\Lambda = 140$ MeV [125] Gasser and Leutwyler found, for $M = 1$ GeV, that the average running quark mass is $\hat{m}(1\text{GeV}) \approx 7 \pm 2$ MeV and the strange quark mass $m_s(1\text{GeV}) \approx 180 \pm 50$ MeV. The up and down current quark masses used in this thesis are taken from figure 2 on p. 121 of reference [19].

We now calculate the strange quark mass by considering the lambda-nucleon mass splitting. In this calculation we consider only the valence quark contribution. Higher order contributions, such as the hyperfine splitting due to one-gluon exchange, is not taken into account. As such, the following is only a first order calculation.

The above up and down quark masses are calculated at the 1 GeV scale. If we are to perform a bag model calculation, we must do so at the bag scale which we take to be 0.5 GeV [94,95]. At this scale, $\hat{m} \approx 12 \pm 3$ MeV.

Given the above average quark mass, we can calculate the non-strange quark contribution to the "average nucleon" mass. This contribution is

$$\frac{3}{2}(\frac{\alpha_u + \alpha_d - 4.08}{R}) \quad \text{MeV} \tag{3.170}$$

with

$$\alpha_{u,d} = [(\omega_{u,d}R)^2 + (m_{u,d}R)^2]^{1/2} \tag{3.171}$$

(we assume $m_d - m_u = 3$ MeV). At 0.6 fm, these non-zero quarks masses contribute 31.4 MeV to the nucleon mass, reducing to 16.2 MeV at 1.2 fm.

Now, the lambda-nucleon mass difference is 177.32 MeV. Hence, at 0.6 fm, the strange quark will contribute 146 MeV to the "average baryon" mass. Calculating the strange quark contribution $((\alpha_s - 2.04)/R)$ gives $m_s = 206$ MeV at 0.6 fm and $m_s = 281$ MeV at 1.2 fm. We show our results in figure 3.5.

We now compare these masses to the value of $m_s = 150 \pm 50$ MeV found by Gasser and Leutwyler. As we are performing our calculation at a smaller mass scale, for a comparison, we will need to increase our mass scale. However, in increasing the scale to 1 GeV, we reduce the average quark mass and must therefore increase $\overline{q}q$, leaving the product $\hat{m}\overline{q}q$ scale invariant. A rough estimate of m_s at 1 GeV is therefore $(7/12) \times 206$ giving a strange quark mass (at 0.6 fm) of approximately 120 MeV (the ratio 7/12 is the ratio $\hat{m}(1\text{GeV})/\hat{m}(0.5\text{ GeV})$). At 1.2 fm, we have $m_s \approx 164$ MeV. These values are within the range of values for the strange quark mass found by Gasser and Leutwyler (remembering that our calculation is a "first- order" approximation).

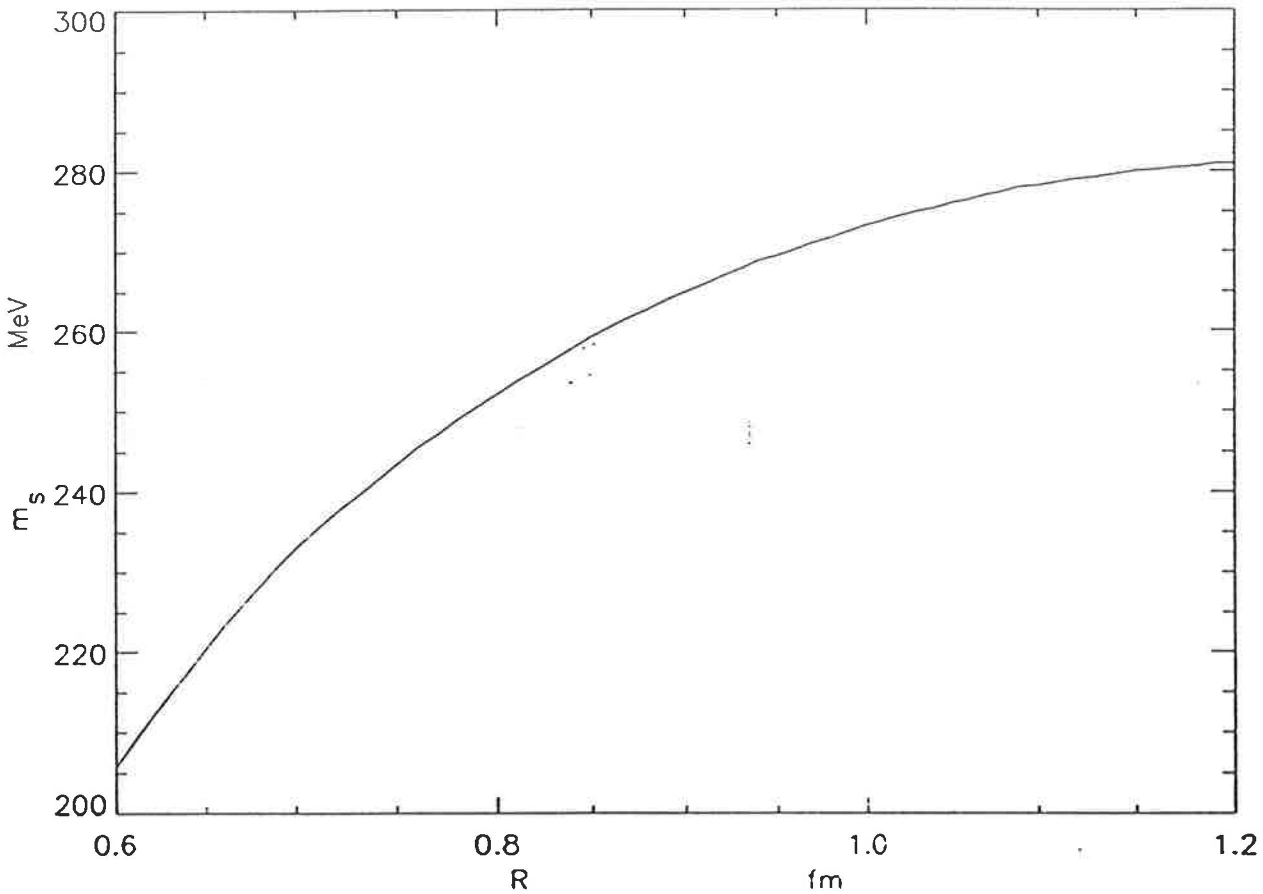


Figure 3.5: Strange Quark Mass

Note that although we give values of m_s in the range $0.6 \leq R \leq 1.2$ fm, because of the result stated on page 88, we are only interested in values within the range $0.8 \leq R \leq 1.1$ fm. This applies to all future calculations.

3.6 Bare Baryon Mass in the Chiral Limit

We are now in a position to calculate the bare baryon mass in the chiral limit. This mass is denoted M_0 and is defined in equation(2.85). In calculating M_0, we have used equation(2.98) and taken $\sigma_{\pi N}(0) = 40$ MeV. The results are given below.

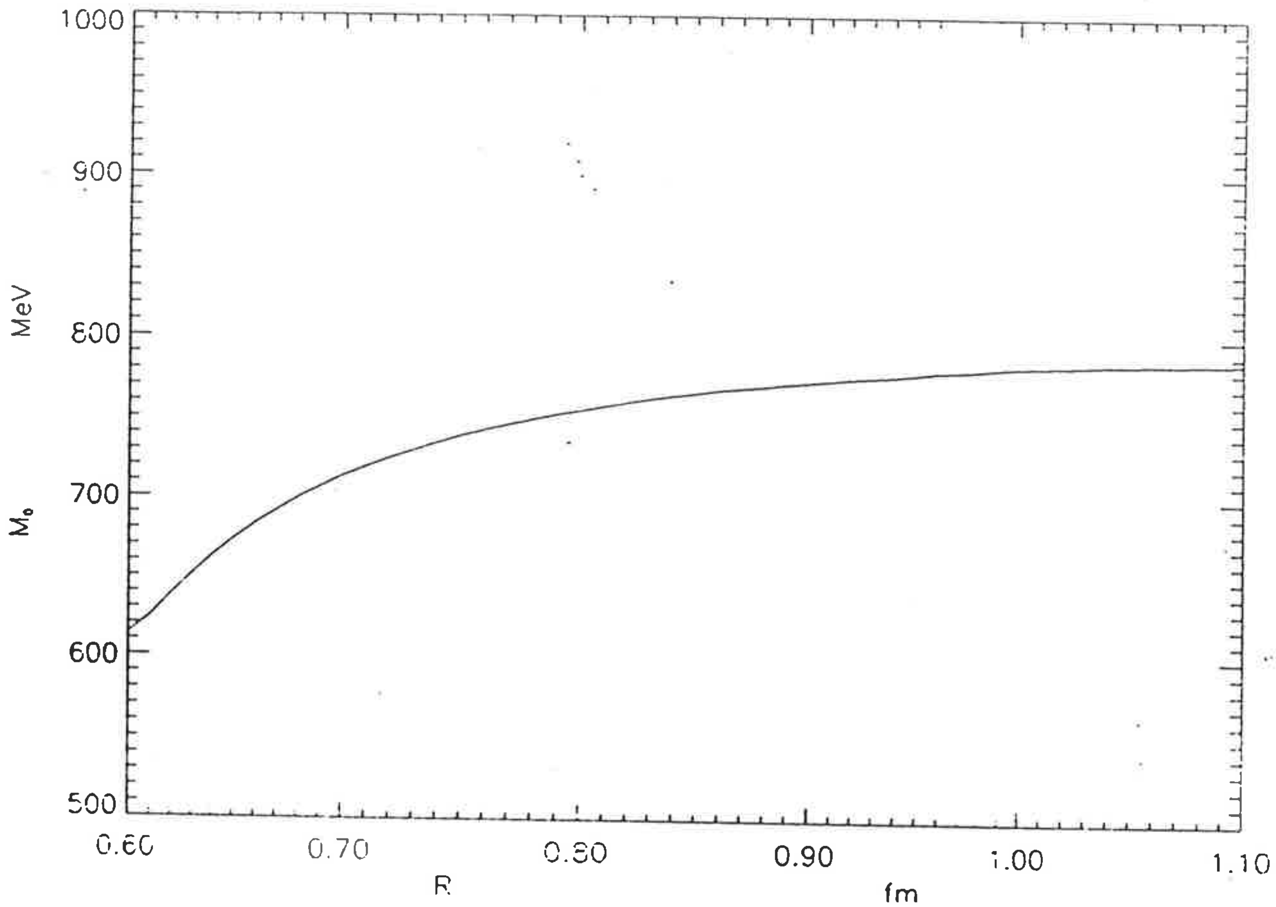


Figure 3.6: Bare Baryon Mass in the Chiral Limit

3.7 The Sigma Term in the CBM

In this section, we calculate quark and meson contributions to the sigma term. The meson cloud contribution is given by[3] $\sigma^{\phi}_{\pi N} = \frac{1}{2}M_\pi^2\langle N|\int d^3x \sum_\phi \phi^2|N\rangle$, with the physical states $|N\rangle$ written as the perturbation expansion given in equation(3.118).

