# A MATHEMATICAL STUDY OF PERISTALTIC TRANSPORT OF PHYSIOLOGICAL FLUIDS

Anacleto Valentino Mernone, B.Sc(Hons), M.Sc(Medical & Health Physics)

*Thesis submitted for the degree of*

*Doctor of Philosophy*

*in*

*Adelaide University*

*(Faculty of Engineering, Computing and Mathematical Sciences)*

Department of Applied Mathematics

June 26, 2000

# TABLE OF CONTENTS

# SUMMARY

In this thesis "A Mathematical Study of Peristaltic Transport of Physiological Fluids", a mathematical investigation of both Newtonian and non-Newtonian fluids are considered.

The Newtonian models are outlined and compared to two non-Newtonian models, namely the power law model and the Casson model. The geometry considered are a planar channel and an axisymmetric tube.

After considering the respective constitutive equations and subsequently the equations of motion and continuity equation, the method of solution is obtained using stream function of zeroth and first order, by performing a perturbation series expansion in amplitude ratio.

Comparisons and implications were made between the Newtonian and non-Newtonian models, in some cases after the introduction of simplifications, without deviating from the context of the problem. The validity of all models were highlighted, in particular that of non-Newtonian fluids when compared to Newtonian fluids.

## SIGNED STATEMENT

This work contains no material which has been accepted for the award of any other degree or diploma in any University or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by any other person, except where due reference has been made in the text.

I consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

A. V. Mernone

26th June 2000

# ACKNOWLEDGEMENTS

I would like to express sincere and deepest gratitude to Dr. Jagannath Mazumdar (A/Professor and Emeritus Director, Centre for Biomedical Engineering of Adelaide University) for his continual and invaluable guidance and understanding given throughout the duration of this research.

Also, thanks must go to the staff of the Department of Applied Mathematics, Adelaide University, for their assistance in all forms during my candidature. I would also like to express my sincere gratitude to Dr. Stephen Lucas of the University of South Australia for his occasional help during the later part of this research.

Finally, as a genuine token of my appreciation, unreserved gratitude goes to my family for their ever persistent encouragement and unconditional understanding from the inception of my academic career and always.

# LIST OF FIGURES

# LIST OF TABLES

# LIST OF PUBLICATIONS

Mernone, A.V and Mazumdar, J.N (1998a)
Mathematical Modelling of Peristaltic Transport on a non-Newtonian Fluid
Australasian Phys. Engng Sci. in Medicine 21:3 pp126-141

Mernone, A.V and Mazumdar, J.N (1998b)
Peristaltic Flow of a non-Newtonian Fluid and its Biomedical Applications
EMAC'98, 3rd Biennial Engineering Mathematics and Applications Conference
pp367-370, published by IEAust

Mernone, A.V and Mazumdar, J.N (2000a)
Biomathematical Modelling of Physiological Fluids using a Casson Fluid with emphasis to Peristalsis(*Journal Australasian Physical & Engineering Sciences in Medicine*- in press)

Mernone, A.V., Mazumdar, J.N and Lucas, S.K (2000b)
A Mathematical Study of Peristaltic Transport of a Casson Fluid
Mathematical and Computer Modelling (in press)

# CHAPTER 1

# INTRODUCTION

## 1.1 General Introduction to Peristaltic Transport and Motivation for Mathematical Modelling.

Peristalsis is the phenomenon in which a circumferential progressive wave of contraction or expansion (or both) propagates along a tube. If the tube is long enough, one might see several identical waves moving along the tube simultaneously. Peristalsis appears in many organisms and a variety of organs.

Peristalsis is well known to physiologists to be one of the major mechanisms for fluid transport in many biological systems. In particular, peristaltic mechanisms may be involved in urine transport from the kidney to the bladder through the ureter, the movement of chyme in the gastrointestinal tract, the transport of spermatozoa in the ductus efferentes of the male reproductive tract and in the cervical canal; the movement of ova in the fallopian tubes; the transport of lymph in the lymphatic vessels and in the vascomotion in small blood vessels.

These flows also provide efficient means for sanitary fluid transport and are thus exploited in industrial peristaltic pumping and medical devices. For example, mechanical roller pumps using viscous fluids are used in the printing industry and the peristaltic transport of noxious fluid in the nuclear industry. In addition, peristaltic pumping occurs in many practical applications involving biomedical systems. Many modern medical devices have been designed on the principle of peristaltic pumping to transport fluids without internal moving parts, for example, the blood in the heart lung machine.

Mathematical studies of peristalsis were initiated by Fung & Yih(1968), Shapiro et al(1969), and others. Most of these analyses are based on the Navier-Stokes equation, considering flow in a circular cylindrical tube or two-dimensional channel with a sinusoidal displacemnt wave travelling in its wall at constant velocity. The objects of these studies are:

(1) to determine the longitudinal pressure pradient that can be generated by the travelling wave;

(2) the flow resulting from peristalsis superposed on pressure differences at the ends; and

(3) conditions of reflux.

To simplify the analysis, various approximations are introduced such as;

(a) small amplitude of the wall displacement compared with the undeformed radius of the tube; or

(b) long wavelength compared with the tube radius ; or

(c) very small Reynolds number so that the non-linear convective acceleration term in the Navier-Stokes equation can be neglected.

In this thesis all these assumptions are applied and in some cases the third assumption is relaxed when considering the power-law fluid and Casson fluid.

One conclusion reached by these studies is that peristalsis is an effective method to move fluid only if the fluid is transported in the form of a series of isolated boluses. If the amplitude of the displacement of the wall is small compared with the tube radius, very little pressure gradient can be generated by the travelling wave. Pressure gradient increases significantly when the radius of the minimum section of the wave approaches zero. This is the reason in normal conditions that peristaltic waves of the ureter, intestine, and the lymphatics are of this mode.

The main motivation for any mathematical analysis of physiological fluid flows is to ultimately have a better understanding of the particular flow being modelled. If there is similarity between the results obtained from the analysis and experimental and clinical data, then the mechanism of flow can at least be explained. Because peristalsis is evident in many physiological flows, an accurate mathematical study can help explain the major contributing factors to many flows in the human body. When comparing results between the mathematical model and the experimental and clinical data it is desirable that the data obtained from experimental research be as close as possible to the actual physiological flow/parameter being analysed. That is to say, it may be necessary to take into account the effect the measuring instrument or device or procedure has on the data obtained. In other words, the results of the mathematical model will be compared with appropriate data of the flow being modelled.

Results obtained from a mathematical analysis of the flow of urine from the kidney, to the bladder, through the ureter are being considered. Data values obtained via an appropriate experiment, indicate the drop in pressure across a certain section of the ureter during peristalsis. Similarity of this data with the corresponding theoretical results of the pressure drop across the same section of the ureter, means that future predictions may be made about the change in pressure drop across the ureter during peristalsis, and possibly the pressure. Therefore the pressure change, can be governed by means of change to the urinary tract environment, and possibly even a slight modification of the urinary tract itself.

Regarding reflux; governing the pressure across the ureter means that if the pressure applied to the ureter is insufficient to pass urine on to the bladder, a minor adjustment to the ureter, causing an increased applied pressure by the ureter on the bladder, can enable the ureter to apply a sufficiently great pressure on the bladder so that urine is successfully passed on to the bladder, without a backflow of urine to the kidney, thus preventing serious kidney diseases. The mathematical analysis may be used to determine a critical pressure, below which reflux occurs, hence, reflux can be diagnosed in its early stages and dealt with accordingly.

Once an accurate mathematical analysis of the flow of urine in the ureter is seen as important, it would be extremely useful to be able to generalise this analysis so that it could be used to model other physiological flows which are considered to be caused by peristalsis. Extending the theoretical model means that one model can be used to obtain theoretical results for various fluid flows in the human body, instead of conducting a completely separate analysis for each physiological flow being modelled.

## 1.2 General Outline of Previous Research in Peristaltic Transport

The study of the mechanisms of peristalsis, in both mechanical and physiological situations, has been the object of scientific research for quite some time. Since the first investigation of Latham(1966) several theoretical and experimental attempts have been made to understand peristaltic action in different situations.

Interest in peristaltic pumping has been stimulated by its relevance to ureteral function. As reliable and accurate urometric measurements became available through the work of Kiil(1967) and Boyarsky(1964), several hydrodynamic models of ureteral function invoking peristalsis were attempted. The earliest models Shapiro(1967), Fung and Yih(1968) and Shapiro et al(1969) were idealised and represented the peristalsis by an infinite train of sinusoidal waves in a two-dimensional channel. Thus they could pretend to only a qualitative relationship with the ureter. These models concerned themselves, in part, with offering an explanation of the biologically and medically important phenomenon of 'reflux'. One manifestation of this reflux is that bacteria sometimes travel from the bladder to the kidney against the mean urine flow. A similar phenomenon has been observed in the small bowel. These observations are puzzling because the travel times are too small to be explained by diffusion and also because retrograde peristaltic waves have not usually been observed.

Later, Lykoudis(1971) and Weinberg et al(1971) proposed models that represent ureteral waves more realistically. Fung(1971) investigated the coupling between the forces of fluid-mechanical origin and the dynamics of the ureteral muscle. Some of these models showed that observed urometric pressure pulses and flow rates could be accounted for by assuming internal dimensions of the ureter which seem physiologically plausible.

But ureteral physiology has not been the only motivation for the study of peristalsis. Burns & Parkes(1967) and Hanin(1968) contributed to the theory of peristaltic pumping without reference to physiological applications. Barton & Raynor(1968) made a calculation based on peristalsis theory of the time required for chyme to traverse the small intestine and found that this calculation compared favourably with observed values. In addition, Fung(1971) studied peristaltic flow taking muscle action in the tube wall into account. Some new examples of peristalsis were given in Liron(1978). Considerable experimental investigations of peristaltic pumping have also been undertaken, for example, Latham(1966), Eckstein(1970), Weinberg et al(1971), Yin & Fung(1971), Hung & Brown(1976).

Most of the theoretical investigations have been carried out by assuming blood and other physiological fluids behave like a Newtonian fluid. Although this approach may provide a satisfactory understanding of the peristaltic mechanism in the ureter, it fails to provide a satisfactory model when the peristaltic mechanism is involved in small blood vessels, lymphatic vessels, intestine, ductus efferentes of the male reproductive transport and in the transport of spermatozoa in the cervical canal. It has now been accepted that most of the physiological fluids behave like non-Newtonian fluids. But it appears that no quantitative rigorous attempt has been made to understand the problem of a non-Newtonian fluid before the investigation of Raju & Devanathan(1972) and Raju(1972) in the case of small wave amplitude.

Srivastava & Srivastava(1984) investigated the problem of peristaltic transport of blood assuming a single layered Casson fluid and ignoring the presence of a peripheral layer. Subsequently, Srivastava & Srivistava(1985) considered the axisymmetric flow of a Casson fluid in a circular non uniform tube. More recently, Siddiqui, et al(1991) investigated peristaltic motion of a non-Newtonian fluid modelled with a constitutive equation for a second order fluid for the case of a planar channel. A perturbation series was used representing parameters such as curvature, inertia and the non-Newtonian character of the fluid. Tang & Rankin(1993) proposed a mathematical model for peristaltic motion of a nonlinear viscous flow where they used an iterative method to solve a free boundary problem.

Das & Batra(1993) studied the fully developed, steady flow of a Casson fluid through a curved tube for small values of Dean Number. A plug core formation at the centre is considered where the shear stress is not sufficient to exceed the yield value. El Misery et al(1996) considered peristaltic flow in a two-dimensional channel of a generalised Newtonian fluid. Under the assumption of creeping motion, the problem is formulated using a perturbation expansion in terms of the variant of the Weissenberg number. Elshehawey et al(1998) consider the problem of peristaltic transport of a non-Newtonian(Carreau) fluid in a non-uniform channel under zero Reynolds number with long wavelength approximation. Again the problem is formulated using a perturbation expansion in terms of a variant Weissenberg number. They find the the pressure rise and friction force are smaller than the corresponding values in the case of uniform geometry. Naidu & Kumar(1995) solve the Navier-Stokes equations of a Newtonian fluid numerically by using a streamline upwinding finite element method on the peristaltic flow, induced by an infinite train of sinusiodal waves in a two-dimensional channel.

## 1.3 Outline of Approach in the Present Study

To achieve the aims stated above, the present study has been organised in the manner described below.

In Chapter 2, a brief description of peristaltic flow is given, in particular, an introduction to the biological occurrence of peristaltic flow. Specifically, fluid flow in the ureter and flow in the gastrointestinal tract, with particular emphasis of flow of chyme in the small intestine is presented. Also, an outline of flow in the Vas deferens is presented.

Chapter 3 examines how the basic laws of physics and laws of fluid dynamics as applied to non-Newtonian fluids may be applied to mathematical modelling of peristalsis. It considers consequences; with particular emphasis on the time independent fluids; power law fluid and Casson fluid.

In Chapters 4 and 5 mathematical models are developed for the understanding of peristalsis.

In Chapter 4, flow in a two-dimensional channel is considered with emphasis on the Newtonian fluid, power law fluid and Casson fluid. Comparisons of models are investigated, outlining possible simplifications and difficulties in finding analytical and numerical solutions to the respective model.

Chapter 5, describes flow in an axisymmetric tube, again considering the Newtonian fluid case and the power law fluid case and Casson fluid case. Comparisons of models are investigated and the validity of an analytical algebraic solution of the power law model is compared with results in the literature.

The concluding chapter, Chapter 6 summarises the work done and discusses the significance of the present study and outlines suggestions and directions for future research in the area of peristalsis as applied to physiological fluids.

# CHAPTER 2

# PERISTALTIC FLOWS

## 2.1 Introduction to Peristaltic Transport and where it occurs.

The word peristalsis stems from the Greek word *peristaltikos,* which means clasping and compressing. Peristalsis is a muscle controlled flow similar to the flow in the cardiovascular system. Peristalsis occurs in may organs. Peristaltic flow is the flow generated in the fluid contained in a distensible tube when a progressive wave travels along the wall of the tube. Although elasticity of the wall does not enter directly into the flow equations, it affects the flow through the progressive wave travelling along its length.

The mathematical problem of peristaltic flow is similar to that of a collapsible tube. In the case of the ureter, it is modelled mathematically assuming that the ureter receives fluid from the kidney at the upper end and passes it down to the bladder against a pressure gradient. Normally, there is more than one wave along the entire length of the ureter, which is of the order of 30cm. The amplitude of the wave is of the order of 5mm and its wave speed is approximately 6 cm/s. The frequency of contractions is of the order of 1-8 per minute. Each contraction lasts about 1.5-9 seconds, where the diastolic (expansion) phase is about twice as long as the systolic (contraction) phase. Pressure during the contraction varies from 2-8mmHg at the pelvis to 2-10 mmHg in the upper portion of the ureter and 2-14 mmHg in the lower portion (Wienberg 1971).

## 2.2 Fluid Flow in the Ureter.

Ureteral peristalsis was described by Aristotle (384-322BC) in his book on animals (Historia animalium). The ureters collect urine from the kidneys and send it to the bladder. The form of the wave propagating down the tube when we consider the ureter is given as Figure 2.1 and Figure 2.2 for anatomical details of ureter and kidney.

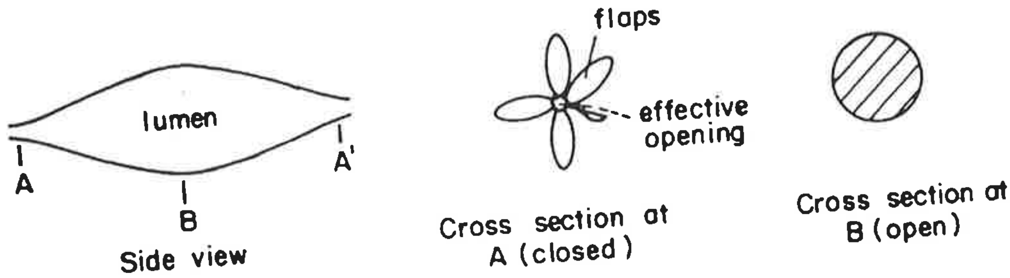


Figure 2.1 Waveform of ureter. Adapted from Liron(1978)

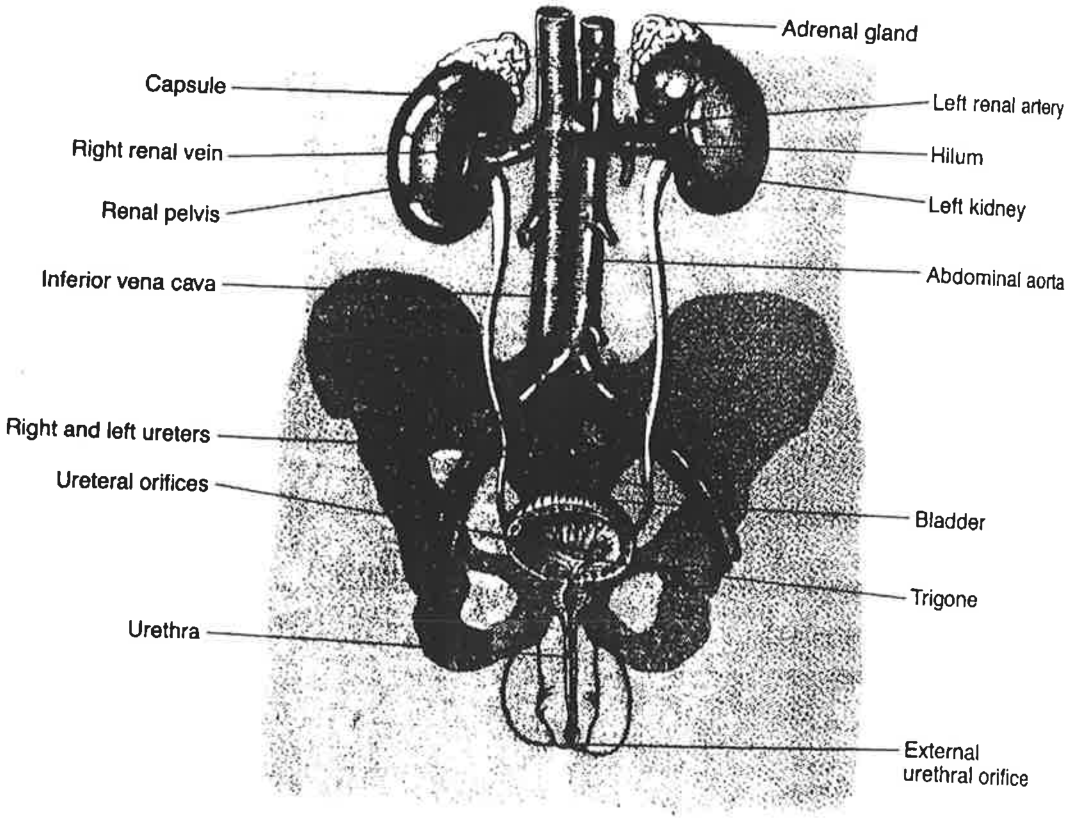


Figure 2.2 Anatomical location of ureter. Adapted from Miller Keane(1992)

At the bladder each ureter passes through a one way valve called the ureterovesicular junction. It works by pressure in the bladder. The kidneys are not capable of producing enough pressure to force urine away from them overcoming the pressure of the expanded bladder.

When the bladder is full, its pressure is high. The bladder presses on the one way valve, which is Z or U shaped, and collapses it, stopping back flow into the ureter. This valve can be opened by each bolus of urine in the ureter if the pressure in the bolus of urine in the ureter is sufficient to exceed the lateral pressure imposed by the urine in the bladder and the muscle in the bladder wall.

If the smooth muscle of the ureter is unable to generate a higher pressure in the bolus of urine or if the ureterovesicular junction is improperly formed then the ureterovesicular junction will not function properly and a disease state called hydroureter results. A hydroureter is a swollen ureter whereby the lumen size is much increased and is filled with urine.

The reason why hydroureter is a disease state is made understandable when one considers hoop stress in a pressurised tube. For example, in a tube of radius, a, with tension T generated by the ureteral smooth muscle, a pressure P is created and given as $P=T/a$ .

Therefore for given T, P can be large if the radius, a, is small. But in a hydroureter the radius, a, becomes so large that the pressure which can be generated by the ureter is insufficient to send urine through the ureterovesicular junction. If the occlusion of the ureter is not complete then depending on the pressure difference between the two ends of the wave, the peristaltic wave may not propel the entire contents of the fluid it contains. That is, some of the fluid propagates forward while other portions proceed in the opposite direction. Then urine remains in the ureter and urine and hence any bacteria backs up to the kidney and eventually causes kidney disease.

One conclusion reached in Chapter 4 and chapter 5 of this thesis is that peristalsis is a effective method to move fluid only if the fluid is transported in the form of a series of isolated boluses. Pressure gradient increases significantly when the radius of the minimum section approaches zero. If the amplitude of displacement of the wall is small compared to the tube radius, very little pressure gradient can be generated by the travelling wave.

## 2.3 Gastrointestinal Flow.

We now consider the flow of an inelastic liquid which is generated by contractions in the intestine. Unlike regular peristaltic motion these contractions occur locally over a finite length and have a finite amplitude, see Figure 2.3 for geometry of wave shape and Figure 2.4 for anatomical details of duodenum and small bowel.

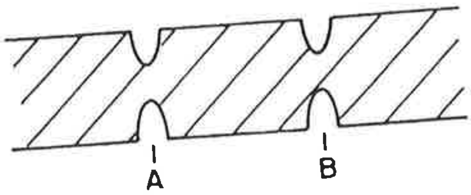


Figure 2.3 Waveform of intestinal tract. Adapted from Liron(1978)

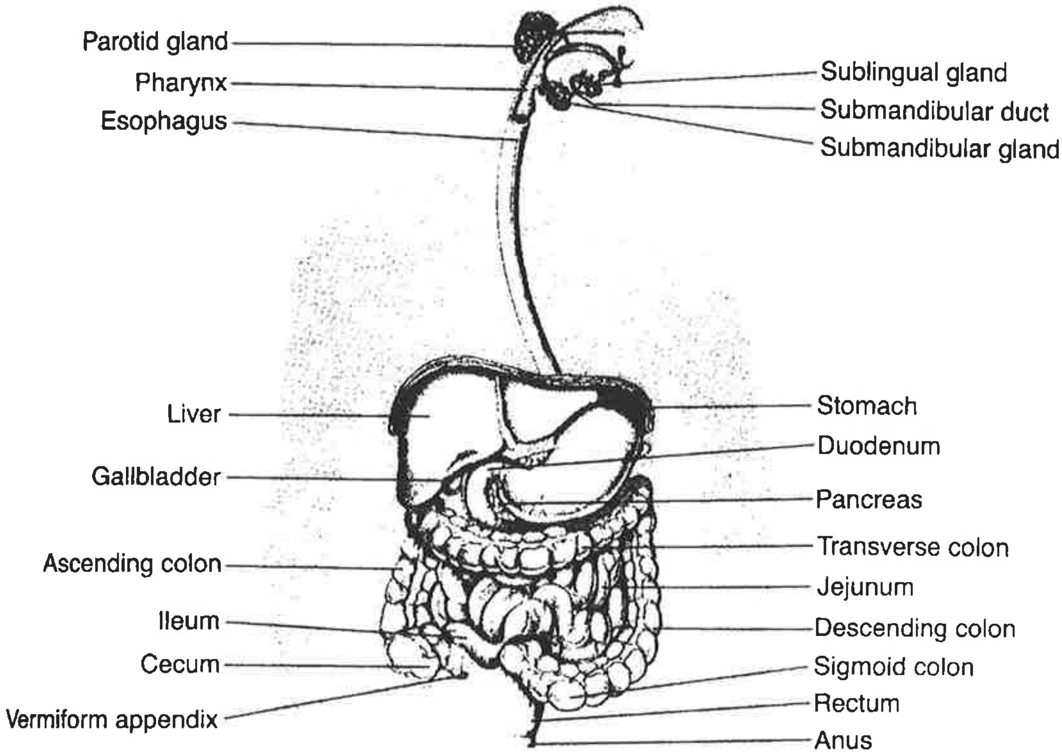


Figure 2.4 Anatomical location of intestinal tract. Adapted from Miller-Keane(1992)

The motor activity of the small bowel is a complex physiological reaction, the internal mechanisms of which are still not well understood. Analysis of this phenomenon is an extremely difficult problem from both theoretical and experimental points of view.

Electromechanical wave processes form the basis of many physiological phenomena. They are inherent in the function of non-linear biological systems that possess the properties of excitability and motor activity. Because of their internal features and specific conditions, the progression from an initial steady state to an excitable state can occur which results in the formation of a propagating peristaltic wave.

Stereotypic and organised small bowel motor activity is a product of dynamically stable neuromuscular regulatory mechanisms charcterised by internal inputs that are functionally excitatory or inhibitory. The neural component consists of an overlapping series of functional modules interconnected by horizontal polysynaptic channels that form the myenteric plexus, a neural network into which reflex pathways are locked and which serves to coordinate motor activity.

Peristalsis is the main mode of propulsion, which enables the passage of solids and liquids in the gastrointestinal tract, and disorganisation of which results in conditions such as paralysis of the intestine and constipation.

Under normal physiological conditions, it begins with a preliminary phase, the gradual reflex, with longitudinal contractions that are followed by the phase of a broad spread of circular contractions.

In the first 30cm or so of the small intestine (the duodenum) a local contraction of some finite length and amplitude occurs, and this contraction depends on the amount of chyme in the alimentary system of the human subjects. Two types of contraction have been distinguished: the stationary and the propagative contraction. In the stationary contraction a bundle of muscle cells contract simultaneously and move fluid contents over certain length on both sides of the contraction. In this mode, the peak of the contraction does not propagate along the axial direction. In the propagative contraction, the peak of the contraction propagates along the axial direction. The propagative model is more realistic and based on experimental data whereby simple mathematical models have been proposed by Macagro and Christensen(1982).

It is known that the content of the intestine (chyme) is not really a Newtonian fluid, however most of the analyses in the literature have assumed it so. Experimental tests on human faeces have revealed that the liquid seems to follow a power-law behaviour, with a power-law index of about 0.25 if the shear rate is above 4 $s^{-1}$, (see power-law mathematical model in chapter 4 and 5, in particular various results for axisymmetric tube case with varying values of power-law index). This power-law index can vary with the pathological condition of the sample. At low shear rates, the liquid has a kinematic viscosity of 5-10 times that of water.

It is found that the pumping action does not depend strongly on the power law index of the liquid. This is a desirable characteristic of any positive-displacement pump and allows us to simplify the analysis by assuming a Newtonian behaviour for the liquid.

It may be noted that the theory of long wavelength and zero-low Reynolds number( see section 2.7) remains applicable as the radius of the small intestine r=1.25cm is small compared with the wavelength $\lambda = 8.01cm$. Barton and Raynor(1968) reported the observed average chyme velocity as 2.54 cm/min on the basis that the male small intestine length was approximately 685cm and the time for the chyme to pass through the small intestine was 4.5hrs.

## 2.4 Flow in Vas deferens.

Vas deferens is a thick walled tube which connects the epididymis, an elongated organ on the posterior surface of the testis, to the ejaculatory duct. Spermatic fluid consisting of spermatozoa and fluid medium flows through the vas Deferens to the ejaculatory duct and is finally expelled during intercourse from the penis by a series of rapid muscle contractions. See Figure 2.5 for anatomical details of Vas deferens.

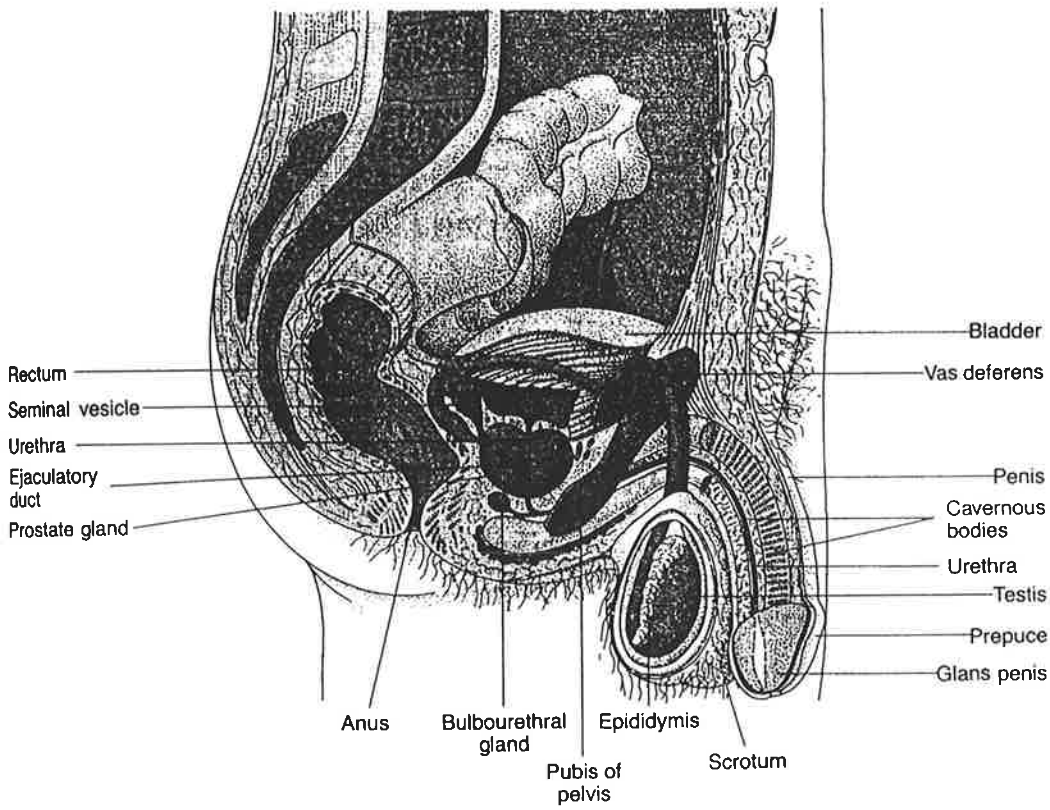


Figure 2.5 Anatomical location of Vas deferens. Adapted from Miller-Keane(1992)

Movement through Vas deferens is accomplished by means of peristaltic action of contractile cells in the duct wall. The Vas deferens in rhesus monkeys is in the form of a diverging tube with a ratio of exit to inlet dimension of approximately four. According to some experimental observations of flows in the Vas deferens of rhesus monkeys made by Guha et al(1975) the period of ejaculation was about two seconds and the average flow rate is 0.02ml/s for 30mmHg pressure rise. The approximate value of various parameters for flow in the vas Deferens of rhesus monkeys based on experimental observations made by Guha et al(1975) are wavelength $\lambda$= 20cm, inlet radius r=0.012cm and viscosity of semen = 4centpoise. The value of Reynolds number is of the order of $10^{-3}$ therefore low Reynolds number mathematical models are viable and applicable.

## 2.5 Biological Significance and Applications, Including Medical Devices.

Peristalsis serves more than one function and that waveshape and its amplitude may be directly related to these various functions. One aspect of particular interest is the question of complete occlusion. If the function is fluid transport, then certainly complete occlusion will do the job as is observed in the ureter. All the fluid bounded by two consecutive constrictions will be carried along with the moving wave. Why then does complete occlusion not occur in the blood vessels? For one thing, one pays for complete occlusion by having to use more energy, in fact much more than if the occlusion was 50% (Liron 1978). Moreover, complete occlusion causes high pressures and high shear rates near the occluded region. In the blood this may cause damage to the erythrocytes, a well known problem in artificial blood pumps. On the other hand, partial occlusion reduces the efficiency of the peristaltic wave in its role as a pump.

The following data about the human ureter may be of interest(Bergman 1967). Normally there are about 3-4 waves along the entire length of the ureter. The wave speed is about 3-6cm/sec. The viscosity of the urine is $0.007 cm^2/sec$ (this is the viscosity of water at 40 degress celsius, since the urine is essentially water. Hence under normal conditions the flow of fluid in the ureter may be considered as Newtonian thus exhibiting a linear relationship between the stess and strain tensors as depicted in Figure 3.1.

It is known that the content of the intestine(chyme) is not really Newtonian; however, most analyses in the literature assume that it is so. Patel et al(1973) have carried out some tests and found that the fluid seems to follow a power-law behaviour, with a power-law index of approximately 0.25 if the shear rate is above 4 $s^{-1}$. This power-law index can vary with the pathological condition of the sample. At low shear rates, the liquid has a kinematic viscosity of about 5-10 times the kinematic viscosity of water). Hence, it is appropriate to model flow of chyme in the intestine as a power-law fluid in an axisymmetric tube, Phan-Thien(1989), as depicted in Figure 3.1.

Srivastava and Srivastava(1989) state that it is difficult to estimate rheological properties of the Vas deferens, however, Guha(1975) found that the viscosity parameter is of the order 0.1 and hence have modelled flow as a non-Newtonian(power-law) fluid.

An examination of available viscometer data(Rand et al, 1964) suggest that the non-newtonian behaviour of blood increases rapidly when the haematocrit rises about 20%, possibly reaching a maximum at between 40 and 70%. It has been established by Merrill et al, that Casson model held satisfactory for blood flowing in tubes 130-1000$\mu$ in diameter, within certain wall shear stress limits. Therefore, for realistic description of blood flow, it is perhaps more appropriate to treat blood as a Casson fluid, Srivastava(1987), thus exhibiting behaviour as depicited in Figure 3.1. Also, Srivastava and Srivastava(1984) investigate the problem of peristaltic transport of blood in a uniform and non-uniform tube under zero Reynolds number and long wavelength approximation. Blood is represented as a two-layered fluid, whereby the central layer is Casson fluid and peripheral layer is Newtonian fluid.

When both mixing and transport are important, one would like to reduce the efficiency of the peristaltic wave as a pump and thus increase the time that a given volume of the fluid stays in the tube, as in the small intestine. Lowering the amplitude of the wave is one way and changing the wave shape is another. Another phenomenon which may be significant biologically and physiologically is that of reflux and trapping. If the occlusion is not complete and even in the ureter to assume that we have complete occlusion is an idealisation; then depending on the pressure difference between the two ends of the wave, the peristaltic wave may not, and usually does not, propel the entire volume of the fluid it contains. It turns out that some of the fluid and material is indeed propagated forward, while other portions proceed in the opposite direction. This is called reflux, back flow.

Under suitable conditions a central blob will form which does not mix with the rest of the fluid and propagates with the wave at the wave speed as seen in Figure 2.6

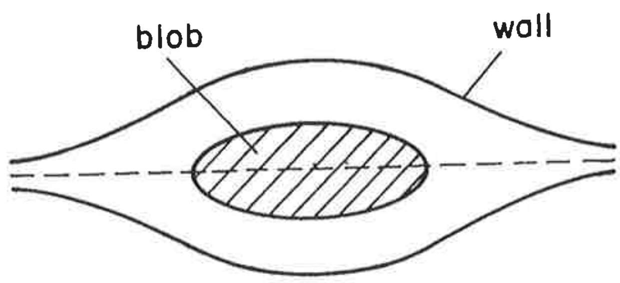


Figure 2.6 Bolus configuration. Adapted from Liron(1978)

If the total flux across any section, that is, the amount of fluid carried in the direction of the propagating wave per unit time, is less than the volume of fluid contained by the wave, then backflow will certainly occur. If this occurs in the ureter, then bacteria may be carried back from the bladder to the kidney - a phenomenon believed to be the mechanism by which some of the bacteria reach and infect the kidneys.

In some instances cilia are found within these organisms and organs, like in the Fallopian tubes. For wave form see Figure 2.7. In some instances peristalsis occurs and in some other instances cilia are used, whereas in other organs both modes of transport are used.

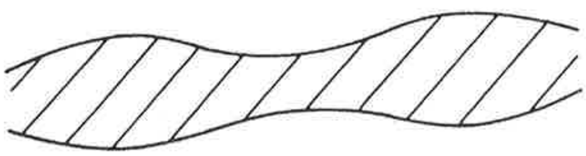

Figure 2.7 Geometry of waveform in Fallopian tube. Adapted from Liron(1978)

It is only natural to inquire under what conditions an organism or organ might find it advantageous to move from cilia transport mode to peristaltic transport mode. For instance, one could suppose that enlarging of the tubes made cilia transport inefficient, forcing the organism or organ to rely on the other mode of transport, peristaltic transport.

### 2.5.1 Understanding and Contributions

The above discussion has pointed out some of the questions that theory may be helpful in determining. So, a fluid dynamically quantitative theory is necessary for a more detailed understanding of physical phenomena such as back flow.

It would be interesting to know under what physical conditions backflow occurs and where it occurs, that is, is it close to the wall where we have postulated a two layered model whereby the peripheral layer is Newtonian? Or is backflow occurring at the centre where we have postulated in our two layered model that the central core of the flow is non-Newtonian? Also, what fraction of the fluid is flowing back, under normal and abnormal conditions?

Several other questions also require theoretical treatment. For example, what sort of peristaltic wave, amplitude, shape and frequency has the organism or organ developed in order to perform a given task? Hence a theory is needed to take these considerations into account.

