Conceptual reasoning

Belief, multiple agents and preference

By
Krzysztof Zbigniew Nowak
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To my Parents—my Mother, who taught me how to love,
and my Father, who taught me how to reason.

Contents

Abstract **xi**

Acknowledgements **xiii**

1 Introduction **1**
1.1 Motivation and goals 1
1.2 Overview and structure 1

2 Background **5**
2.1 Lattices and order 5
2.2 Formal Concept Analysis 8
2.3 Bilattices 11

3 Ontology and belief **15**
3.1 Language consensus 16
3.2 Misrepresentation 18
3.3 Believed worlds, or paradigm shift 21
3.4 Formulae and regions 27
3.5 Language of descriptions 31
3.6 Information ordering on abstract objects 34
3.7 Descriptions and contexts 38

4 Model theory **43**
4.1 Abstract contexts and models 43
4.2 Information ordering on contexts 46
4.3 Validity 51

5 Proof theory **53**
5.1 Formal systems 53
5.2 Entailment, or information ordering on theorems 57
5.3 Proof procedure 61
5.4 Information ordering on theories 65

6 Models and theories **73**
6.1 Contexts and theories revisited 73
6.2 Mappings between contexts and theories 76
6.3 Models revisited 78
6.4 Language revisited 79

6.5 Relating the information orderings 80
6.6 Soundness and completeness 82

7 Multiple agents **85**
7.1 Believed theories 85
7.2 Conceptual worlds and bilattices 87
7.3 Concept lattices of theories 93
7.4 Towards numeric measure 97
7.5 Preference and epistemic states 102

8 Conclusion **105**
8.1 Summary 105
8.2 Future work 106
8.2.1 Dynamic worlds 106
8.2.2 Binary relations 109
8.2.3 Implementation 110
8.3 Related research 112

A Proofs **115**

Bibliography **119**

List of Figures

1	Line diagram for context of Table 1—full labelling	10
2	Line diagram for context of Table 1—reduced labelling	11
3	Bilattices—four-valued and default logics	13
4	World-based bilattice of truth values (2 worlds case)	13
5	Reaching language consensus	17
6	Misrepresentation	19
7	Abandoning perceived worlds	21
8	Restrictions on objects	28
9	World G and its properties	29
10	Formulae, $\|P\| = 3$	30
11	Regions, $\|P\| = 3$	31
12	Restrictions on abstract objects	34
13	Abstract objects, $\|P\| = 3$	35
14	Information ordering on abstract objects	37
15	Formulae, regions and abstract objects, $\|P\| = 2$	38
16	Formulae, regions and abstract objects, $\|P\| = 1$	38
17	Ordering on contexts, initial attempt	47
18	Comparing contexts with total contexts	48
19	Information ordering on partial worlds (saturated contexts)	49
20	Rules of inference	55
21	Deriving provable formulae	57
22	Ordered sets of descriptions, $\|P\| = 2$	59
23	Ordered sets of theorems and axioms (Example 3)	60
24	Inconsistent theory and ignorant theory	60
25	Information ordering on partial worlds—showing theories	65
26	Lattices $\mathbb{T}^{\oplus}$ of $\oplus$-theories and $\mathbb{T}^{\ominus}$ of $\ominus$-theories	66
27	In search for consistent theories—lattice $\mathbb{T}^{\oplus} \times \mathbb{T}^{\ominus}$	67
28	Consistent description sets	68
29	Consistent theories	69
30	Information ordering on consistent theories	70
31	$\boldsymbol{\tau}$ and $\boldsymbol{\kappa}$ mappings	78
32	Some believed theories and their closures, $\|P\| = 1$	86
33	Pairs of believed theories and their closures	86
34	Some believed theories and their closures, $\|P\| > 1$	87
35	Believed theories $\mathbb{B}$ and sets $\mathbb{B}_-$, $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_-$	88
36	Lattice $\mathbb{C}_+$ of believed theories $\mathbb{B}$ of Table 7	88
37	Bilattice truth-values of descriptions	90

38	Sets of bilattice truth-values	91
39	Theories and $\mathbb{B}$-models	94
40	Total theories above $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_3$ and $\mathbf{0}$	94
41	Forming a concept lattice of theories (Example 5)	97
42	Towards numeric measure	99
43	Entropy of $\boldsymbol{\nu}_{\boldsymbol{\varpi}}$	100
44	Maximising entropy	101
45	$\mathbb{B}$-models above $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_3$ and $\mathbf{0}$	101
46	Entropy of $\boldsymbol{\eta}_{\boldsymbol{\varpi}}$	102
47	Finding a theory of a dynamic world	108
48	From properties to binary relations	109
49	Cora, a Tcl/Tk implementation of CR	111

List of Tables

1	Context "Living Beings and Water"	9
2	Paradigm shift	22
3	Introducing descriptions and contexts	38
4	Worlds, description and context (Example 1)	40
5	Worlds, description and context (Example 2)	41
6	Introducing abstract contexts	45
7	Believed theories $\mathbb{B}$ and closure $\mathbb{C}_+$	89
8	Descriptions and their bilattice truth-values	90
9	Theories and their truth-values	92
10	Theories as concepts	95
11	Formal context of the concept lattice of theories (Example 5)	97
12	Measure $\boldsymbol{\nu}^*$ on theories	100
13	Measure $\boldsymbol{\eta}^*$ on $\mathbb{B}$-models and theories	102

Abstract

One of the central issues in Artificial Intelligence (AI) in general—and in the area of knowledge representation and reasoning in particular—is *common sense reasoning.* This includes logics of knowledge and belief, non-monotonic reasoning, truth-maintenance and belief revision. Within these fields the notion of a *consistent belief state* is the crucial one. Additionally, there is a growing interest in *partial information states*, with an *information ordering* being the key notion.

The issues of *inconsistency* and *partiality of information* are central to this work. The thesis proposes a logical knowledge representation formalism employing *partial objects* and *partial worlds* on its semantic side. The syntax includes a language, formulae, and *partial theories*, theories associated with partial worlds. Partial worlds and theories are *consistent*, and *contradictory information* is assumed to arise in *multiple agent* situations. Relevant mathematical structures are discussed, in particular partial theories are related to partial worlds.

A multiple agent case is considered. A set of agents is assumed to provide multiple description sets, where description sets represent *partial* syntactic information about partial worlds. A set of description sets gives rise to a set of theories, equipped with an information ordering—a *lattice* containing all those theories (and some alternatives) can be derived. This demonstrates that partial theories can be partially ordered by an information ordering and the obtained lattice structure facilitates the theory selection process based on *information value* and *truthness* of theories. It is also suggested how to derive a numeric measure (and hence a linear order) on the theories.

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

27 February 1998

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Chapter 1

Introduction

This thesis proposes a framework for knowledge representation and reasoning, with a particular interest in dealing with *contradictory* and *partial* information. The framework allows one to represent and reason about *contexts* involving *objects* having *attributes*. Both *semantic* and *syntactic* information about such contexts are considered. *Models* correspond to contexts, and syntactic information gives rise to *theories* consisting of sentences *provable* in *formal systems*. We then focus on theories and consider related structures that permit dealing with information provided by *multiple agents*.

1.1 Motivation and goals

The motivation for this work was to develop a simple logical framework that would allow one to represent information about agents' *worlds*, with a particular emphasis on *partial* and *contradictory* information. It was also perceived as desirable to be able to relate worlds—or information about the worlds—of different *agents*, compare them, decide how *similar* they are, and which of them should be *preferred*.

The specific goals were as follows.

1. Decide an appropriate *semantics*, or *model theory*.
2. Propose a corresponding *formal system*, or *proof theory*.
3. Relate model theory and proof theory to each other.
4. Analyse a *multiple agent* case.

The first three points result in the logical framework discussed in Chapters 4, 5 and 6. The last one gives rise to Chapter 7.

1.2 Overview and structure

A *formal context* in Formal Concept Analysis (FCA)—see [GW96][1] and a chapter in [DP90]—is a triple (G, M, I), where G is a set of *objects*, M is a set

[1] An English translation is expected to be published by Springer this year.

of *attributes* and $I \subseteq G \times M$ is the *incidence relation* assigning attributes to objects. FCA also defines *formal concepts*, and given a context K its set of concepts $\underline{\mathfrak{L}}(K)$ forms a *complete lattice*, called a *concept lattice* of K. We however also consider *partial contexts* and *abstract contexts*, involving *abstract objects*. *Valid sentences/descriptions* determine which abstract objects are present in the context, and which are not. Given a set of descriptions $\boldsymbol{D}_i \subseteq \boldsymbol{D}$, the set $\boldsymbol{D}_i$ determines a *formal system* with *axioms* $\boldsymbol{D}_i$ and inference rules Φ. Hence, we introduce formal systems, and a set of their theories $\mathbb{T}$ is equipped with an information ordering relation $\leq$. Theories, when associated with *agents*, are called *believed theories* $\mathbb{B}$, and there is a minimal lattice $\mathbb{C}$ of theories that includes the believed theories, together with their *meets* and *joins*, $\mathbb{C} = \mathrm{Cl}_{\wedge,\vee}(\mathbb{B})$. Such lattices provide a framework for common sense knowledge representation and reasoning. In particular, given such a lattice $\mathbb{C}$, it captures *contradictions* and *partiality*. Given $\mathbb{C}$ we can consider $\mathbb{C}_+ = \mathbb{C} \cup \{\mathbf{0}, \mathbf{1}\}$, where $\mathbf{0}$ is an *empty theory* and $\mathbf{1}$ is an "*inconsistent theory.*" Then if $\boldsymbol{T}_2 \geq \boldsymbol{T}_1$ then $\boldsymbol{T}_1$ is more *partial* than $\boldsymbol{T}_2$, and if $\boldsymbol{T}_2 \vee \boldsymbol{T}_1 = \mathbf{1}$ then the theories are contradictory. Further, the lattice $\mathbb{C}_+$ is a *concept lattice*, theories in $\mathbb{C}$ and sentences in $\boldsymbol{D}$ can be *partially ordered*, and a *numeric measure* $\nu : \mathbb{T} \longrightarrow [0, 1]$ can be derived. Hence, given that theories can be ordered (partially ordered, linearly ordered by their numeric measure), we can derive *preference relations* on theories—this allows us to decide which information to accept or reject, or, more generally, how to order information, where in our case information is expressed by theories.

The thesis can be seen as consisting of three main parts.

The first part consists of the background Chapter 2. In that chapter we provide the necessary information on *order* and *lattice* theory. Then a framework of *Formal Concept Analysis (FCA)* is presented. Finally, a *bilattice approach* to truth-values is discussed.

The second part consists of Chapters 3, 4, 5 and 6. Together, the chapters provide the logical framework employed to deal with the multiple agent case of the last part of the thesis.

In particular, Chapter 3 provides some discussion on *objects, language* and *belief*, and can be seen as an informal introduction to Chapter 4, which precisely defines the semantic side of the framework—*worlds*, or *models* are introduced, they are called *abstract contexts*, and are a variant of FCA contexts. Two important aspects of this chapter is an *information ordering* on contexts, and a notion of validity—they are related, because validity is preserved when more information is acquired about the same world.

Chapter 5 is aimed at providing the appropriate syntactic, proof-theoretic *formal system* for reasoning about abstract contexts. Formal systems give rise to *theories* and there is an information ordering on theories. Given a fixed set of properties (attributes) the set of all consistent theories forms a lattice with respect to the information ordering.

Chapter 6 relates abstract contexts and theories to each other. This is an appropriate place to reconsider the issues of models and language. Given that sets of contexts and theories are equipped with their respective information orderings, the chapter includes a short discussion on how the two orderings are related. The chapter concludes with the soundness and completeness result.

The third and last major part of the thesis is constituted by Chapter 7. It introduces *believed theories*—these result from sets of sentences, or *description sets* provided by agents. Given the results of Chapter 5, it is natural to extend the set of believed theories to a lattice generated by the set. Several examples are provided. Then *truth-values* of sentences (theorems, or descriptions) and theories are discussed—this chapter is an attempt to apply the bilattice approach reviewed in Section 2.3 of Chapter 2. Then we consider a question of whether lattices of theories can be seen as *concept lattices* in the FCA sense. Section 7.4 then suggests how a *numeric measure* on theories can be derived, employing a *maximum entropy principle*. Finally, Section 7.5 presents lattices of theories as epistemic states, and a preference relation on theories as *derivable* from the lattice.

Chapter 8 provides a short summary, indicates further research, and discusses some related work other than that reviewed in Chapter 2.

Chapter 2

Background

This chapter provides necessary background on *lattice* and *order theory*, on *Formal Concept Analysis*, and on a *bilattice approach* to reasoning.

In particular, Section 2.1 introduces basic notions in *order theory*, singles out a specific kind of ordered sets known as *lattices*, and presents them as both order-theoretic and algebraic structures. The issue of *information ordering* is discussed.

Section 2.2 reviews the area of *Formal Concept Analysis*. It provides a notion of an FCA *context*, a *concept* of a context, and a *concept lattice* of concepts. Two relevant issues, although less standard, are also mentioned—these are *three-valued contexts*, and the area known as *attribute exploration*.

Section 2.3 discusses the bilattice approach proposed by Ginsberg [Gin88]. In particular, we discuss *world-based* bilattices.

2.1 Lattices and order

In this section we first include the basic definitions of order and lattice theory. Then we discuss *information orderings*. Finally, we recall that lattices can be seen as *ordered sets* and as *algebraic structures*. The section is mainly based on [DP90], but other classic references are [Grä71, Bir48].

Let X be a set and $\leq$ be a binary relation on X, that is a subset of $X \times X$. The relation $\leq$ is called *reflexive* iff$_{\text{def}}$ $\forall_{x \in X}\ x \leq x$, *antisymmetric* iff$_{\text{def}}$ $\forall_{x,y \in X}$ if $x \leq y$ and $y \leq x$ then $x = y$, and *transitive* iff$_{\text{def}}$ $\forall_{x,y,z \in X}$ if $x \leq y$ and $y \leq z$ then $x \leq z$. Let X be a set and $\leq$ be a binary relation on X. The pair $(X, \leq)$, or just X, is referred to as a *partially ordered set*, or *poset* iff$_{\text{def}}$ the relation $\leq$ is reflexive, antisymmetric and transitive; The binary relation $\leq$ itself is called a *partial order* on X, or simply an *order* (cf. [DP90]).

Let $(X, \leq)$ be a poset and let $Y \subseteq X$. Then $a \in Y$ is a *maximal element* of Y iff$_{\text{def}}$ if $a \leq y \in Y$ then $a = y$; $a \in Y$ is a *minimal element* of Y iff$_{\text{def}}$ if $a \geq y \in Y$ then $a = y$; $a \in Y$ is the *greatest element*, or *maximum* of Y, written $a = \max Y$ iff$_{\text{def}}$ $\forall_{y \in Y}\ a \geq y$; $a \in Y$ is the *least element*, or *minimum* of Y, written $a = \min Y$ iff$_{\text{def}}$ $\forall_{y \in Y}\ a \leq y$; a is called the *top* element of X, and is denoted by the symbol $\top$, if $\max X$ exists and $a = \max X$; a is called the *bottom* element of X, and is denoted by the symbol $\bot$, if $\min X$ exists and $a = \min X$; $a \in X$ is an *upper bound* of Y

iff$_{\text{def}}$ $\forall_{y \in Y}\ a \geq y$. $a \in X$ is an *lower bound* of Y iff$_{\text{def}}$ $\forall_{y \in Y}\ a \leq y$. Let $Y^u =_{\text{def}} \{a \in X \mid \forall_{y \in Y} a \geq y\} = \{a \in X \mid a \text{ is an upper bound of } Y\}$, and $Y^l =_{\text{def}} \{a \in X \mid \forall_{y \in Y} a \leq y\} = \{a \in X \mid a \text{ is a lower bound of } Y\}$, Then $a \in X$ is the *least upper bound*, or the *supremum* of Y, written $a = \sup Y$ iff$_{\text{def}}$ $\min Y^u$ exists and $a = \min Y^u$; dually, $a \in X$ is the *greatest lower bound*, or the *infimum* of Y, written $a = \inf Y$ iff$_{\text{def}}$ $\max Y^l$ exists and $a = \max Y^l$;

Let X and Y be posets. A map $\psi : X \longrightarrow Y$ is: *order-preserving* (or *monotone*, or *isotone*) iff$_{\text{def}}$ if $x_1 \leq x_2$ in X then $\psi(x_1) \leq \psi(x_2)$ in Y; *order-embedding* iff$_{\text{def}}$ $x_1 \leq x_2$ in X if and only if $\psi(x_1) \leq \psi(x_2)$ in Y; *order-isomorphism* iff$_{\text{def}}$ it is an order-embedding mapping X onto Y.

We now consider an *information ordering*—an ordering that a set can be equipped with, with the intended meaning that objects higher in the information ordering convey more *information*. Further, given sets equipped with information orderings, we introduce information orderings on the power sets of the initial sets.

There are many situations in which a set X of elements is naturally equipped with an information ordering $\leq$, where given $x_1, x_2 \in X$ the relation $x_1 \leq x_2$ has interpretations such as "x_2 provides more information than x_1," or "is more defined," or "is a better approximation." Some examples—see [DP90]—include approximating a real number with closed intervals, and approximating strings of characters by their initial substrings. An information ordering on maps provides another example, with total maps containing a maximum amount of information and some partial maps being partial determinations of the total ones.

Let $(X, \leq)$ be a set X equipped with an information ordering $\leq$. Does the information ordering $\leq$ on X induce an information ordering on $\mathcal{P}(X)$? Although the question is a reasonable one to ask, it seems that one needs more specific knowledge of X and $\leq$ to introduce an information ordering on $\mathcal{P}(X)$. Indeed, we consider two cases that suggest two different orderings on the power sets of the initial sets.

In the first case we consider, let X be a set of *tokens of information*—each token conveys information about objects of the world. Then $(X, \leq)$ is the set X equipped with an information ordering relation—if $x_1 \leq x_2$ then x_2 represents a more precise information about objects less precisely described by x_1. Clearly, to describe some sets of objects, one might need more than single elements of X. The following paragraphs comment on information ordering on the set of subsets of X.

Let $(X, \leq)$ be a set equipped with an information ordering, where elements of X convey information about objects of the world. Let $Y = \mathcal{P}(X)$ and $y_1, y_2 \in Y$, i.e., $y_1, y_2 \subseteq X$. One might attempt to introduce an *information ordering on* $Y = \mathcal{P}(X)$ as follows. We say that $y_1 \leq y_2$ iff the following two conditions are satisfied:

1. $\forall_{x_2 \in y_2} \exists_{x_1 \in y_1}\ x_1 \leq x_2$,
2. $\forall_{x_1 \in y_1} \exists_{x_2 \in y_2}\ x_2 \geq x_1$.

It seems that this provides a reasonable way of introducing information orderings on the set of subsets of X, with the given interpretation of X. Indeed, condition

(1) requires that if new information appears in y_2 then y_1 must "allow" it, i.e., if $x_2 \in y_2 \setminus y_1$ then there must be an element $x_1 \in y_1$ such that x_2 is a further refinement of x_1—hence moving from y_1 to y_2 never results in the need to *reject/revise* information already conveyed by y_1. The condition (2) in turn requires that the information conveyed by x_1 is not lost when we move up from y_1 to y_2. However, one needs to be careful, as the introduced ordering might fail to be anti-symmetric, and therefore some additional requirements might be needed—see Section 4.2, Definition 13.

In the second case we consider, let there be a single object we want to locate, and let X be a set of *tokens of information* that partially *locate* the object. Then $(X, \leq)$ is the set X equipped with an information ordering relation—if $x_1 \leq x_2$ then x_2 gives a more precise location of the object than x_1 does. Given that one might not be able to provide a single (even if partial) location, but rather a set of alternatives, a set of locations such that the object is in one of them, then one might need to employ subsets of X. The following paragraphs comment on information ordering on the set of subsets of X in this case.

Let there be a single object that needs to be located. Let $(X, \leq)$ be a set equipped with an information ordering, where elements of X are possible, partially specified *locations* of the object. Let $Y = \mathcal{P}(X)$ and $y_1, y_2 \in Y$, i.e., $y_1, y_2 \subseteq X$. It seems that one might attempt to introduce an *information ordering on* $Y = \mathcal{P}(X)$ as follows. We say that $y_1 \leq y_2$ iff the following condition is satisfied:

1. $\forall_{x_2 \in y_2} \exists_{x_1 \in y_1}\ x_1 \leq x_2$.

This seems to provide a reasonable way of introducing information orderings on the power set of X, with the given interpretation of X. Indeed, the condition (1) requires that if new information appears in y_2 then y_1 must "allow" it, i.e., if $x_2 \in y_2 \setminus y_1$ then there must be an element $x_1 \in y_1$ such that x_2 is a further refinement of x_1. Note that some tokens of y_1 can disappear—this corresponds to dismissing some alternative locations for the object we are trying to locate, and corresponds to a gain of information. It is clear that the maximum elements of the power set of X are the singleton sets of the maximum elements of X.

We make use of the above considerations in two different places. We employ the former idea to introduce an information ordering on *contexts* $\mathbb{K}$, given an information ordering on *objects* $\boldsymbol{G}$—this is done in Section 4.2, Definition 13. We employ the latter idea to introduce an information ordering on $\boldsymbol{\kappa}$*-models* $\{\boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}$ given an information ordering on *contexts* $\mathbb{K}$—$\boldsymbol{\kappa}$-models can be identified with sets of total models (total contexts), and a $\boldsymbol{\kappa}$-model carries maximum information if it specifies a singleton set of total models, i.e., it points to exactly one total model of the world—this is done in Section 6.5, Definition 20.

We now present lattices as *ordered sets*, and as *algebraic structures*.

Let X be a poset. X is a *lattice*[1] iff$_{\text{def}}$ $\forall_{x_1, x_2 \in X}$ $\sup\{x_1, x_2\}$ and $\inf\{x_1, x_2\}$ exist.

---

[1] X is a *complete lattice* iff$_{\text{def}}$ $\forall_{Y \subseteq X}$ $\sup Y$ and $\inf Y$ exist.

It is easy to see that every finite lattice is complete—indeed, in any lattice $\sup Y$ and $\inf Y$ exist for any finite $Y \subseteq X$, and hence for any finite lattice X we have that $\forall_{Y \subseteq X}$ $\sup Y$ and $\inf Y$ exist.

Let $(X, \wedge, \vee)$ be an algebra, where X is a set, and $\wedge, \vee : X^2 \longrightarrow X$. The operators $\wedge, \vee$ are: *associative*, if $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, and $(a \vee b) \vee c = a \vee (b \vee c)$; *commutative*, if $a \wedge b = b \wedge a$, and $a \vee b = b \vee a$; *idempotent*, if $a \wedge a = a$, and $a \vee a = a$; satisfy *absorption* identities, if $a \wedge (a \vee b) = a$, and $a \vee (a \wedge b) = a$.

Let $(X, \wedge, \vee)$ be an algebra. X is a *lattice* iff$_{\text{def}}$ X is nonempty, and $\wedge, \vee$ are associative, commutative, idempotent and satisfy the absorption identities.

Lattices can be seen as posets, and as algebras. Let the poset $(X, \leq)$ be a lattice. Put $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$. Then the algebra $(X, \wedge, \vee)$ is a lattice. Let the algebra $(X, \wedge, \vee)$ be a lattice. Put $a \leq b$ iff$_{\text{def}}$ $a \wedge b = a$. Then $(X, \leq)$ is a poset, and the poset is a lattice. Hence, to show that X is a lattice, it suffices to define $\leq$ on X, show that $(X, \leq)$ is a *poset*, and define sup and inf such that $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in X$. Then the algebra $(X, \wedge, \vee)$—where $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$—is a lattice.

Let $(X, \wedge, \vee)$ be a lattice and let $\emptyset \neq Y \subseteq X$. Then Y is a sub-lattice of X if $x_1, x_2 \in Y$ implies $x_1 \vee x_2, x_1 \wedge x_2 \in Y$. Of course, a sub-lattice of a lattice is a lattice.

2.2 Formal Concept Analysis

In this section we provide background information on the area known as *Formal Concept Analysis (FCA)*, developed by a research group of ErnstSchröderZentrum at TH Darmstadt, led by Rudolf Wille. The main, up-to-date reference is [GW96], and its English translation is to be published by Springer soon. Other references include [Wil92], and a chapter in [DP90].

We first give the basics of FCA, including the notions of *contexts, concepts*, and *concept lattices*. To visualise concept lattices one draws *labelled line diagrams*, but usually the labelling is reduced—we get line diagrams with *reduced labelling*. All these basic notions are illustrated with an example—the example is taken from a tutorial on FCA given by Rudolf Wille during the Conceptual Knowledge Processing Conference CKP'96 in Darmstadt, but it can also be found in [GW96]. We then consider *three-valued contexts*, and a knowledge acquisition method called *attribute exploration*.

A *formal context* in FCA is defined as consisting of a set of *objects* G, a set of *attributes* M, and an *incidence relation* $I \subseteq G \times M$ assigning attributes to objects, where gIm means "the object g has the attribute m." Then a *formal context* K is the triple (G, M, I). An example of a formal context is given in Table 1, presenting the context "Living Beings and Water" in a form of a *cross-table*—if g has m then this is denoted by a cross-sign $\times$ in the appropriate row and column of the table.

Derivation operators, denoted simply by the $'$ sign, are employed to find *concepts* of the given context—they map sets of objects to sets of attributes, and sets of attributes to sets of objects. Let $G_1 \subseteq G$ be a set of objects, and $M_1 \subseteq M$ be a set of attributes. The derivation operators are defined as follows.

$$G_1{}' = \{m \in M \mid \forall_{g \in G_1}\ gIm\},$$

As we will be dealing exclusively with finite lattices, those lattices will obviously be complete.

		a	b	c	d	e	f	g	h	i
g_1	LEECH	×	×					×		
g_2	BREAM	×	×					×	×	
g_3	FROG	×	×	×				×	×	
g_4	DOG	×		×				×	×	×
g_5	SPIKE-WEED	×	×		×		×			
g_6	REED	×	×	×	×		×			
g_7	BEAN	×		×	×	×				
g_8	MAIZE	×		×	×		×			

a needs water to live
b lives in water
c lives on land
d needs chlorophyll to prepare food
e two little leaves grow on germinating
f one little leave grows on germinating
g can move around
h has limbs
i suckles its offsprings

Table 1. Context "Living Beings and Water"

$$M_1' = \{g \in G \mid \forall_{m \in M_1}\ gIm\}.$$

There are several useful facts about the derivation operators. Let $G_1, G_2 \subseteq G$ and $M_1, M_2 \subseteq M$. Then we have, for instance, that if $G_1 \subseteq G_2$ then $G_1' \supseteq G_2'$; if $M_1 \subseteq M_2$ then $M_1' \supseteq M_2'$; $G_1 \subseteq G_1''$ and $G_1' = G_1'''$; $M_1 \subseteq M_1''$ and $M_1' = M_1'''$; $(G_1 \cup G_2)' = G_1' \cap G_2'$ and $(M_1 \cup M_2)' = M_1' \cap M_2'$.

(G_1, M_1) is said to be a *formal concept of* K iff $G_1 = M_1'$ and $M_1 = G_1'$—in such a case G_1 is referred to as the *extent* of the concept (G_1, M_1), and M_1 is called its *intent*. Hence, this formal notion of a concept corresponds to a philosophical understanding of a concept as consisting of objects sharing some attributes, and the set of attributes shared by the objects. To obtain a formal concept one can start with an initial set of objects, a subset of the set of all objects G, apply the derivation operator to find a maximal set of attributes shared by the objects, and then apply the derivation operator to that set of attributes to obtain *all* objects (possibly a superset of the initial set of objects) having the set of attributes.

Given a context K, a set of all formal concepts of K is denoted by $\mathcal{L}(K)$. This set can be equipped with an ordering relation $\leq$ given by $(G_1, M_1) \leq (G_2, M_2)$ iff $G_1 \subseteq G_2$; in this case (G_1, M_1) is called a *subconcept* of (G_2, M_2), and $\leq$ is referred to as a subconcept-superconcept relation. Note that, equivalently, we have that $(G_1, M_1) \leq (G_2, M_2)$ iff $M_1 \supseteq M_2$. The ordered set $(\mathcal{L}(K), \leq)$ is denoted by $\underline{\mathcal{L}}(K)$, and called the *concept lattice* of K, as it turns out to be a complete lattice.

Given a concept lattice $\underline{\mathcal{L}}(K)$, the lattice can be visualised by a corresponding *labelled line diagram*, usually drawn with a *reduced labelling.*

Let $\underline{\mathcal{L}}(K)$ be a concept lattice of the context K from Table 1. The corresponding line diagram is presented in Figure 1. The circles of the diagram represent nodes of the concept lattice, i.e., they represent concepts. The line segments between the circles represent the subconcept-superconcept relation, and labels provide the extents and intents of the concepts. Namely, given a concept, the objects (in the extent of the concept) are attached (from below) to the circle representing the concept, and the attributes (in the intent of the concept) are attached (from above) to the circle. Hence, we obtain the labelled line diagram of the context, so far with a *full labelling.*

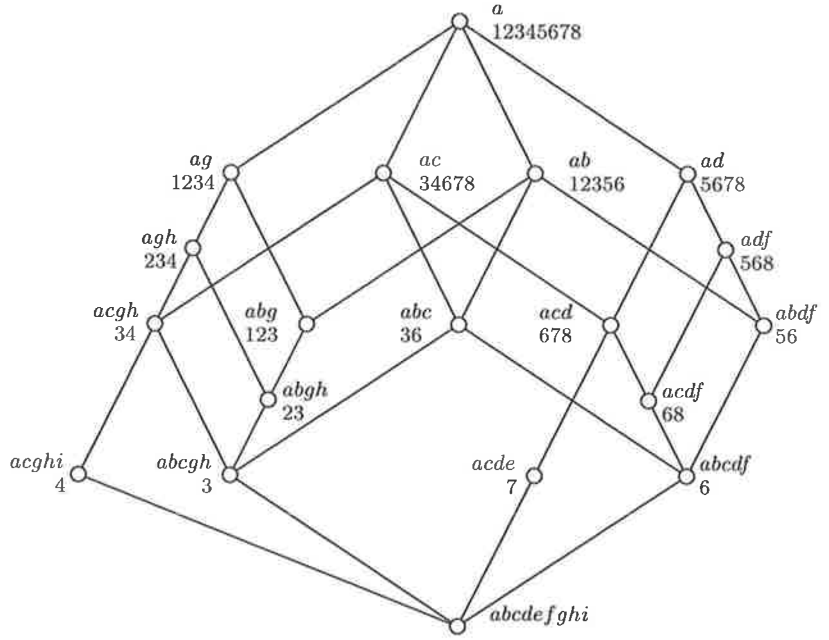

Figure 1. Line diagram for context of Table 1—full labelling

To obtain a diagram with a *reduced labelling*, a slight modification of the above procedure is needed. Let $\gamma g = (\{g\}'', \{g\}')$ be the smallest concept having g in its extent, and let $\mu m = (\{m\}', \{m\}'')$ be the largest concept having m in its intent—γg is called the *object concept* of g, while μm is called the *attribute concept* of m. Then, the name of an object g is attached only to the circle of its object concept γg, and the name of an attribute m is attached only to the circle of its attribute concept μm. This results in a *reduced labelled line diagram*, which still allows us to find extents and intents of all the concepts—the extent of a concept C consists of all objects g whose names can be reached by a descending path starting from the circle of C, and the intent of a concept C consists of all attributes m whose names can be reached by an ascending path starting from the circle of C. A reduced labelled line diagram for our example is presented in Figure 2.

It should also be noted that a reduced labelled line diagram allows us to reconstruct the formal context, because gIm iff $\gamma g \leq \mu m$.

Let us now consider *three-valued contexts.* It is clear that in many cases the incidence relation is not fully known—there can be an object g such that it is not known whether or not it has a given attribute m. As the incidence relation I can be seen as a *function* from $G \times M$ to a two element set, hence the case of *underdetermined objects* requires employing a *partial* incidence function, or a function from $G \times M$ to a three element set. We assume an *epistemic* reading of

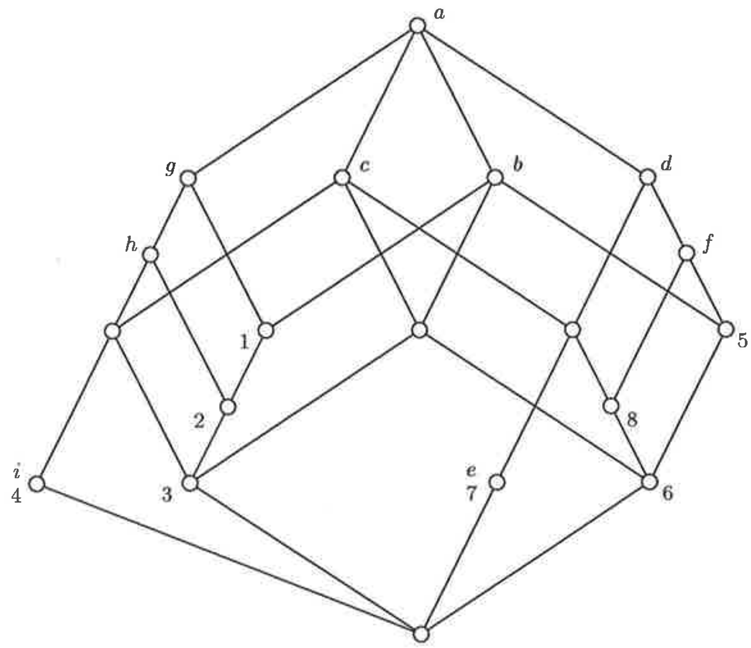

Figure 2. Line diagram for context of Table 1—reduced labelling

the "partiality" of objects, i.e., it is assumed that g either has m or it does not have it, but it is simply *not known* which of the two cases happens. Hence, by *partial contexts*, or three-valued contexts, we mean contexts with an incidence relation given by

$$I(g,m) = \begin{cases} 2 & \text{if } g \text{ is known to have } m, \\ 0 & \text{if } g \text{ is known not to have } m, \\ 1 & \text{otherwise.} \end{cases}$$

Three-valued contexts are discussed by Burmeister in [Bur91]—an English version of it is [Bur89]. We employ partial contexts on the semantic side of our logical formalism, although we assume objects to be *abstract*—see Sections 3.7 and 4.1.

Another interesting aspect of FCA is a knowledge acquisition technique known as *attribute exploration*—see [Gan96a, Gan96b, GW96, Stu96]. The basic idea is that given a context, a certain set of attributes $M_2 \subseteq M$ can *imply* another set $M_1 \subseteq M$. This is related[2] to our $\ominus$-*valid formulae* introduced in Section 4.3.

2.3 Bilattices

In this section we provide some basic information on *bilattices*, an approach developed by Ginsberg—the section is based on [Gin88]. We first consider *truth* and *information ordering* on *truth-values* and state a definition of a structure

[2] As said in [Gan96a], the two approaches were developed independently, and have not been merged, yet.

called a *bilattice.* Then, we focus our attention on *world-based-bilattices*, structures we return to in Section 7.2.

The basic idea behind Ginsberg's bilattices is to label sentences—or rather our degrees of belief in sentences—as more than simply "true" or "false."

Ginsberg mentions previous attempts in this direction. For instance, Dana Scott considers in [Sco82] situations where statements can be partially ordered by their truth and falsity—let such a *truth ordering* be denoted by $\leq_t$. Sandewall introduces in [San85] ordering on statements based on the amount of information they contain[3]—let such an *information ordering* be denoted by $\leq_k$. One can note that if $\boldsymbol{D}$ is a set of statements—sentences in a language we consider—then the orderings $\leq_t$ and $\leq_k$ are orderings on the elements of $\boldsymbol{D}$. However, as many statements can have the same truth-value, it is appropriate to associate the orderings with orderings on the truth-values. Then statements have their corresponding truth-values, and the orderings $\leq_t$ and $\leq_k$ on statements is given by the orderings $\leq_t$ and $\leq_k$ on the statements' truth-values,[4] more precisely, the orderings on truth-values determine the orderings on equivalence classes of sentences, where two sentences are equivalent if they have the same truth-value. Hence, we want to consider an ordered set $(\Gamma, \leq_t, \leq_k)$, the set Γ of truth-values equipped with truth-ordering and information-ordering. Additionally, there is a *negation* operator $\neg : \Gamma \longrightarrow \Gamma$, We expect the following to hold: if $x \leq_t y$ then $\neg x \geq_t \neg y$, if $x \leq_k y$ then $\neg x \leq_k \neg y$, and $\neg\neg x = x$. We want the orderings $\leq_t$ and $\leq_k$ to be such that the corresponding ordered sets are lattices—hence, the corresponding algebraic operations of *meet* and *joins*, denoted by $\wedge_t$, $\vee_t$ and $\wedge_k$, $\vee_k$, respectively. These considerations lead to the following formal definition of a *bilattice*, [Gin88].

Definition 1 *A* bilattice *is a sextuple* $(\Gamma, \wedge_t, \vee_t, \wedge_k, \vee_k, \neg)$, *where* Γ *is the set of truth-values, such that* $(\Gamma, \wedge_t, \vee_t)$ *and* $(\Gamma, \wedge_k, \vee_k)$ *are complete lattices. The negation mapping* $\neg : \Gamma \longrightarrow \Gamma$ *satisfies* $\forall_{x \in \Gamma}\ \neg\neg x = x$, *and* $\neg$ *is a lattice homomorphism from* $(\Gamma, \wedge_t, \vee_t)$ *to* $(\Gamma, \vee_t, \wedge_t)$ *and from* $(\Gamma, \wedge_k, \vee_k)$ *to itself.*

Note that given that the two lattices $(\Gamma, \wedge_t, \vee_t)$ and $(\Gamma, \wedge_k, \vee_k)$ are complete lattices, there are four distinguishing elements of Γ, namely $0_t = \bigwedge_t \Gamma$, $1_t = \bigvee_t \Gamma$, $0_k = \bigwedge_k \Gamma$, and $1_k = \bigvee_k \Gamma$.

Figure 3 presents two bilattices of truth-values. The bilattice on the left consists of truth-values of a four-valued logic, namely the truth values 1_t (true), 0_t (false), 1_k (contradiction/both true and false), and 0_k (unknown). The bilattice on the right consists of truth-values of a default logic, namely the truth values 1_t, 0_t, 1_k, 0_k (representing true/false/contradiction/unknown, respectively), d_t (true by default), d_f (false by default), and d_{tf} (both true and false by default).

It can be noted that the negation operator corresponds to reflection around the axis joining 1_k and 0_k. In Figure 3 we have for instance that $\neg 1_t = 0_t$, $\neg 0_t = 1_t$, $\neg 1_k = 1_k$, and $\neg 0_k = 0_k$.

[3]More precisely, it is assumed that truth-values are subsets of the $[0, 1]$ interval—the truth-value indicates the interval the probability of the statement lies in. Then, any increase in knowledge results in a contraction of the truth-value interval.

[4]Formally, we will have a truth assignment mapping ϱ that maps statements into their truth-values, i.e., $\varrho \colon \boldsymbol{D} \longrightarrow \Gamma$.

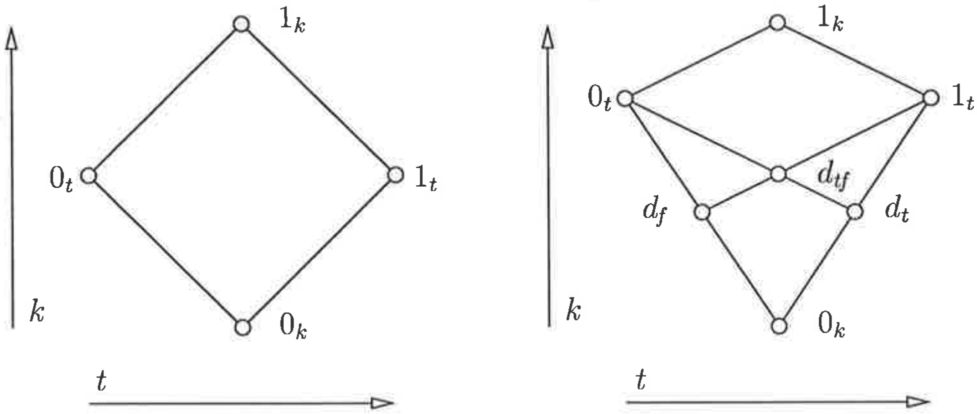

Figure 3. Bilattices—four-valued and default logics

Let us now consider *world-based bilattices*, bilattices with truth-values constructed from *worlds*, see Figure 4.

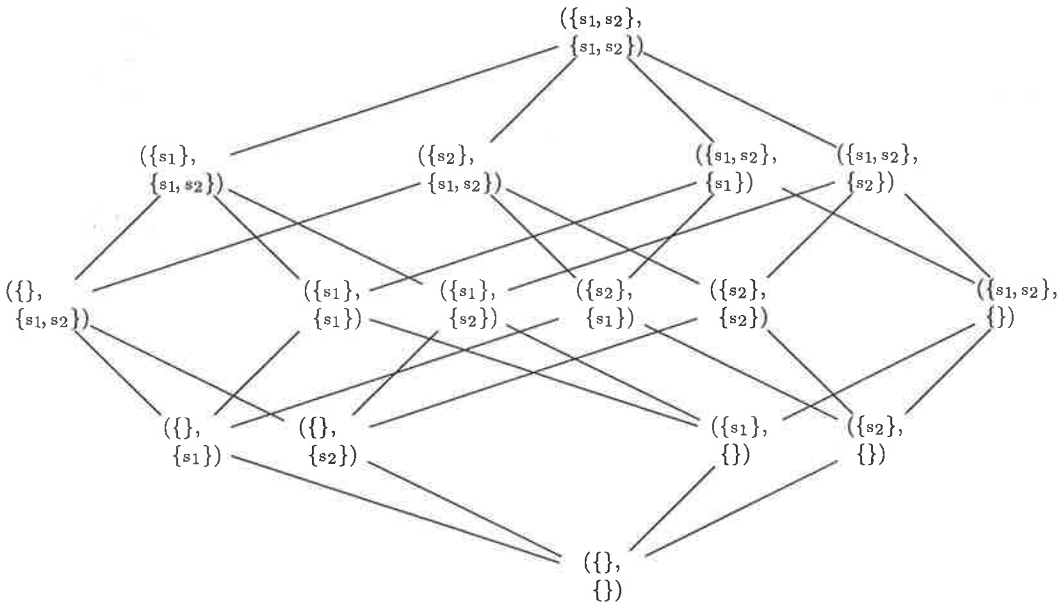

Figure 4. World-based bilattice of truth values (2 worlds case)

Ginsberg points out that a large class of bilattices can be obtained by considering sets of *worlds*. Let S be a set of worlds and let $U, V \subseteq S$. Let $\boldsymbol{D}$ be a set of sentences. If $D \in \boldsymbol{D}$ then we associate with D a pair $(U_D, V_D) \in \mathcal{P}(S) \times \mathcal{P}(S)$, where U_D is a set of worlds in which D is true, and V_D is a set of worlds where D is false. Then given $D \in \boldsymbol{D}$, the pair (U_D, V_D) of sets of worlds is considered to be a *truth-value* of D. Let $\Gamma = \mathcal{P}(S) \times \mathcal{P}(S)$ be the set of all truth-values, and let $\varrho : \boldsymbol{D} \longrightarrow \Gamma$ be a *truth assignment function* given by $\varrho(D) = (U_D, V_D)$.

In general, neither $U_D \cup V_D = S$ nor $U_D \cap V_D = \emptyset$ is required—even though any world in $S \setminus (U_D \cup V_D)$ is "incomplete," or under-determined w.r.t. D, and any world in $(U_D \cap V_D)$ is "inconsistent," or over-determined/contradictory w.r.t. D.

The truth ordering $\leq_t$ and the information ordering $\leq_k$ on truth-values are introduced as follows. Let $(U_1, V_1), (U_2, V_2) \in \Gamma$. Then $(U_1, V_1) \leq_t (U_2, V_2)$ iff $U_1 \subseteq U_2$ and $V_1 \supseteq V_2$, and $(U_1, V_1) \leq_k (U_2, V_2)$ iff $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$.

The four bilattice operations associated with $\leq_t$ and $\leq_k$, i.e., the operations $\wedge_t, \vee_t, \wedge_k$ and $\vee_k$ are given as follows. $(U_1, V_1) \wedge_t (U_2, V_2) = (U_1 \cap U_2, V_1 \cup V_2)$, $(U_1, V_1) \vee_t (U_2, V_2) = (U_1 \cup U_2, V_1 \cap V_2)$, $(U_1, V_1) \wedge_k (U_2, V_2) = (U_1 \cap U_2, V_1 \cap V_2)$, and $(U_1, V_1) \vee_k (U_2, V_2) = (U_1 \cup U_2, V_1 \cup V_2)$.

The four distinguished elements 0_t, 1_t, 0_k and 1_k of the bilattice—i.e., the t-minimal, t-maximal, k-minimal and k-maximal elements of Γ—are as follows. $0_t = (\emptyset, S)$, $1_t = (S, \emptyset)$, $0_k = (\emptyset, \emptyset)$ and $1_k = (S, S)$.

The negation operator $\neg$ on Γ is given by $\neg(U, V) = (V, U)$, i.e., if $\varrho(D) = (U_D, V_D)$ then $\varrho(\neg D) = (V_D, U_D)$.

Given the above, the following result is proven in [Gin88].

Proposition 1 *Let S be a set of worlds. Let $\Gamma = \mathcal{P}(S) \times \mathcal{P}(S)$ be a set of the corresponding truth-values, and let $\wedge_t, \vee_t, \wedge_k, \vee_k, \neg$ be appropriately defined. Then $(\Gamma, \wedge_t, \vee_t, \wedge_k, \vee_k, \neg)$ is a bilattice.*

Figure 4 presents a world-based bilattice, for a set of worlds $S = \{s_1, s_2\}$.

In Section 7.2 we consider world-based bilattices that employ worlds of a specific kind.

Chapter 3

Ontology and belief

The aim of this chapter is to introduce *semantic* entities called *abstract contexts*, and *syntactic* ones called *description sets*.

An abstract context can be identified with a set of *abstract objects*, where a single abstract object represents a set of objects sharing some attributes but indistinguishable with respect to all other attributes (a fixed set of attributes is assumed).

A description set is a set of *sentences* in a *language*, and provides information about an abstract context.

This chapter consists of two parts. In the first part, in Sections 3.1–3.3, some philosophical considerations are provided, In particular, Section 3.1 suggests how a group of agents can *invent* a language of attributes, where an attribute is a "label" the group agrees to associate with some objects, but not with some other ones—when this happens it is appropriate to say that the group has reached a *language consensus* (on the attribute).

There are two distinct language-related phases. Firstly, in a phase of developing the language agents uniquely identify objects and agree on assigning attributes to some of them—this involves the issue of *language consensus* discussed in Section 3.1. Secondly, there is a phase employing the language for *communicating*, or *transferring knowledge*—the language developed in the first phase is employed, but some of the communicating agents may use the language inappropriately, others may be wrong about the objects they report. Hence, in Section 3.2 we consider the issue of *misrepresentation*, which in turn demonstrates that it is possible that the syntactic "pictures" agents have of their worlds might have very little in common with the agents' "real worlds." Having no final authority on how the worlds look, we have to limit ourselves to the pictures provided by the agents, but these correctly describe only agents' "believed worlds"—Section 3.3 discusses this. At the end of Section 3.3 some more detailed linguistic, ontological and epistemological analysis is included, with references to the relevant philosophical literature.

In the second part of this chapter Sections 3.4–3.7 introduce and analyse abstract context and description sets (belonging to abstract worlds and literate worlds respectively, entities discussed in the first part of the chapter). In Section 3.4 we define *formulae* that are sets of *non-contradictory* attributes, and specify *regions* (of the worlds) corresponding to the formulae. A *structure* of the world can be described by specifying which of such regions are empty, and which are not—this can be expressed using "marked formulae," or *descriptions* introduced in Section 3.5. Given a formula—a set of attributes—the corresponding region is a set of objects that all have the attributes collected in the formula—an *abstract object* is a single entity employed to represent such objects, Section 3.6 provides the details. One should note that abstract objects are indeed *abstract* in the sense that they allow us to avoid the need to identify objects uniquely. Furthermore, two agents can have the same abstract object in their contexts, even if their worlds are disjoint. This is the consequence of the fact that abstract objects correspond to sets of "partially identified" objects—an abstract object can turn into another (more specific) abstract object when some new information is provided. Section 3.7 collects the considerations of the chapter, relates abstract contexts to description sets, and provides examples.

3.1 Language consensus

Let $S = \{s_i, \ldots, s_{n_s}\}$ be a finite set of *agents*, and call S a *society*. For every agent $s \in S$ there is a *world*, or set of *objects*, denoted G_s, seen by the agent. Let $G_\cap = \bigcap_{s \in S} G_s$ be a set of objects seen by the society S.

Let $P = \{p_1, \ldots, p_{n_\alpha}\}$ be a finite set of *attributes* the society S invents. When $p \in P$ is an attribute, it is assumed that some objects of the world $G_\cap$ have the attribute, and others do not. Objects that do not have p are assumed to have an attribute associated with p and denoted[1] $\overline{p}$. Hence, $\overline{P} = \{\overline{p_1}, \ldots, \overline{p_{n_\alpha}}\}$, and let $M = P \cup \overline{P}$. Furthermore, if $m \in M$ then $\overline{m}$ is given by:

$$\overline{m} = \begin{cases} \overline{p} & \text{if } m = p \in P, \\ p & \text{if } m = \overline{p} \in \overline{P}. \end{cases}$$

Elements of the set $M = P \cup \overline{P} = \{p_1, \ldots p_n\} \cup \{\overline{p_1}, \ldots \overline{p_n}\}$ are called *attributes*. Given a set of attributes and a world (of objects), attributes are associated with sets of objects that have the attributes. If for every $m \in M$ the following conditions hold: (1) $\mathbf{m} = \{g \in G_\cap \mid S \text{ says}^2 \text{ that } g \text{ has } m\}$, (2) $\overline{\mathbf{m}} = \{g \in G_\cap \mid S \text{ says that } g \text{ has } \overline{m}\}$, (3) $\mathbf{m} \cap \overline{\mathbf{m}} = \emptyset$, and (4) $\mathbf{m} \cup \overline{\mathbf{m}} \subseteq G_\cap$, then we say that S has reached a *language consensus* on M.[3]

We also introduce a *language consensus mapping* $\varepsilon : \mathcal{P}(M) \longrightarrow \mathcal{P}(G_\cap)$ by requiring that $\varepsilon(\{m\}) = \mathbf{m}$, and $\varepsilon(\{\overline{m}\}) = \overline{\mathbf{m}}$. If $\varepsilon(\{m\}) = \mathbf{m}$ then the set of objects $\mathbf{m}$ is called a *property*, and the attribute name m can be seen as the property's name.

Figure 5 shows how agents can reach a language consensus.

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[1]It is appropriate to understand $\overline{p}$ as a negated p, but $\overline{p}$ is treated as a single symbol.

[2]"S says that g has m" means that the agents of S agree, e.g., by voting, that the object g has the attribute m.

[3]One might argue that even if agents agree on the criteria for assigning attributes to objects they might not know for some object whether or not the object has the attribute. This means that—even at the stage of *forming* the language—for some attributes a language consensus is possibly reached at a smaller set of objects. Then—at the stage of *employing* the language—some objects might stay *partial* w.r.t. some attributes.

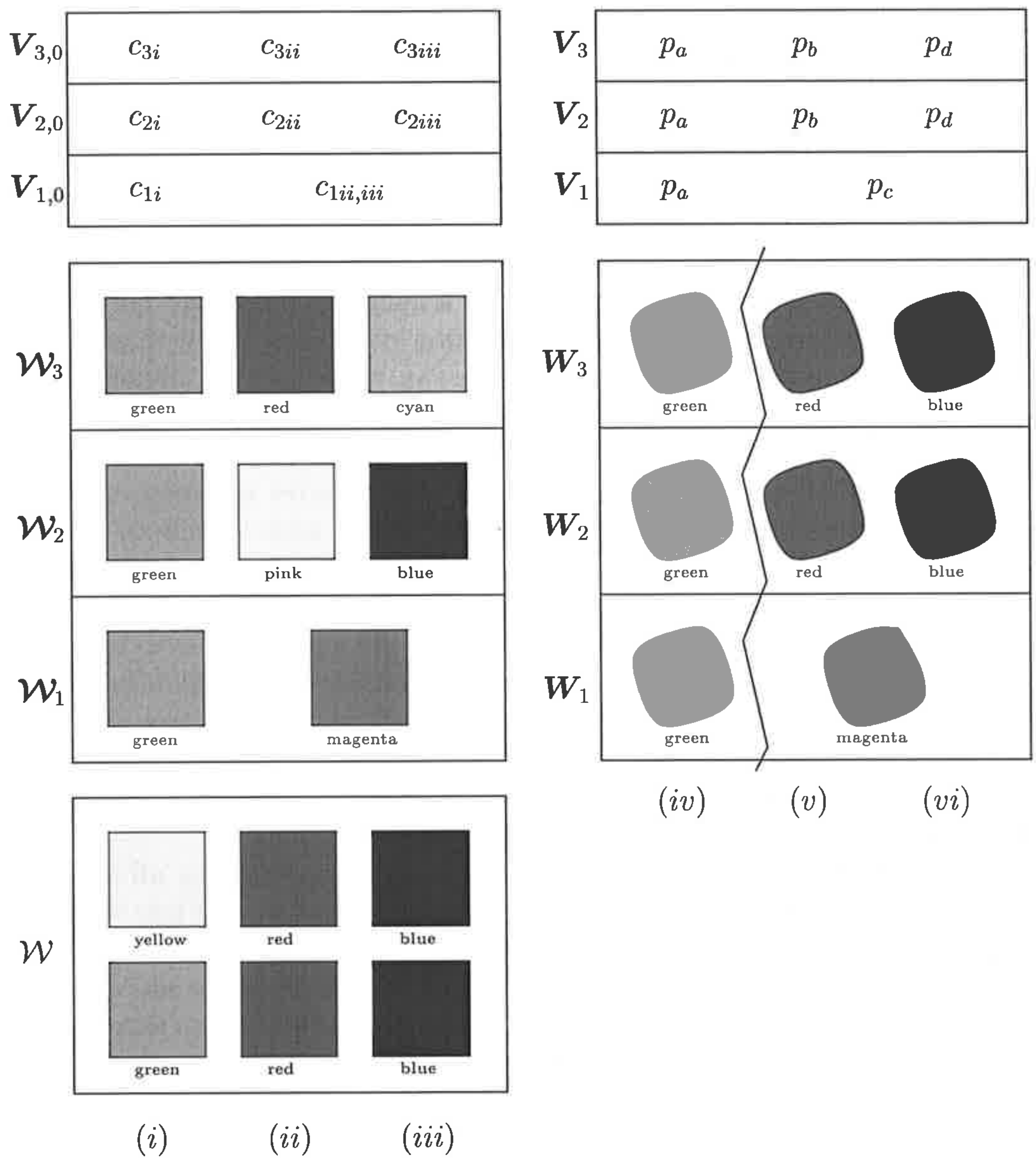

Figure 5. Reaching language consensus

Suppose that three agents s_1, s_2 and s_3 observe the same world $\mathcal{W}$. Call $\mathcal{W}$ a *perceived world* and its objects *perceived objects.* Suppose that—as a matter of fact—the world $\mathcal{W}$ consists of a yellow square, a green square, two red squares and two blue squares, as shown in Figure 5. Neither of the agents can distinguish between the colours green and yellow—all perceive the two colours as green. Moreover, the agent s_1 is unable to distinguish between red and blue—suppose she perceives both of them as magenta. The agent s_2 perceives blue correctly, but red is perceived by her as pink. The agent s_3 perceives red correctly, but blue is perceived by her as cyan. $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_3$ contain the agents' *perceptions*[4] of the perceived objects of $\mathcal{W}$; hence, the worlds are

[4]Note that to know what the perception worlds are, one would need to *inspect* the subjective experiences, or perceptions of the agents, rather than just be informed about them. We limit ourselves to what we are told by the agents.

called the *perception worlds*. Let the agents have their *private* labels for the colours and let $\boldsymbol{V}_{1,0}$, $\boldsymbol{V}_{2,0}$ and $\boldsymbol{V}_{3,0}$ contain the labels. If the agents want to communicate, they need to introduce *common labels*, labels they all accept and use consistently, according to an *agreement* they make. Such common labels are referred to as *attributes* and are employed by the agents s_1, s_2 and s_3 to build their so called *literate worlds* $\boldsymbol{V}_1, \boldsymbol{V}_2$ and $\boldsymbol{V}_3$. The language agreement is referred to as a *language consensus*, and reaching it necessitates using the language. The example demonstrates how a language consensus can be reached. The agents attempt to find such a partition of objects of $\mathcal{W}$ they all accept. Given that they have a *label*—or *attribute*—p_a they want to "attach" the label to the same objects of $\mathcal{W}$ and say that the label must not be attached to any of the remaining objects. Suppose one of the agents first attaches p_a to the yellow square. As they all perceive green and yellow squares as green, they will all agree to continue the process of associating labels with objects in such a way that p_a will be attached to green and yellow squares, but not to any other square of $\mathcal{W}$. Suppose then that s_3 attaches a label p_b to one of the red squares and the agents attempt to reach an agreement again. All agents agree to label the red squares with p_b. Note that the fact that s_2 perceives red as pink is undetectable. However, s_1 attempts to label blue squares with p_b as well—the only solution is that s_1 introduces her own label p_c to label both red and blue squares. The agents s_2 and s_3 introduce p_d to label blue squares—the fact that s_3 perceives blue as cyan is undetectable.

The agents have reached an agreement upon partitioning all objects into those that have p_a, and those that have not.[5] It seems appropriate to say that a language consensus has been reached w.r.t. p_a. Then, *believed worlds* of agents consist of their *believed objects*—objects the agents report to us. If we ignore how many objects indistinguishable w.r.t. the given attributes there are in the believed worlds, then we are dealing with so called *abstract worlds*, containing *abstract objects*. $\boldsymbol{W}_1, \boldsymbol{W}_2$ and $\boldsymbol{W}_3$ of Figure 5 denote the agents' abstract worlds.[6]

Subsequent sections—in particular Sections 3.2 and 3.3—provide some clarifications on perceived, perception, literate, believed and abstract world.

3.2 Misrepresentation

From now on, we understand agents not as members of a society that develops a language—by reaching language consensus—but as *sources of information*. Given a set of information sources S and a source $s \in S$, we often refer to s as an agent that describes her world. Similarly as with agents in Section 3.1, the

[5] Note that the agents' usage of p_b, p_c and p_d is not uniform. But if s_2 and s_3 were the only agents, an agreement would be reached.

[6] Regarding partitioning of objects, the agents partition the perceived objects. Some remarks can be made. Firstly, the perceived objects might be incorrectly partitioned. Secondly, even if they are, it is the agreement, rather than the actual partition that matters. Thirdly, agreement makes everyone happy—this is reflected by a correct partition of the abstract world, even though the abstract world might be an incorrect representation of the perceived world.

sources/agents can misrepresent their worlds. Some clarification on worlds and misrepresentation is offered.

We consider two ways in which agents can misrepresent objects they perceive. Firstly, an agent might misperceive the objects—perceiving them not as they should (e.g. seeing green objects as yellow). Secondly, an agent might *misuse*[7] *the language*—employing improper attributes to refer to her perceptions of the objects (e.g. calling yellow objects the objects she perceives as green).

We assume that misrepresentations are undetectable, i.e. there is no "oracle" that can say whether a misrepresentation has occurred. For the purpose of this section, we introduce an *oracle*, and explicate the notion of misrepresentation by comparing an agent's view of her world with the view the oracle would have of the same world.

Figure 6 shows an example in which some agents misrepresent the objects they perceive.