It is straight foward to show that

$$\int d^3x\,\phi^2 = \int \frac{d^3k}{2\omega_j(k)}[2a_j^\dagger(k)a_j(k) + a_j^\dagger(k)a_j^\dagger(-k) + a_j(k)a_j(-k)] \quad (3.172)$$

the minus sign in the second and third terms ensuring conservation of momentum.

If we define a complete set of meson states

$$|\vec{q}, m; \vec{t}, n\rangle = a_m^\dagger(\vec{q})a_n^\dagger(\vec{t})|0\rangle \quad (3.173)$$

then, to second order, the physical nucleon state

$$|N\rangle = Z_2^N(M_N)^{\frac{1}{2}}\Big[|N_0\rangle + \sum_{B_0',m}|B_0';\vec{q},m\rangle\langle B_0';\vec{q},m|\frac{H_1}{M_N-\tilde{H}_0}|N_0\rangle$$
$$+\sum_{B_0',m,n}|B_0';\vec{q},m;\vec{t},n\rangle\langle B_0';\vec{q},m;\vec{t},n|\frac{H_1}{M_N-\tilde{H}_0}\Lambda\frac{H_1}{M_N-\tilde{H}_0}|N_0\rangle\Big]$$

with

$$\langle B_0';\vec{q},m|\frac{H_1}{M_N-\tilde{H}_0}|N_0\rangle = \int d^3q\,\langle B_0'|\frac{V_{0m}^\dagger(\vec{q})}{M_N-\tilde{H}_0-\omega_j(q)}|N_0\rangle \quad (3.174)$$

In order to calculate the third term on the RHS of the above equation, we will need to insert another complete set of states, giving the matrix element

$$\langle B_0' \;;\; \vec{q},m;\vec{t},n|\frac{H_1}{M_N-\tilde{H}_0}\Lambda\frac{H_1}{M_N-\tilde{H}_0}|N_0\rangle$$
$$= \sum_{B_0',p}\int d^3l\langle B_0';\vec{q},m;\vec{t},n|\frac{H_1}{M_N-\tilde{H}_0}|B_0';\vec{l},p\rangle$$
$$\times\,\langle B_0';\vec{l},p|\frac{H_1}{M_N-\tilde{H}_0}|N_0\rangle \quad (3.175)$$

[3]The pion, kaon and eta are weighted as shown in equation (2.97) on page 47.

With this, we find that (cf. sections 3.3 and 3.4)

$$\int d^3x \langle N|\phi^2|N\rangle$$

$$= Z_2^N(M_N)\sum_{B_0,j}\int \frac{d^3k}{M_j^2\omega_j(k)}\Big[\frac{\langle N_0'|V_{0j}(k)|B_0\rangle}{(M_{N'}-M_B-\omega_j(k))}\frac{\langle B_0|V_{0j}^\dagger(k)|N_0\rangle}{(M_N-M_B-\omega_j(k))}$$

$$+2\frac{\langle N_0'|V_{0j}^\dagger(k)|B_0\rangle}{(-2\omega_j(k))}\frac{\langle B_0|V_{0j}^\dagger(-k)|N_0\rangle}{(M_N-M_N-\omega_j(k))}$$

$$+2\frac{\langle N_0'|V_{0j}(k)|B_0\rangle}{(-2\omega_j(k))}\frac{\langle B_0|V_{0j}(-k)|N_0\rangle}{(M_N-M_N-\omega_j(k))}\Big] \qquad (3.176)$$

the last two terms being identical.

Rather than write out the whole expression, we can simplify the above by noting that contributions from the kaon and eta loops will be small. An explicit calculation shows that these loops contribute less than 1 MeV. We can therefore concentrate on the πNN and $\pi N\Delta$ loops

Figure 3.7: Loops dominating the sigma term

Thus, the pion contribution to the sigma term is

$$\sigma_{\pi N}^{\pi}(0) = \frac{M_\pi^2}{2}\int d^3x\langle N|\phi^2|N\rangle$$

$$= \frac{1}{2}\frac{Z_2^N(M_N)}{12\pi^2}\sum_B\int\frac{dk}{\omega_\pi^2}k^4u^2(kR)\Big\{\frac{\lambda_{\pi NB}^2}{(M_N-M_B-\omega_\pi(k))^2}$$

$$+C_B\frac{4\,\lambda_{\pi BN}\lambda_{\pi NB}}{(-2\omega_\pi(k))(M_N-M_B-\omega_\pi(k))}\Big\} \qquad (3.177)$$

with $B \in (N, \Delta)$, $C_N = 1$ and $C_\Delta = 2$ (these values are are consequence of the symmetry operation given by equation(3.150).

The valence quark contribution to the sigma term is calculated using equation(2.61). We do not assume this value is 26.5 MeV. Instead we use the running quark masses of Gasser and Leutwyler at the scale[4] of 0.5 GeV. Assuming an error of 3 MeV, we see that the average quark mass will be $\hat{m} = 12 \pm 3$ MeV at the bag scale, and hence we find that the valence quarks contribute approximately 17.5 ± 4.5 MeV.

If we consider only the first loop in figure 3.7, we find that $\sigma^{\pi}_{\pi N}(0) \approx 16$ MeV at 0.8 fm and $\sigma^{\pi}_{\pi N}(0) \approx 12$ MeV at 1.1 fm. Adding the valence quark contribution gives us $30 \leq \sigma_{\pi N}(0) \leq 34$ MeV (± 4.5 MeV). This compares well with the value obtained by Gasser [91] using chiral perturbation theory. Adding contributions from the second loop in figure 3.7 increases pion contributions by 7 to 10 MeV. This gives $37 \leq \sigma_{\pi N}(0) \leq 44$ MeV (± 4.5 MeV). This agrees well with the experimental estimate of $\sigma_{\pi N}(0)$ derived in chapter 2.2.2 (page 51), namely 45 ± 12 MeV - see equation (2.113). Results are given in figures 3.8 and 3.9.

Remember, from chapter 2, that the ICPT calculation of the sigma term assumes that the expression is analytic up to, and including, terms of order $M_\pi^4 ln M_\pi^2$. This was done by removing terms of order M_π^3. The LNAC were found by considering the derivative of the nucleon mass equation(2.93). The contributions from terms of order $M_\pi^4 ln M_\pi^2$ are omitted as they are believed to be dominated by LNAC. In the ICPT calculation, it is assumed that these higher order terms are contained in the first loop of figure 3.7.

We can now relate this expression to the work of Gasser et.al. (remember

[4] At these energies, it is usual to use the monopole representation of the form factor when calculating the cutoff mass. It can be shown that $\Lambda_\pi \approx \sqrt{10}/R$.

that the sigma term is scale independent). It is not difficult to show that the first loop in figure 3.7 has a leading nonanalytic piece of order M_π^3. However, the leading nonanalytic term in the second loop is of order $M_\pi^4 ln M_\pi^2$. This is because in the chiral limit, $M_N \neq M_\Delta$. As such, the second loop represents a higher order contribution and is therefore ignored by Gasser and others in the calculation of $\sigma_{\pi N}(0)$ [128] (they do, however, include the effects of the Δ-resonance in their dispersion relation extrapolations of $\Sigma_{\pi N}(0)$ to $\Sigma_{\pi N}(2M_\pi^2)$).

The second loop of figure 3.7 is not considered by ICPT as this theory assumes higher order contributions are contained in the first loop. We feel that this is incorrect and that ICPT therefore neglects contributions from the Delta resonance. By adding these contributions we find that it is no longer necessary to assume that the nucleon has a large strange quark component [129].

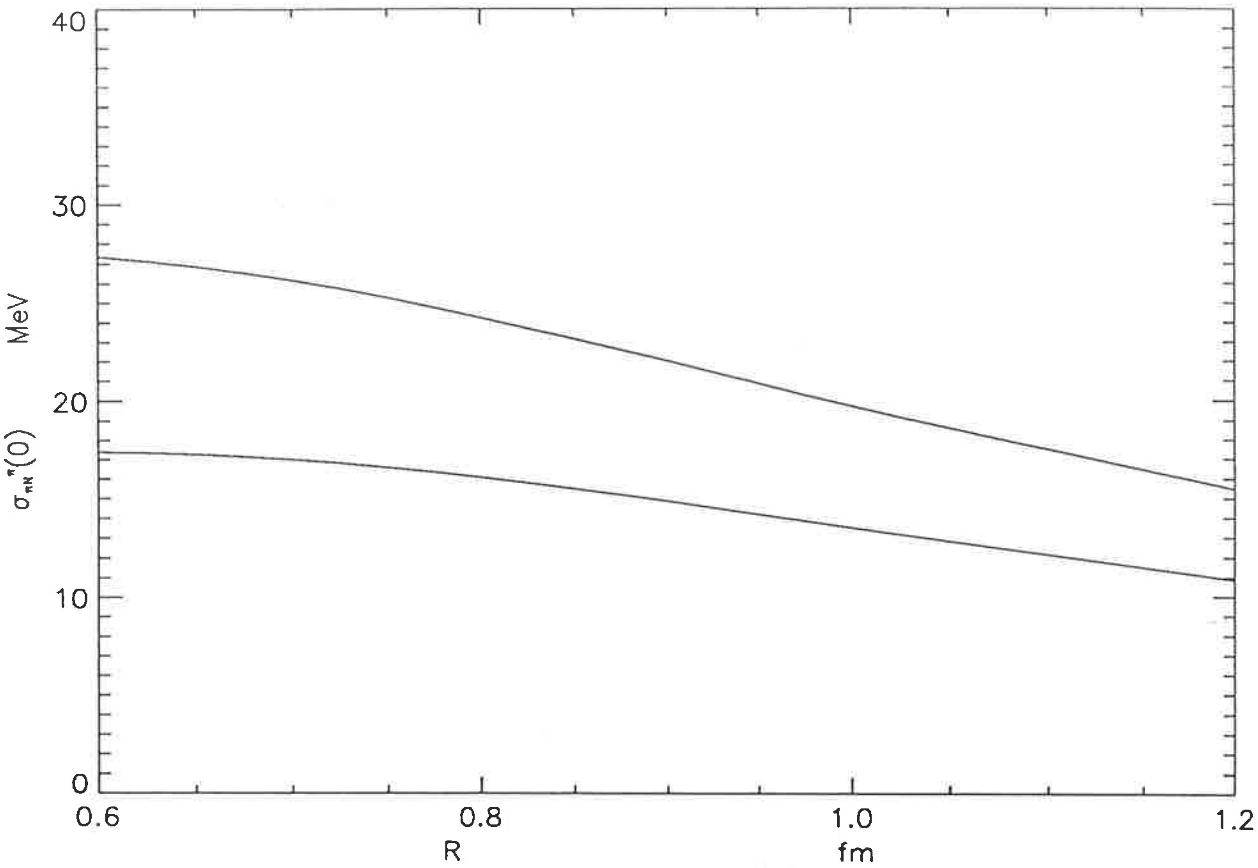


Figure 3.8: Pion contribution to the sigma term.