Hence, it is inherent that peristalsis appears in many tubular organs. Several studies have been undertaken with respect to the peristaltic flow by applying a simple hydrodynamic represented with sinusiodal waves. These studies owing to their physiological emphasis, assume small Reynolds number. A theoretical analysis of peristaltic flow in the range of moderate Reynolds number is extremely difficult because of the non-linearity owing to the interaction between the moving wall and the flow field. However, Fung and Yih(1968) initiated research by considering peristaltic flow with mild non-linear effects and with a small ratio of amplitude to wavelength.

Two interesting phenomena associated with peristaltic flows are fluid trapping and material reflux. The former descibes the development and downstream transport of free eddies, called fluid boluses. The latter refers to net upstream convection of fluid particles against the travelling boundary waves. These two phenomena are of great physiological significance, as they may be responsible for thrombus formation in blood, and pathological transport of bacteria.

From the standpoint of fluid mechanics, these phenomena demonstrate the complexity but also motivate the fundermental study of peristaltic flows. The inclusion of both Newtonian and non-Newtonian models in this thesis serve as a platform to understanding the application of a particular fluid model to the appropriate organ and serve as a comparison between fluid models. For example, chyme behaves as a power-law fluid hence modelling of the material in the intestine as a power-law fluid in an axisymmetric tube. Similarly for the other organs mentioned (Blood as a Casson fluid in two-dimensional channel and urine as Newtonian fluid in axisymmetric and two-dimensional channel and fluid in the Vas defrens as power-law fluid in an axisymmetric tube) and their appropriate fluid dynamical model.

## 2.6 Peristaltic Transport in a Cylindrical Tube.

Consider now peristaltic transport of an idealised two-dimensional flow with an infinite length and constant width cylindrical tube with sinusoidal wave of moderate amplitude travelling along its walls. Assume that the cylindrical tube or the channel is filled with a homogeneous Newtonian viscous fluid, although the fluid involved may be non-Newtonian, and the flow may take place in two layers, ie., a core layer and a peripheral layer. Taking the x-axis along the centre line of the channel, and the y-axis normal to it, the equations governing two-dimensional motion of a viscous fluid, the continuity and Navier-Stokes equations are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u \tag{2.2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v \tag{2.3}$$

Using the stream function $\psi$ , *such* that $u=\psi_y$ , $v=-\psi_x$, and eliminate pressure by cross differeniating equations 2.2 and 3.3 and eliminating pressure p we obtain

$$\frac{\partial^2 u}{\partial y\partial t}-\frac{\partial^2 v}{\partial x\partial t}+u\left(\frac{\partial^2 u}{\partial y\partial x}-\frac{\partial^2 v}{\partial x^2}\right)+v\left(\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial x\partial y}\right)=\nu\left(\frac{\partial}{\partial y}\nabla^2 u-\frac{\partial}{\partial x}\nabla^2 v\right) \tag{2.4}$$

Using the above equation for definition of stream function we obtain

$$\frac{\partial}{\partial t}\left(\psi_{yy}+\psi_{xx}\right)+\psi_y\left(\psi_{yyx}+\psi_{xxx}\right)-\psi_x\left(\psi_{yyy}+\psi_{xxy}\right)=\nu\left(\nabla^2\psi_{yy}+\nabla^2\psi_{xx}\right) \tag{2.5}$$

Defining the Laplacian Operator as

$$\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{2.6}$$

therefore

$$\nabla^2\psi_t+\psi_y\nabla^2\psi_x-\psi_x\nabla^2\psi_y=\nu\nabla^4\psi \tag{2.7}$$

Assume the fluid is subjected to conditions imposed by the symmetric motion of the elastic walls. Let the vertical displacements of the upper and lower walls be $\eta$ *and* $-\eta$ *, respectively.*

Hence, the equations of the walls are given by

$$y = \pm\eta(x,t) = \pm a\left[1 + \varepsilon\cos\frac{2\pi}{\lambda}(x - ct)\right] \qquad (2.8)$$

where $\varepsilon$ *is the amplitude ratio,* $\lambda$ *is the wavelength*, c is the wave speed and a is the undeformed radius if the tube as seen in Figure 2.8

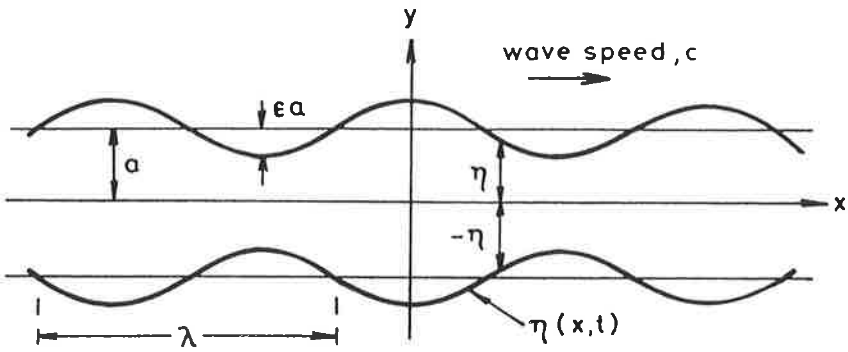


Figure 2.8 Geometry of Cylindrical tube. Adapted from Mazumdar(1992)

The main objectives is to determine the longitudinal pressure gradient that can be generated by the travelling wave, and the flow resulting from peristalsis superimposed on pressure differences at the ends of the tube. In order to solve our equation we assume the following boundary conditions

$$u=0 \qquad (2.9)$$

and

$$v = \pm\frac{2\pi ac\varepsilon}{\lambda}\sin\frac{2\pi}{\lambda}(x - ct) \qquad at\ y = \pm\eta(x,t) \qquad (2.10)$$

Our equation finally reduces; after introducing dimensionless variables

$$X = \frac{x}{\lambda},\quad Y = \frac{y}{\lambda},\quad T = \frac{ct}{\lambda},\quad \Psi = \frac{\psi}{ac},\quad n = \frac{a}{\lambda},\quad R_e = \frac{ac}{\nu} \qquad (2.11)$$

to

$$D_n^{\,2}\Psi_T + \Psi_Y D_n^{\,2}\Psi_X - \Psi_X D_n^{\,2}\Psi_Y = \frac{1}{nR_e} D_n^{\,2}\Psi \tag{2.12}$$

where the operator $D_n$ is given by

$$D_n \equiv n^2 \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \tag{2.13}$$

The corresponding boundary conditions become

$$\Psi_Y = 0 \tag{2.14}$$

and

$$\Psi_X = 2\pi\varepsilon \sin 2\pi(X - T) \quad at \quad Y = \pm\eta(X,T) \tag{2.15}$$

From the above we notice the following:

(a) The Reynolds Number $R_e$ will be small if

(i) the wave speed is small

(ii) the distance between the walls is small

(iii) the viscosity is large

(b) The wave number, n, will be small if the wavelength is large as compared to the distance between the walls.

Also, the amplitude ratio $\varepsilon = \frac{A}{d}$ will be small if the amplitude of the travelling waves is small compared to the distance between the walls. We make various approximations in order to solve the problem, such as

(a) small Reynolds number $R_e$ so that nonlinear convective terms in the Newtonian case in the Navier-Stokes equations may be neglected.

(b) long-wavelength compared with the underformed radius of the tube.

(c) small amplitude of the wall displacement compared with the tube radius.

## 2.7 Longwavelength Approximation.

We now assume small Reynolds Number and small wave number such that

$$R_e \leq 1, \quad n \leq 1 \qquad (2.16)$$

We introduce cylindrical coordinates such that the Z axis is along the centreline of the tube; hence the equation of the tube wall becomes

$$h(Z,t) = a\left(1 + \varepsilon \sin\frac{2\pi}{\lambda}(Z - ct)\right)$$

We now assume that the pressure is independent of the radial coordinate such that

$$p=p(Z,t) \qquad (2.17)$$

If we perform a coordinate transfomation such that

$$r=R \text{ and } z=Z-ct \qquad (2.18)$$

our continuity equation and equation of motion in cylindrical coordinates become if we let u and w be the radial and axial velocity respectively

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0 \qquad (2.19)$$

$$\frac{\partial p}{\partial z} = \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \qquad (2.20)$$

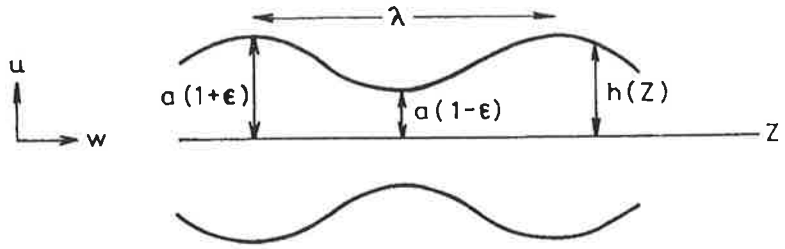


Figure 2.9 Coordinate Transformation. Adapted from Mazumdar(1992)

With the above geometry in mind(Figure 2.9) and using the boundary condition w=-c at r=h

from equation 2.20

$$w = -c - \frac{1}{4\mu}\frac{\partial p}{\partial z}\left(h^2 - r^2\right) \tag{2.21}$$

Given the flow rate is

$$q = 2\pi \int_0^h rwdr \tag{2.22}$$

we obtain by substitution and rearrangement

$$\frac{\partial p}{\partial z} = -\frac{8\mu q}{\pi h^4} - \frac{8\mu c}{h^2} \tag{2.23}$$

hence from above equations 2.19 and 2.21

$$w = -c + 2\left(\frac{q}{\pi h^4} + \frac{c}{h^2}\right)\left(h^2 - r^2\right) \tag{2.24}$$

that is,

$$w = -c + 2\left(\frac{q}{\pi h^2} + c\right)\left(1 - \frac{r^2}{h^2}\right)$$

whose velocity profile will be parabolic-Poiseuille's flow

and

$$u = -\frac{\partial h}{\partial z}\left(\frac{cr^3}{h^3} - \frac{2qr}{\pi h^3} + \frac{2qr^3}{\pi h^5}\right) \tag{2.25}$$

that is,

$$u = \frac{\partial h}{\partial z}\frac{2q}{\pi h^3}r\left[1 - \left(\frac{1}{h^2} + \frac{\pi c}{2q}\right)r^2\right]$$

whose velocity profile is only approximately parabolic

# CHAPTER 3

# OUTLINE OF BIOLOGICAL FLUID DYNAMICS

## 3.1 General Introduction of non-Newtonian Fluids.

The study of fluid dynamics began with an ideal fluid that is incompressible and without viscosity or elasticity and completely frictionless. However, severe limitations in the practical application of frictionless flow to real situations in general led to the development of a dynamical theory for the simpliest class of real fluids-Newtonian fluids.

A Newtonian fluid, by definition, is one in which the coefficient viscosity is constant at all rates of shear as seen in Figure 3. 1. Homogeneous liquids behave closely like Newtonian fluid, however, there are fluids that do not obey the linear relationship between stress and shear strain rate. Hence, these fluids that exhibit a non-linear relationship between shear stress and rate of strain are called non-Newtonian fluids. Many common fluids behave as non-Newtonian, for example, paints, wet clay, solutions of various polymers and many biological fluids like chyme, blood etc.

Although the properties of non-Newtonian fluids do not allow simple and precise analysis as developed for Newtonian fluids, there are some interesting and useful characteristics of non-Newtonian fluids. For example the anomalous behaviour of blood that deviates from Newtonian and exhibit non-Newtonian properties of two types

(a) at low shear rates, the apparent viscosity increases markedly-sometimes a yield stress is required for flow, hence power-law and Casson modelling.

( b )  in small tubes such as capillaries, the apparent viscosity at higher shearing rates is smaller than is for larger tubes. It is thus concluded that the behaviour of blood is almost Newtonian at higher shear rates, while at low shear rates the blood exhibits yield stress and non-Newtonian behaviour.

## 3.2 Classification of non-Newtonian Fluids.

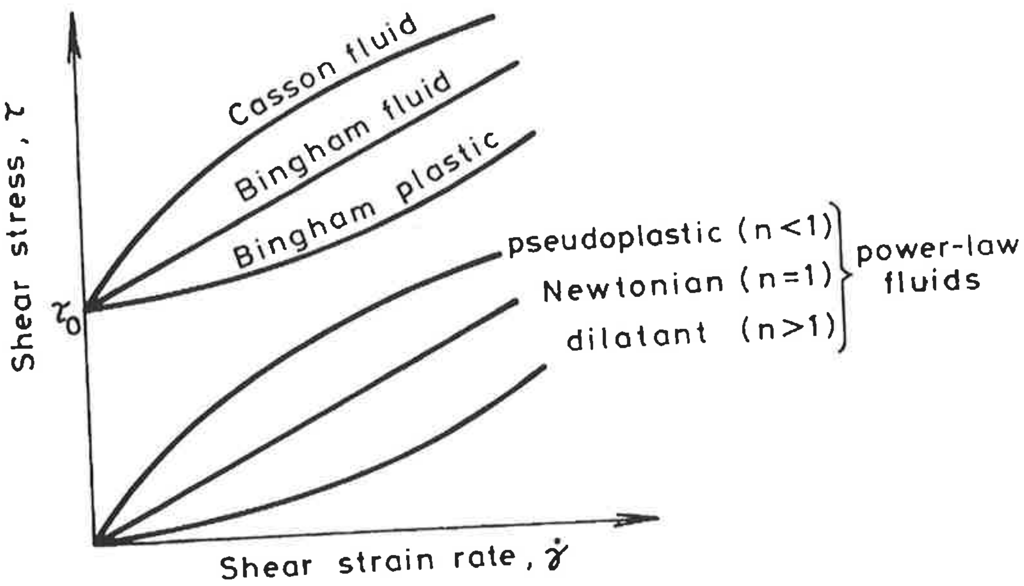


Figure 3.1  Newtonian and non-Newtonian curves.Adapted from Mazumdar(1999)

Fluids which do not obey the linear relationship between shear stress and the rate of shear strain can be grouped into 3 general classifications.

( 1 )  The simpliest of the non-Newtonian fluids is the time independent non-Newtonian fluids in which the shear strain rate is a nonlinear function of the shear stress, independent of shearing time and previous shear stress rate history.

( 2 )  Time dependent non-Newtonian fluids have more complex shearing stress strain rate relationships. Hence the apparent viscosity depends not only on the strain rate but also on the the time shear has been applied.

These can generally be grouped into two categories

(i) thixotropic fluids-printers ink-the shear stress decreases with time as the fluid is sheared

(ii) rheopectic fluids-the shear stress increases with time as the fluid is sheared

( 3 ) Viscoelastic fluids- this is different from truly viscous fluids in which all the energy of deformation is dissipated. Hence the shear strain as well as shear strain rate are related in some way to shear stress.

## 3.3 Time Independent Fluids.

For a time independent non-Newtonian fluid the constitutive equation is

$$\tau = f\left(\dot{\gamma}\right) \tag{3.1}$$

or

$$\dot{\gamma} = f(\tau) \tag{3.2}$$

A Newtonian fluid is a special case of a non-Newtonian fluid where the function $f\left(\dot{\gamma}\right)$ is linear in the form $\mu\dot{\gamma}$

(a) power law fluids

One important class of non-Newtonian fluids is that of power-law fluids which have the constitutive equation

$$\begin{aligned}\tau &= \mu\dot{\gamma}^{n} \\ &= \mu\dot{\gamma}^{n-1}\dot{\gamma}\end{aligned} \tag{3.3}$$

This class of non-Newtonian fluids has effective viscosity coefficient or apparent viscosity $\mu\dot{\gamma}^{n-1}$ and does not have a yield stress.

If n<1, we get a psuedoplastic strain rate. If n>1, we get a dilatant power law fluid in which the apparent viscosity increases progressively with increasing strain rate. If n=1, we obtain the Newtonian fluid as a special case.

(b) Casson fluid

Another important class of non-Newtonian fluids is that of Casson fluid which has the constitutive equation

$$\tau^{\frac{1}{2}} = \mu^{\frac{1}{2}}\dot{\gamma}^{\frac{1}{2}} + \tau_0^{\frac{1}{2}} \quad , \tau \geq \tau_0 \tag{3.4}$$

$$\dot{\gamma} = 0 \quad , \tau \leq \tau_0 \tag{3.5}$$

There exists a yield stress at zero shear rate, followed by a non-linear relationship between shear stress and shear strain rate. If the shearing stress is less than the initial shear rate no flow takes place. The plastic viscosity is non-linear and not constant. The behavior of many real fluids such as slurries, household paint and plastics very closely approximate this concept.

# CHAPTER 4

# FLOW IN A TWO-DIMENSIONAL CHANNEL

## 4.1 Mathematical Modelling of a Newtonian Fluid in a Two-Dimensional Channel.

The basic definition of a Newtonian fluid is given in Chapter 3. This shows that there exists a linear relationship between the shear stress and the shear strain rate where the constant of proportionality is known as the viscosity of the fluid.

It is convenient to model physiological fluids as Newtonian because Newtonian fluids are generally easier to deal with as long as this does not cause over simplification of the particular physiological fluid being modelled.

Much of the research interest of peristaltic flow has concentrated on incompressible Newtonian fluid. The analysis of this type sufficiently explains some physiological flows, such as the flow of fluid from the kidney to the bladder via the ureter against a pressure gradient.

In this section, the peristaltic motion of a Newtonian, viscous, incompressible fluid will be modelled for the case of a two-dimensional channel. The undeformed radius of the channel is given by d and the tube is considered to be infinitely long. The sinusiodal waves travelling along the walls, at wave speed c, have wavelength $\lambda$ and amplitude A. The geometry of the sinusiodal travelling wave is given by G(x,t) where

$$G(x,t) = A\cos\frac{2\pi}{\lambda}(x-ct) \qquad (4.1)$$

We assume the conventional cartesian coordinate system where x is the *abscissa* and y is the *ordinate*, as shown in Figure 4.1

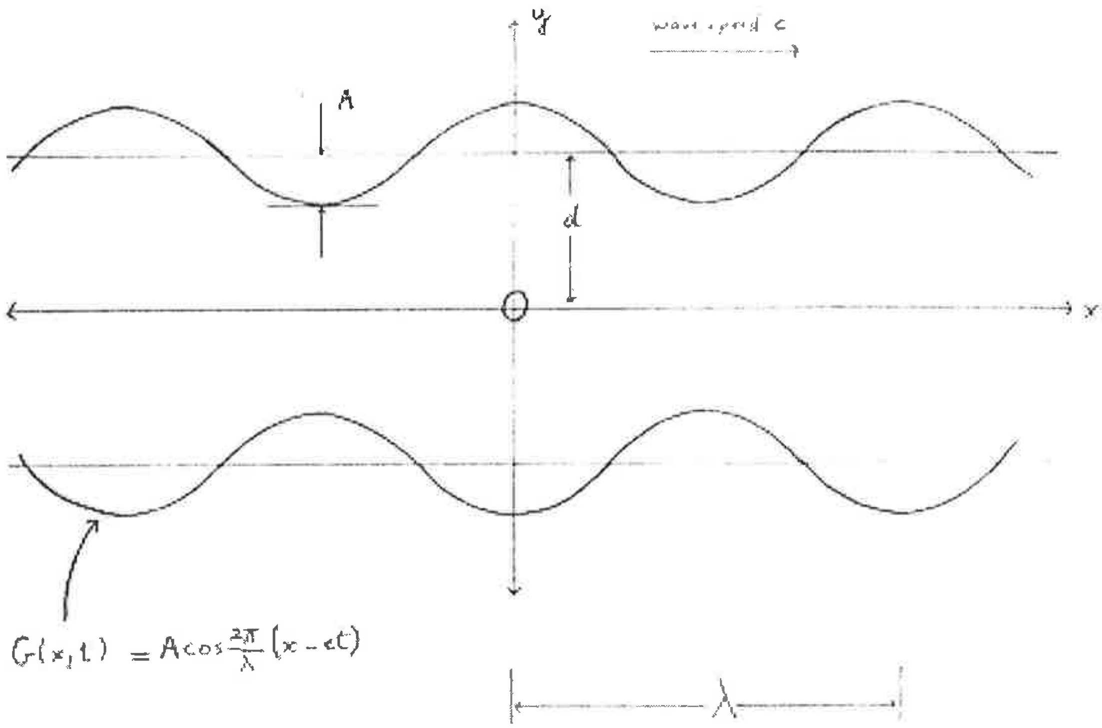


Figure 4.1 Geometry of Peristaltic Flow in a Two-dimensional Channel

The governing equations of motion for the two-dimensional flow of a viscous incompressible Newtonian fluid as discussed in Chapter 2 are given by the Navier-Stokes equations in the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u \qquad (4.2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v \qquad (4.3)$$

where x and y are Cartesian coordinates and u and v are the fluid velocity components in the x and y directions respectively and t is the time, p is the pressure, $\rho$ is the fluid density and $\nu=\frac{\mu}{\rho}$ is the kinematic viscosity of the fluid, $\mu$ as the coefficient of viscosity.

The equation of continuity is given as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (4.4)$$

We may now introduce a stream function $\psi$ that satisties the equation of continuity as

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \tag{4.5}$$

Cross Differeniating equations 4.2 and 4.3 and eliminating pressure p we obtain

$$\frac{\partial^2 u}{\partial y \partial t} - \frac{\partial^2 v}{\partial x \partial t} + u\left(\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 v}{\partial x^2}\right) + v\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y}\right) = \nu\left(\frac{\partial}{\partial y}\nabla^2 u - \frac{\partial}{\partial x}\nabla^2 v\right) \tag{4.6}$$

Using the above equation for definition of stream function we obtain

$$\frac{\partial}{\partial t}\left(\psi_{yy} + \psi_{xx}\right) + \psi_y\left(\psi_{yyx} + \psi_{xxx}\right) - \psi_x\left(\psi_{yyy} + \psi_{xxy}\right) = \nu\left(\nabla^2 \psi_{yy} + \nabla^2 \psi_{xx}\right) \tag{4.7}$$

Defining the Laplacian Operator as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{4.8}$$

and non dimensionalising by introducing non dimensional variables(see Appendix A) we obtain

$$\frac{c^2}{d^2}\left(\frac{\partial}{\partial t}\nabla^2 \psi + \psi_y \nabla^2 \psi_x - \psi_x \nabla^2 \psi_y\right) = \frac{c\nu}{d^3}\nabla^4 \psi \tag{4.9}$$

hence

$$\frac{\partial}{\partial t}\nabla^2 \psi + \psi_y \nabla^2 \psi_x - \psi_x \nabla^2 \psi_y = \frac{1}{R_e}\nabla^4 \psi \tag{4.10}$$

Assuming that during the peristaltic motion that there is no horizontal displacement of the tube walls the boundary conditions are as follows:

( I ) u=0

the no slip condition (particles of a viscous fluid in the vicinity of the surface over which it flows adhere to the surface) (4.11)

( i i ) $v = \pm \frac{\partial}{\partial t} G(x, y)$

the impermeable condition (no fluid penetrates the surface: the fluid transverse velocity component and the surface velocity are equal at the wall) (4.12)

Using G(x,t) as given in equation 4.1 and the above the boundary conditions we find

$$\psi_y = 0 \qquad (4.13)$$

and

$$\psi_x = \mp \frac{2\pi A c}{\lambda} \sin \frac{2\pi}{\lambda}(x - ct) \qquad at \ \ y = \pm(d + G(x,t)) \qquad (4.14)$$

Organs of the human body involved in peristalsis, as described in previous chapters, are of a finite length and hence the end conditions need to be specified. However in this analysis we shall be considering a channel of infinite length, so end conditions need not be specified. What is required is the pressure gradient which is assumed to be of the form,

$$\frac{\partial p}{\partial x} = \left(\frac{\partial p}{\partial x}\right)_0 + A\left(\frac{\partial p}{\partial x}\right)_1 + A^2\left(\frac{\partial p}{\partial x}\right)_2 + \ldots \qquad (4.15)$$

that is,

$$\frac{\rho c^2}{d}\frac{\partial p}{\partial x} = \frac{\rho c^2}{d}\left(\frac{\partial p}{\partial x}\right)_0 + \frac{\rho c^2}{d}\frac{A}{d}\left(\frac{\partial p}{\partial x}\right)_1 + \frac{\rho c^2}{d}\frac{A^2}{d^2}\left(\frac{\partial p}{\partial x}\right)_2 + \ldots \qquad (4.16)$$

and after nondimensionalisation, yields

$$\frac{\partial p}{\partial x} = \left(\frac{\partial p}{\partial x}\right)_0 + \varepsilon\left(\frac{\partial p}{\partial x}\right)_1 + \varepsilon^2\left(\frac{\partial p}{\partial x}\right)_2 + \ldots \qquad (4.17)$$

and the boundary conditions 4.13 and 4.14 are reduced to the form after non-dimensionalisation as follows

$$\psi_y = 0 \qquad (4.18)$$

$$\psi_x = \mp \alpha\varepsilon \sin\alpha(x - t) \qquad at \ y = \pm(1 + \varepsilon\cos\alpha(x - t)) \qquad (4.19)$$

where $\varepsilon$ , $\alpha$ , $R_e$, are the amplitude ratio, wave number and Reynolds Number respectively and are defined in the Appendix.

## 4.2 A Method of Solution of a Newtonian Fluid in a Two-Dimensional Channel.

A method of solution is now required to solve the differential equation(4.10) for the Newtonian case,

The form of the stream function $\psi$ is assumed to be

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots \qquad (4.20)$$

After substituting equation 4.20 into equation 4.10 and equating terms of the same order, that is after collecting like terms in $\varepsilon^0, \varepsilon^1, \varepsilon^2$ *etc* a system of differential equations may be established to solve $\psi_0, \psi_1, \psi_2$ *etc* successively. After some rearrangement of terms of the same order of $\varepsilon$ gives us for unsteady Newtonian flow the following equations to $\varepsilon^2$

$$\frac{\partial}{\partial t}\nabla^2\psi_0 + \psi_{0y}\nabla^2\psi_{0x} - \psi_{0x}\nabla^2\psi_{0y} = \frac{1}{R_e}\nabla^4\psi_0 \tag{4.21}$$

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + \psi_{0y}\nabla^2\psi_{1x} + \psi_{1y}\nabla^2\psi_{0x} - \psi_{0x}\nabla^2\psi_{1y} - \psi_{1x}\nabla^2\psi_{0y} = \frac{1}{R_e}\nabla^4\psi_1 \tag{4.22}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\nabla^2\psi_2 + \psi_{0y}\nabla^2\psi_{2x} + \psi_{1y}\nabla^2\psi_{1x} - \psi_{0x}\nabla^2\psi_{2y} - \psi_{1x}\nabla^2\psi_{1y} + \\ &\psi_{2y}\nabla^2\psi_{0x} - \psi_{2x}\nabla^2\psi_{0y} = \frac{1}{R_e}\nabla^4\psi_2\end{aligned} \tag{4.23}$$

The boundary conditions equations 4.18 and 4.19 can be written using a Taylor series expansion about $y = \pm(1+G)$ *where* $G = \varepsilon\cos\alpha(x-t)$ such that

$$\psi_y(\pm1) \pm G\psi_{yy}(\pm1) + \frac{G^2}{2}\psi_{yyy}(\pm1) \pm \ldots = 0 \tag{4.24}$$

$$\psi_x(\pm1) \pm G\psi_{xy}(\pm1) + \frac{G^2}{2}\psi_{xyy}(\pm1) \pm \ldots = \mp\alpha\varepsilon\sin\alpha(x-t) \tag{4.25}$$

After substituting equation 4.20 into equations 4.24 and 4.25 and collecting and equating terms of the same order in $\varepsilon$ on either side of the equations to $\varepsilon^2$ gives

$$\begin{aligned}&\psi_{0y}(\pm1) = 0\\ &\psi_{1y}(\pm1) \pm \psi_{0yy}(\pm1)\cos\alpha(x-t) = 0\\ &\psi_{2y}(\pm1) \pm \psi_{1yy}(\pm1)\cos\alpha(x-t) + \frac{1}{2}\psi_{0yyy}(\pm1)\cos^2\alpha(x-t) = 0\\ &\psi_{0x}(\pm1) = 0\\ &\psi_{1x}(\pm1) \pm \psi_{oxy}(\pm1)\cos\alpha(x-t) = \mp\alpha\sin\alpha(x-t)\\ &\psi_{2x}(\pm1) \pm \psi_{1xy}(\pm1)\cos\alpha(x-t) + \frac{1}{2}\psi_{0xyy}(\pm1)\cos^2\alpha(x-t) = 0\end{aligned} \tag{4.26(a)-(f)}$$

From equation 4.17, because $\left(\frac{\partial p}{\partial x}\right)_0$ , is a constant $\psi_0$ may be considered as a function of y only. Hence from equation 4.21 and using the condition that equation 4.26 is automatically satisfied we obtain for the steady flow case the following ordinary differential equation

$$\nabla^4 \psi_0 = 0 \quad (4.27)$$

that is

$$\frac{d^4 \psi_0}{dy^4} = 0 \quad (4.28)$$

which has a solution of the form

$$\psi_0(y) = Ay^3 + By^2 + Cy + D \quad (4.29)$$

Using equation 4.2 and a nondimensionalising gives us after making use of equation 4.17 and equation 4.20 and collecting terms of $\varepsilon^0$ the following is obtained

$$\psi_{yt} + \psi_y \psi_{yx} - \psi_x \psi_{yy} = -\frac{\partial p}{\partial x} + \frac{1}{R_e} \nabla^2 \psi_y \quad (4.30)$$

which yields the equation

$$R_e \left(\frac{\partial p}{\partial x}\right)_0 = \frac{\partial^3 \psi_0}{\partial y^3} = \frac{d^3 \psi_0}{dy^3} \quad (4.31)$$

however from equation 4.29

$$\frac{d^3 \psi_0}{dy^3} = 6A \Rightarrow A = \frac{R_e}{6}\left(\frac{\partial p}{\partial x}\right)_0 \quad (4.32)$$

If we use the symmetry condition

$$\frac{\partial u}{\partial y}(y=0) = 0 \Rightarrow \frac{d^2 \psi_0}{dy^2}(y=0) = 0 \quad (4.33)$$

hence from equation 4.29 this implies B=0 and using the condition

$$\psi_0(y=0) = 0 \quad (4.34)$$

$$\Rightarrow D = 0 \quad (4.35)$$

Finally, 4.26(a) implies that

$$3A + C = 0 \quad (4.36)$$

Hence,

$$C = -\frac{R_e}{2}\left(\frac{\partial p}{\partial x}\right)_0 \quad (4.37)$$

Therefore substituting these values in equation 4.29 gives us the solution to $\psi_0(y)$ as

$$\psi_0(y) = -\frac{R_e}{2}\left(\frac{\partial p}{\partial x}\right)_0\left(y - \frac{y^3}{3}\right) \quad where \quad k = \frac{R_e}{2}\left(\frac{\partial p}{\partial x}\right)_0 \quad (4.38)$$

That is,

$$\psi_0(y) = k\left(\frac{y^3}{3} - y\right) \quad (4.39)$$

which coincides with the solution given in the literature, Fung And Yih(1968).

We now seek solutions to $\psi_1(x,y,t)$ and are of the form

$$2\psi_1 = \Phi_1(y)e^{i\alpha(x-t)} + \Phi_1^*(y)e^{-i\alpha(x-t)} \quad (4.40)$$

where * denotes complex conjugate.

The differential equations for $\Phi s$ and their solution, for the case $\left(\frac{\partial p}{\partial x}\right)_0 = 0$ (free pumping) are given in the paper by Fung and Yih(1968) as follows,

$$\Phi_1(y) = A\sinh\alpha y + B\sinh\beta y \qquad where \quad \beta^2 = \alpha^2 - i\alpha R_e$$

*and*

$$A = \frac{-\beta\cosh\beta}{\alpha\cosh\alpha\sinh\beta - \beta\cosh\beta\sinh\alpha} \quad and \quad B = \frac{\alpha\cosh\alpha}{\alpha\cosh\alpha\sinh\beta - \beta\cosh\beta\sinh\alpha}$$

It is found that the non-dimensionalised time average of the pressure gradient is given by

$$\overline{\frac{\partial p}{\partial x}} = \frac{\varepsilon^2}{R_e}K + O(\varepsilon^3)$$

where K is a constant determined from the end conditions.

Hence it is observed that the magnitude of the mean pressure gradient decreases as Reynolds number increases and conversely. This demonstrates the significance of Reynolds number. Pumping against a positive pressure gradient greater than the critical value would result in backflow(reflux).

## 4.3 Mathematical Modelling of a Power Law Fluid in a Two-Dimensional Channel.

In this section, the peristaltic motion of a non-Newtonian fluid will be considered in a two dimensional channel; the fluid will be modelled as a power law fluid. In chapter 3 it was stated that a Newtonian fluid is one whereby its shear stress and shear rate of strain obey a linear relationship, Hence a non-Newtonian fluid is one whereby its shear stress and shear rate of strain are related non-linearly(see section 3.2 for graphical depiction)

The non-Newtonian fluid in this section obeys the following

$$\tau = \mu \dot{\gamma}^{n} \tag{4.41}$$

The stress tensor corresponding to a power law fluid is given by

$$\tau_{ij} = -p\delta_{ij} + m\theta V_{ij} \tag{4.42}$$

where $\tau_{ij}$ is the stress tensor and $V_{ij}$ is the rate of strain tensor and p is the isotropic pressure and m is the flow consistency index.

$$\theta = \left| \frac{1}{2} V_{ij} V_{ij} \right|^{\frac{n-1}{2}} \tag{4.43}$$

where n is the flow behaviour index.

## 4.4 A Method of Solution of the power law fluid in a Two-Dimesional Channel

If the fluid is modelled as an incompressible viscous fluid, the classical equation of motion relating shear stress and shear rate of strain may be used, that is

$$\rho \frac{Dq_i}{Dt} = \frac{\partial \tau_{ij}}{\partial x_j} \tag{4.44}$$

where $\rho$ is the density of the fluid and $q_i$ is the fluid velocity component in the respective ith direction and as stated above $\tau_{ij}$ is the stress tensor. Substituting of equation 4.42 in equation 4.44 gives

$$\rho \frac{Dq_i}{Dt} = \frac{\partial}{\partial x_j}\left\{-p\delta_{ij} + m\theta V_{ij}\right\}$$

$$= -\frac{\partial p}{\partial x_j}\delta_{ij} + m\frac{\partial \theta}{\partial x_j}V_{ij} + m\theta\frac{\partial V_{ij}}{\partial x_j} \tag{4.45}$$

$$= -\frac{\partial p}{\partial x_i} + m\frac{\partial \theta}{\partial x_j}V_{ij} + m\theta\left\{\nabla^2 u_i + \frac{\partial}{\partial x_i}(\nabla.q)\right\}$$

Asuming the incompressibility condition $\nabla.q = 0$, the equation of motion in tensor format becomes

$$\rho \frac{Dq_i}{Dt} = -\frac{\partial p}{\partial x_i} + m\left\{\theta\nabla^2 u_i + \frac{\partial \theta}{\partial x_j}V_{ij}\right\} \tag{4.46}$$

Using the summation convention and setting i=1 and 2 respectively and nondimensionalising in a manner as indicated in the Appendix A we obtain the two momentum equations as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R_e}\left\{\theta\nabla^2 u + 2\frac{\partial u}{\partial x}\frac{\partial \theta}{\partial x} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial \theta}{\partial y}\right\} \tag{4.47}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{R_e}\left\{\theta\nabla^2 v + 2\frac{\partial v}{\partial y}\frac{\partial \theta}{\partial y} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial \theta}{\partial x}\right\} \tag{4.48}$$

where if we use the equation of continuity 4.4 we find from 4.43

$$\theta = \left|2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}$$

$$= \left|2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(-\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}$$

$$= \left|4\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}} \tag{4.49}$$

$$= \left|4\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}$$

and the boundary conditions are given as equations 4.26(a)-(f).

The solution for the stream function $\psi$ is found by assuming a perturbation series in stream function and pressure as

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \qquad (4.50)$$

On substituting equation 4.50 and equation 4.5 into equations 4.47 & 4.48 and equating coeffcients of $\varepsilon$ of equal order on either side of the equations we obtain a series of partial diferrential equations for $\psi_0$, $\psi_1$, $\psi_2$ $etc$

**(i) Solution Procedure (Zeroth Order Approximation)**

The zeroth order equation describing the absence of peristaltic flow(free pumping situation) is found by considering the above substitution in equation 4.47 and collecting coefficients of order $\varepsilon^0$ on either side of the equation and yields, after introducing non dimensional parameters as described in the Appendix A, as follows

$$R_e\left(\frac{\partial p_0}{\partial x}\right) = k = \left(\psi_{0yy}\right)^{n-1}\psi_{0yyy} + (n-1)\left(\psi_{0yy}\right)^{n-1}\psi_{0yy}$$
$$= n\left(\psi_{0yy}\right)^{n-1}\psi_{0yyy} \qquad (4.51)$$

where the solution $\psi_0(y)$ is assumed to be a function of y only because the zeroth order axial pressure gradient, $\frac{\partial p_0}{\partial x}$, is assumed to be constant.