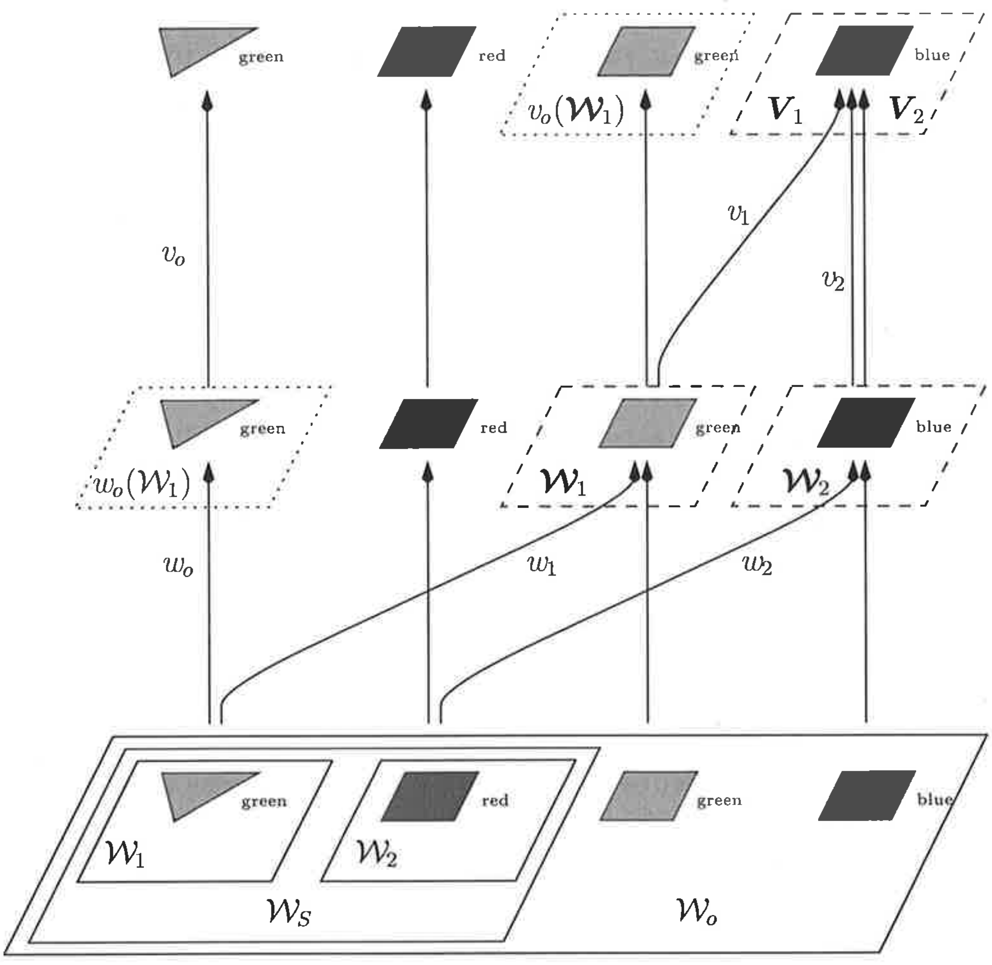

Figure 6. Misrepresentation

Before we analyse the example, let us recall some notation. Let $s \in S$ be an agent. Then the world actually perceived by the agent s is denoted by $\mathcal{W}_s$ and called a *perceived world*. The world of perceptions of s is denoted by $\boldsymbol{\mathcal{W}}_s$,

[7]This amounts to not following the language consensus.

called a *perception world*, and is assumed to be an image of $\mathcal{W}_s$ in a *perception mapping* w_s, i.e., $\boldsymbol{\mathcal{W}}_s = w_s(\mathcal{W}_s)$. The world containing a world description s provides is denoted by $\boldsymbol{V}_s$, called a *literate world*, and is assumed to be an image of $\boldsymbol{\mathcal{W}}_s$ in a *literate mapping* v_s, i.e., $\boldsymbol{V}_s = v_s(\boldsymbol{\mathcal{W}}_s)$. A similar convention applies to an *oracle*, an entity that has access to any worlds and objects and has perfect knowledge of them. Hence, $\mathcal{W}_o$, $\boldsymbol{\mathcal{W}}_o$ and $\boldsymbol{V}_o$ are the oracle's perceived, perception and literate worlds, and w_o and v_o are her (perfect) perception and literate mapping. Given the perception mappings w_s and w_o and the literate mappings v_s and v_o, the corresponding *representation mappings* are naturally defined as $\vartheta_s = v_s \circ w_s$ and $\vartheta_o = v_o \circ w_o$. Thus, the representation mappings send perceived worlds to literate worlds, $\vartheta_s(\mathcal{W}_s) = \boldsymbol{V}_s$ and $\vartheta_o(\mathcal{W}_o) = \boldsymbol{V}_o$. The oracle o forms her worlds $\mathcal{W}_o$, $\boldsymbol{\mathcal{W}}_o$ and $\boldsymbol{V}_o$ in such a way that they are *big enough*. In particular, they contain $\mathcal{W}_S = \bigcup_{s\in S} \mathcal{W}_s$, $\boldsymbol{\mathcal{W}}_S = \bigcup_{s\in S} \boldsymbol{\mathcal{W}}_s$ and $\boldsymbol{V}_S = \bigcup_{s\in S} \boldsymbol{V}_s$. The oracle does not put into the worlds $\mathcal{W}_o$, $\boldsymbol{\mathcal{W}}_o$ and $\boldsymbol{V}_o$ anything that is not related to the agents representational activity. Hence, we have the following.

$$\begin{array}{ccccccc} \mathcal{W}_o & = & \mathcal{W}_S & \cup & w_o^{-1}(\boldsymbol{\mathcal{W}}_S) & \cup & \vartheta_o^{-1}(\boldsymbol{V}_S), \\ \boldsymbol{\mathcal{W}}_o & = & w_o(\mathcal{W}_S) & \cup & \boldsymbol{\mathcal{W}}_S & \cup & v_o^{-1}(\boldsymbol{V}_S), \\ \boldsymbol{V}_o & = & \vartheta_o(\mathcal{W}_S) & \cup & v_o(\boldsymbol{\mathcal{W}}_S) & \cup & \boldsymbol{V}_S. \end{array}$$

We have $w_o(\mathcal{W}_o) = \boldsymbol{\mathcal{W}}_o$, $v_o(\boldsymbol{\mathcal{W}}_o) = \boldsymbol{V}_o$ and $\vartheta_o(\mathcal{W}_o) = \boldsymbol{V}_o$. The oracle can also apply her (perfect) mappings w_o, v_o and ϑ_o to the worlds of agents, and thus can perceive/represent what the agents would see, if they make no mistakes.[8] Let us say that an agent s is:

- *misperceiving*, if $w_o(\mathcal{W}_s) \neq \boldsymbol{\mathcal{W}}_s = w_s(\mathcal{W}_s)$,
- *misusing the language*, if $v_o(\boldsymbol{\mathcal{W}}_s) \neq \boldsymbol{V}_s = v_s(\boldsymbol{\mathcal{W}}_s)$,
- *misrepresenting*, if $\vartheta_o(\mathcal{W}_s) \neq \boldsymbol{V}_s = \vartheta_s(\mathcal{W}_s)$.

We can now analyse the example given in Figure 6. The worlds $\mathcal{W}_1$ and $\mathcal{W}_2$ are perceived by two agents s_1 and s_2. Suppose that the only object of $\mathcal{W}_1$ is a green triangle, and the only object in the world $\mathcal{W}_2$ is a red square. The process of forming the literate worlds $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ of the worlds $\mathcal{W}_1$ and $\mathcal{W}_2$ by the agents s_1 and s_2 is two-step. Firstly, the agents form their perception worlds $\boldsymbol{\mathcal{W}}_1$ and $\boldsymbol{\mathcal{W}}_2$ without using the language. Let s_1 be good at perceiving colours, but bad at perceiving shapes, and thus the agent's internal, language-free perception world $\boldsymbol{\mathcal{W}}_1$ of $\mathcal{W}_1$ be a perception the agent should form while seeing a green square, i.e., s_1 incorrectly perceives the object of $\mathcal{W}_1$ as a square rather than a triangle. Let in turn s_2 be good at perceiving shapes, but bad at perceiving colours, and let the agent's perception world $\boldsymbol{\mathcal{W}}_2$ be a perception the agent should form while seeing a blue square, i.e., s_2 incorrectly perceives the object of $\mathcal{W}_2$ as a blue object rather than red. In the second step, the agents employ the language—the agent s_1 is a bad language user, and although she

[8]Note that the classification is *weak*—an agent might be misperceiving some objects of the perceived world $\mathcal{W}_s$, without misperceiving the world as a whole. Similarly, an agent might be misusing the language on some perceptions of the perception world $\boldsymbol{\mathcal{W}}_s$, without misusing the language on the perception world as a whole. Indeed, we consider only *images* of the mappings. Apart from that, and surprisingly, an agent might "compensate" her misperception by her further language misuse, avoiding misrepresentation.

perceives the object of $\boldsymbol{\mathcal{W}}_1$ as 'green' she associates with the object an *attribute* like *IsBlue*, or *EstBleu*. The agent s_2 uses the language properly. The agents form their literal representations $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ of the worlds $\mathcal{W}_1$ and $\mathcal{W}_2$ they perceive, and the mistakes they make are summarised by and hidden in the literal worlds $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$. Employing the oracle, it is evident that the agents misrepresent their worlds, e.g., the agent s_1 misperceives her perceived world—$\boldsymbol{\mathcal{W}}_1 = w_1(\mathcal{W}_1) \neq w_o(\mathcal{W}_1)$—and she misuses the language when applied to her perception world—$\boldsymbol{V}_1 = v_1(\boldsymbol{\mathcal{W}}_1) \neq v_o(\boldsymbol{\mathcal{W}}_1)$.[9]

Given that agents can misrepresent the worlds they perceive, it is natural to accept *believed* worlds and objects as all that is (correctly) described by the agents, abandoning the perceived objects, the objects that "really are there" (but are possibly misrepresented). Such a shift is addressed in the next section, Section 3.3.

3.3 Believed worlds, or paradigm shift

Taking into account that agents can misrepresent their worlds—as demonstrated in Section 3.2—it is inevitable that the perceived worlds of agents are given up, or abandoned. Instead, what we can reason about, based on world descriptions provided by the agents, are believed and abstract worlds and objects—c.f. Figure 7, that corresponds to Figure 6 of Section 3.2.

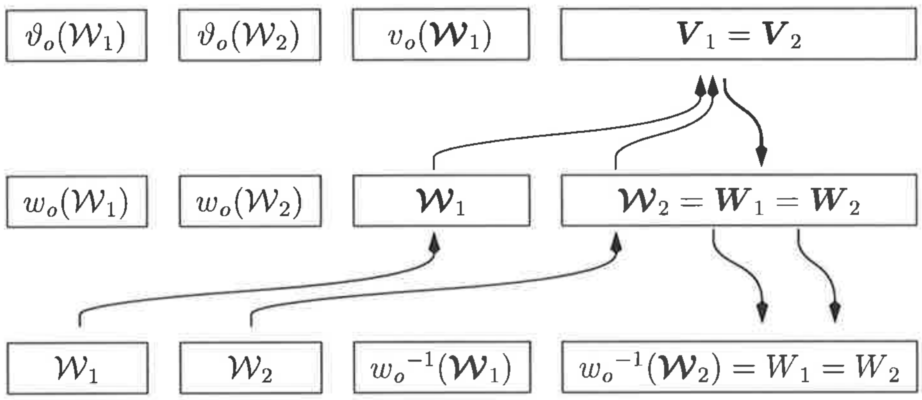

Figure 7. Abandoning perceived worlds

In fact, most of the considerations in this thesis are concerned with the relationship between literate worlds and abstract worlds, where literate worlds can be identified with language formulated descriptions, while abstract worlds can be identified with sets of abstract objects. Hence, we make a *paradigm shift*, as shown in Table 2.

Let $s \in S$ be a source of information (an agent), and let $\mathcal{W}_s$, $\boldsymbol{\mathcal{W}}_s$, $\boldsymbol{V}_s$, $\boldsymbol{W}_s$ and W_s be the perceived, perception, literate, abstract and believed world of s, respectively. $\mathcal{W}_s$ can be identified with a set of perceived objects $\mathcal{G}_s$. If

[9]Note also that *believed objects* of s_1 and s_2 are blue squares, even though there are no such objects in $\mathcal{W}_1$ or $\mathcal{W}_2$. The oracle however can "create" objects indistinguishable with objects of the agents.

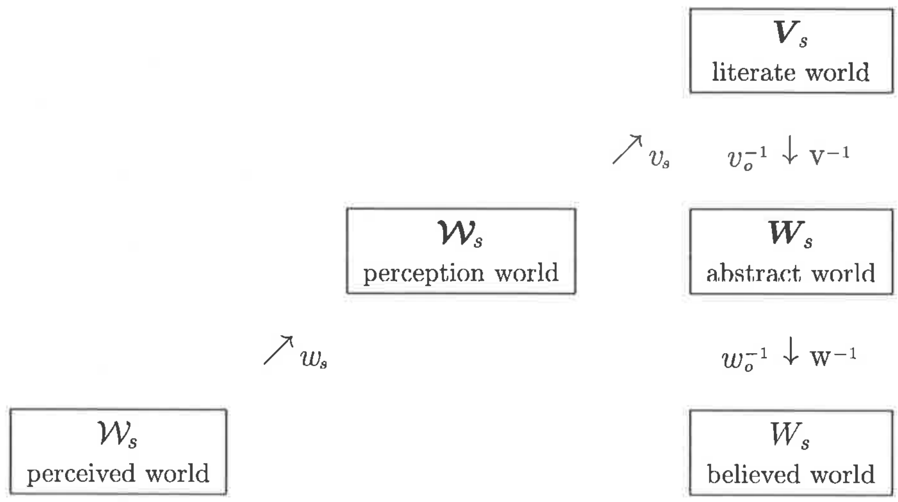

Table 2. Paradigm shift

perceptions of the objects $\mathcal{G}_s$ are denoted by $\boldsymbol{\mathcal{G}}_s$, then $\boldsymbol{\mathcal{W}}_s$ includes the set of perceptions $\boldsymbol{\mathcal{G}}_s$. The literate world $\boldsymbol{V}_s$ can be identified with a pair $(M, \boldsymbol{D}_s)$, where M is a set of attributes employed to report on the perceptions $\boldsymbol{\mathcal{G}}_s$, and $\boldsymbol{D}_s$ is a description set, a set of expressions in a language that forms the s's representation of her world $\mathcal{W}_s$. The abstract world $\boldsymbol{W}_s$ can be identified with a pair $(M, \boldsymbol{G}_s)$, where M is a set of attributes and $\boldsymbol{G}_s$ is a set of abstract objects, a subset of the set of all abstract objects $\boldsymbol{G}$ that can be formed over M. Finally, the believed world W_s contains believed objects G_s.

It was indicated at the beginning of this chapter that agents can perform various, language-related activities. More precisely, the following activities are of interest.

1. The society S of agents *invents* a language L.

2. An agent $s \in S$ *employs* the language L to describe its world.

   (a) A description provided by s is *verified* by S.

   (b) A description provided by s is *not verified.*

Discussing these activities will further clarify the meaning of perceived, perception, literate, abstract and believed worlds.

Inventing a language (activity 1) amounts to inventing attributes, and agreeing on associating them with objects—this is done via language consensus, as discussed in Section 3.1. For a given attribute m the agents of S need to agree which objects have m, and which do not. However, neither all objects need to be labelled (with m or $\overline{m}$), nor, for an object that is labelled, have all the agents to agree uniformly on whether the object has m or $\overline{m}$. All that is required is that they reach agreement (language consensus), as this allows them to make the invented labels *public*, so everyone can understand them. The objects S labels with the attributes are *accessible* to S, and the labels, or attributes, are newly *created* (or *invented*) *symbols*. Note that although the agents' perceptions (sense data) are *private*, the invented attributes (words) are *public*.

Suppose s provides a description of its world, and the description is verified by S (activity 2a). Assume that the perceived world of s—the "real world" s perceives—is the same as the perceived world[10] of S. On the one hand, when this kind of activity is performed, the society S can actually detect misrepresentation, i.e., can detect misperception and misusage of language, as described in Section 3.2. It is true that perceptions (or sense data) are *private*, but misperception can nevertheless be detected. It is not important what the perceptions actually are, but whether they change or not when the objects being perceived change. For instance, given a single object, for an agent s not to misperceive, the agent s should report different colour perceptions iff the colour of the object changes. Hence, the society S can detect that the mapping from the perceived world of s to its perception world is erroneous. On the other hand, when this kind of activity is performed S can actually detect that s is misusing the language. For instance, if S agrees that the only object of the world is red, but s reports a yellow object, it is simply a matter of "educating" s, or making s follow the language consensus, to ensure that the language is not misused. The perceived world gives rise to perceptions (experience data), but is not otherwise accessible, itself. Perceptions are private, but misperception is not undetectable. Attributes (words) are public, as it is assumed that the employed attributes are those on which S has reached language consensus.

Suppose now that s provides a description of its world, but the description is *not* verified[11] by S (activity 2b). In this case neither misperception nor language misusage can be detected. We have to stick to the worlds presented in the rightmost column of Table 2. The literate world of s accounts to the description s provides. The believed world corresponds to the perception world, in the sense that if the agent did not misrepresent, the worlds would be identical. Restrictions imposed on the language unable to differentiate between objects indiscernible w.r.t. the employed set of attributes. Hence, given the language, the semantic entities corresponding to description sets are abstract objects, where a single abstract object corresponds to a set of believed objects that are indiscernible w.r.t. the attributes—it is this indiscernibility relation that converts the believed world of the agent to its abstract world. It is this form of agents' activity (activity 2a) that interest us; furthermore, we want to know the structure of the world (i.e., what kind of objects *are* in the world, and what *are not*, rather than how many indiscernible objects are there) and the restricted language suits us well. The end result is that it is literate worlds (syntactic descriptions) and abstract worlds (abstract objects and contexts) that we focus on. The remaining sections of this chapter provide further clarification on literate and abstract worlds.

One could argue that the perceived world (roughly, the noumenal world in Kantian terms) is being set aside for present purposes, and that it would

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[10]The world perceived by both s and S can consist of a single object, say a red cube.

[11]S might not verify the description provided by s because the world of s might be inaccessible to the other agents of S, or S might fail to verify the description for some other reason.

be a mistake to accord any logical or semantic importance to the perceptual apparatus that mediates between it and the believed and 'literate' worlds. Indeed, noumena are "the external source of experience but are not themselves knowable and can only be inferred from experience of phenomena," [Fle84], see also [Cay95]. Perceived worlds are such noumenal worlds. They are sources of experience for our agents, but given that the agents can misrepresent, inferring the perceived worlds from experience is not guaranteed to be error-free. Therefore, what we infer from agents' *reported* experience are believed worlds, or rather abstract worlds (because of the language we accept), and hence the paradigm shift of Table 2.

In the rest of the thesis we will focus our attention on literate worlds (and the syntactic entities called description sets) and abstract worlds (and the semantic entities called abstract objects and abstract contexts). Summarising, we will be dealing with literate and abstract worlds (cf. Table 2), without any access to the real worlds of the agents. In other words, any imperfections of agents are hidden inside their literate worlds, i.e., within the description sets they provide. This means that agents can possibly lie, can misrepresent (misperceive and misuse the language), and can be placed in and describe distant worlds.[12] This is however what this thesis is aiming at: proposing a framework in which syntactic world descriptions provided by multiple agents, and the corresponding semantic entities, can be investigated and explained. Apart from modest, and not intended to be exhaustive, philosophical considerations, there is a well-defined mathematical content of the thesis. The rest of the thesis demonstrates the mathematics of conceptual reasoning, including defining a language, formal syntactic systems, corresponding semantic structures, and relating syntax to semantics. A concise formulation of these mathematical considerations can be found in [Now98].

Having said this, let us nevertheless include some related philosophical considerations.[13] An agent perceives an external world (perceived world) indirectly through direct experience of essentially private sense data—this can be seen as similar to the traditional empiricism of Locke [Woo83]. Locke is known as attacking innateness of ideas, or rather innateness of knowledge itself, [Woo83], and advocating knowledge derived from experience. Clearly, experience provides us with ideas about the world, and is a significant source of our knowledge of the world, even if there were some innate ideas that could not be obtained from experience. It should be noted that we allow our agents to misperceive, and thus an agent can perceive, e.g., "blue" as "magenta" but we do not intend to detect misperceptions. Regarding the use of language, we accept that the agents' minds are private, but do not require them to be accessible, so that it can be checked whether expressions are used correctly—we do not intend to detect that the language is misused. Misperception in turn is discussed not to suggest that we intend to detect it, but to say that our agents might be imperfect.

[12] If an agent is imperfect in any of these ways, then the reported world is expected to be distant from an "average" or commonly accepted world—and this is the only way we can "verify" the agent, given that we assume no final authority (society, oracle, God) that can uncover agents' imperfections.

[13] What follows is an attempt to clarify our philosophical standpoint.

Certainly, it is evident that we want to "combine" the experience reported by multiple agents, but we do not expect all the reports to be compatible (non-contradictory), and therefore we want to derive a "dominating" view of the world, rather than a perfect one. The problem of privateness of sense data is removed by our language consensus assumption—given that it is publicly known how to associate labels (words, attributes) with objects, we expect the agents to follow the consensus, although we do not assume that agents are perfect language users. Talking about knowledge derived from experience, it should also be noted that we do not assume that agents report pure sense data. Following [SH67], the notion of *experiential data* covers objects of direct experience, such as *sense data*, and experiences themselves (sensations, emotional feelings, thoughts); *psychological states* include such experiential data, but also *mental phenomena* (beliefs, attitudes, moods, intentions, mental abilities), see also [Dan88]. An agent might report not only some raw sense data, but employ attributes denoting some psychological states, e.g., beliefs—the ultimate goal is to "combine" descriptions of multiple agents, see how they are related, whether they are compatible or contradictory, and find the dominating view(s).

The *private language argument* is a related philosophical problem. Usually credited to Wittgenstein ([Aye85, Ken73]), with his *Philosophical Investigations* being the most cited reference, it can take a form of a seemingly simple question: can there be a private language? In a private language, the words (terms) are defined to refer to the private sense data of an agent, and thus the meanings of the terms are only understood by the agent, and no-one else. In [SH67], it is said that Locke "had a theory about words and their meanings that made out public languages to be somehow derived from numerous private languages." Wittgenstein argues in his *Philosophical Investigations* that a logically private language is impossible. Many different opinions on the argument are presented in [Jon71]; according to the Cartesian view all psychological states are private, they are private objects, and there are words for private objects (like toothache), just as there are words for public objects (like chair). Jones, in the introduction to [Jon71] says: "Firstly, it is fairly generally agreed that the existence of a language involves the following of rules [...] Secondly, it is agreed that rule-following presupposes the possibility of checking on the application of words, thereby making sure that the rule is being correctly followed," and then he says: "Could there be any possible check on the application of words used to refer to private objects such as the Cartesians take sensations and feelings to be? Wittgenstein, I think, is suggesting that it would not be possible if the objects were private [...]" In the collection ([Jon71]), Strawson accepts that naming a sensation involves a practice in applying the name, but also believes that an agent can do this privately, and that memory is sufficiently reliable to perform the checkings. Malcolm in turn distinguishes between following a rule and being under the impression that one follows the rule—therefore, memory checks are insufficient. Rhees (same collection) claims that rule-following could not be done privately. Jones himself submits a nice example ([Jon71], pp.19–20):

> Suppose my friend claims that he can recognise a certain property of an iron bar simply by grasping it with his hand. I am told, however,

> that what he recognises is not one of the well-known properties like roughness or shape, but something which he calls 'ponk.' The first difficulty here is not the difficulty about knowing if my friend is right when he says that a particular bar of iron is ponk. There is a more fundamental difficulty that comes before that. The difficulty is that one cannot make any sense of the notion of his being right or wrong;

On the one hand, the example invites to reject the possibility of a private language; however, it also indicates that a private object can become public, with a publicly knowable word, and a possible check on the application of the word.

Kripke [Kri82] also sees the importance of the private language argument in the issue of following the rule (and therefore the ability to perform a check on the rule application). In both [Kri82] and [SH67], the following quote from the Philosophical Investigations is given: "To think one is obeying a rule is not to obey a rule. Hence, it is not possible to obey a rule 'privately;' otherwise, thinking one was obeying a rule would be the same thing as obeying it." The *diary-keeper* argument discussed in [Sme70] is of similar mode—it is said the the diary-keeper has no "acceptable means of distinguishing between correct and incorrect use of [...] sign."

Many instructive comments on the private language problem can be found in a collection edited by Martinich [Mar96], especially in the last part of the book devoted to the nature of language. Martinich says:

> Linguistic communication, [Wittgenstein] believes, is rule-governed behavior, and it does not make sense to say that someone is following a rule unless there is some way of judging whether the rule has been followed or broken. The speaker himself cannot be the final arbiter of this. The judge of whether a rule has been followed or not, like any standard of evaluation, must be separate from and independent of the matter to be decided.

Locke's theory of meaning suggest that language is private—cf. Locke's paper titled "Of Words" in [Mar96]. John Cook, in his "Wittgenstein on Privacy" discusses private and public objects, rejecting the possibility of a private language. Martinich refers to a paper by A. J. Ayer titled "Can There Be a Private Language?" explaining that "[Ayer] said that there is nothing privileged about the publicness of the meaning-verification that Wittgenstein seems to require. All justification and verification of whatever sort must end somewhere. Further, all justification must end with some sense perception—for example, seeing or hearing the judgment of other people that one has or has not followed a rule of speaking correctly—so one may just as well end with one's own private sensation." As Martinich notes, Saul Kripke in "On Rules and Private Language" (reprinted in [Mar96]) replies to Ayer "by saying that Wittgenstein recognizes that all verification ends somewhere and that one might always doubt the veracity of one's perceptions. But Wittgenstein's point is that, skepticism notwithstanding, he has correctly described how human languages work; that humans in fact do end their justification will certain rule-governed publicly observable behavior and not with private sensations; and such a practice does not require any justification."

In [Hac72], it is claimed that a justification for a proper use of the language needs to appeal to something independent, suggesting that memory checks performed privately by a single agent might not be sufficient.

It seems that one of the main points for (or against) the possibility of a private language is whether a check can be performed on the application of the rules for using the language. Certainly, obeying a rule privately is not as good as doing it publicly, but this does not mean that the public, or society, does not make mistakes, failing to check the correctness of the rule application. Maybe there are not just the two extremes: purely private and purely public objects, but the whole spectrum of objects in between—as with justifications, they could be ordered, rather than simply accepted or rejected—as we shift from knowledge (justified beliefs) to beliefs (possibly incorrect), maybe we can shift from public verification to private, even if less secure, verification. This suggests that maybe requiring public verifiability of languages is too strong a request, even though it is clear that verifiability is associated with how good a communication tool the language is.

Referring to our framework, agents are supposed to follow the language consensus, and thus employ public words (attributes). Clearly, public words can be successfully used to describe agents' sense data (primary candidates for private objects)—therefore, it seems that it is a public language that is in use, here. However, even if a private language was possible, it would not affect the framework—agents are not forced to use public words, but by using private ones they risk moving themselves away from the other agents. If there was a private language, if there were private objects, private words for referring to them, private application rules with possible (private, relying on memory) checks, the framework would still do its job, although the produced output would be of similar quality to the private language inputs of the agents, quality of the communication language the agents employ.

3.4 Formulae and regions

Before we introduce formulae and regions, we need to make a connection between a society that develops a language, and a single agent, or source of information that employs the language to describe her world.

Suppose that a language containing attributes M has already been formed, i.e., a language consensus has been reached, and hence it is commonly agreed how attributes associate with objects. Let now S be a set of agents that are sources of information, that employ the language to describe their worlds, but do not modify or develop it any more. Let $s \in S$ be one of such sources of information, and G_s be a world—or set of objects—of s. We assume that s observes the language consensus to the extent that there is no object in G_s believed by s to have both m and $\overline{m}$—i.e., $\varepsilon(m) \cap \varepsilon(\overline{m}) = \emptyset$—but it is *not* assumed that the agent, provided with $m \in M$, partitions the world G_s into complementing subsets—i.e., it is *not* assumed that $\varepsilon(m) \cup \varepsilon(\overline{m}) = G$. In other words, we allow the agent s to have *partial information* of her world. This seems reasonable—when forming a language, the society selects such objects it can agree upon, and can completely classify, or partition the objects, leaving troublesome objects aside. But when a single agent observes her world, it is necessary to account for partiality of the agent's knowledge—if $g \in G_s$ and $m \in M$ then s accepts that g has exactly one of the attributes m and $\overline{m}$, but s might not know which of the two the object has. Consider the situation depicted in Figure 8.

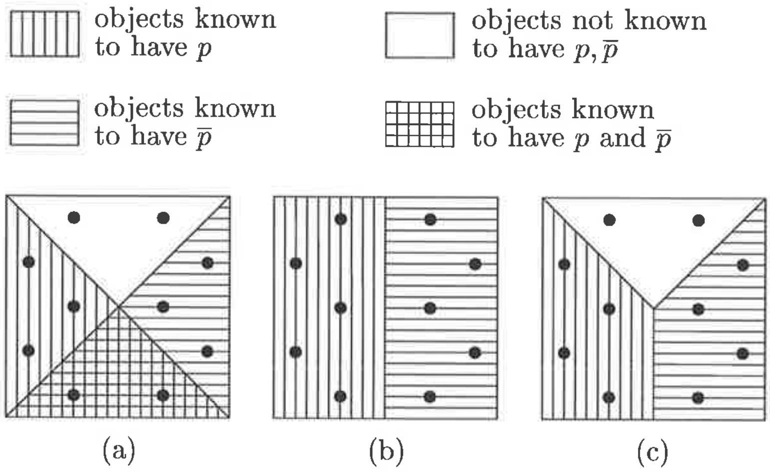


Figure 8. Restrictions on objects

The case (a) of Figure 8 is disallowed—objects must not have both p and $\overline{p}$. If the agent can classify every object w.r.t. which of p and $\overline{p}$ the object has, then she partitions the world G_s as shown in Figure 8 (b). We allow the agent to have partial knowledge of her world: given an attribute p she divides G_s into three regions (some of them might be empty), as shown in Figure 8 (c). One could say that s's knowledge of the objects of G_s is "consistent" but does not need to be "complete."

Note that G_s should be understood as the set of objects believed by s to be in her world, and in "reality" the agent's world might be different than she believes and describes it to be. In particular, she might be misperceiving the world, and misusing the language, as discussed in Section 3.2. Note however, that we have abandoned perceived objects—cf. Figure7—and are satisfied with believed objects. Hence, s describes her believed world G_s, and all we want to know is what this believed world looks like. This allows us to assume that s's knowledge of G_s is unmistaken, even though it might be partial. Hence, the agent can correctly partition her world into three regions, and can correctly report the nonemptiness of the regions—she *knows* her believed world, and her believed world is all we are interested in. As a result, the believed world is not misrepresented, even though it might be distant from the "real" world of s.

Assuming that we are dealing with a single agent, and her corresponding world, we omit the subscript s and denote the world by G and its "regions" by $\boldsymbol{R}$—instead of G_s and $\boldsymbol{R}_s$.

Consider now the situation depicted in Figure 9. Let G be a set of objects, and let $P = \{p_1, p_2, p_3\}$ be a set of (non-negated) attributes. Further, let ε be a mapping from attributes to subsets of G, and hence $\boldsymbol{p}_1 = \varepsilon(p_1), \ldots, \boldsymbol{p}_3 = \varepsilon(p)_3$ and $\overline{\boldsymbol{p}_1} = \varepsilon(\overline{p_1}), \ldots, \overline{\boldsymbol{p}_3} = \varepsilon(\overline{p_3})$.

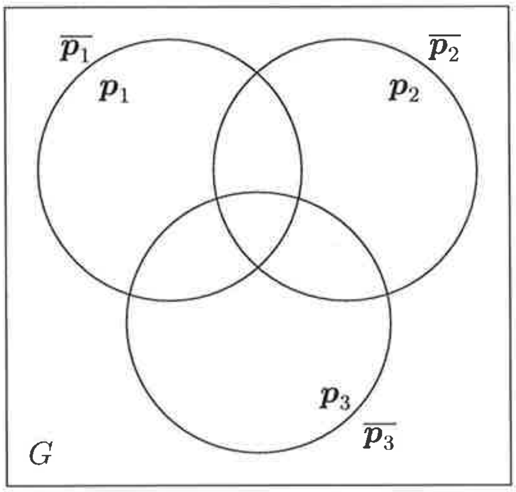


Figure 9. World G and its properties

The subsets $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_3, \overline{\boldsymbol{p}_1}, \ldots, \overline{\boldsymbol{p}_3}$ of G are called *properties*, and constitute some of the "regions" of the world. Let us introduce formal definitions of formulae and regions.

Let $M = P \cup \overline{P}$ be a set of attributes. We define a corresponding set of *formulae* as follows.

Definition 2 *Let M be a set of attributes. Then*

$$\boldsymbol{F} = \{F \subseteq M \mid \forall_{p \in P}\ F \not\supseteq \{p, \overline{p}\}\}$$

is a corresponding set of formulae.

Let α denote the cardinality of P, i.e., $|P| = \alpha$ and $|M| = |P| + |\overline{P}| = 2\alpha$. An example of a set of formulae $\boldsymbol{F}$ for $\alpha = 3$ is presented in Figure 10.

Let G be a world, or set of objects. Given a set of attributes M—and hence also the corresponding set of formulae $\boldsymbol{F}$—one can consider certain subsets of the set of objects G, as determined by $\boldsymbol{F}$.

Definition 3 *Let G, M and $\boldsymbol{F}$ be sets of objects, attributes and formulae, respectively. Let ε be a language consensus mapping singleton subsets of M to subsets of G—if $m \in M$ then $\varepsilon(\{m\}) \subseteq G$. Extend ε as follows:*

$$\text{if } F \cup \{m\} \in \boldsymbol{F} \text{ then } \varepsilon(F \cup \{m\}) = \varepsilon(F) \cap \varepsilon(\{m\}),$$

i.e., $\varepsilon \colon \boldsymbol{F} \longrightarrow \mathcal{P}(G)$. Then define:

$$\boldsymbol{R} = \{R_i \subseteq G \mid F_i \in \boldsymbol{F} \text{ and } R_i = \varepsilon(F_i)\},$$

i.e., $\boldsymbol{R} = \varepsilon(\boldsymbol{F})$. If $R \in \boldsymbol{R}$ then R is called a region *(of G).*

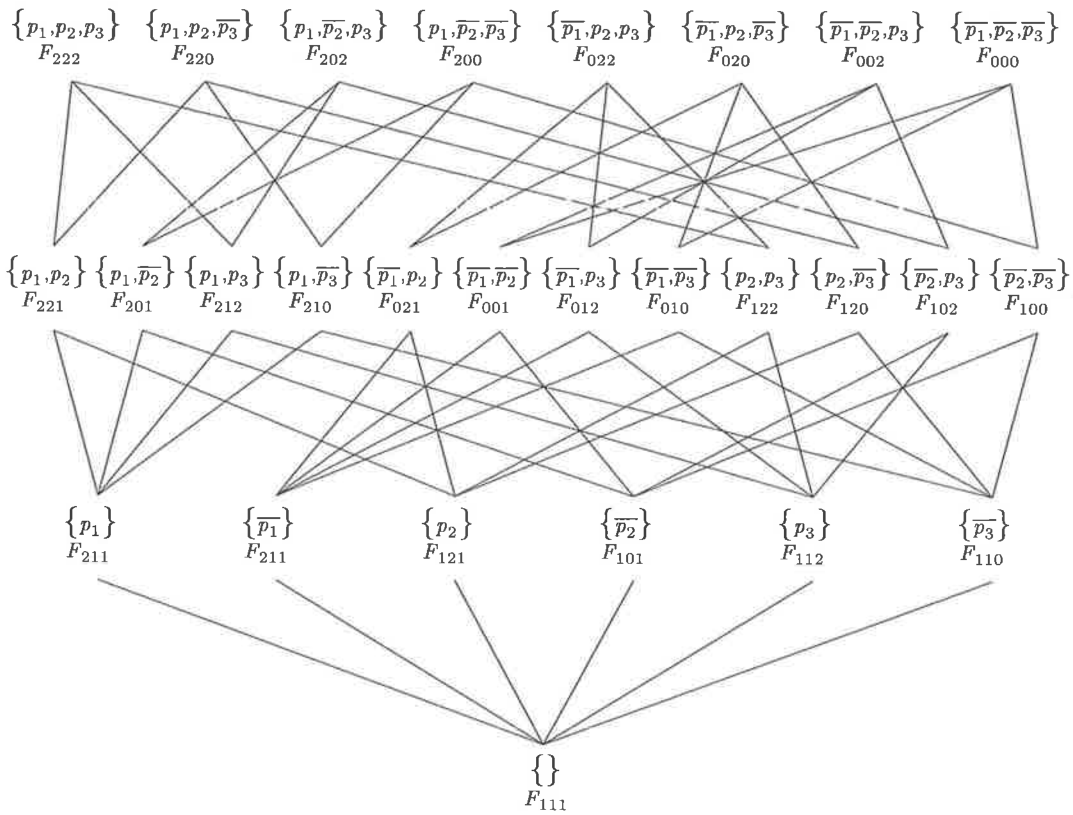


Figure 10. Formulae, $|P| = 3$

An example of a set of regions $\boldsymbol{R}$ for $|P| = \alpha = 3$ is presented in Figure 11.

Note that the extended mapping ε of Definition 3 maps formulae to regions, i.e., $\varepsilon\colon \boldsymbol{F} \longrightarrow \boldsymbol{R}$, and given an $F \in \boldsymbol{F}$ we have $\varepsilon(F) = R_F$, where the region R_F is a set of objects that have all the elements of F as their attributes. Note that if attributes in F are all that matters to us, then R_F consists of indistinguishable—or *indiscernible*—objects. Note also that F not only determines the region R_F but also partitions G into a set of regions, and R_F is one of those regions. To say this formally we introduce a relation of F-*indiscernibility*, characterised by the following property: g_1 and g_2 are F-indiscernible if, for any attribute $m \in F$, g_1 has m iff g_2 has m.

Definition 4 *Let* $g_1, g_2 \in G$ *and* $F \in \boldsymbol{F}$. *We say that* g_1 *and* g_2 *are* F-indiscernible, *denoted* $g_1 \approx_F g_2$ iff$_{\text{def}}$ $\text{attrs}(g_1) \cap F = \text{attrs}(g_2) \cap F$, *where* $\text{attrs}(g)$ *is the set of attributes of the object* g.

Given $F \in \boldsymbol{F}$ *we use the notation* $G/{\approx_F}$ *to denote the set of equivalence classes determined by* $\approx_F$, *i.e.,* $G/{\approx_F} = \{G_i \subseteq G \mid g_1, g_2 \in G_i \text{ iff } g_1 \approx_F g_2\}$.

Note that $G/{\approx_F}$ is indeed a set of equivalence classes, because $\approx_F$ clearly is an equivalence relation (i.e., it is reflexive, symmetric and transitive). We have that $R_F \in G/{\approx_F}$, but $G/{\approx_F}$ is usually not a singleton set. Consider the example of Figure 11—see also Figure 10. Consider the case of $F_{221} = \{p_1, p_2\}$—then $G/{\approx_{F_{221}}} = \{R_{221}, R_{201}, R_{021}, R_{001}\} = G/{\approx_{F_{201}}} = G/{\approx_{F_{021}}} = G/{\approx_{F_{001}}}$. In general, we have that $\boldsymbol{R} = \bigcup_{F \in \boldsymbol{F}} G/{\approx_F}$, and so every region of the world is an equivalence class of indiscernible objects.

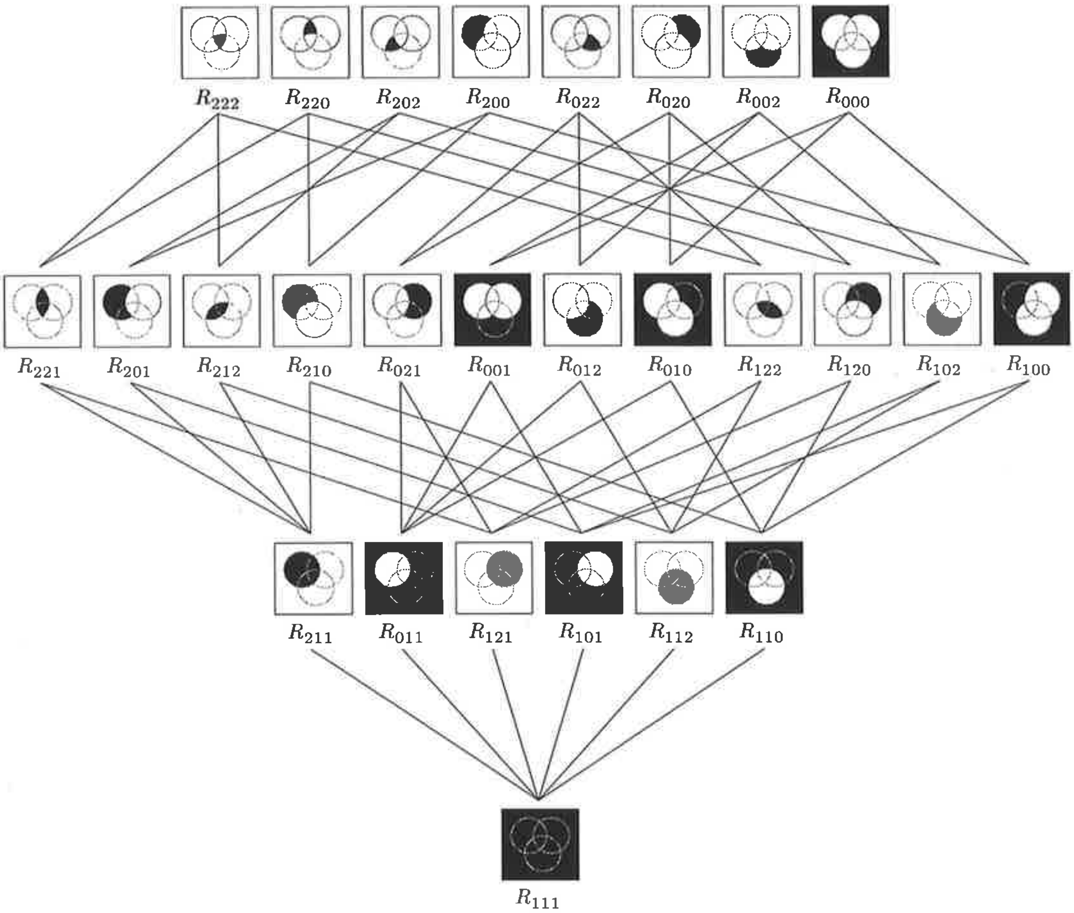


Figure 11. Regions, $|P| = 3$

If G is a set of objects and M is a set of attributes, then there is a set of regions corresponding to formulae over M. If R is a region corresponding to an $F \in \boldsymbol{F}$ then the simplest question one might ask is whether or not R is empty. In fact, information about nonemptiness of regions of the world is all we want to know. In Section 3.5 we provide a language which allows us to formulate claims about the nonemptiness of regions. Such claims, when made about a world, form a description set of the world. In Section 3.6 we introduce *abstract objects.* These correspond to (some) nonempty regions of the world and allow us to provide the same description of the world as the world itself does.

3.5 Language of descriptions

Let $M = P \cup \overline{P}$ be a set of attributes, and let $\boldsymbol{F}$ be the corresponding set of formulae, as given by Definition 2. This pair of M and $\boldsymbol{F}$ gives a *language.*

Definition 5 *Let M and $\boldsymbol{F}$ be attributes and formulae, respectively. Then*

$$\boldsymbol{L} = (M, \boldsymbol{F})$$

is referred to as a language of formulae.

Hence, M itself determines the language, but it is the elements of $\boldsymbol{F}$—these are called formulae—that are seen as elements of the language.

Recall now that if G is a set of objects—the world under consideration—then $\boldsymbol{F}$ determines regions of the world, as given by Definition 3 (but note that two regions corresponding to two different formulae might happen to be exactly the same set of objects of the world). What we want to be able to say about the world is whether or not regions corresponding to formulae are empty. Hence, given a formula $F \in \boldsymbol{F}$ we would like to "mark" the formula with a symbol, say, $\oplus$, if the corresponding region $R_F \in \boldsymbol{R}$ is non-empty, and mark it with a symbol $\ominus$, if the region is empty. Hence the following definition.

Definition 6 *Let $\boldsymbol{F}$ be a set of formulae. Let*

$$\boldsymbol{D} = \boldsymbol{F} \times \{\oplus, \ominus\}.$$

Call elements of $\boldsymbol{D}$ descriptions. *Moreover, when denoting descriptions, employ the following convention,*

$$\oplus F \;=_{\text{notation}}\; (F, \oplus) \quad \text{and} \quad \ominus F \;=_{\text{notation}}\; (F, \ominus),$$

yielding $\boldsymbol{D} = \{\oplus F, \ominus F \mid F \in \boldsymbol{F}\}$. Call

$$\mathbb{L} = (M, \boldsymbol{D})$$

a language of descriptions.

Hence, M itself determines the description language, but it is the elements of $\boldsymbol{D}$ that are seen as elements of the language.

Formally, considering the language $\boldsymbol{L} = (M, \boldsymbol{F})$, the set $M = P \cup \overline{P}$ is a set of symbols, and $\boldsymbol{F}$ is a set of formulae built using the symbols of M according to Definition 2, i.e., if $m \in M$ then $\{m\} \in \boldsymbol{F}$, and recursively, if $F_1, F_2 \in \boldsymbol{F}$ and $\forall_{p \in P}\; F_1 \cup F_2 \not\supseteq \{p, \overline{p}\}$ then $F_1 \cup F_2 \in \boldsymbol{F}$.

Considering the language $\mathbb{L} = (M, \boldsymbol{D})$ the procedure is similar, but instead of bare $\boldsymbol{F}$ we employ two "copies" of $\boldsymbol{F}$ consisting of *marked formulae*, namely $\{\oplus F \mid F \in \boldsymbol{F}\}$ and $\{\ominus F \mid F \in \boldsymbol{F}\}$, and the union of these forms the set of all possible descriptions (over M), i.e., $\boldsymbol{D} = \{\oplus F, \ominus F \mid F \in \boldsymbol{F}\}$. Hence, the elements of $\boldsymbol{D}$ can be seen as "formulae" formed in two stages: at first, $\boldsymbol{F}$ is built, and next every formula of $\boldsymbol{F}$ is preceded with $\oplus$, $\ominus$, or replaced by two pairs $(F, \oplus)$, $(F, \ominus)$.

Note that we treat all elements of M as symbols: if $p, \overline{p} \in M = P \cup \overline{P}$ then not only p but also $\overline{p}$ is treated as a single symbol. Alternatively, and equivalently, we could explicitly employ a *negation operator* $\overline{}$ which, when applied to P gives $\overline{P} = M \setminus P$. Indeed, $\overline{}$ of the symbol $\overline{p}$ can be interpreted as "a kind of negation"—cf. Section 3.1. Suppose we treat $\overline{}$ as such. Then the negation operator only applies to elements of P, or to elements of M given that $\overline{(\overline{p})} = p$, but not to formulae of $\boldsymbol{F}$—we do not intend to have something like $\overline{\{p_1, p_2\}}$ even though we have $\oplus\{p_1, p_2\}$ and $\ominus\{p_1, p_2\}$ in $\boldsymbol{D}$, because formulae of $\boldsymbol{F}$, e.g., $\{p_1, p_2\}$ or $\{\overline{p_1}, p_2\}$ refer to regions of the world, while *descriptions* in $\boldsymbol{D}$ make claims about the nonemptiness of regions. If $\overline{}$ had indeed been employed as a negation operator then such steps would have been taken.

Note also that although *symbols* like p or p_1 are treated as *atomic*, they will inevitably be composed of *characters*, or *symbols of an alphabet*—and so they will be *strings* over the alphabet. For instance, an alphabet $\Sigma = \{A, \ldots, Z, a, \ldots, z\}$ could be employed and then an attribute, say $p = \mathit{IsRed}$, would be a string over Σ, or an element of Σ^*, or Σ^+. This is indeed what we do for most of the examples involving attributes, whenever the attributes are associated with natural language words.

Consider the following instances of attributes, formulae and descriptions. Let $p_1 = \mathit{IsRed}$ and $p_2 = \mathit{IsCar}$ be attributes, $P = \{p_1, p_2\}$. Consider some of the formulae of $\boldsymbol{F}$, e.g., $F_a = \{p_1, p_2\} = \{\mathit{IsRed}, \mathit{IsCar}\}$ and $F_b = \{\overline{p_1}, p_2\} = \{\overline{\mathit{IsRed}}, \mathit{IsCar}\}$. These formulae determine two regions of the world, namely the region of those objects that are "red cars", and the region of those that are "not-red cars," respectively. Consider some of the descriptions of $\boldsymbol{D}$, e.g., $D_c = \ominus F_a = \ominus\{p_1, p_2\} = \ominus\{\mathit{IsRed}, \mathit{IsCar}\}$ and $D_d = \oplus F_b = \oplus\{\overline{p_1}, p_2\} = \oplus\{\overline{\mathit{IsRed}}, \mathit{IsCar}\}$. These descriptions make "nonemptiness claims" about the corresponding regions of the world, namely the claim that the "region of red cars" is empty, and that the "region of not-red cars" is not empty, respectively. Note that e.g., the description $\oplus\{\overline{\mathit{IsRed}}, \mathit{IsCar}\}$ is—by the convention adopted in Definition 6—equivalent to $(\{\overline{\mathit{IsRed}}, \mathit{IsCar}\}, \oplus) \in \boldsymbol{F} \times \{\oplus, \ominus\}$.

Consequently, if $s \in S$ is a source of information, or an agent describing her world, we can say the following about the description s provides. Let $\boldsymbol{D}_s \subseteq \boldsymbol{D}$ be the *description set* of s. Recall that $\boldsymbol{D}_s \subseteq \boldsymbol{F} \times \{\oplus, \ominus\}$. Subsequently, given $\boldsymbol{D}_s$ we can find a corresponding pair of subsets of $\boldsymbol{F}$ as follows. Define $\boldsymbol{F}_s^{\oplus} = \{F \in \boldsymbol{F} \mid \oplus F \in \boldsymbol{D}_s\}$ and $\boldsymbol{F}_s^{\ominus} = \{F \in \boldsymbol{F} \mid \ominus F \in \boldsymbol{D}_s\}$. Clearly, the descriptions set $\boldsymbol{D}_s$ can be identified with the pair $(\boldsymbol{F}_s^{\oplus}, \boldsymbol{F}_s^{\ominus})$. The sets $\boldsymbol{F}_s^{\oplus}$ and $\boldsymbol{F}_s^{\ominus}$ are referred to as the *set of* $\oplus$*-formulae of* s and the *set of* $\ominus$*-formulae of* s, respectively. Conversely, given a pair $(\boldsymbol{F}_s^{\oplus}, \boldsymbol{F}_s^{\ominus})$ of the set of $\oplus$-formulae and the set of $\ominus$-formulae of s, the corresponding description set is given by $\boldsymbol{D}_s = \boldsymbol{F}_s^{\oplus} \times \{\oplus\} \cup \boldsymbol{F}_s^{\ominus} \times \{\ominus\}$. One might think that this is complicating the notation unnecessarily. However, consider a simple example of a description set consisting of the two descriptions of the preceding paragraph, i.e., let $\boldsymbol{D}_s = \{D_c, D_d\} = \{\ominus F_a, \oplus F_b\} = \{\ominus\{p_1, p_2\}, \oplus\{\overline{p_1}, p_2\}\} = \{\ominus\{\mathit{IsRed}, \mathit{IsCar}\}, \oplus\{\overline{\mathit{IsRed}}, \mathit{IsCar}\}\}$ and note that the description set presented in this way is much more readable than the equivalent form of $(\boldsymbol{F}_s^{\oplus}, \boldsymbol{F}_s^{\ominus}) = (\{F_b\}, \{F_a\}) = (\{\{\overline{p_1}, p_2\}\}, \{\{p_1, p_2\}\}) = (\{\{\overline{\mathit{IsRed}}, \mathit{IsCar}\}\}, \{\{\mathit{IsRed}, \mathit{IsCar}\}\})$. However, it is convenient to refer to $\boldsymbol{F}_s^{\oplus}$ and $\boldsymbol{F}_s^{\ominus}$ of $\boldsymbol{D}_s = (\boldsymbol{F}_s^{\oplus}, \boldsymbol{F}_s^{\ominus})$. For instance, we can say that a description set $\boldsymbol{D}_s$ is *consistent* if $\boldsymbol{F}_s^{\oplus} \cap \boldsymbol{F}_s^{\ominus} = \emptyset$, or that s makes *global claims* if $\boldsymbol{F}_s^{\ominus} \neq \emptyset$.

The issue of language—in particular, the question whether a different language should be employed—is taken up in Section 5.1, where specifying a language is a part of defining a formal system, and again in Section 6.4, where a comparison with propositional languages is made.[14]

[14] A propositional language for dealing with sets of attributes is employed by Ganter in [Gan96b].

3.6 Information ordering on abstract objects

In this section we introduce *abstract objects* associated with equivalence classes of indiscernible objects mentioned in Section 3.4. The idea is that abstract objects correspond to nonempty regions—or equivalence classes of indiscernible objects—but we keep abstract objects as "specialised," or as little "partial" as possible.

Consider Figure 12—cf. also Figure 8.

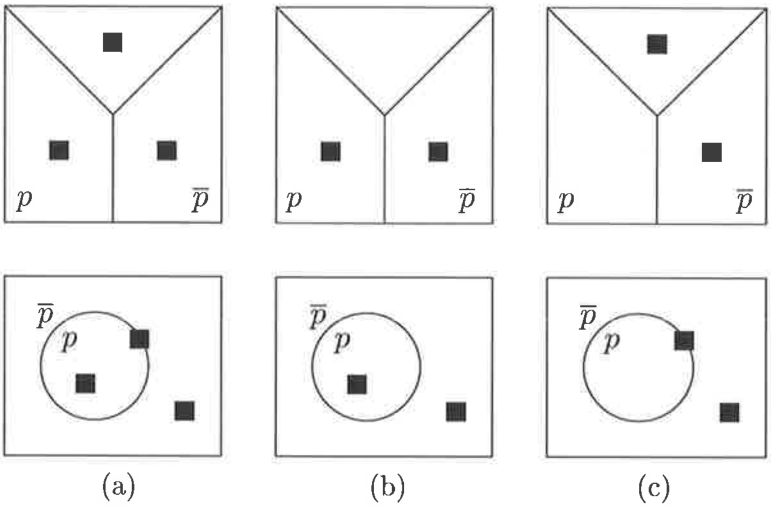


Figure 12. Restrictions on abstract objects

As discussed in Section 3.4, given a single attribute p we divide objects into three sets, as shown in Figure 8 (c). What we are then interested in is which of these sets are nonempty. Knowing this we can replace sets of indiscernible objects with single "abstract" objects; for example, objects of Figure 8 (c) can be replaced with abstract objects as shown in the top line of Figure 8, where if some of the sets are empty then a smaller set of abstract objects is used, as shown in Figure 12 (b,c). The bottom line of Figure 12 shows the same abstract objects, but "partial," or undetermined abstract objects. These are placed on the "border" separating objects that have p from those that have $\overline{p}$—a partial object can fall on either side, when it becomes determined w.r.t. the attribute. For instance, in Figure 12 (a) there are three abstract objects—one has the attribute p, another one has $\overline{p}$, and the last one has an empty set of attributes—nothing is known about it, but it could give rise to a less "partial" abstract object, as soon as we received more information.

Note that restrictions on abstract objects reflect the restrictions on objects as discussed in Section 3.4—abstract objects can be *partial*, but not *inconsistent*.

Formally, we introduce abstract objects as follows. Let M be a set of attributes, and let $\boldsymbol{F}$ be the corresponding set of formulae. Given a formula $F \in \boldsymbol{F}$, we associate with F an *abstract object*, denoted by $\boldsymbol{g}_F$, such that it has

F as its set of attributes.[15] Hence, given M there is a set of abstract objects associated with elements of $\boldsymbol{F}$.

Definition 7 *Let M and $\boldsymbol{F}$ be attributes and formulae, respectively. Define a set of* abstract objects *(over M), denoted by $\boldsymbol{G}$, or $\boldsymbol{G}_M$, as follows:*

$$\mathbf{G} = \{\boldsymbol{g}_F \mid F \in \boldsymbol{F}, \text{ and } \operatorname{attrs}(\boldsymbol{g}_F) = F\}.$$

Note that the construction of $\boldsymbol{F}$ disallows inconsistent abstract objects that have "opposite" attributes m and $\overline{m}$.

Figure 13 shows abstract objects, where $\alpha = |P| = 3$. Note that given a finite set of attributes, there is a finite number of abstract objects determined by the attributes, regardless the number of objects we consider. Indeed, a single abstract object could "represent" an infinite number of objects. (Figure 13 seems to demonstrate that sets of abstract objects, when $|P|$ increases, quickly become difficult to visualise. However, looking ahead to Section 3.7, *abstract contexts* are employed to represent them. The cardinality of P is the number of columns in the table representing the context together with its objects.)

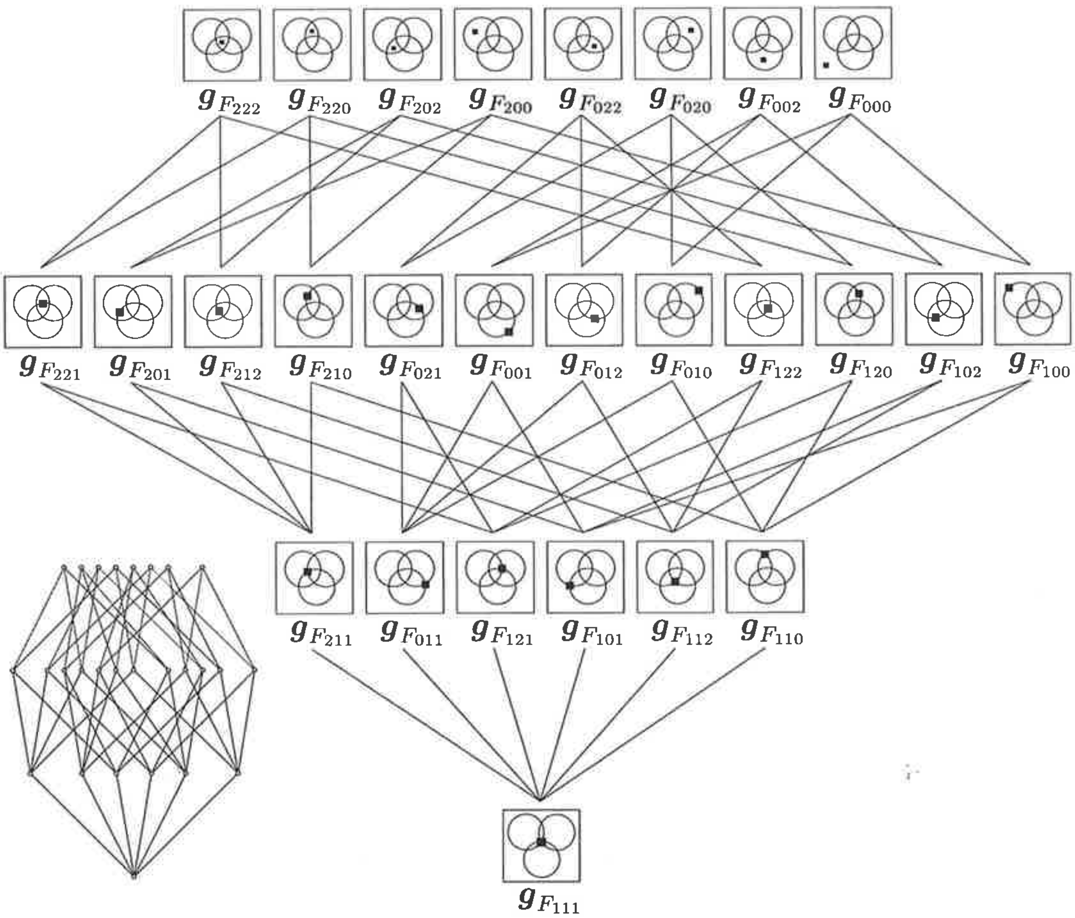


Figure 13. Abstract objects, $|P| = 3$

[15]Given an abstract object $\boldsymbol{g}_F$, the set of attributes of the object is exactly F, for if $\operatorname{attrs}(\boldsymbol{g}_F)$ is a set of attributes of $\boldsymbol{g}_F$ then $|\operatorname{attrs}(\boldsymbol{g}_F)| = |F|$.