The above figure shows the pion contribution to the sigma term. The top curve shows the contribution from both loops shown in figure 3.7 on page 94. The lower curve shows contributions from the πNN loop only.

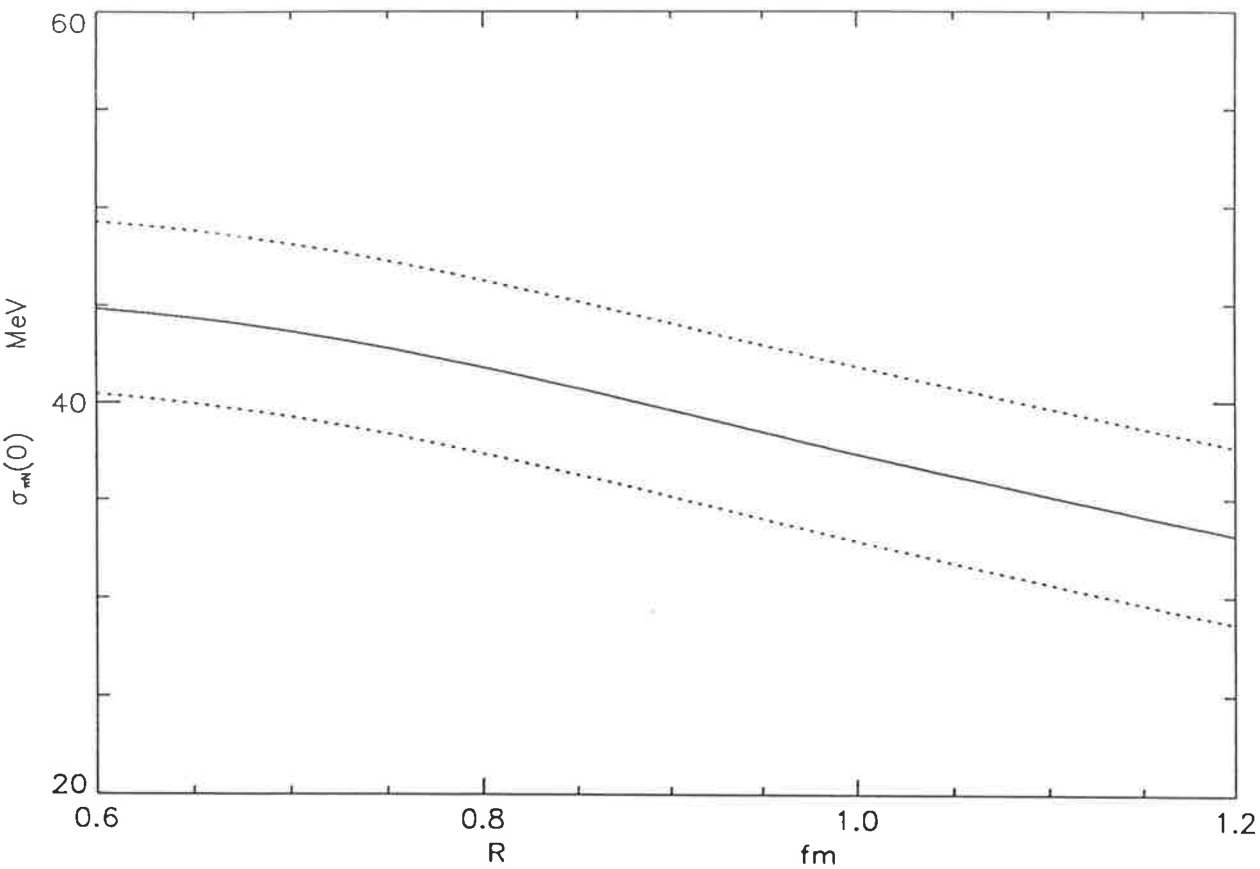


Figure 3.9: The pion-nucleon sigma term with $\hat{m} = 12$ MeV.

The above figure shows the valence quark plus pion contribution to the sigma term. The solid curve is for an average quark mass of 12 MeV. The dashed curves represent our error.

Chapter 4

Conclusions

In chapter 2 we saw that the theoretical value of the pion-nucleon sigma term varied from 26 MeV to 45 MeV. The former value being the valence quark contribution to the sigma term. The latter value may be derived from the former by including LNAC, estimating higher order corrections and assuming the nucleon has a small strange quark component. The experimental value was found to be $\Sigma_{\pi N}(2M_\pi^2) \approx 60 \pm 12$ MeV.

In chapter 3 we considered this problem in the context of the cloudy bag model. We made an explicit calculation of both the quark and meson contributions to the sigma term. Assuming a bag scale of 0.5 GeV, so that our current quark masses are 12 ± 3 MeV, we found the quark contribution to be 17.5 ± 4.5 MeV. The meson contribiution was found to be dominated by pion loops, in particular, πNN and $\pi N\Delta$. Contributions from the kaon and eta loops came to less than 1 MeV. If we consider only the πNN loop, we find the meson contribution, for $0.8 \leq R \leq 1.1$ fm, to be $12 \leq \sigma_{\pi N}^{\pi} \leq 16$ MeV or, adding quark contributions, $30 \leq \sigma_{\pi N}(0) \leq 34$ MeV. This agrees well with the value obtained using chiral perturbation theory.

By adding the $\pi N\Delta$ loop, representing higher order contributions, we found that the meson contribution increased to $20 \leq \sigma^{\pi}_{\pi N} \leq 26$ MeV or, adding quark contributions, $37 \leq \sigma_{\pi N}(0) \leq 44$ MeV for the same range of the bag radius. We conclude that there is no significant strange quark component in the nucleon, and that chiral perturbation theory underestimates higher order contributions by 7 to 10MeV (due to omitting contributions from the Delta resonance). From this we conclude that contributions from the Delta resonance to the sigma term are important, and that Improved Chiral Perturbation Theory neglects such terms.

Appendix A

Kinematics

The following appendicies are included in an attempt to make chapter 2 complete. We do not go into the following in any great detail. We leave that to the references given in the text.

The process given by equation(2.7) is shown in the following diagram with incoming 4-momentum [64] $p_\mu = (E_p, \vec{p})$, $q_\mu = (E_q, \vec{q})$ and outgoing 4-momentum $p'_\mu = (E_{p'}, \vec{p'})$, $q'_\mu = (E_{q'}, \vec{q'})$. Note, $p^\mu = (-E_p, \vec{p})$.

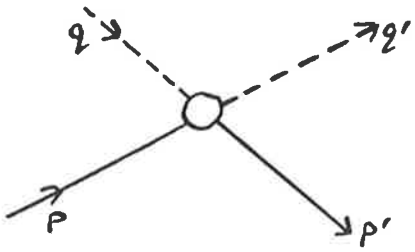


Figure A.1: Off-shell baryon-meson scattering

For this process, we can define an invariant energy

$$\begin{aligned} s &= -(p+q)^2 = -(p'+q')^2 \\ &= -[-E_p^2 + \vec{p}^2 - E_pE_q + \vec{p}\cdot\vec{q} - E_qE_p + \vec{q}\cdot\vec{p} - E_q^2 + \vec{q}^2] \\ &= (E_p+E_q)^2 - (\vec{p}+\vec{q})^2 \end{aligned} \tag{A.1}$$

an invariant momentum transfer

$$\begin{aligned} t &= -(p-p')^2 = -(q-q')^2 \\ &= E_p^2 + E_{p'}^2 - 2E_pE_{p'} - \vec{p}^2 - \vec{p'}^2 + 2pp'\cos\theta \\ &= (E_p - E_{p'})^2 - (\vec{p}-\vec{p'})^2 \end{aligned} \tag{A.2}$$

and an invariant exchange momentum transfer

$$\begin{aligned} u &= -(p-q')^2 = -(q-p')^2 \\ &= (E_p - E_{q'})^2 - (\vec{p}-\vec{q'})^2 \end{aligned} \tag{A.3}$$

The three invariants s, t and u (known as the Mandelstam variables) are related by

$$s + t + u = M_p^2 + M_q^2 + M_{p'}^2 + M_{q'}^2 \tag{A.4}$$

so there are only two independent variables.

Consider the process $\pi^+ + p \rightarrow \pi^+ + p$. In the center of mass (c.m.) frame, $\vec{p}+\vec{q} = \vec{p'} + \vec{q'} = 0$, and hence

$$\sqrt{s} = E_p + E_\pi \tag{A.5}$$

is the energy in the c.m. system.

In this system, the magnitudes of the 3-momenta are equal, i.e., $|q'| = |q| = |p'| = |p| = q_c$, defining q_c to be the c.m. 3-momentum. Therefore $E_p = (M_N^2 + q_c^2)^{\frac{1}{2}}$ is the nucleon energy and $E_\pi = (M_\pi^2 + q_c^2)^{\frac{1}{2}}$ is the pion energy in the c.m. system.

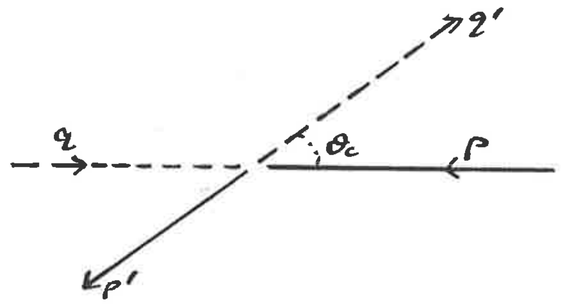


Figure A.2: Center of mass system

From the above we find that ($E_p = E_{p'}$)

$$\begin{aligned} t &= -2q_c^2 + 2q_c^2 \cos_c \theta \\ &= -2q_c^2(1 - \cos_c \theta) \end{aligned} \tag{A.6}$$

with θ_c the c.m. scattering angle. From this we see that $-4q_c^2 \leq t \leq 0$. For physical processes, $-4M_\pi^2 \leq t \leq 0$.