The solution to equation 4.51 is sought in the form

$$\psi_0(y) = A\left(\frac{ny^{\frac{2n+1}{n}}}{2n+1} - y\right) \qquad (4.52)$$

which satisfies the required boundary conditions,that is, equations 4.26(a)-(f) and 4.34. Thus we obtain

$$\psi_0(0) = 0$$
$$\psi_{0y}(1) = 0 \qquad (4.53)$$
$$\psi_{0yy}(0) = 0$$

We are now required to find the constant A appearing in 4.52. Hence substituting equation 4.52 in equation 4.51 we obtain

$$n\left(\frac{A(n+1)}{n}\right)^{n-1} y^{\frac{n-1}{n}} \frac{A(n+1)}{n^2} y^{\frac{1-n}{n}} = k = \left(\frac{A(n+1)}{n}\right)^n \qquad (4.54)$$

$$\Rightarrow A = \frac{n}{n+1} k^{\frac{1}{n}} \qquad (4.55)$$

Therefore the solution for $\psi_0$ becomes

$$\psi_0(y) = \frac{n}{n+1} k^{\frac{1}{n}} \left( \frac{n y^{\frac{2n+1}{n}}}{2n+1} - y \right) \qquad (4.56)$$

We see that for n=1 equation 4.56 reduces to equation 4.39 which is the Newtonian case of peristaltic motion in a two dimensional channel as outlined in the previous section.

## (ii) Solution Procedure (First Order Approximation)

We now seek to develop the equations and hence the solution for $\psi_1$.

On substituting equation 4.5 and equation 4.50 into equations 4.47 and 4.48 and non dimensionalising as shown in the Appendix A and collecting the coefficients of $\varepsilon^1$ we obtain respectively the following partial differential equations.

Equation 4.47 reduces to

$$\psi_{1yt} + \psi_{0y}\psi_{1yx} - \psi_{1x}\psi_{0yy} =$$
$$-\frac{\partial p_1}{\partial x} + \frac{1}{R_e}\left(\psi_{0yy}\right)^{n-1}\left\{\psi_{1yyxx} + \psi_{1yyy} + (n-1)\left(\psi_{1yyy} - \psi_{1xxy}\right) + \frac{2(n-1)}{ny}\left(\psi_{1yy} - \psi_{1xx}\right)\right\} \qquad (4.57)$$

Equation 4.48 reduces to

$$-\psi_{1xt} - \psi_{0y}\psi_{1xx} =$$
$$-\frac{\partial p_1}{\partial y} + \frac{1}{R_e}\left(\psi_{0yy}\right)^{n-1}\left\{-\psi_{1xxx} - \psi_{1xyy} + (n-1)\left(\psi_{1xyy} - \psi_{1xxx}\right) - \frac{2(n-1)}{ny}\psi_{1xy}\right\} \tag{4.58}$$

To obtain a single equation by eliminating the pressure terms, we cross differentiate equation 4.58 and equation 4.59 and subtract to obtain the following

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + \psi_{0y}\nabla^2\psi_{1x} - \psi_{1x}\psi_{0yyy} =$$
$$\frac{1}{R_e}\left(\psi_{0yy}\right)^{n-1}\left\{\begin{matrix} n\psi_{1xxxx} + (4-2n)\psi_{1xxyy} + n\psi_{1yyyy} + \dfrac{2(1-n)}{n^2y^2}\left(\psi_{1yy} - \psi_{1xx}\right) + \\ \dfrac{(n-1)(n-1)}{ny}\psi_{1xxy} + \dfrac{(n+2)(n-1)}{ny}\psi_{1yyy} \end{matrix}\right\} \tag{4.59}$$

Letting n=1, and remembering that $\psi_0$ is a function of y only, equation 4.59 reduces to the governing equation 4.22, $\psi_1$ for the Newtonian case. If we introduce the boundary conditions as given in equation 4.26 and substitute $\psi_0$ as given in equation 4.56 we obtain the boundary condtions as

$$\begin{aligned} \psi_{1y}(1) &= -k^{\frac{1}{n}}\cos\alpha(x-t) \\ \psi_{1x}(1) &= -\alpha\sin\alpha(x-t) \end{aligned} \tag{4.60}$$

From these boundary conditions, it is assumed that $\psi_1$ can be obtained in the form

$$\psi_1(x,y,t) = f(y)\cos\alpha(x-t) + g(y)\sin\alpha(x-t) \tag{4.61}$$

Where f and g are to be determined. By substituting equation 4.61 for $\psi_1$ and equation 4.56 for $\psi_0$ into equation 4.59 and collecting coefficients of $\cos\alpha(x-t)$ *and* $\sin\alpha(x-t)$ on either side of the equation, two differential equations for f(y) and g(y) can be obtained. Due to the complexity of the equations exact solutions in analytical form are not able to be obtained.

Approximate solutions can be obtained by assuming that the wave number is small. However, this approximation will not be performed and analysed in this case but will be considered later in the case of Casson fluid in a two-dimensional channel and a power-law fluid in an axisymmetric tube.

## 4.5 Mathematical Modelling of a Casson Fluid Case in a Two-Dimensional Channel

Consider the peristaltic motion of a non-Newtonian fluid, modelled as a Casson fluid in a two-dimensional channel, where, d, is the undeformed width of the channel and the channel is considered to be infinitely long; A, represents the amplitude of the sinusiodal waves travelling along the channel at velocity c; $\lambda$, is the wavelength (Figure.4.1). A rectangular co-ordinate system is chosen for the channel with x along the centre line and y normal to it. Let u and v be the longitudinal and transverse velocity components, respectively. It is assumed that an infinite train of sinusiodal waves progresses along the walls in the x direction. The vertical displacements for the upper and lower walls are G and -G for peristaltic flow at time t, where G is defined by,

$$G(x,t) = A\cos\frac{2\pi}{\lambda}(x-ct) \qquad (4.62)$$

We assume that there is no motion of the wall in the longitudinal direction (extensible or elastic wall).

For the case of peristaltic pumping of a Casson fluid in a planar channel the stress-strain relationship in tensor format is given by Fung(1981) as

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(J_2)V_{ij} \qquad (4.63)$$

where

$$\mu(J_2) = \left[(\eta^2 J_2)^{\frac{1}{4}} + 2^{-\frac{1}{2}}\tau_y^{\frac{1}{2}}\right]^2 J_2^{-\frac{1}{2}} = \left[\eta^{\frac{1}{2}} + 2^{-\frac{1}{2}}\tau_y^{\frac{1}{2}} J_2^{-\frac{1}{4}}\right]^2 = \left[\alpha + \beta J_2^{-\frac{1}{4}}\right]^2 = \mu(say) \qquad (4.64)$$

Here we have denoted

$$\alpha = \eta^{\frac{1}{2}} \quad : \quad \beta = 2^{-\frac{1}{2}}\tau_y^{\frac{1}{2}} \qquad (4.65)$$

*where* $\eta$ *is the Casson coefficient of viscosity, and* $\tau_y$ *is the yield stress*

Here,

$$V_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (4.66)$$

and

$$J_2 = \frac{1}{2}V_{ij}V_{ij} = \frac{1}{2}\left(V_{11}{}^2 + V_{22}{}^2 + 2V_{12}{}^2\right) \quad (4.67)$$

where

$$V_{11} = \frac{\partial u}{\partial x},\ V_{22} = \frac{\partial v}{\partial y},\ V_{12} = V_{21} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

## 4.6 A Method of Solution of a Casson Fluid in a Two-Dimensional Channel

Substituting equations (4.63-4.67) into the basic equations for continuity and momentum respectively given by

$$div\underline{q} = 0 \quad (4.68)$$

and

$$\rho\frac{Dq_i}{Dt} = \frac{\partial}{\partial x_j}\sigma_{ij} \quad (4.69)$$

we have

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + 2\mu_x\frac{\partial u}{\partial x} + \mu_y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

which, using continuity reduces to

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + 2\mu_x\frac{\partial u}{\partial x} + \mu_y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \mu\nabla^2 u \quad (4.70)$$

Similarly,

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + 2\mu_y\frac{\partial v}{\partial y} + \mu_x\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \mu\nabla^2 v \quad (4.71)$$

Defining as before the stream function $\psi(x,y)$ as $u = \psi_y$ *and* $v = -\psi_x$ we obtain from equations (4.70) and (4.71) respectively

$$\rho\left(\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy}\right) = -\frac{\partial p}{\partial x} + 2\mu_x\psi_{xy} + \mu_y\left(\psi_{yy} - \psi_{xx}\right) + \mu\nabla^2\psi_y \quad (4.72)$$

and

$$\rho\left(-\psi_{xt}-\psi_y\psi_{xx}+\psi_x\psi_{xy}\right)=-\frac{\partial p}{\partial y}-2\mu_y\psi_{xy}+\mu_x\left(\psi_{yy}-\psi_{xx}\right)-\mu\nabla^2\psi_x \qquad (4.73)$$

where $\mu_x=\frac{\partial}{\partial x}\left[\mu\left(J_2\right)\right]$ *and* $\mu_y=\frac{\partial}{\partial y}\left[\mu\left(J_2\right)\right]$

## ( i ) Solution Procedure( Zeroth Order Approximation)

Expressing stream function, $\psi$, pressure, p, and $\mu$ as a series in terms of amplitude ratio $\varepsilon=\frac{A}{d}$, where A is the amplitude and ,d, is the undeformed width of the channel, (Figure 4.1), we have

$$\psi=\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\ldots \qquad (4.74)$$

$$p=p_0+\varepsilon p_1+\varepsilon^2 p_2+\ldots \qquad (4.75)$$

$$\mu=\mu_0+\varepsilon\mu_1+\varepsilon^2\mu_2+\ldots \qquad (4.76)$$

where it is assumed that $\psi_0$ is a function of y only, ie, $\psi_0=\psi_0(y)$, because of zeroth order axial pressure gradient. We finally obtain from equations (4.72 & 4.74-4.76) after collecting coefficents of $\varepsilon^0$.

$$\frac{\partial p_0}{\partial x}=2\mu_{0x}\psi_{0yx}+\mu_{0y}\psi_{0yy}-\mu_{0y}\psi_{0xx}+\mu_0\psi_{0xxy}+\mu_0\psi_{0yyy} \qquad (4.77)$$

that is,

$$\frac{\partial p_0}{\partial x}=\mu_{0y}\psi_{0yy}+\mu_0\psi_{0yyy} \qquad (4.78)$$

Therefore

$$\frac{\partial p_0}{\partial x}=\frac{\partial}{\partial y}\left(\mu_0\psi_{0yy}\right) \qquad (4.79)$$

We now need to find the zeroth order expression for $\mu_0=\mu\left(J_2\right)_0$

From equation (4.67) and expanding and substituting we have

$$J_2=\frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\frac{2}{4}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\}=\frac{1}{2}\left\{\psi_{xy}^{\;2}+\psi_{xy}^{\;2}+\frac{1}{2}\left(\psi_{yy}-\psi_{xx}\right)^2\right\}$$

$$= \left\{ \psi_{xy}{}^2 + \frac{1}{4}\left(\psi_{yy} - \psi_{xx}\right)^2 \right\} \tag{4.80}$$

Therefore we have, after introducing equations (4.74, 4.76, 4.80)

$$\mu\left(J_2\right) = \left\{ \alpha + \beta\left[\frac{1}{4}\left(\psi_{0yy}{}^2 + 2\varepsilon\psi_{0yy}(\psi_{1yy} - \psi_{1xx}\right)\right]^{-\frac{1}{4}} + O(\varepsilon^2) \right\}^2 \tag{4.81}$$

Neglecting second and higher order terms in $\varepsilon$ in equation (4.81) and expanding we have

$$\mu\left(J_2\right) \approx \left\{ \alpha + \beta\left(\frac{1}{4}\right)^{-\frac{1}{4}} \psi_{0yy}{}^{-\frac{1}{2}}\left(1 - \frac{\varepsilon}{2}\psi_{0yy}{}^{-1}(\psi_{1yy} - \psi_{1xx})\right) \right\}^2 \tag{4.82}$$

Which after further expansion and collecting terms in amplitude ratio for the first two terms and using equation(4.82) yield equations for $\mu_0 = \mu\left(J_2\right)_0$ and $\mu_1 = \mu\left(J_2\right)_1$ as

$$\mu_0 = \alpha^2 + 2\sqrt{2}\alpha\beta\psi_{0yy}{}^{-\frac{1}{2}} + 2\beta^2\psi_{0yy}{}^{-1} \tag{4.83}$$

$$\mu_1 = -\alpha\beta\sqrt{2}\psi_{0yy}{}^{-\frac{3}{2}}\left(\psi_{1yy} - \psi_{1xx}\right) - 2\beta^2\psi_{0yy}{}^{-2}\left(\psi_{1yy} - \psi_{1xx}\right) \tag{4.84}$$

Solving equation(4.79) by using equation(4.81) and applying the symmetry boundary condition $\psi_{0yy}(0) = 0$ we have

$$\begin{aligned} &Ky + L = \mu_0\psi_{0yy} \\ &where\ K = \rho\sqrt{c^3}d\,\frac{\partial p_0}{\partial x} \quad and\ L = 2\beta^2 \end{aligned} \tag{4.85}$$

Our equation to solve for $\psi_0(y)$ then becomes

$$\alpha^2\psi_{0yy} + 2\sqrt{2}\alpha\beta\psi_{0yy}{}^{\frac{1}{2}} - Ky = 0 \tag{4.86}$$

If we set

$$\psi_{0yy} = W^2 \tag{4.87}$$

then equation(4.86) becomes a quadratic in W as

$$\alpha^2 W^2 + 2\sqrt{2}\alpha\beta W - Ky = 0 \tag{4.88}$$

whose roots are given by

$$W = -\sqrt{2}\frac{\beta}{\alpha} \pm \frac{1}{\alpha}\sqrt{2\beta^2 + Ky} \tag{4.89}$$

Using equation(4.87 and 4.89) we obtain

$$\psi_{0yy} = \left(-\sqrt{2}\frac{\beta}{\alpha} \pm \frac{1}{\alpha}\sqrt{2\beta^2 + Ky}\right)^2 \tag{4.90}$$

But the symmetry boundary condition $\psi_{0yy}(0) = 0$ demands only the positive sign to be valid, therefore

$$\psi_{0yy} = \left(-\sqrt{2}\frac{\beta}{\alpha} + \frac{1}{\alpha}\sqrt{2\beta^2 + Ky}\right)^2 \tag{4.91}$$

Integrating equation(4.91) twice we obtain

$$\psi_0(y) = \frac{2\beta^2}{\alpha^2}y^2 + \frac{K}{6\alpha^2}y^3 - \frac{8\sqrt{2}\beta}{15K^2\alpha^2}\left(2\beta^2 + Ky\right)^{\frac{5}{2}} + Ay + B \tag{4.92}$$

Where A and B are constants of integration.

Using the boundary conditions $\psi_{0y}(1) = 0 \quad and \quad \psi_0(0) = 0$ we find

$$A = \frac{4\sqrt{2}\beta}{3\alpha^2 K}\left(2\beta^2 + K\right)^{\frac{3}{2}} - \frac{4\beta^2}{\alpha^2} - \frac{K}{2\alpha^2} \tag{4.93}$$

$$B = \frac{8\sqrt{2}\beta}{15K^2\alpha^2}\left(2\beta^2\right)^{\frac{5}{2}} \tag{4.94}$$

If we let $\beta \to 0 \quad that\ is\ \tau_y \to 0$ from equation(4.65), we obtain the Newtonian case in the form

$$\psi_0(y) = -\frac{K}{2\alpha^2}\left(y - \frac{y^3}{3}\right)$$

which coincides with the literature, Fung(1968)

We now seek to determine the dimensionless pressure rise, $\Delta p_0$, where

$$\Delta p_0 = \int_0^1 \frac{\partial p_0}{\partial x} dx \tag{4.95}$$

I will consider two cases( A and B) for pressure rise versus flow rate

(A) The flow rate, q, considering the boundary of the channel given by y=G, is

$$q = \int_0^G u dy = \int_0^G \frac{\partial \psi}{\partial y} dy \tag{4.96}$$
$$\text{where} \quad G = \varepsilon \cos \alpha (x - t)$$

Hence from equations (4.85 and 4.92-4.96) we have

$$\Delta p_0 = \frac{1}{\rho \sqrt{c^3 d}} \int_0^1 \frac{6q\alpha^2 - 12\beta^2 \varepsilon^3 \cos^3 x}{\varepsilon^3 \cos^3 x - 3\varepsilon \cos x} dx \tag{4.97}$$

the solution to which is found by making the substitution

$$z = \tan \frac{1}{2} x$$
$$\text{that is} \tag{4.98}$$
$$\cos x = \frac{1 - z^2}{1 + z^2}, \quad \sin x = \frac{2z}{1 + z^2}, \quad \text{and} \quad dx = \frac{2}{1 + z^2} dz$$

Separating integrands and using the method of partial fractions we obtain the following three integrals to be solved;

$$\frac{12q\alpha^2}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546} \frac{A_1 z + B_1}{\left(z^2 - (\theta + \gamma)\right)} dz\right] + \frac{12q\alpha^2}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546} \frac{C_1 z + D_1}{\left(z^2 - (\theta - \gamma)\right)} dz\right] + \frac{12q\alpha^2}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546} \frac{E_1 z + F_1}{\left(1 - z^2\right)} dz\right] \tag{4.99}$$

$$-\frac{12\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546} \frac{A_2 z + B_2}{\left(z^2 - (\theta + \gamma)\right)} dz\right] - \frac{12\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546} \frac{C_2 z + D_2}{\left(z^2 - (\theta - \gamma)\right)} dz\right] \tag{4.100}$$

$$+\frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546} \frac{A_3 z + B_3}{\left(z^2 - (\theta + \gamma)\right)} dz\right] + \frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546} \frac{C_3 z + D_3}{\left(z^2 - (\theta - \gamma)\right)} dz\right] + \frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546} \frac{E_3 z + F_3}{\left(1 + z^2\right)} dz\right]$$

(4.101)

where

$$\theta = \frac{b}{a} \quad \gamma = \sqrt{\frac{b^2}{a^2} - 1}$$
$$a = \varepsilon^3 - 3\varepsilon \quad b = \varepsilon^3 + 3\varepsilon \tag{4.102}$$

Clearly, in our case, both $a$ and $\theta$ are always negative and both $b$ and $\gamma$ are always positive.

The coefficients $A_i, B_i, C_i, D_i, E_i, F_i, i = 1,2,3$ are found by equating coefficients in equations (4.99-4.101) after replacing $\cos^2 x$ by $1 - \sin^2 x$ in equation(4.97) and applying Gauss-Jordan elimination, and are as follows:

$$A_1 = C_1 = E_1 = 0$$
$$F_1 = -\frac{\{(-\theta+\gamma+1) - 2\theta(2\gamma+2)(-\theta+\gamma-1)\}}{\{(-1+\theta+\gamma)(\theta-\gamma) + 2\theta(-(\theta+\gamma)(\theta-\gamma-1) + 2\theta)(2\gamma+2)\}}$$
$$D_1 = -\frac{(-\theta+\gamma+1)}{2\theta} + \frac{(-1+\theta+\gamma)(\theta-\gamma)}{2\theta} F_1 \tag{4.103}$$
$$B_1 = -1 - D_1 + F_1$$

$$A_2 = C_2 = 0$$
$$D_2 = -\frac{(1+\theta-\gamma)}{2\gamma} \tag{4.104}$$
$$B_2 = \frac{(\theta+\gamma+1)}{2\gamma}$$

$$A_3 = C_3 = E_3 = 0$$
$$F_3 = -\frac{1}{(1+\theta+\gamma)(1-(\theta-\gamma))}$$
$$D_3 = \frac{1}{2\gamma}\left(-1 + \frac{1}{(1-(\theta-\gamma))}\right) \tag{4.105}$$
$$B_3 = -D_3 - F_3$$

The solution to equation(4.97) after considering equation(4.99-4.101) takes the form,

$$\Delta p = \frac{12q\alpha^2}{\rho\sqrt{c^3}d}\left[\frac{B_1}{2\sqrt{\theta+\gamma}}\ln\left(\frac{z-\sqrt{\theta+\gamma}}{z+\sqrt{\theta+\gamma}}\right)+\frac{D_1}{2\sqrt{\theta-\gamma}}\ln\left(\frac{z-\sqrt{\theta-\gamma}}{z+\sqrt{\theta-\gamma}}\right)+\frac{F_1}{2}\tanh^{-1} z\right]$$
$$-\frac{12\beta^2\varepsilon^3}{\rho\sqrt{c^3}d}\left[\frac{B_2}{2\sqrt{\theta+\gamma}}\ln\left(\frac{z-\sqrt{\theta+\gamma}}{z+\sqrt{\theta+\gamma}}\right)+\frac{D_2}{2\sqrt{\theta-\gamma}}\ln\left(\frac{z-\sqrt{\theta-\gamma}}{z+\sqrt{\theta-\gamma}}\right)\right]$$
$$+\frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3}d}\left[\frac{B_3}{2\sqrt{\theta+\gamma}}\ln\left(\frac{z-\sqrt{\theta+\gamma}}{z+\sqrt{\theta+\gamma}}\right)+\frac{D_3}{2\sqrt{\theta-\gamma}}\ln\left(\frac{z-\sqrt{\theta-\gamma}}{z+\sqrt{\theta-\gamma}}\right)+\frac{F_3}{2}\tan^{-1} z\right]$$

**(4.106)**

If we assume $\sqrt{\theta-\gamma} = i\delta_1 \quad \sqrt{\theta+\gamma} = i\delta_2$

since $\theta-\gamma$ *and* $\theta+\gamma$ are always negative, then we have

$$\ln\left|\frac{z-i\delta_j}{z+i\delta_j}\right| = -2i\tan^{-1}\left(\frac{\delta_j}{z}\right) \tag{4.107}$$

$$where \quad i=\sqrt{-1} \quad j=1,2$$

Subsequently,

$$\Delta p_0 = -\frac{12q\alpha^2}{\rho\sqrt{c^3}d}\left[\frac{B_1}{\delta_2}\tan^{-1}\left(\frac{\delta_2}{z}\right)+\frac{D_1}{\delta_1}\tan^{-1}\left(\frac{\delta_1}{z}\right)-\frac{F_1}{2}\tanh^{-1} z\right]+\frac{12\beta^2\varepsilon^3}{\rho\sqrt{c^3}d}\left[\frac{B_2}{\delta_2}\tan^{-1}\left(\frac{\delta_2}{z}\right)+\frac{D_2}{\delta_1}\tan^{-1}\left(\frac{\delta_1}{z}\right)\right]$$
$$-\frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3}d}\left[\frac{B_3}{\delta_2}\tan^{-1}\left(\frac{\delta_2}{z}\right)+\frac{D_3}{\delta_1}\tan^{-1}\left(\frac{\delta_1}{z}\right)-\frac{F_3}{2}\tan^{-1} z\right]$$

**(4.108)**

Graphical representation of dimensionless pressure rise vs flow rate, given the boundary of the channel as y=G is shown in Figures(4.2 & 4.3). Figure(4.2) and Figure(4.3), compare Casson with Newtonian fluids and are depicted for two values of amplitude ratio.

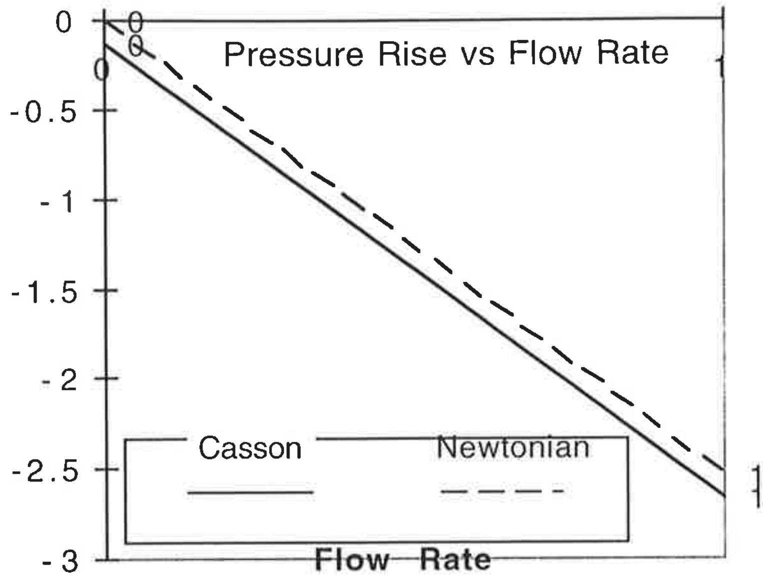


Figure 4.2: Pressure Rise vs Flow Rate, $\varepsilon$=0.2, with waveform G

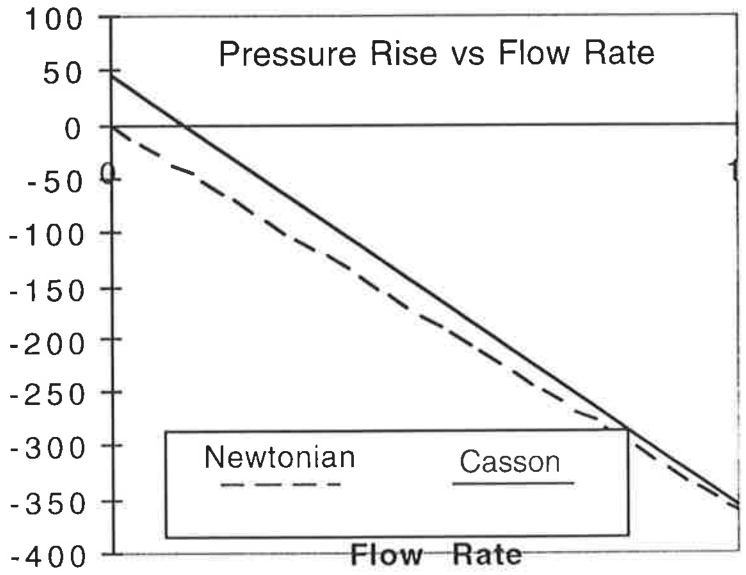


Figure 4.3: Pressure Rise vs Flow Rate, $\varepsilon$=0.8, with waveform G

(B) The flow rate, q, considering the boundary of the channel given by y=1, is

$$q = \int_0^1 u\,dy = \int_0^1 \frac{\partial \psi}{\partial y} dy = \psi(1) - \psi(0) \tag{4.109}$$

therefore from equations(4.92-4.94)

$$\psi(1) - \psi(0) = -\frac{2\beta^2}{\alpha^2} - \frac{K}{3\alpha^2} + \frac{4\sqrt{2}}{3\alpha^2 K}\left(2\beta^2 + K\right)^{\frac{3}{2}} = q \tag{4.110}$$

Applying expansion gives

$$q = \frac{2\beta^2}{\alpha^2} - \frac{K}{3\alpha^2} + \frac{16\beta^4}{3\alpha^2 K} \qquad \textbf{(4.111)}$$

Separating the pressure gradient after solving for quadratic in $K$, and using equation(23) gives

$$\frac{\partial p_0}{\partial x} = \frac{K}{\rho\sqrt{c^3 d}} = \frac{1}{\rho\sqrt{c^3 d}}\left\{-\frac{1}{2}\left(3\alpha^2 q - 6\beta^2\right) \pm \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2}\right\} \qquad \textbf{(4.112)}$$

Hence using equation(4.95) pressure rise is

$$\Delta p_0 = \frac{1}{\rho\sqrt{c^3 d}}\int_0^1\left\{-\frac{1}{2}\left(3\alpha^2 q - 6\beta^2\right) \pm \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2}\right\}dx$$

$$because\ -\frac{1}{2}\left(3\alpha^2 q - 6\beta^2\right) \pm \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2} = constant \qquad \textbf{(4.113)}$$

$$\Delta p_0 = \frac{1}{\rho\sqrt{c^3 d}}\left\{-\frac{1}{2}\left(3\alpha^2 q - 6\beta^2\right) \pm \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2}\right\}$$

However, $\beta = 0$, implies that only the negative sign of the quadratic to be valid, therefore

$$\Delta p_0 = \frac{1}{\rho\sqrt{c^3 d}}\left\{-\frac{1}{2}\left(3\alpha^2 q - 6\beta^2\right) - \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2}\right\} \qquad \textbf{(4.114)}$$

Graphical representation of dimensionless pressure rise vs flow rate, given the boundary of the channel is taken as y=1, is shown in Figure(4.4).

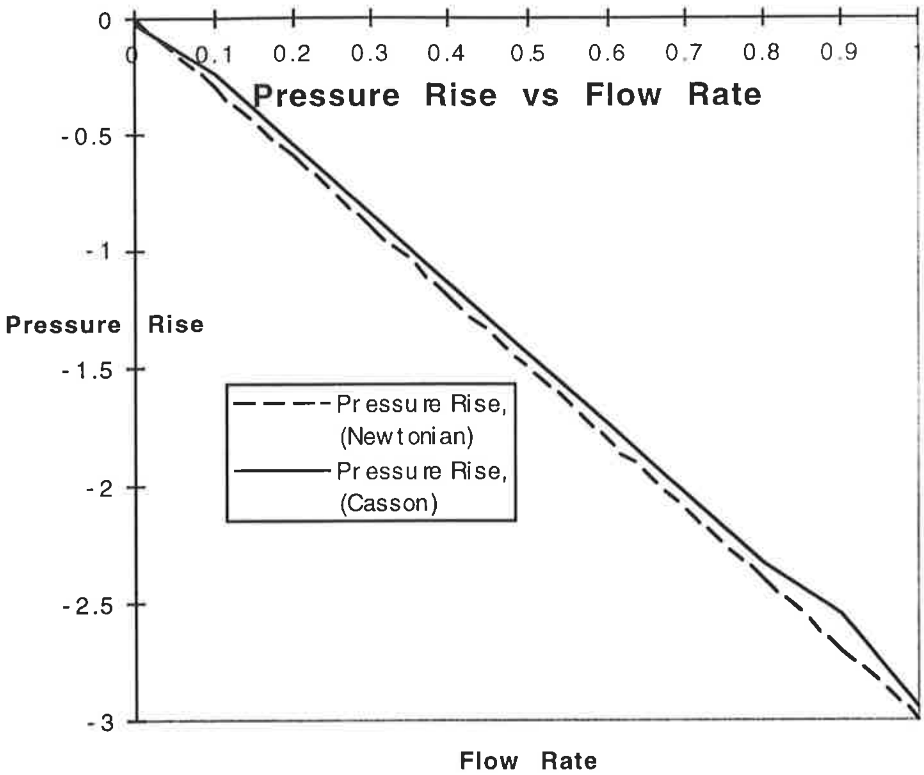


Figure 4.4: Pressure Rise vs Flow Rate, without waveform G

**(ii) Solution Procedure( First Order Approximation )**

We now look at the procedure for determining $\psi_1(x,y,t)$.

The boundary conditions for $\psi_1(x,y,t)$ are derived as follows; Assuming that there is no horizontal displacement of the tube walls during the peristaltic motion, the boundary conditions at the walls are

$$\begin{aligned}&(a)\ \ no-slip\ condition: u = 0\ at\ y = \pm[d + G(x,t)]\\&(b)\ \ impermeable\ condition: v = \pm\frac{\partial}{\partial t}G(x,t)\ at\ y = \pm[d + G(x,t)]\end{aligned} \qquad \textbf{(4.115-4.116)}$$

Using $G(x,t) = A\cos\frac{2\pi}{\lambda}(x - ct)$ and equation (4.74) and non-dimensionalising we obtain

$$\psi_y = 0 \; at \; y = \pm\left[1 + \varepsilon\cos\tilde{\alpha}(x-t)\right]$$
$$\psi_x = \mp\frac{2\pi Ac}{\lambda}\sin\frac{2\pi}{\lambda}(x-ct) = \mp\tilde{\alpha}\,\varepsilon\sin\tilde{\alpha}(x-t) \; at \; y = \pm\left[1+\varepsilon\cos\tilde{\alpha}(x-t)\right] \qquad \textbf{(4.117-4.118)}$$

The boundary conditions (4.117-4.118) can be written, using Taylor series expansions about $y = \pm(1+G)$ *where here* $G = \varepsilon\cos\tilde{\alpha}(x-t)$ as, after equating terms of the same order in $\varepsilon$, on either side of the equations, which gives

$$\psi_y(\pm1) \pm G\psi_{yy}(\pm1) + \frac{G^2}{2}\psi_{yyy}(\pm1) \pm O\left(G^3\right) = 0$$
$$\psi_x(\pm1) \pm G\psi_{xy}(\pm1) + \frac{G^2}{2}\psi_{xyy}(\pm1) \pm O\left(G^3\right) = \mp\tilde{\alpha}\,\varepsilon\sin\tilde{\alpha}(x-t) \qquad \textbf{(4.119-4.120)}$$

Substituting equation (4.74) into equation (4.119-4.120) and collecting terms of the same order in $\varepsilon$, gives

$$\psi_{0y}(\pm1) = 0$$
$$\psi_{1y}(\pm1) \pm \psi_{0yy}(\pm1)\cos\tilde{\alpha}(x-t) = 0$$
$$\psi_{0x}(\pm1) = 0 \qquad \textbf{(4.121)}$$
$$\psi_{1x}(\pm1) \pm \psi_{0xy}(\pm1)\cos\tilde{\alpha}(x-t) = \mp\tilde{\alpha}\sin\tilde{\alpha}(x-t)$$

*and so on for higher order terms in ε.*

Taking the positive sign of the boundary conditions as given in equation (4.121) yields the boundary conditions as

$$\psi_{1y}(1) = -\psi_{0yy}(1)\cos\tilde{\alpha}(x-t) \qquad (4.122)$$
$$\psi_{1x}(1) = -\tilde{\alpha}\sin\tilde{\alpha}(x-t) \qquad (4.123)$$

From these boundary conditions, it can be assumed that $\psi_1(x,y,t)$ can be obtained in the form

$$\psi_1(x,y,t) = f(y)\cos\tilde{\alpha}(x-t) + g(y)\sin\tilde{\alpha}(x-t) \qquad \textbf{(4.124)}$$

Eliminating the pressure terms in equations (4.72) and (4.73) by cross-differentiation and subtraction, the following equation is obtained:

$$\rho(\nabla^2\psi_t + \psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y) = 4\mu_{xy}\psi_{yx} + 2\mu_x\psi_{yyx} + \mu_{yy}(\psi_{yy} - \psi_{xx}) + \mu_y(\psi_{yyy} - \psi_{xxy}) +$$
$$\mu_y\nabla^2\psi_y + \mu\nabla^2\psi_{yy} + 2\mu_y\psi_{xxy} - \mu_{xx}(\psi_{yy} - \psi_{xx}) - \mu_x(\psi_{yyx} - \psi_{xxx}) + \mu_x\nabla^2\psi_x + \mu\nabla^2\psi_{xx} \qquad \textbf{(4.125)}$$

By substituting equation (4.124) for $\psi_1(x,y,t)$ and equation (4.92) for $\psi_0(y)$ into equation (4.125), and collecting coefficients of $\cos\tilde{\alpha}(x-t)$ and $\sin\tilde{\alpha}(x-t)$ on either side of the resulting equation, two differential equations for $f(y)$ and $g(y)$ are obtained.

Due to the length and complexity of these equations approximate solutions are obtained by assuming that the parameter, $\tilde{\alpha}$, which is $\frac{2\pi d}{\lambda}$ is small. As a first approximation, the terms of order $\tilde{\alpha}^2$ and higher can be neglected; as a second approximation, the terms of order $\tilde{\alpha}^3$ and higher can be neglected and so on.