Given the above definition of abstract objects, it is easy to see that abstract objects convey information, and some of them convey more information than others. For instance, using the notation of Figures 13 and 10, the abstract object $\boldsymbol{g}_{F_{111}}$—corresponding to the formula $F_{111} = \emptyset$—tells us nothing but that the world is not empty. Any other abstract object tells us more, as we are also told that some attributes are "instantiated." This leads to the following definition of *information ordering* on abstract objects.

Definition 8 *Let $M = P \cup \overline{P}$ and $\boldsymbol{G}$ be a set of attributes, and a set of all abstract objects (over M), respectively. Define an* information ordering *relation $\leq$ on $\boldsymbol{G}$ by $\boldsymbol{g}_{F_1} \leq \boldsymbol{g}_{F_2}$* iff$_{\text{def}}$ *$F_1 \subseteq F_2$. Then the ordered set $(\boldsymbol{G}, \leq)$ represents information ordering on abstract objects.*

*Let $|P| = \alpha$. If $\boldsymbol{g}_F$ is such that $|F| = \alpha$ then $\boldsymbol{g}_F$ is called M-*total*, otherwise it is called (properly)* partial. *We also say that $\boldsymbol{g}_{F_1}$ is* more partial *than $\boldsymbol{g}_{F_2}$, whenever $\boldsymbol{g}_{F_1} \leq \boldsymbol{g}_{F_2}$.*

If we add a *top* element $1_{\boldsymbol{G}}$ to $\boldsymbol{G}$ then we get a lattice:

Proposition 2 *Let $1_{\boldsymbol{G}}$ be such that $\forall_{g \in \boldsymbol{G}}\ \boldsymbol{g} \leq 1_{\boldsymbol{G}}$. Then $(\boldsymbol{G} \cup \{1_{\boldsymbol{G}}\}, \leq)$ is a lattice.*

The set of abstract objects of Figure 13, when extended by adding a top element $1_{\boldsymbol{G}}$ as said in Proposition 2, forms a lattice. The ordering relation is the information ordering—moving up in the lattice corresponds to further "specialising" objects. Specialisation takes place when additional information about the world becomes available. The lattice is presented in Figure 14.

Our intention is to have sets of abstract objects employed as *abstract worlds*, or *abstract contexts*—this gives us the semantic side of the knowledge representation formalism proposed. Note that on the one hand, if an agent claims the presence of certain abstract objects in her (abstract) world, then we know that the agent believes that some corresponding objects, having certain attributes, are present in the agent's believed world. On the other hand, if some abstract objects are absent then we know that certain objects, so is believed by the agent, must not appear in the world.

Although abstract objects correspond to equivalence classes of indiscernible objects, abstract objects should not be seen as simply sets of objects. This is because two agents with disjoint worlds can still share certain abstract objects.

We conclude this section by providing a method of building a set of abstract objects $\boldsymbol{G}_s$ corresponding to the set of objects G_s believed by the agent s to be in her world. What we really want to know is how to construct $\boldsymbol{G}_s$ out of G_s, so that the corresponding description sets $\boldsymbol{D}_{\boldsymbol{G}_s}$ and $\boldsymbol{D}_{G_s}$ are identical.[16]

Proposition 3 *Let s be an agent, and G_s be a set of believed objects of the agent. Let $\boldsymbol{G}_s$ be a set of corresponding abstract objects. Let $\boldsymbol{D}_{\boldsymbol{G}_s}$ and $\boldsymbol{D}_{G_s}$ be descriptions of the world of s in the language $\mathbb{L}$, as implied by the sets G_s and $\boldsymbol{G}_s$, respectively. Then $\boldsymbol{D}_{\boldsymbol{G}_s} = \boldsymbol{D}_{G_s}$.*

[16]Proof of Proposition 3 contains a method of finding abstract objects, given believed objects (and a set of attributes of interest).

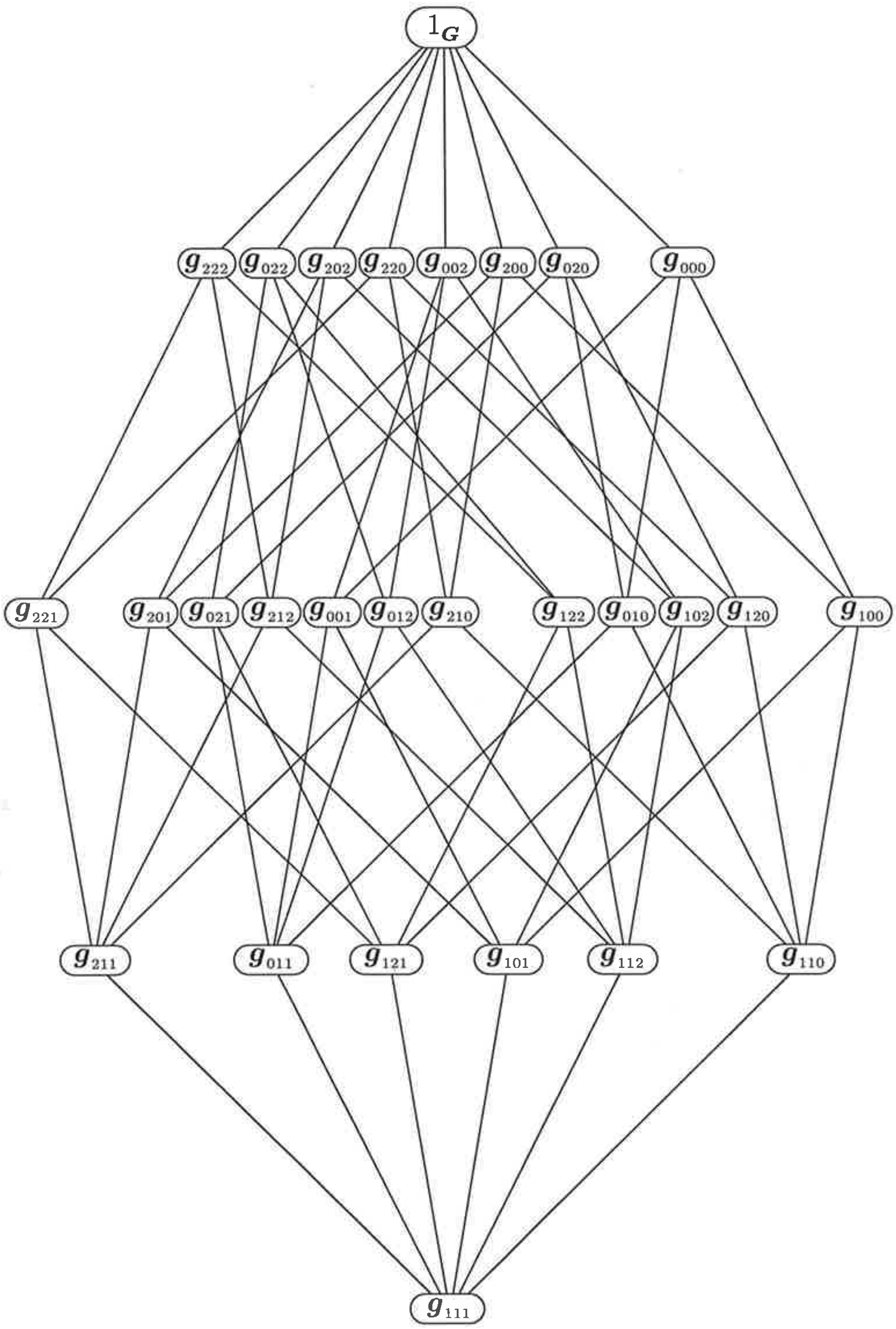


Figure 14. Information ordering on abstract objects

For convenience, we include Figures 15 and 16, containing formulae, the corresponding regions, and the corresponding abstract objects, for $|P| = \alpha = 2$ and $|P| = \alpha = 1$, respectively.

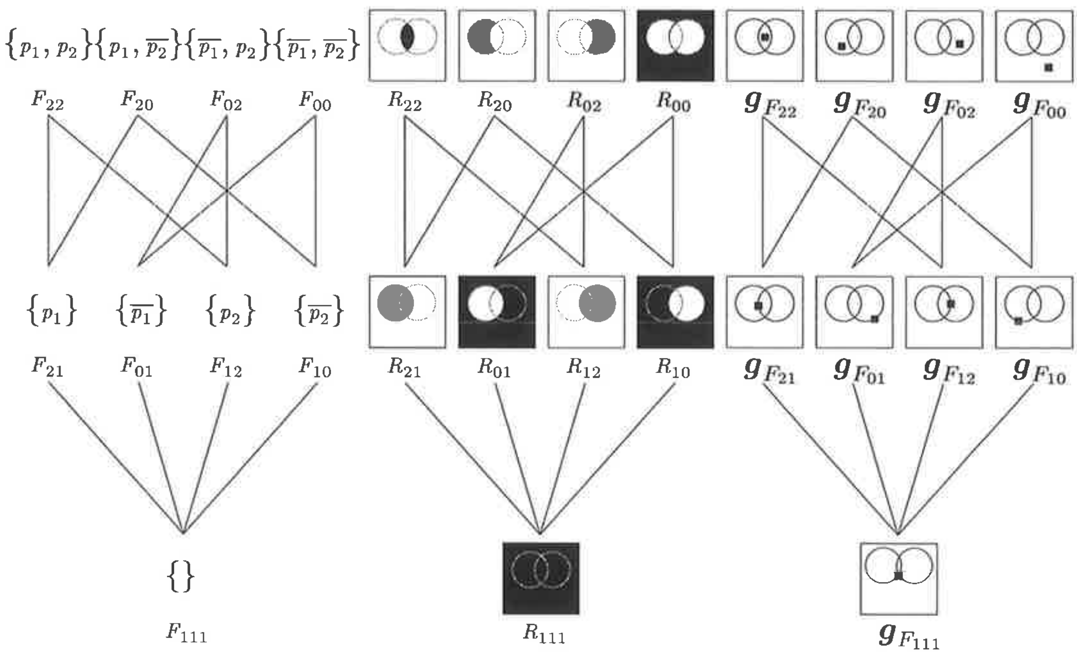


Figure 15. Formulae, regions and abstract objects, $|P| = 2$

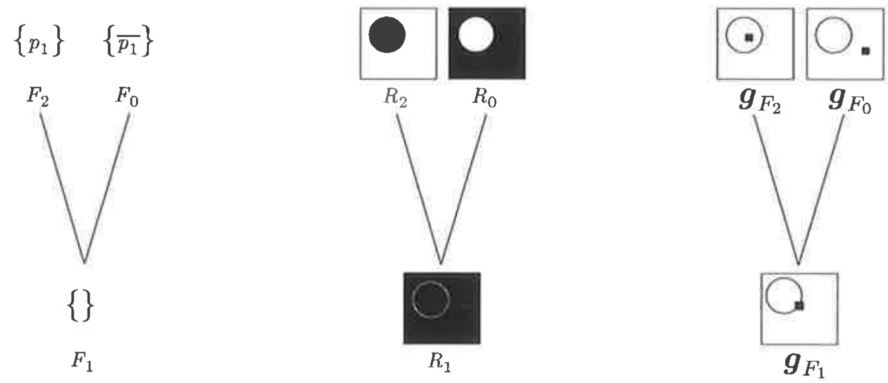


Figure 16. Formulae, regions and abstract objects, $|P| = 1$

3.7 Descriptions and contexts

In this section we summarise the contents of the chapter, introduce descriptions and contexts (of abstract objects) via examples, and sketch the rest of the thesis.

1	2	3	4	5
literate world $\boldsymbol{V}_s = (M, \boldsymbol{D}_s)$	no objects	focus on lang.	lang. consensus $\mathbb{L} = (M, \boldsymbol{D})$	description set $\boldsymbol{D}_s \subseteq \boldsymbol{D}$
abstract world $\boldsymbol{W}_s = (M, \boldsymbol{G}_s)$	abstract objects $\boldsymbol{G}_s$	focus on obj.	abstract objects $\boldsymbol{G}_s \subseteq \boldsymbol{G}$	abstract context $\boldsymbol{K}_s = (\boldsymbol{G}_s, M_s, I_s)$
believed world $W_s = G_s$	believed objects G_s			

Table 3. Introducing descriptions and contexts

Consider Table 3. We have introduced believed worlds and abstract worlds as a result of the paradigm shift of Section 3.3 and the introduction of abstract objects in Section 3.6, respectively. Let $s \in S$ be an agent. Then the believed world W_s of s can be identified with the set G_s of believed objects of s. Given that M is a set of attributes of interest, objects of G_s fall into equivalence classes (regions) of indiscernible (w.r.t. the formulae of $\boldsymbol{F}_M$) objects—abstract objects are associated with such equivalence classes. Hence, if M is fixed then the abstract world $\boldsymbol{W}_s$ of s can be identified with the set $\boldsymbol{G}_s$ of *abstract objects* of s.[17] If M is fixed then the literate world $\boldsymbol{V}_s$ of s can be identified with the set $\boldsymbol{D}_s$ of descriptions[18] of the world of s.

As emphasised by columns 2 and 3 of Table 3, we abandon believed worlds and stick to abstract worlds—the latter provide semantics for our formalism. Regarding literate worlds, or worlds of descriptions, the focus is on language rather than objects, and thus description sets provide *syntactic* information about the world.

On the semantic side, as shown in columns 4 and 5 of Table 3 (middle row), the agent s selects a subset $\boldsymbol{G}_s$ of the set $\boldsymbol{G}$ of all abstract objects over M to form her abstract world. A convenient way of presenting abstract worlds is via *abstract contexts*.[19]

On the syntactic side (the top row of Table 3), we assume a language consensus—this means that language formulated descriptions have the same "meaning" for all agents, cf. Section 3.1. If a set M of attributes is fixed then the set of all possible descriptions $\boldsymbol{D}$ is a set of all *marked formulae*, i.e., $\boldsymbol{D} = \boldsymbol{F} \times \{\oplus, \ominus\}$. A description set $\boldsymbol{D}_s$ the agent s forms of her world is a subset of $\boldsymbol{D}$. Agents employ descriptions to convey information about their worlds, more specifically, information about nonemptiness[20] of regions of the worlds. In other words, descriptions are a syntactic means for communicating information about the semantic entities: worlds and objects.

Consider the following examples involving description sets and contexts.

Example 1 *Let s be an agent, $P = \{p_1, p_2, p_3\}$ and $M = P \cup \overline{P} = \{p_1, \overline{p_1}, p_2, \overline{p_2}, p_3, \overline{p_3}\}$ be a set of attributes, and W_s be the agent's world, as shown in Table 4. The table provides the corresponding abstract world $\boldsymbol{W}_s$, abstract context $\boldsymbol{K}_s$, literate world $\boldsymbol{V}_s$ and description set $\boldsymbol{D}_s$.*

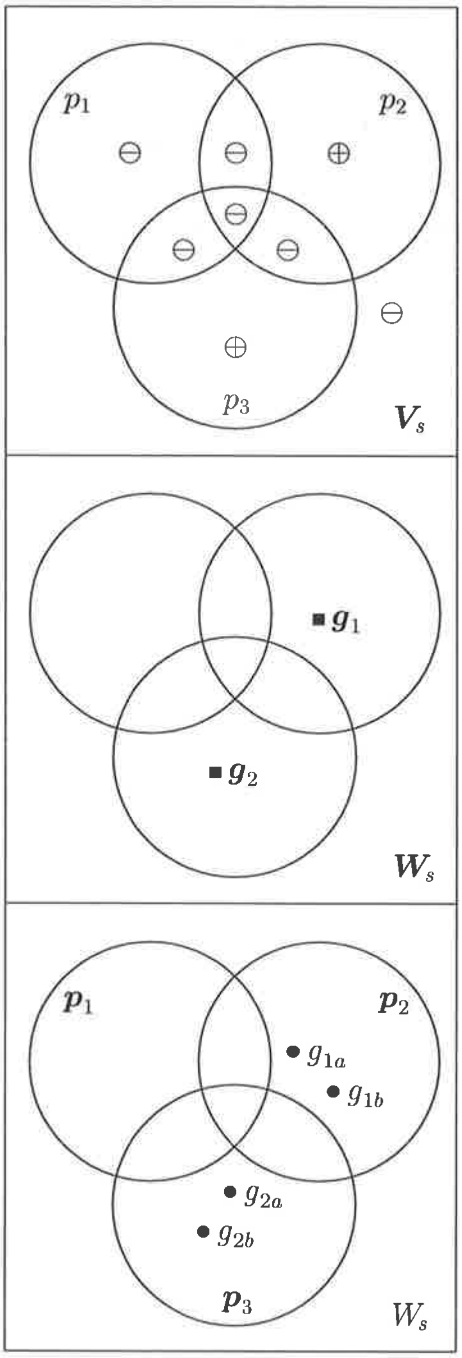
In Example 1—presented in Table 4—some objects of W_s are indiscernible, and hence "collapse" into equivalence classes of indiscernible objects. For example, the objects g_{1a} and g_{1b} "collapse" into a single abstract object $\boldsymbol{g}_1$—leading

[17]Note that $\boldsymbol{G}_s$ is a subset of the set $\boldsymbol{G}$ of all abstract objects (over M).

[18]The descriptions in $\boldsymbol{D}_s$ are correct descriptions of the believed world W_s, but can misrepresent the perceived world $\mathcal{W}_s$ of s, the world of objects that "actually are there."

[19]Abstract contexts are formally introduced in Section 4.1 of Chapter 4. Informally, an abstract context $\boldsymbol{K}_s$ consists of the set of abstract objects $\boldsymbol{G}_s$, the resulting set of attributes M_s, and the *incidence relation* I_s that associates the attributes with the objects.

[20]Let F be a formula and R_F be a region (of the world) determined by the formula. If the region is nonempty then this information about the world takes the form of a description $D = \oplus F$, i.e., the formula is marked with $\oplus$. Similarly, if the region is empty then the information takes the form of $D = \ominus F$, i.e., the formula is marked with $\ominus$.



$$\{ \ominus\{p_1, p_2, p_3\},\ \ominus\{p_1, p_2, \overline{p_3}\},\ \ominus\{p_1, \overline{p_2}, p_3\},\ \ominus\{p_1, \overline{p_2}, \overline{p_3}\},\ \ominus\{\overline{p_1}, p_2, p_3\},\ \oplus\{\overline{p_1}, p_2, \overline{p_3}\},\ \oplus\{\overline{p_1}, \overline{p_2}, p_3\},\ \ominus\{\overline{p_1}, \overline{p_2}, \overline{p_3}\} \}$$

$\boldsymbol{D}_s$

$\boldsymbol{K}_s$	p_1	$\overline{p_1}$	p_2	$\overline{p_2}$	p_3	$\overline{p_3}$
$\boldsymbol{g}_2$	−	+	−	+	+	−
$\boldsymbol{g}_1$	−	+	+	−	−	+

$\boldsymbol{K}_s = (\boldsymbol{G}_s, M_s, I_s)$

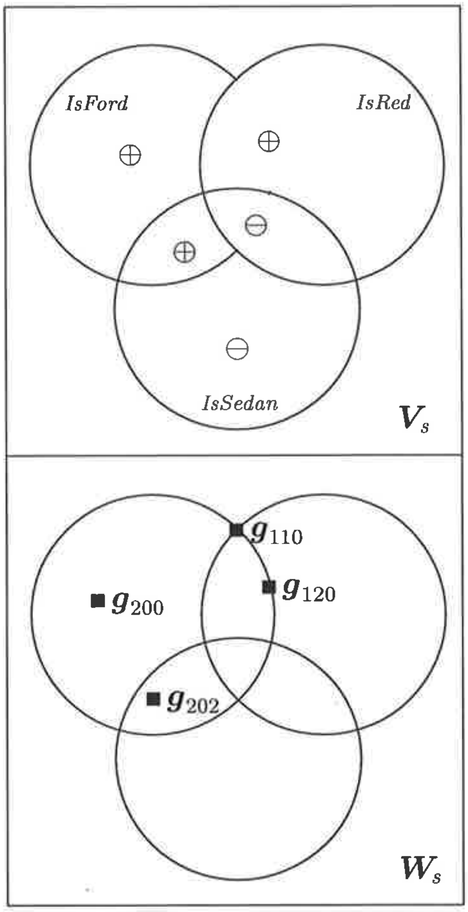
Table 4. Worlds, description and context (Example 1)

to the abstract world $\boldsymbol{W}_s$ with the set $\boldsymbol{G}_s = \{\boldsymbol{g}_1, \boldsymbol{g}_2\}$ of abstract objects. M_s is the set of involved attributes, and it can easily be found, given $\boldsymbol{G}_s$.[21] The incidence relation I_s associates abstract objects with attributes, as shown by the table on the right hand side of middle row of Table 4: the + and − signs are used to show whether the object in the row has the attribute in the column. Given the abstract world $\boldsymbol{W}_s$ one can find the corresponding literate world $\boldsymbol{V}_s$ by marking with $\oplus$ those regions inhabited by abstract objects, and with $\ominus$

[21]More precise, and formal treatment of abstract contexts awaits Section 4.1.

un-inhabited regions, as shown in Table 4. Given the set of attributes, there is an exact correspondence between the literate world $\boldsymbol{V}_s$ and the description set $\boldsymbol{D}_s$—the descriptions in $\boldsymbol{D}_s$ can be read from the diagram showing $\boldsymbol{V}_s$. (One should think of the circles in Table 4 as representing regions of believed objects, but in $\boldsymbol{W}_s$ and $\boldsymbol{V}_s$ they are employed as means to give meaning to abstract objects and descriptions, respectively; indeed, believed objects are neither to be found in $\boldsymbol{W}_s$ nor in $\boldsymbol{V}_s$.)

Example 2 *Let s be an agent. Let $P = \{IsFord, IsRed, IsSedan\}$, and $M = P \cup \overline{P} = \{IsFord, \overline{IsFord}, IsRed, \overline{IsRed}, IsSedan, \overline{IsSedan}\}$ be a set of attributes. Let the abstract world $\boldsymbol{W}_s$ of s be as shown in Table 5. The table provides the corresponding abstract context $\boldsymbol{K}_s$, literate world $\boldsymbol{V}_s$ and description set $\boldsymbol{D}_s$.*



$$\{\ominus\{IsRed, IsSedan\},\ \ominus\{\overline{IsFord}, \overline{IsRed}, IsSedan\},\ \oplus\{IsRed, \overline{IsSedan}\},\ \oplus\{IsFord, \overline{IsRed}, \overline{IsSedan}\},\ \oplus\{IsFord, \overline{IsRed}, IsSedan\}\}$$

$\boldsymbol{D}_s$

g_{110}	•	•	•	•	−	+
g_{120}	•	•	+	−	−	+
g_{200}	+	−	−	+	−	+
g_{202}	+	−	−	+	+	−
$\boldsymbol{K}_s$	*IsFord*	$\overline{IsFord}$	*IsRed*	$\overline{IsRed}$	*IsSedan*	$\overline{IsSedan}$

$$\boldsymbol{K}_s = (\boldsymbol{G}_s, M_s, I_s)$$

Table 5. Worlds, description and context (Example 2)

In Example 2—presented in Table 5—given $\boldsymbol{W}_s$, the corresponding abstract context $\boldsymbol{K}_s$ can be found. Note that this time, in the context table, we also employ the • sign, to denote that it is not known whether the object in the row (of the •) has the attribute in the column—this is necessary to account for (properly) *partial* abstract objects, objects located on borders between regions.[22]

[22]For instance, g_{120} is placed "on the border" between the region **IsFord** (of objects that have the attribute *IsFord*) and the region $\overline{\textbf{IsFord}}$ (of objects that have the attribute $\overline{IsFord}$).

Given the abstract world $\boldsymbol{W}_s$ one can find the corresponding literate world $\boldsymbol{V}_s$ by marking the regions with $\oplus$ or $\ominus$, as shown in Table 4, to account for nonemptiness of the regions. Note that some regions are not marked, e.g., $R_{\{\overline{IsFord}, \overline{IsRed}, \overline{IsSedan}\}} = \overline{\textbf{IsFord}} \cap \overline{\textbf{IsRed}} \cap \overline{\textbf{IsSedan}}$ is not marked, as it is not known—thanks to the presence of $\boldsymbol{g}_{110}$ in $\boldsymbol{G}_s$[23]—whether the region is empty. Again, there is an exact correspondence between the literate world $\boldsymbol{V}_s$ and the description set $\boldsymbol{D}_s$.

The rest of the thesis makes use of abstract contexts and description sets introduced in this chapter. In particular, in Chapter 4 abstract contexts are employed to provide a model theory for the formalism. Chapter 5 treats description sets as axiom sets, and the proof theory developed in the chapter expands description sets into theories—sets of formulae provable from the axiom sets. Chapter 6 further elaborates on models (abstract contexts) and theories (of description sets) by showing how they are related. Chapter 7 employs the formalism built in the other chapters to attack the problem of combining description sets (and theories) of multiple agents.

To follow the notation, treat the digits $1, 2$ and 0 in $\boldsymbol{g}_{120}$ as an encoding that the object is "on the border," "inside" and "outside" of the consecutive regions, respectively. (By consecutive regions here we mean singleton set formulae regions, or properties, namely the regions **IsFord**, **IsRed** and **IsSedan**, with the same ordering as the placement of the attributes in the set P.)

[23]More precisely, thanks to the presence of $\boldsymbol{g}_{110}$, and absence of $\boldsymbol{g}_{000}$. Note that $\boldsymbol{g}_{110}$ can "fall" into $R_{\{\overline{IsFord}, \overline{IsRed}, \overline{IsSedan}\}}$ (or rather a more specialised copy of the object can fall into that region).

Chapter 4

Model theory

This chapter provides a model theory for the logical formalism of the thesis. In Section 4.1 we formally introduce *abstract contexts.* Section 4.2 starts with a set $\mathbb{K}$ of all possible contexts, and introduces an *information ordering* $\leq$ on $\mathbb{K}$. Since contexts can be identified with sets of abstract objects, or with abstract worlds, the information ordering $\leq$ on $\mathbb{K}$ can be understood as information ordering on abstract worlds. In Section 4.3 the notion of *validity* is introduced, and justified. It is usually expected that valid formulae correspond to formulae *provable* in the syntactic, proof-theoretic side of the logic—this is made precise in Chapter 6 (in particular, in Section 6.6), after providing a proof theory in Chapter 5. Consequently, contexts (which validate formulae) are employed as models for theories, where theories are sets of theorems, or provable formulae. A proper account of models and contexts awaits Section 6.3, because there we are equipped with both the model theory of this chapter, and the proof theory of Chapter 5.

4.1 Abstract contexts and models

Let $P = \{p_1, \ldots, p_\alpha\}$, $\overline{P} = \{\overline{p_1}, \ldots, \overline{p_\alpha}\}$, and $M = P \cup \overline{P}$ be a set of attributes. The sets P and $\overline{P}$ were introduced in Section 3.1, refer also to Section 3.5. In particular, $\overline{P}$ can be seen as a result of applying a $^{-}$ operator to P, i.e., for any $p \in P$, there is a corresponding $\overline{p} \in \overline{P}$. The $^{-}$ operator should be seen as a *negation operator*, but as it only applies to single elements of P, rather than to sets or conjunctions of them, thus $\overline{P}$ is treated as a set of symbols. Recall from Section 3.1 that if $m \in M = P \cup \overline{P}$ then $\overline{m}$ is $\overline{p}$, if $m \in P$, and it is p, if $m \in \overline{P}$. Recall from Section 3.6 that a set $\boldsymbol{G}$ of abstract objects (over M) consists of objects with attributes identified with formulae in $\boldsymbol{F}$, i.e., $\mathbf{G} = \{\boldsymbol{g}_F \mid F \in \boldsymbol{F}, \text{ and } \mathrm{attrs}(\boldsymbol{g}_F) = F\}$. The following definition constructs an *abstract context* $\boldsymbol{K}_i$ out of a set $\boldsymbol{G}_i$ of abstract objects, $\boldsymbol{G}_i \subseteq \boldsymbol{G}$.

Definition 9 *Let $\boldsymbol{G}$ be a set of abstract objects, and let $\boldsymbol{G}_i \subseteq \boldsymbol{G}$. An* abstract context $\boldsymbol{K}_i$ *corresponding to the set of abstract objects $\boldsymbol{G}_i$ is constructed as follows.*

1. $M_i = \{m, \overline{m} \mid m \in M_e\}$, *where* $M_e = \bigcup_{\boldsymbol{g} \in \boldsymbol{G}_i} \{m \mid \boldsymbol{g} \text{ has } m\}$.

2. $I_i \colon \boldsymbol{G}_i \times M_i \longrightarrow \{0,1,2\}$ *is given by:*

$$I_i(\boldsymbol{g}, m) =_{\text{def}} \begin{cases} 2 & \text{if } \boldsymbol{g} = \boldsymbol{g}_F \text{ such that } F \ni m, \\ 1 & \text{if } \boldsymbol{g} = \boldsymbol{g}_F \text{ such that } F \cap \{m, \overline{m}\} = \emptyset, \\ 0 & \text{if } \boldsymbol{g} = \boldsymbol{g}_F \text{ such that } F \ni \overline{m}. \end{cases}$$

3. $\boldsymbol{K}_i =_{\text{def}} (\boldsymbol{G}_i, M_i, I_i)$.

An abstract context $\boldsymbol{K}_i = (\boldsymbol{G}_i, M_i, I_i)$ *is represented by a* context table, *with rows corresponding to objects of* $\boldsymbol{G}_i$, *columns corresponding to attributes of* M_i, *and each cell corresponding to a pair* $(\boldsymbol{g}, m) \in \boldsymbol{G}_i \times M_i$*—the cell is filled with* $+$, $\bullet$, *or* $-$ *if* $I_i(\boldsymbol{g}, m)$ *takes the value of* 2, 1, *or* 0, *respectively.*

The incidence function satisfies the following conditions:

- $I_i(\boldsymbol{g}, m) = 2$ *iff* $I_i(\boldsymbol{g}, \overline{m}) = 0$, *and*
- $I_i(\boldsymbol{g}, m) = 1$ *iff* $I_i(\boldsymbol{g}, \overline{m}) = 1$.

Let the set M *of attributes be fixed, and let* $\boldsymbol{G}$ *be the set of all abstract objects (over* M*). Then the set of all subsets of* $\boldsymbol{G}$ *determines the set—denoted by* $\mathbb{K}$*—of all abstract contexts (over* M*), i.e.,* $\mathbb{K} = \{\boldsymbol{K}_i\}_{\boldsymbol{G}_i \subseteq \boldsymbol{G}} = \{(\boldsymbol{G}_i, M, I_i)\}_{\boldsymbol{G}_i \subseteq \boldsymbol{G}}$.

Definition 9 requires some explanation. Firstly, $I_i(\boldsymbol{g}, m)$ takes on the value of $2, 1$, or 0 if $\boldsymbol{g}$ has m, does not have any of m and $\overline{m}$, or has $\overline{m}$, respectively. Notice that—by Definitions 7 and 2—every abstract object has at most one of the attributes m and $\overline{m}$, and hence if it has an attribute m then it must not have $\overline{m}$, i.e., $I_i(\boldsymbol{g}, m) = 2$ iff $I_i(\boldsymbol{g}, \overline{m}) = 0$. By the definition of I_i, we clearly have that $I_i(\boldsymbol{g}, m) = 1$ iff $I_i(\boldsymbol{g}, \overline{m}) = 1$. These constraints allow us to identify an abstract context over M, with the corresponding context over $P = M \setminus \overline{P}$.

Secondly, notice that we define the set $\mathbb{K}$ of all abstract contexts (over M) as $\{(\boldsymbol{G}_i, M, I_i)\}_{\boldsymbol{G}_i \subseteq \boldsymbol{G}}$ rather than $\{(\boldsymbol{G}_i, M_i, I_i)\}_{\boldsymbol{G}_i \subseteq \boldsymbol{G}}$. Indeed, we first fix M and then ask what the possible abstract contexts are. If $\boldsymbol{G}$ is a set of all abstract objects (over M) then every subset $\boldsymbol{G}_i$ of $\boldsymbol{G}$ determines an abstract context $(\boldsymbol{G}_i, M_i, I_i)$—where we must have $M_i \subseteq M$, as we limit ourselves to M. However, the abstract context $(\boldsymbol{G}_i, M_i, I_i)$ can be (trivially) extended to the context $(\boldsymbol{G}_i, M, I_i)$, by putting the $\bullet$ sign in every cell in columns corresponding to attributes in $M \setminus M_i$ (an example of such operation is the move from context $\boldsymbol{K}_1$ to $\boldsymbol{K}_2$ in Table 6).

Recall formal contexts and context tables introduced in Section 2.2. Consider contexts presented in Table 6. $\boldsymbol{K}_1$ is a formal context, in which there is one object "MyNewCar" which has the attribute *IsSedan* and does not have the attribute *IsRed*. If nothing more is known about "MyNewCar" then extending the set of attributes to include another attribute *IsFord* results in a three-valued context $\boldsymbol{K}_2$—"MyNewCar" still has the attribute *IsSedan*, does not have *IsRed*, and it is not known whether it has *IsFord*. The move from $\boldsymbol{K}_2$ to $\boldsymbol{K}_3$ corresponds to explicitly extending the set of attributes from $P = \{$*IsSedan*, *IsRed*, *IsFord*$\}$ to $M = P \cup \overline{P}$, in accordance with Definition 9. In $\boldsymbol{K}_4$, another object—"MyOldCar"—is included, but notice that the objects "MyNewCar" and "MyOldCar" are indiscernible. Hence, in $\boldsymbol{K}_6$ the two objects

$\mathbf{K}_1$	IsSedan	IsRed
MyNewCar	×	

$\mathbf{K}_2$	IsSedan	IsRed	IsFord
MyNewCar	+	−	•

$\mathbf{K}_3$	IsSedan	$\overline{\text{IsSedan}}$	IsRed	$\overline{\text{IsRed}}$	IsFord	$\overline{\text{IsFord}}$
MyNewCar	+	−	−	+	•	•

$\mathbf{K}_4$	IsSedan	$\overline{\text{IsSedan}}$	IsRed	$\overline{\text{IsRed}}$	IsFord	$\overline{\text{IsFord}}$
MyOldCar	+	−	−	+	•	•
MyNewCar	+	−	−	+	•	•

$\mathbf{K}_5$	IsSedan	$\overline{\text{IsSedan}}$	IsRed	$\overline{\text{IsRed}}$
MyOldCar	+	−	−	+
MyNewCar	+	−	−	+

$\mathbf{K}_6$	IsSedan	$\overline{\text{IsSedan}}$	IsRed	$\overline{\text{IsRed}}$	IsFord	$\overline{\text{IsFord}}$
{MyOldCar, MyNewCar}	+	−	−	+	•	•

$\mathbf{K}_7$	IsSedan	$\overline{\text{IsSedan}}$	IsRed	$\overline{\text{IsRed}}$
{MyOldCar, MyNewCar}	+	−	−	+

$\mathbf{K}_8$	IsSedan	$\overline{\text{IsSedan}}$	IsRed	$\overline{\text{IsRed}}$	IsFord	$\overline{\text{IsFord}}$
{MyFutureCar}	•	•	+	−	−	+
{MyOldCar, MyNewCar}	+	−	−	+	•	•

$\mathbf{K}_9$	IsSedan	$\overline{\text{IsSedan}}$	IsRed	$\overline{\text{IsRed}}$	IsFord	$\overline{\text{IsFord}}$
$g_{F_{120}}$	•	•	+	−	−	+
$g_{F_{201}}$	+	−	−	+	•	•

Table 6. Introducing abstract contexts

are replaced by the set consisting of the two (indiscernible) objects. The contexts $\boldsymbol{K}_5$ and $\boldsymbol{K}_7$ are versions of $\boldsymbol{K}_4$ and $\boldsymbol{K}_6$, after eliminating the "redundant" attribute *IsFord*. In $\boldsymbol{K}_8$ we include a (singleton) set of objects—the set consists of a single object "MyFutureCar," which is determined w.r.t. to the attribute *IsFord*, but not w.r.t. *IsSedan*. Finally, in context $\boldsymbol{K}_9$ we replace sets of indiscernible objects with *abstract objects*, finishing up with an *abstract context*. (Note that in the move from $\boldsymbol{K}_8$ to $\boldsymbol{K}_9$ information contained in the "names of the objects"—e.g., information conveyed by using the name "MyNewCar"—is lost. If such information is important, attributes to encode it can be employed.)

Abstract contexts provide semantics for our formalism, and they are employed as *models* (of worlds), whenever syntactic information is received. A clear explication of the issue of models is deferred to Section 6.3.

4.2 Information ordering on contexts

Let M be a set of attributes, and $\boldsymbol{G}$ be the set of all abstract objects (over M). As has been explained in the previous section, sets of abstract objects—i.e., subsets of $\boldsymbol{G}$—determine abstract contexts. If $\mathbb{K}$ is a set of all abstract contexts (over M) then an information ordering relation on $\mathbb{K}$ can be introduced. One would expect that a move up from one abstract context to another, given an information ordering $\leq$, should account to replacing the given set of abstract objects by another set of *more specialised*, or *less partial* objects. If $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are sets of abstract objects of the contexts $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, respectively, and we claim that $\boldsymbol{K}_2$ has greater informational value than $\boldsymbol{K}_1$, then we expect the following. Firstly, for every object $\boldsymbol{g}_2 \in \boldsymbol{G}_2$ there must be an object $\boldsymbol{g}_1 \in \boldsymbol{G}_1$, such that $\boldsymbol{g}_2$ is a specialisation of $\boldsymbol{g}_1$—the new set of abstract objects still corresponds to the same world, but more detailed information is available; in other words, no object that is disallowed by $\boldsymbol{G}_1$ can appear in $\boldsymbol{G}_2$. Secondly, for every object $\boldsymbol{g}_1 \in \boldsymbol{G}_1$ there must be an object $\boldsymbol{g}_2 \in \boldsymbol{G}_2$, such that $\boldsymbol{g}_2$ is a specialisation of $\boldsymbol{g}_1$—the objects of $\boldsymbol{G}_1$ must not just disappear, without leaving (possibly more specialised) copies of themselves in $\boldsymbol{G}_2$; in other words, information already presented by $\boldsymbol{G}_1$ must not be lost. This suggests that one could attempt to define an ordering on abstract contexts as follows.

Let M, $\boldsymbol{G}$ and $\mathbb{K}$ be a set of attributes, the set of abstract objects (over M), and the set of abstract contexts (over $\boldsymbol{G}$). Let $\boldsymbol{K}_1 = (\boldsymbol{G}_1, M, I_1)$ and $\boldsymbol{K}_2 = (\boldsymbol{G}_2, M, I_2)$ be two abstract contexts in $\mathbb{K}$. Then an *ordering* on $\mathbb{K}$ might be introduced by requiring that $\boldsymbol{K}_1 \leq \boldsymbol{K}_2$ iff the following conditions hold:

1. $\forall_{\boldsymbol{g}_2 \in \mathbf{G}_2} \exists_{\boldsymbol{g}_1 \in \mathbf{G}_1} \; \boldsymbol{g}_1 \leq \boldsymbol{g}_2$
2. $\forall_{\boldsymbol{g}_1 \in \mathbf{G}_1} \exists_{\boldsymbol{g}_2 \in \mathbf{G}_2} \; \boldsymbol{g}_1 \leq \boldsymbol{g}_2$

This was indeed our first attempt, but—as is discussed below—an information ordering on abstract contexts (or rather *partial worlds*) should *not* be defined this way. Nevertheless, let us consider an example, and then discuss how a proper information ordering on worlds should be introduced.

Consider the following, simple example of a set of abstract contexts, as presented in Figure 17. Let $P = \{p_1\}$, and $M = \{p_1, \overline{p_1}\}$ be a set of attributes. Then, there are only three possible abstract objects one can consider (refer to Figure 16 for the sets $\boldsymbol{F}$ and $\boldsymbol{G}$ of all formulae and abstract objects over M, respectively). We have that $\boldsymbol{F} = \{F_2, F_1, F_0\} = \{\{p_1\}, \{\overline{p_1}\}, \{\}\}$, and $\boldsymbol{G} = \{\boldsymbol{g}_2, \boldsymbol{g}_1, \boldsymbol{g}_0\}$. The set $\mathbb{K}$ of all abstract contexts (over M) can be determined by considering the set of all subsets of $\boldsymbol{G}$. Figure 17 presents all the contexts of $\mathbb{K}$, and the links between the contexts show an ordering on $\mathbb{K}$, according to our initial, just presented attempt to define the ordering.

A thing to notice is that—see Figure 17—the context $\boldsymbol{K}_{102}$ is strictly below the context $\boldsymbol{K}_{02}$, i.e., $\boldsymbol{K}_{102} < \boldsymbol{K}_{02}$ (we have that $\boldsymbol{K}_{102} \leq \boldsymbol{K}_{02}$ but $\boldsymbol{K}_{02} \not\leq \boldsymbol{K}_{102}$). However, the contexts seem to represent *the same* "world," not in the sense of *abstract worlds* introduced in Chapter 3, but in a sense of *partial worlds* indicating what *possible* and *total* worlds (or "realities") are hidden behind those partial worlds. This invites us to consider a relation between *all* abstract

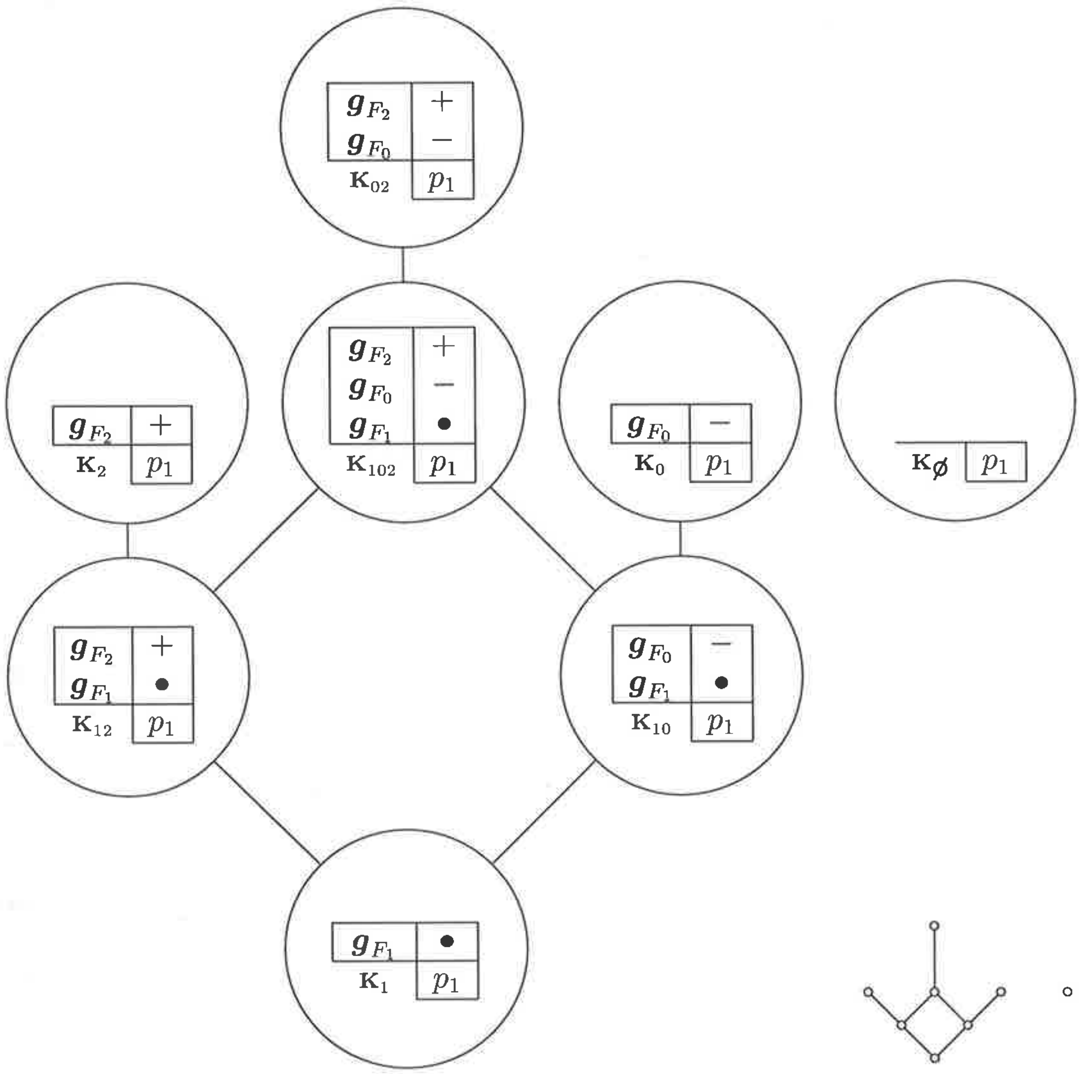


Figure 17. Ordering on contexts, initial attempt

contexts and *total* contexts, as this in turn will allow us to detect pairs of seemingly different contexts which nevertheless are "essentially the same."

Definition 10 *Let M be a set of attributes, $\mathbb{K}$ be the corresponding set of* all *abstract contexts, and $\mathbb{K}_t$ be the corresponding set of all* total *abstract contexts. Define an ordering $\leq$ between $\mathbb{K}$ and $\mathbb{K}_t$, i.e., $\leq \subseteq \mathbb{K} \times \mathbb{K}_t$ as follows. Let $\boldsymbol{K} = (\boldsymbol{G}, M, I) \in \mathbb{K}$ and $\boldsymbol{K}_t = (\boldsymbol{G}_t, M, I_t) \in \mathbb{K}_t$. Then $\boldsymbol{K} \leq \boldsymbol{K}_t$* iff *the following conditions hold:*

1. $\forall_{\boldsymbol{g}_t \in \mathbf{G}_t} \exists_{\boldsymbol{g} \in \mathbf{G}} \;\; \boldsymbol{g} \leq \boldsymbol{g}_t$

2. $\forall_{\boldsymbol{g} \in \mathbf{G}} \exists_{\boldsymbol{g}_t \in \mathbf{G}_t} \;\; \boldsymbol{g} \leq \boldsymbol{g}_t$

Indeed, given Definition 10, we can introduce a mapping tot: $\mathbb{K} \longrightarrow \mathcal{P}(\mathbb{K}_t)$ that allows us to find all total contexts above a given context, and then employ the mapping to define an equivalence relation on the set $\mathbb{K}$ of all abstract contexts.

Definition 11 *A mapping* $\text{tot}: \mathbb{K} \longrightarrow \mathcal{P}(\mathbb{K}_t)$ *is given by* $\text{tot}(\boldsymbol{K}) = \{\boldsymbol{K}_t \in \mathbb{K}_t \mid \boldsymbol{K} \leq \boldsymbol{K}_t\}$. *Further, an equivalence relation* $\approx$ *on* $\mathbb{K}$, *i.e.,* $\approx\ \subseteq \mathbb{K} \times \mathbb{K}$ *is given by* $\boldsymbol{K}_1 \approx \boldsymbol{K}_2$ iff $\text{tot}(\boldsymbol{K}_1) = \text{tot}(\boldsymbol{K}_2)$. *Then* $\mathbb{K}/\!\approx$ *is the set of equivalence classes on* $\mathbb{K}$ *w.r.t.* $\approx$.

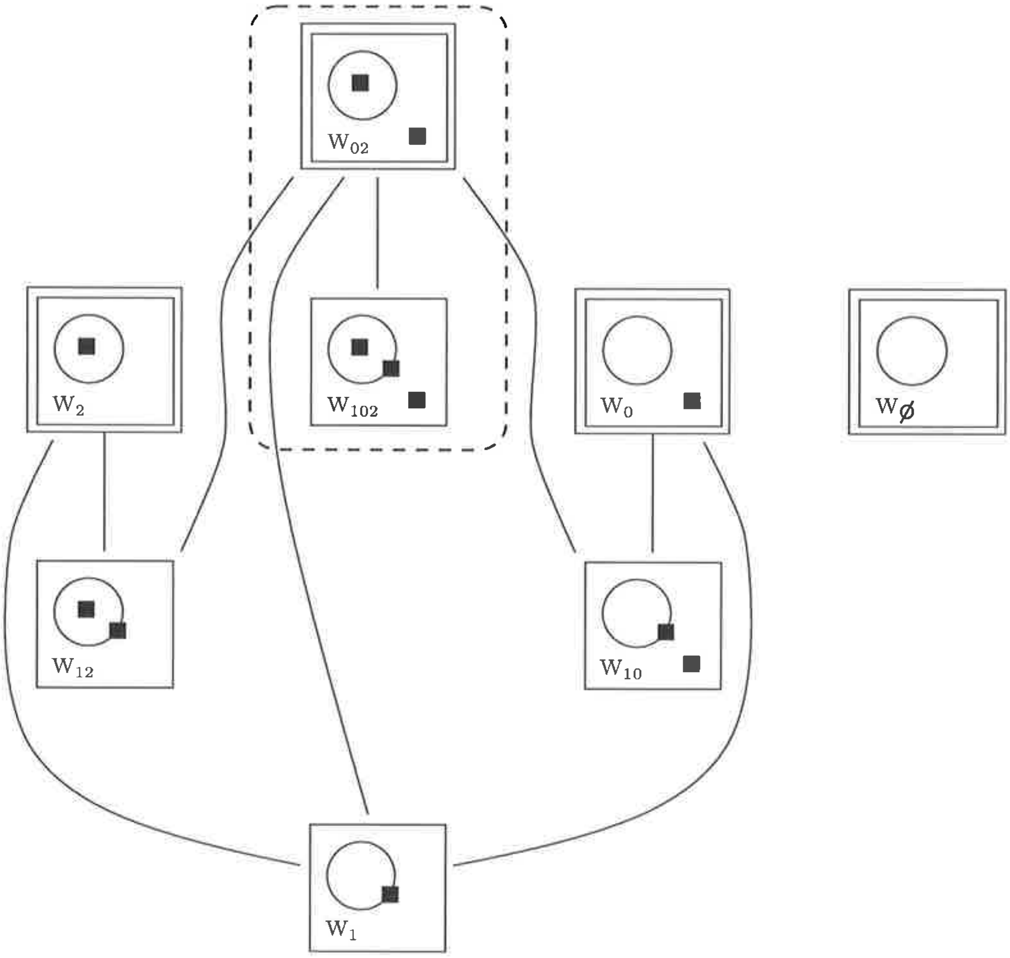


Figure 18. Comparing contexts with total contexts

Figure 18 illustrates the notions introduced in Definition 11. It presents abstract worlds—or, in fact, abstract contexts—and shows total contexts above every context (above in the sense of ordering introduced in Definition 10). Of course, for every total context there is exactly one context above it: the context itself. The lines between the contexts show the relation of Definition 10; total contexts (worlds) are marked with double frames. There is an equivalence class (marked on Figure 18 with a dashed line) of contexts containing two contexts, namely $\{\boldsymbol{K}_{102}, \boldsymbol{K}_{02}\}$; all other equivalence classes are singleton sets of contexts.

It is now clear that on the semantic side of the formalism we should have equivalence classes (w.r.t. $\approx$) of abstract contexts, or single, uniquely determined "representatives" of the equivalence classes. One might decide to "saturate" contexts, and to keep such saturated contexts—call them *partial worlds*—as appropriate semantic entities. For instance, in the equivalence class $\{\boldsymbol{K}_{102}, \boldsymbol{K}_{02}\}$, such a saturation mapping would map $\boldsymbol{K}_{02}$ to $\boldsymbol{K}_{102}$,[1] and would map every other context to itself. Therefore, we expect a proper information ordering on *partial worlds*, or *saturated contexts* to be, in the case of $M = \{p_1, \overline{p_1}\}$, as presented in Figure 19.

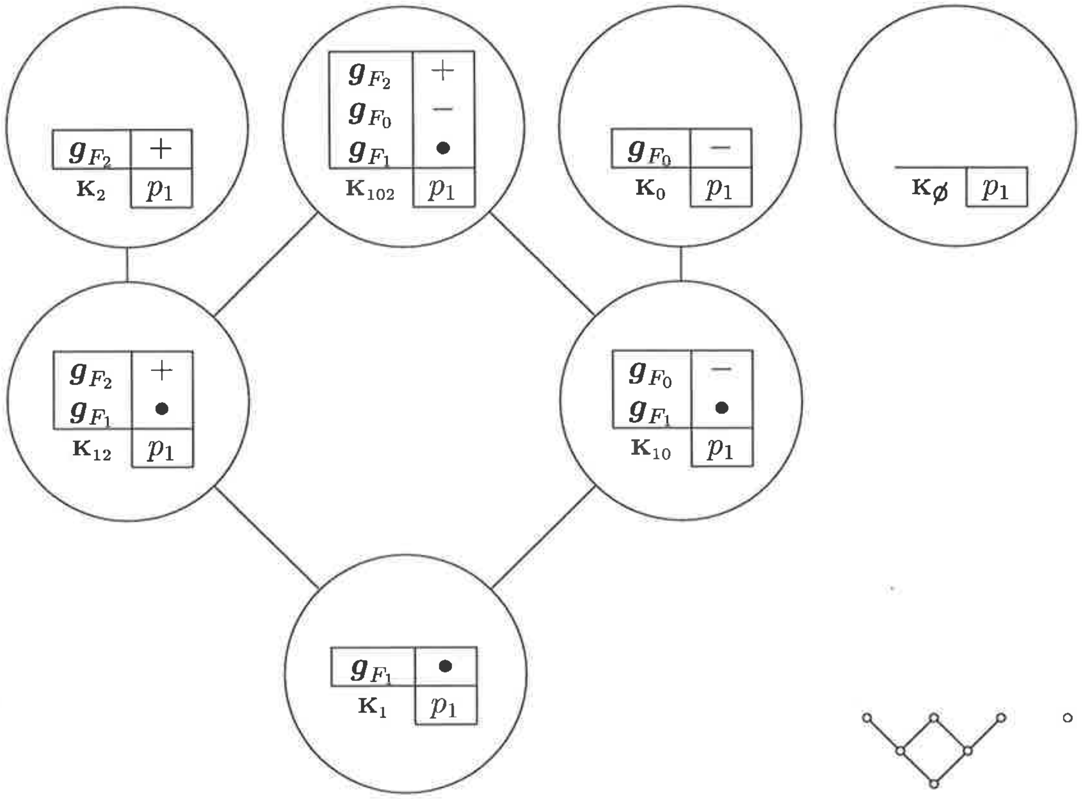


Figure 19. Information ordering on partial worlds (saturated contexts)

Given the equivalence relation of Definition 11, we could stop here, saying that the equivalence classes of contexts are appropriate semantic entities. However, it would be good to have a more constructive approach, for instance, define a saturation mapping and thus be able to detect whether or not two different contexts (different, as sets of objects) are in fact equivalent without finding the sets of total contexts above them (finding total contexts might be expensive). Clearly, a saturated context would be the only such context in its equivalence class, i.e., all contexts of an equivalence class of contexts would saturate to a single element of the class, and hence the saturated contexts would be appropriate representatives of the equivalence classes of contexts. Let us thus attempt to define a saturation mapping.

So far, we have one condition a saturation mapping should satisfy—if, in a context, there is an object having attributes $\psi \cup \{m\}$ and there is an object having the attributes $\psi \cup \{\overline{m}\}$ then the saturated context should also contain the

[1]Note that if we go for saturated contexts, then we loose some total contexts; an "unsaturate" mapping could be introduce to map contexts to their parsimonious (w.r.t. the number of objects) $\approx$-equivalents.

object that has the attributes ψ, see Figure 18. There is another condition saturated contexts should satisfy, the condition of *convexity* w.r.t. the information ordering on objects.

As was said at the beginning of this section, our initial attempt to define information ordering on context is not correct. In fact, it was pointed out that it may be good to place restrictions on which sets can be abstract contexts. For instance, maybe they should be convex under $\leq$. A similar comment was made by Ganter [Gan97], who pointed out the inadequacy of the initial version of ordering on contexts, namely, that the resulting relation would not be anti-symmetric (and therefore, it would treat different contexts as equivalent, and hence we would get not an order, but a quasi-order).[2] Indeed, when a context contains two objects $\boldsymbol{g}_1$ and $\boldsymbol{g}_3$ such that $\boldsymbol{g}_1 \leq \boldsymbol{g}_3$ then the set of total contexts above the given context does not depend on how many objects that are between $\boldsymbol{g}_1$ and $\boldsymbol{g}_3$ are included in the context. Hence, saturated contexts should satisfy the convexity condition.

Summarising, we define a saturation mapping as follows.

Definition 12 *A saturation mapping, $sat\colon \mathbb{K} \longrightarrow \mathbb{K}$ is defined as follows. Let $\boldsymbol{K} \in \mathbb{K}$. Then $sat(\boldsymbol{K})$ is a context (element of $\mathbb{K}$) that satisfies the following conditions:*

1. $sat(\boldsymbol{K}) \supseteq \boldsymbol{K}$
2. *if* $\boldsymbol{g}_1, \boldsymbol{g}_3 \in sat(\boldsymbol{K})$ *and* $\boldsymbol{g}_1 \leq \boldsymbol{g}_2 \leq \boldsymbol{g}_3$ *then* $\boldsymbol{g}_2 \in sat(\boldsymbol{K})$
3. *if* $\boldsymbol{g}_{\psi\cup\{m\}}, \boldsymbol{g}_{\psi\cup\{\overline{m}\}} \in sat(\boldsymbol{K})$ *then* $\boldsymbol{g}_\psi \in sat(\boldsymbol{K})$
4. $sat(\boldsymbol{K})$ *is* $\subseteq$*-minimal*

Given Definition 12, we can introduce *partial worlds* as saturated context, and then define an information ordering on partial worlds in an appropriate, and expected, way.

Definition 13 . *Let $\mathbb{K}$ be a set of all abstract contexts (over M). A set $\mathbb{W}$ of* partial worlds *is the set of* saturated contexts, *i.e.,* $\mathbb{W} = \{\mathrm{sat}(\boldsymbol{K}) \mid \boldsymbol{K} \in \mathbb{K}\}$. *An* information ordering *on partial worlds is introduced as follows.* $\boldsymbol{W}_1 \leq \boldsymbol{W}_2$ iff *the following conditions hold:*

1. $\forall_{\boldsymbol{g}_2 \in \boldsymbol{W}_2} \exists_{\boldsymbol{g}_1 \in \boldsymbol{W}_1} \; g_1 \leq g_2$,
2. $\forall_{\boldsymbol{g}_1 \in \boldsymbol{W}_1} \exists_{\boldsymbol{g}_2 \in \boldsymbol{W}_2} \; g_1 \leq g_2$.

This finishes our considerations on information ordering on contexts, or worlds. [Note that in the remainder of the thesis we might employ contexts, rather than partial worlds, but what we say applies to partial worlds—for a clean presentation on partial worlds and related structures, see [Now98].]

[2] I am grateful to Bernhard Ganter and John Slaney for these, and other critical remarks, which resulted in, I hope, an improved version of the thesis.

4.3 Validity

Let M be a set of attributes, $\boldsymbol{F}$ be the set of formulae over M, and $\boldsymbol{K}_i$ be an abstract context over M. Let $\boldsymbol{G}_i$ be the set of abstract objects of $\boldsymbol{K}_i$. Given a formula $F \in \boldsymbol{F}$, the abstract object $\boldsymbol{g}_F \in \boldsymbol{G}$ either is present or is absent in $\boldsymbol{G}_i$. What should we say about F if $\boldsymbol{g}_F \in \boldsymbol{G}_i$? What should we say about F if $\boldsymbol{g}_F \notin \boldsymbol{G}_i$? Such questions lead us to the following definition of *validity*.

Definition 14 *Let M and $\boldsymbol{F}$ be attributes and the formulae, respectively. Let $\boldsymbol{K}_i = (\boldsymbol{G}_i, M, I_i) \in \mathbb{K}$ be an abstract context (over M). Let $F \in \boldsymbol{F}$. Then:*

$$\boldsymbol{K}_i \models_{\oplus} F \quad \text{iff}_{\text{def}} \quad \exists_{\boldsymbol{g} \in \boldsymbol{G}_i} \forall_{m \in F}\ I_i(\boldsymbol{g}, m) = 2,$$

$$\boldsymbol{K}_i \models_{\ominus} F \quad \text{iff}_{\text{def}} \quad \forall_{\boldsymbol{g} \in \boldsymbol{G}_i} \exists_{m \in F}\ I_i(\boldsymbol{g}, m) = 0.$$

If $\boldsymbol{K}_i \models_{\oplus} F$ then we say that F is $\oplus$-valid, and if $\boldsymbol{K}_i \models_{\ominus} F$ then we say that F is $\ominus$-valid, in $\boldsymbol{K}_i$.