For pion-nucleon scattering in the lab frame,

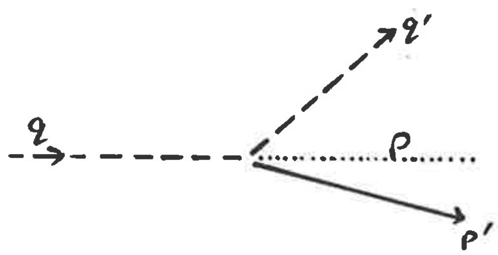


Figure A.3: Lab frame

$$p_\mu = (M_N, 0),\ p'_\mu = (\omega_N, \vec{k}_N)$$
$$q_\mu = (\omega, \vec{k}_L),\ q'_\mu = (\omega_\pi, \vec{k}_\pi)$$

We have defined $E = M_N$ and

$$\begin{aligned} \omega &= (M_\pi^2 + \vec{k}_L^2)^{\frac{1}{2}} \\ \omega_N &= (M_N^2 + \vec{k}_N^2)^{\frac{1}{2}} \\ \omega_\pi &= (M_\pi^2 + \vec{k}_\pi^2)^{\frac{1}{2}} \end{aligned} \tag{A.7}$$

In this frame,

$$s = M_N^2 + M_\pi^2 + 2\omega M_N \tag{A.8}$$

Using this and equation(A.4) we get

$$\begin{aligned} \frac{s-u}{4M_N} &= \frac{2s - 2M_N^2 - 2M_\pi^2 + t}{4M_N} \\ &= \omega + t/4M_N \end{aligned} \tag{A.9}$$

In terms of the 4-momenta

$$s - u = -2(p_\mu + p'_\mu)q^\mu \tag{A.10}$$

and defining the crossing variable

$$\nu = \frac{-(p_\mu + p'_\mu)q^\mu}{2M_N} \tag{A.11}$$

we have

$$\nu = \frac{s-u}{4M_N} \tag{A.12}$$

Alternatively, we can define CGLN variables [130]

$$\begin{aligned} P_\mu &= (\frac{p_\mu + p'_\mu}{2}) \quad , \quad Q_\mu = (\frac{q_\mu + q'_\mu}{2}) \\ s &= -(P+Q)^2 \quad , \quad t = -2(q_\mu - q'_\mu)^2 = 4\kappa^2 \\ u &= -(P-Q)^2 \end{aligned} \tag{A.13}$$

and the crossing variable becomes

$$\nu = \frac{-P \cdot Q}{M_N} \tag{A.14}$$

In the above, we considered the process

$$\pi^+ + p \to \pi^+ + p \tag{A.15}$$

which is called an s-channel process. In this channel, for physical processes

$$\begin{aligned} -4M_\pi^2 \leq \ & t \ \leq 0 \\ & s \ \geq (M_N + M_\pi)^2 \\ & u \ \leq (M_N - M_\pi)^2 \end{aligned} \tag{A.16}$$

There is, however, a t-channel process

$$\pi + \overline{\pi} \to p + \overline{p} \tag{A.17}$$

in which t is defined to be the square of the energy in this channel and

$$\begin{aligned} t &\geq 4M_N^2 \\ s &\leq -(M_N^2 - M_\pi^2) \\ u &\leq -(M_N^2 - M_\pi^2) \end{aligned} \tag{A.18}$$

for physical processes.

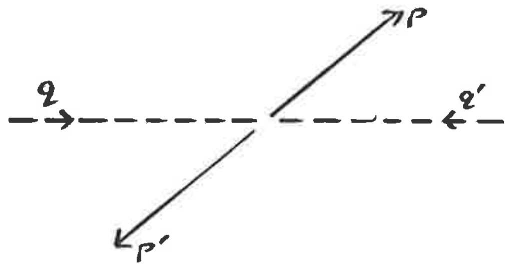
In the t-channel

$$\begin{aligned} p_\mu = (-E_p, -\vec{p}) \quad &, \quad p'_\mu = (E_{p'}, \vec{p'}) \\ q_\mu = (E_q, \vec{q}) \quad &, \quad q'_\mu = (-E_{q'}, -\vec{q'}) \end{aligned} \tag{A.19}$$

so that

$$\begin{aligned} s &= (E_q - E_p)^2 - (\vec{q} - \vec{p})^2 \\ t &= (E_p + E_{p'})^2 - (\vec{p} + \vec{p'})^2 \\ u &= (E_p - E_{q'})^2 - (\vec{p} - \vec{q'})^2 \end{aligned} \tag{A.20}$$

In the c.m. system $\vec{q}+\vec{q'}=\vec{p}+\vec{p'}=0$, $E_q=E_{q'}$ and $E_p=E_{p'}$. The following diagram shows the t-channel process considered here



Hence

$$\begin{aligned} s &= -q_t^2 - p_t^2 + 2q_t p_t \cos\theta_t \\ t &= 4(M_N^2 + p_t^2) = 4(M_\pi^2 + q_t^2) \\ u &= -q_t^2 - p_t^2 - 2q_t p_t \cos\theta_t \end{aligned} \tag{A.21}$$

with q_t the c.m. 3-momentum in the t-channel and θ_t the t-channel scattering angle. From this we see that the total energy of the system is $\sqrt{t}$.

In this channel, the crossing variable is

$$\nu = \frac{p_t q_t \cos\theta_t}{M_N} \tag{A.22}$$

We can also have

$$\pi^- + p \rightarrow \pi^- + p \tag{A.23}$$

which is a u-channel process, with

$$\begin{aligned} u &\geq (M_N + M_\pi)^2 \\ s &\leq (M_N - M_\pi)^2 \\ t &\leq 0 \end{aligned} \tag{A.24}$$

for physical processes.

To obtain this channel,

$$q_\mu \leftrightarrow -q'_\mu \tag{A.25}$$

and in the c.m. system we find that

$$u = (E_p + E_\pi)^2 \tag{A.26}$$

from which we define $\sqrt{u}$ to be the total energy in the u-channel.

The crossing variable is

$$\nu = \frac{u - s}{4M_N} \tag{A.27}$$

Note that if

$$\nu \leftrightarrow -\nu \tag{A.28}$$

then

$$s \leftrightarrow u \tag{A.29}$$

and

$$q_\mu \leftrightarrow -q'_\mu \tag{A.30}$$

That is, the u-channel is the crossed s-channel.

Appendix B

The Transition Amplitude

We define an incoming state consisting of a pion (q, α) and nucleon (p, s) as

$$|q, \alpha; p, s\, in\rangle = a^{\dagger}_{in,\alpha}(q) a^{\dagger}_{in,s}(p)|0\rangle \tag{B.1}$$

with $a^{\dagger}_{in,\alpha}(q)$ creating a pion of energy-momentum q, charge states α and $a^{\dagger}_{in,s}(p)$ creating a nucleon of energy-momentum p and charge-spin states s.

Final states are denoted

$$|q', \alpha'; p', s'\, out\rangle = a^{\dagger}_{out,\alpha'}(q') a^{\dagger}_{out,s'}(p')|0\rangle \tag{B.2}$$

and the amplitude for scattering from the initial to final state is

$$\langle q', \alpha'; p', s'\, out|q, \alpha; p, s\, in\rangle \tag{B.3}$$

We can relate final and initial states through the S-matrix

$$\langle q', \alpha'; p', s'\, out| = \langle q', \alpha'; p', s'\, in|\, S \tag{B.4}$$

and the scattering amplitude becomes

$$\langle q', \alpha'; p', s'\, in|S|q, \alpha; p, s\, in\rangle \tag{B.5}$$

The goal of this appendix is to write the transition amplitude in the form of equation(2.10). To do this we require the following integral representations of the creation and annihilation operators. Writing the meson field as a plane wave expansion (equation(3.38)), it is straight forward to prove that

$$\begin{aligned} a_j^\dagger(k) &= \frac{i}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} \int d^3x \; \phi_j(x) \; \overleftrightarrow{\partial_0} \; e^{ikx} \\ a_j(k) &= \frac{-i}{[2\omega_j(k)(2\pi)^3]^{\frac{1}{2}}} \int d^3x \; \phi_j(x) \; \overleftrightarrow{\partial_0} \; e^{-ikx} \end{aligned} \tag{B.6}$$

assuming $a_j^\dagger(k)$ and $a_j(k)$ are time independent.

Defining

$$f_q(\vec{x},t) = \frac{e^{i(\vec{q}\cdot\vec{x}-\omega_q t)}}{[2\omega_j(q)(2\pi)^3]^{\frac{1}{2}}} \tag{B.7}$$

we can write

$$\begin{aligned} |q',\alpha';p',s'\,out\rangle &= a^\dagger_{out,\alpha'}(q')|p',s'\,out\rangle \\ &= i\int_{(t)} d^3x \; \phi_{out} \; \overleftrightarrow{\partial_0} \; f_{q'}(\vec{x},t)|p',s'\,out\rangle \end{aligned} \tag{B.8}$$

at any time t. As single-particle incoming and outgoing states are identical, we can write $|p',s'\,out\rangle$ as $|p',s'\rangle$ and by letting $\phi_{\alpha'}(x) \to \phi_{out}(x)$ as $t \to \infty$, we find that

$$|q',\alpha';p',s'\,out\rangle = i\int_{t=\infty} d^3x \; \phi_{\alpha'}(\vec{x},t) \; \overleftrightarrow{\partial_0} \; f_q(\vec{x},t)|p',s'\rangle \tag{B.9}$$

Using the relation

$$\int_{\infty} d^3x \; A(x) = \int_{-\infty} d^3x \; A(x) + \int_{-\infty}^{\infty} d^4x \; \frac{\partial A(x)}{\partial t} \tag{B.10}$$

taking the conjugate of (B.9), noting that $f_q^*(\vec{x},t)$ satisfies the Klein-Gordan equation

$$(\Box^2 - M_\pi^2)f_q^*(\vec{x},t) = (-\partial_0^2 + \nabla^2 - M_\pi^2)f_q^*(\vec{x},t) = 0 \tag{B.11}$$

performing double integration by parts and assuming the total divergence vanishes at infinity, we have

$$\langle q', \alpha'; p', s'\, in|\, S\, |q, \alpha; p, s\, in\rangle$$
$$= I - i\int_{-\infty}^{\infty} d^4\, x\, (\Box^2 - M_\pi^2)\langle p', s'|f_q^*(\vec{x}, t)\phi_{\alpha'}(x)|q, \alpha; p, s\, in\rangle \quad (B.12)$$

with the operator $\Box^2$ acting only on $\phi_{\alpha'}(x)$ and

$$I = (2\pi)^4\delta_{\alpha'\alpha}\delta^4(q' - q)(2\pi)^4\delta_{s's}\delta^4(p' - p) \quad (B.13)$$

The procedure used to arrive at the above equation is called "contraction" or "reduction". The process is continued by contracting the incoming meson. Rather than follow the above steps, a simplified expression is obtained by moving the incoming field $\phi_\alpha(y)$ to the left of $\phi_{\alpha'}(x)$. We can do this with the help of the time ordering operator

$$\begin{aligned} T[a(x)b(y)] &= a(x)b(y) \text{ if } t_x > t_y \\ &= b(y)a(x) \text{ if } t_y > t_x \end{aligned} \quad (B.14)$$

assuming $a(x)$ and $b(y)$ commute (the second line will have a minus sign if they do not commute).