Hence the following equation is obtained from equation(4.125) by expanding in a pertubations series as indicated in equations(4.74-4.76) after collecting terms of the first order in amplitude ratio, $\varepsilon$ and remembering $\psi_0 = \psi_0(y)$ only,

$$\rho\sqrt{c^3 d}\left(\psi_{1yyt} + \psi_{0y}\psi_{1yyx} - \psi_{1x}\psi_{0yyy}\right) = \mu_{0yy}\psi_{1yy} + \mu_{1yy}\psi_{0yy} + 2\mu_{0y}\psi_{1yyy} + 2\mu_{1y}\psi_{0yyy} + \mu_0\psi_{1yyyy} + \mu_1\psi_{0yyyy} \tag{4.126}$$

where $\mu_0, \mu_{0y}, \mu_{0yy}$ and $\mu_1, \mu_{1y}, \mu_{1yy}$ are extensive and complicated equations and are obtained from equation (4.83) and equation (4.84) respectively, as follows,

$$\mu_{0y} = -\sqrt{2}\alpha\beta\psi_{0yy}^{-\frac{3}{2}}\psi_{0yyy} - 2\beta^2\psi_{0yy}^{-2}\psi_{0yyy} \tag{4.127}$$

$$\mu_{0yy} = \frac{3\alpha\beta}{\sqrt{2}}\psi_{0yy}^{-\frac{5}{2}}\psi_{0yyy}^{2} - \sqrt{2}\alpha\beta\psi_{0yy}^{-\frac{3}{2}}\psi_{0yyyy} + 4\beta^2\psi_{0yy}^{-3}\psi_{0yyy}^{2} - 2\beta^2\psi_{0yy}^{-2}\psi_{0yyyy} \tag{4.128}$$

$$\mu_{1y} = \frac{3\alpha\beta}{\sqrt{2}}\psi_{0yy}^{-\frac{5}{2}}\psi_{0yyy}\psi_{1yy} - \sqrt{2}\alpha\beta\psi_{0yy}^{-\frac{3}{2}}\psi_{1yyy} + 4\beta^2\psi_{0yy}^{-3}\psi_{0yyy}\psi_{1yy} - 2\beta^2\psi_{0yy}^{-2}\psi_{1yyy} \tag{4.129}$$

$$\begin{aligned}\mu_{1y} = &-\frac{15\alpha\beta}{2\sqrt{2}}\psi_{0yy}^{-\frac{7}{2}}\psi_{0yyy}^{2}\psi_{1yy} + \frac{3\alpha\beta}{\sqrt{2}}\psi_{0yy}^{-\frac{5}{2}}\psi_{0yyyy}\psi_{1yy} + \frac{6\alpha\beta}{\sqrt{2}}\psi_{0yy}^{-\frac{5}{2}}\psi_{0yyy}\psi_{1yyy} - \\ &\sqrt{2}\alpha\beta\psi_{0yy}^{-\frac{3}{2}}\psi_{1yyyy} - 12\beta^2\psi_{0yy}^{-4}\psi_{0yyy}^{2}\psi_{1yy} + 4\beta^2\psi_{0yy}^{-3}\psi_{0yyyy}\psi_{1yy} + \\ &8\beta^2\psi_{0yy}^{-3}\psi_{0yyy}\psi_{1yyy} - 2\beta^2\psi_{0yy}^{-2}\psi_{1yyyy}\end{aligned} \tag{4.130}$$

After substituting for the various terms in equation(4.126) and collecting terms and remembering the approximation made on terms in the parameter, $\tilde{\alpha}$, the following ordinary differential equation is obtained

$$\rho\sqrt{c^3 d}\left\{\begin{array}{l}\tilde{\alpha} f'' \sin\tilde{\alpha}(x-t) - \tilde{\alpha} g'' \cos\tilde{\alpha}(x-t) + \\ \left[\frac{4\beta^2}{\alpha^2}y + \frac{k}{2\alpha^2}y^2 - \frac{4\beta\sqrt{2}}{3k\alpha^2}\left(2\beta^2+ky\right)^{\frac{3}{2}} + A\right]\left(-\tilde{\alpha} f'' \sin\tilde{\alpha}(x-t) + \tilde{\alpha} g'' \cos\tilde{\alpha}(x-t)\right) \\ -\left(-\tilde{\alpha} f \sin\tilde{\alpha}(x-t) + \tilde{\alpha} g \cos\tilde{\alpha}(x-t)\right)\frac{k^2}{4\beta^2\alpha^2}y\end{array}\right\} =$$

$$\left(\frac{k^2 y}{4\beta^2\alpha^2}\right)^2\left(f''\cos\tilde{\alpha}(x-t) + g''\sin\tilde{\alpha}(x-t)\right)\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}} -$$

$$\left(\frac{ky}{4\beta^2\alpha^2}\right)\left(f'''\cos\tilde{\alpha}(x-t) + g'''\sin\tilde{\alpha}(x-t)\right)\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} +$$

$$\alpha^2\left(f^{iv}\cos\tilde{\alpha}(x-t) + g^{iv}\sin\tilde{\alpha}(x-t)\right) -$$

$$\left(\frac{k^2}{4\alpha^2\beta^2} - \frac{3k^3}{16\beta^4\alpha^2}y\right)\frac{\alpha\beta}{\sqrt{2}}\left(f''\cos\tilde{\alpha}(x-t) + g''\sin\tilde{\alpha}(x-t)\right)\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} +$$

$$\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(f^{iv}\cos\tilde{\alpha}(x-t) + g^{iv}\sin\tilde{\alpha}(x-t)\right)$$

**(4.131)**

where the constant A is given in equation(4.93).

Collecting coefficients of $\cos\tilde{\alpha}(x-t)$ in equation(4.131) gives

$$\rho\sqrt{c^3 d}\left\{-\tilde{\alpha} g'' + \left[\frac{4\beta^2}{\alpha^2}y + \frac{k}{2\alpha^2}y^2 - \frac{4\beta\sqrt{2}}{3k\alpha^2}\left(2\beta^2+ky\right)^{\frac{3}{2}} + A\right]\left(\tilde{\alpha} g''\right) - \left(\tilde{\alpha} g\right)\frac{k^2}{4\beta^2\alpha^2}y\right\} =$$

$$\left(\frac{k^2 y}{4\beta^2\alpha^2}\right)^2 (f'')\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}} -$$

$$\left(\frac{ky}{4\beta^2\alpha^2}\right)(f''')\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} + \alpha^2\left(f^{iv}\right) - \qquad \textbf{(4.132)}$$

$$\left(\frac{k^2}{4\alpha^2\beta^2} - \frac{3k^3}{16\beta^4\alpha^2}y\right)\frac{\alpha\beta}{\sqrt{2}}(f'')\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} +$$

$$\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(f^{iv}\right)$$

Collecting coefficients of $\sin\tilde{\alpha}(x-t)$ in equation(4.131) gives

$$\rho\sqrt{c^3 d}\left\{\tilde{\alpha} f'' - \left[\frac{4\beta^2}{\alpha^2}y + \frac{k}{2\alpha^2}y^2 - \frac{4\beta\sqrt{2}}{3k\alpha^2}\left(2\beta^2 + ky\right)^{\frac{3}{2}} + A\right]\left(\tilde{\alpha} f''\right) + \left(\tilde{\alpha} f\right)\frac{k^2}{4\beta^2\alpha^2}y\right\} =$$

$$\left(\frac{k^2 y}{4\beta^2\alpha^2}\right)^2 (g'')\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}} -$$

$$\left(\frac{ky}{4\beta^2\alpha^2}\right)(g''')\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} + \alpha^2\left(g^{iv}\right) - \quad \textbf{(4.133)}$$

$$\left(\frac{k^2}{4\alpha^2\beta^2} - \frac{3k^3}{16\beta^4\alpha^2}y\right)\frac{\alpha\beta}{\sqrt{2}}(g'')\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} +$$

$$\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(g^{iv}\right)$$

The equations for f(y) and g(y) can be simplified by assuming that the Reynolds number associated with the present model is small, where the associated Reynolds number is given as

$$\mathrm{Re} = \rho\sqrt{c^3 d} \quad \textbf{(4.134)}$$

Therefore,

$$\begin{aligned} f(y) &= f_0(y) + \mathrm{Re}^2 f_2(y) + \textit{higher order terms in } \mathrm{Re} \\ g(y) &= \mathrm{Re}\, g_1(y) + \mathrm{Re}^3 g_3(y) + \textit{higher order terms in } \mathrm{Re} \end{aligned} \quad \textbf{(4.135)}$$

Hence, evaluating equation (4.132) and equation(4.133) with equation(4.135) and equating equal terms in Reynolds Number the following ordinary differential equations are obtained for $f_0(y)$ *and* $g_1(y)$ respectively,

$$\left(\frac{k^2 y}{4\beta^2\alpha^2}\right)^2 \left(f_0''\right)\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}} -$$

$$\left(\frac{ky}{4\beta^2\alpha^2}\right)\left(f_0''\right)\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} + \alpha^2\left(f_0^{iv}\right) -$$

$$\left(\frac{k^2}{4\alpha^2\beta^2}-\frac{3k^3}{16\beta^4\alpha^2}y\right)\frac{\alpha\beta}{\sqrt{2}}\left(f_0''\right)\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} + \tag{4.136}$$

$$\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(f_0^{iv}\right) = 0$$

$$\left\{\tilde{\alpha} f_0'' - \left[\frac{4\beta^2}{\alpha^2}y+\frac{k}{2\alpha^2}y^2-\frac{4\beta\sqrt{2}}{3k\alpha^2}\left(2\beta^2+ky\right)^{\frac{3}{2}}+A\right]\left(\tilde{\alpha} f_0''\right)+\left(\tilde{\alpha} f_0\right)\frac{k^2}{4\beta^2\alpha^2}y\right\} =$$

$$\left(\frac{k^2 y}{4\beta^2\alpha^2}\right)^2 \left(g_1''\right)\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}} -$$

$$\left(\frac{ky}{4\beta^2\alpha^2}\right)\left(g_1''\right)\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} + \alpha^2\left(g_1^{iv}\right) - \tag{4.137}$$

$$\left(\frac{k^2}{4\alpha^2\beta^2}-\frac{3k^3}{16\beta^4\alpha^2}y\right)\frac{\alpha\beta}{\sqrt{2}}\left(g_1''\right)\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} +$$

$$\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(g_1^{iv}\right)$$

From equation(4.122 and 4.123) the boundary conditions for $f_0(y)$ and $g_1(y)$ are given as

$$\begin{aligned} &f_0(0)=0 \quad f_0'(0)=0 \quad f_0(1)=1 \quad f_0'(1)=-\psi_{0yy} \\ &g_1(0)=g_1'(0)=g_1(1)=g_1'(1)=0 \end{aligned} \tag{4.138}$$

The analytical solution to equation(4.136) is found by using the results given by Polyanin & Zaitsev(1992)

Hence by reducing equation(4.136) to a second order equation and then integrating twice the solution is found. Comparing our reduced second order equation to and using part 28 on page 134 of Polyanin & Zaitsev(1992) with their notations

$$\alpha = \frac{A}{E}, \beta = \frac{B}{E}, \gamma = -\frac{C}{E}$$

where, $A = \dfrac{k^4 3\alpha\beta\left(\dfrac{4\beta^2}{\alpha^2}\right)^{-\frac{5}{2}}\dfrac{14}{4}}{2\sqrt{2}\left(4\beta^2\alpha^2\right)^2}$ $B = \dfrac{3k^3\alpha\beta\left(\dfrac{4\beta^2}{\alpha^2}\right)^{-\frac{3}{2}}\dfrac{10}{4}}{16\beta^4\alpha^2 2\sqrt{2}}$ $C = \dfrac{k^2\alpha\beta\left(\dfrac{4\beta^2}{\alpha^2}\right)^{-\frac{3}{2}}\dfrac{10}{4}}{4\sqrt{2}\beta^2\alpha^2}$

$$D = \frac{k\sqrt{2}\alpha\beta\left(\dfrac{4\beta^2}{\alpha^2}\right)^{-\frac{3}{2}}\dfrac{10}{4}}{4\beta^2\alpha^2} \qquad E = \alpha^2 + \sqrt{2}\alpha\beta\left(\frac{4\beta^2}{\alpha^2}\right)^{-\frac{1}{2}}\frac{6}{4} \tag{4.139}$$

Consequently, with $w_0 = f_0''$ equation(4.136) reduces to

$$w_0'' - \frac{D}{E}yw_0' + \frac{1}{E}\left\{Ay^2 + By - C\right\}w_0 = 0 \tag{4.140}$$

Then following through the analysis they describe where

$$s = \frac{D}{4E} \pm \frac{1}{2}\sqrt{\frac{D^2}{4E^2} - \frac{A}{E}} \tag{4.141}$$

is the root of the quadratic(see page134 of Polyanin & Zaitsev(1992))

$$4s^2 + 2as + \alpha = 0 \, , \, a = -\frac{D}{E} \, , \, b = 0 \tag{4.142}$$

It is found that if we consider only the first two terms of the series,

$w_0 = f_0'' = \exp(hy)\exp(sy^2)z(\xi)$ *where* $z(\xi)$ *is found in Table2.2 page 143 of Polyanin*

$$= \exp(hy)\exp(sy^2)\left\{C_1\left[1 + \sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n}\frac{\left(k'\zeta^2\right)^n}{n!}\right] + C_2 y^{\frac{1}{2}}\left[1 + \sum_{n=1}^{\infty}\frac{\left(a+\frac{1}{2}\right)_n}{(b)_n}\frac{\left(k'\zeta^2\right)^n}{n!}\right]\right\} \tag{4.143}$$

where

$$z(\zeta) = \xi(a, \frac{1}{2}, k'\zeta^2)$$
$$\zeta = \frac{y-\mu}{\lambda}, \mu = -\frac{2b_2 h + b_1}{a_1}, \lambda = 1, \tag{4.144}$$

$\xi\left(a, \frac{1}{2}, k'\zeta^2\right)$ is the degenerate hypergeometric solution and is found on page143, part 103 and page137, part 65 of Polyanin & Zaitsev(1992).

Subsequently, the solution to equation(4.136) and hence equation(4.143 and 4.144), after applying symbolic integration twice using MATLAB v5.3 is very intricate and given in Appendix C.

Numerical solutions to equations (4.136) for $f_0(y)$ and (4.137) for $g_1(y)$ results the plots in Figures(4.5-4.9). Figure (4.5) shows a comparison of $f_0(y)$ with other models, in particular, Raju and Devanathan(1972). Figure (4.6) shows the curves for $f_0(y)$ and $f_0'(y)$ with varying values of yield stress. That is, $\beta$ is gradually varied between zero and unity. Figures (4.7-4.9) show curves for $g_1(y)$ and $g_1'(y)$ with varying values of yield stress and various values of wave number. Figure(4.10) gives the streamfunction plot, as derived in this research from equations(4.74, 4.92-4.94), and 4.124 and 4.135 which are very similar to plots given in the paper by Raju and Devanthan(1972).

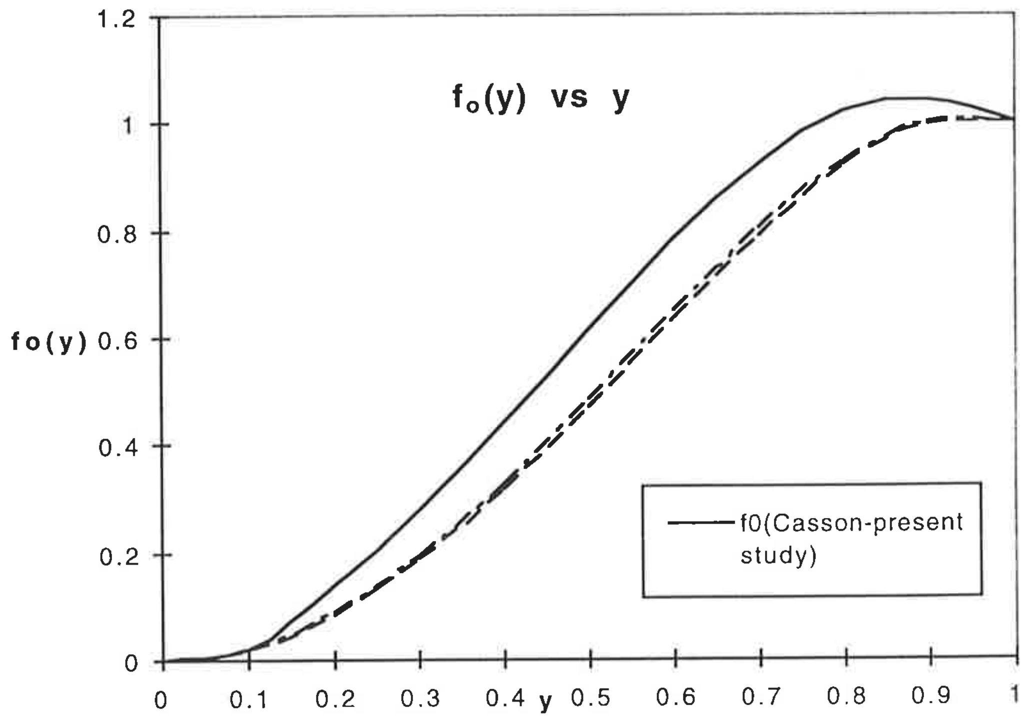


Figure 4.5: Comparing $f_0$ vs y with Raju & Devanathan(1972)

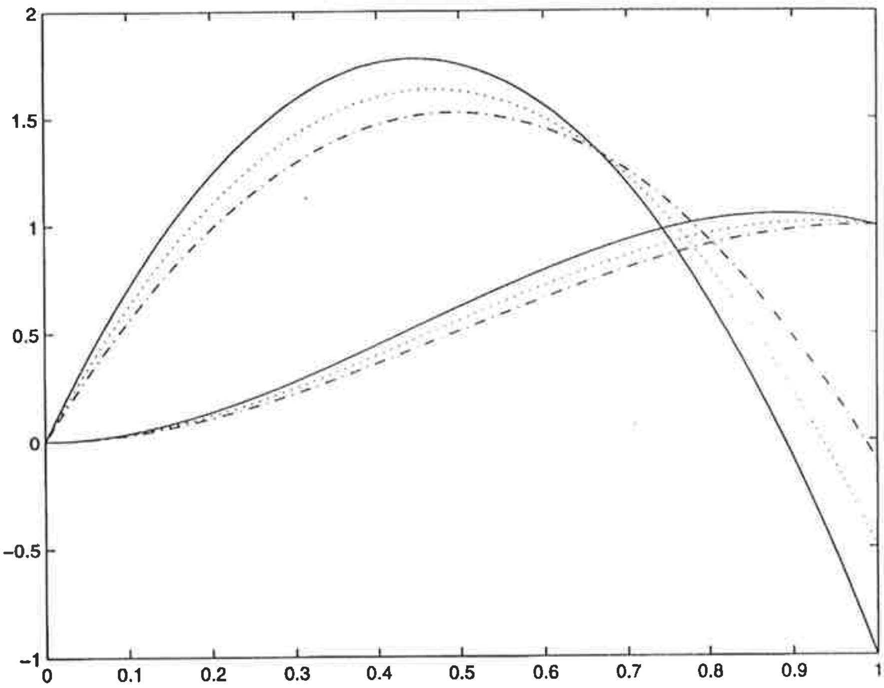


Figure 4.6: $f_0(y)$ & $f_0'(y)$ *with*, ______ $f_0'(1)=-1(\beta=0)$, .........$f_0'(1)=-0.5$, –. – . – $f_0'(1)=-0.1(\beta=1)$

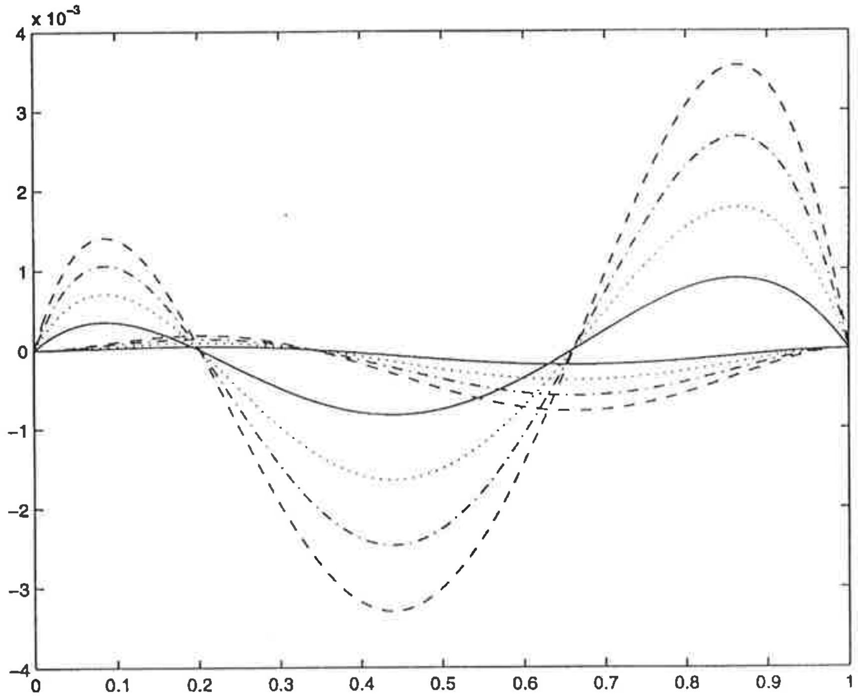


Figure 4.7: $g_1(y)$ & $g_1'(y)$ *with* $f_0'(1)=-1$, ____ $\tilde{\alpha}=0.2$, ......$\tilde{\alpha}=0.4$, –. – $\tilde{\alpha}=0.6$, – – –$\tilde{\alpha}=0.8$

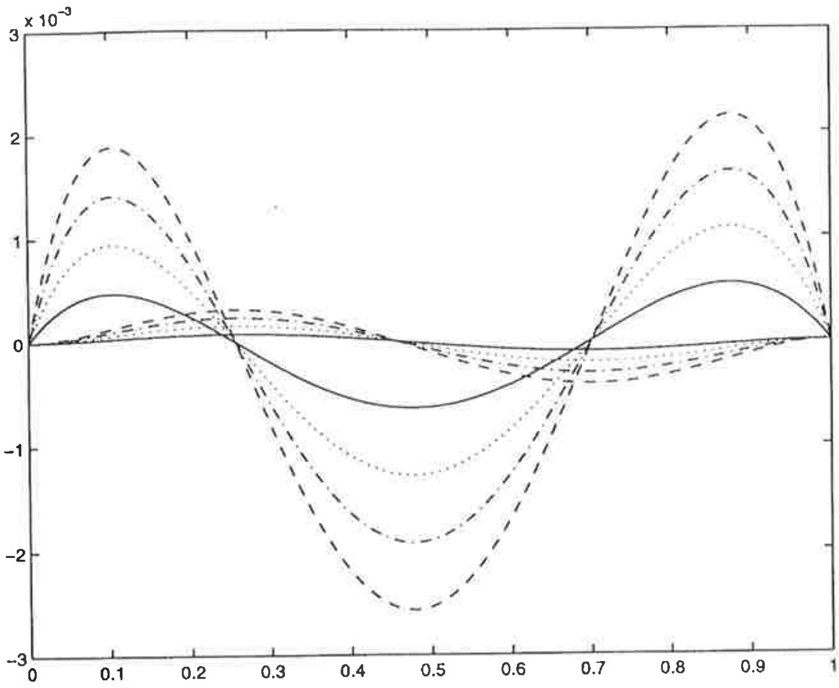


Figure 4.8: $g_1(y)$ & $g_1^{'}(y)$ *with* $f_0^{'}(1) = -0.5$, ____ $\tilde{\alpha} = 0.2$, ...... $\tilde{\alpha} = 0.4$, -. - $\tilde{\alpha} = 0.6$, - - - $\tilde{\alpha} = 0.8$

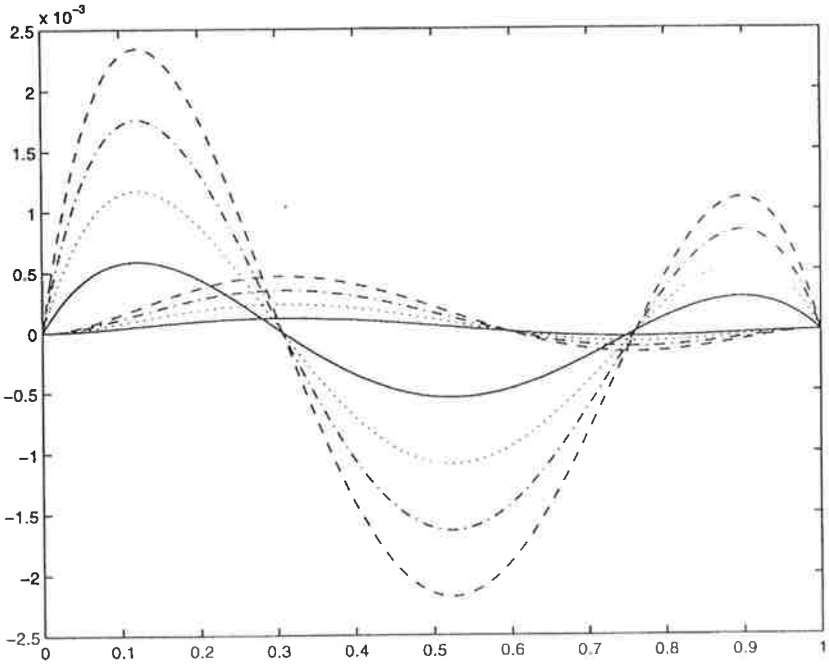


Figure 4.9: $g_1(y)$ & $g_1^{'}(y)$ *with* $f_0^{'}(1) = -0.1$ : ____ $\tilde{\alpha} = 0.2$, ...... $\tilde{\alpha} = 0.4$, -. - $\tilde{\alpha} = 0.6$, - - - $\tilde{\alpha} = 0.8$

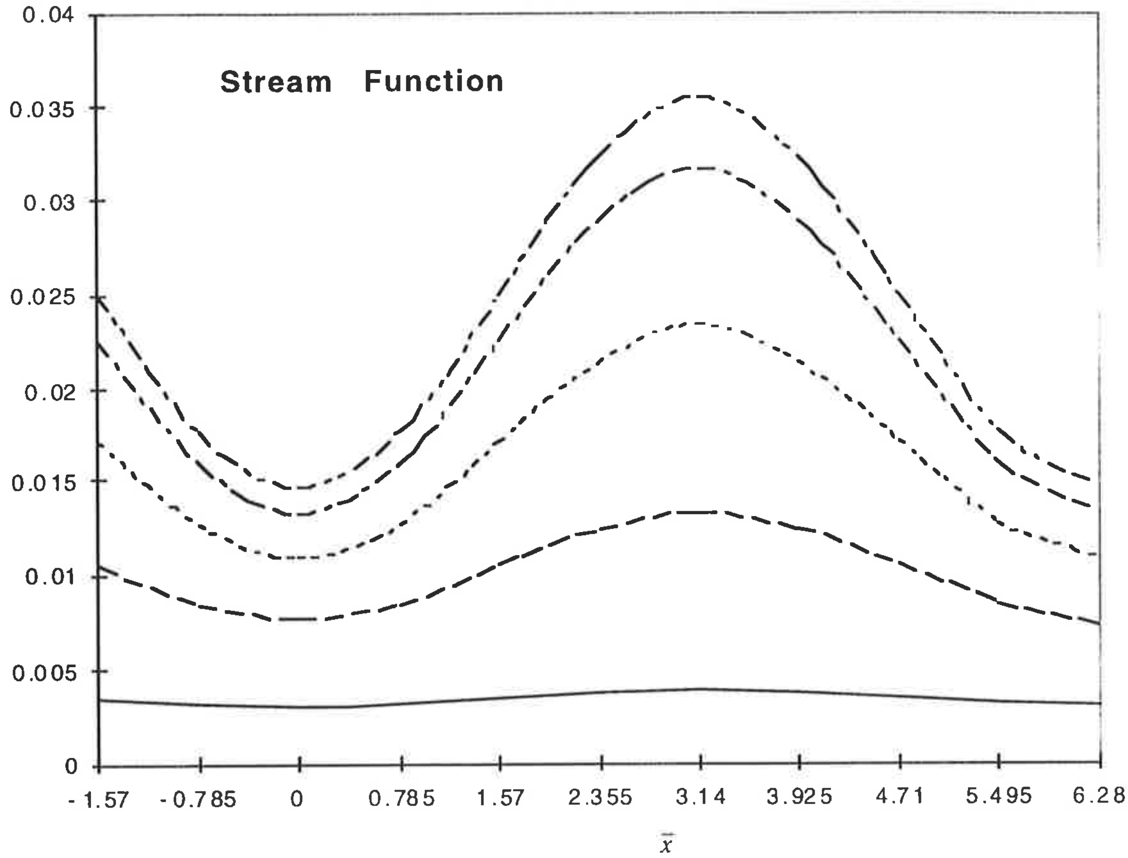


Figure 4.10: Plot of function $\psi$ vs $\bar{x}$

## 4.7 Discussion of Results

In this study it was found that for the Casson model, the governing partial differential equations are indeed extensive and complicated. If however it is considered that the zeroth order perturbation in stream function is a function of the axial coordinate only, we find that the Casson model may be quantitatively expressed as a Newtonian model (Figures 4.2 - 4.4).

It was found that in the zeroth order approximation in stream function that there was a dependence on the Casson coefficient of viscosity, yield stress, the density of the fluid, the wave speed and the dimensions of the channel.

When considering this approximation in the zeroth order stream function, results show for Figures(4.2 & 4.3), representing Pressure rise vs flow rate derived with the upper limit at y=G for lower values of amplitude ratio the difference between Newtonian(dashed line) and non-Newtonian(bold line) in Figure 4.2 seems to be consistent with the literature, and very similar to Figures(4.2 and 4.3). However, for higher values of amplitude ratio represented by Figure 4.3, the pressure gradient is noticeably affected by the non-Newtonian character of the fluid. The effect appears to increase as the occlusion gets larger. Also, in Figure(4.4), representing pressure rise vs flow rate derived using upper limit y=1, results are also consistent with the literature.

However, we see that for the first order in stream function the differential equation to be solved is complex, and the analytical solution derived from symbolic integration is more so. When comparing the values of our Casson model in Figure 4.5, obtained from numerical integration, of the first order in stream function, with those of the power-law model of Mernone & Mazumdar(1998a) and Raju & Devanathan(1972) the results are similar (that is, the curves almost coincide) when comparing the two power-law models(Raju's numerical and this thesis' analytical). However, noticably different when comparing the Casson model with the power-law, but similar if form. The Casson model indicating the effects of the yield stress and Casson viscosity on the stream function. When considering $f_0(y)$and $f_0'(y)$in Figure 4.6 it is found that as the yield stress $\beta$ is gradually varied between zero and unity the effects on both $f_0(y)$and $f_0'(y)$ are noticeable. It appears that the maximum value for $f_0'(y)$ in decreased and shifted slightly to the right. Similarly, in Figures 4.7-4.9 when considering the functions $g_1(y)$ and $g_1'(y)$ we find that the wave number $\tilde{\alpha}$ has considerable affect on the curves.

It appears that as the yield stress $\beta$ is gradually varied between zero and unity, and therefore the value of $f_0'(y) = -\psi_{0yy}(y)$, there is a shift in the size and shape of the left side and right side in the curve representing $g_1'(y)$. There seems to be a reversal in the location of peaks between the right side and left side. It is of interest to note that the points of inflection occur in exactly the same location when considering each of the respective graph of $g_1(y)$ and $g_1'(y)$. As the yield stress $\beta$ is gradually varied between zero and unity the points of inflection are shifted slightly to the right.

The numerical values obtained for $f_0(y)$ and $f_0'(y)$, and $g_1(y)$ and $g_1'(y)$ are indicative of the validity of the perturbation analysis used throughout this research as indicated in equation (4.74 and 4.124). It is seen that the order in magnitude of $f_0(y)$ is very much greater than that of $g_1(y)$ as is suggested and expected by the perturbation method.

From the numerical calculations we find that the change in behaviour of the streamfunction patterns occur depending on many parameters, including $K,\ \tilde{\alpha}, \alpha, \beta, R_e, and\ \varepsilon$. When we consider Figure (4.10), which is a plot of the stream function given by equations(4.74 and 4.92) and equations(4.124 and 4.135) and selecting $\varepsilon = 0.01$; for the case of high pressure gradient, with ___ representing $\psi_{0.1}$(y=0.1), - - - - representing $\psi_{0.3}$, ..... representing $\psi_{0.5}$, -.-.-. representing $\psi_{0.7}$ and -..-..- representing $\psi_{0.9}$(y=0.9), it is found that the streamfunction curves run parallel to the axis of the channel when considered near the axis(y=0.1), whereas considerable deformation is observed when they are considered near the boundary(y=0.9).

Perhaps a possible explanation for this sort of behaviour of the streamlines can be given by considering the region as consisting of two parts - a central core and a boundary layer region. As the pressure gradient increases, we find that the curves in the central region are more influenced by it, than by the motion of the boundary and hence the curves run parallel to the axis, while in the region near the boundary the flow is influenced by both the wave and the pressure gradient(see explanation later, Chapter 5).

This modelling is appropriate as it may allow insight into the validity of the reduction of the complexity of modelling some non-Newtonian fluids like flow of urine in the ureter and blood flow in the blood vessels under certain physiological conditions.

## 4.8 Comparisons and Implications

Finally, we note that the governing equation for all three mathematical models show some similarity to the Newtonian case when certain assumptions and simplification are made, particularly for the power law and Casson case.

It was found that for the power law and Casson model the governing partial differential equations were indeed extensive. If however, the fact was used that the zeroth order perturbation in stream function is a function of the axial coordinate only,( because the zeroth order axial pressure gradient is constant), it was found that in the power law fluid, if the flow behaviour index was set to unity, the model may be considered and quantitatively expressed as a Newtonian model.

Similarly, if we consider the same assumption for the zeroth order axial pressure gradient for the Casson model; It was found that the Casson model was also a natural extension and may be physically modelled as a Newtonian model. When we considered the first order perturbation in stream function the resulting equations were very complicated and did not allow analytical solutions without considerable effort and manipulation, if at all.

It was found for the power law case that the governing equations reduced to those of the Newtonian model if the power law index was set to unity. Also it was found for the Casson model if we carry out a Binomial expansion on the viscosity term and the derivatives of the apparent viscosity term we find that the Casson model reduces to a Newtonian model in the case of the zeroth order perturbation in stream function. It was also found that the solution for the first order in stream function agreed with that in the literature, after simplifications and assumptions, and the results of numerical analysis performed indicate the validity in the derivation and assumptions made in deriving the complex differential equations. That is to say, the order of magnitude betwwen $f_0(y)$ and $g_1(y)$ is consistent with current knowledge in fluid flow.