Sets of valid formulae of the context $\boldsymbol{K}_i$ are denoted as follows:

$$\boldsymbol{F}^{\oplus}_{\boldsymbol{K}_i} = \{F \in \boldsymbol{F} \mid \boldsymbol{K}_i \models_{\oplus} F\},$$

$$\boldsymbol{F}^{\ominus}_{\boldsymbol{K}_i} = \{F \in \boldsymbol{F} \mid \boldsymbol{K}_i \models_{\ominus} F\}.$$

Some comments are now in order. Suppose that we said that F is "instantiatable," or "has an instance" (in $\boldsymbol{G}_i$, or $\boldsymbol{K}_i$), if $\exists_{\boldsymbol{g} \in \boldsymbol{G}_i} \forall_{m \in F}\ I_i(\boldsymbol{g}, m) = 2$. Then, on the one hand $\oplus$-validity would be identical with *instantiatability*. On the other hand, $\ominus$-validity would mean *guaranteed non-instantiatability*. An important point to note is that validity is defined in such a way that valid formulae remain valid in any context above the context $\boldsymbol{K}_i$ in the information ordering $\leq$ and this is what justifies our use of the term "validity." If a formula is ($\oplus$- or $\ominus$-) valid then we do *not* claim that it holds (is valid) in *every* context of $\mathbb{K}$. However, it *does* hold in every context $\boldsymbol{K} \in \mathbb{K}$ such that $\boldsymbol{K} \geq \boldsymbol{K}_i$. Indeed, if we move up in the information ordering from $\boldsymbol{K}_i$ to $\boldsymbol{K}$ then if a formula has an instance in $\boldsymbol{K}_i$ then it also has an instance in $\boldsymbol{K}$—the instances are not lost, and even if they are replaced by more specialised objects then those objects remain to be instances. Similarly, if a formula has no instance in $\boldsymbol{K}_i$, the formula does not acquire an instance when we move to $\boldsymbol{K}$, simply because objects in $\boldsymbol{K}$ are specialisations of objects in $\boldsymbol{K}_i$, and hence the former objects preserve the attributes of the latter ones.

For examples of valid formulae of contexts, refer back to Tables 4 and 5. For instance, in Table 5 every description of the description set $\boldsymbol{D}_s$ gives a valid formula. For instance, if $\oplus F \in \boldsymbol{D}_s$ then F is $\oplus$-valid in $\boldsymbol{K}_s$. There are however more valid formulae in $\boldsymbol{K}_s$ that can be read from $\boldsymbol{D}_s$ in this way. Indeed, although a description set might uniquely identify the corresponding abstract context, description sets rarely list all valid formulae. If we define:

$$\boldsymbol{F}^{\oplus}_{\boldsymbol{D}_i} = \{F \in \boldsymbol{F} \mid \oplus F \in \boldsymbol{D}_i\}, \text{ and}$$

$$\boldsymbol{F}^{\ominus}_{\boldsymbol{D}_i} = \{F \in \boldsymbol{F} \mid \ominus F \in \boldsymbol{D}_i\},$$

and $\boldsymbol{D}_i$ does describe $\boldsymbol{K}_i$, then we clearly have $\boldsymbol{F}^{\oplus}_{\boldsymbol{D}_i} \subseteq \boldsymbol{F}^{\oplus}_{\boldsymbol{K}_i}$, and $\boldsymbol{F}^{\ominus}_{\boldsymbol{D}_i} \subseteq \boldsymbol{F}^{\ominus}_{\boldsymbol{K}_i}$.

Note that if, on the one hand, a context $\boldsymbol{K}_i$ is given, then the set of all formulae valid in the context can be found by checking whether a formula F is $\oplus$- or $\ominus$-valid, for every $F \in \boldsymbol{F}$.

On the other hand, if a description set is given, one can employ a *proof theory* that can be used to find a set, or *theory*, of all formulae that can be inferred from the description set—this is the subject of Chapter 5. We will come back to contexts and models in Chapter 6, where we relate contexts to theories.

Chapter 5

Proof theory

This chapter provides a *proof theory* for the logical formalism of the thesis. Section 5.1 defines a *formal system*—this includes specifying language, axioms, inference rules and provability. In Section 5.2, it is shown that theories—sets of formulae provable within the formal system—are equipped with an *information ordering* on their theorems, and the ordering can be associated with an *entailment relation.* Section 5.3 provides a *proof procedure* showing how to find, given a set of axioms, all the theorems. Section 5.4 equips the set of all (consistent) theories (over attributes M) with an information ordering.

5.1 Formal systems

This section first provides a definition of a *formal system*, then we comment on the definition, and finally an example of such a system is presented.

Definition 15 *A* formal system $\mathcal{H}_i$ *consists of the following.*

- ***Language***
Let $P = \{p_1, \ldots, p_\alpha\}$ *and* $\overline{P} = \{\overline{p_1}, \ldots, \overline{p_\alpha}\}$. *Then* $M = P \cup \overline{P}$ *is a set of* attribute symbols, $\boldsymbol{F} = \{F \subseteq M \mid \forall_{p \in P}\ F \not\supseteq \{p, \overline{p}\}\}$ *is a set of* formulae, *and* $\boldsymbol{D} = \boldsymbol{F} \times \{\oplus, \ominus\}$ *is a set of* descriptions, *or* description formulae. $\boldsymbol{L} = (M, \boldsymbol{F})$ *and* $\mathbb{L} = (M, \boldsymbol{D})$ *are referred to as a* language of formulae, *and a* language of descriptions, *respectively.*

- ***Axioms***
There are no logical axioms. *A given set* $\boldsymbol{D}_i \subseteq \boldsymbol{D}$ *determines the formal system* $\mathcal{H}_i$ *with* $\boldsymbol{D}_i$ *accepted as the set of* proper axioms.

- ***Rules of inference***
The set $\Phi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ *is a set of* inference rules, *where:*

$$\varphi_1 : \frac{\oplus F \cup \{m\}}{\oplus F} \qquad \varphi_2 : \frac{\ominus F}{\ominus F \cup \{m\}}$$

$$\varphi_3 : \frac{\ominus F \cup \{m\},\ \ominus F \cup \{\overline{m}\}}{\ominus F} \qquad \varphi_4 : \frac{\oplus F,\ \ominus F \cup \{m\}}{\oplus F \cup \{\overline{m}\}}$$

- ***Provability***
  There are two provability relations *on formulae, $\oplus$-provability, denoted $\vdash_\oplus$, and $\ominus$-provability, denoted $\vdash_\ominus$. Axioms are provable, i.e., if $D = \oplus F \in \boldsymbol{D}_i$ then F is $\oplus$-provable in $\mathcal{H}_i$, denoted $\mathcal{H}_i \vdash_\oplus F$, or simply $\vdash_\oplus F$. Similarly, if $D = \ominus F \in \boldsymbol{D}_i$ then F is $\ominus$-provable in $\mathcal{H}_i$, denoted $\mathcal{H}_i \vdash_\ominus F$, or simply $\vdash_\ominus F$. A formula $F \in \boldsymbol{F}$ is $\oplus$-provable ($\ominus$-provable) if $\oplus F$ ($\ominus F$) can be obtained from the axiom set $\boldsymbol{D}_i$ using the inference rules Φ. A description $D = \oplus F$ ($D = \ominus F$) is* provable *in $\mathcal{H}_i$, denoted $\mathcal{H}_i \vdash D$, if F is $\oplus$-provable ($\ominus$-provable).*

A logical consequence *operator* Cn *maps a description set $\boldsymbol{D}_i$ into a set of all provable descriptions. Provable descriptions are called* theorems, *and the set of all theorems is a* theory, *denoted $\boldsymbol{T}_i$. Hence,* $\mathrm{Cn} : \mathcal{P}(\boldsymbol{D}) \longrightarrow \mathcal{P}(\boldsymbol{D})$, *and* $\mathrm{Cn}(\boldsymbol{D}_i) = \boldsymbol{T}_i$. *A theory $\boldsymbol{T}_i$ is* consistent *if there is no formula $F \in \boldsymbol{F}$ such that $\{\oplus F, \ominus F\} \subseteq \boldsymbol{T}_i$. A description set $\boldsymbol{D}_i$ is said to be* consistent *if the corresponding theory is consistent, i.e., if $\boldsymbol{T}_i = \mathrm{Cn}(\boldsymbol{D}_i)$ is consistent. Given an axiom set $\boldsymbol{D}_i \subset \boldsymbol{D}$, and the theory $\boldsymbol{T}_i = \mathrm{Cn}(\boldsymbol{D}_i)$, there is a "minimal and unique axiom set," called a* generator *of the theory $\boldsymbol{T}_i$, and denoted $\boldsymbol{A}_i = \mathrm{gen}(\boldsymbol{T}_i)$. The generator $\boldsymbol{A}_i$ of $\boldsymbol{T}_i$ satisfies the following: (1) $\mathrm{Cn}(\boldsymbol{A}_i) = \boldsymbol{T}_i$, (2) for no proper subset $\boldsymbol{A}_j$ of $\boldsymbol{A}_i$ it holds that $\mathrm{Cn}(\boldsymbol{A}_j) = \boldsymbol{T}_i$, and (3) $\boldsymbol{A}_i$ is built from $\subseteq$-maximal $\oplus$-formulae and $\subseteq$-minimal $\ominus$-formulae of $\boldsymbol{T}_i$.*

Some comments about the language, axioms, rules of inference, and provability should be made.

Given a set of formulae $\boldsymbol{F}$ it is natural to ask what operations can be applied to formulae, to obtain other formulae. Note that it is known what the set of all formulae is—it is exactly $\boldsymbol{F}$, and it is finite, whenever the attribute symbol set M is finite. Note also that formulae are sets of attributes (attribute symbols), and hence set-theoretic operations can be applied to formulae. If $F, F_1, F_2 \in \boldsymbol{F}$ are formulae, then $F_1 \cap F_2$ and $F_1 \setminus F_2$ are formulae. However, $F_1 \cup F_2$ is a formula only if it belongs to $\boldsymbol{F}$ (rather than to $\mathcal{P}(M) \setminus \boldsymbol{F}$). If $F' \subseteq M$ is a set complement of F, i.e., $F' = M \setminus F$ then F' is a formula iff F is a formula and $|F| = |P|$ in which case we have that $F' = \overline{F} =_{\mathrm{def}} \{\overline{m} \mid m \in F\}$.

Regarding axioms, the definition only requires that a description set $\boldsymbol{D}_i$ taken as a set of axioms for the formal system $\mathcal{H}_i$ satisfies $\boldsymbol{D}_i \subseteq \boldsymbol{D}$, or $\boldsymbol{D}_i \in \mathcal{P}(\boldsymbol{D})$. However, as only consistent theories will be considered, the set of description sets of interest is smaller than $\mathcal{P}(\boldsymbol{D})$.

Rules of inference are simple. Given a set of formulae (sets of attributes) ordered by the (set-theoretic) $\subseteq$ relation—with bigger formulae higher in the line diagram, cf. e.g. Figure 10—the inference rules $\varphi_1, \ldots, \varphi_4$ can be represented symbolically as shown in Figure 20. Thus, φ_1 says that if a formula is $\oplus$-provable then any formula below it is also $\oplus$-provable, and hence φ_1 allows one to "go down" with $\oplus$-provable formulae—$\oplus$-provable formulae form a *down-set*, taking $\subseteq$ as the ordering relation. Similarly, φ_2 allows one to "go up" with $\ominus$-provable formulae—$\ominus$-provable formulae form an *up-set*, again with $\subseteq$ ordering (but note that we do not consider $\subseteq$ as an information-ordering relation—c.f., Section 5.2). Finally, φ_3 allows one to go down with $\ominus$-provable formulae, and φ_4 to go up with $\oplus$-provable ones. Note that the rules φ_2 and φ_4 could "produce" sets of attribute symbols not in $\boldsymbol{F}$—it is assumed that the rules are not applied in such situations. This relates to our choice of formulae and axioms. Note that one could take $\mathcal{P}(M)$ as the set of formulae, and "classify" the elements of $\mathcal{P}(M) \setminus \boldsymbol{F}$ as $\ominus$-formulae by taking $\{\ominus\{p, \overline{p}\}\}_{p \in P}$ as a set of *logical* $\ominus$-axioms. We however limit ourselves to $\boldsymbol{F}$, i.e., we simply rule out—by the way $\boldsymbol{F}$ is defined—"contradictory" sets of attribute symbols.

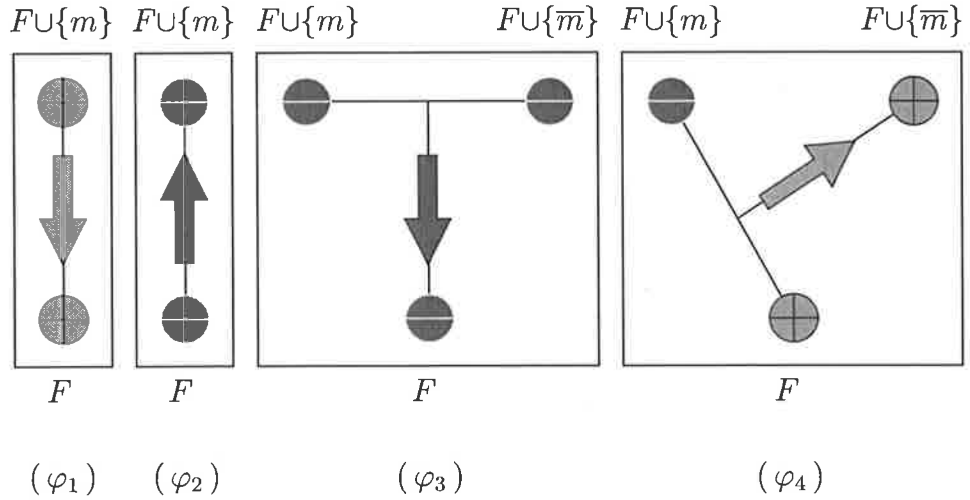


Figure 20. Rules of inference

The rules φ_1 and φ_2 will be justified in Section 5.2, after introducing the notion of *entailment*, and hence φ_3 and φ_4 will also be explained there.

Regarding provability, let $\boldsymbol{D}_s$ be a set of axioms used to determine the formal system $\mathcal{H}_s$, and let $\boldsymbol{T}_s = \mathrm{Cn}(\boldsymbol{D}_s)$. It clearly is only a matter of convenience, whether formulae of $\boldsymbol{F}$, or descriptions of $\boldsymbol{D} = \boldsymbol{F} \times \{\oplus, \ominus\}$ are employed. In particular, the set of descriptions $\boldsymbol{D}_s$ determines two sets of formulae—denoted $\boldsymbol{F}^{\oplus}_{\boldsymbol{D}_s}$ and $\boldsymbol{F}^{\ominus}_{\boldsymbol{D}_s}$—as follows:

$$\boldsymbol{F}^{\oplus}_{\boldsymbol{D}_s} =_{\mathrm{def}} \{F \in \boldsymbol{F} \mid \oplus F \in \boldsymbol{D}_s\} =_{\mathrm{notation}} \boldsymbol{D}^{\oplus}_s;$$

$$\boldsymbol{F}^{\ominus}_{\boldsymbol{D}_s} =_{\mathrm{def}} \{F \in \boldsymbol{F} \mid \ominus F \in \boldsymbol{D}_s\} =_{\mathrm{notation}} \boldsymbol{D}^{\ominus}_s.$$

Conversely, given two sets of formulae $\boldsymbol{D}^{\oplus}_s$ and $\boldsymbol{D}^{\ominus}_s$, the corresponding set of descriptions is as follows:

$$\boldsymbol{D}_s = \boldsymbol{D}^{\oplus}_s \times \{\oplus\} \cup \boldsymbol{D}^{\ominus}_s \times \{\ominus\}.$$

(Recall that—as was said in Definition 6—we employ a notational convention of identifying pairs $(F, \oplus)$ and $(F, \ominus)$ with $\oplus F$ and $\ominus F$, respectively.)

Analogous definitions and notational conventions are applied to the theory $\boldsymbol{T}_s = \mathrm{Cn}(\boldsymbol{D}_s)$ and the generator $\boldsymbol{A}_s = \mathrm{gen}(\boldsymbol{T}_s)$:

$$\boldsymbol{F}^{\oplus}_{\boldsymbol{T}_s} =_{\mathrm{def}} \{F \in \boldsymbol{F} \mid \oplus F \in \boldsymbol{T}_s\} =_{\mathrm{notation}} \boldsymbol{T}^{\oplus}_s;$$

$$\boldsymbol{F}^{\ominus}_{\boldsymbol{T}_s} =_{\mathrm{def}} \{F \in \boldsymbol{F} \mid \ominus F \in \boldsymbol{T}_s\} =_{\mathrm{notation}} \boldsymbol{T}^{\ominus}_s;$$

$$\boldsymbol{T}_s = \boldsymbol{T}_s^{\oplus} \times \{\oplus\} \ \cup \ \boldsymbol{T}_s^{\ominus} \times \{\ominus\};$$
$$\boldsymbol{F}_{\boldsymbol{A}_s}^{\oplus} =_{\text{def}} \{F \in \boldsymbol{F} \mid \oplus F \in \boldsymbol{A}_s\} =_{\text{notation}} \boldsymbol{A}_s^{\oplus};$$
$$\boldsymbol{F}_{\boldsymbol{A}_s}^{\ominus} =_{\text{def}} \{F \in \boldsymbol{F} \mid \ominus F \in \boldsymbol{A}_s\} =_{\text{notation}} \boldsymbol{A}_s^{\ominus};$$
$$\boldsymbol{A}_s = \boldsymbol{A}_s^{\oplus} \times \{\oplus\} \ \cup \ \boldsymbol{A}_s^{\ominus} \times \{\ominus\}.$$

Regarding consistency, we can now say that a theory $\boldsymbol{T}_s$ is consistent, if $\boldsymbol{T}_s^{\oplus} \cap \boldsymbol{T}_s^{\ominus} = \emptyset$. A description set $\boldsymbol{D}_s$ can be inconsistent, even if there is no formula $F \in \boldsymbol{F}$ such that $\{\oplus F, \ominus F\} \subseteq \boldsymbol{D}_s$—using the above notation, it can happen that $\boldsymbol{D}_s^{\oplus} \cap \boldsymbol{D}_s^{\ominus} = \emptyset$, but $\boldsymbol{T}_s^{\oplus} \cap \boldsymbol{T}_s^{\ominus} \neq \emptyset$, and hence the theory $\boldsymbol{T}_s = \mathrm{Cn}(\boldsymbol{D}_s)$ is inconsistent, even though the description set $\boldsymbol{D}_s$ does not explicitly show that it is. Regarding *generators*, note that—given a $\boldsymbol{D}_s$, and $\boldsymbol{T}_s = \mathrm{Cn}(\boldsymbol{D}_s)$ and $\boldsymbol{A}_s = \mathrm{gen}(\boldsymbol{T}_s)$—it can easily happen that $\boldsymbol{D}_s$ and $\boldsymbol{A}_s$ are "disjoint," in the sense that $\boldsymbol{D}_s^{\oplus} \cap \boldsymbol{A}_s^{\oplus} = \emptyset$ and $\boldsymbol{D}_s^{\ominus} \cap \boldsymbol{A}_s^{\ominus} = \emptyset$.

If the set of attribute symbols M is fixed, then a set of all consistent theories (over M) is denoted by $\mathbb{T}$. We limit ourselves to $\mathbb{T}$, i.e., we do not consider inconsistent theories. Indeed, notice that our semantics, as shown in Chapter 4, is fairly standard—given an abstract context, its set of valid formulae is consistent.[1] A set of all consistent description sets is denoted by $\mathbb{D}$—we certainly have that $\mathbb{T} \subseteq \mathbb{D} \subseteq \mathcal{P}(\boldsymbol{D})$. Further, given the fixed M, there is a family of all formal systems, namely $\{(\mathcal{H}_s, \boldsymbol{D}_s)\}_{\boldsymbol{D}_s \in \mathbb{D}}$, and the corresponding family of theories, $\mathbb{T} = \{\boldsymbol{T}_s \mid \boldsymbol{T}_s = \mathrm{Cn}(\boldsymbol{D}_s),\ \boldsymbol{D}_s \in \mathbb{D}\} = \{\mathrm{Cn}(\boldsymbol{D}_s)\}_{\boldsymbol{D}_s \in \mathbb{D}}$.

Consider the following example.

Example 3 *Let* $\boldsymbol{D}_s = \{\oplus\{p_1\}, \ominus\{p_1, p_2, p_3\}, \ominus\{p_1, p_2, \overline{p_3}\}\}$. *Then* $\boldsymbol{T}_s = \{\oplus\{p_1, \overline{p_2}\},\ \oplus\{p_1\}, \oplus\{\overline{p_2}\}, \oplus\{\}, \ominus\{p_1, p_2\}, \ominus\{p_1, p_2, p_3\}, \ominus\{p_1, p_2, \overline{p_3}\}\}$, *and* $\boldsymbol{A}_s = \{\oplus\{p_1, \overline{p_2}\},\ \ominus\{p_1, p_2\}\}$.

Figure 21 shows how the theory resulting from the description set given in Example 3 can be found. An ordered set of formulae from Figure 10 is used. When provable formulae are derived, appropriate nodes of the ordered set are coloured, or marked.

Indeed, the descriptions in $\boldsymbol{T}_s \setminus \boldsymbol{D}_s$ are obtained by applying the inference rules as follows: $\ominus\{p_1, p_2\}$ by φ_3, $\oplus\{p_1, \overline{p_2}\}$ by φ_4, and finally $\oplus\{\overline{p_2}\}$ and $\oplus\{\}$ by φ_1.

Given $\boldsymbol{D}_s$, and the resulting $\boldsymbol{T}_s$ and $\boldsymbol{A}_s$ of Example 3, if we employ the notation of this section, we have:

$\boldsymbol{D}_s^{\oplus} = \{\{p_1\}\}$, $\quad \boldsymbol{D}_s^{\ominus} = \{\{p_1, p_2, p_3\}, \{p_1, p_2, \overline{p_3}\}\}$,
$\boldsymbol{T}_s^{\oplus} = \{\{p_1, \overline{p_2}\}, \{p_1\}, \{\overline{p_2}\}, \{\}\}$, $\quad \boldsymbol{T}_s^{\ominus} = \{\{p_1, p_2\}, \{p_1, p_2, p_3\}, \{p_1, p_2, \overline{p_3}\}\}$,
$\boldsymbol{A}_s^{\oplus} = \{\{p_1, \overline{p_2}\}\}$, $\quad \boldsymbol{A}_s^{\ominus} = \{\{p_1, p_2\}\}$.

We certainly have that $\boldsymbol{D}_s = \boldsymbol{D}_s^{\oplus} \times \{\oplus\} \cup \boldsymbol{D}_s^{\ominus} \times \{\ominus\}$, and similarly for $\boldsymbol{T}_s$ and $\boldsymbol{A}_s$. The theory $\boldsymbol{T}_s$ of Example 3 is consistent, because $\boldsymbol{T}_s^{\oplus} \cap \boldsymbol{T}_s^{\ominus} = \emptyset$, and hence the description set $\boldsymbol{D}_s$ itself is also consistent. Thus, we have that $\boldsymbol{T}_s \in \mathbb{T}$, and $\boldsymbol{D}_s \in \mathbb{D}$. Note also—and this is true for any theory—that $\boldsymbol{A}_s^{\oplus}$ is a set of $\subseteq$-maximal elements of $\boldsymbol{T}_s^{\oplus}$, and $\boldsymbol{A}_s^{\ominus}$ is a set of $\subseteq$-minimal elements of $\boldsymbol{T}_s^{\ominus}$. Also, $\boldsymbol{T}_s = \mathrm{Cn}_{\{\varphi_1, \varphi_2\}}(\boldsymbol{A}_s)$, i.e., only φ_1 and φ_2 need to be employed to find the theory, given its generator.

[1]This is not to say that it is assumed every description set we get is consistent—but if it is not, it needs to be replaced by its consistent subsets.

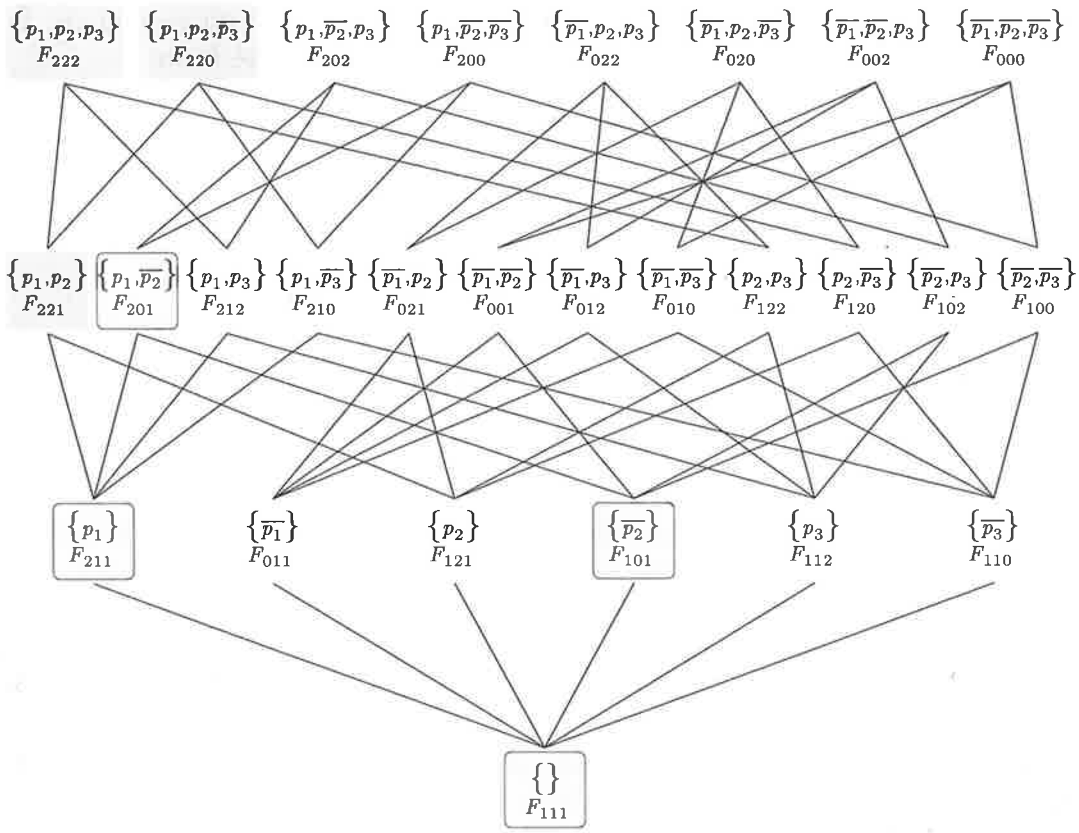


Figure 21. Deriving provable formulae

5.2 Entailment, or information ordering on theorems

In this section we introduce *entailment relations* on descriptions. If $\boldsymbol{D}$ is a set of all descriptions (for a given set of attributes M) then the entailment relations provide *information orderings* on $\boldsymbol{D}$. We will have that descriptions entailed by theorems are theorems. The entailment relations also justify the inference rules φ_1 and φ_2. Some examples are given; in particular, Example 3 of Section 5.1 is reconsidered.

Definition 16 *Let* $D_1 = \oplus F_1, D_2 = \oplus F_2, D_3 = \ominus F_3$ *and* $D_4 = \ominus F_4$ *be descriptions.* D_2 $\oplus$-entails D_1, *denoted* $D_2 \geq_\oplus D_1$ iff $F_2 \supseteq F_1$. *Similarly,* D_3 $\ominus$-entails D_4, *denoted* $D_3 \geq_\ominus D_4$ iff $F_3 \subseteq F_4$.

The entailment relations can indeed be interpreted as information orderings. If D_a ($\oplus$- or $\ominus$-) entails D_b then D_a contains more information than D_b, for if D_a is a theorem, then so is D_b, by the inference rules φ_1 and φ_2. (Note that formally $\geq_\oplus$ and $\geq_\ominus$ are relations on descriptions—they are sets of pairs of descriptions—but one could apply them to formulae, e.g., by saying that $F_2 \geq_\oplus F_1$ iff $F_2 \supseteq F_1$. Then we could say that $D_2 \geq_\oplus D_1$ iff $F_2 \geq_\oplus F_1$. Even further, we could then drop the subscript $\oplus$ from $\geq_\oplus$ and say $D_2 \geq D_1$ instead of $D_2 \geq_\oplus D_1$—clearly, if $D_2 \geq_\oplus D_1$ then D_2 and D_1 are $\oplus$-provable, and hence $D_2 \geq D_1$ would not be ambiguous.)

The introduced entailment relations also shed some light on φ_1 and φ_2. Let $D_1 = \oplus F_1$ and $D_2 = \oplus F_2$ be descriptions. Then φ_1 takes the form:

$$\frac{D_2,\ D_2 \geq_\oplus D_1}{D_1}.$$

Similarly, let $D_3 = \ominus F_3$ and $D_4 = \ominus F_4$ be descriptions—then φ_2 takes the form:

$$\frac{D_3,\ D_3 \geq_\ominus D_4}{D_4}.$$

Hence, φ_1 and φ_2 are analogous to *modus ponens*. In contrast, the rules φ_3 and φ_4 are much stronger, as they are associated with the assumptions we make about objects. Finally, the soundness and completeness result of Section 6.6 demonstrates the appropriateness of the chosen set Φ of inference rules, in particular, it will be shown that the inference rules are semantically sound.

As the definition puts it, $\oplus$-entailment on descriptions agrees with $\supseteq$ on the corresponding formulae, but $\ominus$-entailment agrees with $\subseteq$ (rather than $\supseteq$) on the corresponding formulae. Hence, a line diagram of $\boldsymbol{D}$ equipped with the information ordering $\leq_\ominus$ is a reversal of the diagram of the set ordered by $\leq_\oplus$—Figure 22 demonstrates this for $\alpha = |P| = 2$.

Recall from Section 5.1 that $\oplus$-provable formulae and $\ominus$-provable formulae form a *down-set* and *up-set*, respectively, taking $\subseteq$ on the corresponding formulae as the ordering relation. It seems however more appropriate to employ the information ordering relations $\leq_\oplus$ and $\leq_\ominus$ (on descriptions), respectively.

Figure 22 shows descriptions, or marked formulae, ordered by the information orderings. The descriptions that are $\oplus$-marked formulae are shown in the left half of the picture, and they are ordered by the $\leq_\oplus$ information ordering, while the right half shows descriptions that are $\ominus$-marked formulae ordered by the $\leq_\ominus$ information ordering. We lift the description sets by adding to formulae elements denoted by $1^\oplus, 0^\oplus, 1^\ominus$, and $0^\ominus$. This makes any two description sets share at least the top and bottom elements of the ordered sets of descriptions. We have the following:

$$\begin{array}{ll} \mathcal{F}^\oplus = \boldsymbol{F} \cup \{1^\oplus, 0^\oplus\}, & \mathcal{F}^\ominus = \boldsymbol{F} \cup \{1^\ominus, 0^\ominus\}, \\ \mathcal{D}^\oplus = \mathcal{F}^\oplus \times \{\oplus\}, & \mathcal{D}^\ominus = \mathcal{F}^\ominus \times \{\ominus\}, \\ \underline{\mathcal{D}}^\oplus = (\mathcal{D}^\oplus,\ \leq_\oplus), & \underline{\mathcal{D}}^\ominus = (\mathcal{D}^\ominus,\ \leq_\ominus). \end{array}$$

Note that $\underline{\mathcal{D}}^\oplus$ contains all descriptions that are $\oplus$-marked formulae, and $\underline{\mathcal{D}}^\ominus$ all $\ominus$-marked formulae. If we want to consider ordered sets of descriptions resulting from a description set $\boldsymbol{D}_s \subseteq \boldsymbol{D}$ (recall that $\boldsymbol{D}_s = \boldsymbol{D}_s^\oplus \times \{\oplus\} \cup \boldsymbol{D}_s^\ominus \times \{\ominus\}$) then we get the following:

$$\begin{array}{ll} \mathcal{F}_s^\oplus = \boldsymbol{D}_s^\oplus \cup \{1^\oplus, 0^\oplus\}, & \mathcal{F}_s^\ominus = \boldsymbol{D}_s^\ominus \cup \{1^\ominus, 0^\ominus\}, \\ \mathcal{D}_s^\oplus = \mathcal{F}_s^\oplus \times \{\oplus\}, & \mathcal{D}_s^\ominus = \mathcal{F}_s^\ominus \times \{\ominus\}, \\ \underline{\mathcal{D}}_s^\oplus = (\mathcal{D}_s^\oplus,\ \leq_\oplus), & \underline{\mathcal{D}}_s^\ominus = (\mathcal{D}_s^\ominus,\ \leq_\ominus). \end{array}$$

The same can be done for a theory $\boldsymbol{T}_s = \boldsymbol{T}_s^\oplus \times \{\oplus\} \cup \boldsymbol{T}_s^\ominus \times \{\ominus\}$ and its generator $\boldsymbol{A}_s = \boldsymbol{A}_s^\oplus \times \{\oplus\} \cup \boldsymbol{A}_s^\ominus \times \{\ominus\}$, as they both are also description sets. Example 3 is now considered again—see Figure 23.

Figure 23 shows ordered sets $\underline{\mathcal{T}}_s^\oplus$ and $\underline{\mathcal{T}}_s^\ominus$ of descriptions (theorems) of $\boldsymbol{T}_s$, and ordered sets $\underline{\mathcal{A}}_s^\oplus$ and $\underline{\mathcal{A}}_s^\ominus$ of descriptions of $\boldsymbol{A}_s$. Note that—and this is the case for every theory—description sets of $\underline{\mathcal{A}}_s^\oplus$ and $\underline{\mathcal{A}}_s^\ominus$ are anti-chains (assuming

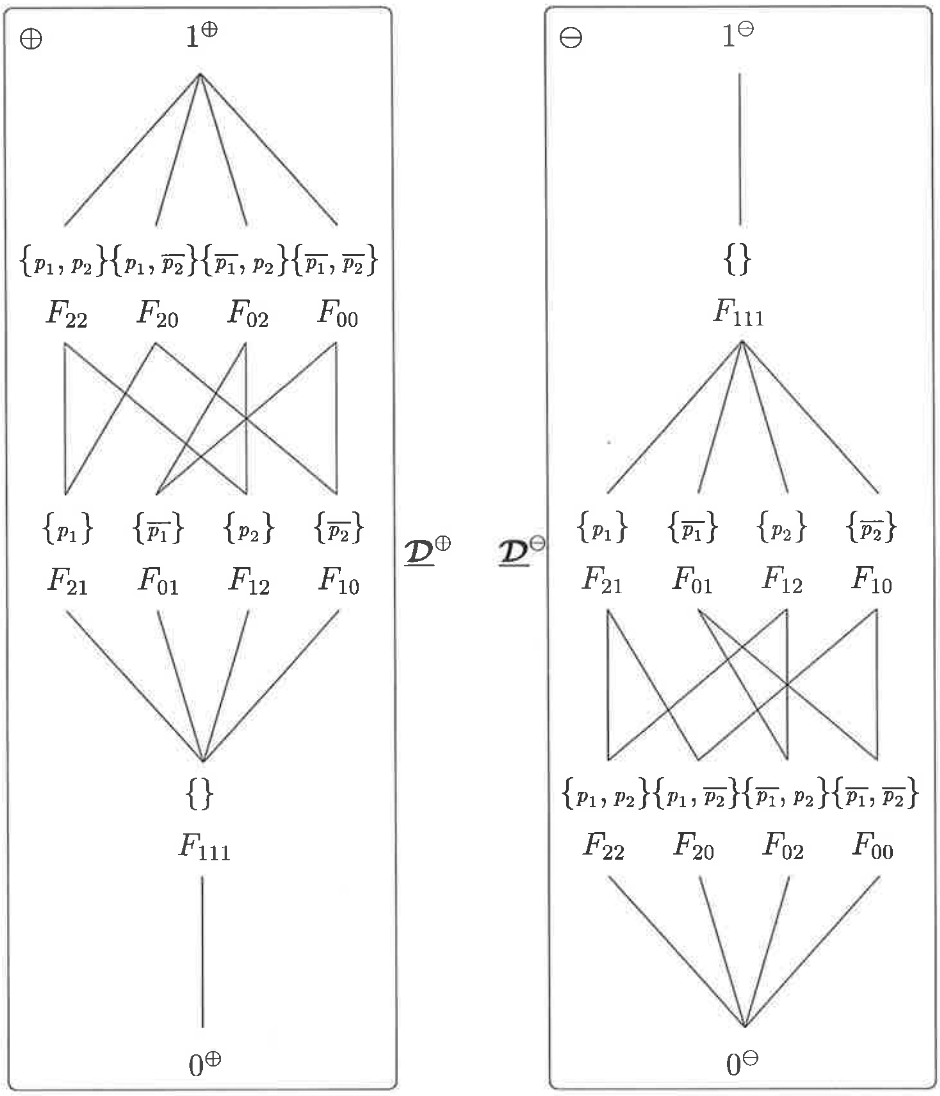


Figure 22. Ordered sets of descriptions, $|P| = 2$

that we ignore the top and bottom elements $1^{\oplus}, 0^{\oplus}, 1^{\ominus}$ and $0^{\ominus}$). Recall that we said in Section 5.1—discussing Example 3—that $\boldsymbol{A}_s^{\oplus}$ and $\boldsymbol{A}_s^{\ominus}$ are sets of $\subseteq$-maximal elements of $\boldsymbol{T}_s^{\oplus}$ and $\subseteq$-minimal elements of $\boldsymbol{T}_s^{\ominus}$, respectively. We can now say that the corresponding descriptions—see Figure 23—are *entailment maximal.* Every theory in turn consists of two sets of descriptions, or theorems, that are *down-sets*, w.r.t. the entailment orderings.

Finally, consider Figure 24.

Let $\mathcal{D}^{\oplus}$—the ordered set of all formulae marked $\oplus$—be denoted by $\mathbf{1}^{\oplus}$, and $\mathcal{D}^{\ominus}$—the set of $\ominus$-marked formulae—by $\mathbf{1}^{\ominus}$, Also, let $(\{1^{\oplus}, 0^{\oplus}\}, \leq_{\oplus})$ and $(\{1^{\ominus}, 0^{\ominus}\}, \leq_{\ominus})$ be denoted by $\mathbf{0}^{\oplus}$ and $\mathbf{0}^{\ominus}$, respectively. Clearly, $(\mathbf{1}^{\oplus}, \mathbf{1}^{\ominus})$ corresponds to an (utterly) *inconsistent theory*, denoted by $\mathbf{1}$, and $(\mathbf{0}^{\oplus}, \mathbf{0}^{\ominus})$ to an *ignorant theory*, denoted by $\mathbf{0}$—theory that contains no information. Whether theories and description sets are treated as ordered sets, or just sets (of descriptions) depends only on which aspects we want to emphasise—hence, from now on, we will denote a theory by $\boldsymbol{T}_s$ rather than $(\mathcal{T}_s^{\oplus}, \mathcal{T}_s^{\ominus})$, even if order on

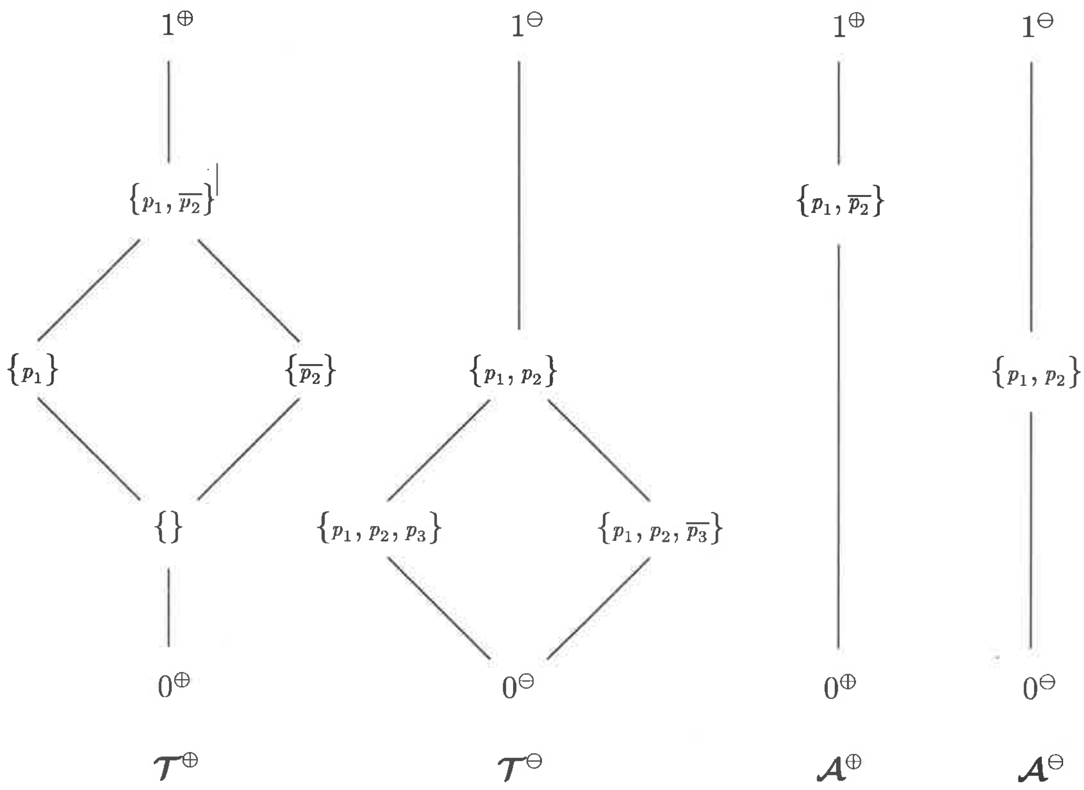


Figure 23. Ordered sets of theorems and axioms (Example 3)

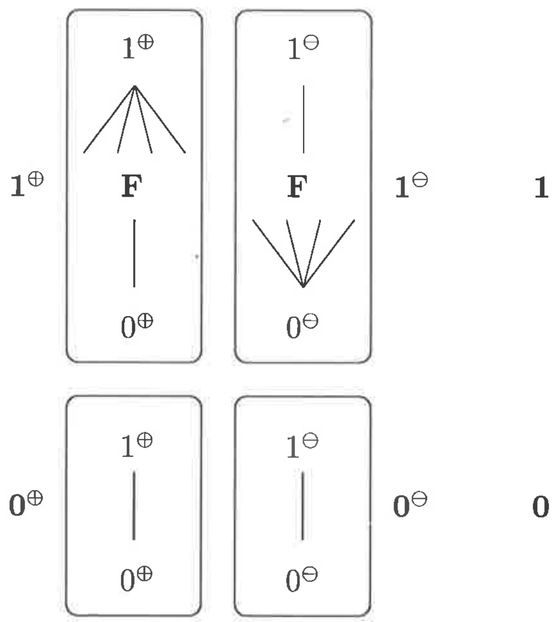


Figure 24. Inconsistent theory and ignorant theory

theorems is not to be neglected. We also use the symbols **1** and **0** to denote the inconsistent theory (the only inconsistent theory we consider), and the ignorant theory (the empty theory), respectively. Recall that $\mathbb{T}$ denotes the set of all consistent theories—clearly, **0** is in $\mathbb{T}$ but **1** is not. In Section 5.4 we will consider a set of all consistent theories, together with the inconsistent theory $\mathbf{1}$, i.e., the set $\mathbb{T} \cup \{\mathbf{1}\}$.

5.3 Proof procedure

In this section we provide a proof procedure that allows us to find the theory $\boldsymbol{T}_s = \mathrm{Cn}(\boldsymbol{D}_s)$, given a description set $\boldsymbol{D}_s$—recall that $\boldsymbol{D}_s$ is the set of axioms the corresponding formal system $\boldsymbol{\mathcal{H}}_s$ starts with. In fact, the proof procedure will allow us to find—given $\boldsymbol{D}_s$—a *generator*, or *minimal and unique axiom set* $\boldsymbol{A}_s$ of the theory $\boldsymbol{T}_s$, without (unnecessarily) constructing $\boldsymbol{T}_s$ itself. Note that $\boldsymbol{A}_s$ is a parsimonious representation of $\boldsymbol{T}_s$, and having $\boldsymbol{A}_s = \mathrm{gen}(\boldsymbol{T}_s)$ the theory can easily be found by using the inference rules φ_1 and φ_2, i.e., $\boldsymbol{T}_s = \mathrm{Cn}_{\{\varphi_1,\varphi_2\}}(\boldsymbol{A}_s)$.

As shown in Figure 20 of Section 5.1, the inference rules $\varphi_1, \varphi_2, \varphi_3$ and φ_4 allow us to go down with $\oplus$-provable, up with $\ominus$-provable, down with $\ominus$-provable, and up with $\oplus$-provable formulae, respectively. Note that "up" and "down" relate to the $\subseteq$ relation on formulae, as shown in Figure 10, and *not* to the entailment orderings of Section 5.2. In fact, it seems easier to see how a description set $\boldsymbol{D}_s$ "turns into" its theory $\boldsymbol{T}_s$ by using a $\subseteq$-ordered sets of formulae—like those in Figure 10 and Figures 15 and 16—and hence this is the method[2] we will employ to visualise inference. (In a sense, Section 5.2 has done its work—of introducing information ordering on theorems—and from now on we assume that it is clear how sets of descriptions are *ordered*.)

Consider the rule φ_1. Definition 15 says that φ_1 is given by $\frac{\oplus F \cup \{m\}}{\oplus F}$. Using the provability relation $\vdash_\oplus$, this can be expressed by saying that if $F \cup \{m\}$ is $\oplus$-provable then F is $\oplus$-provable, i.e., if $\vdash_\oplus F \cup \{m\}$ then $\vdash_\oplus F$. Using the entailment relation $\geq_\oplus$, this can be expressed by saying that $F \cup \{m\}$ $\oplus$-entails F, i.e., $F \cup \{m\} \geq_\oplus F$, or rather $\oplus F \cup \{m\} \geq_\oplus \oplus F$, to be formally correct.[3] Note however that φ_1 in this formulation is "local." Indeed, a formula $\oplus$-entails another formula directly below it, but a "global" version of φ_1 can easily be given, e.g., in the form that if $F_2 \supseteq F_1$ then $F_2 \geq_\oplus F_1$, simply because the local version of φ_1 can be applied several times in sequence. This suggests that the proof procedure should involve global forms of the inference rules Φ.

Given a description set $\boldsymbol{D}_s$ and the resulting theory $\boldsymbol{T}_s = \mathrm{Cn}(\boldsymbol{D}_s)$ and its generator $\boldsymbol{A}_s = \mathrm{gen}(\boldsymbol{T}_s)$, employ the following notation:

$$\boldsymbol{D}_s = \boldsymbol{D}_s^\oplus \times \{\oplus\} \cup \boldsymbol{D}_s^\ominus \times \{\ominus\},$$

$$\boldsymbol{T}_s = \boldsymbol{T}_s^\oplus \times \{\oplus\} \cup \boldsymbol{T}_s^\ominus \times \{\ominus\},$$

$$\boldsymbol{A}_s = \boldsymbol{A}_s^\oplus \times \{\oplus\} \cup \boldsymbol{A}_s^\ominus \times \{\ominus\}.$$

[2] Descriptions are marked formulae. Hence, line diagrams of formulae can be employed to show which descriptions are axioms, and which theorems follow from the axioms. For instance, if $|P| = 3$ then the line diagram of Figure 10 can be used to show theorems, by marking some formulae with $\oplus$ (or colouring them green), and some other formulae with $\ominus$ (or colouring them red). Inferences are then locally applied to nodes in the diagram, and spread over it as much as the axioms allow.

[3] There is no need to be that strict—c.f., Definition 16 together with the comment that follows it.

The question is how to find $\boldsymbol{A}_s^{\oplus}$ and $\boldsymbol{A}_s^{\ominus}$, given $\boldsymbol{D}_s^{\oplus}$ and $\boldsymbol{D}_s^{\ominus}$, as this gives $\boldsymbol{A}_s$, which in turn can be employed to easily find $\boldsymbol{T}_s = \mathrm{Cn}_{\{\varphi_1,\varphi_2\}}(\boldsymbol{A}_s)$.

Consider $\ominus$-provable formulae first. Given $\boldsymbol{D}_s^{\ominus}$, how can $\boldsymbol{T}_s^{\ominus}$ be found? The first thing to notice is that the inference rules φ_2 and φ_3 are *the only* rules that needed to be employed to derive $\ominus$-provable formulae, and furthermore, they operate on $\ominus$-provable formulae only.[4] Having in mind that we want to construct a parsimonious representation of $\boldsymbol{T}_s$, we replace $\boldsymbol{D}_{s,1}^{\ominus} = \boldsymbol{D}_s^{\ominus}$ with a set, denoted $\boldsymbol{D}_{s,2}^{\ominus}$, of $\subseteq$-minimal formulae of $\boldsymbol{D}_{s,1}^{\ominus}$. Note that nothing is really lost by applying this step, as the inference rule φ_2 can be used to recover $\boldsymbol{D}_{s,1}^{\ominus}$. Obviously, the formulae of $\boldsymbol{D}_{s,2}^{\ominus}$ form an anti-chain because they are minimal. However, we can still have[5] that $\boldsymbol{T}_s^{\ominus} \neq \mathrm{Cn}_{\{\varphi_2\}}(\boldsymbol{D}_{s,2}^{\ominus})$, because some $\ominus$-provable formulae might require φ_3 to be applied.

Suppose that such a situation occurs. That is, there are F_1, F_2 in $\boldsymbol{T}_s^{\ominus}$, but not in $\boldsymbol{D}_{s,2}^{\ominus}$, and that φ_3 can be applied to F_1, F_2 to produce F_3, but there are no F_4, F_5 in $\boldsymbol{D}_{s,2}^{\ominus}$ such that φ_3 can be applied to them to give F_3 or a subset of F_3. Assume also that F_1, F_2 are "optimal," in the sense that they produce F_3, but no two other formulae produce a subset of F_3.

1. $F_1 = F_3 \cup \{m\}$ and $F_2 = F_3 \cup \{\overline{m}\}$.
   It is assumed that φ_3 applies to F_1, F_2, and hence there is an appropriate $m \in M$.

2. $F_4 \subseteq F_1$ and $F_5 \subseteq F_2$.
   Assuming the first application of φ_3 operates on F_1 and F_2, these must be derivable—via φ_2—from some $F_4, F_5 \in \boldsymbol{D}_{s,2}^{\ominus}$.

3. $F_4 \setminus F_5 \neq \emptyset$ and $F_5 \setminus F_4 \neq \emptyset$.
   It is so, because $\boldsymbol{D}_{s,2}^{\ominus}$ is an anti-chain.

4. $F_4 \not\subseteq F_3$ and $F_5 \not\subseteq F_3$.
   Otherwise, F_3 could be derived from F_4 or F_5 by employing φ_2.

5. $F_4 \ni m$ and $F_5 \ni \overline{m}$.
   This obtains from (1,2,4).

6. $F_4 = F_6 \cup \{m\}$, where $F_6 \not\ni m$ and $F_5 = F_7 \cup \{\overline{m}\}$, where $F_7 \not\ni \overline{m}$.
   Subsets of F_4 and F_5 disjoint with $\{m, \overline{m}\}$ are selected.

7. $F_6 \subseteq F_4 \subseteq F_1$ and $F_7 \subseteq F_5 \subseteq F_2$.
   This obtains from (6,2).

8. $F_1 \cup F_2 = F_3 \cup \{m, \overline{m}\}$, by (1).

9. $F_6 \cup F_7 \subseteq F_3$, by (6,7).

[4]This means that $\oplus$-provable formulae are useless in deriving $\ominus$-provable ones. Semantically, knowing that there are *some* objects in the world does not help to determine what objects are not present.

[5]We abuse the notation slightly, by applying inference rules to ($\ominus$-provable) formulae, rather than to the corresponding descriptions—this should cause no confusion, and simplifies the notation.

10. $F_6 \cup F_7 = F_3$.
Indeed, assume $F_6 \cup F_7 \neq F_3$. But then by (9) we have that $F_6 \cup F_7 \subset F_3$. Then by (6) and φ_2 we would find both $F_6 \cup F_7 \cup \{m\}$ and $F_6 \cup F_7 \cup \{\overline{m}\}$ to be $\ominus$-provable, and hence φ_3 would apply to them to produce $F_6 \cup F_7$ $\ominus$-provable. But note that it would mean that the choice of F_1 and F_2 would not be optimal in the abovementioned sense. Hence contradiction, and thus (10) holds.

11. $F_3 = F_4 \cup F_5 \setminus \{m, \overline{m}\}$, by (6,10).
Hence, given F_4 and F_5 one can find the desired F_3. Note that $F_4 \cup F_5$ must contain exactly one pair of "opposite" attribute symbols for φ_3 to be applied.

The above analysis justifies the steps *(i–v)* of Procedure 1.

Consider now $\oplus$-provable formulae. Given $\boldsymbol{D}_s^{\oplus}$, and the already obtained $\boldsymbol{A}_s^{\ominus}$, how can $\boldsymbol{T}_s^{\oplus}$ be found? The inference rules φ_1 and φ_1 are *the only* rules that are to be used. Having in mind that we want to construct a parsimonious representation of $\boldsymbol{T}_s$, we replace $\boldsymbol{D}_{s,1}^{\oplus} = \boldsymbol{D}_s^{\oplus}$ with a set, denoted $\boldsymbol{D}_{s,2}^{\oplus}$, of $\subseteq$-maximal formulae of $\boldsymbol{D}_{s,1}^{\oplus}$. Again, note that nothing is really lost by applying this step, as the inference rule φ_1 can be used to recover $\boldsymbol{D}_{s,1}^{\oplus}$. Obviously, the formulae of $\boldsymbol{D}_{s,2}^{\oplus}$ form an anti-chain because they are maximal. However, we can still have that $\boldsymbol{T}_s^{\oplus} \neq \mathrm{Cn}_{\{\varphi_1\}}(\boldsymbol{D}_{s,2}^{\oplus})$, because some $\oplus$-provable formulae might require φ_4 to be applied.

Suppose that such a situation occurs. That is, there is F_1 in $\boldsymbol{T}_s^{\oplus}$ (but not in $\boldsymbol{D}_{s,2}^{\oplus}$) and there is F_2 in $\boldsymbol{T}_s^{\ominus}$, such that φ_4 can be applied to F_1, F_2 to derive F_3 to be $\oplus$-provable, but there are no $F_4 \in \boldsymbol{D}_{s,2}^{\oplus}$ and $F_5 \in \boldsymbol{A}_s^{\ominus}$ such that φ_4 can be applied to them to give $\oplus$-provable F_3 or a subset of F_3. Assume also that F_1, F_2 are "optimal" in the sense that they produce F_3, but no two other formulae produce a superset of F_3.

1. $F_2 = F_1 \cup \{m\}$ and $F_3 = F_1 \cup \{\overline{m}\}$.
It is assumed that φ_4 applies to F_1, F_2, i.e., F_1 is $\oplus$-provable, F_2 is $\ominus$-provable, and F_3 is derived as a $\oplus$-provable formula.

2. $F_4 \supseteq F_1$ and $F_5 \subseteq F_2$.
Assuming that the first application of φ_4 operates on F_1 and F_2, these must be derivable—via φ_1 and φ_2, respectively—from some $F_4 \in \boldsymbol{D}_{s,2}^{\oplus}$, and some $F_5 \in \boldsymbol{A}_s^{\ominus}$.

3. $F_4 \not\ni \overline{m}$.
Otherwise, we would have $F_4 \supseteq F_3$, and hence one could derive F_3 as $\oplus$-provable, by applying φ_1 to F_4.

4. $F_4 \not\ni m$.
Otherwise, we would have $F_4 \supseteq F_1 \cup \{m\}$, and hence $F_1 \cup \{m\}$ would be $\oplus$-provable, while in fact it is $\ominus$-provable.

5. $F_4 \cap \{m, \overline{m}\} = \emptyset$, by (3,4).

6. $F_5 \ni m$.
   This results from the following. $F_5 \not\subseteq F_1$, because otherwise F_1 would be $\ominus$-provable. Hence, we can put $m \in F_5 \setminus F_1$, and then $F_5 = F_7 \cup \{m\}$, with $F_7 \not\ni m$.

7. $F_7 \subseteq F_1$.
   Because—by (6,2,1)—$F_7 \subseteq F_5 \subseteq F_2 = F_1 \cup \{m\}$ and $F_7 \not\ni m$.

8. $F_7 = F_1$.
   Otherwise—by (7)—F_7 would be a proper subset of F_1, and F_1 and F_2 would not be optimal, as φ_4 would apply to F_7 and F_5 to produce a proper subset of F_3 as a $\oplus$-provable formula. (To see that φ_4 would apply to F_7 and F_5, notice that F_7 is $\oplus$-provable, and F_5 is $\ominus$-provable and $F_5 = F_7 \cup \{m\}$.)

9. Hence, $F_3 = F_7 \cup \{\overline{m}\}$, where $F_7 = F_5 \setminus \{m\}$, where $\{m\} = F_5 \setminus F_4$. Thus, given F_4 and F_5, the rule φ_4 applies iff $F_5 \setminus F_4 = \{m\}$, and the resulting F_3 can be computed from F_4 and F_5.

The above analysis justifies the steps *(vi–x)* of Procedure 1.

Procedure 1 *Let $\boldsymbol{D}_s$ be a description set, and $\boldsymbol{D}_s^{\oplus}$ and $\boldsymbol{D}_s^{\ominus}$ be the corresponding $\oplus$- and $\ominus$-formulae, respectively. The following procedure finds the generator $\boldsymbol{A}_s$, of the theory $\boldsymbol{T}_s$ the description set $\boldsymbol{D}_s$ implies.*

(i) $\boldsymbol{D}_{s,1}^{\ominus} = \boldsymbol{D}_s^{\ominus}$,

(ii) $\boldsymbol{D}_{s,2}^{\ominus} = \{F \in \boldsymbol{D}_{s,1}^{\ominus} \mid F \text{ is } \subseteq\text{-}\textit{minimal}\}$,

(iii) $\boldsymbol{D}_{s,3}^{\ominus} = \boldsymbol{D}_{s,2}^{\ominus} \cup \{F_3 \in \boldsymbol{F} \mid \exists_{m \in M}\ F_3 \cup \{m, \overline{m}\} = F_4 \cup F_5,\ \text{where } F_4, F_5 \in \boldsymbol{D}_{s,2}^{\ominus}\}$ *(performed recursively)*,

(iv) $\boldsymbol{D}_{s,4}^{\ominus} = \{F \in \boldsymbol{D}_{s,3}^{\ominus} \mid F \text{ is } \subseteq\text{-}\textit{minimal}\}$,

(v) $\boldsymbol{A}_s^{\ominus} = \boldsymbol{D}_{s,4}^{\ominus}$,

(vi) $\boldsymbol{D}_{s,1}^{\oplus} = \boldsymbol{D}_s^{\oplus}$,

(vii) $\boldsymbol{D}_{s,2}^{\oplus} = \{F \in \boldsymbol{D}_{s,1}^{\oplus} \mid F \text{ is } \subseteq\text{-}\textit{maximal}\}$,

(viii) $\boldsymbol{D}_{s,3}^{\oplus} = \boldsymbol{D}_{s,2}^{\oplus} \cup \{F_3 \in \boldsymbol{F} \mid F_4 \in \boldsymbol{A}_s^{\ominus},\ F_5 \in \boldsymbol{D}_s^{\oplus},\ F_5 \setminus F_4 = \{m\}, \text{ and } F_3 = (F_5 \setminus \{m\}) \cup \{\overline{m}\}\}$ *(performed recursively)*,

(ix) $\boldsymbol{D}_{s,4}^{\oplus} = \{F \in \boldsymbol{D}_{s,3}^{\oplus} \mid F \text{ is } \subseteq\text{-}\textit{maximal}\}$,

(x) $\boldsymbol{A}_s^{\oplus} = \boldsymbol{D}_{s,4}^{\oplus}$.