Using (B.10) and noting that at $t = \infty, t_x > t_y$, we have

$$\langle p', s'|\phi_{\alpha'}(x)|\vec{q}, \alpha; p, s\, in\rangle$$
$$= i\int_{-\infty} d^3y\langle p', s'|\phi_{\alpha'}(x)[\phi_\alpha(\vec{y}, t)\, \overleftrightarrow{\partial_0}\, f_q(y)]|p, s\rangle$$
$$= i\int_{\infty} d^3y\langle p', s'|[\phi_\alpha(\vec{y}, t)\, \overleftrightarrow{\partial_0}\, f_q(\vec{x}, t)]\phi_{\alpha'}(x)|p, s\rangle$$
$$-i\int_{-\infty}^{\infty} d^4y\frac{\partial}{\partial t}\{\langle p', s'|T[\phi_{\alpha'}(x)\phi_\alpha(y)]\, \overleftrightarrow{\partial_0}\, f_q(y)|p, s\rangle\} \quad (B.15)$$

By definition, the first term contains

$$a_{out,\alpha}(q)|p', s'\rangle \quad (B.16)$$

which vanishes. By writing

$$T[a(x)b(y)] = \theta(t_x - t_y)a(x)b(y) + \theta(t_y - t_x)b(y)a(x) \qquad \text{(B.17)}$$

and using

$$\frac{\partial}{\partial t}\theta(t) = \delta(t) \qquad \text{(B.18)}$$

we find that, upon expanding the second line on the right of (B.15),

$$\begin{aligned} &i\int_{-\infty} d^3y\, \langle p', s'|\phi_{\alpha'}(x)[\phi_\alpha(\vec{y},t)\, \overleftrightarrow{\partial_0}\, f_q(\vec{x},t)]|p,s\rangle \\ &= -\, i\int_0 d^3y\, f_q(y)\langle p', s'|[\partial_0\phi_\alpha(y), \phi_{\alpha'}(x)]|p,s\rangle \\ &-\, i\int_{-\infty}^{\infty} d^4y\, (\Box_y^2 - M_\pi^2)\{\langle p', s'|f_q(\vec{x},t)T[\phi_{\alpha'}(x)\phi_\alpha(\vec{y},t)]|p,s\rangle \qquad \text{(B.19)} \end{aligned}$$

The commutator term will in general be a polynomial of finite order q. It will be dropped for now.

Thus, substituting (B.19) into (B.15) and (B.12), we have

$$\begin{aligned} &\langle q', \alpha'; p', s'\, in|\, S\, |q, \alpha; p, s\, in\rangle \\ &= I - \int_{-\infty}^{\infty} d^4x\, d^4y\, f_{q'}^*(x)f_q(y)(\Box_x^2 - M_\pi^2)(\Box_y^2 - M_\pi^2)\langle p', s'|T[\phi_{\alpha'}(x)\phi_\alpha(y)]|p,s\rangle \end{aligned}$$

It is possible to write the Lorentz invariant S-matrix in the form

$$\begin{aligned} &\langle q', \alpha'; p', s'\, in|S|q, \alpha; p, s\, in\rangle \\ &= I + i(2\pi)^4\delta^4(p_f - p_i)\left(\frac{M_{p'}M_p}{E_{p'}E_p}\right)^{\frac{1}{2}} \frac{\langle q', \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle}{[2\omega_q(2\pi)^3]^{\frac{1}{2}}[2\omega_{q'}(2\pi)^3]^{\frac{1}{2}}} \end{aligned}$$

from which we define the transition amplitude

$$\begin{aligned} &\left(\frac{M_{p'}M_p}{E_{p'}E_p[2\omega_q(2\pi)^3]^{\frac{1}{2}}[2\omega_{q'}(2\pi)^3]^{\frac{1}{2}}}\right)^{\frac{1}{2}}\delta^4(p_f - p_i)\langle q', \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle \\ &= i(q^2 - M_\pi^2)(q'^2 - M_\pi^2)\int_{-\infty}^{\infty} d^4x d^4y\, f_{q'}^*(x)f_q(y)\langle p', s'|T[\phi_{\alpha'}(x)\phi_\alpha(y)]|p,s\rangle \end{aligned}$$

(as the Fourier transform of $\Box_x^2 Tf(x) = (q^2 - M_\pi^2)\times$ Fourier transform of $Tf(x)$).

If we make use of

$$F(x+a) = e^{-iPx}F(a)e^{iPx} \tag{B.20}$$

with

$$\begin{aligned} \langle p', s'|e^{-iPx} &= e^{-ip'x}\langle p', s'| \\ e^{iPx}|p, s\rangle &= |p, s\rangle e^{ipx} \end{aligned} \tag{B.21}$$

we find that the transition amplitude becomes

$$\begin{aligned} &\langle q', \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle \\ &= i\,(q^2 - M_\pi^2)(q'^2 - M_\pi^2)(\frac{E_{p'}E_p}{M_{p'}M_p})^{\frac{1}{2}} \int_{-\infty}^{\infty} d^4z\, e^{-iq'z}\langle p', s'|T[\phi_{\alpha'}(z)\phi_\alpha(0)]|p, s\rangle \end{aligned}$$

from which we obtain equation(2.10) up to a normalization factor.

If we had contracted the incoming meson first, then the outgoing meson, we would have derived the following

$$\begin{aligned} &\langle q', \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle \\ &= i\,(q^2 - M_\pi^2)(q'^2 - M_\pi^2)(\frac{E_{p'}E_p}{M_{p'}M_p})^{\frac{1}{2}} \int_{-\infty}^{\infty} d^4z\, e^{iqz}\langle p', s'|T[\phi_\alpha(z)\phi_{\alpha'}(0)]|p, s\rangle \end{aligned}$$

From these two equations we find the Gell-Mann - Goldberger crossing symmetry law

$$\langle q', \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle = \langle -q, \alpha; p', s'\, in|T| - q', \alpha'; p, s\, in\rangle \tag{B.22}$$

i.e., if the nucleon states are unaltered, the amplitudes transform into each other with

$$\begin{aligned} \alpha &\leftrightarrow \alpha' \\ q &\leftrightarrow -q' \end{aligned} \tag{B.23}$$

An alternative formulation can be found if we make use of the source equation

$$(\Box^2 - M_\pi^2)\phi_\alpha(x) = -j_\alpha(x) \tag{B.24}$$

where $j_\alpha(x)$ has the quantum numbers of the pion.

It is found that

$$\begin{aligned}&\langle q' \ , \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle \\ &\quad = i\Big(\frac{E_{p'}E_p}{M_{p'}M_p}\Big)^{\frac{1}{2}} \int_{-\infty}^{\infty} d^4z\, e^{iqz}\langle p', s'|T[j_\alpha(z)j_{\alpha'}(0)]|p, s\rangle\end{aligned} \tag{B.25}$$

We could also have used the retarded commutator

$$R[A(x)B(y)] = \theta(x_0 - y_0)[A(x), B(y)] \tag{B.26}$$

which vanishes for $x_0 < y_0$, in place of the time ordered operator.

It can be found that

$$\begin{aligned}&\langle q' \ , \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle \\ &\quad = -i\Big(\frac{E_{p'}E_p}{M_{p'}M_p}\Big)^{\frac{1}{2}} \int_{-\infty}^{\infty} d^4z\, e^{iqz}\langle p', s'|R[j_\alpha(z)j_{\alpha'}(0)]|p, s\rangle\end{aligned} \tag{B.27}$$

plus a polynomial of order q.

Appendix C

The Pion - Nucleon Interaction

The pion has isospin 1, and the nucleon has isospin 1/2. Therefore, we can form states of total isospin 1/2 and 3/2. If these states have scattering amplitudes $T_{1/2}$ and $T_{3/2}$, then [64]

$$T_{\beta\alpha} = T_{1/2}(P_{1/2})_{\beta\alpha} + T_{3/2}(P_{3/2})_{\beta\alpha} \tag{C.1}$$

with $P_{1/2}$ and $P_{3/2}$ projection operators which we now find.

Define a total isospin operator

$$\vec{I} = \vec{T} + \frac{1}{2}\vec{\tau} \tag{C.2}$$

where $\vec{\tau}$ are the Pauli matrices, and

$$(T_k)_{ij} = -i\varepsilon_{ijk} \tag{C.3}$$

generate the regular representation of $SU(2)$.

Squaring (C.2) and rearranging, we have

$$\vec{T}\cdot\vec{\tau} = I^2 - T^2 - \frac{1}{4}\tau^2 \tag{C.4}$$

and hence

$$\vec{T}\cdot\vec{\tau}|it\tau\rangle = \big(i(i+1) - t(t+1) - \frac{3}{4}\big)|it\tau\rangle \tag{C.5}$$

Thus

$$\begin{aligned} \vec{T}\cdot\vec{\tau}|3/2,1,1/2\rangle &= |3/2,1,1/2\rangle \equiv |3/2\rangle \\ \vec{T}\cdot\vec{\tau}|1/2,1,1/2\rangle &= -2|1/2,1,1/2\rangle \equiv -2|3/2\rangle \end{aligned} \tag{C.6}$$

The projection operators we wish to define must satisfy the following

$$\begin{aligned} P_i|i\rangle &= |i\rangle \\ P_i|j\rangle &= 0 \;\; i \neq j \\ P_i^2 &= P_i \\ \sum_i P_i &= 1 \\ P_iP_j &= 0 \;\; i \neq j \end{aligned} \tag{C.7}$$

where i and j refer to the total isospin of the state. The following operators satisfy the above

$$P_{1/2} = \frac{1-\vec{T}\cdot\vec{\tau}}{3} \;,\; P_{3/2} = \frac{2+\vec{T}\cdot\vec{\tau}}{3} \tag{C.8}$$

Therefore, (C.1) becomes

$$\begin{aligned} T_{\beta\alpha} &= \langle\beta|T|\alpha\rangle \\ &= \frac{1}{3}(T_{1/2} + 2T_{3/2})\delta_{\beta\alpha} + \frac{1}{3}(T_{3/2} - T_{1/2})\langle\beta|\vec{T}\cdot\vec{\tau}|\alpha\rangle \end{aligned} \tag{C.9}$$

From (C.3) we find a traceless 3×3 matrix

$$\vec{T}\cdot\vec{\tau} = -i\varepsilon_{ijk}\tau_k \tag{C.10}$$

and making use of the products $\tau_1\tau_2 = i\tau_3$ etc., and $\{\tau_i,\tau_j\} = 2\delta_{ij}$, it is straight forward to show that

$$\langle\beta|\vec{T}\cdot\vec{\tau}|\alpha\rangle = -\frac{1}{2}[\tau_\beta,\tau_\alpha] \tag{C.11}$$

Thus

$$T_{\beta\alpha} = \frac{1}{3}T_1\delta_{\beta\alpha} + \frac{1}{6}T_2[\tau_\beta, \tau_\alpha] \tag{C.12}$$

defining

$$T_1 = T_{1/2} + 2T_{3/2} \tag{C.13}$$

the no isospin-flip amplitude, and

$$T_2 = T_{1/2} - T_{3/2} \tag{C.14}$$

the isospin-flip amplitude.