# CHAPTER 5:

## FLOW IN AN AXISYMMETRIC TUBE

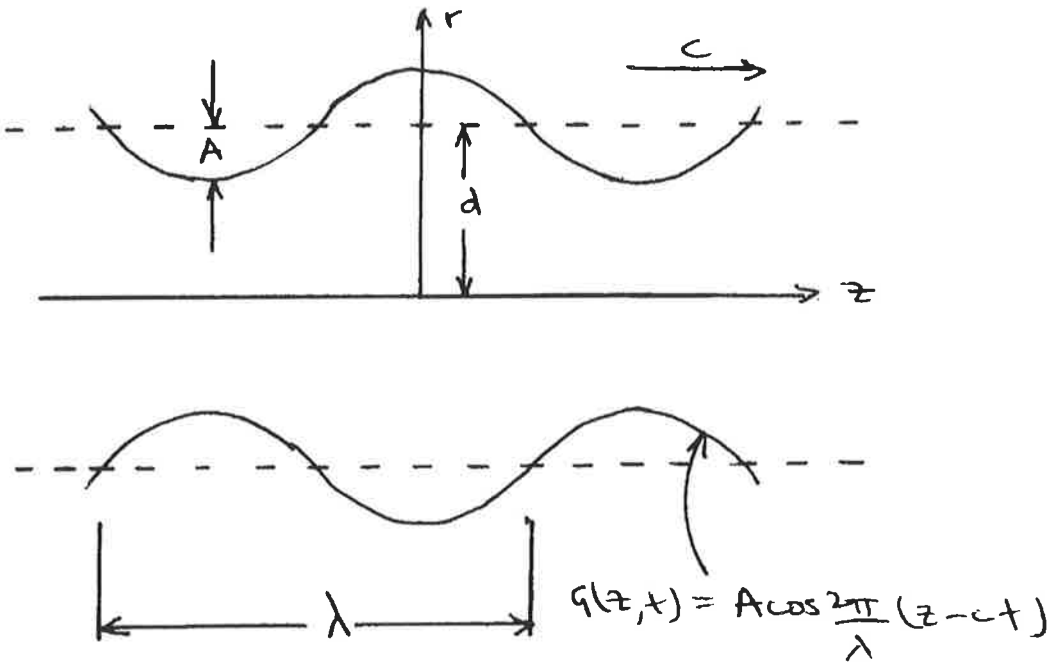


Figure 5.1: Geometry of Peristaltic Flow in an Axisymmetric Tube

### 5.1 Mathematical modelling of a Newtonian fluid in an Axisymmetric Tube

The governing equations for the flow of a Newtonian, incompressible fluid in a circular cylindrical tube in cylindrical coordinates is given by the Navier-Stokes equations and continuity equation in cylindrical coordinates as

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u^2}{r^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad (5.1)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right) \quad (5.2)$$

$$\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}+\frac{u}{r}=0 \quad (5.3)$$

There exists a stream function $\psi$ which relates to the velocity components u and w, satisfying equation 5.3 such that

$$u=\frac{1}{r}\frac{\partial\psi}{\partial z}=\frac{1}{r}\psi_z \quad and \quad w=-\frac{1}{r}\frac{\partial\psi}{\partial r}=-\frac{1}{r}\psi_r \quad (5.4)$$

(The signs above are opposite to those used in Raju & Devanathan(1972))

Eliminating the pressure term by cross differentiating the momentum equations 5.1 and 5.2 and subtracting gives the following

$$\frac{\partial^2 u}{\partial z\partial t}-\frac{\partial^2 w}{\partial r\partial t}-\frac{u}{r}\frac{\partial u}{\partial z}+u\left(\frac{\partial^2 u}{\partial r\partial z}-\frac{\partial^2 w}{\partial r^2}\right)+w\left(\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 w}{\partial r\partial z}\right)+\frac{u}{r}\frac{\partial w}{\partial r}=$$

$$\nu\left\{\frac{\partial^3 u}{\partial r^2\partial z}-\frac{\partial 3w}{\partial r^3}+\frac{1}{r}\left(\frac{\partial^2 u}{\partial r\partial z}-\frac{\partial^2 w}{\partial r^2}\right)-\frac{1}{r^2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\right)-\frac{\partial^3 w}{\partial z^2\partial r}+\frac{\partial^3 u}{\partial z^3}\right\} \quad (5.5)$$

Substituting for stream function $\psi$ in place of velocity components u and w by using equation 5.4 we obtain from (5.5)

$$\frac{\partial}{\partial t}\left(\nabla_1{}^2\psi\right)+\frac{1}{r}\frac{\partial\psi}{\partial z}\left(\nabla_1{}^2\psi_r-\frac{2}{r}\nabla_1{}^2\psi+\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right)-\frac{1}{r}\frac{\partial\psi}{\partial r}\left(\nabla_1{}^2\psi_z\right)=\nu\left(\nabla_1{}^4\psi\right) \quad (5.6)$$

where

$$\nabla_1{}^2\equiv\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r} \quad (5.7)$$

We assume that there is no horizontal displacement of the walls and consider an idealised tube of infinite length where the end conditions need not be specified because we assume the pressure gradient in the axial direction is specified. The boundary conditions for an axisymmetric case are therefore given as follows after using equation(5.4)

$$w=-\frac{1}{r}\psi_r=0 \quad at \quad r=d+G(z,t) \quad where \quad G(z,t)=A\cos\frac{2\pi}{\lambda}(z-ct) \quad (5.8)$$

$$u=\frac{1}{r}\psi_z=\frac{\partial}{\partial t}G(z,t)=\frac{\partial}{\partial t}\left(A\cos\frac{2\pi}{\lambda}(z-ct)\right)=\frac{2\pi Ac}{\lambda}\sin\frac{2\pi}{\lambda}(z-ct) \quad at \quad r=d+G(z,t) \quad (5.9)$$

The pressure gradient in the axial direction (z) is of the form

$$\frac{\partial p}{\partial z} = \left(\frac{\partial p}{\partial z}\right)_0 + A\left(\frac{\partial p}{\partial z}\right)_1 + A^2\left(\frac{\partial p}{\partial z}\right)_2 + \ldots \qquad (5.10)$$

where A is the amplitude of the sinusoidal waves travelling along the walls of the tube and

$$\left(\frac{\partial p}{\partial z}\right)_0 = constant \quad and \quad \left(\frac{\partial p}{\partial z}\right)_i = F(r,z,t) \quad i = 1,2,3.... \qquad (5.11)$$

If we introduce non-dimensional quantities as outlined in Appendix B, the boundary conditions become

$$\psi_r = 0 \quad at \quad r = 1 + G(z,t) \quad where\ G(z,t) = \varepsilon\cos\alpha(z-t) \qquad (5.12)$$

$$\psi_z = r\alpha\varepsilon\sin\alpha(z-t) \quad at \quad r = 1 + G(z,t) \qquad (5.13)$$

and

$$\frac{\partial p}{\partial z} = \left(\frac{\partial p}{\partial z}\right)_0 + \varepsilon\left(\frac{\partial p}{\partial z}\right)_1 + \varepsilon^2\left(\frac{\partial p}{\partial z}\right)_2 + \ldots \qquad (5.14)$$

$$where \quad \varepsilon\ is\ the\ amplitude\ ratio,\ \varepsilon = \frac{A}{d}$$

Substitution of non dimensional variables as shown in Appendix B equation 5.6 becomes

$$\frac{c^2}{d}\left\{\frac{\partial}{\partial t}\left(\nabla_1^2\psi\right) + \frac{1}{r}\frac{\partial\psi}{\partial z}\left(\nabla_1^2\psi_r - \frac{2}{r}\nabla_1^2\psi + \frac{1}{r^2}\frac{\partial\psi}{\partial r}\right) - \frac{1}{r}\frac{\partial\psi}{\partial r}\left(\nabla_1^2\psi_z\right)\right\} = \frac{\nu c}{d^2}\left(\nabla_1^4\psi\right) \qquad (5.15)$$

which becomes

$$\frac{\partial}{\partial t}\left(\nabla_1^2\psi\right) + \frac{1}{r}\frac{\partial\psi}{\partial z}\left(\nabla_1^2\psi_r - \frac{2}{r}\nabla_1^2\psi + \frac{1}{r^2}\frac{\partial\psi}{\partial r}\right) - \frac{1}{r}\frac{\partial\psi}{\partial r}\left(\nabla_1^2\psi_z\right) = \frac{1}{R_e}\left(\nabla_1^4\psi\right) \qquad (5.16)$$

## 5.2 A Method of Solution of a Newtonian Fluid in an Axisymmetric Tube

The method of solution for solving equation 5.16 is in much the same manner as solving that for the two dimensional channel case. That is, we substitute a perturbation series for the stream function $\psi$ in terms of the amplitude ratio $\varepsilon$ of the form

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots \qquad (5.17)$$

Therefore equation 5.17 in equation 5.16 gives

$$\frac{\partial}{\partial t}\left(\nabla_1^{\,2}\psi_0+\varepsilon\nabla_1^{\,2}\psi_1+\ldots_1\right)+$$
$$\frac{1}{r}\left(\psi_{0z}+\varepsilon\psi_{1z}+\ldots\right)\left(\nabla_1^{\,2}\psi_{0r}+\varepsilon\nabla_1^{\,2}\psi_{1r}+\ldots-\frac{2}{r}\left(\nabla_1^{\,2}\psi_0+\varepsilon\nabla_1^{\,2}\psi_1+\ldots\right)+\frac{1}{r^2}\left(\psi_{0r}+\varepsilon\psi_{1r}+\ldots\right)\right)-$$
$$\frac{1}{r}\left(\psi_{0r}+\varepsilon\psi_{1r}+\ldots\right)\left(\nabla_1^{\,2}\psi_{0z}+\varepsilon\nabla_1^{\,2}\psi_{1z}+\ldots\right)=$$
$$\frac{1}{R_e}\left(\nabla_1^{\,4}\psi_0+\varepsilon\nabla_1^{\,2}\psi_1\right)$$

(5.18)

Equating coefficients of equal power of $\varepsilon$ we obtain for the first two terms

$$\frac{\partial}{\partial t}\left(\nabla_1^{\,2}\psi_0\right)+\frac{1}{r}\frac{\partial\psi_0}{\partial z}\left(\nabla_1^{\,2}\psi_{0r}-\frac{2}{r}\nabla_1^{\,2}\psi_0+\frac{1}{r^2}\frac{\partial\psi_0}{\partial r}\right)-\frac{1}{r}\frac{\partial\psi_0}{\partial r}\left(\nabla_1^{\,2}\psi_{0z}\right)=\frac{1}{R_e}\left(\nabla_1^{\,4}\psi_0\right)$$

(5.19)

and

$$\frac{\partial}{\partial t}\left(\nabla_1^{2}\psi_1\right)+\frac{1}{r}\frac{\partial\psi_1}{\partial z}\left(\nabla_1^{2}\psi_{0r}-\frac{2}{r}\nabla_1^{2}\psi_0+\frac{1}{r^2}\frac{\partial\psi_0}{\partial r}\right)-\frac{1}{r}\frac{\partial\psi_1}{\partial r}\left(\nabla_1^{2}\psi_{0z}\right)+$$
$$\frac{1}{r}\frac{\partial\psi_0}{\partial z}\left(\nabla_1^{2}\psi_{1r}-\frac{2}{r}\nabla_1^{2}\psi_1+\frac{1}{r^2}\frac{\partial\psi_1}{\partial r}\right)-\frac{1}{r}\psi_{0r}\nabla_1^{2}\psi_{1z} \quad (5.20)$$
$$=\frac{1}{R_e}\left(\nabla_1^{4}\psi_1\right)$$

Applying a Taylor series expansion (as before ) about r=1+G(z,t) implies the boundary conditions, equations 5.12 and 5.13 become

$$\psi_r(1)+G(z,t)\psi_{rr}(1)+\frac{(G(z,t))^2}{2}\psi_{rrr}+\ldots=0 \quad (5.21)$$

$$\psi_z(1)+G(z,t)\psi_{zr}(1)+\frac{(G(z,t))^2}{2}\psi_{zrr}+\ldots=\alpha\varepsilon\sin\alpha(z-t) \quad (5.22)$$

Using the form for $\psi$ as given in equation 5.17 in equations 5.21and 5.22 we obtain

$$\psi_{0r}(1)+\varepsilon\psi_{1r}(1)+\varepsilon\cos\alpha(z-t)\psi_{0rr}(1)+\ldots=0 \quad (5.23)$$

$$\psi_{oz}(1)+\varepsilon\psi_{1z}(1)+\varepsilon\cos\alpha(z-t)\psi_{0zr}(1)+\ldots=\alpha\varepsilon\sin\alpha(z-t) \quad (5.24)$$

Equating coefficients of the same order of amplitude ratio $\varepsilon$ on both sides of the boundary condition equations we obtain

$$\psi_{0r}(1)=0 \tag{5.25}$$

$$\psi_{1r}(1)+\cos\alpha(z-t)\psi_{0rr}(1)=0 \tag{5.26}$$

$$\psi_{0z}(1)=0 \tag{5.27}$$

$$\psi_{1z}(1)+\psi_{0zr}(1)\cos\alpha(z-t)=\alpha\sin\alpha(z-t) \tag{5.28}$$

## ( i ) Solution Procedure(Zeroth Order Approximation)

We now consider the zeroth order perturbation, where if we consider that $\psi_0=\psi_0(r)$ only because of the constant zeroth order pressure gradient, we obtain from equation 5.19

$$\nabla_1^{\,4}\psi_0=0 \tag{5.29}$$

which becomes

$$\frac{d^4\psi_0}{dr^4}-\frac{2}{r}\frac{d^3\psi_0}{dr^3}+\frac{3}{r^2}\frac{d^2\psi_0}{dr^2}-\frac{3}{r^3}\frac{d\psi_0}{dr}=0$$
$$\text{that is} \tag{5.30}$$
$$r^4\frac{d^4\psi_0}{dr^4}-2r^3\frac{d^3\psi_0}{dr^3}+3r^2\frac{d^2\psi_0}{dr^2}-3r\frac{d\psi_0}{dr}=0$$

Which is a homogeneous linear equation of order four, whose solution is akin to that of the Euler or Cauchy equation, that is, the solution is of the form

$$\psi_0(r)=Ar^4+Br^3+Cr^2+Dr+E \tag{5.31}$$

where A,B,C,D.E are constant coefficients

Using the boundary condition as in previous analysis,

$$\psi_0(0)=0\Rightarrow E=0 \tag{5.32}$$

Using equation 5.2 and non-dimensionalising we obtain

$$R_e\left(\frac{\partial p}{\partial z}\right)_0=-\frac{1}{r}\frac{d^3\psi_0}{dr^3}+\frac{1}{r^2}\frac{d^2\psi_0}{dr^2}+\frac{1}{r^3}\frac{d\psi_0}{dr}=-16A-\frac{3B}{r}+\frac{D}{r^3} \tag{5.33}$$

Hence using boundary conditions 5.25 and 5.27

$$B = D = 0 \quad and \quad A = -\frac{R_e}{16}\left(\frac{\partial p}{\partial z}\right)_0$$

$$and \qquad (5.34)$$

$$4A + 2C = 0 \Rightarrow C = \frac{R_e}{8}\left(\frac{\partial p}{\partial z}\right)_0$$

Hence collecting values for the coefficients of the fourth order Ordinary Differential Equation 5.31 we obtain from equation 5.32-5.34 the solution to $\psi_0(r)$ as

$$\psi_0(r) = \frac{R_e}{8}\left(\frac{\partial p}{\partial z}\right)_0\left(r^2 - \frac{r^4}{2}\right) \qquad (5.35)$$

## 5.3 Mathematical Modelling of a Power Law Fluid Case in an Axisymmetric Tube

Consider the peristaltic motion of a non-Newtonian fluid, modelled as a power law fluid, which is viscous and incompressible in an axisymmetric cylindrical tube,where, d, is the undeformed radius of the tube and the tube is considered to be infinitely long. As before, A, represents the amplitude of the sinusoidal waves travelling along the walls of the tube, $\lambda$, is the wavelength and they are travelling at speed, c,( as shown in Fig. 5.1).

As indicated previously, the geometry of the sinusoidal travelling waves is given by G(z,t) where the vertical displacements for the upper and lower walls are G and -G for peristaltic flow at time t,

$$G(z,t) = A\cos\frac{2\pi}{\lambda}(z - ct) \qquad (5.36)$$

The non-Newtonian power law fluid is characterised by the well-known constitutive equations 5.42 and 5.43

$$\sigma_{ij} = -p\delta_{ij} + m\theta V_{ij} \qquad (5.37)$$

$$\theta = \left| V_{ij}V_{ij} \right|^{\frac{n-1}{2}} \qquad (5.38)$$

where, $\sigma_{ij}$, and $V_{ij}$, are the stress and the deformation tensors respectively, p denotes the isotropic pressure and, m and n are respectively flow consistency index and the flow behaviour index.

Because the fluid being modelled is assumed to be incompressible the following equation of motion may be used,

$$\rho \frac{Dq_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} \tag{5.39}$$

Here D/Dt is the material derivative of a particle following the fluid and is given as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{q}.\underline{\nabla} \tag{5.40}$$

where, $\rho$, is the density of the fluid and, $q_i$, is the velocity component in the respective direction.

Substitution of equations 5.37 and 5.38 into equation 5.39 finally gives

$$\rho \frac{Dq_i}{Dt} = -\frac{\partial p}{\partial x_i} + m\frac{\partial \theta}{\partial x_j} V_{ij} + m\theta\left\{\nabla^2 u_i + \frac{\partial}{\partial x_i}\left(\nabla.\underline{q}\right)\right\} \tag{5.41}$$

which after using the incompressibility condition $\nabla.\underline{q} = 0$ yields

$$\rho \frac{Dq_i}{Dt} = -\frac{\partial p}{\partial x_i} + m\left\{\theta \nabla^2 u_i + V_{ij}\frac{\partial \theta}{\partial x_j}\right\} \tag{5.42}$$

By setting i=1 and then i=2 and using summation convention with dimensionless variables and parameters (shown in the AppendixB ) the following equations of motion are obtained from equation 5.42

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial r} + \frac{1}{\mathrm{Re}}\left\{\theta\left(\nabla^2 u - \frac{u}{r^2}\right) + 2\frac{\partial u}{\partial r}\frac{\partial \theta}{\partial r} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\frac{\partial \theta}{\partial z}\right\} \tag{5.43}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\left\{\theta \nabla^2 w + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\frac{\partial \theta}{\partial r} + 2\frac{\partial w}{\partial z}\frac{\partial \theta}{\partial z}\right\} \tag{5.44}$$

where $\theta$ appearing in equations 5.43 and 5.44 is given by

$$\theta = \left| 4\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)^2 \right|^{\frac{n-1}{2}} \tag{5.45}$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{5.46}$$

The equation of continuity in cylindrical coordinates is given by

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0 \tag{5.47}$$

There exists a stream function $\Psi$ satisfying 5.47 such that

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial z} \tag{5.48}$$

$$w = -\frac{1}{r}\frac{\partial \Psi}{\partial r} \tag{5.49}$$

where u = radial velocity, w = axial velocity

## 5.4 A Method of Solution of a Power Law Fluid in an Axisymmetric Tube

We now express $\Psi$ and $p$ as a power series in the parameter $\varepsilon$ as

$$\Psi = \Psi_0 + \varepsilon \Psi_1 + \varepsilon^2 \Psi_2 + \ldots \tag{5.50}$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots \tag{5.51}$$

Hence, if we consider equations 5.50 and 5.51 and make the substitution given in equation 5.48 and 5.49 and collect like terms associated with powers of $\varepsilon$ on either side of the equation 5.44, this yields to the differential equation for the zeroth order term $\Psi_0(r)$, given by

$$K_0 = \frac{n}{r}\frac{d^3\Psi_0}{dr^3}\left\{\frac{1}{r}\frac{d^2\Psi_0}{dr^2} - \frac{1}{r^2}\frac{d\Psi_0}{dr}\right\}^{n-1} - \frac{(2n-1)}{r}\left\{\frac{1}{r}\frac{d^2\Psi_0}{dr^2} - \frac{1}{r^2}\frac{d\Psi_0}{dr}\right\}^{n} \tag{5.52}$$

where $K_0$ is defined as follows(the opposite sign to Raju & Devanathan(1972))

$$-K_0 = K = -\mathrm{Re}\left(\frac{\partial p_0}{\partial z}\right) \tag{5.52a}$$

We now assume that $\Psi_0(r)$ is a function of the radial direction only, due to the constant zeroth order axial pressure gradient, the solution for equation 5.52 and thus the solution for $\Psi_0(r)$ is is given as

$$\Psi_0(r) = \left(\frac{k}{2}\right)^{\frac{1}{n}} \frac{n}{n+1} \left\{ \frac{nr^{3+\frac{1}{n}}}{3n+1} - \frac{r^2}{2} \right\} \tag{5.53}$$

This is derived assuming the solution to(5.52) is of the form

$$\Psi_0(r) = A\left\{ \frac{nr^{3+\frac{1}{n}}}{3n+1} - \frac{r^2}{2} \right\} \quad \text{where A is a constant.} \tag{5.53a}$$

which satisfies the boundary conditions $\Psi_0(0) = 0$, $\Psi_{or}(1) = 0$,

Hence substituting(5.53a) into 5.52 yields,

$$K_0 = \frac{n}{r} A \frac{(n+1)(2n+1)}{n^2} r^{\frac{1}{n}} \left[ \frac{(n+1}{n} A \right] r^{\frac{n-1}{n}} - \frac{(2n-1)}{r} \left[ \frac{(n+1}{n} A \right] r$$

hence

$$A = \left(\frac{k}{2}\right)^{\frac{1}{n}} \frac{n}{n+1}$$

For n = 1, equation 5.53 reduces to the case of a Newtonian flow for axisymmetric peristaltic flow, (Raju and Devanathan, 1972).

## ( i ) Solution Procedure(First Order Approximation)

We shall now consider the 1st order perturbation in $\Psi$.

By substituting equations 5.48 and 5.49 and collecting coefficients of order $\varepsilon$ in equation 5.43 the following equation is obtained

$$\frac{1}{r}\Psi_{1zt} - \frac{1}{r^2}\Psi_{or}\Psi_{1zz} = -\frac{\partial p_1}{\partial r} -$$

$$\frac{1}{R_e}\left\{\left(\frac{1}{r}\Psi_{0rr} - \frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}} \left\{ -\frac{n}{r}\Psi_{1zzz} + \frac{(n-2)}{r}\Psi_{1zrr} + \frac{n(2-n)+2(1-n)}{nr^2}\Psi_{1zr} + \frac{2(n-1)}{nr^3}\Psi_{1z} \right\} \tag{5.54}$$

which becomes

$$\frac{1}{r}\Psi_{1zt} - \frac{1}{r^2}\Psi_{or}\Psi_{1zz} =$$
$$-\frac{\partial p_1}{\partial r} - \frac{1}{R_e}\left\{\left(\frac{1}{r}\Psi_{0rr} - \frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}}\left\{-\frac{n}{r}\Psi_{1zzz} + \frac{(n-2)}{r}\Psi_{1zrr} + \frac{2-n^2}{nr^2}\Psi_{1zr} + \frac{2(n-1)}{nr^3}\Psi_{1z}\right\} \tag{5.55}$$

Similarly for equation 5.44, substituting equations 5.48 and 5.49 and collecting coefficients of order $\varepsilon$ in equation 5.44 the following equation is obtained

$$\frac{1}{r}\Psi_{1rt} - \frac{1}{r^2}\Psi_{or}\Psi_{1zr} + \frac{1}{r^2}\Psi_{0rr}\Psi_{1z} - \frac{1}{r^3}\Psi_{0r}\Psi_{1z} =$$
$$\frac{\partial p_1}{\partial z} + \frac{1}{R_e}\left\{\left(\frac{1}{r}\Psi_{0rr} - \frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}}\left\{\begin{aligned}&\frac{n}{r}\Psi_{1rrr} + \frac{(2-n)}{r}\Psi_{1zzr} + \frac{n^2-n+2-2n}{nr^2}\Psi_{1zz} + \\ &\frac{n-2n^2+2n-2}{nr^2}\Psi_{1z} + \frac{2n^2-n+2-2n}{nr^3}\Psi_{1r}\end{aligned}\right\} \tag{5.56}$$

which becomes

$$\frac{1}{r}\Psi_{1rt} - \frac{1}{r^2}\Psi_{or}\Psi_{1zr} + \frac{1}{r^2}\Psi_{0rr}\Psi_{1z} - \frac{1}{r^3}\Psi_{0r}\Psi_{1z} =$$
$$\frac{\partial p_1}{\partial z} + \frac{1}{R_e}\left\{\left(\frac{1}{r}\Psi_{0rr} - \frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}}\left\{\begin{aligned}&\frac{n}{r}\Psi_{1rrr} + \frac{(2-n)}{r}\Psi_{1zzr} + \frac{3n-2n^2-2}{nr^2}\Psi_{1zz} + \\ &\frac{(n-2)(n-1)}{nr^2}\Psi_{1zz} + \frac{2n^2+2-3n}{nr^3}\Psi_{1r}\end{aligned}\right\} \tag{5.57}$$

To form one equation from these, we differentiate equation 5.55 w.r.t. z and equation 5.57 w.r.t. r and adding we obtain,

$$-\Psi_{0r}\left\{\frac{1}{r^2}\Psi_{1zzz} + \frac{1}{r^2}\Psi_{1zrr} - \frac{1}{r^3}\Psi_{1zr} - \frac{3}{r^4}\Psi_{1z}\right\} - \frac{3}{r^3}\Psi_{0rr}\Psi_{1z} + \frac{1}{r^2}\Psi_{0rrr}\Psi_{1z}$$
$$-\frac{1}{r^2}\Psi_{1rt} + \frac{1}{r}\Psi_{1rrt} + \frac{1}{r}\Psi_{1zzt} =$$
$$\frac{1}{\mathrm{Re}}\left(\frac{k}{2}\right)^{\frac{n-1}{n}} r^{\frac{n-1}{n}}\left\{\begin{aligned}&\frac{n}{r}\Psi_{1rrrr} - \frac{2}{nr^2}\left(n^2-n+1\right)\Psi_{1rrr} + \frac{\left(4n^3-4n^2+n+2\right)}{n^2r^3}\Psi_{1rr} + \\ &\frac{\left(4n^2-4n^3-n-2\right)}{n^2r^4}\Psi_{1r} + \frac{\left(2n^2-2n-2\right)}{nr^2}\Psi_{1rzz} + \frac{\left(3n-n^2-2\right)}{n^2r^3}\Psi_{1zz} + \\ &\frac{(4-2n)}{r}\Psi_{1zzrr} + \frac{n}{r}\Psi_{1zzzz}\end{aligned}\right\} \tag{5.58}$$

By letting n = 1, equation 5.58 reduces to the governing equation for $\Psi$ in the case of a Newtonian fluid undergoing peristaltic motion in an axisymmetric tube. Remembering that $\Psi_0(r)$ is a function of r only, the above equation thus becomes

$$\nabla^2\Psi_{1r} - \frac{2}{r}\nabla^2\Psi_1 + \frac{1}{r^2}\Psi_{1r} - \frac{1}{r}\Psi_{0r}\nabla^2\Psi_{1z} = \frac{1}{\mathrm{Re}}\nabla_1^{\ 4}\Psi_1 \tag{5.59}$$

where, $\nabla_1^{\ 4} = \nabla_1^{\ 2}\nabla_1^{\ 2}$ (5.60)

and $$\nabla_1^{\ 2} \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{5.61}$$

Taking the boundary conditions as follows

$$\begin{aligned} &\Psi_{0r}(1) = 0 \\ &\Psi_{1r}(1) + \Psi_{0rr}(1)\cos\alpha(z-t) = 0 \\ &\Psi_{0z}(1) = 0 \\ &\Psi_{1z}(1) + \Psi_{0zr}(1)\cos\alpha(z-t) = \alpha\sin\alpha(z-t) \end{aligned} \tag{5.62}$$

and substitution for $\Psi_0(r)$ as given in equation 5.53 yields the boundary conditions in the form

$$\begin{aligned} &\Psi_{1r}(1) = -\left(\frac{k}{2}\right)^{\frac{1}{n}} \cos\alpha(z-t) \\ &\Psi_{1z}(1) = \alpha\sin\alpha(z-t) \end{aligned} \tag{5.63}$$

From these boundary conditions, $\Psi_1$ can be assumed to be of the form,

$$\Psi_1(r,z,t) = F(r)\cos\alpha(z-t) + G(r)\sin\alpha(z-t) \tag{5.64}$$

where *F(r)* and *G(r)* are to be determined.

Substituting for, $\Psi_0(r)$, as given in equation 5.53 and, $\Psi_1$, as given in equation 5.64 into equation 5.58 and collecting coefficients of, $\cos\alpha(z-t)$, on either side of the resulting equation after multiplying both sides by $\mathrm{Re}\left(\frac{k_0}{2}\right)^{\frac{1-n}{n}} r^{\frac{1+4n}{n}}$ yields

$$nr^4F^{IV}-\frac{2}{n}\left(n^2-n+1\right)r^3F'''+\left\{\frac{\left(4n^3-4n^2+n+2\right)}{n^2}r^2+(2n-4)\alpha^2r^4\right\}F''-$$

$$\left\{\frac{\left(4n^3-4n^2+n+2\right)}{n^2}r+\left(\frac{2n^2-2n-2}{n}\right)\alpha^2r^3\right\}F'+\left\{n\alpha^4r^4+\left(\frac{n^3-3n+2}{n^2}\right)\alpha^2r^2\right\}F=$$

$$-\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}}-r^{\frac{1+3n}{n}}\right)G''\right\}+\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{1+2n}{n}}-r^{\frac{2+3n}{n}}\right)G'\right\}+$$

$$\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\left(\frac{n-1}{n}\right)r^{\frac{2+2n}{n}}G\right\}+\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{1+3n}{n}}-r^{\frac{2+4n}{n}}\right)\alpha^2G\right\}+$$

$$\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{-r^{\frac{1+3n}{n}}G''+r^{\frac{1+2n}{n}}G'\right\}+\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{\alpha^2r^{\frac{1+3n}{n}}G\right\}$$

(5.65)

Substituting for, $\Psi_0(r)$, as given in equation 5.53 and, $\Psi_1$, as given in equation 5.64 into equation 5.58 and collecting coefficients of, $\sin\alpha(z-t)$, after multiplying both sides by , $\mathrm{Re}\left(\frac{k_0}{2}\right)^{\frac{1-n}{n}}r^{\frac{1+4n}{n}}$ yields,

$$nr^4G^{IV}-\frac{2}{n}\left(n^2-n+1\right)r^3G'''+\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r^2\right\}G''+$$

$$\left\{(2n-4)\alpha^2r^4\right\}G''-\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r\right\}G'+\left\{+\left(\frac{2n^2-2n-2}{n}\right)\alpha^2r^3\right\}G'$$

$$+\left\{n\alpha^4r^4+\left(\frac{n^3-3n+2}{n^2}\right)\alpha^2r^2\right\}G=$$

$$-\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}}-r^{\frac{1+3n}{n}}\right)F''+\frac{n}{n+1}\left(r^{\frac{1+2n}{n}}-r^{\frac{3+2n}{n}}\right)F'+\frac{n-1}{n}r^{\frac{2+2n}{n}}F\right.$$

$$\left.+\frac{n}{n+1}\left(r^{\frac{1+3n}{n}}-r^{\frac{2+4n}{n}}\right)\alpha^2F\right\}+\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{r^{\frac{1+3n}{n}}F''-r^{\frac{1+2n}{n}}F'-\alpha^2r^{\frac{1+3n}{n}}F\right\}$$

(5.66)

The corresponding boundary conditions from equation(5.63) are

$$F(1)=-1 \quad F'(1)=-\left(\frac{K}{2}\right)^{\frac{1}{n}}$$
$$G(0)=G(1)=G'(0)=G'(1)=0 \tag{5.66a}$$
$$F(0)=F'(0)=0$$

It is not possible to find closed form solutions to the differential equations 5.65 and 5.66, so approximate solutions will be sought.

Firstly the equations for *F(r)* and *G(r)* can be simplified by assuming that the Reynolds number associated with the present model is small, and consequently the forms of *F(r)* and *G(r)* are assumed to be given as

$$F(r)=F_0(r)+\mathrm{Re}^2 F_2(r)+\ldots$$
$$G(r)=\mathrm{Re}\,G_1(r)+\mathrm{Re}^3 G_3(r)+\ldots \tag{5.67}$$

Substituting these forms for *F(r)* and *G(r )* into equation 5.65 and equation 5.66 and collecting terms of equal order in Re on either side of the equations yield the following differential equations for the first terms $F_0$ *(r)* and $G_1$ *(r)*.

$$nr^4F_0^{IV}-\frac{2}{n}\left(n^2-n+1\right)r^3F_0'''+\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r^2+(2n-4)\alpha^2r^4\right\}F_0''$$
$$-\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r+\left(\frac{2n^2-2n-2}{n}\right)\alpha^2r^3\right\}F_0'+\left\{n\alpha^4r^4+\left(\frac{n^3-3n+2}{n^2}\right)\alpha^2r^2\right\}F_0=0 \tag{5.68}$$

and

$$nr^4G_1^{IV}-\frac{2}{n}\left(n^2-n+1\right)r^3G_1'''+\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r^2+(2n-4)\alpha^2r^4\right\}G_1''$$
$$-\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r+\left(\frac{2n^2-2n-2}{n}\right)\alpha^2r^3\right\}G_1'+\left\{n\alpha^4r^4+\left(\frac{n^3-3n+2}{n^2}\right)\alpha^2r^2\right\}G_1=$$
$$-\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}}-r^{\frac{1+3n}{n}}\right)F_0''+\frac{n}{n+1}\left(r^{\frac{1+2n}{n}}-r^{\frac{3+2n}{n}}\right)F_0'+\frac{n-1}{n}r^{\frac{2+2n}{n}}F_0\right.$$
$$\left.+\frac{n}{n+1}\left(r^{\frac{1+3n}{n}}-r^{\frac{2+4n}{n}}\right)\alpha^2F_0\right\}+\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{r^{\frac{1+3n}{n}}F_0''-r^{\frac{1+2n}{n}}F_0'-\alpha^2r^{\frac{1+3n}{n}}F_0\right\} \tag{5.69}$$

Once $F_0(r)$ and $G_1(r)$ and subsequently $F(r)$ and $G(r)$ have been solved $\Psi_1$ is immediately found from equation 5.64.

If we assume $\alpha$ is very small such that higher order terms in $\alpha$ may be neglected, equations 5.68 and 5.69 may be simplified as

$$nr^4F_0^{IV} - \frac{2}{n}\left(n^2 - n + 1\right)r^3F_0''' + \left\{\frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}r^2\right\}F_0'' - \left\{\frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}r\right\}F_0' = 0 \tag{5.70}$$

and

$$nr^4G_1^{IV} - \frac{2}{n}\left(n^2 - n + 1\right)r^3G_1''' + \left\{\left(\frac{4n^3 - 4n^2 + n + 2}{n^2}\right)r^2\right\}G_1'' - \left\{\left(\frac{4n^3 - 4n^2 + n + 2}{n^2}\right)r\right\}G_1' =$$
$$-\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}} - r^{\frac{1+3n}{n}}\right)F_0'''\right\} + \left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{1+2n}{n}} - r^{\frac{3+2n}{n}}\right)F_0'\right\} +$$
$$\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n-1}{n}r^{\frac{2+2n}{n}}F_0\right\} - \left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{r^{\frac{1+3n}{n}}F_0'' - r^{\frac{1+2n}{n}}F_0'\right\} \tag{5.71}$$

The solution to equation 5.70 can be sought as the form

$$\hat{F}(r) = Ar^{m_1} + Br^{m_2} + Cr^{m_3} + Dr^{m_4} \tag{5.72}$$

where $m_i$ , i = 1-4 are to be determined by making the substitution

$$\hat{F}(r) = r^m \tag{5.73}$$

On substitution (5.73) into equation 5.70 yields

$$r^m\left\{nm(m-1)(m-2)(m-3) - \frac{2}{n}\left(n^2 - n + 1\right)m(m-1)(m-2) + \frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}(m-1)m - \frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}m\right\} = 0 \tag{5.74}$$

This finally reduces to,

$$m(m-2)\left[nm^2 - \left(6n - 2 + \frac{2}{n}\right)m\right] + m(m-2)\left[9n - 6 + \frac{3}{n} + \frac{2}{n^2}\right] = 0 \tag{5.75}$$

Where the roots $m_1$ and $m_2$ are given by

$$m_1 = 0$$
$$m_2 = 2$$

The other two roots are evaluated by determining the determinant of (5.75) to see if all the roots are real, hence

$$\Delta = \frac{4}{n^2}(2n-1)^2 \qquad (5.76)$$

$$\Delta \geq 0$$

$$so, \quad m_3 = 3 + \frac{1}{n}, \quad m_4 = 3 - \frac{3}{n} + \frac{2}{n^2}$$

our general solution to (5.77) is

$$\hat{F}_0(r) = A + Br^2 + Cr^{m_3} + Dr^{m_4} \qquad (5.77)$$

where the values of $m_3$ and $m_4$ are given above.

There is now sufficient information to find $F_0(r)$ and $G_1(r)$

For the Newtonian case n = 1 (5.77) gives the general solution to $F_0(r)$ as

$$\hat{F}_0(r) = A + Br^2 + Cr^4 + Dr^2 \ln r \qquad (5.78)$$

where lnr is introduced because $m_2 = m_4 = 2$ in this particular case.

From (5.78)

$$\hat{F}_0''(r) = 2B + 3D + 2D \ln r \qquad (5.79)$$

keeping this second derivative bounded at r= 0 means,

D = 0

Care needs to be taken by taking into consideration the raising of a negative value to the power of 1/n in equation(5.66a) due to 5.52a.

This problem can be avoided by re-writing the solution to as given in (5.53) as

$$\Psi_0(r) = \left(\frac{k}{2}\right)^{\frac{1}{n}} \frac{n}{n+1} \left\{ \frac{r^2}{2} - \frac{nr^{\frac{3n+1}{n}}}{3n+1} \right\}$$

Hence, also, the corresponding boundary conditions are given by

$$
\begin{aligned}
&F_0(1) = -1 \qquad F_0'(1) = \left(\frac{k}{2}\right)^{\frac{1}{n}} \\
&G_1(1) = 0 \qquad G_1'(1) = 0 \\
&F_0(0) = 0 \qquad F_0'(0) = 0 \\
&G_1(0) = 0 \qquad G_1'(0) = 0
\end{aligned}
\tag{5.80}
$$

Hence the boundary conditions (5.80) enable the constant coefficients to be determined as

$$
\begin{aligned}
&A = 0 \\
&B + C = -1 \\
&2B + 4C = \frac{k}{2}
\end{aligned}
\tag{5.81}
$$

For the case k=1, the solution to (5.81) is

C=5/4

B=-9/4

Thus, for n=k=1, the approximating function for $F_0(r)$ is

$$\hat{F}_0(r) = \frac{1}{4}\left(5r^4 - 9r^2\right) \tag{5.82}$$

For, n = k = 1 the homogeneous solution to equation 5.71 again keeping the second derivative bounded at r=0 is

$$G'_n(r) = A_1 + B_1 r^2 + C_1 r^4 \tag{5.83}$$

A particular solution to equation 5.74 can be sought of the form,

$$\hat{G}_p(r) = D_1 r^6 + E_1 r^8 \tag{5.84}$$

Therefore substitution of (5.84) into equation 5.71 with n = 1 yields

$$192Dr^6 + 1152Er^8 = \frac{5\alpha}{2}\left[5r^6 - r^8\right]$$

Hence $D_1 = \frac{25\alpha}{384}$ $\qquad E_1 = \frac{-5\alpha}{2304}$ (5.85) so from (5.83) and (5.84) the solution to equation 5.71 is

$$\hat{G}_1(r) = A_1 + B_1 r^2 + C_1 r^4 + \frac{25\alpha}{384} r^6 - \frac{5\alpha}{2304} r^8 \tag{5.86}$$

The boundary conditions imply from equation 5.80

$$\begin{aligned}
A_1 &= 0 \\
B_1 + C_1 &= \frac{5\alpha}{2304} - \frac{25\alpha}{384} \\
2B_1 + 4C_1 &= \frac{5\alpha}{288} - \frac{25\alpha}{64} \\
\therefore 2C_1 &= -\frac{95\alpha}{384} \\
\therefore B_1 &= \frac{275\alpha}{2304}
\end{aligned} \tag{5.87}$$

For the non-Newtonian case n = 0.8, k = 1

$$\hat{F}(r) = A + Br^2 + Cr^{4.25} + Dr^{2.375} \tag{5.88}$$

Therefore using the boundary conditions in equation 5.80

A = 0

B + C + D = 1

2B + 4.25C + 2.375D = 0.420448 (5.89)

Hence another boundary condition is required, therefore using the condition of symmetry, ie, $\Psi_{1rr} = 0$, at r = 0

C = 1.490906, D = -2.490906
B=0 (5.90)

therefore,

$$\hat{F}_0(r) = 1.490906 r^{4.25} - 2.490906 r^{2.375} \tag{5.91}$$

In Raju and Devanathan(1972) they have not specified the values of the parameter $\alpha$, except for an example using $\alpha$=1, to obtain numerical solutions to $F_0(r)$ and $G_1(r)$, so the only meaningful comparison that can be made is between approximate values obtained for $F_0(r)$ in the present analysis and the corresponding solution given in Raju and

Devanathan(1972). In spite of this, the approximating function for $G_1(r)$ will be specified for the special case of n =1.