Applying Procedure 1 to Example 3 we get the following.

(i) $\boldsymbol{D}_{s,1}^{\ominus} = \boldsymbol{D}_s^{\ominus} = \{\{p_1, p_2, p_3\}, \{p_1, p_2, \overline{p_3}\}\}$,

(ii) $\boldsymbol{D}_{s,2}^{\ominus} = \{F \in \boldsymbol{D}_{s,1}^{\ominus} \mid F \text{ is } \subseteq\text{-minimal}\} = \{\{p_1, p_2, p_3\}, \{p_1, p_2, \overline{p_3}\}\} = \boldsymbol{D}_{s,1}^{\ominus}$,

(iii) $\boldsymbol{D}^{\ominus}_{s,3} = \boldsymbol{D}^{\ominus}_{s,2} \cup \{F_3 \in \boldsymbol{F} \mid \exists_{m \in M}\ F_3 \cup \{m, \overline{m}\} = F_4 \cup F_5,$ where $F_4, F_5 \in \boldsymbol{D}^{\ominus}_{s,2}\} = \{\{p_1, p_2, p_3\}, \{p_1, p_2, \overline{p_3}\}, \{p_1, p_2\}\}$,

(iv) $\boldsymbol{D}^{\ominus}_{s,4} = \{F \in \boldsymbol{D}^{\ominus}_{s,3} \mid F$ is $\subseteq$-minimal$\} = \{\{p_1, p_2\}\}$,

(v) $\boldsymbol{A}^{\ominus}_s = \boldsymbol{D}^{\ominus}_{s,4} = \{\{p_1, p_2\}\}$,

(vi) $\boldsymbol{D}^{\oplus}_{s,1} = \boldsymbol{D}^{\oplus}_s = \{\{p_1\}\}$,

(vii) $\boldsymbol{D}^{\oplus}_{s,2} = \{F \in \boldsymbol{D}^{\oplus}_{s,1} \mid F$ is $\subseteq$-maximal$\} = \{\{p_1\}\}$,

(viii) $\boldsymbol{D}^{\oplus}_{s,3} = \boldsymbol{D}^{\oplus}_{s,2} \cup \{F_3 \in \boldsymbol{F} \mid F_4 \in \boldsymbol{A}^{\ominus}_s,\ F_5 \in \boldsymbol{D}^{\oplus}_s,\ F_5 \setminus F_4 = \{m\},$ and $F_3 = (F_5 \setminus \{m\}) \cup \{\overline{m}\}\} = \{\{p_1\}, \{p_1, \overline{p_2}\}\}$,

(ix) $\boldsymbol{D}^{\oplus}_{s,4} = \{F \in \boldsymbol{D}^{\oplus}_{s,3} \mid F$ is $\subseteq$-maximal$\} = \{\{p_1, \overline{p_2}\}\}$,

(x) $\boldsymbol{A}^{\oplus}_s = \boldsymbol{D}^{\oplus}_{s,4} = \{\{p_1, \overline{p_2}\}\}$.

5.4 Information ordering on theories

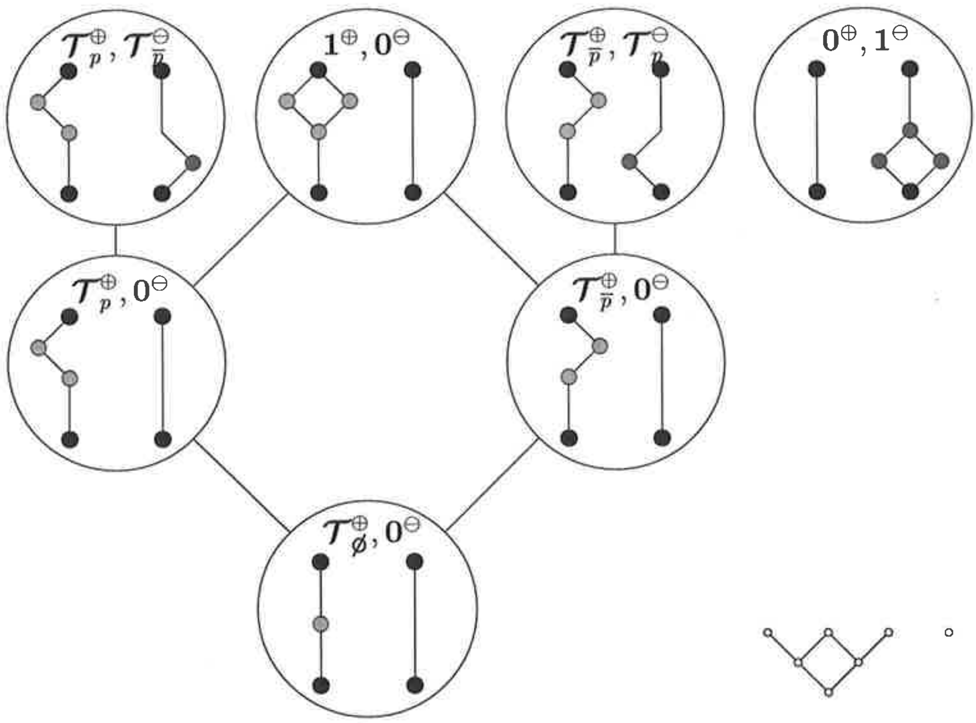


Figure 25. Information ordering on partial worlds—showing theories

In this section we look at consistent theories equipped with an information ordering. A natural place to start is to consider *theories of contexts*—Figure 25 presents those theories that correspond to contexts (or rather *partial worlds*, see Section 4.2) over P, where $|P| = 1$.

To find all consistent theories we consider a set $\boldsymbol{D}$ of all possible descriptions (over a fixed set of attributes M) and single out a set of those description sets (subsets of $\boldsymbol{D}$) which are *consistent theories.* The set of all consistent theories is denoted by $\mathbb{T}$, and it is equipped with a natural *information ordering* $\leq$. To explicitly show $\mathbb{T}$ for the simple case of $|P| = 1$, we include a sequence of figures—Figure 26 shows a set $\mathbb{T}^{\oplus}$ of $\oplus$-*theories* and $\mathbb{T}^{\ominus}$ of $\ominus$-*theories*, Figure 27 starts a search for consistent theories with $\mathbb{T}^{\oplus} \times \mathbb{T}^{\ominus}$, Figure 28 contains only those description sets in $\mathbb{T}^{\oplus} \times \mathbb{T}^{\ominus}$ which are consistent (but not necessarily closed under Φ, so some of the description sets are not theories), Figure 29 stops the search by providing all consistent theories, and finally Figure 30 shows the consistent theories in detail.

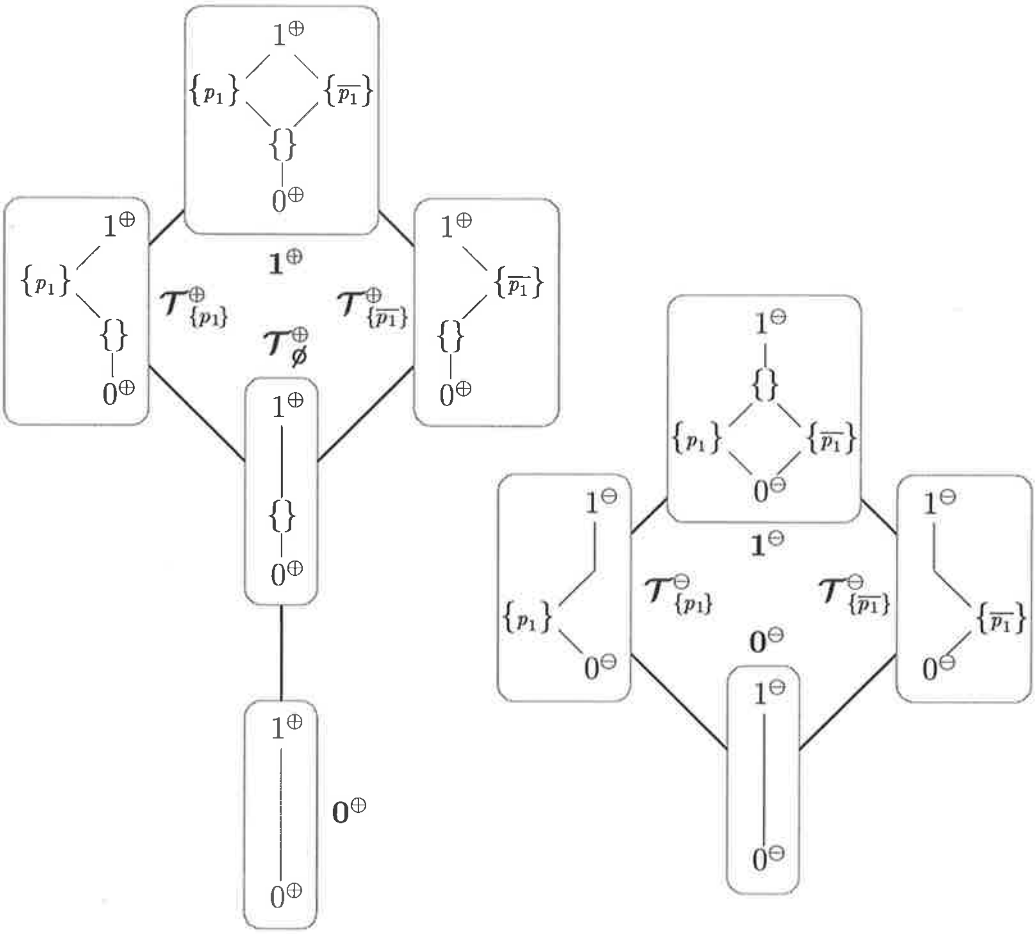


Figure 26. Lattices $\mathbb{T}^{\oplus}$ of $\oplus$-theories and $\mathbb{T}^{\ominus}$ of $\ominus$-theories

Let $\boldsymbol{D}_s$ be a description set—the resulting theory is denoted by $\boldsymbol{T}_s$, i.e. $\boldsymbol{T}_s = \mathrm{Cn}(\boldsymbol{D}_s)$. Let $\mathbb{T}$ denote the set of all consistent theories. In Section 5.2 we considered two specific theories, **1** and **0**, treating these as simply sets of descriptions, rather than as ordered sets, there is no need to keep $1^{\oplus}, 0^{\oplus}, 1^{\ominus}$, and $0^{\ominus}$. Then **0**, the ignorant theory, is simply an empty theory, or empty set of descriptions (note that $\boldsymbol{0} \in \mathbb{T}$). The "inconsistent theory" **1** contains all possible descriptions, i.e., all formulae are marked with both $\oplus$ and $\ominus$—all formulae are marked as both $\oplus$-provable, and as $\ominus$-provable. Hence, $\boldsymbol{1} \notin \mathbb{T}$, and given our semantics of Chapter 4, **1** does not fit to our framework. Indeed, although **1**

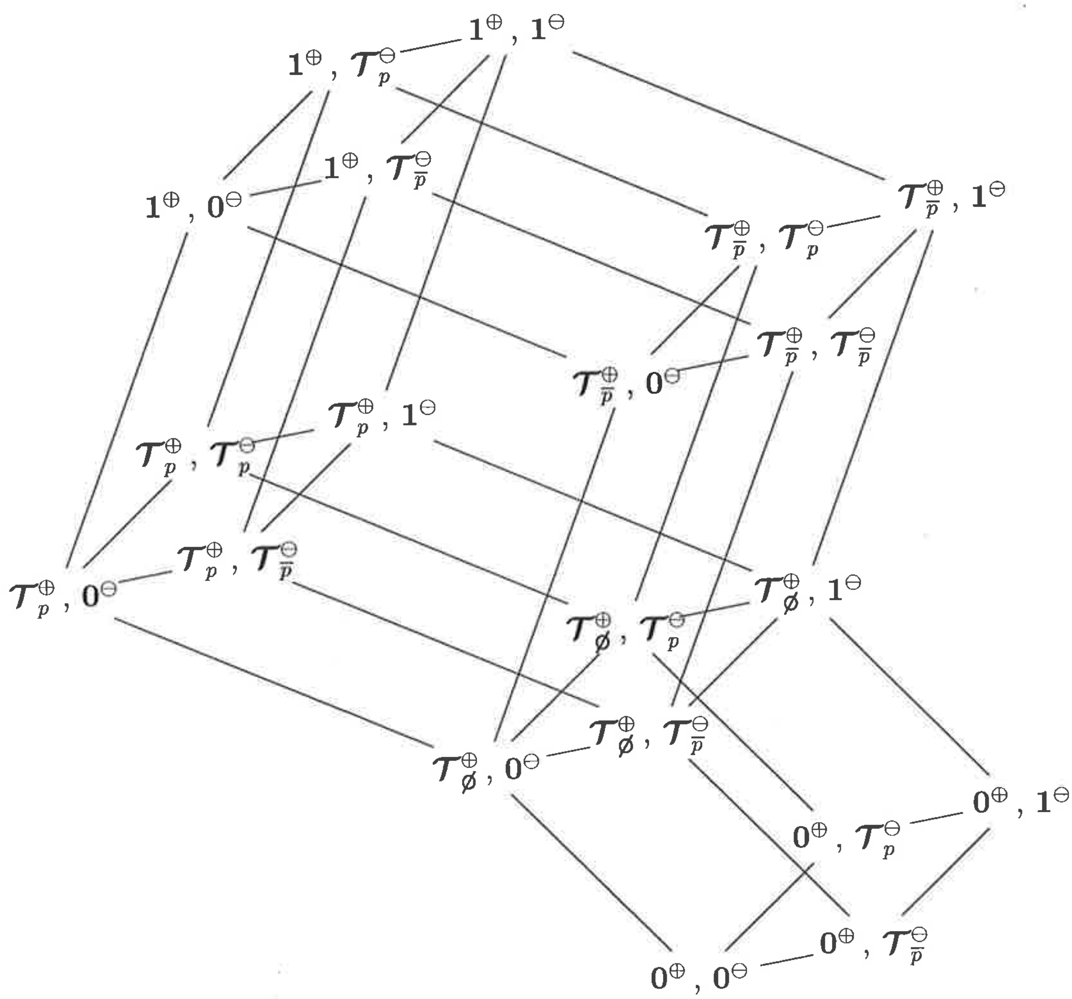


Figure 27. In search for consistent theories—lattice $\mathbb{T}^{\oplus} \times \mathbb{T}^{\ominus}$

can be associated with the "most inconsistent theory," or be associated with the set of inconsistent theories, the way we employ **1** is purely conventional. More precisely, we will use **1** to say that two theories are mutually exclusive. If two theories $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ cannot be "merged," in the sense that accepting both would allow us to derive both $\oplus F$ and $\ominus F$, then we will have that $\boldsymbol{T}_1 \vee \boldsymbol{T}_2 = \mathbf{1}$—this will soon be explained.

Not surprisingly, an information ordering relation $\leq$ comes into play, and the relation is a natural one: theorems represent some information, and so do theories, or sets of theorems. Hence, it is appropriate to have $\leq$ coinciding with the subset relation $\subseteq$ on theories seen as sets of theorems (provable descriptions).

Definition 17 *Let M be a set of attributes, and let $\mathbb{T}$ be a set of all consistent theories (over M). An* information ordering relation *$\leq$ on $\mathbb{T}$ is defined as follows. Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in \mathbb{T}$. Then $\boldsymbol{T}_1 \leq \boldsymbol{T}_2$ iff $\boldsymbol{T}_1 \subseteq \boldsymbol{T}_2$.*

The information ordering $\leq$ on theories allows us to treat $\mathbb{T}$ as a *lattice*, after adding a top element **1** to it. The corresponding *meet* and *join* operations on

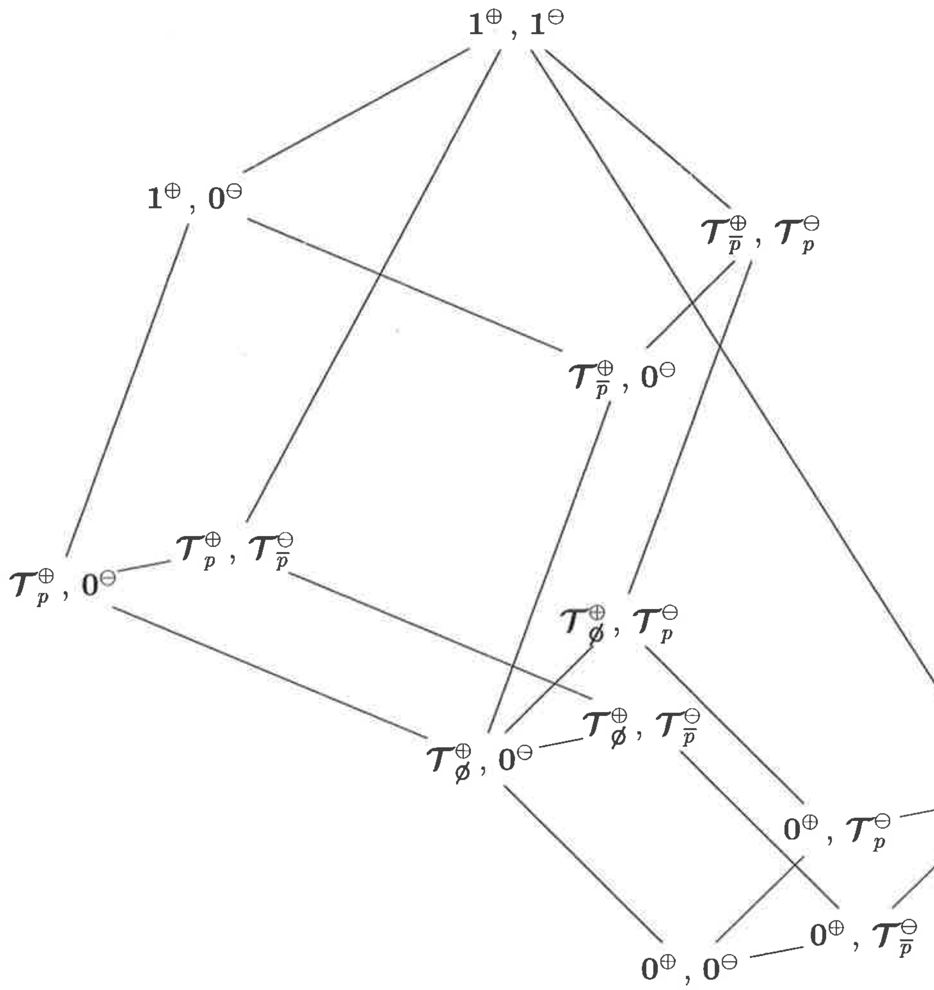


Figure 28. Consistent description sets

$\mathbb{T} \cup \{\mathbf{1}\}$ are defined in an obvious way. The meet on two theories is simply the intersection of the theories. To find their join, one needs to find the union of the theories, and then compute its consequences—if the resulting description set is a consistent theory, then it is the join of the theories, otherwise $\mathbf{1}$ is taken to be the join. Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in \mathbb{T}$. Then:

$$\boldsymbol{T}_1 \wedge \boldsymbol{T}_2 = \boldsymbol{T}_1 \cap \boldsymbol{T}_2,$$

$$\boldsymbol{T}_1 \vee \boldsymbol{T}_2 = \begin{cases} \mathrm{Cn}(\boldsymbol{T}_1 \cup \boldsymbol{T}_2) & \text{if } \boldsymbol{T}_1 \cup \boldsymbol{T}_2 \text{ is consistent,} \\ \mathbf{1} & \textit{otherwise.} \end{cases}$$

Considering the theory $\mathbf{1}$, we have:

$$\boldsymbol{T}_1 \wedge \mathbf{1} = \boldsymbol{T}_1,$$

$$\boldsymbol{T}_1 \vee \mathbf{1} = \mathbf{1}.$$

Hence, the following proposition.

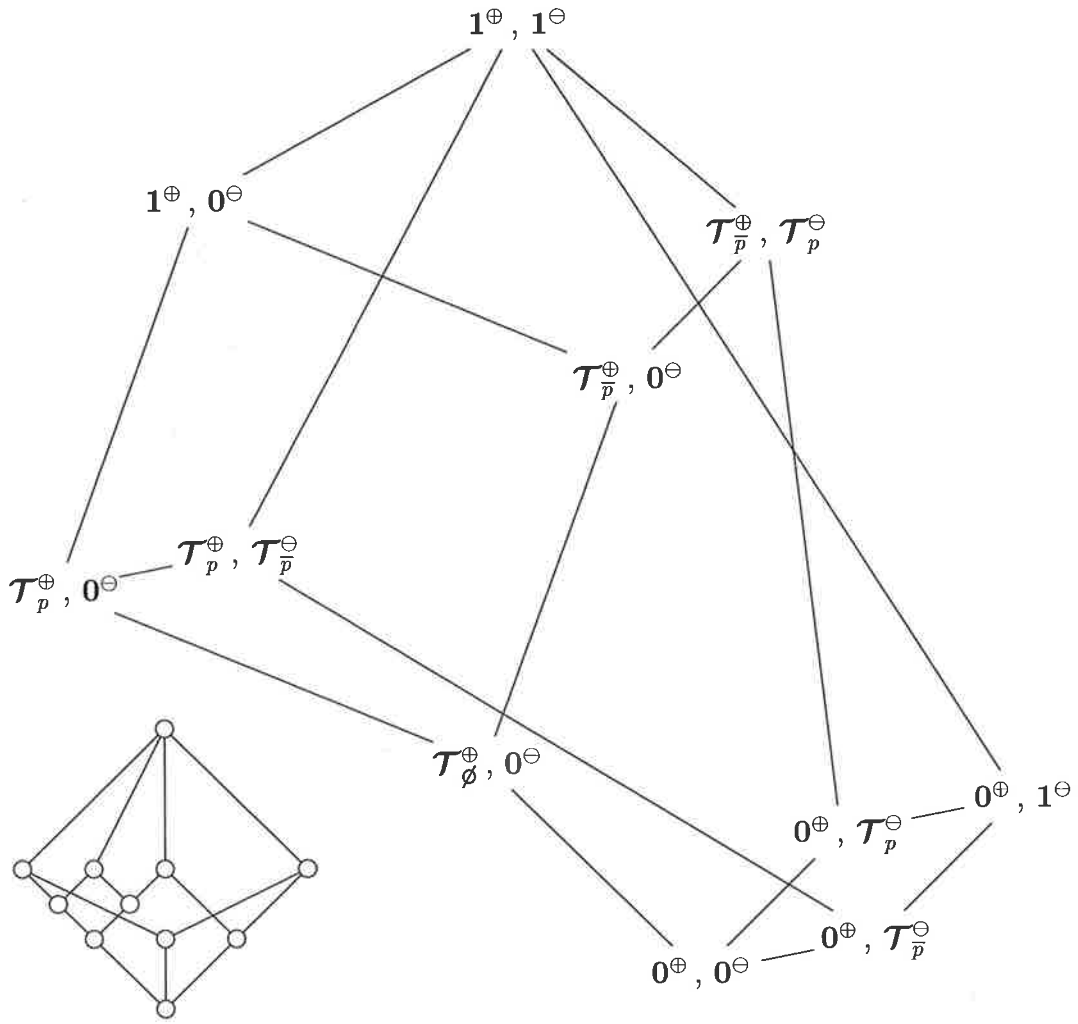


Figure 29. Consistent theories

Proposition 4 *Let* $\mathbb{T}$ *be the set of all consistent theories equipped with the information ordering relation* $\leq$*. Extend the ordered set* $\mathbb{T}$ *by adding an additional element—denoted* $\mathbf{1}$*—and extending the ordering relation by requesting that for all* $\boldsymbol{T} \in \mathbb{T}$ *it holds that* $\boldsymbol{T} \leq \mathbf{1}$*. Then* $(\mathbb{T} \cup \{\mathbf{1}\}, \leq)$ *is a* lattice.

In Figure 25 we depict the information ordering on (abstract) *contexts*—refer back to Figure 17—but showing *theories*, rather than contexts. The theories are built-up from the $\oplus$-theories and $\ominus$-theories of Figure 26. The shown theories are—in the sense made precise in Chapter 6—*theories of the contexts.*

In Figure 26 we consider a simple case of $P = \{p_1\}$, and hence $M = \{p_1, \overline{p_1}\}$. The left hand part of the figure shows the ordered set (lattice) of all possible theories that consist of $\oplus$-provable theorems only. The right hand part in turn shows the lattice of all possible theories that consist of $\ominus$-provable theorems only. Clearly, any theory over M will be built-up from theories of Figure 26. Although no doubt the case of $|P| = 1$ is very simple, it still allows us to make some points, and this is what we do in considering the other figures.

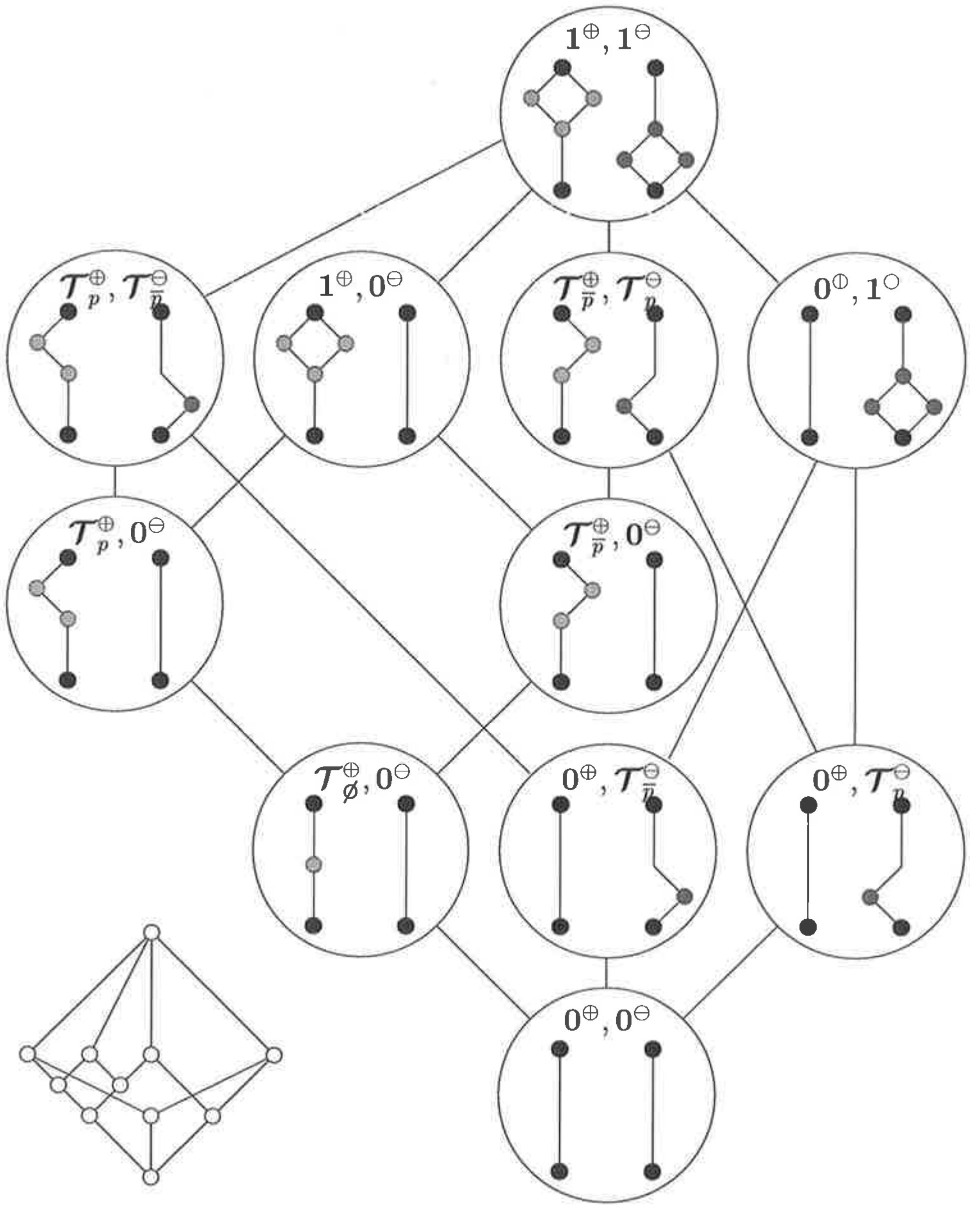


Figure 30. Information ordering on consistent theories

As one can suspect, the theories of Figure 25 are possibly *not* all the consistent theories we could consider. We start with sets of descriptions—as presented in Figure 27—that result from considering all possible combinations of theories of Figure 26 (we get this by taking a product of the lattices of theories of Figure 26).

After taking a closer look at Figure 27 it is clear we have there some inconsistent description sets (apart from our top element **1**)—hence, it seems appropriate to get rid of them. The resulting set of consistent description sets is presented in Figure 28

This is not good enough yet, because Figure 28 contains description sets that are not theories. Some of them, when closed under the inference rules Φ, disappear or rather merge with some other description sets which *are* (consistent) theories. This leads us to Figure 29.

Indeed, Figure 29 presents $\mathbb{T} \cup \{\mathbf{1}\}$. This ordered set of consistent theories forms a lattice, and the smaller inset diagram shows an equivalent but more readable version of the line diagram of the lattice. Links correspond to information ordering, and in Figure 30 we show this ordered set of consistent theories with more detail.

It is instructive to compare Figure 30 with Figure 25. Both show consistent theories, but the former shows all of them, while the latter only those "corresponding to" contexts. This will be further elaborated. For now, notice that—although the orderings on theories and contexts will be shown in Section 6.5 to agree with each other—there is no *1–1 onto* correspondence between theories and contexts. Indeed, by comparing Figures 30 and 25 and 17 we can see that there can be different contexts with the same theory, and that there are theories with no corresponding contexts. For instance, one can say something about a world without committing oneself to the nonemptiness of the world, but this is not possible if the information about the world is to be presented in its semantic form, as a context, or set of abstract objects.

In Chapter 6 we relate models to theories. This includes relating the information orderings, and also addresses the just mentioned point that for some consistent theories there are no contexts.

In Chapter 7 we consider subsets of the set of all consistent theories, namely theories *believed* by agents. Given a set of believed theories, we can form a corresponding lattice of theories—usually a tiny sub-lattice of the lattice of all consistent theories—by taking a closure on believed theories, where the closure operation just adds to the set of believed theories all their joins and meets.

Chapter 6

Models and theories

This chapter relates abstract contexts and theories to one another. In Section 6.1 we consider contexts and theories again, including *total* contexts and theories. Cardinalities of the sets of contexts and theories are considered. Section 6.2 defines mappings between contexts and theories. In Section 6.3 *models* are revisited—given a theory, a set of total contexts in which the theory is valid can be seen as a set of standard models for the theory. The section however also discusses *partial models*—if a theory is a theory of a context then that context (possibly partial) is an obvious candidate for a model of the theory. A theory however does not need to be a theory of a context, i.e., there might be no context with its set of valid sentences identical to the theory—a set of minimal contexts in which the theory is valid is then taken to be a model for the theory. Our choice of language is considered again in Section 6.4—in particular, some alternative propositional languages are demonstrated to be inappropriate. Given that the sets of contexts and theories are equipped with their respective information ordering, Section 6.5 considers how the orderings are related. Finally, Section 6.6 gives the soundness and completeness result.

6.1 Contexts and theories revisited

In this section we recall some facts about sets of contexts and theories, and consider cardinalities of the sets. We also provide examples of such sets, together with their cardinalities.

Let $M = P \cup \overline{P}$ be a fixed set of attributes. Further, let $\boldsymbol{F}$ and $\boldsymbol{G}$ be formulae and abstract objects (over M), respectively. There is a *1–1 onto* mapping between formulae and abstract objects—indeed, $\boldsymbol{G} = \{\boldsymbol{g}_F \mid F \in \boldsymbol{F}\}$. Let $\mathbb{G} = \mathcal{P}(\boldsymbol{G})$. Clearly, if $\boldsymbol{G}_i \subseteq \boldsymbol{G}$, i.e., $\boldsymbol{G}_i \in \mathbb{G}$, then $\boldsymbol{G}_i$ determines an abstract context $\boldsymbol{K}_i = (\boldsymbol{G}_i, M, I_i)$. It suffices to require that $I_i(\boldsymbol{g}_F, m)$ takes the value of $2, 1$, and 0, if $m \in F$, $\{m, \overline{m}\} \cap F = \emptyset$ and $\overline{m} \in F$, respectively. Let $\mathbb{K}$ be a set of all abstract contexts (over M). Then $\mathbb{K} = \{\boldsymbol{K}_i = (\boldsymbol{G}_i, M, I_i) \mid \boldsymbol{G}_i \in \mathbb{G}\}$. Clearly, there is a *1–1 onto* mapping between $\mathbb{G}$ and $\mathbb{K}$.

Regarding the size of $\mathbb{K}$, we have the following. Let $|P| = \alpha$. Then $|\boldsymbol{F}| = 3^\alpha$, for formulae can be associated with mappings from P to a three-element set, say $\{+, \bullet, -\}$. Of course, the same can be said about abstract objects $\boldsymbol{G}$, and hence we have that $|\boldsymbol{G}| = |\boldsymbol{F}| = 3^\alpha$. Given that $\mathbb{G} = \mathcal{P}(\boldsymbol{G})$, we have that

$|\mathbb{G}| = 2^{3^\alpha}$. As contexts are determined by sets of abstract objects, we have that $|\mathbb{K}| = |\mathbb{G}| = 2^{3^\alpha}$.

Recall that an abstract object can either be (properly) partial, or total (more precisely, M-partial, or M-total). An object $\boldsymbol{g}_F \in \boldsymbol{G}$ is total if for every $p \in P$ it has exactly one of the attributes p and $\overline{p}$, and hence $|F| = |P| = \alpha$, as F—in $\boldsymbol{g}_F$—is the set of attributes of $\boldsymbol{g}_F$. Let $\boldsymbol{F}^{(\alpha)}$ be a set of formulae of length α, i.e., $\boldsymbol{F}^{(\alpha)} = \{F \in \boldsymbol{F} \mid |F| = \alpha\}$. Then, the set of total abstract objects—denoted $\boldsymbol{G}^{(\alpha)}$—is the set of abstract objects corresponding to the formulae of $\boldsymbol{F}^{(\alpha)}$, i.e., $\boldsymbol{G}^{(\alpha)} = \{\boldsymbol{g}_F \mid F \in \boldsymbol{F}^{(\alpha)}\}$. Let $\mathbb{G}^{(\alpha)} = \mathcal{P}(\boldsymbol{G}^{(\alpha)})$. Clearly, if $\boldsymbol{G}_i^{(\alpha)} \subseteq \boldsymbol{G}^{(\alpha)}$, i.e., $\boldsymbol{G}_i^{(\alpha)} \in \mathbb{G}^{(\alpha)}$, then $\boldsymbol{G}_i^{(\alpha)}$ determines a *total* abstract context $\boldsymbol{K}_i^{(\alpha)} = (\boldsymbol{G}_i^{(\alpha)}, M, I_i)$—this time the incidence relation takes values in the set $\{2, 0\}$, as every object of the context is total, and hence it is determined w.r.t. all the attributes. Let $\mathbb{K}^{(\alpha)}$ be a set of all *total* abstract contexts (over M). Then $\mathbb{K}^{(\alpha)} = \{\boldsymbol{K}_i^{(\alpha)} = (\boldsymbol{G}_i^{(\alpha)}, M, I_i) \mid \boldsymbol{G}_i^{(\alpha)} \in \mathbb{G}^{(\alpha)}\}$. Clearly, there is a *1–1 onto* mapping between $\mathbb{G}^{(\alpha)}$ and $\mathbb{K}^{(\alpha)}$.

Regarding the size of $\mathbb{K}^{(\alpha)}$, we have the following. First note that $|\boldsymbol{F}^{(\alpha)}| = 2^\alpha$, for formulae in $\boldsymbol{F}^{(\alpha)}$ can be associated with mappings from P to a two-element set $\{+, -\}$. Of course, the same can be said about total abstract objects $\boldsymbol{G}^{(\alpha)}$, and hence we have that $|\boldsymbol{G}^{(\alpha)}| = |\boldsymbol{F}^{(\alpha)}| = 2^\alpha$. Given that $\mathbb{G}^{(\alpha)} = \mathcal{P}(\boldsymbol{G}^{(\alpha)})$, we have that $|\mathbb{G}^{(\alpha)}| = 2^{2^\alpha}$. As total contexts are determined by sets of total abstract objects, we have that $|\mathbb{K}^{(\alpha)}| = |\mathbb{G}^{(\alpha)}| = 2^{2^\alpha}$. Abstract contexts and total abstract contexts are associated with *models*—this is discussed in Section 6.3.

Consider now theories. Let $\mathbb{T}$ be a set of all consistent theories (over the fixed M). Recall that if $\boldsymbol{T} \in \mathbb{T}$ then $\boldsymbol{T}$ is a (consistent) description set, and it is closed under the inference rules Φ, i.e., $\boldsymbol{T} = \mathrm{Cn}(\boldsymbol{T})$. Further, given $\boldsymbol{T} \in \mathbb{T}$ there is a *minimal unique axiom set* $\boldsymbol{A}$ of $\boldsymbol{T}$, referred to as a *generator* of $\boldsymbol{T}$, i.e., $\boldsymbol{A} = \mathrm{gen}(\boldsymbol{T})$. Given $\boldsymbol{A}$, there is a corresponding pair of sets of provable formulae, namely $(\boldsymbol{A}^{\oplus}, \boldsymbol{A}^{\ominus})$. Recall that if $\boldsymbol{A}^{\oplus}$ and $\boldsymbol{A}^{\ominus}$ are seen as sets ordered by the $\oplus$-entailment $\geq_{\oplus}$ and $\ominus$-entailment $\geq_{\ominus}$ relations respectively, then they are anti-chains. This means that consistent theories can be associated with pairs of such anti-chains. Hence, let us first consider two specific subsets of $\mathbb{T}$. Let $\mathbb{T}^{\oplus}$ be a set of consistent theories with an empty set of $\ominus$-provable formulae, and similarly let $\mathbb{T}^{\ominus}$ be a set of theories with an empty set of $\oplus$-provable formulae. Certainly, $|\mathbb{T}^{\oplus}| \leq |\mathcal{P}(\boldsymbol{F})|$ and $|\mathbb{T}^{\ominus}| \leq |\mathcal{P}(\boldsymbol{F})|$—this is a step towards an upper limit on number of consistent theories. It should be noted that cardinalities of $\mathbb{T}^{\oplus}$ and $\mathbb{T}^{\ominus}$ are usually much smaller than $|\mathcal{P}(\boldsymbol{F})|$. Indeed, we have that $|\mathbb{T}^{\oplus}| = |\{\boldsymbol{T}_i \mid \boldsymbol{T}_i \in \mathbb{T}^{\oplus}\}| = |\{(\boldsymbol{A}_i^{\oplus}, \emptyset)\}_i|$, and hence there are as many theories in $\mathbb{T}^{\oplus}$ as there are anti-chains in $\boldsymbol{F}$ ordered by $\oplus$-entailment $\geq_{\oplus}$, and similarly for $\mathbb{T}^{\ominus}$. The precise assessment of the cardinality of $\mathbb{T}$ seems complicated, so let us contend ourselves with an upper limit. As $|\boldsymbol{F}| = 3^\alpha$ and $|\mathcal{P}(\boldsymbol{F})| = 2^{3^\alpha}$ we have that $|\mathbb{T}^{\oplus}| \leq 2^{3^\alpha}$ and $|\mathbb{T}^{\ominus}| \leq 2^{3^\alpha}$. A number of all pairs in $\mathbb{T}^{\oplus} \times \mathbb{T}^{\ominus}$ is $|\mathbb{T}^{\oplus}| \cdot |\mathbb{T}^{\ominus}|$. Now, for any $\boldsymbol{T} \in \mathbb{T}$ there is a corresponding pair in $\mathbb{T}^{\oplus} \times \mathbb{T}^{\ominus}$, although many pairs do *not* correspond to consistent theories. Hence, we clearly have that $|\mathbb{T}| \leq |\mathbb{T}^{\oplus}| \cdot |\mathbb{T}^{\ominus}| \leq 2^{3^\alpha} \cdot 2^{3^\alpha} = 2^{2 \cdot 3^\alpha}$. Although it is true that $|\mathbb{T}| \leq 2^{2 \cdot 3^\alpha}$, one should remember that $2^{2 \cdot 3^\alpha}$ is usually much bigger than $|\mathbb{T}|$. Firstly, while searching for theories in $\mathbb{T}^{\oplus}$ and $\mathbb{T}^{\ominus}$ one should consider anti-chains rather than arbitrary subsets of $\boldsymbol{F}$, and secondly, while searching for consistent theories many pairs in $\mathbb{T}^{\oplus} \times \mathbb{T}^{\ominus}$ should be dismissed, as many of them correspond to inconsistent theories, and many of the remaining ones to description sets which are not closed under the inference rules Φ.

It is easy to derive *a* lower limit on $|\mathbb{T}|$. Let $\mathbb{T}^{(\alpha)}$ be a set of *total* consistent theories—a theory $\boldsymbol{T} \in \mathbb{T}$ is total, if for every $F \in \boldsymbol{F}^{(\alpha)}$ we have that either $\oplus F \in \boldsymbol{T}$ or $\ominus F \in \boldsymbol{T}$.[1] Total consistent theories can be associated—in a *1–1 onto* fashion—with mappings from $\boldsymbol{F}^{(\alpha)}$ to a two-element set, say $\{\oplus, \ominus\}$.[2] With $|\boldsymbol{F}^{(\alpha)}| = 2^{\alpha}$, there are $2^{2^{\alpha}}$ such mappings, and therefore $|\mathbb{T}^{(\alpha)}| = 2^{2^{\alpha}}$. Obviously, $|\mathbb{T}| \geq 2^{2^{\alpha}}$, because $\mathbb{T} \supseteq \mathbb{T}^{(\alpha)}$.

Summarising, we have derived the following results on cardinalities of sets of contexts and theories. $|\mathbb{K}| = 2^{3^{\alpha}}$, $|\mathbb{K}^{(\alpha)}| = 2^{2^{\alpha}}$, $2^{2^{\alpha}} \leq |\mathbb{T}| \leq 2^{2 \cdot 3^{\alpha}}$ and $|\mathbb{T}^{(\alpha)}| = 2^{2^{\alpha}}$.

The fact that $|\mathbb{K}^{(\alpha)}| = |\mathbb{T}^{(\alpha)}|$ is not surprising—it is easy to find mappings $\boldsymbol{\tau}_{(\alpha)} : \mathbb{K}^{(\alpha)} \longrightarrow \mathbb{T}^{(\alpha)}$ and $\boldsymbol{\kappa}_{(\alpha)} : \mathbb{T}^{(\alpha)} \longrightarrow \mathbb{K}^{(\alpha)}$, such that $\boldsymbol{\tau}_{(\alpha)}$ and $\boldsymbol{\kappa}_{(\alpha)}$ are *1–1 onto* and $\boldsymbol{\kappa}_{(\alpha)} = \boldsymbol{\tau}_{(\alpha)}^{-1}$. Let $\boldsymbol{K}_i^{(\alpha)} = (\boldsymbol{G}_i^{(\alpha)}, M, I_i) \in \mathbb{K}^{(\alpha)}$—note that elements of $\boldsymbol{G}_i^{(\alpha)}$ are *total* abstract objects. Then $\boldsymbol{A}_i^{\oplus} = \{F \in \boldsymbol{F}^{(\alpha)} \mid \boldsymbol{g}_F \in \boldsymbol{G}_i^{(\alpha)}\}$, and all $\oplus$-provable formulae are $\boldsymbol{T}_i^{\oplus} = \mathrm{Cn}(\boldsymbol{A}_i^{\oplus})$, with all the remaining formulae being $\ominus$-provable, i.e., $\boldsymbol{T}_i^{\ominus} = \boldsymbol{F} \setminus \boldsymbol{T}_i^{\oplus}$. The pair $(\boldsymbol{T}_i^{\oplus}, \boldsymbol{T}_i^{\ominus})$ determines a theory $\boldsymbol{T}_i^{(\alpha)} = \boldsymbol{\tau}_{(\alpha)}(\boldsymbol{K}_i^{(\alpha)}) \in \mathbb{T}^{(\alpha)}$. Going in the opposite direction, if $\boldsymbol{T}_i^{(\alpha)} \in \mathbb{T}^{(\alpha)}$ then $\boldsymbol{A}_i^{\oplus}$ allows us to derive all abstract objects $\boldsymbol{G}_i^{(\alpha)}$ of the corresponding context $\boldsymbol{K}_i^{(\alpha)}$—indeed, $\boldsymbol{G}_i^{(\alpha)} = \{\boldsymbol{g}_F \in \boldsymbol{G} \mid F \in \boldsymbol{A}_i^{\oplus}\}$. Clearly, the objects of $\boldsymbol{G}_i^{(\alpha)}$ are total, and they determine a total abstract context $\boldsymbol{K}_i^{(\alpha)} = \boldsymbol{\kappa}_{(\alpha)}(\boldsymbol{T}_i^{(\alpha)}) \in \mathbb{K}^{(\alpha)}$. Mappings between the sets $\mathbb{K}$ and $\mathbb{T}$ of all—rather than only total—abstract contexts and consistent theories are considered in Section 6.2.

Consider the simple case of $\alpha = 1$. Given our results on cardinalities of sets of contexts and theories, we have $|\mathbb{K}| = 2^{3^{\alpha}} = 8$, $|\mathbb{K}^{(\alpha)}| = 2^{2^{\alpha}} = 4$, $2^{2^{\alpha}} = 4 \leq |\mathbb{T}| \leq 2^{2 \cdot 3^{\alpha}} = 64$ and $|\mathbb{T}^{(\alpha)}| = 2^{2^{\alpha}} = 4$. Referring back to Figure 17 showing abstract contexts for $\alpha = 1$, there are indeed 8 abstract contexts, and 4 total abstract contexts. Referring back to Figure 30 showing—for $\alpha = 1$—all consistent theories (plus an additional element $(\mathbf{1}^{\oplus}, \mathbf{1}^{\ominus})$), there are 10 consistent theories, and 4 total consistent theories. Regarding the number of consistent theories, we can see from Figure 26 that $|\mathbb{T}^{\oplus}| = 5$ (rather than $2^{3^{\alpha}} = 8$) and $|\mathbb{T}^{\ominus}| = 4$ (rather than 8).[3] Further, description sets corresponding to pairs in $\mathbb{T}^{\oplus} \times \mathbb{T}^{\ominus}$—there are $5 \cdot 4 = 20$ such pairs—are shown in Figure 27. Some of them are inconsistent, and those which are consistent—there are 12 of them—are shown in Figure 28. Those of the consistent description sets which actually *are* theories, i.e., they are closed under the inference rules Φ—there are 10 of them—are shown in Figure 29.

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[1] If a theory is total then also every formula in $\boldsymbol{F}$ is either $\oplus$-provable or $\ominus$-provable.

[2] It is easy to see that *any* such mapping determines a *consistent* theory—indeed, regions (of objects) corresponding to formulae in $\boldsymbol{F}^{(\alpha)}$ form a partition of the set of all objects, and hence the regions are "atomic," or "independent," in the sense that a claim about nonemptiness of a specific region can be made independently on claims made about all the remaining regions. We return to this point in Section 6.4.

[3] One can also note that $\mathbb{T}^{\oplus}$ and $\mathbb{T}^{\ominus}$ overlap on the empty theory, and thus there are 8 theories in $\mathbb{T}^{\oplus} \cup \mathbb{T}^{\ominus}$. For the case of $\alpha = 1$ there are only two consistent theories outside of $\mathbb{T}^{\oplus} \cup \mathbb{T}^{\ominus}$.

The results on cardinalities of sets of contexts and theories seem to be discouraging. Regarding theories, given that $2^{2^\alpha} \leq |\mathbb{T}| \leq 2^{2 \cdot 3^\alpha}$ and $|\mathbb{T}^{(\alpha)}| = 2^{2^\alpha}$, it is evident that a number of theories one might consider is *large*. However, as discussed in Section 7.1, we focus on theories that are actually *believed* by some agents. (There are many possible ways the world could be, there is only one way the world really is, and there are at most as many ways the world is *believed* to be as there are believing agents we consider—believed theories are descriptions of such believed worlds of agents.)

Regarding contexts, it seems appropriate—given a theory $\boldsymbol{T}$—to associate with the theory a set of those total contexts in which $\boldsymbol{T}$ is valid. Such total contexts could be employed as "models" of $\boldsymbol{T}$. Indeed, if $\boldsymbol{T}$ describes a world, then the world must correspond to exactly one of the total contexts in which $\boldsymbol{T}$ is valid. There are three relevant points to be made here. Firstly, the number of total contexts is 2^{2^α}, and hence it might be impossible in practice to inspect them to find those total contexts in which $\boldsymbol{T}$ is valid. Secondly, the theory $\boldsymbol{T}$ we obtain is usually partial, and hence it only partially describes the world. Hence, it seems appropriate to be satisfied with *partial* contexts corresponding to $\boldsymbol{T}$—indeed, if such partial contexts are found then the set of total contexts in which $\boldsymbol{T}$ is valid is simply the set of those total contexts that are above—w.r.t. the information ordering $\leq$ on contexts—the partial contexts. Thirdly, given a theory $\boldsymbol{T}$, there can be more than one partial context corresponding to $\boldsymbol{T}$. A more precise account of these issues is given in Section 6.3. What can be said now is that given some theories we can associate with them *partial*—rather than *total*—contexts, and hence there is no need to consider 2^{2^α} total contexts.

The above results shed some light on how many contexts and theories there are, and this is relevant if one wants to derive *all* contexts or theories over a given set of attributes. Indeed, one might want to do this, but only to gain some intuition about the mathematical structures involved. Normally, one would only be interested in contexts and theories resulting from the provided description sets—cf. Chapter 7—and therefore limit oneself to a small number of interesting contexts and theories.

6.2 Mappings between contexts and theories

Let M be a fixed set of attributes, and let $\mathbb{K}$ and $\mathbb{T}$ be the set of all contexts and the set of all theories (over M), respectively. In this section we define mappings between $\mathbb{K}$ and $\mathbb{T}$.

Let $\boldsymbol{K}_i \in \mathbb{K}$ be an abstract context. There is an obvious choice for a theory associated with $\boldsymbol{K}_i$—the set $\boldsymbol{T}_{\boldsymbol{K}_i}$ of all descriptions that are valid in $\boldsymbol{K}_i$ forms a consistent theory, i.e., $\boldsymbol{T}_{\boldsymbol{K}_i} = \{D \in \boldsymbol{D} \mid \boldsymbol{K}_i \models D\} \in \mathbb{T}$. (Indeed, to see that $\boldsymbol{T}_{\boldsymbol{K}_i}$ is a consistent theory, it is sufficient to note that the set of descriptions valid in $\boldsymbol{K}_i$ is consistent and closed under Φ—it is consistent by the definition of validity, and it is closed under Φ because semantic equivalents of the inference rules hold, cf. Section 6.6.)

Let $\boldsymbol{T}_i \in \mathbb{T}$. If there is a $\boldsymbol{K}_i \in \mathbb{K}$ such that $\boldsymbol{T}_{\boldsymbol{K}_i} = \boldsymbol{T}_i$ then $\boldsymbol{K}_i$ is the best candidate for a context associated with $\boldsymbol{T}_i$. Unfortunately, it can easily happen that there is no such context $\boldsymbol{K}_i$. Given $\boldsymbol{T}_i$, one can in principle find a set of total theories which are above $\boldsymbol{T}_i$, i.e., $\mathbb{T}_i^{(\alpha)} = \{\boldsymbol{T} \in \mathbb{T}^{(\alpha)} \mid \boldsymbol{T} \geq \boldsymbol{T}_i\}$. There is a set $\mathbb{K}_i^{(\alpha)}$ of total contexts corresponding to the total theories of $\mathbb{T}_i^{(\alpha)}$, namely $\mathbb{K}_i^{(\alpha)} = \{\boldsymbol{K} \in \mathbb{K}^{(\alpha)} \mid \boldsymbol{T_K} \in \mathbb{T}_i^{(\alpha)}\}$. (One would normally treat $\mathbb{K}_i^{(\alpha)}$ as a set of (total) models for $\boldsymbol{T}_i$. We consider this issue in Section 6.3.)

There is however a set—denoted by $\mathcal{K}_{\boldsymbol{T}_i}$—of *partial* contexts associated with $\boldsymbol{T}_i$. The set $\mathcal{K}_{\boldsymbol{T}_i}$ is usually much smaller than $\mathbb{K}_i^{(\alpha)}$—the idea is to pick-up those contexts in which $\boldsymbol{T}_i$ is valid, but which are also minimal w.r.t. the information ordering $\leq$ on contexts.

Let $\mathbb{D}$ be now a set of all consistent *description sets*, i.e., $\mathbb{D} = \{\boldsymbol{D}_i \subseteq \boldsymbol{D} \mid \mathrm{Cn}(\boldsymbol{D}_i) \in \mathbb{T}\}$. It is important to be able to determine a $\boldsymbol{\kappa}$-model for any description set. Fortunately, it is obvious how to extend $\boldsymbol{\kappa}$ to the domain $\mathbb{D} \supseteq \mathbb{T}$. Given a $\boldsymbol{D}_i \in \mathbb{D}$, we can simply pre-process $\boldsymbol{D}_i$ to get the corresponding theory $\boldsymbol{T}_i = \mathrm{Cn}(\boldsymbol{D}_i)$, and then apply $\boldsymbol{\kappa}$ to $\boldsymbol{T}_i$

The above considerations lead to the following definition.

Definition 18 *Let M be a set of attributes. Let $\mathbb{K}$ and $\mathbb{T}$ be the set of all contexts and the set of all consistent theories (over M), respectively. Define mappings $\boldsymbol{\tau}: \mathbb{K} \longrightarrow \mathbb{T}$ and $\boldsymbol{\kappa}: \mathbb{T} \longrightarrow \mathcal{P}(\mathbb{K})$ as follows.*

If $\boldsymbol{K}_i \in \mathbb{K}$ then $\boldsymbol{\tau}(\boldsymbol{K}_i) = \boldsymbol{T}_{\boldsymbol{K}_i} = \{D \in \boldsymbol{D} \mid \boldsymbol{K}_i \models D\}$.

If $\boldsymbol{T}_i \in \mathbb{T}$ then $\boldsymbol{\kappa}(\boldsymbol{T}_i) = \mathcal{K}_{\boldsymbol{T}_i} = \{\boldsymbol{K} \in \mathbb{K} \mid \boldsymbol{T_K} \geq \boldsymbol{T}_i$ and $\boldsymbol{K}$ is $\leq$-minimal$\}$.
Let $\mathbb{D}$ be the set of all consistent description sets (over M). A mapping $\boldsymbol{\kappa}: \mathbb{D} \longrightarrow \mathcal{P}(\mathbb{K})$ extends the mapping $\boldsymbol{\kappa}$ defined on $\mathbb{T}$.

If $\boldsymbol{D}_i \in \mathbb{D}$ then $\boldsymbol{\kappa}(\boldsymbol{D}_i) = \{\boldsymbol{K} \in \mathbb{K} \mid \boldsymbol{T_K} \geq \mathrm{Cn}(\boldsymbol{D}_i)$ and $\boldsymbol{K}$ is $\leq$-minimal$\}$.
i.e., $\boldsymbol{\kappa}(\boldsymbol{D}_i) = \boldsymbol{\kappa}(\mathrm{Cn}(\boldsymbol{D}_i))$. We denote the set $\boldsymbol{\kappa}(\boldsymbol{D}_i)$ by $\mathcal{K}_{\boldsymbol{D}_i}$.

Consider the case of $|P| = \alpha = 1$. Figure 31 presents contexts $\mathbb{K}$ and theories $\mathbb{T}$ (as already shown in details in Figures 17 and 30, respectively), and the mappings $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}$ between the contexts and theories.

Regarding the mapping $\boldsymbol{\tau}$, it is easy to see that $\boldsymbol{\tau}$ is not *onto*. Using the example of Figure 31, there are no contexts with a theory $(\mathbf{0}^{\oplus}, \boldsymbol{T}_p^{\ominus})$, or $(\mathbf{0}^{\oplus}, \boldsymbol{T}_{\overline{p}}^{\ominus})$, or $(\mathbf{0}^{\oplus}, \mathbf{0}^{\ominus})$.

Regarding the mapping $\boldsymbol{\kappa}$, recall that it maps theories to *sets* of contexts. If a theory is a *theory of a context*, then the theory is mapped to a singleton set of that context. However, if a theory is not a theory of a context then it is mapped to a set of minimal contexts in which the theory is valid. This is what happens in the bottom row of Figure 31. Some theories are unproblematically mapped into singleton sets of contexts—these cases are omitted in Figure 31. Some other theories are mapped to (non-singleton) sets of contexts, e.g., as shown in Figure 31, the empty theory $(\mathbf{0}^{\oplus}, \mathbf{0}^{\ominus})$ is mapped to $\{\boldsymbol{K}_1, \boldsymbol{K}_{\emptyset}\}$, and the theory $(\mathbf{0}^{\oplus}, \boldsymbol{T}_{\overline{p}}^{\ominus})$ to $\{\boldsymbol{K}_2, \boldsymbol{K}_{\emptyset}\}$, and the theory $(\mathbf{0}^{\oplus}, \boldsymbol{T}_p^{\ominus})$ to $\{\boldsymbol{K}_0, \boldsymbol{K}_{\emptyset}\}$.

Given that $\mathbb{K}$ and $\mathbb{T}$ are ordered sets, it is natural to ask whether the mappings preserve the orderings. We come back to this question in Section 6.5, after clarifying the issues of models and language.

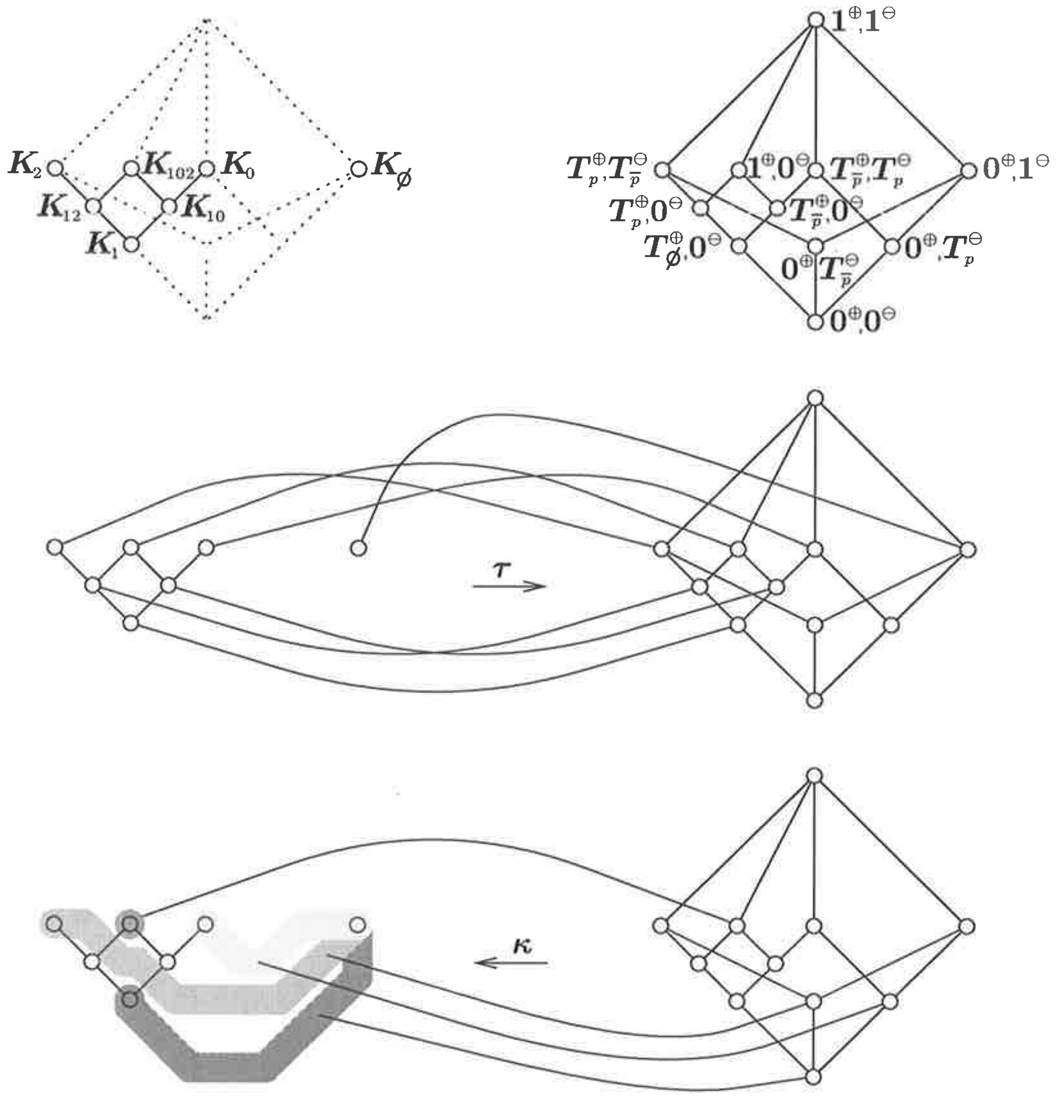


Figure 31. $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}$ mappings

6.3 Models revisited

Let $\boldsymbol{T}_i \in \mathbb{T}$ be a consistent theory. How should *models* for the theory be defined? Let $\mathbb{K}^{(\alpha)}$ and $\mathbb{T}^{(\alpha)}$ be a set of total contexts and total theories, respectively. If we decided to select *total* contexts in which $\boldsymbol{T}_i$ is valid as models, then we would have $\text{MOD}(\boldsymbol{T}_i) = \mathbb{K}_i^{(\alpha)} = \{\boldsymbol{K} \in \mathbb{K}^{(\alpha)} \mid \boldsymbol{T}_{\boldsymbol{K}} \in \mathbb{T}_i^{(\alpha)}\}$, where $\mathbb{T}_i^{(\alpha)} = \{\boldsymbol{T} \in \mathbb{T}^{(\alpha)} \mid \boldsymbol{T} \geq \boldsymbol{T}_i\}$, as already mentioned in Section 6.2.