Let T_+ denote the scattering amplitude for the process $\pi^+ + p \to \pi^+ + p$ and T_- the amplitude for $\pi^- + p \to \pi^- + p$. As $\pi^+ p$ is a pure isospin 3/2 state,

$$T_+ = T_{3/2} = T_1 - T_2 \tag{C.15}$$

Now, defining

$$|\pi^+\rangle = \frac{-1}{\sqrt{2}}|\pi_1 + i\pi_2\rangle\ ,\ \ |\pi^-\rangle = \frac{1}{\sqrt{2}}|\pi_1 - i\pi_2\rangle\ ,\ \ |\pi^0\rangle = |\pi_3\rangle \tag{C.16}$$

we find

$$\begin{aligned}\langle\pi^- p|T|\pi^- p\rangle &= \frac{1}{2}\langle p|T_{11} + T_{22} + iT_{21} - iT_{12}|p\rangle \\ &= \langle p|T_1 + T_2|p\rangle \\ &= \langle p|T_-|p\rangle \end{aligned} \tag{C.17}$$

having used (C.12). We have defined

$$T_- = T_1 + T_2 = \frac{1}{3}T_{3/2} + \frac{2}{3}T_{1/2} \tag{C.18}$$

From these amplitudes, we can define isospin-even and isospin-odd scattering amplitudes

$$T^{(+)} = \frac{1}{2}(T_- + T_+) \tag{C.19}$$

and

$$T^{(-)} = \frac{1}{2}(T_- - T_+) \tag{C.20}$$

C.1 Crossing relations

Under the crossing operation $\alpha \leftrightarrow \alpha'$, $q \leftrightarrow -q'$, the states

$$\begin{aligned}
|\pi^+\rangle &\leftrightarrow -\langle\pi^-| \\
|\pi^-\rangle &\leftrightarrow -\langle\pi^+| \\
|\pi^0\rangle &\leftrightarrow \langle\pi^0|
\end{aligned}$$

using (C.16). Hence, using the Gell-Mann - Goldberger crossing symmetry law

$$\langle q', \alpha'; p', s'|T_+(s,t,u)|q, \alpha; p, s\rangle = \langle -q, \alpha; p', s'|T_-(u,t,s)| - q', \alpha'; p, s\rangle$$
$$\langle q', \alpha'; p', s'|T_-(s,t,u)|q, \alpha; p, s\rangle = \langle -q, \alpha; p', s'|T_+(u,t,s)| - q', \alpha'; p, s\rangle$$

and so

$$\langle q', \alpha'; p', s'|T^{(+)}(s,t,u)|q, \alpha; p, s\rangle = \langle -q, \alpha; p', s'|T^{(+)}(u,t,s)| - q', \alpha'; p, s\rangle$$
$$\langle q', \alpha'; p', s'|T^{(-)}(s,t,u)|q, \alpha; p, s\rangle = \langle -q, \alpha; p', s'|T^{(+)}(u,s,t)| - q', \alpha'; p, s\rangle$$

and thus, writing the amplitudes in terms of ν and t, (we can substitute ν for s or u because, in the lab frame, ν reduces to the pion energy ω)

$$T^{(+)}(\nu, t) = T^{(+)}(-\nu, t) \tag{C.21}$$
$$T^{(-)}(\nu, t) = -T^{(-)}(-\nu, t) \tag{C.22}$$

C.2 Invariant Amplitudes

In the previous appendix, we defined scattering amplitudes between πN states. We now write these states in the form

$$\langle q', \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle = \overline{u}(p', s')Tu(p, s) \tag{C.23}$$

with $u(p, s)$ and $u(p', s')$ Dirac spinors (positive energy solutions of the Dirac equation).

As $\overline{u}(p', s')Tu(p, s)$ is a Lorentz invariant, it will be a function of the 4-momentum p, p', q, q' and the 16 Dirac matrices $I, \gamma^\mu, \gamma^\mu\gamma^\nu, \gamma^\mu\gamma_5$ and γ_5. Therefore, T can be written [131]

$$T = A + B_\mu\gamma^\mu + C_{\mu\nu}\gamma^\mu\gamma^\nu + D_\mu\gamma^\mu\gamma_5 + E\gamma_5 \tag{C.24}$$

with A, B, C, D and E transforming like a scalar, vector, tensor, pseudovector and pseudoscalar respectively. They will also be functions of p, p', q and q'. Due to conservation of energy-momentum, $p + q = p' + q'$, only three of these momenta will be independent. In the third term, $\mu \neq \nu$.

As A is a scalar, it will be a function of the Mandelstam variables s and t only. The second term must be $\gamma^\mu q_\mu$ or $\gamma^\mu q'_\mu$. However, conservation of energy-momentum requires that it have the form $\gamma^\mu(q_\mu + q'_\mu)$.

It can be shown that, between Dirac spinors, the third term reduces to the sum of a scalar and vector, while the fourth term reduces to a term which transforms as a vector. Because of energy-momentum conservation, $E = 0$.

Thus, we can write

$$T = A(s, t) + \gamma \cdot QB(s, t) \tag{C.25}$$

with $Q_\mu = (q_\mu + q'_\mu)/2$. The amplitudes A and B are the invariant scattering amplitudes.

Between Dirac spinors, using the Gordon decomposition [132]

$$\overline{u}(p',s')\gamma^\mu u(p,s) = \overline{u}(p',s')\Big[\frac{P^\mu}{M_N} + i\sigma^{\mu\nu}\frac{k_\nu}{M_N}\Big]u(p,s) \qquad \text{(C.26)}$$

we find

$$\overline{u}(p',s')\gamma^\mu Q_\mu u(p,s) = \overline{u}(p',s')\Big[\nu - \frac{\not{Q}\,\not{k}}{M_N}\Big]u(p,s) \qquad \text{(C.27)}$$

with P, Q and k defined by (A.13).

Thus, the transition amplitude can be written

$$T(s,t) = A(s,t) + \nu B(s,t) - \frac{\not{Q}\,\not{k}}{M_N}B(s,t) \qquad \text{(C.28)}$$

Choosing a system in which $q = q'$, so that $k = 0$, in terms of the isospin-even amplitude

$$T^{(+)}(s,t) = A^{(+)}(s,t) + \nu B^{(+)}(s,t) \qquad \text{(C.29)}$$

From (C.22) we find the crossing relations for $A^{(+)}$ and $B^{(+)}$

$$A^{(+)}(\nu,t) = A^{(+)}(-\nu,t)\ ,\ B^{(+)}(\nu,t) = -B^{(+)}(-\nu,t) \qquad \text{(C.30)}$$

These crossing relations will be useful when writing dispersion relations.

Appendix D

Dispersion Relations

In equation(B.25), $j_\alpha(z)$ has the quantum numbers of the pion. Hence, it will not alter the baryon number or strangeness. We may therefore consider a complete set of states $|n\rangle$ with baryon number unity and strangeness zero. These states are N, $N+\pi$, $N+2\pi$, ...

Since they form a complete set we can write

$$I = \sum_{|n\rangle} |n\rangle\langle n| \tag{D.1}$$

and therefore

$$\begin{aligned}
&\int d^4z\, e^{iqz} \;\langle p', s'|T[j_\alpha(z) j_{\alpha'}(0)]|\vec{p}, s\rangle \\
&= (2\pi)^3 \sum_{|n\rangle} \Big[\, \delta^3(\vec{p}_n - (\vec{p'} - \vec{q})) \int_{-\infty}^{\infty} dz_0\, \theta(z_0) e^{i(E_{p'} - \omega_q - E_n) z_0} \langle p', s'|j_\alpha(0)|n\rangle\langle n|j_{\alpha'}(0)|p, s\rangle \\
&\qquad + \delta^3(\vec{p}_n - (\vec{p} + \vec{q})) \int_{-\infty}^{\infty} dz_0\, \theta(-z_0) e^{i(-E_p - \omega_q + E_n) z_0} \langle p', s'|j_{\alpha'}(0)|n\rangle\langle n|j_\alpha(0)|p, s\rangle \Big]
\end{aligned}$$

with $E_n = [M_n^2 + \vec{p}_n^2]^{\frac{1}{2}}$ the energy of the intermediate state $|n\rangle$ and $\vec{p}_n$ its total three-momentum.

Using

$$\int_{-\infty}^{\infty} dt\, \theta(\pm t) e^{-i\omega t} = \frac{\mp i}{\omega \mp i\varepsilon} \tag{D.2}$$

for small $\varepsilon > 0$, we have [133]

$$\begin{aligned}
&\langle q', \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle \\
&= (2\pi)^3 \sum_{|n\rangle} \Big[\delta^3(\vec{p}_n - (\vec{p}\,' - \vec{q})) \frac{\langle p', s'|j_\alpha(0)|n\rangle\langle n|j_{\alpha'}(0)|p, s\rangle}{E_n - E_{p'} + \omega_q - i\varepsilon} \\
&\qquad - \delta^3(\vec{p}_n - (\vec{p} + \vec{q})) \frac{\langle p', s'|j_{\alpha'}(0)|n\rangle\langle n|j_\alpha(0)|p, s\rangle}{E_p - E_n + \omega_q + i\varepsilon} \Big]
\end{aligned} \quad \text{(D.3)}$$

From these expressions we see that the intermediate states only exist if the total momentum $\vec{p}_n = \vec{p}\,' - \vec{q}$ and $\vec{p}_n = \vec{p} + \vec{q}$ in the first and second line of the above equation respectively.

Can we write a dispersion relation from equation(D.3) ? In order to answer this question we consider foward scattering

$$\begin{aligned}
&\vec{p} = \vec{p}\,' \,,\; \vec{q} = \vec{q}\,' \,,\; E_p = E_{p'} \\
&\omega_q = \omega_{q'} = \omega
\end{aligned} \quad \text{(D.4)}$$

The energy denominators in (D.3) are

$$E_n - E_{p'} + \omega - i\varepsilon \quad \text{and} \quad E_p - E_n + \omega + i\varepsilon \quad \text{(D.5)}$$

As $\varepsilon > 0$ these lead to singularities in the scattering amplitude when

$$\omega = E_{p'} - E_n + i\varepsilon \quad \text{(D.6)}$$

so that

$$Im\,\omega > 0 \quad \text{(D.7)}$$

and

$$\omega = E_n - E_p - i\varepsilon \quad \text{(D.8)}$$

so

$$Im\,\omega < 0 \quad \text{(D.9)}$$

That is, the scattering amplitude has singularities in the half-plane $Im\,\omega \geq 0$ and $Im\,\omega \leq 0$.