From equations(5.83 - 5.87) the approximate solution for the function given by $\hat{G}_1(r)$ for the case, n = k = 1, is given by

$$\hat{G}_1(r) = \frac{275\alpha r^2}{2304} - \frac{95\alpha r^4}{768} + \frac{25\alpha r^6}{384} - \frac{5\alpha r^8}{2304} \qquad (5.92)$$

## 5.6 Discussion

If we define an error term as

$$E_{F_0(r)} = \left\| \hat{F}_0(r) - F_0(r) \right\| \qquad (5.93)$$

a measure of whether the approximate function $\hat{F}_0(r)$ from 5.77 gives values which are a good approximation to $F_0(r)$, the numerical solution ( note, these values taken from Raju & Devanathan 1972).

From Tables 5.1-5.4 where we are considering various values of flow behaviour index, n, and a function of pressure gradient (k), it is evident that the values are very good. This is graphically depicted in figures 5.2-5.5

| r | $\hat{F}_0(r)$ | $F_0(r)$ | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.022 | 0.022 | 0.000 |
| 0.2 | 0.088 | 0.085 | 0.003 |
| 0.3 | 0.192 | 0.186 | 0.006 |
| 0.4 | 0.328 | 0.318 | 0.010 |
| 0.5 | 0.484 | 0.471 | 0.013 |
| 0.6 | 0.648 | 0.634 | 0.014 |
| 0.7 | 0.802 | 0.790 | 0.012 |
| 0.8 | 0.928 | 0.920 | 0.008 |
| 0.9 | 1.002 | 0.999 | 0.003 |
| 1.0 | 1.000 | 1.000 | 0.000 |

Table 5.1: n=k=1, where n is flow behaviour index, showing error term

For Newtonian case n=k=1, the approximating function $\hat{F}_0(r)$ is given by (5.82) as

$$\hat{F}_0(r) = \frac{1}{4}\left(5r^4 - 9r^2\right) \qquad (5.94)$$

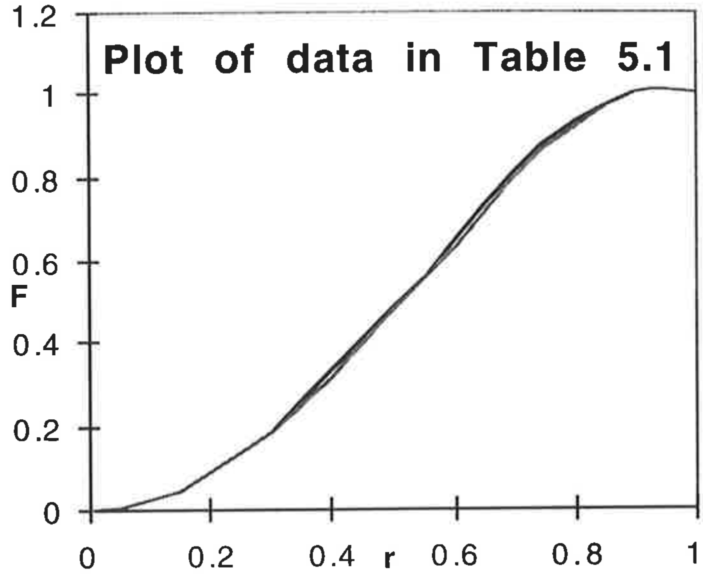


Figure 5.2: Plot of data in Table 5.1, showing comparison between the approximate solution $\hat{F}_0(r)$ to the numerical solution $F_0(r)$

For the non-Newtonian pseudoplastic case n = 0.8, k = 1, the approximating function for $\hat{F}_0(r)$ is given by (5.91) as

$$\hat{F}_0(r) = 1.490906r^{4.25} - 2.490906r^{2.375} \qquad (5.95)$$

A comparison between values obtained from equation 5.95 and values as recorded in Raju and Devanathan(1978) is given in Table 5.2

| r | $\hat{F}_0(r)$ | $F_0(r)$ | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.010 | 0.019 | 0.009 |
| 0.2 | 0.053 | 0.076 | 0.023 |
| 0.3 | 0.134 | 0.168 | 0.034 |
| 0.4 | 0.252 | 0.292 | 0.040 |
| 0.5 | 0.402 | 0.439 | 0.037 |
| 0.6 | 0.570 | 0.600 | 0.030 |
| 0.7 | 0.740 | 0.759 | 0.019 |
| 0.8 | 0.889 | 0.897 | 0.008 |
| 0.9 | 0.987 | 0.989 | 0.002 |
| 1.0 | 1.000 | 1.000 | 0.000 |

Table 5.2: n=0.8, k=1, where n is the flow behaviour index

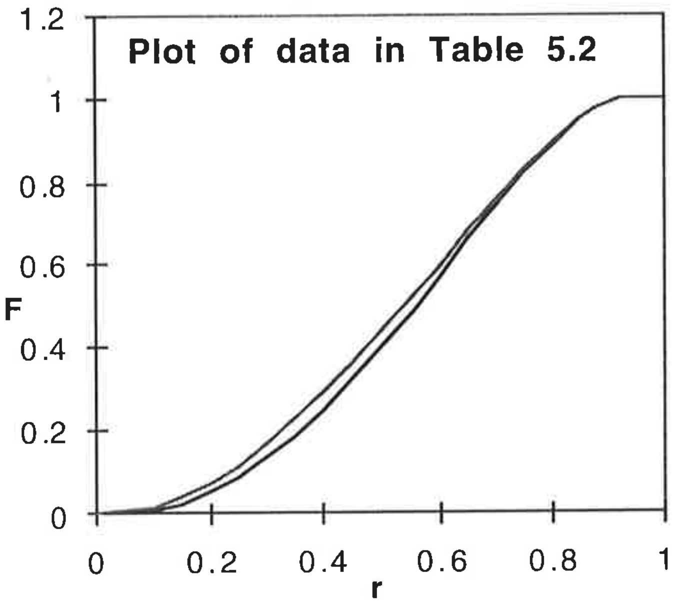


Figure 5.3: Plot of data in Table 5.2, showing comparison between the approximate solution $\hat{F}_0(r)$ to the numerical solution $F_0(r)$

Similarly, for the non-Newtonian dilatant case n = 1.2, k = 1, the approximating function for $\hat{F}_0(r)$ is given by (5.77) after using boundary conditions (5.80) as

$$\hat{F}_0(r) = 1.391035r^{\frac{23}{6}} - 2.397035r^2 \tag{5.96}$$

A comparison between values obtained from equation 5.96 and values as recorded in Raju and Devanathan(1972) is given in Table 5.3

| r | $\hat{F}_0(r)$ | $F_0(r)$ | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.024 | 0.024 | 0.000 |
| 0.2 | 0.093 | 0.092 | 0.001 |
| 0.3 | 0.202 | 0.199 | 0.003 |
| 0.4 | 0.342 | 0.337 | 0.005 |
| 0.5 | 0.501 | 0.494 | 0.007 |
| 0.6 | 0.666 | 0.658 | 0.008 |
| 0.7 | 0.819 | 0.811 | 0.008 |
| 0.8 | 0.940 | 0.935 | 0.005 |
| 0.9 | 1.009 | 1.007 | 0.002 |
| 1.0 | 1.000 | 1.000 | 0.000 |

Table 5.3: n=1.2, k=1,where n is flow behaviour index, showing error term

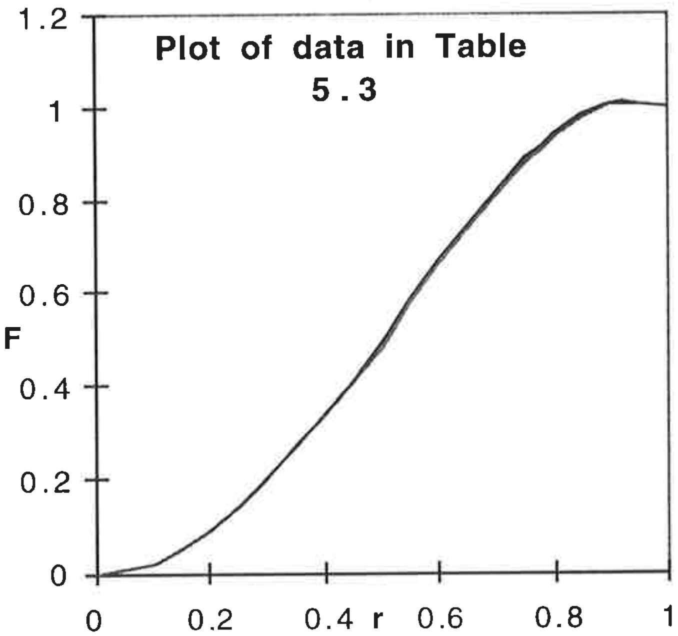


Figure 5.4: Plot of data in Table 5.3, showing comparison between the approximate solution $\hat{F}_0(r)$ to the numerical solution $F_0(r)$

Also, for the non-Newtonian dilatant case n =1.4, k = 1, the approximating function for $\hat{F}_0(r)$ is given by (5.77) after using boundary conditions (5.80) as

$$\hat{F}_0(r) = 1.522212r^{\frac{26}{7}} - 2.52212r^2 \tag{5.97}$$

A comparison between values obtained from equation 5.97 and values as recorded in Raju and Devanathan(1972) is given in Table 5.4

| r | $\hat{F}_0(r)$ | $F_0(r)$ | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.025 | 0.025 | 0.000 |
| 0.2 | 0.097 | 0.098 | 0.001 |
| 0.3 | 0.210 | 0.209 | 0.001 |
| 0.4 | 0.353 | 0.351 | 0.002 |
| 0.5 | 0.515 | 0.512 | 0.003 |
| 0.6 | 0.680 | 0.676 | 0.004 |
| 0.7 | 0.831 | 0.828 | 0.003 |
| 0.8 | 0.950 | 0.947 | 0.003 |
| 0.9 | 1.014 | 1.013 | 0.001 |
| 1.0 | 1.000 | 1.000 | 0.000 |

Table 5.4: n=1.4, k=1, where n is flow behaviour index, showing error term

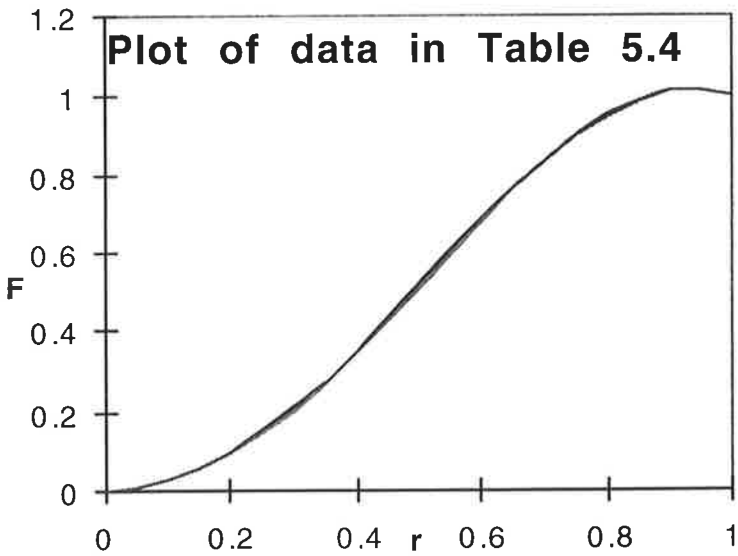


Figure 5.5: Plot of data in Table 5.4, showing comparison between the approximate solution $\hat{F}_0(r)$ to the numerical solution $F_0(r)$

Also, for the non-Newtonian pseudoplastic case n = 0.5, k= 1, the approximating function for $\hat{F}_0(r)$ is given by (5.77) after using boundary conditions (5.80) as

$$\hat{F}_0(r) = 0.25\left(3r^5 - 7r^2\right) \qquad (5.98)$$

A comparison between values obtained from equation 5.98 and values as recorded in Raju and Devanathan(1972) is given in Table 5.5, where error term is approximate.

| r | $\hat{F}_0(r)$ n=0.5 | $F_0(r)$ n=0.6 | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.017 | 0.015 | 0.002 |
| 0.2 | 0.070 | 0.062 | 0.018 |
| 0.3 | 0.156 | 0.140 | 0.016 |
| 0.4 | 0.272 | 0.249 | 0.023 |
| 0.5 | 0.414 | 0.385 | 0.029 |
| 0.6 | 0.572 | 0.542 | 0.030 |
| 0.7 | 0.731 | 0.707 | 0.024 |
| 0.8 | 0.874 | 0.860 | 0.014 |
| 0.9 | 0.975 | 0.972 | 0.003 |
| 1.0 | 1.000 | 1.000 | 0.000 |

Table 5.5: n=0.5, n=0.6, k=1, where n is flow behaviour index, showing error term

It is worth mentioning again that a graphical plot of the closeness of solutions for $F_0(r)$ is graphically depicted in Figures 5.2-5.5 for behaviour index (n)=$n_i$ and k=1

From these Tables and Figures it can be seen that the approximate solutions are very accurate for n = 1.0, 1.2 & 1.4 and less accurate for n = 0.8 & n=0.5, but still good enough to be used to find approximate solutions for the stream function and pressure gradient for those values of n.

From equation 5.50 it is clear that for $\varepsilon$ very small the stream function, can be approximated as

$$\Psi = \Psi_0 + \varepsilon \Psi_1 \tag{5.99}$$

Thus, defining, $\Psi^N$ to be $\Psi$ in the case n = N and defining $\hat{F}^N(r)$ as the approximate solution for $F(r)$ in the case n = N, equation 5.99 gives the approximation to the stream function, along with equations 5.53 and 5.64 as

$$\Psi^N \approx \left(\frac{k}{2}\right)^{\frac{1}{N}} \frac{N}{N+1}\left(\frac{r^2}{2} - \frac{Nr^{\frac{3N+1}{N}}}{3N+1}\right) + \varepsilon\left[\hat{F}^N(r)\cos\alpha(z-t) + \hat{G}^N(r)\sin\alpha(z-t)\right] \qquad \textbf{(5.100)}$$

where $\hat{G}^N(r)$, is the approximate function to $G(r)$ in the case n = N.

From equation 5.67 it can be seen that

$$\varepsilon\left[\hat{F}^N(r)\cos\alpha(z-t) + \hat{G}^N(r)\sin\alpha(z-t)\right] =$$

$$\cos\alpha(z-t)\left[\varepsilon F_0^N(r) + O\left(\varepsilon \mathrm{Re}^2\right)\right] + \sin\alpha(z-t)\left[\varepsilon \mathrm{Re}\, G_1^N(r) + O\left(\varepsilon \mathrm{Re}^3\right)\right] \qquad \textbf{(5.101)}$$

Under the assumption, $\varepsilon$, & Re small and defining

$$\alpha(z-t) = \bar{z} \qquad \textbf{(5.102)}$$

equations 5.100-5.101 give an approximation to the stream function as,

$$\Psi^N \approx \left(\frac{k}{2}\right)^{\frac{1}{N}} \frac{N}{N+1}\left(\frac{r^2}{2} - \frac{Nr^{\frac{3N+1}{N}}}{3N+1}\right) + \varepsilon\left[\hat{F}_0^N(r)\cos(\bar{z})\right] \qquad \textbf{(5.103)}$$

Streamlines for , n = N = 0.5, 1.0, 1.2 are given in Figures 5.6-5.8 respectively, where k = 1.0 and $\varepsilon$ =0.01.

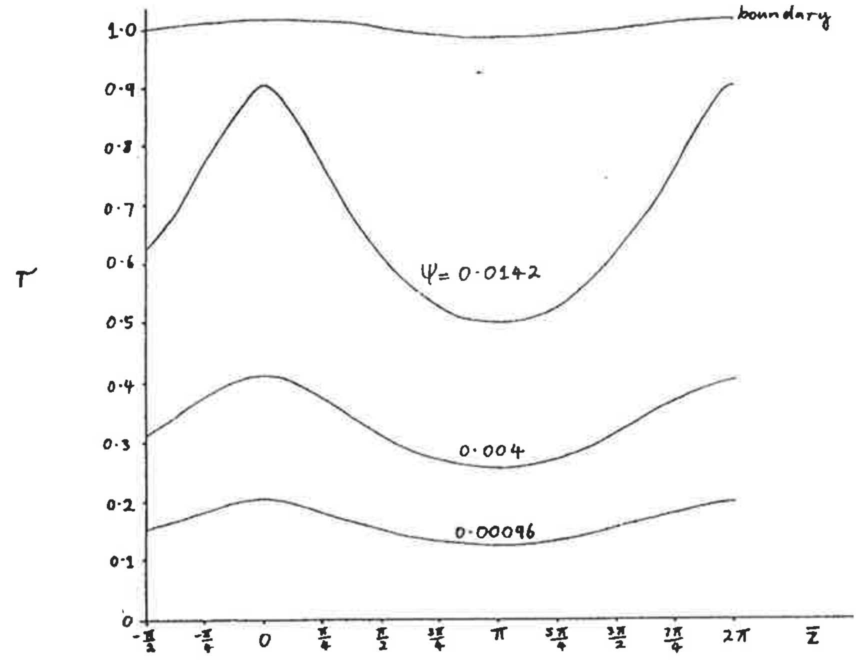


Figure 5.6: Streamline Ψ for n=0.5, k=1,where n is flow behaviour index

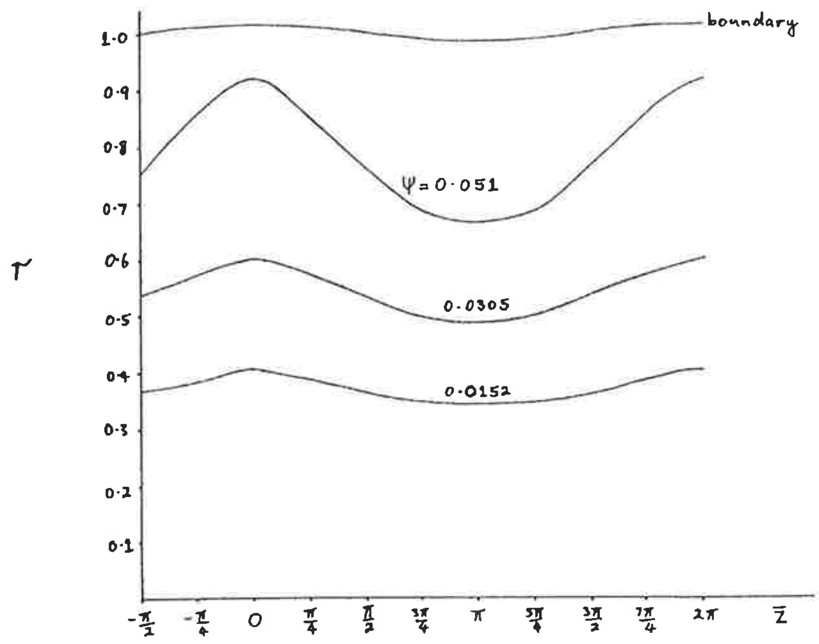


Figure 5.7: Streamline Ψ for n=1,k=1,where n is flow behaviour index

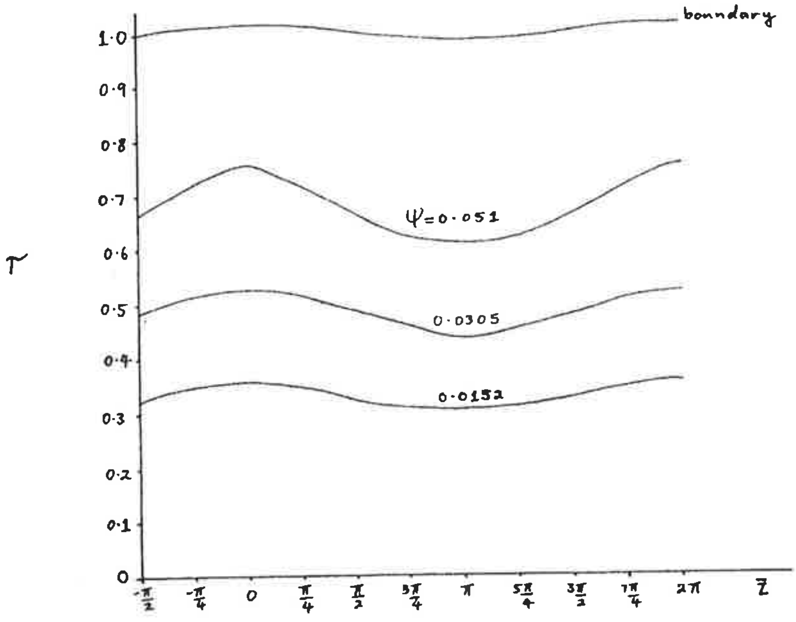


Figure 5.8: Streamline $\Psi$ for n=1.2, k=1, where n is flow behaviour index

The approximating functions for $F_0(r)$ in the case k=0.01 and n=0.5, 0.8, 1.0 and 1.2 are found in the same manner as the approximating function $F_0(r)$ k=1, and are given by

$$\hat{F}_0^{0.5}(r) = 0.666675r^5 - 1.666675r^2 \quad \textbf{(5.104)}$$

$$\hat{F}_0^{0.8}(r) = 1.267376r^{4.25} - 2.267376r^{2.375} \quad \textbf{(5.105)}$$

$$\hat{F}_0^{1.0}(r) = 1.0025r^4 - 2.0025r^2 \quad \textbf{(5.106)}$$

$$\hat{F}_0^{1.2}(r) = 1.097504r^{\frac{23}{6}} - 2.097504r^2 \quad \textbf{(5.107)}$$

Streamlines for n = 0.5, 0.8, 1.0 & 1.2 in the case k = 0.01 are plotted in Figures 5.9-5.12 respectively where $\varepsilon = 0.01$: and are derived from equation 5.103 using 5.104-5.107

It can be seen that the streamlines plotted in (Fig 5.6-5.8 )(high pressure gradient) and Fig. (5.9-5.12) (low pressure gradient ) are similar to those established in Raju and Devanathan(1972).

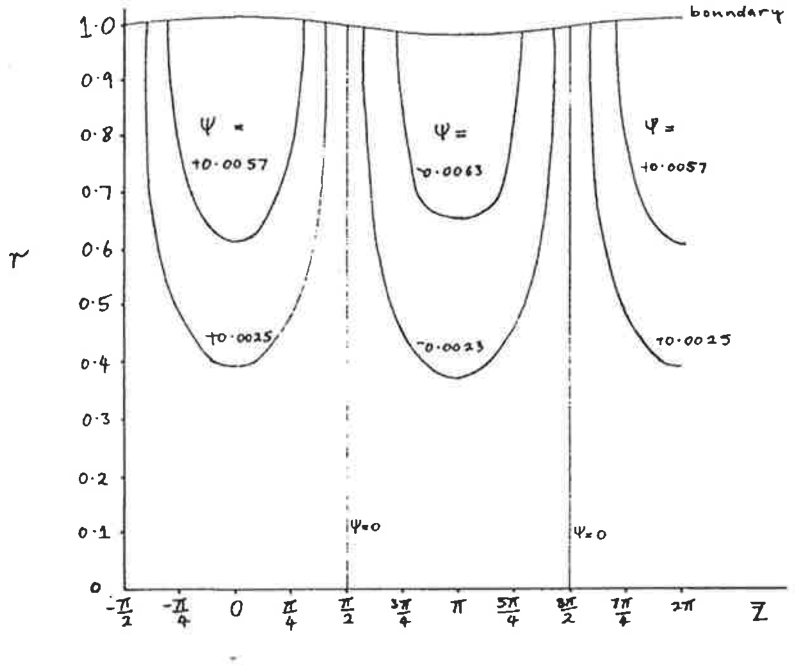


Figure 5.9: Streamline $\Psi$ for n=0.5, k=0.01, where n is flow behaviour index

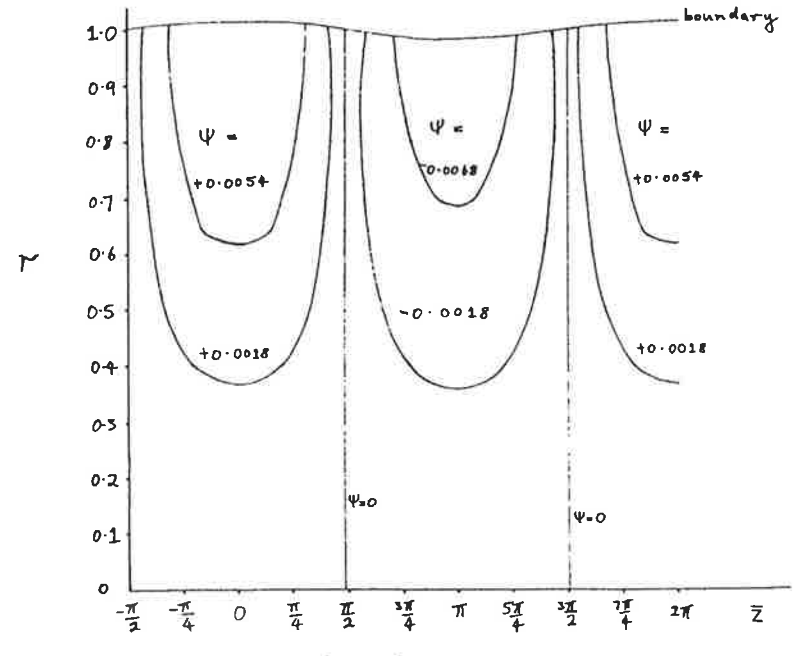


Figure 5.10: Streamline $\Psi$ for n=0.8, k=0.01,where n is flow behaviour index

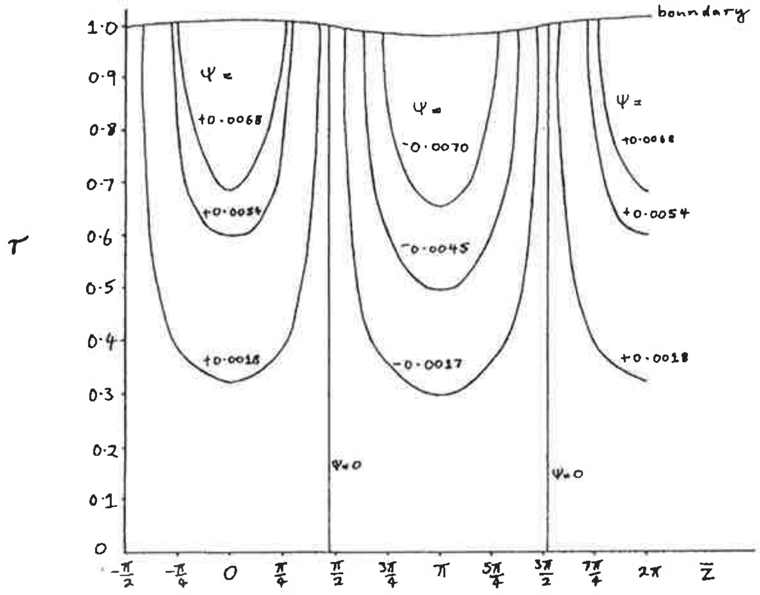


Figure 5.11: Streamline $\Psi$ for n=1.0, k=0.01,where n is flow behaviour index

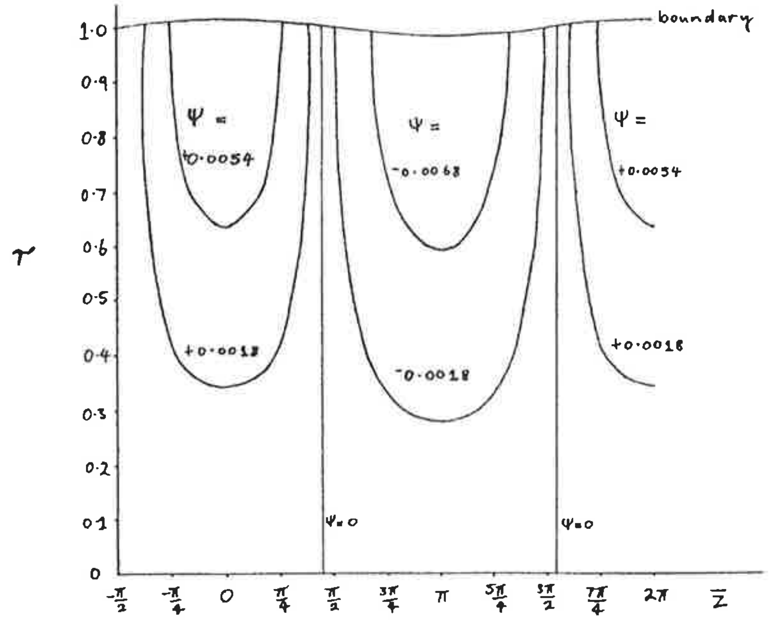


Figure 5.12: Streamline $\Psi$ for n=1.2, k=0.01,where n is flow behaviour index

The difference between streamlines for high and low pressure is apparent due to the boundary of the tube has a more significant impact with regard to low pressure gradient. All approximate results have been obtained by assuming that the terms of the order $\alpha^2$ and higher are negligible and the parameters $\varepsilon$ and Re are small. More accurate results could be obtained by neglecting terms $\alpha^3$ and higher but keeping order $\alpha^2$ and $\alpha$ as well as $\alpha^0$ .

## 5.7 Mathematical Modelling of a Casson Fluid in an Axisymmetric Tube

In this section mathematical equations are developed for the case of a Casson fluid in an axisymmetric tube. Due to the immense complexities of the equations and time restrictions involved in finding solutions for both the zeroth order approximation and first order approximation in streamfunction, only a brief outline will be given. Perhaps this may be a matter of possible future research. There may be possible similarities with both the Newtonian and power law models as developed earlier in this thesis.

As before, the vertical displacements for the upper and lower walls are given by G and -G, where the geometry G is defined by equation(5.36)

As with the case of a Casson fluid in a two-dimensional channel the stress-strain relationship is given by equations(4.63-4.67). In this case, we use the co-ordinate system where w and z are defined in the axial direction and u and r are defined as the radial direction.

The varying viscosity term is still defined as equation(4.64), where equations(4.65-4.67) still hold. However, due to the axisymmetric system,

$$V_{11} = \frac{\partial w}{\partial z}, V_{22} = \frac{\partial u}{\partial r}, V_{12} = \frac{1}{2}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)$$

As before, we consider the equation of continuity as equation(5.3) and develop the equations of motion as

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + 2\mu\frac{\partial^2 w}{\partial z^2} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\frac{\partial \mu}{\partial r} + 2\frac{\partial w}{\partial z}\frac{\partial \mu}{\partial z} + \mu\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \quad \textbf{(5.108)}$$

and

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} + 2\mu\frac{\partial^2 u}{\partial r^2} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\frac{\partial \mu}{\partial z} + 2\frac{\partial u}{\partial r}\frac{\partial \mu}{\partial r} + \mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \quad \textbf{(5.109)}$$

where

$$J_2 = \frac{\Psi_{rz}^{\ 2}}{r^2} + \frac{1}{4}\left\{\frac{\Psi_{rr}}{r} - \frac{\Psi_r}{r^2} - \frac{\Psi_{zz}}{r}\right\} \quad \textbf{(5.110)}$$

As before equations(4.74-4.76) express the streamfunction, pressure and viscosity in terms of amplitude ratio.

Therefore, as defined by (4.64) and taking only zeroth power in amplitude ratio

$$\mu_0 = \left\{ \alpha + \beta\left(\frac{1}{4}\right)^{-\frac{1}{4}} \left[ \frac{\Psi_{0rr}{}^2}{r^2} + \frac{\Psi_{0r}{}^2}{r^4} - 2\frac{\Psi_{0rr}\Psi_{0r}}{r^3} \right]^{-\frac{1}{4}} \right\}^2 \qquad \textbf{(5.111)}$$

We finally obtain after introducing the streamfunction as defined by (5.48 and 5.49) and substituting equations(4.74-4.76) and collecting coefficents of $\varepsilon^0$ in equation(5.108) and remember that the zeroth order in stream function is a function of r only, due to that constant axial pressure gradient

$$\left(\frac{\partial p}{\partial z}\right)_0 = \mu_{0r}\frac{\Psi_{0rr}}{r} - \mu_{0r}\frac{\Psi_{0r}}{r^2} + \mu_0\frac{\Psi_{0rrr}}{r} - \mu_0\frac{\Psi_{0rr}}{r^2} + \mu_0\frac{\Psi_{0r}}{r^3} \qquad \textbf{(5.112)}$$

which simplified is

$$r\frac{\partial p_0}{\partial z} = \frac{\partial}{\partial r}\left(r\mu_0\frac{\partial}{\partial r}\left(\frac{\Psi_{0r}}{r}\right)\right) \qquad \textbf{(5.113)}$$

and integrating once gives

$$\frac{r^2}{2}\frac{\partial p_0}{\partial z} + constant = r\mu_0\frac{\partial}{\partial r}\left(\frac{\Psi_{0r}}{r}\right) \qquad \textbf{(5.114)}$$

There are many complexities involved in trying to solve equation(5.112). The first of which are the difficulties collecting the derivatives of equation(5.111) and substituting them into equation(5.112) thus creating a very complex equation due to the radial dependency. The first order equation in streamfunction is many times complicated and may be a possible resaerch area for the future

## 5.8 Comparisons and Implications

The results obtained in this analytical study of a non-Newtonian power law fluid are consistent with those established by Raju and Devanathan(1972) in their study of peristaltic motion of a non-Newtonian fluid.

It should be noted that Raju and Devanathan(1972) have sought an approximate numerical solution of the non-Newtonian power-law model in an axisymmetric tube. They have made use of the Runge-Kutta Gill integration. In this thesis the solution to the same problem/model(non-Newtonian power law model) is achieved by deriving an analytical solution based on the assumptions and simplifications made by Raju and Devanathan in their paper “Peristaltic motion of a non-Newtonian fluid”, thus allowing a direct comparison between a numerical solution and an analytical solution. Figures 5.2 –5.5 and the error terms in Tables 5.1 –5.5 verify and validate the correctness of the analytical mathematical procedure undertaken in this study.

It was seen that the equation of motion for the non-Newtonian (power law) model of peristaltic motion reduced to the governing equations for the Newtonian model in the case of flow behaviour index, n = 1.

Approximate results for the stream function $\Psi$ were obtained and is seen for n >= 1 are close to the exact results given in the literature. For the case n <= 1, the results were less accurate, but nevertheless to an acceptable level, if we consider the magnitude of the errors.

The streamlines obtained from the approximate results were found to be very similar to those found in Raju and Devanathan(1972), thus demonstrating the effect of pressure gradient and boundary structure and type. In fact, it is found that for low pressure gradient the streamlines form closed loops and for higher presure gradient the streamlines run parallel to the axis of the tube when considered near the axis , whereas a considerable deformation is noted near the boundary. The streamline plots for low pressure gradient and high pressure gradient are entirely different, as shown in Figures 5.6-5.8 and 5.9-5.12. An explanation for this is that the region may be considered to be consisting of two parts; one the central core region and the other near the boundary, the boundary region. The physical expalanation for this sort of behaviour of the streamlines can be given as that in the case of a rigid tube with sinusiodal deformation, (Raju and Devanathan, 1972). Hence, highlighting the validity of current modelling of this type of a two layered fluid with a non-Newtonian core region and Newtonian peripheral region.

In the case of low pressure gradient, the effects of the wave travelling along the boundary of the tube are more dominant. But as the pressure gradient increases, we find that the streamlines in the central part of the region are more influenced by it, than the motion of the boundary, hence run approximately parallel to the axis.

It is also seen that for the study of a Casson fluid in an axisymmetric tube that for the zeroth order perturbation in stream function, the model reduces to the Newtonian case contains parameters involving fluid density, geometry, wave speed and Casson viscosity coefficient. A similar result is obtained for the case of a Casson fluid in a two dimensional channel under certain simplifications.

One conclusion reached by these studies was that peristalsis is an effective method to move fluid only if the fluid is transported in the form of a series of isolated boluses. If the amplitude of displacement of the wall is small compared to the tube radius, very little pressure gradient can be generated by the travelling wave. Pressure gradient increases significantly when the radius of minimum section approaches zero, (complete occlusion). It is thus understandable why peristalsis is a common phenomena in the lymphatics, intestine, ureter,and many other biological systems and peristaltic pumps (dialysis machines and heart-lung machines).