Indeed, contexts in $\mathbb{K}_i^{(\alpha)}$ are total, and hence they can be identified with total, or completely—w.r.t. all the attributes in M—determined worlds. The world actually described by $\boldsymbol{T}_i$ must correspond to exactly one of the contexts of $\mathbb{K}_i^{(\alpha)}$. Hence, $\text{MOD}(\boldsymbol{T}_i) = \mathbb{K}_i^{(\alpha)}$ seems an appropriate choice for models of $\boldsymbol{T}_i$.

However, finding all total contexts in which $\boldsymbol{T}_i$ is valid might be expensive. An attractive alternative is to consider only *minimal partial models* of the world described by $\boldsymbol{T}_i$. It is $\mathcal{K}_{\boldsymbol{T}_i} = \boldsymbol{\kappa}(\boldsymbol{T}_i)$ which gives the set of such partial models of $\boldsymbol{T}_i$. Let us use the notation $\text{mod}(\boldsymbol{T}_i)$ for the set of partial models (contexts) of

$\boldsymbol{T}_i$, i.e., $\text{mod}(\boldsymbol{T}_i) = \boldsymbol{\mathcal{K}}_{\boldsymbol{T}_i} = \boldsymbol{\kappa}(\boldsymbol{T}_i)$, with $\boldsymbol{\kappa}$ given by Definition 18. We certainly have that if $\boldsymbol{K} \in \text{mod}(\boldsymbol{T}_i)$ then $\boldsymbol{T}_{\boldsymbol{K}} \geq \boldsymbol{T}_i$, and also that $\boldsymbol{T}_i = \bigcap_{\boldsymbol{K} \in \text{mod}(\boldsymbol{T}_i)} \boldsymbol{T}_{\boldsymbol{K}}$. We refer to the set $\text{mod}(\boldsymbol{T}_i) = \boldsymbol{\mathcal{K}}_{\boldsymbol{T}_i} = \boldsymbol{\kappa}(\boldsymbol{T}_i)$ as a $\boldsymbol{\kappa}$-model of $\boldsymbol{T}_i$.

Hence the following definition.

Definition 19 *Let $\boldsymbol{T}_i \in \mathbb{T}$ be a theory.*

The set of total models *for $\boldsymbol{T}_i$—denoted by* $\text{MOD}(\boldsymbol{T}_i)$*—is given by:*

$$\text{MOD}(\boldsymbol{T}_i) = \mathbb{K}_i^{(\alpha)} = \{\boldsymbol{K} \in \mathbb{K}^{(\alpha)} \mid \boldsymbol{T}_{\boldsymbol{K}} \in \mathbb{T}_i^{(\alpha)}\},$$

where $\mathbb{T}_i^{(\alpha)} = \{\boldsymbol{T} \in \mathbb{T}^{(\alpha)} \mid \boldsymbol{T} \geq \boldsymbol{T}_i\}$.

The set of minimal partial models *for $\boldsymbol{T}_i$—denoted by* $\text{mod}(\boldsymbol{T}_i)$, *or* $\boldsymbol{\mathcal{K}}_{\boldsymbol{T}_i}$*—is given by:*

$$\text{mod}(\boldsymbol{T}_i) = \boldsymbol{\kappa}(\boldsymbol{T}_i) = \{\boldsymbol{K} \in \mathbb{K} \mid \boldsymbol{T}_{\boldsymbol{K}} \geq \boldsymbol{T}_i \text{ and } \boldsymbol{K} \text{ is } \leq\text{-minimal}\}.$$

The set $\text{mod}(\boldsymbol{T}_i)$ *is referred to as a $\boldsymbol{\kappa}$-model of $\boldsymbol{T}_i$.*

It is clear that $\text{mod}(\boldsymbol{T}_i)$ is all we need to capture the "meaning" of $\boldsymbol{T}_i$. Indeed, elements of $\text{mod}(\boldsymbol{T}_i)$ are contexts, and hence they determine the worlds the theory $\boldsymbol{T}_i$ might be describing. Moreover, given $\text{mod}(\boldsymbol{T}_i)$, i.e., the $\boldsymbol{\kappa}$-model of $\boldsymbol{T}_i$, the set of total models $\text{MOD}(\boldsymbol{T}_i)$ can be found, namely $\text{MOD}(\boldsymbol{T}_i) = \{\boldsymbol{K} \in \mathbb{K}^{(\alpha)} \mid \exists_{\boldsymbol{K}_j \in \text{mod}(\boldsymbol{T}_i)} \boldsymbol{K} \geq \boldsymbol{K}_j\}$, and using the notion of an *upset* of an ordered set, this takes the form of $\text{MOD}(\boldsymbol{T}_i) = \uparrow\text{mod}(\boldsymbol{T}_i) \cap \mathbb{K}^{(\alpha)}$.

Summarising, given a theory $\boldsymbol{T}_i \in \mathbb{T}$, it is sufficient to find $\text{mod}(\boldsymbol{T}_i)$ to determine the meaning of $\boldsymbol{T}_i$, or to determine worlds the theory $\boldsymbol{T}_i$ possibly describes.

Simple examples of total models and $\boldsymbol{\kappa}$-models for theories can be read from Figure 31. For instance, the empty theory $\mathbf{0}$, i.e., the theory given by $(\mathbf{0}^{\oplus}, \mathbf{0}^{\ominus})$, has the following models—$\text{MOD}(\mathbf{0}) = \{\boldsymbol{K}_2, \boldsymbol{K}_{02}, \boldsymbol{K}_0, \boldsymbol{K}_{\emptyset}\}$, and $\text{mod}(\mathbf{0}) = \{\boldsymbol{K}_1, \boldsymbol{K}_{\emptyset}\}$.

6.4 Language revisited

Recall from Section 3.5, that we employ a *language of formulae* $\boldsymbol{L} = (M, \boldsymbol{F})$, and a *language of descriptions* $\mathbb{L} = (M, \boldsymbol{D})$, where $\boldsymbol{D} = \boldsymbol{F} \times \{\oplus, \ominus\}$, or $\boldsymbol{D} = \{\oplus F, \ominus F \mid F \in \boldsymbol{F}\}$ is a set of *descriptions*—they can be seen as pairs *(formula, marker)*, or simply as marked formulae. If then $\boldsymbol{K}$ is a context, we classify formulae of $\boldsymbol{F}$ into those which are $\oplus$-valid, $\ominus$-valid, or have undetermined validity. In the language of descriptions, we select a set of $\oplus$-descriptions (formulae marked with $\oplus$) and a set of $\ominus$-descriptions, possibly leaving some descriptions unselected. Hence, it seems we could employ a standard, three-valued propositional logic—with propositional symbols denoting attributes—to get an equivalent language. Let us attempt to provide such a language, denoted by $\boldsymbol{\mathcal{L}}_p$.

Let $P = \{p_1, \ldots, p_\alpha\}$ be employed as a set of propositional symbols. To get formulae equivalent to those of $\boldsymbol{F}$, one can decide for $\boldsymbol{\mathcal{L}}_p = (\{p_1, \ldots, p_\alpha\}, \neg, \wedge)$—indeed, formulae of $\boldsymbol{F}$ can be seen as single disjuncts in a disjunctive normal form, i.e., as conjunctions of possibly negated propositional symbols $p_1, \ldots, p_\alpha$. Further, one might want to say that formulae are *true* or *false*, rather than

$\oplus$-valid, or $\ominus$-valid, and hence employ e.g., $\models_{\text{true}} p_1$, $\models_{\text{true}} p_1 \wedge p_2$, or $\models_{\text{false}} \neg p_1$, instead of $\models_\oplus \{p_1\}$, $\models_\oplus \{p_1, p_2\}$, or $\models_\ominus \{\overline{p_1}\}$, respectively. Although the inference rules $\Phi = \{\varphi_1, \ldots, \varphi_4\}$ look fine in that alternative notation, one might be tempted to e.g., infer $\models_{\text{true}} p_1 \wedge p_2$ from $\models_{\text{true}} p_1$ and $\models_{\text{true}} p_2$. But such an inference can easily go wrong. Hence, the alternative notation is counterintuitive. Indeed, if we say that F is *instantiated* whenever F is $\oplus$-valid, then our logic is aimed at dealing with *instantiatibility*[4]—we do not ask whether F is *true* in $\boldsymbol{K}$, but rather whether F is instantiated in $\boldsymbol{K}$, or $\oplus$-valid in it. (Recall that our definition of validity ensures that if a formula is $\oplus/\ominus$-valid in $\boldsymbol{K}$ then although it does not need to be valid also in every other context, it is valid in every context above $\boldsymbol{K}$.) Furthermore, given that the negation operator $\neg$ of $\boldsymbol{\mathcal{L}}_p$ should only apply to single propositional symbols $p_1, \ldots, p_\alpha$, our language $\mathbb{L}$ of descriptions is exactly what we need.

There is another language one might consider—let us denote it by $\boldsymbol{\mathcal{L}}_f$. Recall that $\boldsymbol{F}^{(\alpha)} = \{F \in \boldsymbol{F} \mid |F| = \alpha\}$, and elements of $\boldsymbol{F}^{(\alpha)}$ could be called *total formulae*, as they correspond to total objects. Given elements of $\boldsymbol{F}^{(\alpha)}$, the corresponding regions of the world are *atomic*, or *independent*, in the sense that any of them can be empty or not, independently of all the other atomic regions—then *any* region is a union of some of those atomic regions. Hence, given any region we could associate with it a "disjunction" of formulae in $\boldsymbol{F}^{(\alpha)}$. Employ a set of propositional symbols $f_1, \ldots, f_{2^\alpha}$—these correspond to $F_1, \ldots, F_{2^\alpha} \in \boldsymbol{F}^{(\alpha)}$. Thus, we get $\boldsymbol{\mathcal{L}}_f = (\{f_1, \ldots, f_{2^\alpha}\}, \vee)$. Using $\boldsymbol{\mathcal{L}}_f$, for any region of objects there is a corresponding formula that is a disjunction of the propositional symbols. Note however, that for a given α, instead of e.g., $F = \{p_1\}$ we would need to employ a disjunction of $2^{\alpha-1}$ propositional symbols of $\boldsymbol{\mathcal{L}}_f$, and extending the set of attributes would require the formulae to be recomputed. Given that more often than not the information we get is *partial*, our description language $\mathbb{L}$ seems to be a much better alternative than $\boldsymbol{\mathcal{L}}_f$.

Finally, our language is similar to the language of *state descriptions* Carnap employs in his work on inductive reasoning [Car50].

6.5 Relating the information orderings

Let $\mathbb{K}$ and $\mathbb{T}$ be a set of all contexts and a set of all consistent theories (over a fixed M), respectively. In Section 6.2 we have defined mappings between $\mathbb{K}$ and $\mathbb{T}$, namely $\boldsymbol{\tau}\colon \mathbb{K} \longrightarrow \mathbb{T}$ and $\boldsymbol{\kappa}\colon \mathbb{T} \longrightarrow \mathcal{P}(\mathbb{K})$. Clearly, $\boldsymbol{\tau}$ is *total*, but neither *1-1* nor *onto*, and $\boldsymbol{\kappa}$ is *total*[5] and *1-1*[6] but not *onto*—the image of $\boldsymbol{\kappa}$ is $\boldsymbol{\kappa}(\mathbb{T}) = \{\boldsymbol{\mathcal{K}}_{\boldsymbol{T}} \mid \boldsymbol{T} \in \mathbb{T}\} \subset \mathcal{P}(\mathbb{K})$. However, $\mathbb{K}$ and $\mathbb{T}$ are *ordered* sets, and thus it is natural to ask whether $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}$ are *order-preserving*. We consider the mapping $\boldsymbol{\tau}$ first, and then $\boldsymbol{\kappa}$.

[4]An alternative to $\models_\oplus F$ would be a much less convenient $\models_{\text{true}}$ instantiatable(F).

[5]Note that $\boldsymbol{\kappa}$ is defined on $\mathbb{T}$, but not on $\mathbb{T} \cup \{\mathbf{1}\}$, i.e., $\boldsymbol{\kappa}$ is not being applied to the inconsistent theory $\mathbf{1}$.

[6]To see that $\boldsymbol{\kappa}$ is *1-1*, note the following. If $\boldsymbol{T}_1 \neq \boldsymbol{T}_2$ then there is a description D in, say, $\boldsymbol{T}_1 \setminus \boldsymbol{T}_2$. But then D is valid in every element of $\boldsymbol{\mathcal{K}}_1 = \boldsymbol{\kappa}(\boldsymbol{T}_1)$, but there are elements of $\boldsymbol{\mathcal{K}}_2 = \boldsymbol{\kappa}(\boldsymbol{T}_2)$ in which D is not valid. Hence, $\boldsymbol{\mathcal{K}}_1 \neq \boldsymbol{\mathcal{K}}_2$.

Given that $\boldsymbol{\tau}\colon \mathbb{K} \longrightarrow \mathbb{T}$ is defined by $\boldsymbol{\tau}(\boldsymbol{K}) = \boldsymbol{T_K} = \{D \in \boldsymbol{D} \mid \boldsymbol{K} \models D\}$, and the fact that $(\mathbb{K}, \leq)$ and $(\mathbb{T}, \leq)$ are sets ordered by the corresponding information ordering relations, we expect that $\boldsymbol{\tau}$ *is* order-preserving. Indeed, one would suspect that there is something wrong with our definition of the information ordering on $\mathbb{K}$ if $\boldsymbol{\tau}$ was not order-preserving. There is no surprise here—we get the following result.

Proposition 5 *Let $(\mathbb{K}, \leq)$ and $(\mathbb{T}, \leq)$ be the sets of contexts and theories equipped with the corresponding information orderings, and let $\boldsymbol{\tau}$ be the mapping from $\mathbb{K}$ to $\mathbb{T}$. Let $\boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathbb{K}$ and let $\boldsymbol{T}_1 = \boldsymbol{\tau}(\boldsymbol{K}_1), \boldsymbol{T}_2 = \boldsymbol{\tau}(\boldsymbol{K}_2) \in \mathbb{T}$. We have that if $\boldsymbol{K}_1 \leq \boldsymbol{K}_2$ then $\boldsymbol{T}_1 \leq \boldsymbol{T}_2$.*

Recall that $\boldsymbol{\kappa} : \mathbb{T} \longrightarrow \mathcal{P}(\mathbb{K})$ is given by $\boldsymbol{\kappa}(\boldsymbol{T}_i) = \boldsymbol{\mathcal{K}}_{\boldsymbol{T}_i} = \{\boldsymbol{K} \in \mathbb{K} \mid \boldsymbol{T_K} \geq \boldsymbol{T}_i$ and $\boldsymbol{K}$ is $\leq$-minimal$\}$. The domain of $\boldsymbol{\kappa}$, i.e., the set of theories $\mathbb{T}$ is ordered by the information ordering $\leq$ on theories. However, before we ask whether $\boldsymbol{\kappa}$ is order-preserving, we need an (information) ordering on $\mathcal{P}(\mathbb{K})$—indeed, the codomain of $\boldsymbol{\kappa}$ is $\mathcal{P}(\mathbb{K})$, rather than just $\mathbb{K}$. It is suggested in Section 2.1 tells us how to define an information ordering on $\mathcal{P}(\mathbb{K})$. We limit ourselves to the image $\boldsymbol{\kappa}(\mathbb{T}) = \{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}$ of $\boldsymbol{\kappa}$.

Definition 20 *Let $(\mathbb{K}, \leq)$ be the set of contexts equipped with the information ordering $\leq$. Let $\{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}$ be a set of sets of contexts. An* information ordering *on $\{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}$ is defined as follows. Let $\boldsymbol{\mathcal{K}}_1 = \boldsymbol{\mathcal{K}}_{\boldsymbol{T}_1} = \boldsymbol{\kappa}(\boldsymbol{T}_1)$ and $\boldsymbol{\mathcal{K}}_2 = \boldsymbol{\mathcal{K}}_{\boldsymbol{T}_2} = \boldsymbol{\kappa}(\boldsymbol{T}_2)$, and hence $\boldsymbol{\mathcal{K}}_1, \boldsymbol{\mathcal{K}}_2 \in \{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}$. We say that $\boldsymbol{\mathcal{K}}_1 \leq \boldsymbol{\mathcal{K}}_2$* iff *the following condition is satisfied:*

1. $\forall_{\boldsymbol{K}_2 \in \boldsymbol{\mathcal{K}}_2} \exists_{\boldsymbol{K}_1 \in \boldsymbol{\mathcal{K}}_1}\ \boldsymbol{K}_1 \leq \boldsymbol{K}_2$.

Hence, Definition 20 turns $\{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}$ into an ordered set $(\{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}, \leq)$. This is the set of $\boldsymbol{\kappa}$-models of theories in $\mathbb{T}$, now equipped with the information ordering $\leq$.

Given the ordered sets $(\mathbb{T}, \leq)$ and $(\{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}, \leq)$, we can now consider whether the mapping $\boldsymbol{\kappa}\colon \mathbb{T} \longrightarrow \mathcal{P}(\mathbb{K})$, or rather $\boldsymbol{\kappa}\colon \mathbb{T} \longrightarrow \{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}$, is order-preserving.

Proposition 6 *Let $(\mathbb{T}, \leq)$ and $(\{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}, \leq)$ be the set of theories, and the set of $\boldsymbol{\kappa}$-models of theories, respectively, both equipped with the corresponding information orderings. Let $\boldsymbol{\kappa}$ be the mapping from $\mathbb{T}$ to $\{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}$. Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in \mathbb{T}$. Let $\boldsymbol{\mathcal{K}}_1 = \boldsymbol{\mathcal{K}}_{\boldsymbol{T}_1} = \boldsymbol{\kappa}(\boldsymbol{T}_1)$ and $\boldsymbol{\mathcal{K}}_2 = \boldsymbol{\mathcal{K}}_{\boldsymbol{T}_2} = \boldsymbol{\kappa}(\boldsymbol{T}_2)$, and hence $\boldsymbol{\mathcal{K}}_1, \boldsymbol{\mathcal{K}}_2 \in \{\, \boldsymbol{\mathcal{K}_T} \mid \boldsymbol{T} \in \mathbb{T}\}$. We have that if $\boldsymbol{T}_1 \leq \boldsymbol{T}_2$ then $\boldsymbol{\mathcal{K}}_1 \leq \boldsymbol{\mathcal{K}}_2$.*

Note that although Proposition 6 seems complicated, it addresses an intuitively simple and relevant question—if theories $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are such that $\boldsymbol{T}_1 \leq \boldsymbol{T}_2$, is the ordering preserved when we look at the theories *semantically*, i.e., when we consider their corresponding $\boldsymbol{\kappa}$-models $\boldsymbol{\mathcal{K}}_1$ and $\boldsymbol{\mathcal{K}}_2$.

6.6 Soundness and completeness

Let $\mathcal{H}_i = \mathcal{H}_{D_i}$ be a formal system with axioms $\boldsymbol{D}_i$, cf., Section 5.1. Given such a description set $\boldsymbol{D}_i$ we want to show that any sentence, i.e., any description in the set $\boldsymbol{D}$ of all descriptions (possibly outside of $\boldsymbol{D}_i$) is provable in the formal system $\mathcal{H}_{D_i}$ iff it is valid in the $\boldsymbol{\kappa}$-model of $\boldsymbol{D}_i$.

Given a theory $\boldsymbol{T}_i \in \mathbb{T}$, the following procedure[7] allows us to find its $\boldsymbol{\kappa}$-model, denoted $\mathcal{K}_{T_i}$.

Procedure 2 *Let $\boldsymbol{T}_i \in \mathbb{T}$ be a consistent theory, with the provable formulae given by $(\boldsymbol{T}_i^{\oplus}, \boldsymbol{T}_i^{\ominus})$, and the axiom formulae by $(\boldsymbol{A}_i^{\oplus}, \boldsymbol{A}_i^{\ominus})$. To find the $\boldsymbol{\kappa}$-model $\mathcal{K}_i$ of $\boldsymbol{T}_i$ we proceed as follows.*

1. $\boldsymbol{G}_i = \{\boldsymbol{g}_A \mid A \in \boldsymbol{A}_i^{\oplus}\}$,
2. $\boldsymbol{G}_i^{\bullet} = \boldsymbol{G}_i \cup \{\boldsymbol{g}_{\{\}}\}$,
3. $\boldsymbol{K}_i^{\bullet} = (\boldsymbol{G}_i^{\bullet}, M_i, I_i)$,
4. $\mathcal{K}_{i,0} = \{\boldsymbol{K}_i^{\bullet}\}$,
5. *for every formula* $F_j \in \{F_j\}_{j=1,,\dots,n} = \boldsymbol{A}_i^{\ominus}$
   (a) *for every context* $\boldsymbol{K}_{i,j-1,k} \in \mathcal{K}_{i,j-1}$
      i. *for every object* $\boldsymbol{g}_l \in \boldsymbol{G}_{i,j-1,k}$ *such that* $\{\boldsymbol{g}_l\} \not\models_{\ominus} F_j$
         A. *find* $\boldsymbol{G}_{l,j} = \{\boldsymbol{g} \in \boldsymbol{G} \mid \boldsymbol{g} \geq \boldsymbol{g}_l$ and $\{\boldsymbol{g}\} \models_{\ominus} F_j$ and $\boldsymbol{g}$ is $\leq$ -minimal$\}$, *and* $\mathbb{G}_{l,j} = \{\boldsymbol{G}_{\lambda} \mid \emptyset \neq \boldsymbol{G}_{\lambda} \subseteq \boldsymbol{G}_{l,j}\} = \{\boldsymbol{G}_{\lambda}\}_{\lambda}$,
         B. *find* $\{\boldsymbol{G}_{i,j-1,k,\lambda}\}_{\lambda} = \{\boldsymbol{G}_{i,j-1,k} \setminus \{\boldsymbol{g}_l\} \cup \boldsymbol{G}_{\lambda} \mid \boldsymbol{G}_{\lambda} \in \mathbb{G}_{l,j}\}$, *and* $\{\boldsymbol{K}_{i,j-1,k,\lambda}\}_{\lambda}$, *the corresponding contexts,*
         C. *replace* $\{\boldsymbol{K}_{i,j-1,k}\}$ *with* $\{\boldsymbol{K}_{i,j-1,k,\lambda}\}_{\lambda}$,
   (b) *the resulting set of contexts is* $\mathcal{K}_{i,j}$
6. *the resulting set of contexts is* $\mathcal{K}_i$.

Consider the following example.

Example 4 *Let $\boldsymbol{T}_i$ be such that $\boldsymbol{A}_i^{\oplus} = \{\{p_1\}\}$ and $\boldsymbol{A}_i^{\ominus} = \{\{p_1\}, \{p_1, p_2, p_3\}, \{p_1, \overline{p_2}, \overline{p_3}\}\}$. We have the following.*

1. $\boldsymbol{G}_i = \{\boldsymbol{g}_{\{p_1\}}\}$,
2. $\boldsymbol{G}_i^{\bullet} = \{\boldsymbol{g}_{\{p_1\}}, \boldsymbol{g}_{\{\}}\}$,
3. $\boldsymbol{K}_i^{\bullet} = (\{\boldsymbol{g}_{\{p_1\}}, \boldsymbol{g}_{\{\}}\}, \{p_1, \overline{p_1}, p_2, \overline{p_2}, p_3, \overline{p_3}, \}, I_i)$,
4. $\mathcal{K}_{i,0} = \{\boldsymbol{K}_i^{\bullet}\}$,
5. $\{F_j\}_j = \{\{p_1\}, \{p_1, p_2, p_3\}, \{p_1, \overline{p_2}, \overline{p_3}\}\}$,

[7] The procedure is elaborated in Appendix A.

- $F_1 = \{p_1\}$
  (a) *for every context* $\boldsymbol{K}_{i,0,k} \in \boldsymbol{\mathcal{K}}_{i,0}$
    i. *for every object* $\boldsymbol{g}_l \in \boldsymbol{G}_{i,0,k}$ *such that* $\{\boldsymbol{g}_l\} \not\models_\ominus F_j$
      A. *find* $\boldsymbol{G}_{l,j} = \{\boldsymbol{g} \in \boldsymbol{G} \mid \boldsymbol{g} \geq \boldsymbol{g}_l$ and $\{\boldsymbol{g}\} \models_\ominus F_j$ and $\boldsymbol{g}$ is $\leq$ -minimal$\}$, *and* $\mathbb{G}_{l,j} = \{\boldsymbol{G}_\lambda \mid \emptyset \neq \boldsymbol{G}_\lambda \subseteq \boldsymbol{G}_{l,j}\} = \{\boldsymbol{G}_\lambda\}_\lambda$,
      B. *find* $\{\boldsymbol{G}_{i,0,k,\lambda}\}_\lambda = \{\boldsymbol{G}_{i,0,k} \setminus \{\boldsymbol{g}_l\} \cup \boldsymbol{G}_\lambda \mid \boldsymbol{G}_\lambda \in \mathbb{G}_{l,j}\}$, *and* $\{\boldsymbol{K}_{i,0,k,\lambda}\}_\lambda$, *the corresponding contexts,*
      C. *replace* $\{\boldsymbol{K}_{i,0,k}\}$ *with* $\{\boldsymbol{K}_{i,0,k,\lambda}\}_\lambda$,
  (b) *the resulting set of contexts is* $\boldsymbol{\mathcal{K}}_{i,1}$
- $F_2 = \{p_1, p_2, p_3\}$
  (a) *for every context* $\boldsymbol{K}_{i,1,k} \in \boldsymbol{\mathcal{K}}_{i,1}$
    i. *for every object* $\boldsymbol{g}_l \in \boldsymbol{G}_{i,1,k}$ *such that* $\{\boldsymbol{g}_l\} \not\models_\ominus F_j$
      A. *find* $\boldsymbol{G}_{l,j} = \{\boldsymbol{g} \in \boldsymbol{G} \mid \boldsymbol{g} \geq \boldsymbol{g}_l$ and $\{\boldsymbol{g}\} \models_\ominus F_j$ and $\boldsymbol{g}$ is $\leq$ -minimal$\}$, *and* $\mathbb{G}_{l,j} = \{\boldsymbol{G}_\lambda \mid \emptyset \neq \boldsymbol{G}_\lambda \subseteq \boldsymbol{G}_{l,j}\} = \{\boldsymbol{G}_\lambda\}_\lambda$,
      B. *find* $\{\boldsymbol{G}_{i,1,k,\lambda}\}_\lambda = \{\boldsymbol{G}_{i,1,k} \setminus \{\boldsymbol{g}_l\} \cup \boldsymbol{G}_\lambda \mid \boldsymbol{G}_\lambda \in \mathbb{G}_{l,j}\}$, *and* $\{\boldsymbol{K}_{i,1,k,\lambda}\}_\lambda$, *the corresponding contexts,*
      C. *replace* $\{\boldsymbol{K}_{i,1,k}\}$ *with* $\{\boldsymbol{K}_{i,1,k,\lambda}\}_\lambda$,
  (b) *the resulting set of contexts is* $\boldsymbol{\mathcal{K}}_{i,2}$
- $F_3 = \{p_1, \overline{p_2}, \overline{p_3}\}$
  (a) *for every context* $\boldsymbol{K}_{i,2,k} \in \boldsymbol{\mathcal{K}}_{i,2}$
    i. *for every object* $\boldsymbol{g}_l \in \boldsymbol{G}_{i,2,k}$ *such that* $\{\boldsymbol{g}_l\} \not\models_\ominus F_j$
      A. *find* $\boldsymbol{G}_{l,j} = \{\boldsymbol{g} \in \boldsymbol{G} \mid \boldsymbol{g} \geq \boldsymbol{g}_l$ and $\{\boldsymbol{g}\} \models_\ominus F_j$ and $\boldsymbol{g}$ is $\leq$ -minimal$\}$, *and* $\mathbb{G}_{l,j} = \{\boldsymbol{G}_\lambda \mid \emptyset \neq \boldsymbol{G}_\lambda \subseteq \boldsymbol{G}_{l,j}\} = \{\boldsymbol{G}_\lambda\}_\lambda$,
      B. *find* $\{\boldsymbol{G}_{i,2,k,\lambda}\}_\lambda = \{\boldsymbol{G}_{i,2,k} \setminus \{\boldsymbol{g}_l\} \cup \boldsymbol{G}_\lambda \mid \boldsymbol{G}_\lambda \in \mathbb{G}_{l,j}\}$, *and* $\{\boldsymbol{K}_{i,2,k,\lambda}\}_\lambda$, *the corresponding contexts,*
      C. *replace* $\{\boldsymbol{K}_{i,2,k}\}$ *with* $\{\boldsymbol{K}_{i,2,k,\lambda}\}_\lambda$,
  (b) *the resulting set of contexts is* $\boldsymbol{\mathcal{K}}_{i,3}$

6. *the resulting set of contexts* $\boldsymbol{\mathcal{K}}_{i,3}$ *is the* $\boldsymbol{\kappa}$*-model of* $\boldsymbol{T}_i$*, i.e., the model* $\boldsymbol{\mathcal{K}}_i = \boldsymbol{\kappa}(\boldsymbol{T}_i)$.

We can now formulate a proposition concerning soundness and completeness of the logical formalism.

Proposition 7 *Let* $\boldsymbol{D}_i \in \mathbb{D}$ *and* $D \in \boldsymbol{D}$*. Let* $\mathcal{H}_i = \mathcal{H}_{\boldsymbol{D}_i}$ *be a formal system with axioms* $\boldsymbol{D}_i$*. Let* $\mathcal{K}_i = \mathcal{K}_{\boldsymbol{D}_i}$ *be a* $\boldsymbol{\kappa}$*-model of* $\boldsymbol{D}_i$*.*

$$\mathcal{K}_i \models D \text{ iff } \mathcal{H}_i \vdash D$$

Hence, theories are syntactic equivalents of $\boldsymbol{\kappa}$-models.

In Chapter 7, agent-related considerations are carried out employing theories, rather than models. Two comments are in place. Firstly, information provided by agents has a form of description sets, and theories *are* description sets, and hence it seems appropriate to compute theories corresponding to description sets, and operate on them. Secondly however, if one wants to consider the *meaning* of theories, then κ-models of theories can be found. The problem with such a semantic approach is that information obtained from an agent might be partial to such an extent that no single partial context corresponds to the theory of the agent, but a set of contexts should be found. Dealing in Chapter 7 mostly with theories, we limit ourselves to syntactic considerations, but we have the corresponding semantics at our disposal.

Chapter 7

Multiple agents

This chapter deals with a *multiple agent* case, employing the developed logical framework.

Section 7.1 introduces *believed theories* resulting from sets of sentences, or *description sets* provided by agents. The set of believed theories gives rise to a lattice of theories. Then *truth-values* of descriptions and theories are considered in Section 7.2. The question whether lattices of theories can be seen as FCA *concept lattices* is addressed in Section 7.3. Then Section 7.4 proposes a *numeric measure* on theories. In Section 7.5 we comment on *preference* and *epistemic states.*

7.1 Believed theories

Let S be a set of *agents* and let $\{\boldsymbol{D}_s\}_{s \in S}$ be a set of description sets provided by the agents. Two agents might provide the same description set, and even if their description sets differ they might produce the same theory. Let $\mathbb{B}$ be the set of theories of the agents, or *believed theories*—certainly, $\mathbb{B} \subseteq \mathbb{T}$. Assume believed theories are *nonempty* and *consistent* (agents are assumed to be consistent), so $\mathbb{B} \cap \{\mathbf{0}, \mathbf{1}\} = \emptyset$.

More precisely, let S be a set of *agents*, or *sources of information.* Then, there is a mapping β from S to the set of all consistent theories $\mathbb{T}$, i.e., $\beta \colon S \longrightarrow \mathbb{T}$, and the mapping is given by $\beta(s) = \mathrm{Cn}(\boldsymbol{D}_s)$. Let $\beta(S) = \mathbb{B}$, so $\mathbb{B}$ is a set of theories that are actually *believed* by some agents. It is natural to consider a related equivalence relation $\approx_\beta$ on S given by: $s_1 \approx_\beta s_2$ iff $\beta(s_1) = \beta(s_2)$. Then we get a set $\boldsymbol{S} = S/\approx_\beta$ of $\approx_\beta$-equivalence classes—an element of $\boldsymbol{S}$ is simply a set of agents that believe in the same theory. It is hence appropriate to think about a mapping $\boldsymbol{\beta} \colon \boldsymbol{S} \longrightarrow \mathbb{B}$ defined as follows: if $\mathcal{S} \in \boldsymbol{S}$ and $\mathcal{S} = [s] \ni s$ then $\boldsymbol{\beta}(\mathcal{S}) = \beta(s)$—we simply take the sets of equivalent agents as the elements of the domain. Let $\boldsymbol{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$. Then $\mathbb{B} = \{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_n\}$, where $\boldsymbol{B}_i = \boldsymbol{\beta}(\mathcal{S}_i)$. It is the set $\mathbb{B}$ this section is concerned with.

Furthermore, define $\mathbb{C} = \mathrm{Cl}_{\wedge,\vee}(\mathbb{B})$, where $\wedge$ and $\vee$ are operations already defined—recall $(\mathbb{T}, \wedge, \vee)$ of Section 5.4.

Figure 32 presents six examples for the case $M = \{p_1, \overline{p_1}\}$, or $P = \{p_1\}$. Each example gives a set $\mathbb{B}$ of believed theories and its closure $\mathbb{C}$, and $\mathbb{B}, \mathbb{C} \subseteq \mathbb{T}$, where $\mathbb{T}$ is a set of all consistent theories for the case of $|P| = \alpha = 1$, this set $\mathbb{T}$ of theories is presented in Figures 29 and 30.

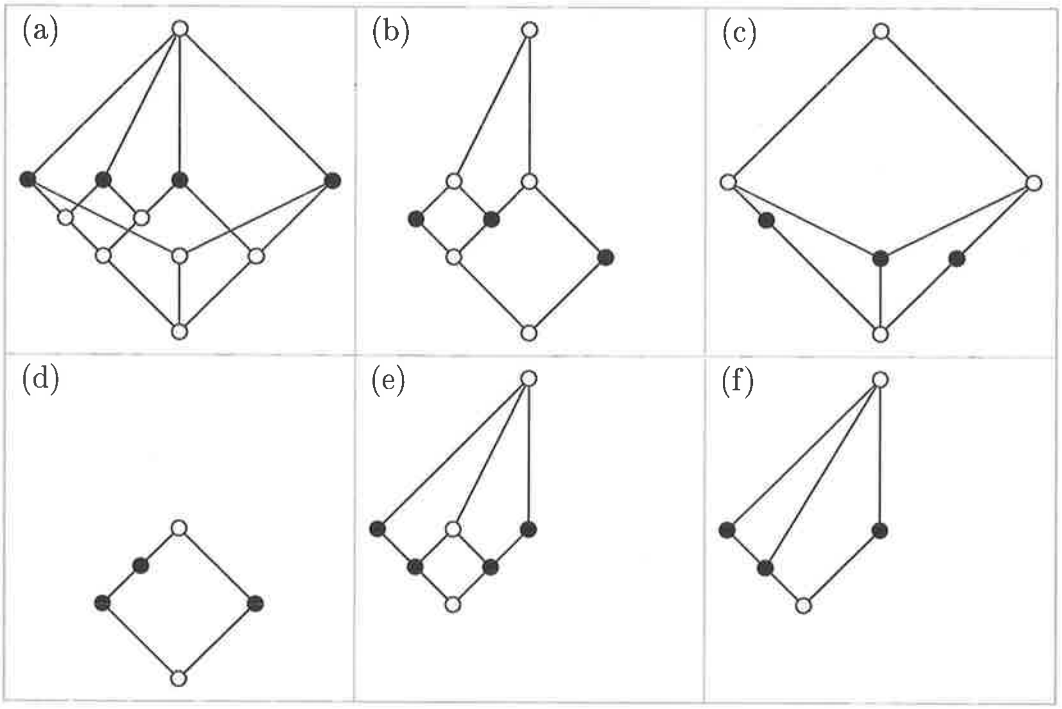


Figure 32. Some believed theories and their closures, $|P| = 1$

Clearly, it is interesting to see whether believed theories have a non-empty meet, and whether they can be joined consistently. Hence, we define $\mathbb{C}_+ = \mathbb{C} \cup \{\mathbf{0}, \mathbf{1}\}$, so forming $\mathbb{C}_+$ simply amounts to adding—unless they are already there—a *bottom* $\mathbf{0}$ and a *top element* $\mathbf{1}$ to $\mathbb{C}$—for every $\boldsymbol{C} \in \mathbb{C}$ require $\mathbf{0} < \boldsymbol{C} < \mathbf{1}$. Certainly, $\mathbb{C}$ and $\mathbb{C}_+$ are *lattices.*

Proposition 8 *Let $\mathbb{B}$ be a set of believed theories. Then $\mathbb{C}$ and $\mathbb{C}_+$ are* lattices.

The possible cases of two believed theories and their closures are presented in Figure 33.

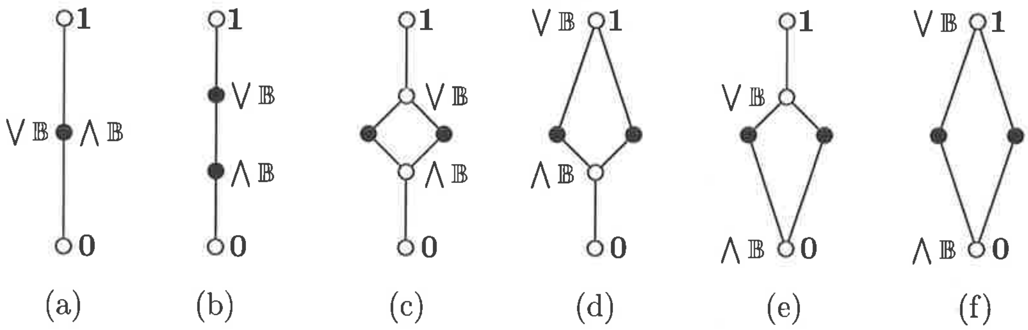


Figure 33. Pairs of believed theories and their closures

Figure 33 allows us to see whether believed theories have a nonempty intersection (meet), and whether or not they contradict each other (join at the top element **1**).

More interesting examples, with $|P| = \alpha > 1$ are presented in Figure 34.

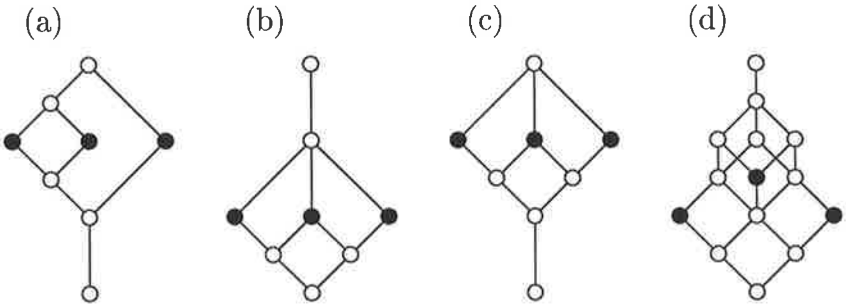


Figure 34. Some believed theories and their closures, $|P| > 1$

The examples of Figure 34 are as follows. Let $\mathbb{B}_a, \ldots, \mathbb{B}_d$ denote the sets of believed theories of the examples presented in Figure 34(a), ..., 34(d), where $\mathbb{B}_a = \{\boldsymbol{B}_{a,1}, \boldsymbol{B}_{a,2}, \boldsymbol{B}_{a,3}\}$, with $\boldsymbol{A}_{a,1} = \text{gen}(\boldsymbol{B}_{a,1}) = \{\oplus\{p_1, p_2\}\}$, $\boldsymbol{A}_{a,2} = \text{gen}(\boldsymbol{B}_{a,2}) = \{\oplus\{p_1, p_3\}\}$, $\boldsymbol{A}_{a,3} = \text{gen}(\boldsymbol{B}_{a,3}) = \{\ominus\{p_1\}, \oplus\{p_4, \overline{p_1}\}\}$, $\mathbb{B}_b = \{\boldsymbol{B}_{b,1}, \boldsymbol{B}_{b,2}, \boldsymbol{B}_{b,3}\}$, with $\boldsymbol{A}_{b,1} = \text{gen}(\boldsymbol{B}_{b,1}) = \{\ominus\{p_1\}, \ominus\{p_3\}\}$, $\boldsymbol{A}_{b,2} = \text{gen}(\boldsymbol{B}_{b,2}) = \{\ominus\{p_1\}, \ominus\{p_2\}\}$, $\boldsymbol{A}_{b,3} = \text{gen}(\boldsymbol{B}_{b,3}) = \{\ominus\{p_2\}, \ominus\{p_3\}\}$, $\mathbb{B}_c = \{\boldsymbol{B}_{c,1}, \boldsymbol{B}_{c,2}, \boldsymbol{B}_{c,3}\}$, with $\boldsymbol{A}_{c,1} = \text{gen}(\boldsymbol{B}_{c,1}) = \{\oplus\{p_1, p_2\} \ominus \{\overline{p_1}\}\}$, $\boldsymbol{A}_{c,2} = \text{gen}(\boldsymbol{B}_{c,2}) = \{\oplus\{p_1\}, \oplus\{\overline{p_1}\}, \oplus\{\overline{p_2}, \overline{p_3}\}\}$, $\boldsymbol{A}_{c,3} = \text{gen}(\boldsymbol{B}_{c,3}) = \{\oplus\{\overline{p_1}, p_3\} \ominus \{p_1\}\}$, $\mathbb{B}_d = \{\boldsymbol{B}_{d,1}, \boldsymbol{B}_{d,2}, \boldsymbol{B}_{d,3}\}$, with $\boldsymbol{A}_{d,1} = \text{gen}(\boldsymbol{B}_{d,1}) = \{\oplus\{p_1\}\}$, $\boldsymbol{A}_{d,2} = \text{gen}(\boldsymbol{B}_{d,2}) = \{\oplus\{\}\ominus\{p_2, p_3\}\}$, $\boldsymbol{A}_{d,3} = \text{gen}(\boldsymbol{B}_{d,3}) = \{\ominus\{p_2, p_4\}\}$. For each of the examples, the top and bottom elements of the lattices are **1** and **0**, i.e., for each example the lattice $\mathbb{C}_+$ is shown.

Given a set of believed theories $\mathbb{B}$, we thus usually consider its closure $\mathbb{C}$, and a lattice $\mathbb{C}_+$ that results from adding **1** and **0** to $\mathbb{C}$. It is sometimes also useful to consider a minimal subset of $\mathbb{B}$ that generates the same lattice $\mathbb{C}$ as $\mathbb{B}$ itself does, i.e., $\mathbb{B}_-$ is a minimal subset of $\mathbb{B}$ such that $\text{Cl}(\mathbb{B}_-) = \text{Cl}(\mathbb{B})$—such a set is denoted by $\mathbb{B}_-$. We certainly have that $\mathbb{B}_- \subseteq \mathbb{B} \subseteq \mathbb{C}_- \subseteq \mathbb{C} \subseteq \mathbb{C}_+$, the only guaranteed proper inclusion being $\mathbb{C}_- \subset \mathbb{C}_+$. The sets $\mathbb{B}_-, \mathbb{B}, \mathbb{C}_-, \mathbb{C}$ and $\mathbb{C}_+$ are presented in Figure 35.

We finish the section with a more involved example, as presented in Table 7. The lattice of theories of Table 7 is presented in Figure 36.

7.2 Conceptual worlds and bilattices

Given that $\mathbb{C}_+$ is a lattice, one might ask whether it is a *concept lattice* in the sense defined by FCA. Consider the example of Figure 37.

In FCA, concepts are certain pairs (*extent, intent*), where a concept's extent is a set of *objects*, and its *intent* is a set of *attributes*. Further, concepts are ordered by a *subconcept/superconcept* relation $\leq$, and moving up in the lattice of concepts makes the concept's extent (set of objects) bigger, and its intent

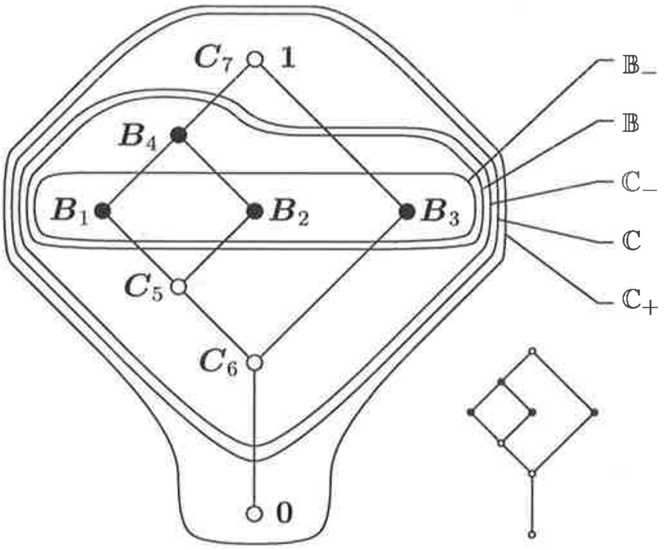


Figure 35. Believed theories $\mathbb{B}$ and sets $\mathbb{B}_-$, $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_-$

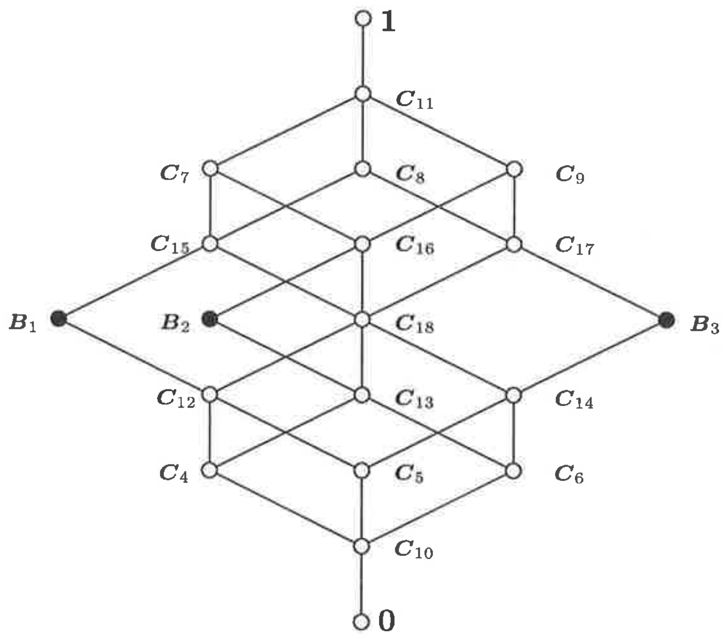


Figure 36. Lattice $\mathbb{C}_+$ of believed theories $\mathbb{B}$ of Table 7

(set of attributes) smaller. Given a lattice of theories like that of Figure 37, moving up corresponds to expanding the set of theorems. Hence, theorems, or descriptions/provable sentences should be taken as objects, to form extents of theories (concepts). It is appropriate to consider only those descriptions that are axioms of some theories in $\mathbb{C}$. In the *labelled line diagram*—see Figure 37 (b)—we then place descriptions in such a way that they appear at or below the node corresponding to the theory, e.g., in Figure 37 the theory $\boldsymbol{B}_2$ has the descriptions

theory $\boldsymbol{C} \in \mathbb{C}_+$	$\boldsymbol{A} = \text{gen}(\boldsymbol{C})$	sentence, $\{\beta_i\}_i$	models
$\boldsymbol{B}_1$	$\{\oplus\{p_1 p_2\}\}$	β_1	$\{i_1 i_2 i_3 i_5\}$
$\boldsymbol{B}_2$	$\{\oplus\{p_1 p_3\}\}$	β_2	$\{i_1 i_2 i_4 i_6\}$
$\boldsymbol{B}_3$	$\{\oplus\{p_2 p_3\}\}$	β_3	$\{i_1 i_3 i_4 i_7\}$
$\boldsymbol{C}_4 = \boldsymbol{B}_1 \wedge \boldsymbol{B}_2$	$\{\oplus\{p_1\}\}$	$\beta_1 \vee \beta_2$	$\{i_1 i_2 i_3 i_4 i_5 i_6\}$
$\boldsymbol{C}_5 = \boldsymbol{B}_1 \wedge \boldsymbol{B}_3$	$\{\oplus\{p_2\}\}$	$\beta_1 \vee \beta_3$	$\{i_1 i_2 i_3 i_4 i_5 i_7\}$
$\boldsymbol{C}_6 = \boldsymbol{B}_2 \wedge \boldsymbol{B}_3$	$\{\oplus\{p_3\}\}$	$\beta_2 \vee \beta_3$	$\{i_1 i_2 i_3 i_4 i_6 i_7\}$
$\boldsymbol{C}_7 = \boldsymbol{B}_1 \vee \boldsymbol{B}_2$	$\{\oplus\{p_1 p_2\}, \oplus\{p_1 p_3\}\}$	$\beta_1 \wedge \beta_2$	$\{i_1 i_2\}$
$\boldsymbol{C}_8 = \boldsymbol{B}_1 \vee \boldsymbol{B}_3$	$\{\oplus\{p_1 p_2\}, \oplus\{p_2 p_3\}\}$	$\beta_1 \wedge \beta_3$	$\{i_1 i_3\}$
$\boldsymbol{C}_9 = \boldsymbol{B}_2 \vee \boldsymbol{B}_3$	$\{\oplus\{p_1 p_3\}, \oplus\{p_2 p_3\}\}$	$\beta_2 \wedge \beta_3$	$\{i_1 i_4\}$
$\boldsymbol{C}_{10} =$ $\boldsymbol{B}_1 \wedge \boldsymbol{B}_2 \wedge \boldsymbol{B}_3$	$\{\oplus\{\}\}$	$\beta_1 \vee \beta_2 \vee \beta_3$	$\{i_1 i_2 i_3 i_4 i_5 i_6 i_7\}$
$\boldsymbol{C}_{11} =$ $\boldsymbol{B}_1 \vee \boldsymbol{B}_2 \vee \boldsymbol{B}_3$	$\{\oplus\{p_1 p_2\}, \oplus\{p_1 p_3\},$ $\oplus\{p_2 p_3\}\}$	$\beta_1 \wedge \beta_2 \wedge \beta_3$	$\{i_1\}$
$\boldsymbol{C}_{12} = \boldsymbol{C}_4 \vee \boldsymbol{C}_5$	$\{\oplus\{p_1\}, \oplus\{p_2\}\}$	$(\beta_1 \vee \beta_2) \wedge (\beta_1 \vee \beta_3)$	$\{i_1 i_2 i_3 i_4 i_5\}$
$\boldsymbol{C}_{13} = \boldsymbol{C}_4 \vee \boldsymbol{C}_6$	$\{\oplus\{p_1\}, \oplus\{p_3\}\}$	$(\beta_1 \vee \beta_2) \wedge (\beta_2 \vee \beta_3)$	$\{i_1 i_2 i_3 i_4 i_6\}$
$\boldsymbol{C}_{14} = \boldsymbol{C}_5 \vee \boldsymbol{C}_6$	$\{\oplus\{p_2\}, \oplus\{p_3\}\}$	$(\beta_1 \vee \beta_3) \wedge (\beta_2 \vee \beta_3)$	$\{i_1 i_2 i_3 i_4 i_7\}$
$\boldsymbol{C}_{15} = \boldsymbol{C}_7 \wedge \boldsymbol{C}_8$	$\{\oplus\{p_1, p_2\}, \oplus\{p_3\}\}$	$(\beta_1 \wedge \beta_2) \vee (\beta_1 \wedge \beta_3)$	$\{i_1 i_2 i_3\}$
$\boldsymbol{C}_{16} = \boldsymbol{C}_7 \wedge \boldsymbol{C}_9$	$\{\oplus\{p_1, p_3\}, \oplus\{p_2\}\}$	$(\beta_1 \wedge \beta_2) \vee (\beta_2 \wedge \beta_3)$	$\{i_1 i_2 i_4\}$
$\boldsymbol{C}_{17} = \boldsymbol{C}_8 \wedge \boldsymbol{C}_9$	$\{\oplus\{p_2, p_3\}, \oplus\{p_1\}\}$	$(\beta_1 \wedge \beta_3) \vee (\beta_2 \wedge \beta_3)$	$\{i_1 i_3 i_4\}$
$\boldsymbol{C}_{18} =$ $\boldsymbol{C}_{12} \vee \boldsymbol{C}_{13} \vee \boldsymbol{C}_{14} =$ $\boldsymbol{C}_{15} \wedge \boldsymbol{C}_{16} \wedge \boldsymbol{C}_{17}$	$\{\oplus\{p_1\}, \oplus\{p_2\},$ $\oplus\{p_3\}\}$	$(\beta_1 \vee \beta_2) \wedge (\beta_1 \vee \beta_3)$ $\wedge(\beta_2 \vee \beta_3) =$ $(\beta_1 \wedge \beta_2) \vee (\beta_1 \wedge \beta_3)$ $\wedge(\beta_2 \vee \beta_3)$	$\{i_1 i_2 i_3 i_4\}$
0	$\{\}$	$\beta_1 \vee \neg\beta_1$	$\{i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8\}$
1	$\{\oplus 1, \ominus 1\}$	$\beta_1 \wedge \neg\beta_1$	$\{\}$

Table 7. Believed theories $\mathbb{B}$ and closure $\mathbb{C}_+$

D_2, D_3 and D_4, and they can be found by traversing the lattice "down" from the theory's node. Finding concepts' intents is more complicated, but the idea is simple—if you go up with theories, the set of descriptions gets bigger, but "truthness" of the theories decreases, and hence we should take as concepts' intents sets of "models" of the theories. We first look at Ginsberg's world-based bilattices [Gin88], at the bilattice approach provides a method of finding *world-based truth-values* and thus it might help us to decide about *truthness* of theories.

Let the set of believed theories $\mathbb{B}$ be seen as a set of *worlds*. We can use the bilattice approach to associate *truth-values* with descriptions. The set of truth-values Γ is given by $\Gamma = \mathcal{P}(\mathbb{B}) \times \mathcal{P}(\mathbb{B})$ i.e., a truth value is a pair of sets of worlds. A truth-valuation function $\varrho : \boldsymbol{D} \longrightarrow \Gamma$ is given by $\varrho(D) = (U_D, V_D)$, where U_D is a set of worlds where D is *true*, and V_D where it is *false*. In our case it is appropriate to define U_D and V_D as follows. Given $D \in \boldsymbol{D}$, let $\boldsymbol{T}_D = \text{Cn}(\{D\})$—this is the smallest theory that contains D. Then,

$$U_D = \{\boldsymbol{B}_i \in \mathbb{B} \mid \boldsymbol{B}_i \geq \boldsymbol{T}_D\},$$

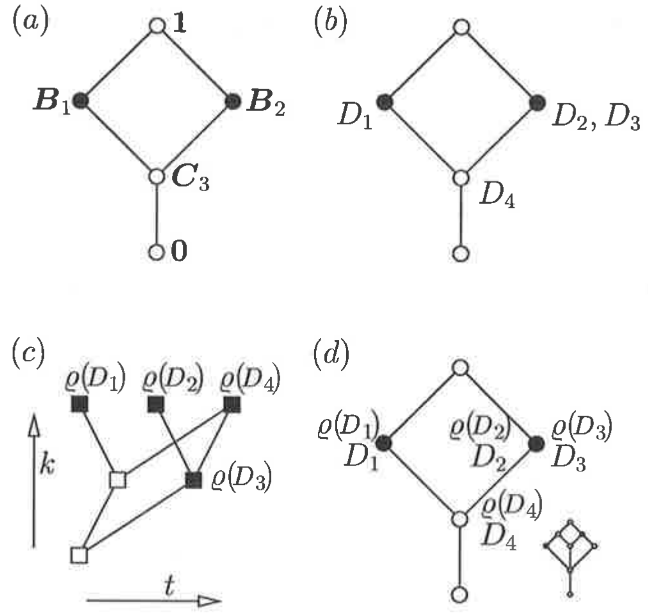


Figure 37. Bilattice truth-values of descriptions

$$V_D = \{\boldsymbol{B}_i \in \mathbb{B} \mid \boldsymbol{B}_i \vee \boldsymbol{T}_D = \mathbf{1}\}.$$

Points to note now are the following two. Firstly, for any $D \in \boldsymbol{D}$, we have that $U_D \cap V_D = \emptyset$, because worlds are consistent. Secondly, for any $D \in \boldsymbol{D}$, $U_D \cup V_D \subseteq \mathbb{T}$, and the containment is strict in all cases apart from $D = \ominus\{\}$, as $D = \ominus\{\}$ is the only description which entails—more precisely, $\ominus$-entails—all the formulae of $\boldsymbol{F}$).

Given the example of Figure 37, the truth values for the descriptions are presented in Table 8.

theory	theory's axioms	axioms's truth-value $\varrho(D) = (U_D, V_D)$
$\boldsymbol{B}_1$	$D_1 = \oplus\{p_1, p_2\}$	$(\{\boldsymbol{B}_1\}, \{\boldsymbol{B}_2\})$
$\boldsymbol{B}_2$	$D_2 = \ominus\{p_2\}$	$(\{\boldsymbol{B}_2\}, \{\boldsymbol{B}_1\})$
	$D_3 = \oplus\{p_1, \overline{p_2}\}$	$(\{\boldsymbol{B}_2\}, \emptyset)$
$\boldsymbol{C}_3$	$D_4 = \oplus\{p_1\}$	$(\{\boldsymbol{B}_1, \boldsymbol{B}_2\}, \emptyset)$

Table 8. Descriptions and their bilattice truth-values

Figure 37 (c) shows the bilattice-based truth values,[1] together with the bilattice orderings t (truth-ordering on truth-values) and k (information ordering). In Figure 37 (d), the truth values are placed next to the corresponding descriptions.

Following the bilattice approach, we can now consider truth and information ordering on truth-values, corresponding bilattice meet and join operations, and discuss *negation*. After doing this, we then try to extend the method of finding truth-values of descriptions to derive truth-values of theories.

[1]Only some of the truth-values are shown, c.f., Figure 4.