However, by causality, the scattering amplitude must be singularity free in one half-plane. Therefore, we cannot use equation(B.25).

On the other hand, if we had used (B.27)

$$\int d^4z\, e^{iqz}\langle p', s'|R[j_\alpha(z)j_{\alpha'}(0)]|\vec{p}, s\rangle$$
$$= (2\pi)^3 \sum_{|n\rangle} \Big[\delta^3(\vec{p}_n - (\vec{p'} - \vec{q}))\frac{\langle p', s'|j_\alpha(0)|n\rangle\langle n|j_{\alpha'}(0)|p, s\rangle}{E_n - E_{p'} + \omega_q - i\varepsilon}$$
$$-\delta^3(\vec{p}_n - (\vec{p} + \vec{q}))\frac{\langle p', s'|j_{\alpha'}(0)|n\rangle\langle n|j_\alpha(0)|p, s\rangle}{E_p - E_n + \omega_q - i\varepsilon}\Big] \qquad \text{(D.10)}$$

If we again consider foward scattering the energy denominators in equation(D.10) lead to singularities in the scattering amplitude when

$$\omega = E_{p'} - E_n + i\varepsilon \ \text{ and } \ \omega = E_n - E_p + i\varepsilon \qquad \text{(D.11)}$$

so that

$$Im\,\omega > 0 \qquad \text{(D.12)}$$

That is, the half-plane $Im\,\omega \leq 0$ is singularity free and hence from equation(B.27) we can write down dispersion relations for the scattering amplitude.

The form of the dispersion relation can be found from the mass spectrum of the intermediate states. If the intermediate state $|n\rangle$ is a single particle, which, in our case, is the nucleon, then the mass $M_n = M_N$. If $|n\rangle = N + \pi, N + 2\pi, \ldots$, then M_n will have a continuous spectrum $M_N + M_\pi \leq M_n \leq \infty$, etc. Hence, the dispersion relation will be in two parts - a term due to the discrete case $|n\rangle$ = nucleon, plus an energy integral from each line of equation(D.10).

The scattering amplitude is usually written as a function of an invariant energy E and scattering angle θ. From this point on, we use the notation

$$T_{fi}(E,\theta) = \langle q', \alpha'; p', s'\, in|T|q, \alpha; p, s\, in\rangle \tag{D.13}$$

D.1 Fixed-t dispersion relations

From (D.10) we see that $T_{fi}(E,\theta)$ is an analytic function of ω_q with poles at

$$\omega_q = [M_N^2 + (\vec{p}+\vec{q})^2]^{\frac{1}{2}} - E_p + i\varepsilon \tag{D.14}$$

and

$$\omega_q = E_{p'} - [M_N^2 + (\vec{p'}-\vec{q})^2]^{\frac{1}{2}} + i\varepsilon \tag{D.15}$$

and cuts in the ω_q plane along the straight lines

$$[(M_N + M_\pi)^2 + (\vec{p}+\vec{q})^2]^{\frac{1}{2}} - E_p + i\varepsilon \text{ to } \infty + i\varepsilon \tag{D.16}$$

and

$$-\infty + i\varepsilon \text{ to } E_{p'} - [(M_N + M_\pi)^2 + (\vec{p'}-\vec{q})^2]^{\frac{1}{2}} + i\varepsilon \tag{D.17}$$

In Appendix A we defined the Mandelstam variables s, t and u satisfying

$$s + t + u = 2M_N^2 + 2M_\pi^2 \tag{D.18}$$

For the first pole (D.14)

$$s = M_N^2 \tag{D.19}$$

after substituting $\omega_q = E_q$ and dropping the $+i\varepsilon$ term. The second pole leads to

$$u = M_N^2 \tag{D.20}$$

Thus, for fixed-θ, the contribution of the single particle to the dispersion relation is

$$\frac{a(\theta)}{s - M_N^2} + \frac{b(\theta)}{u - M_N^2} \tag{D.21}$$

with $a(\theta)$ and $b(\theta)$ to be determined by lowest order perturbation theory.

The cuts (D.16) and (D.17) give

$$(M_N + M_\pi)^2 \leq s \leq \infty \tag{D.22}$$

and

$$(M_N + M_\pi)^2 \leq u \leq \infty \tag{D.23}$$

Before we continue, we note that in the c.m frame, the Mandlestam variable s was identified as the square of the energy in that frame. We also showed that $t = -2q_c(1 - \cos\theta_c)$ with θ_c the scattering angle in the c.m. frame. Hence, instead of writing the transition amplitude as a function of E and θ, we write it in terms of s and t. Thus,

$$T(s,t) = \frac{a(t)}{s - M_N^2} + \frac{b(t)}{u - M_N^2} + \frac{1}{2\pi i}\int_{s_1}^{\infty} ds' \frac{T(s',t)}{s' - s} + \frac{1}{2\pi i}\int_{u_1}^{\infty} du' \frac{T(u',t)}{u' - u} \tag{D.24}$$

for fixed-t and defining $s_1 = (M_N + M_\pi)^2$ and $u_1 = (M_N + M_\pi)^2$. The denoninators should be read as $s' - s - i\varepsilon$ and $u' - u - i\varepsilon$.

From lowest order perturbation theory, it is found that

$$\begin{aligned} a(t) &= -g^2 \\ b(t) &= g^2 \end{aligned}$$

with $g^2/4\pi$ the pion-nucleon coupling constant.

We now write the denominators $s - M_N^2$ and $u - M_N^2$ in terms of the crossing variable ν given by equation(A.12).

$s - M_N^2$:

In the lab frame

$$\vec{p} = \vec{p'} = 0 \ , \ \vec{q} = \vec{q'} \tag{D.25}$$

the crossing variable

$$\nu = \omega + \frac{t}{4M_N} \tag{D.26}$$

and at the first pole (D.14)

$$\omega + M_N = [M_N^2 + q^2]^{\frac{1}{2}} + \ i\varepsilon \tag{D.27}$$

Squaring both sides and defining ε' to be small and positive, we find that

$$\omega = \frac{-M_\pi^2}{2M_N} + i\varepsilon' \tag{D.28}$$

If we denote ν at the pole by ν_B, then

$$\nu_B = \omega_{pole} + \frac{t}{4M_N} = \frac{-M_\pi^2}{2M_N} + \frac{t}{4M_N} \tag{D.29}$$

Using equation(A.12) and (D.18)

$$\nu = \frac{s - M_N^2 - M_\pi^2 + t/2}{2M_N} \tag{D.30}$$

and hence

$$s - M_N^2 = 2M_N(\nu - \nu_B) \tag{D.31}$$

Thus

$$ds = 2M_N d\nu \tag{D.32}$$

and

$$\frac{1}{s - M_N^2} = \frac{1}{2M_N^2}\frac{1}{\nu - \nu_B} \tag{D.33}$$

$u - M_N^2$:

At the second pole

$$\omega - M_N = M_N^2 + q^2 + i\varepsilon \tag{D.34}$$

which leads to

$$\omega = \frac{M_\pi^2}{2M_N} + i\varepsilon' \tag{D.35}$$

In the lab frame

$$u = M_N^2 + M_\pi^2 - 2\omega_\pi M_N \tag{D.36}$$

and so, for foward scattering (D.4)

$$\omega = \frac{M_N^2 + M_\pi^2 - u}{2M_N} \tag{D.37}$$

and by (A.12)

$$\nu = \omega - \frac{t}{4M_N} \tag{D.38}$$

Hence, at the second pole (D.15)

$$\nu_{B'} = \frac{M_\pi^2}{2M_N} - \frac{t}{4M_N} \tag{D.39}$$

Thus, at the poles

$$\nu = \pm\nu_B \tag{D.40}$$

From the above we find

$$u - M_N^2 = -2M_N(\nu + \nu_B) \tag{D.41}$$

Therefore, the pole terms in (D.24) are

$$\frac{a(t)}{s - M_N^2} + \frac{b(t)}{u - M_N^2} = \frac{-g^2}{2M_N(\nu - \nu_B)} + \frac{g^2}{-2M_N(\nu + \nu_B)} \tag{D.42}$$

If we denote the nucleon pole (Born) term by $B_N^+(\nu, t)$, then

$$B_N^+(\nu, t) = \frac{g^2}{2M_N}\left[\frac{1}{\nu_B - \nu} - \frac{1}{\nu_B + \nu}\right] \tag{D.43}$$

with the + refering to the process $\pi^+ + N \rightarrow \pi^+ + N$.

The cuts are given by (D.16) and (D.17). The first cut is along

$$\omega_q = [M_n^2 + q^2]^{\frac{1}{2}} - M_N + i\varepsilon \tag{D.44}$$

for $M_N + M_\pi \leq M_n \leq \infty$. This leads to

$$\omega_q = \frac{M_n^2 - M_N^2 - M_\pi^2}{2M_N} + i\varepsilon' \tag{D.45}$$

and if the lowest mass of the intermediate state is $M_n = M_N + M_\pi$, then the lowest energy will be $\omega_q = M_\pi + i\varepsilon'$. Hence

$$M_\pi + i\varepsilon' \leq \omega_q \leq \infty + i\varepsilon' \tag{D.46}$$

Thus, using (D.26), the first cut in the ν plane is along

$$M_\pi + \frac{t}{4M_N} \leq \nu \leq \infty \tag{D.47}$$

In a similar manner the second cut is along

$$-\infty \leq \nu \leq -M_\pi - \frac{t}{4M_N} \tag{D.48}$$

If we denote

$$\nu_1 = M_\pi + \frac{t}{4M_N} \tag{D.49}$$

then the cuts are along

$$\begin{aligned} \nu_1 \leq \ & \nu \leq \infty \\ -\infty \leq \ & \nu \leq -\nu_1 \end{aligned} \tag{D.50}$$

and overlap when $\nu_1 < 0$, or $t < -4M_\pi M_N$.

Finally, we write the energy denominators under the integrals in (D.24) in terms of ν.

The lower bound of the first cut is

$$s_1 = (M_N + M_\pi)^2 \tag{D.51}$$

From equation(D.30)

$$s = 2M_N\nu + (M_N + M_\pi)^2 - 2M_NM_\pi - t/2 \tag{D.52}$$

so

$$s_1 - s = 2M_N(\nu_1 - \nu) \tag{D.53}$$

with ν_1 defined by equation(D.49). Continuing this to higher contributions to the first cut, we see that

$$s' - s = 2M_N(\nu' - \nu) \tag{D.54}$$

and thus

$$\frac{ds'}{s' - s} = \frac{d\nu'}{\nu' - \nu} \tag{D.55}$$

The lower bound of the second cut is at

$$u_1 = (M_N + M_\pi)^2 \tag{D.56}$$

From equations(D.37) and (D.38) we find

$$u_1 - u = 2M_N(\nu + \nu_1) \tag{D.57}$$

and therefore

$$u' - u = 2M_N(\nu' + \nu) \tag{D.58}$$

Thus

$$\frac{du'}{u' - u} = \frac{d\nu'}{\nu' + \nu} \tag{D.59}$$

and using the above, the dispersion relation (D.24) becomes

$$T(\nu,t)=B_N^+(\nu,t)+\frac{1}{2\pi i}\int_{\nu_1}^{\infty}d\nu'\,\frac{T(\nu',t)}{\nu'-\nu-i\varepsilon}+\frac{1}{2\pi i}\int_{-\infty}^{-\nu_1}d\nu'\,\frac{T(\nu',t)}{\nu'+\nu-i\varepsilon} \tag{D.60}$$

for fixed-t.