# CHAPTER 6

# CONCLUSION

## 6.1 Brief Summary

The first chapter of this research outlined why a mathematical perspective of peristaltic motion may be helpful in the understanding of physiological flow and why complications may arise.

In this study, mathematical models were developed and implemented for the case of a Newtonian fluid, power-law and Casson fluid with the geometry modelled as a two dimensional channel and axisymmetric tube. In the case of an axisymmetric tube, this may be approximated to flow in many physiological organs in the human body. However, the literature provides many examples of flow in a two-dimensional channel. This provides a very accurate insight into the dynamics and workings of the models of peristaltic flow, and hence it is important to consider both geometries not only for completeness but also serve as a comparison between the two geometries, thus providing continuity of results.

The study then involved the development of mathematical models for the study of peristaltic flow for the case of non Newtonian fluids; in particular the study analysed the constitutive equation for the power law model in both a two dimensional channel and axisymmetric tube. Also this research considered the non Newtonian case represented as a Casson fluid in the geometry of a two dimensional channel and axisymmetric tube.

Although the constitutive equations governing the fluid motion for the non Newtonian models of power law fluid and Casson fluid differ substantially, this research found that under certain conditions, (when considering the zeroth order approximation in stream function), both models reduce to that of the Newtonian case. In particular, it is seen that for the non Newtonian case of the power law fluid in an axisymmetric tube and two dimensional channel, the governing equations reduce to the Newtonian case when the power law index approached unity. The study also finds that for the case of an axisymmetric tube for the power law that the developed algebraic analytical solutions agree both graphically and numerically with that published in the literature when streamline functions are considered.

The study also shows the equations for the first order approximation in stream function for the Casson two dimensional channel case are very complicated and extensive, however an analytical and numerical solution is found which seem to agree in form with research in the literature.

## 6.2 Clinical Significance of Present Study

Consider results obtained from a mathematical analysis of flow of urine from the kidney to the bladder via the ureter. Assuming that the actual flow provides some cohesion between data obtained from experiment( Fung 1981,1984,1990) and the results of mathematical analysis, the present study validate that peristalsis explains the flow of urine from the kidney to the bladder.

For example, experiment(Fung, 1981, 1984, 1990) indicates a pressure drop in the ureter during peristalsis. Similarity of this data with corresponding theoretical results of pressure drop, means that future predictions may be made about the change in pressure across the ureter, during peristalsis; and probably the pressure and subsequent pressure change, can be governed by means of change to the urinary environment.

To generalise the analysis of flow of urine, assumptions which seem reasonable for urine, may and are not necessarily correct for other physiological fluids; but however may be relaxed and replaced by less restrictive assumptions.

With regard to reflux, control of the pressure across the ureter means that if the pressure applied by the ureter is sufficient to allow urine to flow into the bladder, a minor change to the ureter, causing increased applied pressure by the ureter on the bladder, can enable the ureter to then apply a sufficiently great pressure on the bladder so that urine passes into the bladder without backflow of urine to the kidney.

The mathematical analysis may be used to determine a critical pressure, below which reflux occurs. Thus reflux may be diagnosed at an early stage and dealt with by the medical urologists accordingly.

It has been seen that an accurate mathematical analysis of flow of urine in the ureter is an important issue; It may be useful to generalise this analysis to other physiological flows like that in the intestine and others whereby peristalsis occurs.

The non-Newtonian models provide insight to the complexity of these models and the phenomena of peristalsis, but ultimately in the first instance a Newtonian approach is viable for a basic understanding of peristalsis.

For example, it may be reasonable to assume that flow of urine is that of Newtonian flow, however by allowing the inclusion on non-Newtonian effects into the analysis, it becomes a special case of a generalised fluid. On this basis the assumption of Newtonian fluid is a special case of a generalised non-Newtonian fluid whereby theoretical results are a significant step when mathematically modelling peristalsis.

## 6.3 Recommendations for Future Study

In this research a study has been made concerning mathematical modelling of peristalsis as applied to Newtonian and non-Newtonian fluids. It is seen that their are similarities between all models when certain assumptions and simplifications are considered.

Perhaps in future work, it would be possible to generalise these mathematical models, by relaxing the assumptions and simplifications and attempt to obtain and compare solutions that are arrived at using a numerical solution, based on the existing equations and boundary conditions. And if apparent, compare these solutions and models to research that appears in the literature in the future.

Also as a possible area of future work would to consider a Casson model in an axisymmetric tube and extend research beyond what has been established in this thesis and compare the results with those within. Cohesion with analytical and numerical solutions will provide an inevitable quality control of the models in question.

# APPENDIX A

Non dimensional variables and parameters are as follows

### Two-dimensional Channel

$$x' = \frac{x}{d} \quad y' = \frac{y}{d} \quad u' = \frac{u}{c} \quad v' = \frac{v}{c} \quad R_e = \frac{cd}{\nu} \quad \nu = \frac{\mu}{\rho} \quad \psi' = \frac{\psi}{cd}$$

# APPENDIX B

Non dimensional variables and parameters are as follows

### Axisymmetric Tube

$$r' = \frac{r}{d} \quad , z' = \frac{z}{d} \quad , w' = \frac{w}{c} \quad , u' = \frac{u}{c}$$

$$G' = \frac{G}{d} \quad , \Psi' = \frac{\Psi}{cd^2} \quad , t' = \frac{ct}{d} \quad , p' = \frac{p}{\rho c^2}$$

$$\varepsilon = \frac{A}{d} \quad , \alpha = \frac{2\pi d}{\lambda}$$

$$\text{Re} = \frac{c^{2-n} d^n}{m} \rho$$

$$\theta' = \left( \frac{d^2}{c^2} \right)^{\frac{n-1}{2}} \theta$$

$$\left( \frac{\partial p}{\partial z} \right)_0' = \frac{d}{\rho c^2} \left( \frac{\partial p}{\partial z} \right)_0$$

## APPENDIX C

Analytical solution to Casson model in two-dimensional channel

$$f_0 = \frac{C_1\left(1 - \frac{2B^2Ea_1}{\Omega b_1}\right)}{2}\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)\frac{1}{s}\left(\left(s^{\frac{1}{2}}y - \frac{h}{2\sqrt{s}}\right)erf\left(s^{\frac{1}{2}}y - \frac{h}{2\sqrt{s}}\right) + \frac{\exp\left(-\left(s^{\frac{1}{2}}y - \frac{h}{2s^{\frac{1}{2}}}\right)^2\right)}{\sqrt{\pi}}\right)$$

$$-\frac{C_1}{2}\frac{a_1\Omega^3}{2Eb_1 s}\left\{-\frac{\left(-\frac{\exp\left(-sy^2+hy\right)}{2s} + \frac{h\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)erf(\%1)}{4s^{\frac{3}{2}}}\right)}{2s} + \frac{1}{2hs}\left(-\frac{\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)erf(\%1)}{4s^{\frac{3}{2}}} + \frac{h\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)\left(\%1.erf(\%1) + \frac{\exp(\%1^2)}{\sqrt{\pi}}\right)}{4s^2}\right) + \frac{\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)\left(\%1.erf(\%1) + \frac{\exp(\%1^2)}{\sqrt{\pi}}\right)}{4s^2}\right\}$$

$$+\frac{C_1a_1B\sqrt{\pi}}{b_1}\exp\left(\frac{h^2}{4s}\right)\frac{1}{s^{\frac{3}{2}}}\left(\frac{erf(\%1)\%1^2}{2} + \frac{erf(\%1)h\%1}{2s^{\frac{1}{2}}} - 2\frac{-\frac{\%1}{4\exp(\%1^2)} + \frac{\sqrt{\pi}erf(\%1)}{8} - \frac{h}{4\sqrt{s}\exp(\%1^2)}}{\sqrt{\pi}}\right)$$

$$-\frac{C_1a_1B\sqrt{\pi}}{b_1}\exp\left(\frac{h^2}{4s}\right)\frac{1}{s^{\frac{3}{2}}}\left(\frac{erf(\%1)\%1^2}{2} - \frac{-\frac{\%1}{2\exp(\%1^2)} + \frac{\sqrt{\pi}erf(\%)}{4}}{\sqrt{\pi}} + \frac{erf(\%1)}{2\sqrt{s}}\right)$$

$$+C_2\left(1 - \frac{2B^2E(a+\frac{1}{2})}{\Omega b_1}\right)\sum_{k_2=0}^{\infty}\left(\frac{1}{3}\exp\left(\frac{h^2}{4s}\right)(-1)^{k_2}2^{(-1-2k_2)}\left(-\frac{h}{\sqrt{s}}\right)^{(-1+2k_2)}y^{\frac{3}{2}}\frac{{}_2F_1([3/2,-1-2k_2],[5/2],2\frac{ys}{h})}{(\sqrt{s}(1/2+k_2)\Gamma(1+k_2))} + \frac{1}{3}2^{(-1-2k_2)}\exp\left(\frac{h^2}{4s}\right)h(-1)^{k_2}\left(-\frac{h}{\sqrt{s}}\right)^{(2k_2)}y^{\frac{3}{2}}\frac{{}_2F_1([1/2,-1-2k_2],[5/2],2\frac{ys}{h})}{(s(1/2+k_2)\Gamma(1+k_2))}\right)$$

$$+C_2\left(\frac{\Omega^3(a+\frac{1}{2})}{2Eb_1}\right)\sum_{k_2=0}^{\infty}\left(\begin{array}{l}\frac{1}{7}\exp\left(\frac{h^2}{4s}\right)(-1)^{k_2}2^{(-1-2k_2)}\left(-\frac{h}{\sqrt{s}}\right)^{(-1+2k_2}y^{\frac{7}{2}}\frac{{}_2F_1([7/2,-1-2k_2],[9/2],2\frac{ys}{h})}{(\sqrt{s}(1/2+k_2)\Gamma(1+k_2))}\\+\frac{1}{7}2^{(-1-2k_2)}\exp\left(\frac{h^2}{4s}\right)h(-1)^{k_2}\left(-\frac{h}{\sqrt{s}}\right)^{(2k_2)}y^{\frac{7}{2}}\frac{{}_2F_1([5/2,-1-2k_2],[9/2],2\frac{ys}{h})}{(s(1/2+k_2)\Gamma(1+k_2))}\end{array}\right)$$

$$+C_2\left(\frac{2B(a+\frac{1}{2})}{b_1}\right)\sum_{k_2=0}^{\infty}\left(\begin{array}{l}\frac{1}{5}\exp\left(\frac{h^2}{4s}\right)(-1)^{k_2}2^{(-1-2k_2)}\left(-\frac{h}{\sqrt{s}}\right)^{(-1+2k_2}y^{\frac{5}{2}}\frac{{}_2F_1([5/2,-1-2k_2],[7/2],2\frac{ys}{h})}{(\sqrt{s}(1/2+k_2)\Gamma(1+k_2))}\\+\frac{1}{5}2^{(-1-2k_2)}\exp\left(\frac{h^2}{4s}\right)h(-1)^{k_2}\left(-\frac{h}{\sqrt{s}}\right)^{(2k_2)}y^{\frac{7}{2}}\frac{{}_2F_1([3/2,-1-2k_2],[7/2],2\frac{ys}{h})}{(s(1/2+k_2)\Gamma(1+k_2))}\end{array}\right)$$

$$+C_3+C_4$$

where $C_1, C_2, C_3, C_4$ are constants of integration determined by the boundary conditions

where $\Omega = D - 4sE$ and $\%1 = y\sqrt{s} - \frac{h}{2\sqrt{s}}$

# BIBLIOGRAPHY


Barton, C and Raynor, S (1968), Peristaltic Flow in Tubes
*Bull. Math. Biophys. 30*, 663-680

Bugiarello, G., Kapur, C., Hsiao, G. (1965) The profile viscosity and other characteristics of blood flow in a non-uniform shear fluid. Proc. 4th Int. Congr. On Rheol. 4, Symp. On Biorheol. (ed. Copley, A.L) Interscience, New York, 351-370

Burns. J.C. and Parkes, T. (1967)
Peristaltic motion, *J. Fluid Mech.29* (4),731-743

Boyarsky, S. (1964) Surgical physiology of the renal pelvis, *Monogr. Surg. Sci*, 1:173-213

Chien, S., Usami, S., Taylor, H.M., Lundberg, J.L and Gregersons, M.I. (1966) Effects of Hematocrit and plasma protiens on human rheology at low shear rates. J. Appl. Physiol. 21, 81-7

Eckstein, E.C (1970) Experiment & Theoretical pressure studies of peristaltic pumping, S.M Thesis, MIT

El Misery, A.M, Elshehawey, E.F, and Hakeem, A.A(1996) Peristaltic Motion of an Incompressible Generalised Newtonian Fluid in a Planar Channel
*J.Phys. Soc. Japan* 65:11,3524-3529

Elshehawey, E.F, El Misery, A.M and El Naby, A.N(1998) Peristaltic Motion of Generalised Newtonian Fluid in a Non-Uniform Channel
*J.Phys. Soc. Japan* 67:2,434-440

Fung, Y. C & Yih, C. S. (1968)
Peristaltic Transport. J. Applied Mechanics, Trans. of the ASME
V35, pp669-675

Fung, Y.C. (1971)
Peristaltic Pumping:A Bioengineering Model, Urodynamics
*Proc.Workshop Hydrodynam. Upper Urinary Tract.* Acad. Sci.,Wash.,D.C

Fung. Y.C. (1981). Biomechanics, Mechanical Properties of Living Tissues
Springer-Verlag, New York

Fung. Y.C. (1984). Biodynamics Circulation, Springer-Verlag, New York

Fung. Y.C. (1990). Biomechanics: Motion, Flow, Stress and Growth Mechanical Properties of Living Tissues, Springer-Verlag, New York

Guha, S.K, Kaur,H and Ahmed,A.M(1975) Mechanism of Spermatic Fluid in Vas deferens, *Med. Biol. Engng*, 13:518-522

Jaffrin, M.Y and Shapiro, A.H. (1971)
Pertistaltic Pumping, *Annual review of Fluid Mechanics*, 43: 4, 13-36

Hanin, M (1968) The flow through a channel due to traversely oscillating walls, *Israel J. Tech.* 6:67-71

Hung, T.K & Brown, T.,D, (1976) Solid particle motion in two-dimensional peristaltic flows. *J. Fluid Mech.* 73:77-96

Kiil, F. (1967) The function of the ureter and renal pelvis, Philadelphia(Saunders)

Kumar, B.V and Naidu, K.B (1995) A Numerical study of peristaltic flows, *Comp & Fluids*, 24:161-176

Latham, T.W(1966) Fluid motion in a peristaltic pump, M.S Thesis, MIT(Cambridge)

Liron, N. (1978)
A new look at peristalsis and its functions.Horizons in Biochemistry and Biophysics
5, 161-182

Lykoudis, P.S(1971) Peristaltic pumping : A bioengineering bodel. *Proc. W/Shop Hydrdyn. Upper Urinay tract.* Nat. Acad. Sci.

Macagno, E.O, Christensen et al (1982)
Modelling the Effects of Wall Movement on Absorbtion in the Intestine
*Am. J. Physiol.* G541-G550

Mazumdar, J. (1992). Biofluid Mechanics
World Scientific Publishing

Mazumdar, J. (1999) An Introduction to Mathematical Physiology and Biology.
Australian Mathematical Society, Cambridge University Press(Second Edition)

Mernone, A.V and Mazumdar, J.N, (1998a)
Mathematical Modelling of Peristaltic Transport of a non-Newtonian Fluid
*Aust. Phys. & Eng. Sci. in Medicine* 21: 3, 126-141

Mernone, A.V and Mazumdar, J.N, (1998b)
Peristaltic Flow of a non-Newtonian Fluid and its Biomedical Applications
*EMAC '98, 3rd Biennial Engineering Mathematics and Applications Conference*, 367-370

Merrill, E.W., Benis, A.M., Gilliland, E.R., Sherwood, T.K., and Salzman, E.W(1965)
Pressure-flow relations of human blood. J. Appl. Physiol., 20, 954-967

Miller-Keane, (1992)
Encyclopedia & Dictionary of Medicine, Nursing and Applied Health, 5th Edition

Misery, A.M, Elshehawey, F and Hakeem, A. (1996)
Peristaltic Motion of an Incompressible Generalised Newtonian Fluid in a Planar Channel
*J. Phys. Soc. of Japan* : 65: 11, 3524-3529

Patel, P.D., Picologlone, B.F., and Lykoudis, P.S(1973) Biorheological Aspects of Colonic Activity. Pt.1 Theoretical considerations, "Biorheology, 10, 431-440

Phan-Thien, N and Low, H.T(1989) On the flow of a non-Newtonian liquid induced by intestine like contractions, J.Biomech. Eng, 111, 1-8

Polyanin, A.D and Zaitsev, V.F(1992), Handbook of Exact solution to Ordinary Differential Equations

Raju, K. K & Devanathan, R. (1972)
Peristaltic motion of a non-Newtonian Fluid.
Rheol. Acta 11, 170-178

Raju, K.K. (1972)
Flow of a non-Newtonian Fluid in a tube with Sinusoidal Deformation J.Phys. Soc. of Japan V33, No.1, pp225-231

Shapiro, A.H(1967) Pumping and retrograde diffusion in peristaltic waves. *Proc. Workshop Uteteral Reflux Children*. Nat Acad. Sci., Wash. D.C

Shapiro A.H, Jaffrin M.Y and Weinberg S.L. (1969)
Peristaltic pumping with long wavelengths at Low Reynolds number.
*J.Fluid Mech.*, 37,799-825

Siddiqui, A.M, Provost, A and Schwarz, W.H (1991) Peristaltic Pumping of a Second Order Fluid in a Planar Channel,*Rheol Acta* 30, 249-262

Srivastava, L.M., Peristaltic Transport of a Casson fluid(1987) Nig. J. Sci. Res., 1, 71-8 2

Srivastava, L.M and Srivastava, V.P (1984 )Peristaltic Transport of Blood: Casson Model II *J Biomechanics* ,17:11, 821-829

Srivastava, L.M and Srivastava, V.P (1985) Interaction of Peristaltic Flow with Pulsatile Flow in a Circular Cylindrical Tube
*J. Biomechanics* , 18: 4, 247-253

Srivastava, L.M and Srivastava, V.P (1988) Peristaltic transport of a power-law fluid:application to Ductus Efferentes of the reproductive tract
*Rheol. Acta*, 27:,428-433

Tang, D and Rankin, S (1993) Numerical and Asymptotic Solutions for Peristaltic Motion of Non Linear Viscous Flows with Elastic Free Boundaries
*SIAM J Sci. Comput.* 14 : 6, 1300-1319

Weinberg, A.H, Jaffrin, M.Y and Shapiro, A.H (1971)
Urodynamics ( S. Boyarsky, ed) Academic Press New Nork

Weinberg, A.H (1970), Theretical and experimental treatment of peristaltic piumping and its relation to ureteral function. PhD thesis, Mit, Cambridge

Yin, F & Fung, Y. C. (1969)
Peristaltic waves in circular cylindrical tubes.
*J. Applied Mechanics, Trans. of the ASME*
V36,pp579-587

Yin, F & Fung, Y. C. (1971)
Comparison of theory and experiment in peristalic transport.
*J. Fluid Mech.* 47:93-112

# MATHEMATICAL MODELLING OF PERISTALTIC TRANSPORT OF A NON-NEWTONIAN FLUID.

A. Mernone* and J. Mazumdar

Dept. of Applied Mathematics, The University of Adelaide, Australia, 5005

## Abstract

The paper considers the phenomena of peristaltic transport of a non-Newtonian fluid represented as a power law fluid. The governing equations are the modified Navier-Stokes equations and the continuity equation in axisymmetric form. A solution is sought in terms of a perturbation series and it is shown the close proximity between analytical and numerical solutions when considering stream functions for various values of the flow behaviour index.

## Keywords

Peristalsis, Ureter, non-Newtonian power law fluid, Perturbation method.

## Introduction

Peristalsis is the phenomenon in which a circumferential progressive wave of contraction or expansion travels along a tube. Peristalsis appears in many organisms and in a variety of organs. Some examples include swallowing of food through the oesophagus, transport of urine through the ureter, transport of the egg down the fallopian tube and transport of chyme through the small intestine. The initial wave may be followed by similar or identical waves after a short time has elapsed(4).

The above examples of when and where peristalsis occurs seem to warrant the inference that the role of peristalsis is transport. This is definitely clear in the ureter, where the tube connects the kidney to the bladder and is responsible for the transport of urine. The kidneys are not capable of producing enough pressure to force urine away from them by overcoming the pressure caused by the expanded bladder. However, in the case of the small intestine, the existence of segmental contraction could indicate that mixing is important

* Postgraduate student, Dept. Applied Mathematics, University of Adelaide
email:amernone@maths.adelaide.edu.au

and indeed food is absorbed by the tube walls during the passage of chyme through the small intestine(4).

Therefore, if the function is fluid transport, then certainly complete occlusion will do the job, as is observed in the ureter. But, complete occlusion is an idealisation, and depending on the pressure difference between the two ends of the wave, the peristaltic wave usually does not propel the entire volume of the fluid, resulting in reflux. If this occurs in the ureter, then bacteria may be carried back from the bladder to the kidney; a phenomenon believed to be the mechanism by which bacteria reach and infect the kidneys(4,13) (see Fig. 1)

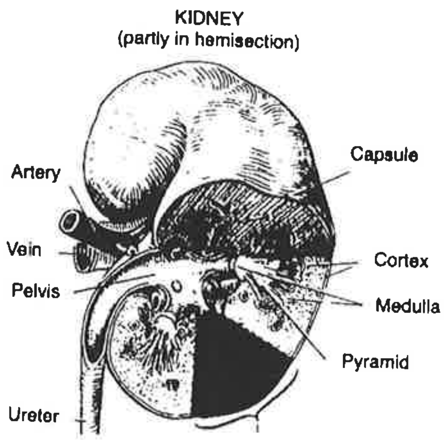


**Figure 1**, *Kidney with ureter*

The action of a healthy ureter is one whereby the amplitude of the travelling wave on the elastic wall is so large that at the narrowest point the walls press against each other. Some typical dimensions for the ureter may be of interest. The entire length of the ureter is of the order of 30cm; normally, there are 3-4 waves along this length. The amplitude (average maximum inflated diameter) of the wave is of the order of 5mm, thus the amplitude ratio, is approximately unity. The wavelength ratio is approximately 0.04. The speed of the wave is between 1-6 cm/s whereby the frequency of contraction varies from one individual to the next and is about 1-8 per min(4). Because of the equivalence of inertial to viscous forces the Reynolds number is approximately equal to unity.

It is known that physiological flows are not only maintained by pressure gradient but are supported by the motion of the boundaries. Many authors(7,9-12) including the pioneering work by C.S. Yih &Y.C. Fung(5) and F. Yin & Y.C. Fung(6) where they have considered peristalsis applied to a two dimensional channel and asymmetric geometry respectively; whereby a perturbation method of solution in terms of wave amplitude to the tube radius is considered taking into account non linear convective terms. However, these papers have considered the case of a Newtonian fluid and it is not until the paper by K.K. Raju & R. Devanathan(3) and others(8,9,14) that a non-Newtonian fluid is considered.

Therefore, following this analysis we shall obtain the solution for the stream function as a power series in terms of the amplitude of deformation. We will show the close proximity between numerical and analytic forms for the first order perturbation in stream function and consider the effect of flow behaviour index on streamlines

## Statement of Problem

Consider the peristaltic motion of a non-Newtonian fluid, modelled as a power law fluid, which is viscous and incompressible in an axisymmetric cylindrical tube,where, d, is the undeformed radius of the tube and the tube is considered to be infinitely long. A, represents the amplitude of the sinusoidal waves travelling along the walls of the tube, $\lambda$, is the wavelength and they are travelling at speed, c,( as shown in Fig. 2).

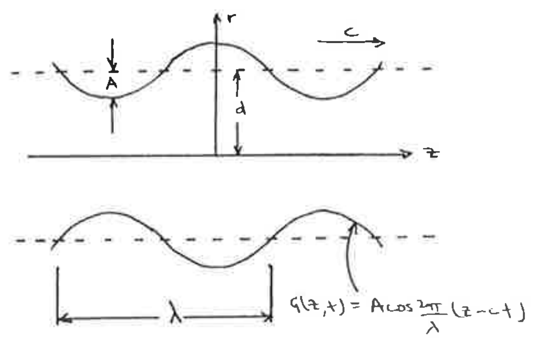


**Figure 2**, *Geometry of Peristaltic Flow*

The geometry of the sinusoidal travelling waves is given by G(z,t) where

$$G(z,t) = A\cos\frac{2\pi}{\lambda}\left(z - ct\right) \quad (1.1)$$

The non-Newtonian power law fluid is characterised by the constitutive equation(1,2)

$$\sigma_{ij} = -p\delta_{ij} + m\theta V_{ij} \qquad (1.2)$$

where, $\sigma_{ij}$, and $V_{ij}$, are the stress and the deformation tensors respectively, p denotes the isotropic pressure and, m and n are respectively flow consistency index and the flow behaviour index.

In equation (1.2), $\theta$, is given by

$$\theta = \left| V_{ij} V_{ij} \right|^{\frac{n-1}{2}} \qquad (1.3)$$

Because the fluid being modelled is assumed to be incompressible the following equation of motion may be used,

$$\rho \frac{Dq_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} \qquad (1.4)$$

Here D/Dt is the material derivative of a particle following the fluid and is given as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{q}.\underline{\nabla}$$

where, $\rho$, is the density of the fluid and, $q_i$, is the velocity component in the respective direction.
Substitution of (1.2) in (1.4) finally gives

$$\rho \frac{Dq_i}{Dt} = -\frac{\partial p}{\partial x_i} + m \frac{\partial \theta}{\partial x_j} V_{ij} + m\theta \left\{ \nabla^2 u_i + \frac{\partial}{\partial x_i} \left( \nabla.\underline{q} \right) \right\}$$

which after using the incompressibility condition $\nabla.\underline{q} = 0$ yields

$$\rho \frac{Dq_i}{Dt} = -\frac{\partial p}{\partial x_i} + m \left\{ \theta \nabla^2 u_i + V_{ij} \frac{\partial \theta}{\partial x_j} \right\} \qquad (1.5)$$

By setting i=1 and then i=2 and using summation convention with dimensionless variables and parameters (shown in the Appendix C) the following equations of motion are obtained from (1.4) and (1.5)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial r} + \frac{1}{\mathrm{Re}}\left\{ \theta\left( \nabla^2 u - \frac{u}{r^2} \right) + 2\frac{\partial u}{\partial r}\frac{\partial \theta}{\partial r} + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)\frac{\partial \theta}{\partial z} \right\} \qquad (1.6)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\left\{ \theta \nabla^2 w + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)\frac{\partial \theta}{\partial r} + 2\frac{\partial w}{\partial z}\frac{\partial \theta}{\partial r} \right\} \qquad (1.7)$$

where in equation (1.6) and (1.7), $\theta$, is given by

$$\theta = \left| 4\left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right)^2 \right|^{\frac{n-1}{2}} \qquad (1.8)$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \qquad (1.9)$$

The equation of continuity in cylindrical coordinates is given by

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0 \qquad (1.10)$$

There exists a stream function satisfying (1.10) such that

$$u = -\frac{1}{r}\frac{\partial \Psi}{\partial z}$$

$$w = \frac{1}{r}\frac{\partial \Psi}{\partial r} \qquad (1.11)$$

where u = radial velocity,
w = axial velocity

## Boundary Conditions

Assuming that there is no horizontal displacement and using (1.11) we obtain the non-dimensional boundary conditions,

$$\Psi_r = 0 \quad at \quad r = d + G$$

$$\Psi_z = r\alpha\varepsilon \sin\alpha(z-t) \qquad (2.1)$$

$$where \quad G = \varepsilon\cos\alpha(z-t)$$

and

$$\frac{\partial p}{\partial z} = \left(\frac{\partial p}{\partial z}\right)_0 + \varepsilon\left(\frac{\partial p}{\partial z}\right)_1 + \varepsilon^2\left(\frac{\partial p}{\partial z}\right)_2 + \mathrm{K}$$

(2.2)

(For explicit details see Appendix B)

## Method of Solution

The solution for, $\Psi$, the stream function, is sought via the expansions given as a power series, in the form,

$$\Psi = \Psi_0 + \varepsilon\,\Psi_1 + \varepsilon^2\Psi_2 + \mathrm{K} \qquad (3.1a)$$

$$p = p_0 + \varepsilon\,p_1 + \varepsilon^2 p_2 + \mathrm{K} \qquad (3.2a)$$

Hence, if we consider equations (3.1a) and (3.2a) and make the substitution given in (1.11) and collect like terms associated with powers of $\varepsilon$ on either side of the equation (1.7), this yields to the differential equation for the zeroth order term $\Psi_0(r)$, given by

$$K_0 = \frac{n}{r}\frac{d^3\Psi_0}{dr^3}\left\{\frac{1}{r}\frac{d^2\Psi_0}{dr^2} - \frac{1}{r^2}\frac{d\Psi_0}{dr}\right\}^{n-1} - \frac{(2n-1)}{r}\left\{\frac{1}{r}\frac{d^2\Psi_0}{dr^2} - \frac{1}{r^2}\frac{d\Psi_0}{dr}\right\}^{n}$$

(3.1)

We now assume that $\Psi_0(r)$ is a function of the radial direction only, due to the constant zeroth order axial pressure gradient, and thus the solution for equation (3.1) is obtained as given in Appendix B (3.1B-3.3aB),

Hence the solution for $\Psi_0(r)$ is

$$\Psi_0(r) = \left(\frac{k}{2}\right)^{\frac{1}{n}} \frac{n}{n+1}\left\{\frac{nr^{3+\frac{1}{n}}}{3n+1} - \frac{r^2}{2}\right\} \qquad (3.2)$$

For n = 1, (3.2) reduces to the case of a Newtonian flow for axisymmetric peristaltic flow, (6).

We shall now consider the 1st order perturbation in $\Psi$. We collect coefficients of order $\varepsilon$ in equations (1.6) and (1.7) after substitution of (1.11) and to form one equation from these we differentiate (1.6) w.r.t. z and (1.7) w.r.t. r and adding we obtain,

$$-\Psi_{0r}\left\{\frac{1}{r^2}\Psi_{1zzz} + \frac{1}{r^2}\Psi_{1zrr} + \frac{1}{r^3}\Psi_{1zr}\right\}$$

$$-\Psi_{0r}\left\{\frac{3}{r^4}\Psi_{1z}\right\}$$

$$-\frac{3}{r^3}\Psi_{0rr}\Psi_{1z} + \frac{1}{r^2}\Psi_{0rrr}\Psi_{1z}$$

$$-\frac{1}{r^2}\Psi_{1rt} + \frac{1}{r}\Psi_{1rrt} + \frac{1}{r}\Psi_{1zzt} =$$

$$\frac{1}{\mathrm{Re}}\left(\frac{k}{2}\right)_r^{\frac{n-1}{n}}\left\{\frac{n}{r}\Psi_{1rrrr} - \right.$$

$$\frac{2}{nr^2}\left(n^2 - n + 1\right)\Psi_{1rrr}$$

$$+\frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2r^3}\Psi_{1rr} +$$

$$\frac{\left(4n^2 - 4n^3 - n - 2\right)}{n^2r^4}\Psi_{1r}$$

$$+\frac{\left(2n^2 - 2n - 2\right)}{nr^2}\Psi_{1rzz} + \frac{\left(3n - n^2 - 2\right)}{n^2r^3}\Psi_{1zz} +$$

$$\left.\frac{(4-2n)}{r}\Psi_{1zzrr} + \frac{n}{r}\Psi_{1zzzz}\right\}$$

(3.3)

(For extensive derivation see Appendix B)

By letting n = 1, equation (3.3) reduces to the governing equation for $\Psi$ in the case of a Newtonian fluid undergoing peristaltic motion in an axisymmetric tube (6). Remembering that $\Psi_0(r)$ is a function of r only, the equation becomes

$$\nabla^2\Psi_{1r} - \frac{2}{r}\nabla^2\Psi_1 + \frac{1}{r^2}\Psi_{1r} - \frac{1}{r}\Psi_{0r}\nabla^2\Psi_{1z}$$

$$= \frac{1}{Re}\nabla_1^4\Psi_1$$

(3.4)

where, $\nabla_1^4 = \nabla_1^2\nabla_1^2$

and $$\nabla_1^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \qquad (3.5)$$

Taking the boundary conditions are as follows

$$\Psi_{0r}(1) = 0$$

$$\Psi_{1r}(1) + \Psi_{0rr}(1)\cos\alpha(z-t) = 0$$

$$\Psi_{0z}(1) = 0$$

$$\Psi_{1z}(1) + \Psi_{0zr}(1)\cos\alpha(z-t)$$

$$= \alpha\sin\alpha(z-t)$$

(3.6)

and substitution for $\Psi_0(r)$ as given in (3.2) yields the boundary conditions

$$\Psi_{1r}(1) = -\left(\frac{k}{2}\right)^{\frac{1}{n}}\cos\alpha(z-t)$$

$$\Psi_{1z}(1) = \alpha\sin\alpha(z-t)$$

(3.7)

From these boundary conditions, $\Psi_1$ can be assumed to be of the form,

$$\Psi_1(r,z,t) = F(r)\cos\alpha(z-t) + G(r)\sin\alpha(z-t)$$

(3.8)

where *F(r)* and *G(r)* are to be determined.

In the Appendix B, lengthy differential equations were derived in order to solve the first order term for the stream function (see Appendix B). Once *F(r)* and *G(r)* have been solved $\Psi_1$ is immediately found from ( 3.8).

It is not possible to find closed form solutions to the differential equations 3.6B and 3.7B, so approximate solutions will be sought. Firstly the equations for *F(r)* and *G(r)* can be simplified by assuming that the Reynolds number associated with the present model is small, and consequently the forms of *F(r)* and *G(r)* are assumed to be given as

$$F(r) = F_0(r) + \mathrm{Re}^2 F_2(r) + \mathrm{K}$$
$$G(r) = \mathrm{Re}\,G_1(r) + \mathrm{Re}^3 G_3(r) + \mathrm{K} \qquad (3.9)$$

Substituting these forms for *F(r)* and *G(r )* into (3.6B) and (3.7B), and collecting terms of equal order in Re on either side of the equations yield the following differential equations for $F_0(r)$ and $G_1(r)$.

If we assume $\alpha$ is very small such that higher order terms in $\alpha$ may be neglected, equations (3.8B) and (3.9B) may be simplified as

$$nr^4F_0^{IV} - \frac{2}{n}\left(n^2 - n + 1\right)r^3F_0''' +$$

$$\left\{\frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}r^2\right\}F_0''$$

$$-\left\{\frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}r\right\}F_0' = 0$$

(3.10)

and

$$nr^4 G_1^{IV} - \frac{2}{n}\left(n^2 - n + 1\right)r^3 G_1''' +$$

$$\left\{\left(\frac{4n^3 - 4n^2 + n + 2}{n^2}\right)r^2\right\}G_1''$$

$$-\left\{\left(\frac{4n^3 - 4n^2 + n + 2}{n^2}\right)r\right\}G_1' =$$

$$-\left(\frac{k}{2}\right)^{\frac{2-n}{n}} \alpha\left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}} - r^{\frac{1+3n}{n}}\right)F_0''\right\} +$$

$$\left(\frac{k}{2}\right)^{\frac{2-n}{n}} \alpha\left\{\frac{n}{n+1}\left(r^{\frac{1+2n}{n}} - r^{\frac{3+2n}{n}}\right)F_0'\right\} +$$

$$\left(\frac{k}{2}\right)^{\frac{2-n}{n}} \alpha\left\{\frac{n-1}{n} r^{\frac{2+2n}{n}} F_0\right\} -$$

$$\left(\frac{k}{2}\right)^{\frac{1-n}{n}} \alpha\left\{r^{\frac{1+3n}{n}} F_0'' - r^{\frac{1+2n}{n}} F_0'\right\}$$

(3.11)

The solution to (3.10) and the homogeneous solution to (3.11) can be sought as the form

$$\hat{F}(r) = Ar^{m_1} + Br^{m_2} + Cr^{m_3} + Dr^{m_4}$$

(see Appendix A1) (3.12)

It follows from (3.12) that the general solution to equation (3.10) is given by

$$\hat{F}_0(r) = A + Br^2 + Cr^{3+\frac{1}{n}} + Dr^{\frac{3n^2 - 3n + 2}{n^2}}$$

(3.13)

In Raju and Devanathan(3) they have not specified the values of the parameter $\alpha$ they have used to obtain numerical solutions to $F_0(r)$ and $G_1(r)$, so the only meaningful comparison that can be made is between approximate values obtained for $F_0(r)$ in the present analysis and the corresponding exact solution given in (3). In spite of this, the approximating function for $G_1(r)$ will be specified for the special case of n = 1.

This means that the differential equations for approximating $F_0(r)$ and $G_1(r)$ are now given as (3.10) and (3.11).