Let $(U_1,V_1),(U_2,V_2) \in \Gamma$. Then the truth ordering $\leq_t$ and the information ordering $\leq_k$ on Γ are given by $(U_1,V_1) \leq_t (U_2,V_2)$ iff $U_1 \subseteq U_2$ and $V_1 \supseteq V_2$ and $(U_1,V_1) \leq_k (U_2,V_2)$ iff $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$. Further, the bilattice operations $\wedge_t, \vee_t, \wedge_k, \vee_k$ are given by $(U_1,V_1) \wedge_t (U_2,V_2) = (U_1 \cap U_2, V_1 \cup V_2)$, $(U_1,V_1) \vee_t (U_2,V_2) = (U_1 \cup U_2, V_1 \cap V_2)$, $(U_1,V_1) \wedge_k (U_2,V_2) = (U_1 \cap U_2, V_1 \cap V_2)$, and $(U_1,V_1) \vee_k (U_2,V_2) = (U_1 \cup U_2, V_1 \cup V_2)$. The four distinguished elements 0_t, 1_t, 0_k, 1_k of the bilattice are $0_t = (\emptyset, S)$, $1_t = (S, \emptyset)$, $0_k = (\emptyset, \emptyset)$, and $1_k = (S, S)$.

Introducing a negation operator $\neg$ is interesting. Firstly, we need to define a negation operator on descriptions, $\neg\colon \boldsymbol{D} \longrightarrow \boldsymbol{D}$.

Definition 21 *The* negation operator *on descriptions* $\neg\colon \boldsymbol{D} \longrightarrow \boldsymbol{D}$ *is given by*

$$\neg D = \begin{cases} \ominus F & \text{if } D = \oplus F \\ \oplus F & \text{if } D = \ominus F \end{cases}$$

We can now formulate the following proposition.

Proposition 9 *Let $\boldsymbol{T} \in \mathbb{T}$ be a consistent theory and $D \in \boldsymbol{D}$ be a description. If $\boldsymbol{T}_i \cap \{D, \neg D\} = \emptyset$ then $\boldsymbol{T}_i \cup \{D\}$ is consistent. Thus, we also have that if $\boldsymbol{T}_i \vee \boldsymbol{T}_D = \mathbf{1}$ then $\boldsymbol{T}_i \ni \neg D$.*

An immediate consequence of the proposition is that if $\boldsymbol{T}_i \cap \{D, \neg D\} = \emptyset$ then $\boldsymbol{T}_i \cup \{D\}$ is consistent.

The bilattice approach requires the negation operator on truth values to satisfy $\neg(U,V) = (V,U)$. Nothing goes wrong here—the two negations agree in the following sense.

Proposition 10 *Let $\neg\colon \boldsymbol{D} \longrightarrow \boldsymbol{D}$ be given by Definition 21, and let $\neg\colon \Gamma \longrightarrow \Gamma$ be given by $\neg(U,V) = (V,U)$. Then $\neg\varrho(D) = \varrho(\neg D)$, where $\varrho(D) = (U_D, V_D)$.*

Equipped with the negation operator, we can also relate locations of descriptions in the lattice of theories to their locations in the bilattice of truth values. This is shown in Figure 38.

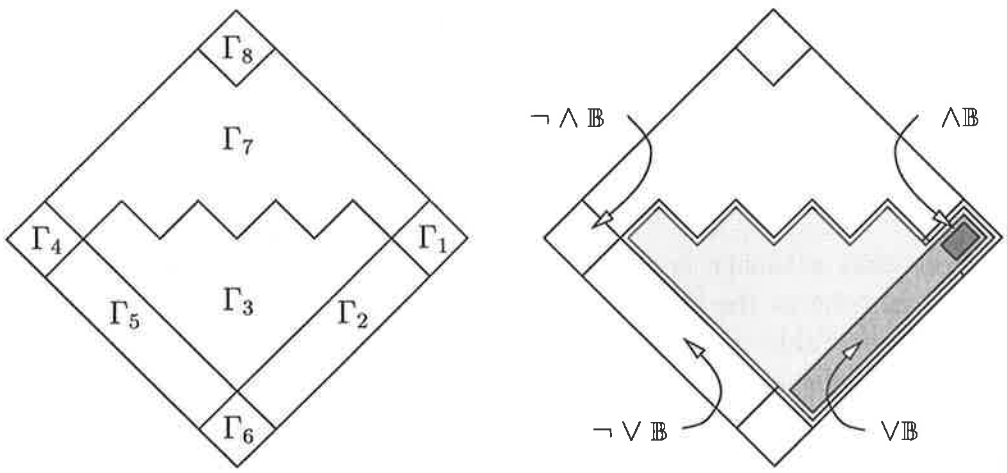


Figure 38. Sets of bilattice truth-values

Denote some subsets of the set Γ of truth values as follows. $\Gamma_1 = \{(U,V) \mid U = \mathbb{B}, V = \emptyset\}$, $\Gamma_2 = \{(U,V) \mid U \neq \emptyset, V = \emptyset\}$, $\Gamma_3 = \{(U,V) \mid U \neq \emptyset, V \neq \emptyset\}$, $\Gamma_4 = \{(U,V) \mid U = \emptyset, V = \mathbb{B}\}$, $\Gamma_5 = \{(U,V) \mid U = \emptyset, V \neq \emptyset\}$, $\Gamma_6 = \{(U,V) \mid U = \emptyset, V = \emptyset\}$, $\Gamma_7 = \{(U,V) \mid U \cap V \neq \emptyset\}$, and $\Gamma_8 = \{(U,V) \mid U = V = \mathbb{B}\}$, Then, if $\wedge\mathbb{B} > \mathbf{0}$ then the truth value of every description in $\wedge\mathbb{B}$ is in the (singleton) set Γ_1. If $\vee\mathbb{B} < \mathbf{1}$ then the truth value of any description in $\vee\mathbb{B} \setminus \wedge\mathbb{B}$ is in the set $\Gamma_2 \setminus \Gamma_1$. Negation reverses truth-values of descriptions in the sense that the truth value of the negated description is the negated truth value, i.e., it is the truth-value one obtains by reflection w.r.t. the vertical axis of Γ—in Figure 38 $\neg\wedge\mathbb{B}$ denotes the set of negations of the elements of $\wedge\mathbb{B}$. It is important to note that we are more interested in the right-hand side of the bilattice, so it is clearly undesirable to bother with elements of $\neg \wedge \mathbb{B}$. Indeed, in our lattices of theories, we consider the meet $\wedge\mathbb{B}$ of all the theories, but there is no point to explicitly deal with negations of descriptions *all* agents accept.

Let us now try to find truth values of theories, or sets of descriptions, rather than single descriptions. One can note that a single theory can contain several descriptions with different truth values. If this happens, then one can separate such descriptions by presenting a theory as a join of its two sub-theories (a small version of such modified lattice is included in Figure 37 (d)). However, what we need is truth-values on theories, rather than single descriptions. Modify the truth valuation function ϱ, so that it takes theories from $\mathbb{C}$ as its arguments, $\varrho : \mathbb{C} \longrightarrow \Gamma$, and let it be given by $\varrho(\boldsymbol{C}) = (U_{\boldsymbol{C}}, V_{\boldsymbol{C}})$, where $U_{\boldsymbol{C}}$ is a set of worlds where $\boldsymbol{C}$ is *true*, and $V_{\boldsymbol{C}}$ where it is *false*, and $U_{\boldsymbol{C}}$ and $V_{\boldsymbol{C}}$ are similar to U_D and V_D defined previously:

$$U_{\boldsymbol{C}} = \{\boldsymbol{B}_i \in \mathbb{B} \mid \boldsymbol{B}_i \geq \boldsymbol{C}\},$$

$$V_{\boldsymbol{C}} = \{\boldsymbol{B}_i \in \mathbb{B} \mid \boldsymbol{B}_i \vee \boldsymbol{C} = \mathbf{1}\}.$$

The resulting truth-values on theories are presented in Table 9.

theory in $\mathbb{C}_+$	theory's truth-value
$\boldsymbol{B}_1$	$(\{\boldsymbol{B}_1\}, \{\boldsymbol{B}_2\})$
$\boldsymbol{B}_2$	$(\{\boldsymbol{B}_2\}, \{\boldsymbol{B}_1\})$
$\boldsymbol{C}_3$	$(\{\boldsymbol{B}_1, \boldsymbol{B}_2\}, \emptyset)$
$\mathbf{0}$	$(\{\boldsymbol{B}_1, \boldsymbol{B}_2\}, \emptyset)$
$\mathbf{1}$	$(\emptyset, \{\boldsymbol{B}_1, \boldsymbol{B}_2\})$

Table 9. Theories and their truth-values

Note that a theory is at most as true as its theorems (descriptions), and at least as false as the descriptions are. The truth values of theories of $\mathbb{C}_+$ are presented in Table 9. Trivially, the empty theory $\mathbf{0}$ is true in every believed theory, and false in none, and the reverse applies to $\mathbf{1}$.

The problem however is that we must *not* be satisfied with truth-values of theories so obtained. Indeed, if we associate a propositional symbol β_i with a statement "the theory (under consideration) is *true* in $\boldsymbol{B}_i$" (i.e., it is below $\boldsymbol{B}_i$), then we would e.g., associate a sentence $\beta_1 \vee \beta_2$ with $\boldsymbol{C}_3$. Propositional models over $\{\beta_i\}_i$ are functions from $\{\beta_i\}_i$ to $\{true, false\}$, so j is a model if $j : \{\beta_i\}_i \longrightarrow \{true, false\}$, and let us use a convention that e.g., j_{01} denotes a model which satisfies $j(\beta_1) = false$ and $j(\beta_2) = true$. Then however, we would get $\{j_{11}, j_{10}, j_{01}\}$ as a set of models associated with $\boldsymbol{C}_3$—but firstly, no theory can be true in both $\boldsymbol{B}_1$ *and* $\boldsymbol{B}_2$ (because they contradict each other), so the model j_{11} needs to be excluded. Secondly, it is not obvious whether j_{00} should be excluded—maybe it is possible that both $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are "false," but $\boldsymbol{C}_3 = \boldsymbol{B}_1 \wedge \boldsymbol{B}_2$ nevertheless *is* "true"?

We propose a different method of obtaining *models* for theories. The reason that bilattice-based results are non-satisfactory is that world-based bilattices treat worlds as indistinguishable (see [Gin88], Sections 4 and 7). Our worlds (theories) are however structured, and in the process of deriving models for theories we should make use of this structure. This is addressed in Section 7.3.

7.3 Concept lattices of theories

In Section 7.2 we suggested that given a set of believed theories $\mathbb{B}$ the corresponding lattice $\mathbb{C}_+$ of theories could be seen as an FCA *concept lattice*. In this section it will be shown how such a concept lattice can be obtained. It was already said in Section 7.2 that descriptions would be employed as objects, to find extents of concepts, and it was clear that *truthness* of theories, or their *models* should be used to form concepts' intents (sets of attributes). Given that nodes of the lattice $\mathbb{C}_+$ are theories, one would also expect that *concepts* would simply be the *theories* of $\mathbb{C}_+$—this also provides some justification for calling the *worlds* of Section 7.2 *conceptual worlds*. The results of Section 7.2 were not however fully satisfactory. Although it was a step in the right direction, in this section we refine our view on truthness and models of theories.

An attempt to present a lattice $\mathbb{C}_+$ of theories as a concept lattice can be seen as a new formulation for the old enquiry—given a theory in $\mathbb{C}_+$, what is the theory's *informational value* and its *truthness*? Hence, our search for attributes is clearly aimed at finding *models* representing *truthness* of theories. It also seems that presenting $\mathbb{C}_+$ as a concept lattice would permit to use existing software packages for displaying concept lattices to display lattices of theories.

This section is structured as follows. Firstly, we recall the example of Section 7.2. Secondly, we show formally how to build a concept lattice of theories, and address some related issues. Finally, we provide a more involved example than the one we start with.

As was noted in Section 7.2, finding extents of concepts, or theories is unproblematic, and precise details will soon be provided. Finding intents is more complicated, but it will now be clarified. Consider the example of Section 7.2. Figure 39 is a slight refinement of the top part of Figure 37, and recall that the descriptions are given by Table 8.

As shown in Figure 39, we associate propositional symbols β_1 and β_2 with the believed theories $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, respectively. So called $\mathbb{B}$-models for theories will simply be propositional models over the propositional symbols $\{\beta_i\}_i$; for instance, it should be read from Figure 39 that the set of $\mathbb{B}$-models for $\boldsymbol{B}_1$ is

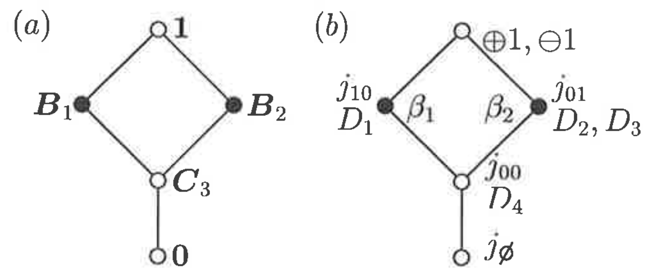


Figure 39. Theories and $\mathbb{B}$-models

$\{j_{10}\}$, while the set of models for $\boldsymbol{C}_3$ is $\{j_{00}, j_{10}\}$, where e.g., j_{10} is a propositional model over $\{\beta_1, \beta_2\}$, and it is a $\mathbb{B}$-model for $\boldsymbol{B}_1$ but not for $\boldsymbol{B}_2$ Furthermore, an additional attribute, denoted $j_\emptyset$, is assigned to the empty theory $\mathbf{0}$, to ensure that the concepts of $\mathbf{0}$ and $\boldsymbol{C}_3$ are distinct. We will now specify what the intended meaning of the propositional symbols $\{\beta_i\}_i$ is, and how to obtain $\mathbb{B}$-*models* for theories.

Let $\mathbb{E}$ be a set of total theories, i.e., $\mathbb{E}$ is a set of $\leq$-maximal elements of $\mathbb{T}$. We first propose how to find models for total theories, and then extend this to all theories. Considering our example of Figure 39 (a), the theories $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_3$ and $\mathbf{0}$ and their corresponding total theories, i.e., total theories above them, are presented in Figure 40.

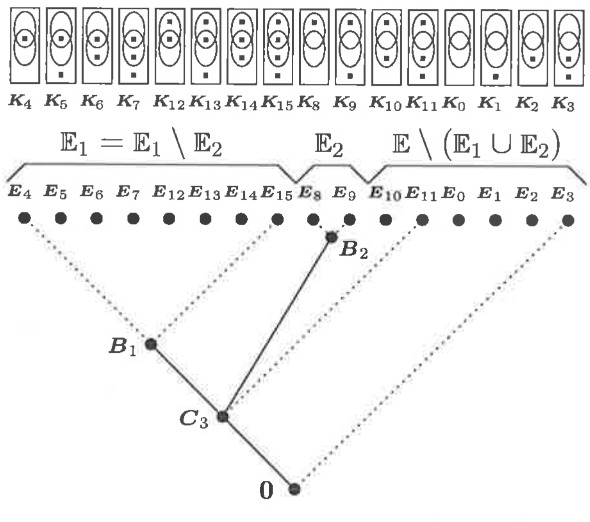


Figure 40. Total theories above $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_3$ and $\mathbf{0}$

In Figure 40, $\mathbb{E}$ is a set of all total theories (recall from Table 8 that P is a two element set $\{p_1, p_2\}$), $\mathbb{E}_1 = \{\boldsymbol{E}_4, \boldsymbol{E}_5, \boldsymbol{E}_6, \boldsymbol{E}_7, \boldsymbol{E}_{12}, \boldsymbol{E}_{13}, \boldsymbol{E}_{14}, \boldsymbol{E}_{15}\}$ is a set of total theories above $\boldsymbol{B}_1$, and $\mathbb{E}_2 = \{\boldsymbol{E}_8, \boldsymbol{E}_9\}$ are total theories above $\boldsymbol{B}_2$. If there are two believed theories, there are in general four sets to consider, namely $\mathbb{E}_1 \setminus \mathbb{E}_2$, $\mathbb{E}_2 \setminus \mathbb{E}_1$, $\mathbb{E}_1 \cap \mathbb{E}_2$ and $\mathbb{E} \setminus (\mathbb{E}_1 \cup \mathbb{E}_2)$, but in our example $\mathbb{E}_1 \cap \mathbb{E}_2$ is empty. (To be more precise about the total theories of Figure 40, let $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3$ and $\boldsymbol{g}_4$ be the objects of the contexts $\boldsymbol{K}_8.\boldsymbol{K}_4.\boldsymbol{K}_2$ and $\boldsymbol{K}_1$. Then the contexts $\boldsymbol{K}_0, \ldots, \boldsymbol{K}_{15}$ are all total contexts, where $\boldsymbol{K}_i$ is a total context, with the subscript i being a binary code for the appropriate function from $\{g_1, g_2, g_3, g_4\}$ to $\{0,1\}$. The theories $\boldsymbol{E}_0, \ldots, \boldsymbol{E}_{15}$ are theories of the contexts $\boldsymbol{K}_0, \ldots, \boldsymbol{K}_{15}$.)

concept (theory)	concept's extent	concept's intent
$\boldsymbol{B}_1$	$\{D_1, D_4\}$	$\{j_{10}\}$
$\boldsymbol{B}_2$	$\{D_2, D_3, D_4\}$	$\{j_{01}\}$
$\boldsymbol{C}_3$	$\{D_4\}$	$\{j_{10}, j_{01}, j_{00}\}$
$\mathbf{0}$	$\emptyset$	$\{j_{10}, j_{01}, j_{00}, j_{\emptyset}\}$
$\mathbf{1}$	$\{D_1, D_2, D_3, D_4, \oplus 1, \ominus 1\}$	$\emptyset$

Table 10. Theories as concepts

Let $\boldsymbol{E} \in \mathbb{E}$ be a *total theory* and $\boldsymbol{B}_i \in \mathbb{B}$ be a *believed theory*. We say that $\boldsymbol{E}$ is *possibly true* at $\boldsymbol{B}_i$ iff $\boldsymbol{B}_i \leq \boldsymbol{E}$, otherwise we say that $\boldsymbol{E}$ is *necessarily false* at $\boldsymbol{B}_i$—i.e., $\boldsymbol{E}$ is *necessarily false* at $\boldsymbol{B}_i$ iff $\boldsymbol{B}_i \not\leq \boldsymbol{E}$. Let now consider *all* theories, including *partial* theories, let $\boldsymbol{T} \in \mathbb{T}$ be an arbitrary consistent theory. We say that $\boldsymbol{T}$ is *possibly true* at $\boldsymbol{B}_i$ iff there is a total theory $\boldsymbol{E} \in \mathbb{E}$ such that $\boldsymbol{T} \leq \boldsymbol{E}$ and $\boldsymbol{E}$ is possibly true at $\boldsymbol{B}_i$. Similarly, we say that $\boldsymbol{T}$ is *necessarily false* at $\boldsymbol{B}_i$ iff there is no $\boldsymbol{E} \in \mathbb{E}$ such that $\boldsymbol{T} \leq \boldsymbol{E}$ and $\boldsymbol{E}$ is possibly true at $\boldsymbol{B}_i$. This is equivalent to the following (simply by referring back to the case of total theories)—$\boldsymbol{T}$ is *possibly true* at $\boldsymbol{B}_i$ iff there is $\boldsymbol{E} \in \mathbb{E}$ such that $\boldsymbol{T} \leq \boldsymbol{E}$ and $\boldsymbol{B}_i \leq \boldsymbol{E}$. Similarly, $\boldsymbol{T}$ is *necessarily false* at $\boldsymbol{B}_i$ iff there is no $\boldsymbol{E} \in \mathbb{E}$ s.t. $\boldsymbol{T} \leq \boldsymbol{E}$ and $\boldsymbol{B}_i \leq \boldsymbol{E}$. Let $\{\beta_i\}_i$ be a set of propositional symbols associated with $\{\boldsymbol{B}_i\}_i$ and let $M_\beta = \{j \colon \{\beta_i\}_i \longrightarrow \{\mathit{true}, \mathit{false}\}\}$ be a set of propositional models over $\{\beta_i\}_i$. Let $\boldsymbol{T} \in \mathbb{T}$—then $j \in M_\beta$ is a $\mathbb{B}$-model for $\boldsymbol{T}$ iff

$$j(\beta_i) = \begin{cases} \mathit{true} & \text{if } \boldsymbol{T} \text{ is possibly true at } \boldsymbol{B}_i, \\ \mathit{false} & \text{if } \boldsymbol{T} \text{ is necessarily false at } \boldsymbol{B}_i. \end{cases}$$

Applying this definition of a model to the theories of the example of Figure 39 (*a*), we find the models of the theories as shown in Figure 39 (*b*). Note that Figure 40 gives all we need to find the models.

Collecting the descriptions (objects forming extents of the theories) and models (attributes that give intents of the theories) we can see the theories as concepts being *(extent, intent)* pairs, see Table 10. This finishes our initial example, and we now formally summarise how concept lattices of theories are constructed.

Let $\mathbb{C}_+$ be a lattice of theories resulting from a set $\mathbb{B}$ of believed theories. We want to show that $\mathbb{C}_+$ is a *concept lattice*, i.e., that there is a *formal context* $K_{\mathbb{C}_+} = (G_{\mathbb{C}_+}, M_{\mathbb{C}_+}, I_{\mathbb{C}_+})$ such that $\underline{\mathfrak{L}}(K_{\mathbb{C}_+}) \cong \mathbb{C}_+$.

Consider *objects* first. Define $G_{\mathbb{C}_+}$ in two steps. Let $\boldsymbol{A}_{\mathbb{C}_-} = \bigcup_{\boldsymbol{C} \in \mathbb{C}_-} \operatorname{gen}(\boldsymbol{C})$, and then put $G_{\mathbb{C}_+} = \boldsymbol{A}_{\mathbb{C}_-} \cup \{\oplus 1, \ominus 1\}$. Hence, $G_{\mathbb{C}_+}$ is a set of objects of the desired context $K_{\mathbb{C}_+}$, and it just collects those descriptions that are axioms for at least some of the theories of $\mathbb{C}_+$. The elements $\oplus 1$ and $\ominus 1$ are added to ensure that the inconsistent theory $\mathbf{1}$, i.e., the top element of $\mathbb{C}_+$ will be a concept of the context $K_{\mathbb{C}_+}$.

Consider now *attributes.* Define $M_{\mathbb{C}_+}$ as follows. Let $\{\beta_i\}_i = \{\beta_1, \ldots, \beta_{n_1}\}$ be a set of propositional symbols associated with $\mathbb{B} = \{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_{n_1}\}$. Let $M_\beta = \{j \mid j : \{\beta_i\}_i \longrightarrow \{\mathit{true,false}\}$ is a total function$\}$. Further, let $\boldsymbol{K} \in \mathbb{K}$ be a context, and define $M_{\mathbb{B}}^{\boldsymbol{K}} = \{j \in M_\beta \mid j$ is a $\mathbb{B}-$model for $\boldsymbol{T_K}\}$. Then $M_{\mathbb{C}_+}$ is given by $\{j \in M_\beta \mid \exists_{\boldsymbol{K} \in \mathbb{K}} \ M_{\mathbb{B}}^{\boldsymbol{K}} \ni j\}$., plus $j_\emptyset$, added to ensure that the empty theory $\mathbf{0}$ is a concept.

To define $I_{\mathbb{C}_+}$, consider two mappings, $\xi_G : \mathbb{C}_+ \longrightarrow \mathcal{P}(G_{\mathbb{C}_+})$ and $\xi_M : \mathbb{C}_+ \longrightarrow \mathcal{P}(M_{\mathbb{C}_+})$. The mapping ξ_G is given by,

$$\xi_G(\boldsymbol{C}_i) = \begin{cases} \{A \in \boldsymbol{A}_{\mathbb{C}_-} \mid \exists_{A_i \in \boldsymbol{A}_i} \ A_i \geq A\} & \text{if } \boldsymbol{C}_i < \mathbf{1}, \\ G_{\mathbb{C}_+} & \text{if } \boldsymbol{C}_i = \mathbf{1}. \end{cases}$$

where $\boldsymbol{A}_i = \mathrm{gen}(\boldsymbol{C}_i)$. Regarding ξ_M, let $\xi_M(\boldsymbol{C})$ be the set of $\mathbb{B}$-models for $\boldsymbol{C}$, and hence we have that $\xi_M(\mathbf{0}) = M_{\mathbb{C}_+}$ and $\xi_M(\mathbf{1}) = \emptyset$.

Let now $m \in M_{\mathbb{C}_+}$ and $g \in G_{\mathbb{C}_+}$, and let $\boldsymbol{C}_g$ be the (unique) element of $\mathbb{C}_+$ such that $\mathrm{gen}(\boldsymbol{C}_g) \ni g$. We then define,

$$I_{\mathbb{C}_+}(g, m) = \begin{cases} 1 & \text{if } m \in \xi_M(\boldsymbol{C}_g), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have defined the context $K_{\mathbb{C}_+} = (G_{\mathbb{C}_+}, M_{\mathbb{C}_+}, I_{\mathbb{C}_+})$ corresponding to the lattice $\mathbb{C}_+$ of theories.

To show that $(\mathbb{C}_+, \leq) \cong (\underline{\mathcal{L}}(K_{\mathbb{C}_+}), \leq)$, i.e., that the two lattices are isomorphic, it is sufficient to note that if $\boldsymbol{C}_i \in \mathbb{C}_+$ is a theory, then $\boldsymbol{C}_i^* = (\xi_G(\boldsymbol{C}_i), \xi_M(\boldsymbol{C}_i))$ is the corresponding concept of the context $K_{\mathbb{C}_+}$.

It should be noted that finding the set of all possible $\mathbb{B}$-models can be non-trivial. In particular, regarding the model $j_{0\ldots0}$, i.e., the model which demonstrates that it is possible that all the believed theories are wrong, we can note that if $\wedge\mathbb{B} > \mathbf{0}$ then $j_{0\ldots0}$ is a $\mathbb{B}$-model for some theories of $\mathbb{C}_+$—at least for $\mathbf{0}$—but otherwise, finding it can be more demanding.

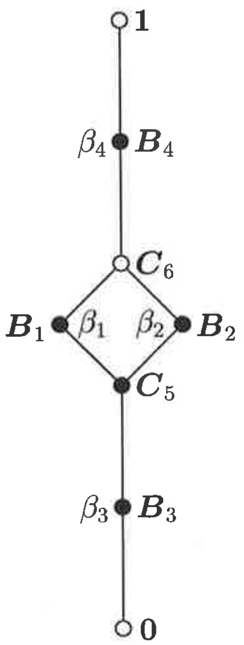
Consider now another example.

Example 5 *Let a set $\mathbb{B}$ of believed theories be $\mathbb{B} = \{\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3, \boldsymbol{B}_4\}$, where $\boldsymbol{B}_1 = \mathrm{Cn}(\oplus\{p_1, p_2\})$, $\boldsymbol{B}_2 = \mathrm{Cn}(\oplus\{p_1, p_3\})$, $\boldsymbol{B}_3 = \mathrm{Cn}(\oplus\{\})$ and $\boldsymbol{B}_4 = \mathrm{Cn}(\oplus\{p_1, p_2, p_3\})$.*

The lattice $\mathbb{C}_+$ resulting from $\mathbb{B}$ of Example 5 is given in the left part of Figure 41, and the figure demonstrates how the corresponding concept lattice can be found.

Given the concept lattice of theories shown in the right-hand side of Figure 41 it is possible to find a context for which the lattice is its concept lattice. The context is presented in Table 11.

It would be interesting to further explore the ideas presented in this section, and to implement appropriate algorithms in software. It is clear that, given a set of believed theories, finding all total theories quickly becomes intractable, but finding $\mathbb{B}$-models is much more feasible. Total theories and $\mathbb{B}$-models are examined again in Section 7.4, where numeric measures on theories are proposed.



	β_1	β_2	β_3	β_4
$\dot{j}_{0000}$	0	0	0	0
~~$\dot{j}_{0001}$~~	~~0~~	~~0~~	~~0~~	~~1~~
$\dot{j}_{0010}$	0	0	1	0
~~$\dot{j}_{0011}$~~	~~0~~	~~0~~	~~1~~	~~1~~
~~$\dot{j}_{0100}$~~	~~0~~	~~1~~	~~0~~	~~0~~
~~$\dot{j}_{0101}$~~	~~0~~	~~1~~	~~0~~	~~1~~
$\dot{j}_{0110}$	0	1	1	0
~~$\dot{j}_{0111}$~~	~~0~~	~~1~~	~~1~~	~~1~~
~~$\dot{j}_{1000}$~~	~~1~~	~~0~~	~~0~~	~~0~~
~~$\dot{j}_{1001}$~~	~~1~~	~~0~~	~~0~~	~~1~~
$\dot{j}_{1010}$	1	0	1	0
~~$\dot{j}_{1011}$~~	~~1~~	~~0~~	~~1~~	~~1~~
~~$\dot{j}_{1100}$~~	~~1~~	~~1~~	~~0~~	~~0~~
~~$\dot{j}_{1101}$~~	~~1~~	~~1~~	~~0~~	~~1~~
$\dot{j}_{1110}$	1	1	1	0
$\dot{j}_{1111}$	1	1	1	1

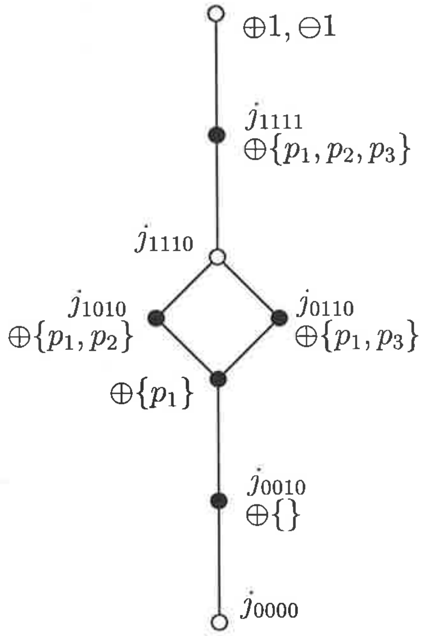


Figure 41. Forming a concept lattice of theories (Example 5)

	$\dot{j}_{0000}$	$\dot{j}_{0010}$	$\dot{j}_{1010}$	$\dot{j}_{0110}$	$\dot{j}_{1110}$	$\dot{j}_{1111}$
$\ominus 1$						
$\oplus 1$						
$\oplus\{p_1, p_2, p_3\}$						×
$\oplus\{p_1, p_3\}$				×	×	×
$\oplus\{p_1, p_2\}$			×		×	×
$\oplus\{p_1\}$			×	×	×	×
$\oplus\{\}$		×	×	×	×	×

Table 11. Formal context of the concept lattice of theories (Example 5)

7.4 Towards numeric measure

In this section we show how to derive *numeric measures* on theories.

Let $\mathbb{B} = \{\boldsymbol{B}_i\}_i$ be a set of believed theories. let $\mathbb{C} = \mathrm{Cl}(\mathbb{B})$ and $\mathbb{C}_+ = \mathbb{C} \cup \{\mathbf{0}, \mathbf{1}\}$. Let $\mathbb{E}$ be the set of all total theories, and let $\mathbb{E}_i = \mathbb{E} \cap \uparrow\boldsymbol{B}_i$ be the set of total theories above $\boldsymbol{B}_i$. Note that $\mathbb{E}_i = \{\boldsymbol{E} = \boldsymbol{\tau}(\boldsymbol{K}) \in \mathbb{E} \mid \boldsymbol{K} \in \mathrm{MOD}(\boldsymbol{B}_i)\}$, i.e., $\mathbb{E}_i$ is a set of (total) theories of *total models* of $\boldsymbol{B}_i$. Associate with $\boldsymbol{B}_i$ its *weight* $\varpi_i \in [0, 1]$, and let $\boldsymbol{\varpi} = (\varpi_i)_i$ denote all such weights. The weights can be seen as values given by a *weight function* $\varpi \colon \mathbb{B} \longrightarrow [0, 1], \varpi(\boldsymbol{B}_i) = \varpi_i$.

A single $\boldsymbol{B}_i$ gives its *measure* $\nu_i : \mathbb{E} \longrightarrow [0, 1]$, determined as follows: if $\boldsymbol{E}_a \in \mathbb{E}_i$ and $\boldsymbol{E}_b \in \mathbb{E} \setminus \mathbb{E}_i$ then $\nu_i(\boldsymbol{E}_a)/\nu_i(\boldsymbol{E}_b) = \varpi_i/(1 - \varpi_i)$—as a consequence we get that if $\boldsymbol{E}_{a_1}, \boldsymbol{E}_{a_2} \in \mathbb{E}_i$ then $\nu_i(\boldsymbol{E}_{a_1}) = \nu_i(\boldsymbol{E}_{a_2})$, and if $\boldsymbol{E}_{b_1}, \boldsymbol{E}_{b_2} \in \mathbb{E} \setminus \mathbb{E}_i$ then $\nu_i(\boldsymbol{E}_{b_1}) = \nu_i(\boldsymbol{E}_{b_2})$. Further, we require $\sum_{\boldsymbol{E} \in \mathbb{E}} \nu_i(\boldsymbol{E}) = 1$.

Given $\{\nu_i\}_i$, define $\boldsymbol{\nu} : \mathbb{E} \longrightarrow [0, 1]$ as follows—if $\boldsymbol{E} \in \mathbb{E}$ then $\boldsymbol{\nu}(\boldsymbol{E}) = (1/|\mathbb{B}|) \sum_{\boldsymbol{B}_i \in \mathbb{B}} \nu_i(\boldsymbol{E})$ (but notice that $\boldsymbol{\nu}$ depends on $\boldsymbol{\varpi}$).

An obvious problem is that $\boldsymbol{\nu}$ takes $\boldsymbol{\varpi}$ as its parameters, i.e., $\boldsymbol{\nu} = \boldsymbol{\nu}_{\boldsymbol{\varpi}}$. Hence the question—how can we determine weights in $\boldsymbol{\varpi}$, assuming that they are *not* given? As suggested by Kyburg in [Kyb94], a *maximum entropy principle* provides a solution—hence, we first recall the principle, and then formulate it for the case of our theories.

Assume that—given a problem—the set of possible answers is $\mathbb{E} = \{\boldsymbol{E}_i\}_i$. The *maximum entropy principle* says that given the problem, and the set of possible answers to the problem, we should assign prior probabilities to the answers in such a way as to maximise *entropy*. Let $\boldsymbol{\pi}\colon \mathbb{E} \longrightarrow [0,1]$ be a probability distribution on $\mathbb{E}$. Certainly, there are many probability distributions in the set $[0,1]^{\mathbb{E}}$ of functions. Then *entropy*, or *uncertainty of information*, is given by,

$$-\sum_{\boldsymbol{E}_i \in \mathbb{E}} \boldsymbol{\pi}(\boldsymbol{E}_i) \log \boldsymbol{\pi}(\boldsymbol{E}_i),$$

and the *maximum entropy principle* says that we should maximise entropy, i.e., we should select a probability distribution $\boldsymbol{\pi}^*$ that satisfies,

$$-\sum_{\boldsymbol{E}_i \in \mathbb{E}} \boldsymbol{\pi}^*(\boldsymbol{E}_i) \log \boldsymbol{\pi}^*(\boldsymbol{E}_i) = \max_{\boldsymbol{\pi}}(-\sum_{\boldsymbol{E}_i \in \mathbb{E}} \boldsymbol{\pi}(\boldsymbol{E}_i) \log \boldsymbol{\pi}(\boldsymbol{E}_i)).$$

Considering the entropy principle as applied to our theories, we get the following. If we are given a question "which total theory should be selected as the *correct* one?" then $\mathbb{E} = \{\boldsymbol{E}_i\}_i$ is the set of possible answers. Let $\boldsymbol{\nu}_{\boldsymbol{\varpi}} : \mathbb{E} \longrightarrow [0,1]$ be a probability distribution on $\mathbb{E}$. The point is that such probability distributions must satisfy the restrictions imposed by the *structure* of $\mathbb{C}_+$. Entropy is given by

$$-\sum_{\boldsymbol{E}_i \in \mathbb{E}} \boldsymbol{\nu}_{\boldsymbol{\varpi}}(\boldsymbol{E}_i) \log \boldsymbol{\nu}_{\boldsymbol{\varpi}}(\boldsymbol{E}_i),$$

and we should select a probability distribution $\boldsymbol{\nu}^*_{\boldsymbol{\varpi}}$ that maximises entropy. Hence, the maximising $\boldsymbol{\varpi}^*$, but clearly, to obtain $\boldsymbol{\nu}^*_{\boldsymbol{\varpi}}$ we first determine $\boldsymbol{\varpi}^*$—this accounts to maximising entropy on $\boldsymbol{\varpi}$,

$$\max_{\boldsymbol{\varpi}}(-\sum_{\boldsymbol{E}_i \in \mathbb{E}} \boldsymbol{\nu}_{\boldsymbol{\varpi}}(\boldsymbol{E}_i) \log \boldsymbol{\nu}_{\boldsymbol{\varpi}}(\boldsymbol{E}_i)),$$

and this gives $\boldsymbol{\varpi}^*$, which in turn determines $\boldsymbol{\nu}^*$.

It is clear how to extend $\boldsymbol{\nu}$ from $\mathbb{E}$ to $\mathbb{T}$, i.e., how to find $\boldsymbol{\nu}\colon \mathbb{T} \longrightarrow [0,1]$—if $\boldsymbol{T} \in \mathbb{T}$ then $\boldsymbol{\nu}(\boldsymbol{T}) = \sum_{\boldsymbol{E} \geq \boldsymbol{T}} \boldsymbol{\nu}(\boldsymbol{E})$, where $\boldsymbol{E} \in \mathbb{E}$, i.e., $\boldsymbol{E}$ is a total theory.

Consider now the example presented in Figure 42—this is the same example as already presented in the preceding two sections, cf., Figure 37 with descriptions given in Table 8.

We have $\mathbb{B} = \{\boldsymbol{B}_1, \boldsymbol{B}_2\}$, $\mathbb{C} = \mathrm{Cl}(\mathbb{B}) = \{\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{C}_3, \mathbf{1}\}$ and $\mathbb{C}_+ = \mathbb{C} \cup \{\mathbf{0}, \mathbf{1}\} = \{\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{C}_3, \mathbf{0}, \mathbf{1}\}$. Regarding total theories, refer back to Figure 40 where both total contexts and total theories are given—we have the following. The set of all total theories is $\mathbb{E} = \mathbb{E}_0 = \{\boldsymbol{E}_0, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{E}_4, \boldsymbol{E}_5, \boldsymbol{E}_6, \boldsymbol{E}_7, \boldsymbol{E}_8, \boldsymbol{E}_9, \boldsymbol{E}_{10}, \boldsymbol{E}_{11}, \boldsymbol{E}_{12}, \boldsymbol{E}_{13}, \boldsymbol{E}_{14}, \boldsymbol{E}_{15}\}$. The total theories above $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are $\mathbb{E}_1 = \{\boldsymbol{E}_4, \boldsymbol{E}_5, \boldsymbol{E}_6, \boldsymbol{E}_7, \boldsymbol{E}_{12}, \boldsymbol{E}_{13}, \boldsymbol{E}_{14}, \boldsymbol{E}_{15}\}$, and $\mathbb{E}_2 = \{\boldsymbol{E}_8, \boldsymbol{E}_9\}$. Note that $\mathbb{E}_1 \cup \mathbb{E}_2 =$

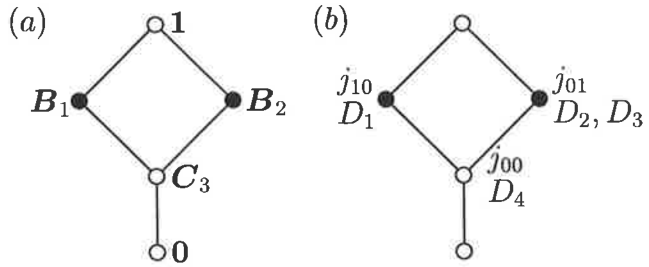


Figure 42. Towards numeric measure

$\{\boldsymbol{E}_4, \boldsymbol{E}_5, \boldsymbol{E}_6, \boldsymbol{E}_7, \boldsymbol{E}_8, \boldsymbol{E}_9, \boldsymbol{E}_{12}, \boldsymbol{E}_{13}, \boldsymbol{E}_{14}, \boldsymbol{E}_{15}\}$, but $\mathbb{E} \cap \boldsymbol{C}_3 = \{\boldsymbol{E}_4, \boldsymbol{E}_5, \boldsymbol{E}_6, \boldsymbol{E}_7, \boldsymbol{E}_8, \boldsymbol{E}_9, \boldsymbol{E}_{10}, \boldsymbol{E}_{11}, \boldsymbol{E}_{12}, \boldsymbol{E}_{13}, \boldsymbol{E}_{14}, \boldsymbol{E}_{15}\}$. With $\mathbb{B} = \{\boldsymbol{B}_1, \boldsymbol{B}_2\}$ associate *weights* $\varpi_1, \varpi_2 \in [0,1]$, and let $\boldsymbol{\varpi} = (\varpi_1, \varpi_2)$.

We can associate with $\mathbb{B} = \{\boldsymbol{B}_1, \boldsymbol{B}_2\}$ *measures* $\nu_1, \nu_2 \colon \mathbb{E} \longrightarrow [0,1]$. Consider the measure ν_1 first. Now, $\nu_1(\boldsymbol{E}_4) = \nu_1(\boldsymbol{E}_5) = \nu_1(\boldsymbol{E}_6) = \nu_1(\boldsymbol{E}_7) = \nu_1(\boldsymbol{E}_{12}) = \nu_1(\boldsymbol{E}_{13}) = \nu_1(\boldsymbol{E}_{14}) = \nu_1(\boldsymbol{E}_{15})$, and $\nu_1(\boldsymbol{E}_0) = \nu_1(\boldsymbol{E}_1) = \nu_1(\boldsymbol{E}_2) = \nu_1(\boldsymbol{E}_3) = \nu_1(\boldsymbol{E}_8) = \nu_1(\boldsymbol{E}_9) = \nu_1(\boldsymbol{E}_{10}) = \nu_1(\boldsymbol{E}_{11})$, and thus e.g., $\nu_1(\boldsymbol{E}_4)/\nu_1(\boldsymbol{E}_0) = \varpi_1/(1-\varpi_1)$. We also have that $8 \cdot \nu_1(\boldsymbol{E}_4) + 8 \cdot \nu_1(\boldsymbol{E}_0) = 1$. The last two equations give us the values of ν_1 on $\boldsymbol{E}_4$ and $\boldsymbol{E}_0$, namely, we get $\nu_1(\boldsymbol{E}_4) = \varpi_1/8$ and $\nu_1(\boldsymbol{E}_0) = (1-\varpi_1)/8$. This of course gives us the measure ν_1 on all total theories.

Similarly, considering the case of ν_2, we have $\nu_2(\boldsymbol{E}_8) = \nu_2(\boldsymbol{E}_9)$ and $\nu_2(\boldsymbol{E}_0) = \nu_2(\boldsymbol{E}_1) = \nu_2(\boldsymbol{E}_2) = \nu_2(\boldsymbol{E}_3) = \nu_2(\boldsymbol{E}_4) = \nu_2(\boldsymbol{E}_5) = \nu_2(\boldsymbol{E}_6) = \nu_2(\boldsymbol{E}_7) = \nu_2(\boldsymbol{E}_{10}) = \nu_2(\boldsymbol{E}_{11}) = \nu_2(\boldsymbol{E}_{12}) = \nu_2(\boldsymbol{E}_{13}) = \nu_2(\boldsymbol{E}_{14}) = \nu_2(\boldsymbol{E}_{15})$. Hence, $\nu_2(\boldsymbol{E}_8)/\nu_2(\boldsymbol{E}_0) = \varpi_2/(1-\varpi_2)$ and $2 \cdot \nu_2(\boldsymbol{E}_8) + 14 \cdot \nu_2(\boldsymbol{E}_0) = 1$. The resulting measure ν_2 takes the values $\nu_2(\boldsymbol{E}_8) = \varpi_2/(14-12\varpi_2)$ $\nu_2(\boldsymbol{E}_0) = (1-\varpi_2)/(14-12\varpi_2)$—this gives $\nu_2 \colon \mathbb{E} \longrightarrow [0,1]$.

Given $\{\nu_1, \nu_2\}_i$, define $\boldsymbol{\nu} : \mathbb{E} \longrightarrow [0,1]$ as follows. If $\boldsymbol{E} \in \mathbb{E}$ then $\boldsymbol{\nu}(\boldsymbol{E}) = \frac{1}{2}(\nu_1(\boldsymbol{E}) + \nu_2(\boldsymbol{E}))$. It is easy to see that $\boldsymbol{\nu}(\boldsymbol{E})$ depends on whether or not $\boldsymbol{E}$ belongs to $\mathbb{E}_1, \mathbb{E}_2$. Given a two element set $\mathbb{E} = \{\boldsymbol{E}_1, \boldsymbol{E}_2\}$, there are in general four sets to consider, these sets partition $\mathbb{E}$ into sets of total theories with the same measure, namely the sets $\mathbb{E}_1 \cap \mathbb{E}_2$, $\mathbb{E}_1 \setminus \mathbb{E}_2$, $\mathbb{E}_2 \setminus \mathbb{E}_1$ and $\mathbb{E} \setminus (\mathbb{E}_1 \cup \mathbb{E}_2)$. In our case, we have $\mathbb{E}_1 \cap \mathbb{E}_2 = \emptyset$ and hence there is no $\boldsymbol{E} \in \mathbb{E}_1 \cap \mathbb{E}_2$. For the remaining three sets we have the following. If $\boldsymbol{E} \in \mathbb{E}_1 = \mathbb{E}_1 \setminus \mathbb{E}_2$ then $\boldsymbol{\nu}(\boldsymbol{E}) = \frac{1}{2}(\varpi_1/8 + (1-\varpi_2)/(14-12\varpi_2))$, if $\boldsymbol{E} \in \mathbb{E}_2 = \mathbb{E}_2 \setminus \mathbb{E}_1$ then $\boldsymbol{\nu}(\boldsymbol{E}) = \frac{1}{2}((1-\varpi_1)/8 + \varpi_2/(14-12\varpi_2))$, and if $\boldsymbol{E} \in \mathbb{E} \setminus (\mathbb{E}_1 \cup \mathbb{E}_2)$ then $\boldsymbol{\nu}(\boldsymbol{E}) = \frac{1}{2}((1-\varpi_1)/8 + (1-\varpi_2)/(14-12\varpi_2))$. For instance, $\boldsymbol{\nu}(\boldsymbol{E}_4) = \frac{1}{2}(\varpi_1/8 + (1-\varpi_2)/(14-12\varpi_2))$, $\boldsymbol{\nu}(\boldsymbol{E}_8) = \frac{1}{2}((1-\varpi_1)/8 + \varpi_2/(14-12\varpi_2))$ and $\boldsymbol{\nu}(\boldsymbol{E}_0) = \frac{1}{2}((1-\varpi_1)/8 + (1-\varpi_2)/(14-12\varpi_2))$. Hence, we have a measure $\boldsymbol{\nu} \colon \mathbb{E} \longrightarrow [0,1]$, but recall that $\boldsymbol{\nu}$ still depends on $\boldsymbol{\varpi}$, i.e., we have $\boldsymbol{\nu}_{\boldsymbol{\varpi}} \colon \mathbb{E} \longrightarrow [0,1]$. We can now apply the maximum entropy principle to determine the maximising probability distribution $\boldsymbol{\nu}^*_{\boldsymbol{\varpi}}$. The entropy is given by,

$$\sum_{\boldsymbol{E}_i \in \mathbb{E}_1,\ \boldsymbol{E}_i \in \mathbb{E}_2,\ \boldsymbol{E}_i \in \mathbb{E} \setminus (\mathbb{E}_1 \cup \mathbb{E}_2)} -\boldsymbol{\nu}_{\boldsymbol{\varpi}}(\boldsymbol{E}_i) \log \boldsymbol{\nu}_{\boldsymbol{\varpi}}(\boldsymbol{E}_i),$$

which is equal to,

$$-8 \cdot \boldsymbol{\nu}(\boldsymbol{E}_4) \log \boldsymbol{\nu}(\boldsymbol{E}_4) - 2 \cdot \boldsymbol{\nu}(\boldsymbol{E}_8) \log \boldsymbol{\nu}(\boldsymbol{E}_8) - 6 \cdot \boldsymbol{\nu}(\boldsymbol{E}_0) \log \boldsymbol{\nu}(\boldsymbol{E}_0),$$

and is a function of $\boldsymbol{\varpi}$, namely
$-8 \cdot \frac{1}{2}(\varpi_1/8 + (1-\varpi_2)/(14-12\varpi_2)) \cdot \log \frac{1}{2}(\varpi_1/8 + (1-\varpi_2)/(14-12\varpi_2)) +$
$-2 \cdot \frac{1}{2}((1-\varpi_1)/8 + \varpi_2/(14-12\varpi_2)) \cdot \log \frac{1}{2}((1-\varpi_1)/8 + \varpi_2/(14-12\varpi_2)) +$
$-6 \cdot \frac{1}{2}((1-\varpi_1)/8 + (1-\varpi_2)/(14-12\varpi_2)) \cdot \log \frac{1}{2}((1-\varpi_1)/8 + (1-\varpi_2)/(14-12\varpi_2))$.
This gives entropy as a function of (ϖ_1, ϖ_2)—the function for $(\varpi_1, \varpi_2) \in [0.1, 0.9] \times [0.1, 0.9]$ is presented in Figure 43.

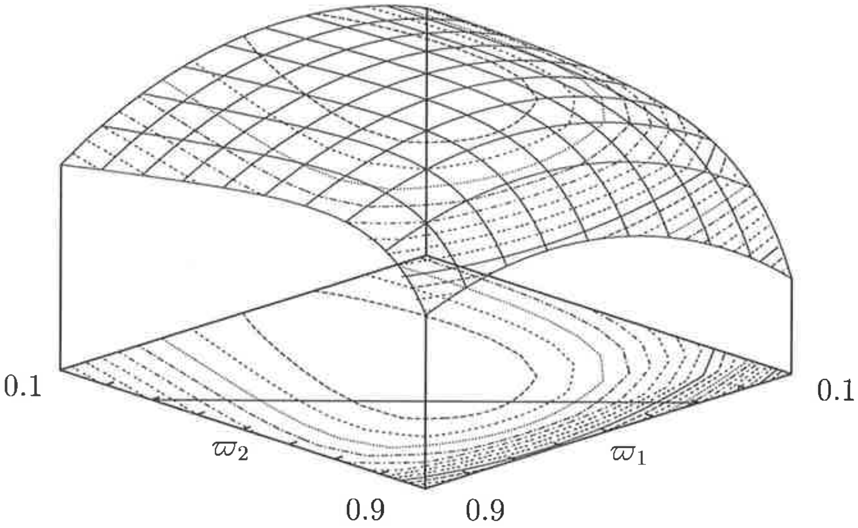


Figure 43. Entropy of $\boldsymbol{\nu}_{\boldsymbol{\varpi}}$

The shape of the entropy function is determined by the restrictions imposed by $\mathbb{C}_+$. There are usually other restrictions on (ϖ_1, ϖ_2) one might be willing to impose, for instance, it seems reasonable to assume that an average weight (confidence level, source's reliability, or trustworthiness) is above 0.5. Suppose we decide to assume that $(\varpi_1 + \varpi_2)/2 \geq 0.6$—the horizontal line drawn in Figure 43 selects the appropriate fragment of the domain. A more precise picture on the entropy function is given in Figure 44

In Figure 44 the domain is taken to be $[0.55, 0.65] \times [0.55, 0.65]$, as the entropy function takes its maximum there. In our example, the function—given the accepted restriction on average weight—reaches maximum at $((\varpi_1, \varpi_2) = (0.585, 0.615)$. Table 12 gives the values of $\boldsymbol{\nu}^*$ on total theories, and on theories in $\mathbb{C}_+$—recall that a measure of a theory is simple a sum of the measures of all the total theories above the given theory.

total theory	measure $\boldsymbol{\nu}^*(\boldsymbol{E})$	theory in $\mathbb{C}$	measure $\boldsymbol{\nu}^*(\boldsymbol{C})$
$\boldsymbol{E}_4$	0.06564105	$\boldsymbol{B}_1$	0.52512840
$\boldsymbol{E}_8$	0.07238765	$\boldsymbol{B}_2$	0.14477530
$\boldsymbol{E}_0$	0.05501605	$\boldsymbol{C}_3$	0.77993580

Table 12. Measure $\boldsymbol{\nu}^*$ on theories

An obvious drawback of this method is that one needs to find all consistent theories, and this is usually expensive. We now consider an alternative.

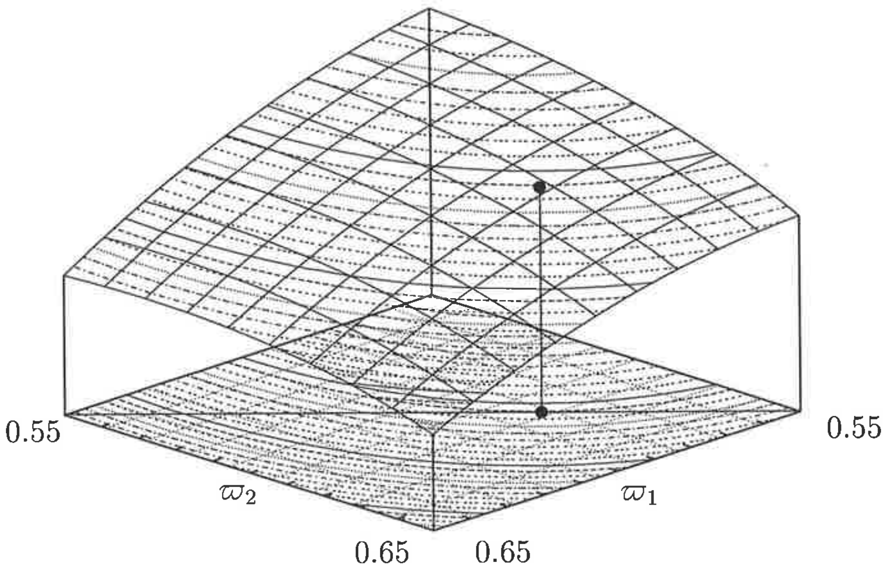


Figure 44. Maximising entropy

Consider Figure 45—we continue with the same example.

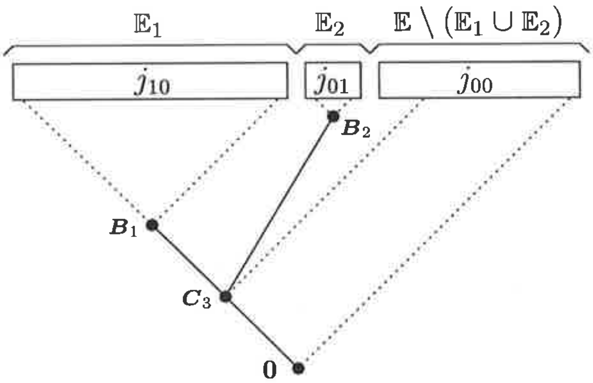


Figure 45. $\mathbb{B}$-models above $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_3$ and $\mathbf{0}$

Figure 45 presents the theories of $\mathbb{C}_+$ and their $\mathbb{B}$-models, which "replace" the total theories above the theories of $\mathbb{C}_+$—cf., Figure 40. It is evident that $\mathbb{B}$-models represent equivalence classes of total theories.[2] In an exactly analogous way as with $\boldsymbol{\nu}$ we now obtain a measure $\boldsymbol{\eta}$ on $\mathbb{B}$-models. We get $\eta_1(j_{10}) = \varpi_1/(2-\varpi_1)$ and $\eta_1(j_{01}) = \eta_1(j_{00}) = (1-\varpi_1)/(2-\varpi_1)$. Similarly, $\eta_2(j_{01}) = \varpi_2/(2-\varpi_2)$ and $\eta_2(j_{10}) = \eta_1(j_{00}) = (1-\varpi_2)/(2-\varpi_2)$. Then $\boldsymbol{\eta} = (\eta_1 + \eta_2)/2$ giving the entropy as a function of $\boldsymbol{\varpi}$, is presented in Figure 46.

The symmetry of $\boldsymbol{\eta}_{\boldsymbol{\varpi}}$ w.r.t. $\boldsymbol{\varpi}$ is not surprising—$\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ of our example have the same number of $\mathbb{B}$-models. Because of the symmetry we get—assuming the same restriction on the average weight—that the entropy function reaches its maximum at $((\varpi_1, \varpi_2) = (0.6, 0.6)$. Table 13 gives the values of $\boldsymbol{\eta}^*$ on total theories, and on theories in $\mathbb{C}_+$.

[2]Recall that total theories can be mapped to total contexts, or total models, in a *1–1 onto* fashion.

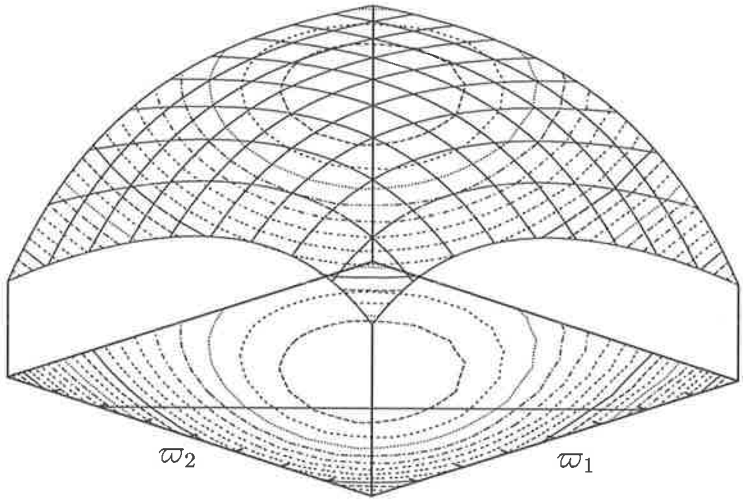


Figure 46. Entropy of η_{ϖ}

$\mathbb{B}$-model	measure $\eta^*(j)$	theory in $\mathbb{C}$	measure $\eta^*(C)$
j_{10}	0.35714286	B_1	0.35714286
j_{01}	0.35714286	B_2	0.35714286
j_{00}	0.28571428	C_3	1.00000000

Table 13. Measure η^* on $\mathbb{B}$-models and theories

There are two comments in place. Firstly, the second measure η avoids the difficulty of dealing with all consistent theories (although $\mathbb{B}$-models need to be found)—this is important regarding implementability. But secondly, the two measures would usually give different results—it is not obvious which measure is the *right* one, and although it seems that total theories provide a good starting point for measure related considerations, it invites to treat all total theories in a uniform manner, even though some total theories might be very different from the believed theories of agents. Nevertheless, numeric measures seem interesting, and it would be desirable to explore this possibility more extensively, and relate the numeric ordering the measures imply to the partial ordering given by the information ordering $\leq$ on theories.

7.5 Preference and epistemic states

Given a set $\mathbb{B}$ of believed theories, the lattice $\mathbb{C}_+$ can be seen as an *epistemic state.* Further, the partial order $\leq$ and numeric measures of Section 7.4 can be employed to derive a preference relation on theories.

Given the information ordering $\leq$ on theories, let $\preceq_k \; = \leq$ be a $\leq$-related *information ordering preference relation* on theories. A corresponding $\leq$-related *truthness preference relation* $\preceq_t$ would be determined by a subset relation on $\mathbb{B}$-models of theories, i.e., if $C_1, C_2 \in \mathbb{C}_+$ and the sets of $\mathbb{B}$-models of C_1, C_2 are J_1, J_2 respectively, then $C_2 \preceq_t C_1$ iff $J_2 \subseteq J_1$.

Let $\boldsymbol{\nu}$ be a numeric measure on theories. Then $\sqsubseteq_t$ defined as $\boldsymbol{C}_2 \sqsubseteq_t \boldsymbol{C}_1$ iff $\boldsymbol{\nu}(\boldsymbol{C}_2) \leq \boldsymbol{\nu}(\boldsymbol{C}_1)$ would be a $\boldsymbol{\nu}$-related *truthness preference relation.* Further, if $\boldsymbol{\nu}$ is a probability measure (numeric measure) on theories, then $-\log \boldsymbol{\nu}$ is usually considered to be an information measure—recall considerations of Section 7.4—so, if $\boldsymbol{\nu}(\boldsymbol{C})$ is a probability (numeric measure) of $\boldsymbol{C}$, then $-\log \boldsymbol{\nu}(\boldsymbol{C})$ is its information measure. Hence, a $\boldsymbol{\nu}$-related *information preference relation*, denoted $\sqsubseteq_k$, would be defined as $\boldsymbol{C}_1 \sqsubseteq_k \boldsymbol{C}_2$ iff $-\log \boldsymbol{\nu}(\boldsymbol{C}_1) \leq -\log \boldsymbol{\nu}(\boldsymbol{C}_2)$.