Using the isospin even amplitude of the previous appendix, we write

$$T^+(\nu,t)=B_N^+(\nu,t)+\frac{1}{2\pi i}\int_{\nu_1}^{\infty}d\nu'\,\frac{T^+(\nu',t)}{\nu'-\nu-i\varepsilon}+\frac{1}{2\pi i}\int_{-\infty}^{-\nu_1}d\nu'\,\frac{T^+(\nu',t)}{\nu'+\nu-i\varepsilon} \tag{D.61}$$

then, by (C.22), we have

$$T^+(\nu,t)=B_N^+(\nu,t)+\frac{1}{2\pi i}\int_{\nu_1}^{\infty}d\nu'\,T^+(\nu',t)\Big[\frac{1}{\nu'-\nu-i\varepsilon}+\frac{1}{\nu'+\nu-i\varepsilon}\Big] \tag{D.62}$$

The pole term B_N^+ was defined for $\pi^+ + N \rightarrow \pi^+ + N$, so that, when writing an expression for T^-, a new pole term B_N^- will have to be defined.

We saw from (D.10) that the lower half-plane is singularity free. By Cauchy's theorem

$$0=\frac{1}{2\pi i}\int_{\nu_1}^{\infty}d\nu'\,T^+(\nu',t)\left[\frac{1}{\nu'-\nu+i\varepsilon}+\frac{1}{\nu'+\nu+i\varepsilon}\right] \tag{D.63}$$

taking the complex conjugate and subtracting from (D.62) we find

$$T^+(\nu,t)=B_N^+(\nu,t)+\frac{1}{2\pi i}\int_{\nu_1}^{\infty}d\nu'\,2iImT^+(\nu',t)\Big[\frac{1}{\nu'-\nu-i\varepsilon}+\frac{1}{\nu'+\nu-i\varepsilon}\Big] \tag{D.64}$$

Now write

$$\frac{1}{\nu' - \nu - i\varepsilon} = P\frac{1}{\nu' - \nu} + i\delta(\nu' - \nu) \tag{D.65}$$

where P is the principle value of the integral. Then

$$\int_{\nu_1}^{\infty} d\nu'\, ImT^{+}(\nu', t)\Big[\frac{1}{\nu' - \nu - i\varepsilon} + \frac{1}{\nu' + \nu - i\varepsilon}\Big]$$

$$= P\int_{\nu_1}^{\infty} d\nu'\, ImT^{+}(\nu', t)\Big[\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu}\Big] + \frac{i}{2}ImT^{+}(\nu, t) + \frac{i}{2}ImT^{+}(-\nu, t)$$

$$= P\int_{\nu_1}^{\infty} d\nu'\, 2\nu'\, \frac{ImT^{+}(\nu', t)}{\nu'^2 - \nu^2} + iImT^{+}(\nu, t) \tag{D.66}$$

Substituting the above equation into (D.62) and writing

$$T^{+} = ReT^{+} + iImT^{+} \tag{D.67}$$

we have the dispersion relation

$$ReT^{+}(\nu, t) = B_N^{+}(\nu, t) + \frac{2}{\pi}P\int_{\nu_1}^{\infty} d\nu'\, \nu'\, \frac{ImT^{+}(\nu', t)}{\nu'^2 - \nu^2} \tag{D.68}$$

We now use the form $T^{+} = A^{+} + \nu B^{+}$. As $A^{+}(\nu, t)$ has no Born terms,

$$ReA^{+}(\nu, t) = \frac{2}{\pi}P\int_{\nu_1}^{\infty} d\nu' \nu' \frac{ImA^{+}(\nu', t)}{\nu'^2 - \nu^2} \tag{D.69}$$

$B^{+}(\nu, t)$, however, has a Born term and using (C.30)

$$ReB^{+}(\nu, t) = \frac{g^2}{M_N}\frac{\nu}{\nu^2 - \nu_B^2} + \frac{2\nu}{\pi}P\int_{\nu_1}^{\infty} d\nu' \nu' \frac{ImB^{+}(\nu', t)}{\nu'^2 - \nu^2} \tag{D.70}$$

Combining these dispersion relations, we find

$$ReT^{+}(\nu, t) = \frac{g^2}{M_N}\frac{\nu^2}{\nu^2 - \nu_B^2} + ReA^{+}(0, t) + \frac{2\nu^2}{\pi}P\int_{\nu_1}^{\infty} \frac{d\nu'}{\nu'}\frac{ImT^{+}(\nu', t)}{\nu'^2 - \nu^2} \tag{D.71}$$

a once-subtracted dispersion relation [134]. Subtractions are made when there is some doubt about the convergence of the integral (and are related to the inclusion of the polynomial in (B.19)). The above expression is called "once-subtracted" because, if we had subtracted the dispersion relation for $T^{+}(0, t)$ from the RHS of (D.68), we would get the above.

D.2 Fixed-ν Dispersion Relations

Due to a property of the amplitude for the s-channel process $\pi + N \rightarrow \pi + N$ known as generalized crossing, the amplitude for the t-channel process $\pi + \overline{\pi} \rightarrow N + \overline{N}$ will be the same as the s-channel amplitude.

A fixed-u dispersion relation is obtained if we contract on a pion and a nucleon, with

$$|p, s\rangle = \left(\frac{M_N}{E_p}\right) \int_t d^3x \, e^{i(\vec{p}\cdot\vec{x} - E_p t)} \overline{\psi}_{in}(\vec{x}, t)\gamma^0 u(\vec{p})|0\rangle \tag{D.72}$$

the nucleon state vector. The nucleon field obeys

$$\overline{\psi}(\partial_\mu \gamma^\mu - M_N) = \overline{a}(x_\mu) \tag{D.73}$$

where $\overline{a}(x)$ is the nucleon source operator.

The spinor $u(\vec{p})$ is a solution of the free Dirac equation

$$(i\gamma^\mu p_\mu - M_N)u(p) = 0 \tag{D.74}$$

It can be found that

$$\begin{aligned}
&\langle q', \alpha'; p', s' \, in|T|q, \alpha; p, s \, in\rangle \\
&\quad = (2\pi)^3 \sum_{|n\rangle} \Big[\delta^3(\vec{p}_n - (\vec{p} + \vec{q})) \frac{\langle p', s'|j_{\alpha'}(0)|n\rangle\langle n|\overline{a}_u(0)|p, s\rangle}{E_n - E_p - \omega_q - i\varepsilon} \\
&\qquad - \delta^3(\vec{p}_n - (\vec{p'} + \vec{p})) \frac{\langle p', s'|\overline{a}_u(0)|n\rangle\langle n|j_{\alpha'}(0)|p, s\rangle}{E_{p'} - E_p - E_n - i\varepsilon} \Big]
\end{aligned} \tag{D.75}$$

with $\overline{a}_u(0) = \overline{a}(0) \cdot u(p)$.

Note that as the single particle state $|\pi\rangle$ does not exist, there will be no Born terms in the strict sense.

Parity dictates that there must be an even number of pions in the intermediate state (or $N\overline{N}$ pairs). Hence $|n\rangle = \pi + \pi, 4\pi, \ldots$, and therefore

$$2M_\pi \leq \mu_n \leq \infty \, , \ldots , 2M_N \leq \mu_n \leq \infty \tag{D.76}$$

(at the pole, with $E_n = [\mu_n^2 + (\vec{p} - \vec{p'})^2]^{\frac{1}{2}}$) and as $t = \mu_n^2$, t takes the values

$$4M_\pi^2 \le t \le \infty \tag{D.77}$$

Thus, for fixed-u,

$$T(s,t) = \frac{1}{2\pi i}\int_{(M_N+M_\pi)^2}^{\infty} ds' \frac{T(s',u)}{s'-s-i\varepsilon} + \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty} dt' \frac{T(t',u)}{t'-t-i\varepsilon} \tag{D.78}$$

We now write the first integral in terms of t.

As $s + t + u = 2M_N^2 + 2M_\pi^2$, for $s \ge (M_N + M_\pi)^2$ and $u \ge (M_N + M_\pi)^2$, we have $t \le -4M_N M_\pi$ and therefore

$$T(s,t) = \frac{1}{2\pi i}\int_{-\infty}^{-4M_N M_\pi} dt' \frac{T(t',u)}{t'-t-i\varepsilon} + \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty} dt' \frac{T(t',u)}{t'-t-i\varepsilon} \tag{D.79}$$

for fixed-u.

In terms of the even-isospin amplitude and the crossing variable

$$T^+(\nu,t) = \frac{1}{2\pi i}\int_{-\infty}^{-4M_N M_\pi} dt' \frac{T^+(\nu,t')}{t'-t-i\varepsilon} + \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty} dt' \frac{T^+(\nu,t')}{t'-t-i\varepsilon} \tag{D.80}$$

for fixed-ν.

Using the same procedure as was used to derive a dispersion relation for fixed-t, we find that

$$ReT^+(\nu,t) = \frac{2}{\pi}P\int_{-\infty}^{-4M_N M_\pi} dt' \frac{ImT^+(\nu,t')}{t'-t} + \frac{2}{\pi}P\int_{4M_\pi^2}^{\infty} dt' \frac{ImT^+(\nu,t')}{t'-t} \tag{D.81}$$

Setting $\nu = 0$ and subtracting at $t = t_1$, we obtain the once-subtracted dispersion relation

$$\begin{aligned} ReT^+(0,t) = ReT^+(0,t_1) &+ \frac{2(t-t_1)}{\pi}P\int_{-\infty}^{-4M_N M_\pi} dt' \frac{ImT^+(0,t')}{(t'-t)(t'-t_1)} \\ &+ \frac{2(t-t_1)}{\pi}P\int_{4M_\pi^2}^{\infty} dt' \frac{ImT^+(0,t')}{(t'-t)(t'-t_1)} \end{aligned} \tag{D.82}$$

Note that Born terms are contained in $B^+(\nu,t)$. But, $B^+(\nu,t) = -B^+(-\nu,t)$ and hence must vanish at $\nu = 0$. Therefore, any Born terms that may exist in the above dispersion relation will vanish.

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