Let $\preceq$ represent the $\leq$-related preference relations $\preceq_t$ and $\preceq_k$. Let $\sqsubseteq$ represent the $\boldsymbol{\nu}$-related preference relations $\sqsubseteq_t$ and $\sqsubseteq_k$. The set of believed theories $\mathbb{B}$ can be seen as an *epistemic state*, given that the theories resulting from description sets we obtain from the agents have already been found. Given that $\mathbb{B}$ determines the lattice $\mathbb{C}_+$ of theories, and that the preference relations $\preceq$ and $\sqsubseteq$ can be derived, it seems appropriate to see the triple $(\mathbb{C}_+, \preceq, \sqsubseteq)$ as the epistemic state resulting from $\mathbb{B}$.

The preference relation $\sqsubseteq$, although attractive and deserving further research, can be perceived as problematic, so let us assume that the preference relation we have at our disposal is $\preceq$. Given $\preceq$, we can decide which theory or description[3] to prefer (recall however that $\preceq$ is a partial order, so some theories might be non-comparable). It can also be noted that the preference relation on theories allows us to consider *similarity* between theories, and the related idea of *systems of spheres.* We omit details here, but some preliminary discussions on these issues can be found in [NE94a, NE94b, Now96a, Now96b]. The idea of similarity spheres imposed on worlds comes from Mormann [Mor92], where so called *combinatorial worlds* he considers are standard FCA contexts. In our case, similarity spheres would be imposed on *theories* of *(partial) abstract contexts.* In Mormann's paper, similarity spheres are derived by employing so called *critical pairs* of attributes. The point now is that firstly, critical pairs can be obtained from the preference relation on descriptions, and secondly, Mormann assumes that the source we obtain critical pairs from is "total science," but we are unwilling to accept such an assumption. We derive our critical pairs from (our preference relation derived from) the information we get from the agents. Additionally, critical pairs are preferences on $\ominus$-valid formulae, and worlds are modified by "suspending" such descriptions in an order specified by the critical pairs. Given that descriptions include $\ominus$-valid formulae and $\oplus$-valid formulae, be can also modify worlds by operating (suspending, adding) descriptions which are $\oplus$-valid ($\oplus$-provable) formulae. More detailed exposition is left to future research.

[3] Given a preference relation on theories, one can introduce a preference relation on single descriptions by associating descriptions with theories they imply, i.e., D would be associated with $\mathrm{Cn}(\{D\})$.

Chapter 8

Conclusion

In Section 8.1 a short summary is provided. Section 8.2 addresses further research, namely it considers dynamic, rather than static worlds, and binary, rather than unary, relations. Section 8.3 identifies references to related work other than presented in Chapter 2.

8.1 Summary

This thesis provides a framework for reasoning about partial abstract contexts, and applies the framework to deal with information provided by multiple agents.

The logical framework defines simple semantic structures—these are a variant of FCA contexts. A proof-theoretic part of the framework is provided. A set of sentences valid in a context forms a set of axioms and, given a set Φ of simple inference rules, the corresponding formal system gives a method of finding all formulae which are guaranteed to be valid in the context. The analysis of the framework includes relating theories to contexts.

Given an agent, it is assumed that the agent communicates information about a context by providing a set of sentences valid in the context. Such a set of sentences is called a description set, and it is used as a set of axioms for the corresponding formal system, to derive the theory induced by the description set. The analysis of multiple agents provides methods for deriving preference relations on theories. There is no doubt that the issue of preference is crucial in common sense reasoning, where information can be partial, contradictory and uncertain. The approach taken in this work is standard, in the sense that preference relations operate on theories, and hence, on consistent (and closed under Φ) sets of sentences. Semantically, contexts, or worlds, are standard in the same sense. Deriving a preference relation thus permits selecting best, or most preferred theories, or (sets of) worlds.

Although the thesis is self-contained, and complete in the sense that it both provides the logical framework and specifies how most preferred theories can be selected, there are many issues that were not addressed in this work. Comments on future research and some related work form Sections 8.2 and 8.3.

8.2 Future work

There are many issues that deserve a more detailed examination. In this section we consider including time into the framework, and extending from attributes (unary relations) to binary relations; additionally, we remark on implementing the framework. Some other related research issues are listed in Section 8.3.

8.2.1 Dynamic worlds

In our search for preferred theories, the contexts (worlds) the theories describe are *static*—time is not involved, so worlds do not change. As a result, if two theories contradict one another then the theories describe two different contexts, rather then describing the same context that has changed. Recall that if a preference relation on theories is derived, then it captures informational value and truthness of the theories. As theories come from agents, the preference relation permits evaluating informational value and the *trustworthiness* of agents. If worlds are *dynamic*, then the informational value and trustworthiness changes with time. [Then, if a given agent provides contradictory information at different time points, there are two possibilities—either the world has changed, or the agent has moved (in the believed world; this covers the case of revising beliefs). Keeping track of the agent's performance, and observing how the most preferred world changes, one has to decide whether the agent's trustworthiness (level of performance) persists, or the world itself persists.]

Presenting partial theories of dynamic worlds is similar to presenting multiple theories of static worlds—but this time, the theories describe temporal snapshots of a single, dynamic world, rather than multiple, temporally unrelated worlds. Hence, most of the results carry over to the dynamic world case, and all that is needed is to temporally order the information obtained about the world.

Let α_0 be a discrete, linear, unbounded in both directions ordered set of time points, i.e., α_0 is isomorphic to the set of integers. Then α_i is a *time interval* if it is a convex subset of α_0, and so a time interval α_i is a sequence of consecutive time points, $\alpha_i = \{t_i^-, t_i^- + 1, \ldots, t_i^+\}$ with the inherited ordering, and t_i^- and t_i^+ are the endpoints of α_i.

A dynamic world can be seen as a sequence $(W_t)_t$, where t is a time moment and W_t is the world at time t, or as a sequence $(W_\alpha)_\alpha$, where α is a (maximal) time interval at which the world undergoes no change—the set of intervals is a temporally ordered sequence of meeting intervals. Then, validity is defined as follows. We say that φ holds at t_i in the dynamic world $(W_t)_t$, denoted $(W_t)_t \models \varphi @ t_i$, just in case when $W_{t_i} \models \varphi$. Employing intervals, φ holds at α_i (i.e., in every time point of the interval α_i) in the dynamic world $(W_\alpha)_\alpha$, denoted $(W_\alpha)_\alpha \models \varphi @ \alpha_i$, just in case when $W_{\alpha_i} \models \varphi$.

Therefore, description sets of dynamic worlds consists of formulae of the following form: $\varphi @ t$, $\varphi @ \alpha$, $D @ t$ and $D @ \alpha$, where t is a time point, α is a time interval, and $\varphi @ \alpha = \{\varphi @ t \mid t \in \alpha\}$, $D @ t = \{\varphi @ t \mid \varphi \in D\}$, and $D @ \alpha = \{\varphi @ \alpha \mid \varphi \in D\}$. We usually employ formulae of the form $D @ \alpha$, as this also covers the case of single formulae (with a singleton set D) and single time points (with a singleton interval α).

Let a set $\{D_i \,@\, \alpha_i\}_i$ describe a (single) dynamic world. Then a theory of the dynamic world is formed as follows. Firstly, for every D_i we compute the corresponding theory $A_i = \mathrm{Cn}(D_i)$, and hence we obtain $\{A_i \,@\, \alpha_i\}_i = \{\mathrm{Cn}(D_i) \,@\, \alpha_i\}_i$. Secondly, so far there are no restrictions on the intervals $\{\alpha_i\}_i$, in particular they can overlap—but if α_1 and α_2 overlap and in the dynamic world the formulae $A_1 \,@\, \alpha_1$ and $A_2 \,@\, \alpha_2$ are valid in the world, then so is the formula[1] $A_1 \vee A_2 \,@\, \alpha_1 \cap \alpha_2$—therefore, it is desirable to find intersections of intervals and consider a sequence of meeting intervals.

Let $\{\alpha_i\}_i$ be a set of intervals. Then $\{\alpha_i\}_i = \{\{t_i^-, \ldots, t_i^+\}\}_i$, where t_i^- and t_i^+ are endpoints of the interval α_i. Let $\{b_j\}_j = \bigcup_i \{t_i^-, t_i^+\}$, i.e., $\{b_j\}_j$ collects all the endpoints of the intervals $\{\alpha_i\}_i$. As $\{b_j\}_j$ is a subset of a linearly ordered set α_0, so it can itself be linearly ordered—let $(b_j)_j$ be such an ordered sequence of time points. The sequence $(b_j)_j$ determines the sequence[2] of meeting intervals, $(\beta_j)_j = (\{b_j, \ldots, b_{j+1}\})_j$. It is the intervals in $(\beta_j)_j$ that interest us, and all that remains to be done is to find theories that hold at those intervals. Given that a β_j is an intersection of some intervals from $\{\alpha_i\}_i$, i.e., $\beta_j = \bigcap_k \alpha_k$, the theory that holds at β_j is a join of the theories that hold at the intervals $\{\alpha_k\}_k$. Summarising, if a description set of a dynamic world contains $\{A_k \,@\, \alpha_k\}_k$ and $\beta_j = \bigcap_k \alpha_k$ then the dynamic theory (logical consequences of the description set) of the world contains the formula $B_j \,@\, \beta_j = \bigvee_k A_k \,@\, \bigcap_k \alpha_k$.

Consider an example,[3] where a description set consists of formulae of the form $D \,@\, \alpha$—suppose that the description set is $\{D_1 \,@\, \alpha_1, \ldots, D_5 \,@\, \alpha_5\}$. Firstly, we compute logical consequences $A_1 = \mathrm{Cn}(D_1), \ldots, A_5 = \mathrm{Cn}(D_5)$ of the descriptions $D_1, \ldots, D_5$, so we obtain $\{A_1 \,@\, \alpha_1, \ldots, A_5 \,@\, \alpha_5\}$. The bottom half of Figure 47 shows the intervals $\alpha_1, \ldots, \alpha_5$ and the theories $A_1, \ldots, A_5$ (the theories $A_1, \ldots, A_5$ are marked with the filled circles). The endpoints of $\alpha_1, \ldots, \alpha_5$ determine a sequence of temporally ordered meeting intervals $\beta_1, \ldots, \beta_5$, and the joins of the appropriate theories from $\{A_1, \ldots, A_5\}$ determine the theories $B_1, \ldots, B_5$ that hold at the intervals $\beta_1, \ldots, \beta_5$. The intervals $\beta_1, \ldots, \beta_5$ and the theories $B_1, \ldots, B_5$ are presented in the upper half of Figure 47. [The thick links between nodes (theories) in the rightmost lattice of the upper half of Figure 47 shows how the dynamic theory of the world evolves through the temporal sequence of meeting intervals.]

Some comments are now in place. Firstly, when a dynamic world is considered, theories holding at meeting intervals describe snapshots of the world. Given several sources of information, and multiple temporal descriptions of the sources' dynamic worlds, multiple dynamic theories result—this would require combining the results of this section with the results of Chapter 7.

Secondly, temporal information is not always precise [BCT95]. When *precise absolute times* are used as time stamps, then validity intervals have precise endpoints on the timeline. *Imprecise absolute times* are used to express that $\{t_1, \ldots, t_2\}$ is the validity interval, but the timepoints t_1 and t_2 are only known to

---

[1] $A_1 \vee A_2$ is a join of the theories A_1 and A_2, while $\alpha_1 \cap \alpha_2$ is an intersection of the (convex) intervals α_1 and α_2.

[2] Of course, the number of meeting intervals is smaller by one than the number of endpoints.

[3] To instantiate the example, assume that the description sets and theories involved in the current example are those of Figure 34(d) of Section 7.1.

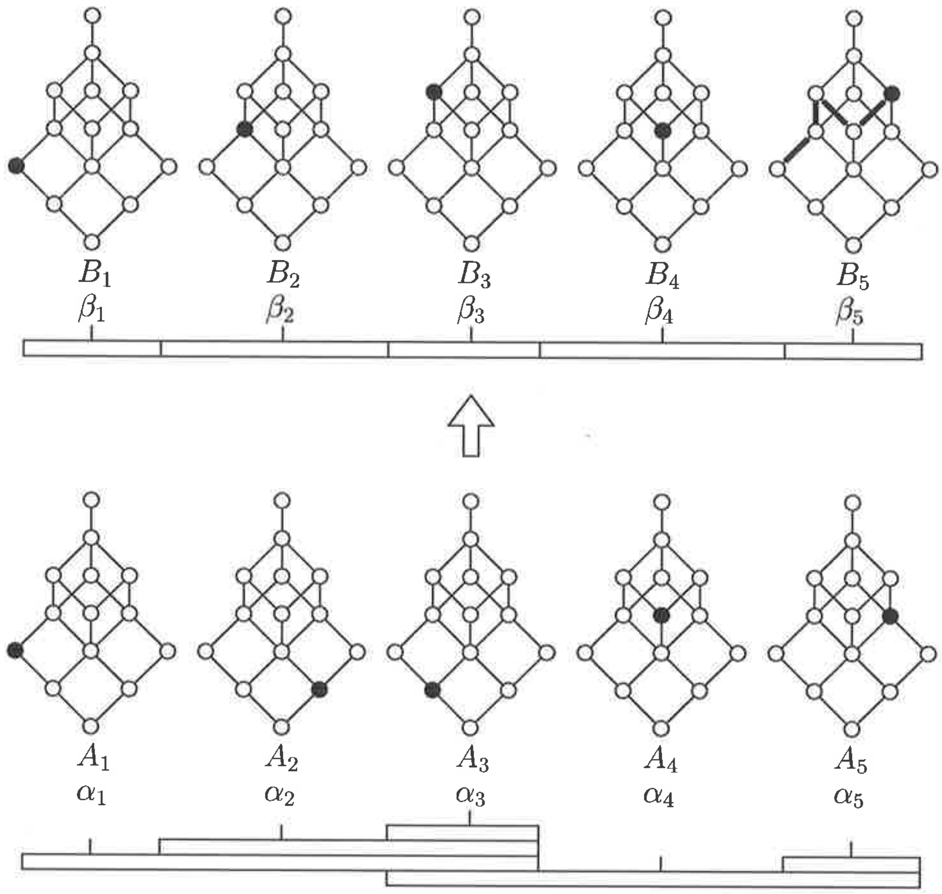


Figure 47. Finding a theory of a dynamic world

belong to some specified intervals. Finally, some algebras have been proposed to deal with *qualitative times*, e.g., Allen's interval algebra [All83]. The presented framework can be applied in all cases, including the case of only qualitative information about the validity intervals; for instance, the dynamic theory presented in Figure 47 can be obtained from purely qualitative information about the relationships between the intervals, namely, to obtain the dynamic theory of Figure 47 it suffices to know that α_3 `ends` α_2, α_2 `ends` α_1, α_3 `starts` α_4, α_5 `ends` α_4 and α_3 `precedes` α_5.

Thirdly, information might be not only temporally incomplete, but some data might be missing. For instance, in [TCG+93] *partial temporal elements* are introduced to represent the fact that at some time intervals validity of a formula is not known. Given a formula φ, a partial temporal element for φ is a pair (l, u), where $l \subseteq u$ and it is known that φ holds at l but does not hold outside of u, i.e., the validity of φ in $u \setminus l$ is unknown. In our framework, we would have a theory $\mathrm{Cn}(\{\neg\varphi\})$ valid outside of u, a theory $\mathrm{Cn}(\{\varphi\})$ valid at l, and an empty theory at $u \setminus l$ where the theory drops in the information hierarchy because of the missing data. It seems that the best method to reflect the fact that some data is missing is to employ partial theories ordered by their information contents.

8.2.2 Binary relations

It is clear that if only *properties* of objects are considered, then the corresponding language is less expressive than a language that permits representation of *relations* between objects. Consider Figure 48.

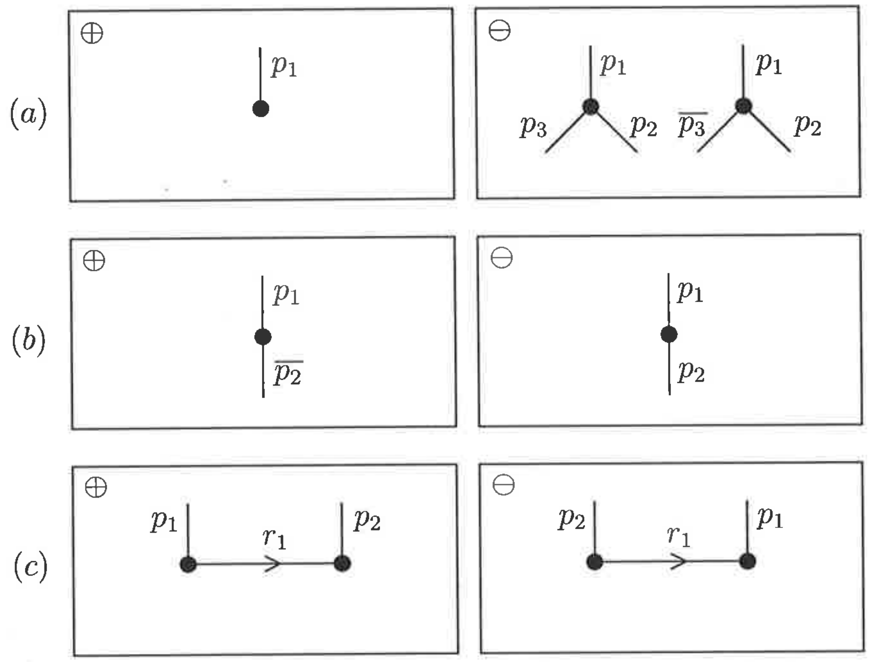


Figure 48. From properties to binary relations

Let Figure 48 (a) represent ⊕- and ⊖-formulae of the description set $\boldsymbol{D}_s$ of Example 3. Let Figure 48 (b) represent the resulting generator $\boldsymbol{A}_s$ of the theory obtained from $\boldsymbol{D}_s$. Hence, formulae presented on the left hand side of the figure (⊕-formulae) can be identified with abstract objects, but the right hand side of the figure shows those objects which are "prohibited." Suppose now that all we know about the world forms Figure 48 (c)—there is an object that has p_1 and is in a relation r_1 to an object (possibly the same object) that has p_2, but we also know that r_1 must not go in the opposite direction, because, as the right hand side of the figure shows, there must be no object with p_2 being in relation r_1 to an object with p_1.

It should be noted that in the binary relation case—cf. Figure 48 (c)—the ⊕- and ⊖-descriptions are collections of graphs (collections of sets of points with attribute links, but also with binary links between some of the points). This is different to the unary relation case, where the descriptions are single, separated points, with attribute links, but with no links between the points.

The resulting difficulty is that in the binary relation case, descriptions would not correspond to objects, or specify those objects that are present in the context, but would need to be partial descriptions of the graphs, and hence collections of objects could be involved in a single description. In the binary relation case, it is not difficult to give (essentially syntactic) world descriptions in terms of collections of graphs, but providing partial, *language formulated* descriptions that approximate graph-expressed descriptions is troublesome. It seems that a solution might require making provisional assumptions (guesses), that the objects we talk about are those that have been talked about most recently, unless a proliferation of graphs is allowed. These issues require more detailed analysis that can be offered here.

[Figure 48 invites some further comments—these are left for Section 8.3.]

8.2.3 Implementation

Although the framework of *conceptual reasoning* (CR), as presented in this thesis was developed with an implementation in mind, implementation efforts were not seen as a significant aspect of the performed research. It was understood that formal analysis of appropriate mathematical structures was more important than software engineering efforts.

Nevertheless, a skeleton implementation in a scripting language Tcl/Tk has been attempted, with a graphical user interface, as shown in Figure 49, and some functions for performing desired operations implemented in C, under the Unix operating system (given the interface, to perform a desired function one needs to click on the corresponding button, as the buttons' labels indicate). This initial Tcl/Tk implementation was also ported to Java, to allow access via internet—porting from Tcl/Tk to Java is a trivial programming exercise.

The aim of this section is not to report on an implemented system, as this was not intended, but rather demonstrate our interest in an eventual implementation and suggest what one might expect from it.

The implementation of CR is named "Cora"—see Figure 49. The top level window contains (apart from the *info* and *quit* buttons) a button labelled $\boldsymbol{B}_i$, and a button labelled $\mathbb{B}_j$; the buttons are used two invoke two other windows, a $\boldsymbol{B}_i$ window (ascii-titled `BB`), and a $\mathbb{B}_j$ window (ascii-titled `BBB`).

The $\boldsymbol{B}_i$ window is used to perform operations on a single description set, more precisely, given a description set $\boldsymbol{D}_i$ (provided by an agent, or source of information) the corresponding theory $\boldsymbol{T}_i = \mathrm{Cn}(\boldsymbol{D}_i)$, the generator $\boldsymbol{A}_i = \mathrm{gen}(\boldsymbol{T}_i)$, and the model $\boldsymbol{\mathcal{K}}_i = \boldsymbol{\kappa}(\boldsymbol{T}_i)$, as investigated in Chapters 4–6, can be computed. The top-row buttons $\boldsymbol{D}_i$, $\boldsymbol{A}_i$, $\boldsymbol{T}_i$, $\boldsymbol{\mathcal{K}}_i$ of the window $\boldsymbol{B}_i$ invoke the functions of reading a description set (from a file), computing the generator, the theory, and the model, respectively. The second row buttons allow to open editors with the corresponding files. The third row buttons are used to graphically display the corresponding sets of (provable) formulae/theorems—cf. Figure 21 of Section 5.1 (when magnified, the icons employed in $\boldsymbol{B}_i$ actually show $\oplus$- and $\ominus$-provable formulae of the description set, generator and theory of Example 3, displayed in Figure 21, Section 5.1). The bottom row buttons allow to visualise $\oplus$- and $\ominus$-provable formulae as "objects" and "counter-objects"—cf. Figure 48 of Section 8.2.2.

For instance, when employing Example 3 of Section 5.1, clicking on the buttons labelled $\boldsymbol{D}_i$, $\boldsymbol{A}_i$, and $\boldsymbol{T}_i$, would result in computing (lines of output are shown) the following sets:

```
{+{p1},-{p1,p2,p3},-{p1,p2,-p3}},

{+{p1,-p2},+{p1},+{-p2},+{},-{p1,p2},-{p1,p2,p3},-{p1,p2,-p3}},
```

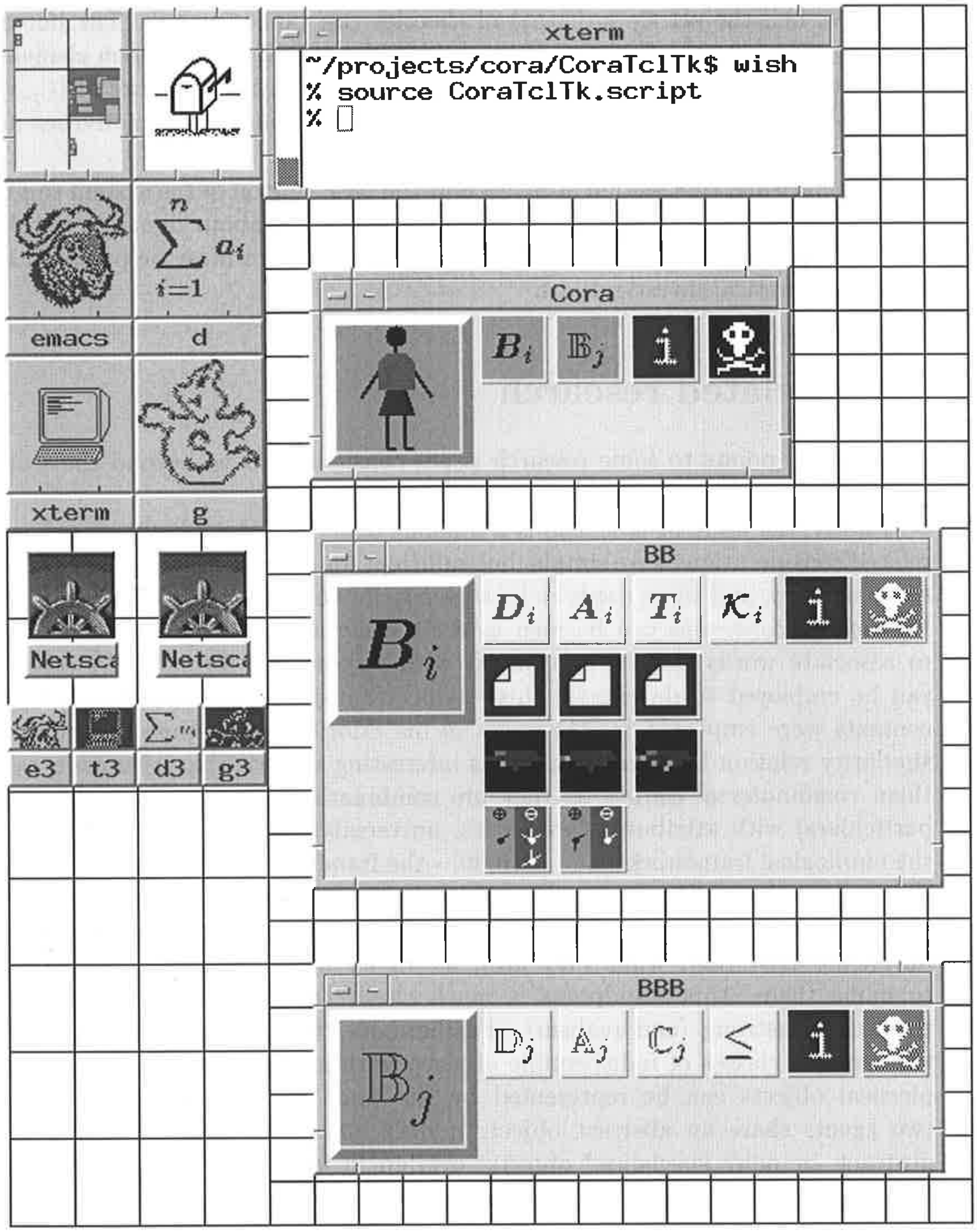


Figure 49. Cora, a Tcl/Tk implementation of CR

```
{+{p1,-p2},-{p1,p2}},
```

respectively.

The $\mathbb{B}_j$ window is used to perform operations on multiple description sets. More precisely, given a set $\mathbb{D}_j = \{\boldsymbol{D}_{i,j}\}_j$ of description sets (provided by multiple agents) the corresponding set $\mathbb{A}_j = \{\text{gen}(\text{Cn}(\boldsymbol{D}_{i,j}))\}_j$ of generators of the resulting theories can be computed. [Assume that in the multiple agents case we limit ourselves to generators of theories.] Given the set $\mathbb{A}_j$, its closure under

$\wedge$ and $\vee$, i.e., the set $\mathbb{C}_j = \mathrm{Cl}(\mathbb{A}_j)$ of theories can be computed. The button labelled $\leq$ invokes the function of computing the ordering relation on elements of the set $\mathbb{C}_j$, and (when implemented) to show the lattice structure of $(\mathbb{C}_j, \leq)$ (the task of drawing lattices is expected to be laborious). Such activities are theoretically investigated in Chapter 7.

Summarising, this section sketches how the base version of the system can be structured. However, the issues of temporal reasoning (about dynamic worlds) and binary relations are left aside, as implementing them must be preceded by further theoretical investigations.

8.3 Related research

This section points to some research issues considered to be beyond the scope of the thesis.

Firstly, the issue of *language* is a difficult one—c.f. [Qui60]. Our language-related considerations were simple, but sufficient for our purposes. Language is a social activity, and there needs to be a *society* for the language to be developed. A *language consensus* can be seen as a generally accepted agreement on how to associate words (labels, attributes) with objects. A formal FCA context can be employed to do exactly this—associate objects with attributes. Such contexts were employed by Mormann in his [Mor92], to propose a structural similarity relation between them. It is interesting to note that Mormann calls them *combinatorial worlds*, as they are combinations of objects (individuals, particulars) with attributes (properties, universals). This suggests *realism* as the ontological framework—c.f., [Arm89]—the framework that requires not only particulars (objects) but also universals (properties, relations) to exist. It seems that we take a less demanding approach—we ignore the issue whether or not universals exist (and what they are), as all we need are objects and words to name them—this standpoint is much closer to Quine (and nominalism) than to Armstrong (and realism). Furthermore, we employ abstract objects to represent classes of indiscernible objects, with a consequence that different physical objects can be represented by the same abstract object—hence, if two agents share an abstract object, it does not mean their worlds (of less abstract, or more specialised objects) overlap. Clearly, our understanding of abstract objects agrees with the usual one—abstract objects might lack space-time location, [Hal87].

Furthermore, our abstract contexts permit the representation of partial objects, and such non-standard objects as fictional, future and nonexistent objects. There is no doubt that existence assumptions limit representational capabilities of knowledge representation frameworks—cf., [Hir91]—hence it is important to avoid imposing unnecessary ontological restrictions (while at the same time keeping the set of objects of interest small, by employing partial objects). There is no consensus on whether *existence* is a property [Hir91, Dju96], and how to deal with nonexistent objects [Par80]—but as it is easy to talk about them in natural language, so it seems that a knowledge representation formalism should also have such capabilities. This suggests an *intensional*—rather than extensional—approach. The idea of employing Meinong's theory of objects [Mei60], and other related issues are mentioned in [Now94]. Meinongian semantics is proposed in [Rap85] and employed to provide semantics for the SNePS formalism [SR87]. However, revising beliefs in SNePS is not satisfactory—the user intervention is required to deal with contradictions [SR87].

Referring to Figure 48 of Section 8.2, the figure not only connects our approach to the SNePS formalism, but also to *terminological reasoning*, and more precisely to the *hybrid representation formalism*, see [Neb90]—indeed, the left and right hand sides of Figure 48 (*c*) resemble A-Box and T-Box of the hybrid terminological framework. However, terminological logics, now called *description logics*, have an extensional semantics, and hence do not seem appropriate to deal with *partial* objects.

Consider multiple agents, and hence multiple theories. Given a set of theories it is natural to look at their meets and joins—this is for instance done by Levi in [Lev91], but his exposition is not a detailed one. (Note that his notation is different, as he is primarily concerned with *truth* rather than *informational value*—hence, **1** is the top element of the lattice, it is judged error-free, and represents a maximum ignorance state. However, Levi assumes that **1** is a consequence of all other "potential states" and hence does not need to be empty and corresponds to our $\bigwedge \mathbb{B}$, rather than **0**. Subsequently, his **0**—strongest potential state that has every other state as a consequence—corresponds to our inconsistent theory **1**.)

Ordered sets of theories connect to Ginsberg's world-based bilattices, as discussed in Section 7.2. It is interesting to note that Popper's *verisimilitude* is closely related to world-based bilattice truth-values—for precise definitions of verisimilitude, and a related discussion, see Mortensen's [Mor83]. Mortensen also comments on a connection between the theory of verisimilitude and Lewis's theory of counterfactuals [Lew73] and a measure of *similarity* on worlds. There is a question whether Popper's definition can be used to define Lewis's similarity measure. Note that we have attempted to employ the bilattice approach to address the issue of *truthness* of theories, but then introduced $\mathbb{B}$-models in Section 7.3. Employing *systems of spheres* is suggested in Section 7.5, but is seen as a step towards modifying epistemic states, rather than evaluating theories' nearness to the *truth*. Much more work is needed to clarify these issues.

Regarding the mathematical tools employed in the thesis, there is a place for improvement, too. For instance, lattices were mainly used, but in several cases a top element was added to the structure to make it a lattice—some alternatives are provided in Scott [Sco82]. Related mathematical structures include *intersection structures*, *domains* and *information systems*, all thoroughly discussed in [DP90].

The logical framework itself also deserves more attention. It is claimed that the logic is exactly what was needed—for instance, *negation* is not introduced until Section 7.2, where bilattices are introduced, but then it is shown there that, given a nonempty $\bigwedge \mathbb{B}$, it is better to ignore negations of the elements of $\bigwedge \mathbb{B}$. Clearly, our logic is multi-valued—recall two notions of validity, namely $\oplus$- and $\ominus$-validity. It should be noted that multi-value logics were recommended by Belnap in [Bel77, Bel76] as appropriate for reasoning by computers. An interesting exposition of multi-valued logics is given in [Urq86]. *Partial logics* are in turn treated in detail in [Bla86]. *Partial information states* are important from our perspective—a good survey is provided in [Doh91].

Regarding epistemological standpoint, and relating to AGM *belief revision* framework [AGM85] and *truth-maintenance systems* [Doy79], neither pure coherence theory, nor foundations theory is accepted. Theories and contexts correspond to *consistent epistemic states*, but multiple of these are employed, and hence contradictions are not simply removed. Although we derive a preference relation, no information is treated as an absolutely certain, and hence there are no foundational beliefs. It should be noted that an *epistemic entrenchment* relation [GM88] moves AGM beyond pure coherence—c.f., Doyle's [Doy92]. Regarding systems of spheres mentioned in Section 7.5, the idea has been applied to AGM—see e.g., [Gro88, Spo88].

Finally, the issue of *preference* is an interesting one, see e.g., [Sho88] which is an attempt to unify nonmonotonic formalisms employing the notions of preference and rational decision making. However, as it is pointed out in [DW89], Arrow's result about the impossibility of universal social choice rules is an impediment to such a unification. It seems that preference causes no trouble in this work, but more detailed investigations seem desired. There also is a significant amount of work on *preference structures*—see [RV85] for a survey.

The above is a non-exhaustive list of related research issues that could result in improving the work presented here. However, it seems that even in its current shape, the work provides a perspective on how one can deal with partial and contradictory information coming from multiple agents.

Appendix A

Proofs

Proposition 2 Let $1_{\boldsymbol{G}}$ be such that $\forall_{g \in \boldsymbol{G}}\ \boldsymbol{g} \leq 1_{\boldsymbol{G}}$. Then $(\boldsymbol{G} \cup \{1_{\boldsymbol{G}}\}, \leq)$ is a lattice.

Proof To show that $\boldsymbol{G} \cup \{1_{\boldsymbol{G}}\}$ is a lattice define two algebraic operations $\wedge, \vee$: $\boldsymbol{G} \cup \{1_{\boldsymbol{G}}\} \times \boldsymbol{G} \cup \{1_{\boldsymbol{G}}\} \longrightarrow \boldsymbol{G} \cup \{1_{\boldsymbol{G}}\}$. Let $F_1, F_2 \in \boldsymbol{F}$ and let:

$$\boldsymbol{g}_{F_1} \wedge \boldsymbol{g}_{F_2} = \boldsymbol{g}_{F_1 \cap F_2}$$

$$\boldsymbol{g}_{F_1} \vee \boldsymbol{g}_{F_2} = \begin{cases} \boldsymbol{g}_{F_1 \cup F_2} & \text{if } F_1 \cup F_2 \in \boldsymbol{F} \\ 1_{\mathbf{G}} & \textit{otherwise} \end{cases}$$

$$\boldsymbol{g}_{F_1} \wedge 1_{\mathbf{G}} = \boldsymbol{g}_{F_1}$$

$$\boldsymbol{g}_{F_1} \vee 1_{\mathbf{G}} = 1_{\mathbf{G}}$$

Then $(\boldsymbol{G} \cup \{1_{\boldsymbol{G}}\}, \wedge, \vee)$ is a lattice, because $\wedge$ and $\vee$ are inf and sup, respectively. Indeed, $\boldsymbol{g}_{F_1} \wedge \boldsymbol{g}_{F_2} = \inf\{\boldsymbol{g}_{F_1}, \boldsymbol{g}_{F_2}\}$ (because $F_1 \cap F_2 = \inf(F_1, F_2)$, w.r.t. $\subseteq$) and $\boldsymbol{g}_{F_1} \vee \boldsymbol{g}_{F_2} = \sup\{\boldsymbol{g}_{F_1}, \boldsymbol{g}_{F_2}\}$ (because if $F_1 \cup F_2 \in \boldsymbol{F}$ then $F_1 \cup F_2 = \sup(F_1, F_2)$, in $\boldsymbol{F}$, w.r.t. $\subseteq$, else the element $1_{\boldsymbol{G}}$ is employed). Obviously, $\boldsymbol{g}_{F_1} = \inf\{\boldsymbol{g}_{F_1}, 1_{\boldsymbol{G}}\}$ and $1_{\boldsymbol{G}} = \sup\{\boldsymbol{g}_{F_1}, 1_{\boldsymbol{G}}\}$, because $\boldsymbol{g}_{F_1} \leq 1_{\boldsymbol{G}}$ (by the definition of $1_{\boldsymbol{G}}$). Hence, $(\boldsymbol{G} \cup \{1_{\boldsymbol{G}}\}, \leq)$ is a lattice. ■

Proposition 3 Let s be an agent, and G_s be a set of believed objects of the agent. Let $\boldsymbol{G}_s$ be a set of corresponding abstract objects. Let $\boldsymbol{D}_{\boldsymbol{G}_s}$ and $\boldsymbol{D}_{G_s}$ be descriptions of the world of s in the language $\mathbb{L}$, as implied by the sets G_s and $\boldsymbol{G}_s$, respectively. Then $\boldsymbol{D}_{\boldsymbol{G}_s} = \boldsymbol{D}_{G_s}$.

Proof Given a fixed M, let $\boldsymbol{F}$ be the set of corresponding formulae and let $\boldsymbol{R}$ be the set of regions—recall that $\varepsilon \colon \boldsymbol{F} \longrightarrow \boldsymbol{R}$. All we need to do is to visit every region and check whether or not it is empty—while visiting the region we *mark* the corresponding formula. Let $\alpha = |P|$. For every $\iota = \alpha, \ldots, 0$ the set $\boldsymbol{F}^{(\iota)} \subseteq \boldsymbol{F}$ is the set of formulae of cardinality ι. The following procedure generates the corresponding description sets $\boldsymbol{D}_{\boldsymbol{G}_s}$ and $\boldsymbol{D}_{G_s}$, while also producing the set of corresponding abstract objects $\boldsymbol{G}_s$.

1. None of the formulae in $\boldsymbol{F}$ is marked (none of the regions has been visited).
2. Set $\mathbf{D}^{\oplus}_{\mathbf{G}_s} = \mathbf{D}^{\oplus}_{G_s} = \emptyset$, $\mathbf{D}^{\ominus}_{\mathbf{G}_s} = \mathbf{D}^{\ominus}_{G_s} = \emptyset$, $\boldsymbol{G}_s = \emptyset$.
3. For every $\iota = \alpha$ downto 0, perform (ι):

(ι) For every un-marked $F \in \boldsymbol{F}^{(\iota)}$, consider $\varepsilon(F)$ and perform at most one of the two cases:

- It is known that $\varepsilon(F) \neq \emptyset$. Perform $(a_\oplus$–$d_\oplus)$:
  - $(a_\oplus)$ Add $\oplus F$ to $\mathbf{D}^{\oplus}_{G_s}$.
  - $(b_\oplus)$ Add $\oplus F$ to $\mathbf{D}^{\oplus}_{\mathbf{G}_s}$.
  - $(c_\oplus)$ Add $\boldsymbol{g}_F$ to $\boldsymbol{G}_s$.
  - $(d_\oplus)$ For every F_1 that satisfies $F_1 < F$ perform $(da_\oplus$–$dc_\oplus)$:
    - $(da_\oplus)$ Add $\oplus F_1$ to $\mathbf{D}^{\oplus}_{G_s}$.
    - $(db_\oplus)$ Add $\oplus F_1$ to $\mathbf{D}^{\oplus}_{\mathbf{G}_s}$.
    - $(dc_\oplus)$ Mark F_1.
- It is known that $\varepsilon(F) = \emptyset$. Perform $(a_\ominus$–$b_\ominus)$:
  - $(a_\ominus)$ Add $\ominus F$ to $\mathbf{D}^{\ominus}_{G_s}$.
  - $(b_\ominus)$ Add $\ominus F$ to $\mathbf{D}^{\ominus}_{\mathbf{G}_s}$.

The procedure generates $\mathbf{D}_{G_s}$ and $\mathbf{D}_{\mathbf{G}_s}$. Evidently, $\mathbf{D}_{G_s} = \mathbf{D}_{\mathbf{G}_s}$. ■

Proposition 4 Let $\mathbb{T}$ be the set of all consistent theories equipped with the information ordering relation $\leq$. Extend the ordered set $\mathbb{T}$ by adding an additional element—denoted $\mathbf{1}$—and extending the ordering relation by requesting that for all $\boldsymbol{T} \in \mathbb{T}$ it holds that $\boldsymbol{T} \leq \mathbf{1}$. Then $(\mathbb{T} \cup \{\mathbf{1}\}, \leq)$ is a *lattice.*

Proof Let $\mathbb{T}_1 = \mathbb{T} \cup \{\mathbf{1}\}$. Define two algebraic operations $\wedge, \vee \colon \mathbb{T}_1 \times \mathbb{T}_1 \longrightarrow \mathbb{T}_1$. Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in \mathbb{T}$ and let:

$$\boldsymbol{T}_1 \wedge \boldsymbol{T}_2 = \boldsymbol{T}_1 \cap \boldsymbol{T}_2,$$

$$\boldsymbol{T}_1 \vee \boldsymbol{T}_2 = \begin{cases} \mathrm{Cn}(\boldsymbol{T}_1 \cup \boldsymbol{T}_2) & \text{if } \boldsymbol{T}_1 \cup \boldsymbol{T}_2 \text{ is consistent,} \\ \mathbf{1} & \textit{otherwise,} \end{cases}$$

$$\boldsymbol{T}_1 \wedge \mathbf{1} = \boldsymbol{T}_1,$$

$$\boldsymbol{T}_1 \vee \mathbf{1} = \mathbf{1}.$$

Then $(\mathbb{T}_1, \wedge, \vee)$ is a lattice, because $\wedge$ and $\vee$ are inf and sup, respectively. Indeed, $\boldsymbol{T}_1 \wedge \boldsymbol{T}_2 = \inf\{\boldsymbol{T}_1, \boldsymbol{T}_2\}$ (because the ordering relation $\leq$ is simply a set-theoretic subset relation $\subseteq$, and $\boldsymbol{T}_1 \cap \boldsymbol{T}_2 = \inf\{\boldsymbol{T}_1, \boldsymbol{T}_2\}$ w.r.t. $\subseteq$) and $\boldsymbol{T}_1 \vee \boldsymbol{T}_2 = \sup\{\boldsymbol{T}_1, \boldsymbol{T}_2\}$ (because if $\boldsymbol{T}_1 \vee \boldsymbol{T}_2 = \mathrm{Cn}(\boldsymbol{T}_1 \cup \boldsymbol{T}_2) \in \mathbb{T}$ then $\boldsymbol{T}_1 \vee \boldsymbol{T}_2 = \mathrm{Cn}(\boldsymbol{T}_1 \cup \boldsymbol{T}_2) = \sup(\boldsymbol{T}_1, \boldsymbol{T}_2)$, in $\mathbb{T}$, w.r.t. $\subseteq$, else the element $\mathbf{1}$ is employed). Obviously, $\boldsymbol{T}_1 = \inf\{\boldsymbol{T}_1, \mathbf{1}\}$ and $\mathbf{1} = \sup\{\boldsymbol{T}_1, \mathbf{1}\}$, because $\boldsymbol{T}_1 \leq \mathbf{1}$ (by the definition of $\mathbf{1}$). Hence, $(\mathbb{T}_1, \leq)$ is a lattice. ■

Proposition 5 Let $(\mathbb{K}, \leq)$ and $(\mathbb{T}, \leq)$ be the sets of contexts and theories equipped with the corresponding information orderings, and let $\boldsymbol{\tau}$ be the mapping from $\mathbb{K}$ to $\mathbb{T}$. Let $\boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathbb{K}$ and let $\boldsymbol{T}_1 = \boldsymbol{\tau}(\boldsymbol{K}_1), \boldsymbol{T}_2 = \boldsymbol{\tau}(\boldsymbol{K}_2) \in \mathbb{T}$. We have that if $\boldsymbol{K}_1 \leq \boldsymbol{K}_2$ then $\boldsymbol{T}_1 \leq \boldsymbol{T}_2$.

Proof Let $\boldsymbol{G}_1, \boldsymbol{G}_2$ be the sets of objects of $\boldsymbol{K}_1, \boldsymbol{K}_2$. Let $\boldsymbol{K}_1 \leq \boldsymbol{K}_2$ and let $\oplus F \in \boldsymbol{T}_1$. Then, there is $F_1 \supseteq F$ such that $\boldsymbol{g}_{F_1} \in \boldsymbol{G}_1$. By Definition 13, there is $\boldsymbol{g}_{F_2} \in \boldsymbol{G}_2$ such that $\boldsymbol{g}_{F_2} \geq \boldsymbol{g}_{F_1}$—thus, $F_2 \supseteq F_1 \supseteq F$, and $\oplus F \in \boldsymbol{T}_2$.

Similarly, Let $\ominus F \in \boldsymbol{T}_1$. Then, $\boldsymbol{g}_F \notin \boldsymbol{G}_1$, and there is no object below $\boldsymbol{g}_F$ and in $\boldsymbol{G}_1$. By Definition 13 the same applies to $\boldsymbol{G}_2$—hence, $\ominus F \in \boldsymbol{T}_2$. ■

Proposition 6 Let $(\mathbb{T}, \leq)$ and $(\{\, \mathcal{K}_{\boldsymbol{T}} \mid \boldsymbol{T} \in \mathbb{T}\}, \leq)$ be the set of theories, and the set of $\boldsymbol{\kappa}$-models of theories, respectively, both equipped with the corresponding information orderings. Let $\boldsymbol{\kappa}$ be the mapping from $\mathbb{T}$ to $\{\, \mathcal{K}_{\boldsymbol{T}} \mid \boldsymbol{T} \in \mathbb{T}\}$. Let $\boldsymbol{T}_1, \boldsymbol{T}_2 \in \mathbb{T}$. Let $\mathcal{K}_1 = \mathcal{K}_{\boldsymbol{T}_1} = \boldsymbol{\kappa}(\boldsymbol{T}_1)$ and $\mathcal{K}_2 = \mathcal{K}_{\boldsymbol{T}_2} = \boldsymbol{\kappa}(\boldsymbol{T}_2)$, and hence $\mathcal{K}_1, \mathcal{K}_2 \in \{\, \mathcal{K}_{\boldsymbol{T}} \mid \boldsymbol{T} \in \mathbb{T}\}$. We have that if $\boldsymbol{T}_1 \leq \boldsymbol{T}_2$ then $\mathcal{K}_1 \leq \mathcal{K}_2$.

Proof Let $\mathcal{K}_1 = \{\boldsymbol{K}_i\}_i$ and $\boldsymbol{D}_3 = \boldsymbol{T}_2 \setminus \boldsymbol{T}_1$. In an attempt to derive a contradiction, suppose that $\boldsymbol{T}_1 \leq \boldsymbol{T}_2$ but $\mathcal{K}_1 \not\leq \mathcal{K}_2$, i.e., there is $\boldsymbol{K}_0 \in \mathcal{K}_1$ such that for every $\boldsymbol{K}_i \in \mathcal{K}_1$ we have that $\boldsymbol{K}_i \not\leq \boldsymbol{K}_0$.

Consider the following. Given $\boldsymbol{K}_i \in \mathcal{K}_1$, let $\boldsymbol{K}_i^*$ be a minimal context above $\boldsymbol{K}_i$ such that the descriptions in $\boldsymbol{D}_3$ are valid in $\boldsymbol{K}_i^*$. Then $\{\boldsymbol{K}_i^* \mid \boldsymbol{K}_i \in \mathcal{K}_1\} \supseteq \mathcal{K}_2 \ni \boldsymbol{K}_0$. Thus, there is $\boldsymbol{K}_i \in \mathcal{K}_1$, say $\boldsymbol{K}_1$, such that $\boldsymbol{K}_1^* = \boldsymbol{K}_0$. However, $\boldsymbol{K}_1^* \geq \boldsymbol{K}_1$, so we have that $\boldsymbol{K}_0 \geq \boldsymbol{K}_1$. Hence, there is a context $\boldsymbol{K}_i \in \mathcal{K}_1$—namely, $\boldsymbol{K}_1$—such that $\boldsymbol{K}_i \leq \boldsymbol{K}_0$. Contradiction. ■

Proposition 7 Let $\boldsymbol{D}_i \in \mathbb{D}$ and $D \in \boldsymbol{D}$. Let $\mathcal{H}_i = \mathcal{H}_{\boldsymbol{D}_i}$ be a formal system with axioms $\boldsymbol{D}_i$. Let $\mathcal{K}_i = \mathcal{K}_{\boldsymbol{D}_i}$ be a $\boldsymbol{\kappa}$-model of $\boldsymbol{D}_i$.

$$\mathcal{K}_i \models D \text{ iff } \mathcal{H}_i \vdash D$$

Proof
Soundness. Suppose $\mathcal{H}_i \vdash D$. We want to show that $\mathcal{K}_i \models D$. It is sufficient to notice that the syntactic proof of D in $\mathcal{H}_i$ can be carried out semantically—because semantic equivalents of Φ preserve validity—in every context of $\mathcal{K}_i$. Hence, D is valid in every context of $\mathcal{K}_i$, and so it it valid in $\mathcal{K}_i$.K, i.e., $\mathcal{K}_i \models D$.
Completeness (sketch). Suppose $\mathcal{K}_i \models D$. We want to show that $\mathcal{H}_i \vdash D$. Notice that $\mathcal{K}_i = \boldsymbol{\kappa}(\boldsymbol{D}_i) = \boldsymbol{\kappa}(\mathrm{Cn}(\boldsymbol{D}_i)) = \boldsymbol{\kappa}(\boldsymbol{T}_i)$. Hence, given $\boldsymbol{T}_i$, generate the $\boldsymbol{\kappa}$-model $\boldsymbol{\kappa}(\boldsymbol{T}_i)$ of $\boldsymbol{T}_i$ using Procedure 2. Firstly, formulae valid in $\boldsymbol{\kappa}(\boldsymbol{T}_i)$ so obtained are provable in $\mathcal{H}_i$—by tracing the procedure one can see that the obtained contexts do not jointly violate the syntactic system $\mathcal{H}_i$. Secondly, D is valid in $\boldsymbol{\kappa}(\boldsymbol{T}_i)$—because $\boldsymbol{\kappa}(\boldsymbol{T}_i) = \boldsymbol{\kappa}(\boldsymbol{D}_i)$. Hence, D is provable in $\mathcal{H}_i$, i.e., $\mathcal{H}_i \vdash D$. ■

Proposition 8 Let $\mathbb{B}$ be a set of believed theories. Then $\mathbb{C}$ and $\mathbb{C}_+$ are *lattices*.

Proof Given a set $\mathbb{B}$ of believed theories, let M be the appropriate set of attributes. Further, let $\mathbb{T}$ be the set of all consistent theories (over M), and recall that $(\mathbb{T}_1, \wedge, \vee)$ is a lattice, where $\mathbb{T}_1 = \mathbb{T} \cup \{\mathbf{1}\}$. Certainly, $\mathbb{C}$ and $\mathbb{C}_+$ are subsets of $\mathbb{T}_1$, and they are closed under $\wedge$ and $\vee$—hence, they are sub-lattices of $(\mathbb{T}_1, \wedge, \vee)$, and so lattices. ■

Proposition 9 Let $\boldsymbol{T} \in \mathbb{T}$ be a consistent theory and $D \in \boldsymbol{D}$ be a description. If $\boldsymbol{T}_i \cap \{D, \neg D\} = \emptyset$ then $\boldsymbol{T}_i \cup \{D\}$ is consistent. Thus, we also have that if $\boldsymbol{T}_i \vee \boldsymbol{T}_D = \mathbf{1}$ then $\boldsymbol{T}_i \ni \neg D$.

Proof Let $\boldsymbol{\kappa}(\boldsymbol{T}) = \boldsymbol{\mathcal{K}}_{\boldsymbol{T}}$ be the $\boldsymbol{\kappa}$-model of $\boldsymbol{T}$. There is a context $\boldsymbol{K}_a \in \boldsymbol{\mathcal{K}}_{\boldsymbol{T}}$ such that $\boldsymbol{K}_a \models \boldsymbol{T}$ (because $\boldsymbol{T}$ is valid in every element of $\boldsymbol{\mathcal{K}}_{\boldsymbol{T}}$) and $\boldsymbol{K}_a \not\models \neg D$ (because otherwise we would have that the description $\neg D$ is valid in every element of $\boldsymbol{\mathcal{K}}_{\boldsymbol{T}}$, and hence we would have that $\boldsymbol{T} \ni \neg D$). Given $\boldsymbol{K}_a$ we will construct a context $\boldsymbol{K}_b$ such that $\boldsymbol{K}_b \models \boldsymbol{T}$ and $\boldsymbol{K}_b \models D$.

1. Case of $D = \oplus F$.
   Let $\boldsymbol{G}_a$ be the set of objects of $\boldsymbol{K}_a$. Construct $\boldsymbol{G}_b = \boldsymbol{G}_a \cup \{\boldsymbol{g}_F\}$, and then $\boldsymbol{K}_b$ with $\boldsymbol{G}_b$ as its objects. We have that

   (i) formulae $\oplus$-valid in $\boldsymbol{K}_a$ remain $\oplus$-valid in $\boldsymbol{K}_b$, because $\boldsymbol{G}_b \supseteq \boldsymbol{G}_a$,

   (ii) formulae $\ominus$-valid in $\boldsymbol{K}_a$ remain $\ominus$-valid in $\boldsymbol{K}_b$, because $\boldsymbol{g}_F$ does not invalidate any of the formulae $\ominus$-valid in $\boldsymbol{K}_a$ (because $\boldsymbol{K}_a \not\models \ominus F$), and

   (iii) $\boldsymbol{K}_b \models \oplus F$, because $\boldsymbol{g}_F \in \boldsymbol{G}_b$.

   Hence, $\boldsymbol{K}_b \models \boldsymbol{T} \cup \{\oplus F\}$, and thus $\boldsymbol{K}_b \models \boldsymbol{T} \cup \{D\}$, where $D = \oplus F$.

2. Case of $D = \ominus F$.
   Let $\boldsymbol{G}_a$ be the set of objects of $\boldsymbol{K}_a$. We will modify $\boldsymbol{G}_a$ to obtain $\boldsymbol{G}_b$ such that the corresponding $\boldsymbol{K}_b$ is as desired. For every $\boldsymbol{g}_{F_i} \in \boldsymbol{G}_a$—recall that F_i is the set of attributes of $\boldsymbol{g}_{F_i}$—we either leave $\boldsymbol{g}_{F_i}$ unchanged, or replace it with another object, depending on the set of attributes of $\boldsymbol{g}_{F_i}$. There are two cases to consider.

   (a) $F_i \cap \{F \cup \overline{F}\} \not\subseteq F$, i.e.,[1] there is an $m \in F_i$ such that $m \in \overline{F}$—in this case we have that[2] $\{\boldsymbol{g}_{F_i}\} \models \ominus F$, and thus there is no need to modify $\boldsymbol{g}_{F_i}$.

   (b) $F_i \cap \{F \cup \overline{F}\} \subseteq F$—but in this case we also have that $F_i \cap \{F \cup \overline{F}\} \subset F$, because otherwise we would have that $F_i \supseteq F$, and thus also $\boldsymbol{K}_a \models \oplus F$. Hence, there is an $m \in F$ such that $F_i \cap \{m, \overline{m}\} = \emptyset$. Replace $\boldsymbol{g}_F$ with $\boldsymbol{g}_{F \cup \{\overline{m}\}}$.

---

[1] $\overline{F} = \{\overline{m} \in M \mid m \in F\}$.

[2] $\{\boldsymbol{g}_{F_i}\} \models \ominus F$ is a shorter way of saying that $\ominus F$ is valid in the sub-context of $\boldsymbol{K}_a$ determined by the set of objects $\{\boldsymbol{g}_{F_i}\}$.

Resulting $\boldsymbol{G}_b$ determines the context $\boldsymbol{K}_b$. We have that

(i) formulae $\oplus$-valid in $\boldsymbol{K}_a$ remain $\oplus$-valid in $\boldsymbol{K}_b$, because no objects of $\boldsymbol{G}_a$ were lost when constructing $\boldsymbol{G}_b$—some of them remained unchanged, other ones were replaced by more specialised versions,

(ii) formulae $\ominus$-valid in $\boldsymbol{K}_a$ remain $\ominus$-valid in $\boldsymbol{K}_b$, by construction of $\boldsymbol{G}_b$, and

(iii) $\boldsymbol{K}_b \models \ominus F$, by construction of $\boldsymbol{G}_b$.

Hence, $\boldsymbol{K}_b \models \boldsymbol{T} \cup \{\ominus F\}$, and thus $\boldsymbol{K}_b \models \boldsymbol{T} \cup \{D\}$, where $D = \ominus F$.

Thus, whether D is a $\oplus$-valid or a $\ominus$-valid formula, there is a context $\boldsymbol{K}_b$ such that $\boldsymbol{K}_b \models \boldsymbol{T} \cup \{D\}$. Hence, $\boldsymbol{T}_{\boldsymbol{K}_b} \supseteq \boldsymbol{T} \cup \{D\}$, where $\boldsymbol{T}_{\boldsymbol{K}_b}$ is a theory of the context $\boldsymbol{K}_b$, the set of all descriptions valid in $\boldsymbol{K}_b$. As $\boldsymbol{T}_{\boldsymbol{K}_b}$ is a consistent theory, we also have that $\boldsymbol{T}_{\boldsymbol{K}_b} \supseteq \mathrm{Cn}(\boldsymbol{T} \cup \{D\})$, and thus $\boldsymbol{T} \cup \{D\}$ is consistent.

Given that if $\boldsymbol{T}_i \cap \{D, \neg D\} = \emptyset$ then $\boldsymbol{T}_i \cup \{D\}$ is consistent, it is easy to show that if $\boldsymbol{T}_i \vee \boldsymbol{T}_D = \mathbf{1}$ then $\boldsymbol{T}_i \ni \neg D$. Indeed, in an attempt to derive a contradiction, suppose that $\boldsymbol{T}_i \vee \boldsymbol{T}_D = \mathbf{1}$ but $\boldsymbol{T}_i \not\ni \neg D$. Then there are two cases to consider. If $\boldsymbol{T}_i \ni D$ then $\boldsymbol{T}_i \vee \boldsymbol{T}_D = \boldsymbol{T}_i$ is consistent—contradiction. If $\boldsymbol{T}_i \not\ni D$ then $\boldsymbol{T}_i \cap \{D, \neg D\} = \emptyset$ and hence—by the first part of the theorem—$\boldsymbol{T}_i \cup \{D\}$ is consistent, which in turn means that $\mathrm{Cn}(\boldsymbol{T}_i \cup \{D\})$ is consistent. This however, because $\boldsymbol{T}_i \vee \boldsymbol{T}_D = \mathrm{Cn}(\boldsymbol{T}_i \cup \{D\})$, gives us that $\boldsymbol{T}_i \vee \boldsymbol{T}_D$ is consistent—contradiction. ■

Proposition 10 Let $\neg : \boldsymbol{D} \longrightarrow \boldsymbol{D}$ be given by Definition 21, and let $\neg : \Gamma \longrightarrow \Gamma$ be given by $\neg(U, V) = (V, U)$. Then $\neg\varrho(D) = \varrho(\neg D)$, where $\varrho(D) = (U_D, V_D)$.

Proof Let $\varrho(D) = (U_D, V_D)$. Then $\varrho(\neg D) = (U_{\neg D}, V_{\neg D})$ and $\neg\varrho(D) = (V_D, U_D)$. Hence, we need to show that $U_{\neg D} = V_D$ and $V_{\neg D} = U_D$. To show that $U_{\neg D} = V_D$ notice that $\boldsymbol{T} \in U_{\neg D}$ iff $\boldsymbol{T} \geq \boldsymbol{T}_{\neg D}$ iff $\boldsymbol{T} \ni \neg D$ iff $\boldsymbol{T} \vee \boldsymbol{T}_D = \mathbf{1}$ iff $\boldsymbol{T} \in V_D$,the second to last step holds by Proposition 9. Replacing D with $\neg D$ in $U_{\neg D} = V_D$ yields $V_{\neg D} = U_D$. ■

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