The corresponding boundary conditions are given by

$$F_0'(1) = -\left(\frac{k_0}{2}\right)^{\frac{1}{n}} \qquad F_0(1) = 1$$

$$G_1(1) = 0$$

$$G_1'(1) = 0 \qquad F_0(0) = 0$$

$$G_1(0) = 0 \qquad F_0'(0) = 0$$

$$G_1'(0) = 0$$

(3.14)

For the Newtonian case n = 1, (3.13) (A1) gives the general solution to (3.10) as

$$\hat{F}_0(r) = A + Br^2 + Cr^4 + Dr^2 \ln r$$

(3.15)

(See Appendix A7 - A9)

From Appendix A, the approximate solution for the function given by $\hat{G}_1(r)$ for the case, n = k = 1, is given by

$$\hat{G}_1(r) = \frac{275\alpha r^2}{2304} - \frac{95\alpha r^4}{768} + \frac{25\alpha r^6}{384} - \frac{5\alpha r^8}{2304}$$

(3.16)

**Discussion**

If we define an error term as

$$E_{F_0(r)} = \left|\hat{F}_0(r) - F_0(r)\right| \qquad (4.1)$$

a measure of whether the approximate function $\hat{F}(r)$ gives values which are a good approximation to $F_0(r)$, the exact solution

( note, these values taken from Raju & Devanathan (3) ). From Tables 3.1-3.4, where we are considering various values of flow behaviour index (n) and a function of pressure gradient (k) it is evident that the values are very good. This is graphically depicted in figures 3.1-3.4

| r | $\hat{F}_0(r)$ | $F_0(r)$ | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.022 | 0.022 | 0.000 |
| 0.2 | 0.088 | 0.085 | 0.003 |
| 0.3 | 0.192 | 0.186 | 0.006 |
| 0.4 | 0.328 | 0.318 | 0.010 |
| 0.5 | 0.484 | 0.471 | 0.013 |
| 0.6 | 0.648 | 0.634 | 0.014 |
| 0.7 | 0.802 | 0.790 | 0.012 |
| 0.8 | 0.928 | 0.920 | 0.008 |
| 0.9 | 1.002 | 0.999 | 0.003 |
| 1.0 | 1.000 | 1.000 | 0.000 |

**Table 3.1**, *n=k=1, where n is flow behaviour index, showing error term*

For the non-Newtonian case n=k=1, the approximating function $\hat{F}_0(r)$ is given by

$$\hat{F}_0(r) = \frac{1}{4}\left(5r^4 - 9r^2\right) \qquad (4.2)$$

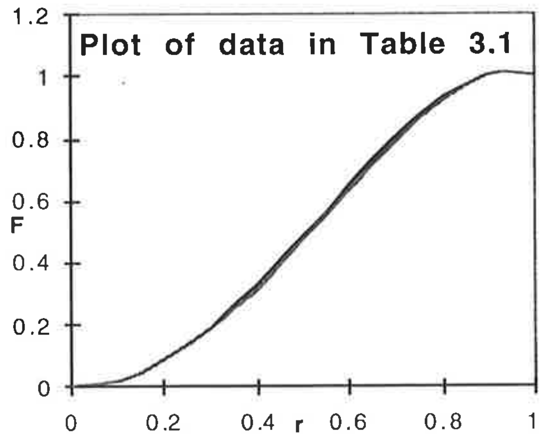


**Figure 3.1,** *Plot of data in Table 3.1, showing comparison between the approximate solution $\hat{F}_0(r)$ to the numerical solution $F_0(r)$*

For the non-Newtonian case n = 0.8, k = 1, the approximating function for $\hat{F}_0(r)$ found by combining, A15-A18, is given by

$$\hat{F}_0(r) = 1.490906r^{4.25} - 2.490906r^{2.375} \qquad (4.3)$$

A comparison between values obtained from (4.3) and exact values as recorded in (3) is given in Table 3.2

| r | $\hat{F}_0(r)$ | $F_0(r)$ | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.010 | 0.019 | 0.009 |
| 0.2 | 0.053 | 0.076 | 0.023 |
| 0.3 | 0.134 | 0.168 | 0.034 |
| 0.4 | 0.252 | 0.292 | 0.040 |
| 0.5 | 0.402 | 0.439 | 0.037 |
| 0.6 | 0.570 | 0.600 | 0.030 |
| 0.7 | 0.740 | 0.759 | 0.019 |
| 0.8 | 0.889 | 0.897 | 0.008 |
| 0.9 | 0.987 | 0.989 | 0.002 |
| 1.0 | 1.000 | 1.000 | 0.000 |

**Table3.2**, *n=0.8,k=1, where n is flow behaviour index, showing error term*

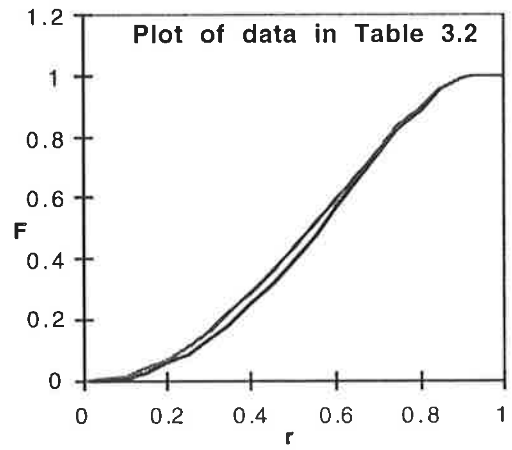


**Figure 3.2**, *Plot of data in Table 3.2, showing comparison between the approximate solution $\hat{F}_0(r)$ to the numerical solution $F_0(r)$*

Similarly, for the non-Newtonian case n = 1.2, k = 1, the approximating function for $\hat{F}_0(r)$ is given by

$$\hat{F}_0(r) = 1.391035 r^{\frac{23}{6}} - 2.397035 r^2 \quad (4.4)$$

A comparison between values obtained from (4.4) and exact values as recorded in (3) is given in Table 3.3

| r | $\hat{F}_0(r)$ | $F_0(r)$ | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.024 | 0.024 | 0.000 |
| 0.2 | 0.093 | 0.092 | 0.001 |
| 0.3 | 0.202 | 0.199 | 0.003 |
| 0.4 | 0.342 | 0.337 | 0.005 |
| 0.5 | 0.501 | 0.494 | 0.007 |
| 0.6 | 0.666 | 0.658 | 0.008 |
| 0.7 | 0.819 | 0.811 | 0.008 |
| 0.8 | 0.940 | 0.935 | 0.005 |
| 0.9 | 1.009 | 1.007 | 0.002 |
| 1.0 | 1.000 | 1.000 | 0.000 |

**Table 3.3**, *n=1.2, k=1,where n is flow behaviour index, showing error term*

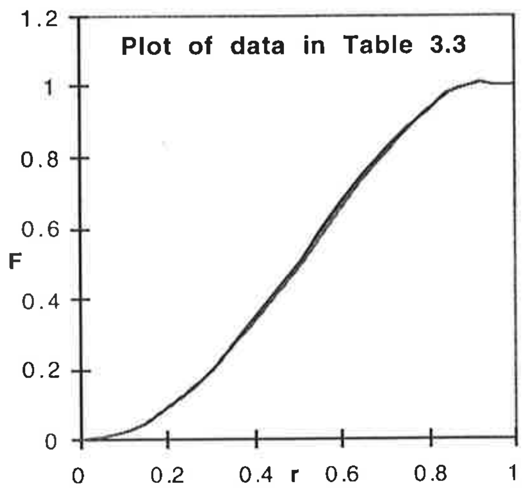


**Figure 3.3,** *Plot of data in Table 3.3, showing comparison between the approximate solution* $\hat{F}_0(r)$ *to the numerical solution* $F_0(r)$

Also, for the non-Newtonian case n =1.4, k = 1, the approximating function for $\hat{F}_0(r)$ is given by

$$\hat{F}_0(r) = 1.522212 r^{\frac{26}{7}} - 2.52212 r^2 \quad (4.5)$$

A comparison between values obtained from (4.5) and exact values as recorded in (3) is given in Table 3.4.

| r | $\hat{F}_0(r)$ | $F_0(r)$ | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.025 | 0.025 | 0.000 |
| 0.2 | 0.097 | 0.098 | 0.001 |
| 0.3 | 0.210 | 0.209 | 0.001 |
| 0.4 | 0.353 | 0.351 | 0.002 |
| 0.5 | 0.515 | 0.512 | 0.003 |
| 0.6 | 0.680 | 0.676 | 0.004 |
| 0.7 | 0.831 | 0.828 | 0.003 |
| 0.8 | 0.950 | 0.947 | 0.003 |
| 0.9 | 1.014 | 1.013 | 0.001 |
| 1.0 | 1.000 | 1.000 | 0.000 |

**Table 3.4**, *n=1.4, k=1, where n is flow behaviour index, showing error term*

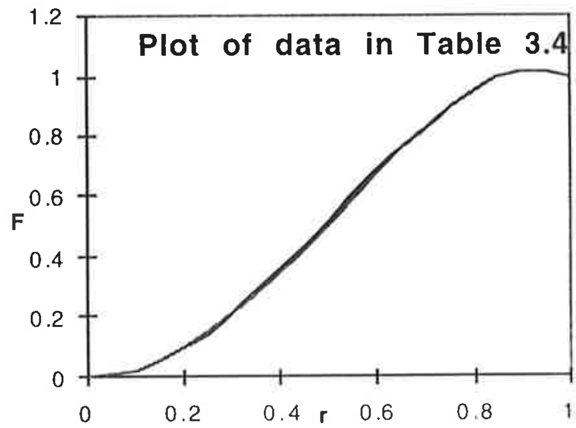


**Figure 3.4**, *Plot of data in Table 3.4, showing comparison between the approximate solution* $\hat{F}_0(r)$ *to the numerical solution* $F_0(r)$

Also, for the non-Newtonian case n = 0.5, k= 1, the approximating function for $\hat{F}_0(r)$ is given by

$$\hat{F}_0(r) = 0.25\left(3r^5 - 7r^2\right) \qquad (4.6)$$

A comparison between values obtained from (4.6) and exact values as recorded in (3) is given in Table 3.5, where error term is approximate.

| r | $\hat{F}_0(r)$ n=0.5 | $F_0(r)$ n=0.6 | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.017 | 0.015 | 0.002 |
| 0.2 | 0.070 | 0.062 | 0.018 |
| 0.3 | 0.156 | 0.140 | 0.016 |
| 0.4 | 0.272 | 0.249 | 0.023 |
| 0.5 | 0.414 | 0.385 | 0.029 |
| 0.6 | 0.572 | 0.542 | 0.030 |
| 0.7 | 0.731 | 0.707 | 0.024 |
| 0.8 | 0.874 | 0.860 | 0.014 |
| 0.9 | 0.975 | 0.972 | 0.003 |
| 1.0 | 1.000 | 1.000 | 0.000 |

**Table 3.5,** *n=0.5, n=0.6, k=1, where n is flow behaviour index, showing error term*

It is worth mentioning again that a graphical plot of the closeness of solutions for $F_0(r)$ is graphically depicted in Figs. 3.1 - 3.4 for behavior index (n)= $n_i$ and k=1

From these tables and Figs. it can be seen that the approximate solutions are very accurate for n = 1.0, 1.2 & 1.4 and less accurate for n = 0.8 & 0.5, but still good enough to be used to find approximate solutions for the stream function and pressure gradient for those values of n.

From (3.1a) it is clear that for $\varepsilon$ very small the stream function, can be approximated as

$$\Psi = \Psi_0 + \varepsilon\Psi_1 \qquad (4.7)$$

Thus, defining, $\Psi^N$ to be $\Psi$ in the case n = N and defining $\hat{F}^N(r)$ as the approximate solution for *F(r)* in the case n = N, (4.7) gives the approximation to the stream function, along with (3.2) and (3.8) as

$$\Psi^N \approx \left(\frac{k}{2}\right)^{\frac{1}{N}} \frac{N}{N+1}\left(\frac{r^2}{2} - \frac{N r^{\frac{3N+1}{N}}}{3N+1}\right) +$$

$$\varepsilon\left[\hat{F}^N(r)\cos\alpha(z-t) + \hat{G}^N(r)\sin\alpha(z-t)\right]$$

(4.8)

where $\hat{G}^N(r)$, is the approximate function to *G(r)* in the case n = N.

From (3.9) it can be seen that

$$\varepsilon\left[\hat{F}^N(r)\cos\alpha(z-t) + \hat{G}^N(r)\sin\alpha(z-t)\right]$$

$$=$$

$$\cos\alpha(z-t)\left[\varepsilon F_0^N(r) + G\,\mathrm{Re}^2\right] +$$

$$\sin\alpha(z-t)\left[\varepsilon\,\mathrm{Re}\,G^N(r) + O\left(\varepsilon\,\mathrm{Re}^3\right)\right]$$

(4.9)

under the assumption, $\varepsilon$, & Re small and defining

$$\alpha(z-t) = \bar{z} \qquad (4.10)$$

(4.8) , (4.9)&(4.10) give an approximation to the stream function as,

$$\Psi^N \approx \left(\frac{k}{2}\right)^{\frac{1}{N}} \frac{N}{N+1}\left(\frac{r^2}{2} - \frac{N r^{\frac{3N+1}{N}}}{3N+1}\right) +$$

$$\varepsilon\left[\hat{F}_0^N(r)\cos(\bar{z})\right]$$

(4.11)

Streamlines for , n = N = 0.5, 1.0, 1.2 are given in Figs. 3.5, 3.6 and 3.7 respectively, where k = 1.0 and $\varepsilon$ =0.01.

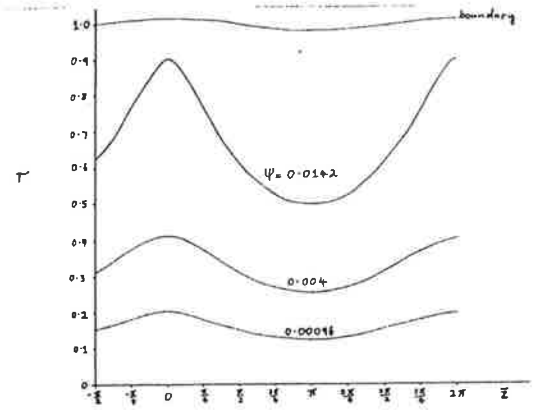


**Figure 3.5**,*Streamline* $\Psi$ *for n=0.5, k=1,where n is flow behaviour index*

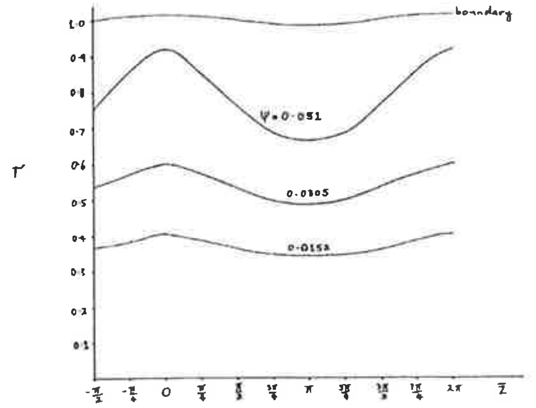


**Figure 3.6**,*Streamline* $\Psi$ *for n=1, k=1,where n is flow behaviour index*

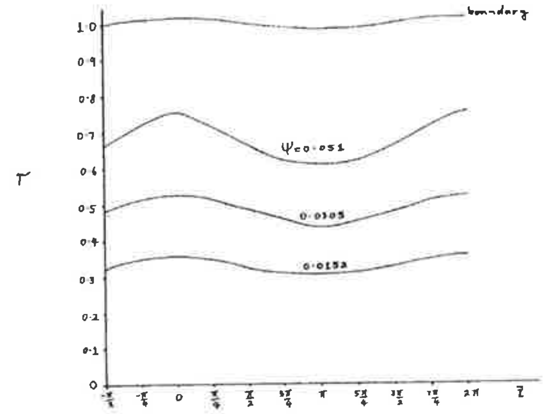


**Figure 3.7**,*Streamline* $\Psi$ *for n=1.2, k=1, where n is flow behaviour index*

The approximating functions for $F_0(r)$ in the case k=0.01 and n=0.5, 0.8, 1.0 and 1.2 are found in the same manner as the approximating function $F_0(r)$k=1, and are given by

$$\hat{F}_0^{0.5}(r) = 0.666675r^5 - 1.666675r^2 \tag{4.12}$$

$$\hat{F}_0^{0.8}(r) = 1.267376r^{4.25} - 2.267376r^{2.375} \tag{4.13}$$

$$\hat{F}_0^{1.0}(r) = 1.0025r^4 - 2.0025r^2 \tag{4.14}$$

$$\hat{F}_0^{1.2}(r) = 1.097504r^{\frac{23}{6}} - 2.097504r^2 \tag{4.15}$$

Streamlines for n = 0.5, 0.8, 1.0 & 1.2 in the case k = 0.01 are plotted in Figs. 3.8 - 3.11 respectively where $\varepsilon$ =0.01; and are derived from (4.11) using (4.12) - (4.15).

It can be seen that the streamlines plotted in (Fig 3.5-3.7 )( high pressure gradient ) and Fig. (3.8 - 3.11) (low pressure gradient ) are similar to those established in Raju and Devanathan(3).

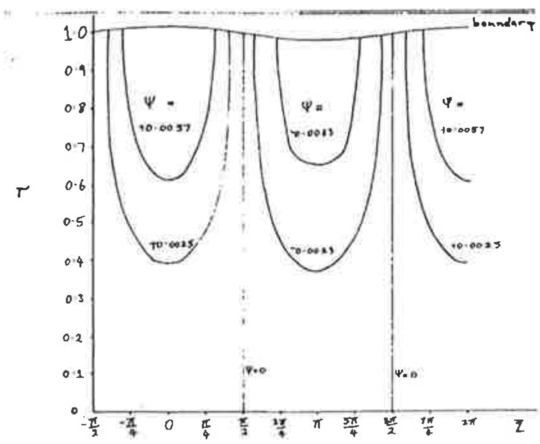


**Figure 3.8**,*Streamline* $\Psi$ *for n=0.5, k=0.01, where n is flow behaviour index*

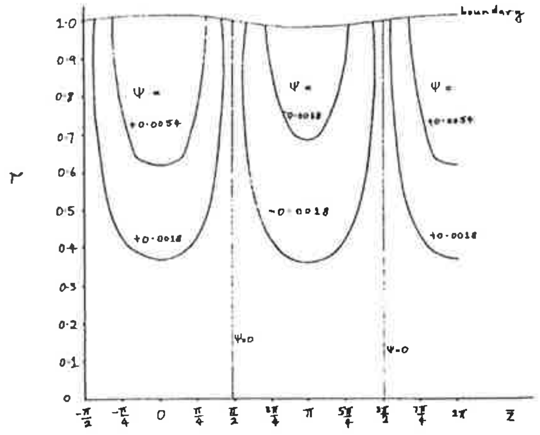


**Figure 3.9,** *Streamline* $\Psi$ *for n=0.8, k=0.01, where n is flow behaviour index*

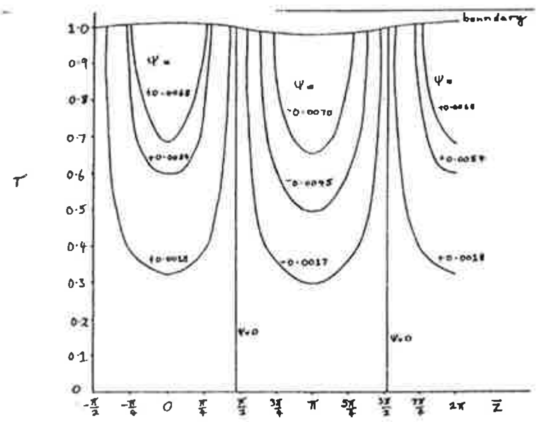


**Figure 3.10,** *Streamline* $\Psi$ *for n=1.0, k=0.01, where n is flow behaviour index*

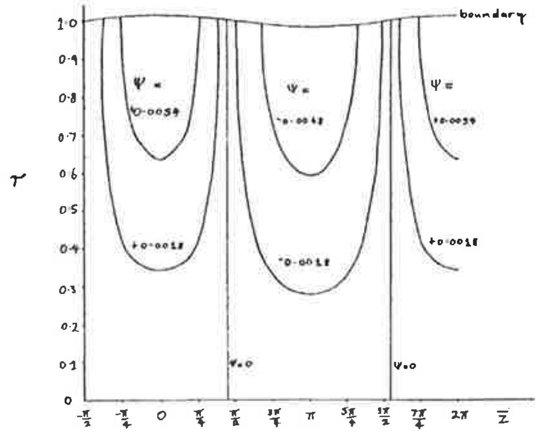


**Figure 3.11,** *Streamline* $\Psi$ *for n=1.2, k=0.01, where n is flow behaviour index*

The difference between streamlines for high and low pressure is apparent due to the boundary of the tube has a more significant impact with regard to low pressure gradient.

All approximate results have been obtained by assuming that the terms of the order $\alpha^2$ and higher are negligible and the parameters and Re are small.

More accurate results could be obtained by neglecting terms $\alpha^3$ and higher but keeping order $\alpha^2$ and $\alpha$ as well as $\alpha^0$.

**Conclusion**

The results obtained in this study are consistent with those established by Raju and Devanathan(3) in their study of peristaltic motion of a non-Newtonian fluid.

It was seen that the equation of motion for the non-Newtonian (power law) model of peristaltic motion reduced to the governing equations for the Newtonian model in the case of flow behaviour index, n = 1.

Approximate results for the stream function $\Psi$ were obtained and is seen for n >= 1 are close to the exact results given in the literature. For the case n <= 1, the results were less accurate, but nevertheless to an acceptable level, if we consider the magnitude of the errors.

The streamlines obtained from the approximate results were found to be very similar to those found in Raju and Devanathan(3), thus demonstrating the effect of pressure gradient and boundary structure. In fact, it is found that for low pressure gradient the streamlines form closed loops and for higher presure gradient the streamlines run parallel to the axis of the tube when considered near the axis , whereas a considerable deformation is noted near the boundary. An explanation for this is that the region may be considered to be consisting of two regions; one the central core region and the other near the boundary, the boundary layer region. Highlighting the validity of current modelling of this type. In the case of low pressure gradient, the effects of the wave travelling along the boundary of the tube are more dominant. But as the pressure gradient increases, we find that the streamlines in the central part of the region are more influenced by it, than the motion of the boundary, hence run approximately parallel to the axis. One conclusion reached by these studies is that peristalsis is an effective method to move fluid only if the fluid is transported in the form of a series of isolated boluses. If the amplitude of displacement of the wall is small compared to the tube radius, very little pressure gradient can be generated by the travelling wave. Pressure gradient increases significantly when the radius of minimum section approaches zero, (complete occlusion). It is thus understandable why peristalsis is a common phenomena in the lymphatics, intestine, ureter,and many other biological systems and peristaltic pumps (dialysis machines and heart-lung machines).

## Appendix A

Solution to (3.10) and (3.11) may now be sought as

$$\hat{F}(r) = Ar^{m_1} + Br^{m_2} + Cr^{m_3} + Dr^{m_4} \qquad \text{(A1)}$$

where $m_{ii}$, i = 1-4 are to be determined by making the substitution

$$\hat{F}(r) = r^m \qquad \text{(A2)}$$

On substitution (A1) into (3.10) yields

$$r^m \Big\{ nm(m-1)(m-2)(m-3) - \frac{2}{n}\left(n^2 - n + 1\right)m(m-1)(m-2) + \frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}(m-1)m - \frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}m \Big\} = 0 \qquad \text{(A3)}$$

This finally reduces to,

$$m(m-2)\left[nm^2 - \left(6n - 2 + \frac{2}{n}\right)m\right] + m(m-2)\left[9n - 6 + \frac{3}{n} + \frac{2}{n^2}\right] = 0 \qquad \text{(A4)}$$

Where

$$m_1 = 0$$

$$m_2 = 2$$

The other two roots are evaluated by determining the determinate of (A4) to see if all the roots are real, hence

$$\Delta = \frac{4}{n^2}(2n-1)^2 \qquad \text{(A5)}$$

$$\Delta \geq 0$$

$$so, \quad m_3 = 3 + \frac{1}{n}, \quad m_4 = 3 - \frac{3}{n} + \frac{2}{n^2}$$

our general solution to (3.10) is

$$\hat{F}_0(r) = A + Br^2 + Cr^{3+\frac{1}{n}} + Dr^{\frac{3n^2-3n+2}{n^2}} \qquad \text{(A6)}$$

There is now sufficient information to find $F_0(r)$ and $G_1(r)$

For the Newtonian case n = 1 (3.13) gives the general solution to equation (3.10) as

$$\hat{F}_0(r) = A + Br^2 + Cr^4 + Dr^2 \ln r \qquad \text{(A7)}$$

where lnr is introduced because $m_2 = m_4 = 2$

From (A8)

$$\hat{F}_0''(r) = 2B + 3D + 2D\ln r \qquad \text{(A8)}$$

keeping this second derivative bounded at r= 0 means,

$$D = 0$$

The boundary conditions (3.27) imply

$$A = 0$$

$$B + C = -1$$

$$2B + 4C = \frac{k}{2} \qquad \text{(A9)}$$

For, n = k = 1 the homogeneous solution to (3.11) again keeping the second derivative bounded at r=0 is

$$G'_n(r) = A_1 + B_1 r^2 + C_1 r^4 \qquad (A10)$$

A particular solution to (3.26) can be sought of the form,

$$\hat{G}_p(r) = D_1 r^6 + E_1 r^8 \qquad (A11)$$

Therefore substitution of (A13) into (3.11) with n = 1 yields

$$192Dr^6 + 1152Er^8 = \frac{5\alpha}{2}\left[5r^6 - r^8\right]$$

Hence $D = \frac{25\alpha}{384}$ $E = \frac{-5\alpha}{2304}$ (A12)

so from (A12) and (A13) the solution to (3.11) is

$$\hat{G}_1(r) = A_1 + B_1 r^2 + C_1 r^4 + \frac{25\alpha}{384} r^6 - \frac{5\alpha}{2304} r^8 \qquad (A13)$$

The boundary conditions imply from (3.14)

$$A_1 = 0$$

$$B_1 + C_1 = \frac{5\alpha}{2304} - \frac{25\alpha}{384}$$

$$2B_1 + 4C_1 = \frac{5\alpha}{288} - \frac{25\alpha}{64}$$

$$\therefore 2C_1 = -\frac{95\alpha}{384}$$

$$\therefore B_1 = \frac{275\alpha}{2304} \qquad (A14)$$

For the non-Newtonian case n = 0.8, k = 1

$$\hat{F}(r) = A + Br^2 + Cr^{4.25} + Dr^{2.375} \qquad (A15)$$

therefore using the boundary conditions (3.14)

A = 0

B + C + D = 1

2B + 4.25C + 2.375D = 0.420448 (A16)

Hence another boundary condition is required, therefore using the condition of symmetry, ie, $\Psi_{1rr} = 0$, at r = 0

C = 1.490906, D = -2.490906
B=0 (A17)
therefore,

$$\hat{F}_0(r) = 1.490906r^{4.25} - 2.490906r^{2.375} \qquad (A18)$$

## Appendix B

w = 0 at r = d + G(z,t)

$$u = \frac{\partial}{\partial t}G(z,t) = Ac\frac{2\pi}{\lambda}\sin\frac{2\pi}{\lambda}(z - ct)$$
at r = d + G(z,t)

therefore using (1.11)

$$-\frac{1}{r}\frac{\partial \Psi}{\partial z} = \frac{2\pi}{\lambda}Ac\sin\frac{2\pi}{\lambda}(z - ct)$$
at r = d +G(z,t)

and $\frac{1}{r}\frac{\partial \Psi}{\partial r} = 0$ at r = d + G(z,t)

(2.1B)

Because we are dealing with an infinite tube the end conditions are not specified; instead the pressure gradient in the longitudinal (z) direction is specified and is assumed to be of the form,

$$\frac{\partial p}{\partial z} = \left(\frac{\partial p}{\partial z}\right)_0 + A\left(\frac{\partial p}{\partial z}\right)_1 + A^2\left(\frac{\partial p}{\partial z}\right)_2 + \mathrm{K}$$

(2.2B)

$$\left(\frac{\partial p}{\partial z}\right)_0 = \text{constant and} \quad \left(\frac{\partial p}{\partial z}\right)_i = F(z,r,t)$$

(2.3B)

where A is specified in Figure 2.

$$-\mathrm{Re}\left(\frac{\partial p}{\partial z}\right)_0 =$$

$$\frac{n}{r}\frac{d^3\Psi_0}{dr^3}\left\{\frac{1}{r}\frac{d^2\Psi_0}{dr^2} - \frac{1}{r^2}\frac{d\Psi_0}{dr}\right\}^{n-1}$$

$$-\frac{(2n-1)}{r}\left\{\frac{1}{r}\frac{d^2\Psi_0}{dr^2} - \frac{1}{r^2}\frac{d\Psi_0}{dr}\right\}^{n}$$

(3.1B)

if we now let

$$k = \mathrm{Re}\left(\frac{\partial p}{\partial z}\right)_0 \qquad (3.2\mathrm{B})$$

(3.3)gives us the differential equation for, $\Psi_0(r)$, as

$$\Psi_0(r) = A\left\{\frac{nr^{3+\frac{1}{n}}}{3n+1} - \frac{r^2}{2}\right\}$$

(3.3B)

which satisfies the required boundary conditions,

$$\Psi_0(0) = 0$$
$$\Psi_{0r}(1) = 0$$

On substitution of (3.3B) into (3.1) yields

$$A = \left(\frac{k}{2}\right)^{\frac{1}{n}} \frac{n}{n+1} \qquad (3.3\mathrm{aB})$$

Collection of coefficents in $\mathcal{E}$ in equation (1.6) after substitution of (1.11) we obtain

$$\frac{1}{r}\Psi_{1zt} - \frac{1}{r^2}\Psi_{0r}\Psi_{1zz} = -\frac{\partial p}{\partial r} +$$

$$\frac{1}{\mathrm{Re}}\left\{\left(\frac{1}{r}\Psi_{0rr} - \frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}} X$$

$$\left\{\frac{-n}{r}\Psi_{1zzz} + \left(\frac{n-2}{r}\right)\Psi_{1zrr}\right\} +$$

$$\frac{1}{\mathrm{Re}}\left\{\left(\frac{1}{r}\Psi_{0rr} - \frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}} X$$

$$\left\{\left(\frac{2-n^2}{nr^2}\right)\Psi_{1zr}\right\} +$$

$$\frac{1}{\mathrm{Re}}\left\{\left(\frac{1}{r}\Psi_{0rr} - \frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}} X$$

$$\left\{\left(\frac{2(n-1)}{nr^3}\right)\Psi_{1z}\right\}$$

(3.4B)

Similarly collection ofcoefficients in $\mathcal{E}$ in equation (1.7) after substitution of (1.11) we obtain,

$$\frac{1}{r}\Psi_{1rt} - \frac{1}{r^2}\Psi_{0r}\Psi_{1rz} + \frac{1}{r^2}\Psi_{0rr}\Psi_{1z} -$$

$$\frac{1}{r^3}\Psi_{0r}\Psi_{1z} =$$

$$-\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}\left\{\left(\frac{1}{r}\Psi_{0rr} - \frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}} X$$

$$\frac{n}{r}\Psi_{1rrr} + \left(\frac{2-n}{r}\right)\Psi_{1zzr} + \left(\frac{3n-2-2n^2}{nr^2}\right)\Psi_{1rr}$$

$$+\frac{(n-1)(n-2)}{nr^2}\Psi_{1zz} + \left(\frac{2n^2+2-3n}{nr^3}\right)\Psi_{1r}$$

(3.5B)

Substituting for, $\Psi_0(r)$, as given in (3.2) and, $\Psi_1$, as given in (3.8) and collecting coefficients of, $\cos\alpha(z-t)$, on either side of the resulting equation after multiplying both sides by,

$$\mathrm{Re}\left(\frac{k_0}{2}\right)^{\frac{1-n}{n}} r^{\frac{1+4n}{n}} \quad \text{yields}$$

$$nr^4F^{IV} - \frac{2}{n}\left(n^2-n+1\right)r^3F''' +$$
$$\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r^2 + (2n-4)\alpha^2 r^4\right\}F'' -$$
$$\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r + \left(\frac{2n^2-2n-2}{n}\right)\alpha^2 r^3\right\}F'$$
$$+\left\{n\alpha^4 r^4 + \left(\frac{n^3-3n+2}{n^2}\right)\alpha^2 r^2\right\}F =$$
$$-\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}} - r^{\frac{1+3n}{n}}\right)G''\right\} +$$
$$\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{1+2n}{n}} - r^{\frac{2+3n}{n}}\right)G'\right\} +$$
$$\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\left(\frac{n-1}{n}\right)r^{\frac{2+2n}{n}}G\right\} +$$
$$\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{1+3n}{n}} - r^{\frac{2+4n}{n}}\right)\alpha^2 G\right\} +$$
$$\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{-r^{\frac{1+3n}{n}}G'' + r^{\frac{1+2n}{n}}G'\right\} +$$
$$\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{\alpha^2 r^{\frac{1+3n}{n}}G\right\}$$

(3.6B)

making the substitution as before for, $\Psi_0(r)$, as given in (3.2) and, $\Psi_1$ as given in (3.8) and collecting coefficients of, $\sin\alpha(z-t)$, after multiplying both sides by ,

$$\mathrm{Re}\left(\frac{k_0}{2}\right)^{\frac{1-n}{n}} r^{\frac{1+4n}{n}} \quad \text{yields,}$$

$$nr^4G^{IV} - \frac{2}{n}\left(n^2-n+1\right)r^3G''' +$$
$$\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r^2\right\}G'' +$$
$$\left\{(2n-4)\alpha^2 r^4\right\}G''$$
$$-\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r\right\}G' +$$
$$\left\{+\left(\frac{2n^2-2n-2}{n}\right)\alpha^2 r^3\right\}G'$$
$$+\left\{n\alpha^4 r^4 + \left(\frac{n^3-3n+2}{n^2}\right)\alpha^2 r^2\right\}G =$$
$$-\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}} - r^{\frac{1+3n}{n}}\right)F'' +\right.$$
$$\frac{n}{n+1}\left(r^{\frac{1+2n}{n}} - r^{\frac{3+2n}{n}}\right)F' + \frac{n-1}{n}r^{\frac{2+2n}{n}}F$$
$$\left.+\frac{n}{n+1}\left(r^{\frac{1+3n}{n}} - r^{\frac{2+4n}{n}}\right)\alpha^2 F\right\} +$$
$$\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{r^{\frac{1+3n}{n}}F'' - r^{\frac{1+2n}{n}}F' - \alpha^2 r^{\frac{1+3n}{n}}F\right\}$$

(3.7B)

$$n r^4 F_0^{IV} - \frac{2}{n}\left(n^2 - n + 1\right) r^3 F_0''' + \left\{ \frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2} r^2 + (2n-4)\alpha^2 r^4 \right\} F_0''$$

$$- \left\{ \frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2} r + \left(\frac{2n^2 - 2n - 2}{n}\right)\alpha^2 r^3 \right\} F_0'$$

$$+ \left\{ n\alpha^4 r^4 + \left(\frac{n^3 - 3n + 2}{n^2}\right)\alpha^2 r^2 \right\} F_0 = 0 \qquad (3.8B)$$

and

$$n r^4 G_1^{IV} - \frac{2}{n}\left(n^2 - n + 1\right) r^3 G_1''' + \left\{ \left(\frac{4n^3 - 4n^2 + n + 2}{n^2}\right) r^2 + (2n-4)\alpha^2 r^4 \right\} G_1''$$

$$- \left\{ \left(\frac{4n^3 - 4n^2 + n + 2}{n^2}\right) r + \left(\frac{2n^2 - 2n - 2}{n}\right)\alpha^2 r^3 \right\} G_1'$$

$$+ \left\{ n\alpha^4 r^4 + \left(\frac{n^3 - 3n + 2}{n^2}\right)\alpha^2 r^2 \right\} G_1 =$$

$$-\left(\frac{k}{2}\right)^{\frac{2-n}{n}} \alpha \left\{ \frac{n}{n+1}\left( r^{\frac{2+4n}{n}} - r^{\frac{1+3n}{n}} \right) F_0'' + \right.$$

$$\frac{n}{n+1}\left( r^{\frac{1+2n}{n}} - r^{\frac{3+2n}{n}} \right) F_0' + \frac{n-1}{n} r^{\frac{2+2n}{n}} F_0$$

$$\left. + \frac{n}{n+1}\left( r^{\frac{1+3n}{n}} - r^{\frac{2+4n}{n}} \right) \alpha^2 F_0 \right\} +$$

$$\left(\frac{k}{2}\right)^{\frac{1-n}{n}} \alpha \left\{ r^{\frac{1+3n}{n}} F_0''' - r^{\frac{1+2n}{n}} F_0' - \alpha^2 r^{\frac{1+3n}{n}} F_0 \right\} \qquad (3.9B)$$

## Appendix C

- Dimensionless variables and parameters are defined as follows

$$r' = \frac{r}{d}, \quad z' = \frac{z}{d}, \quad w' = \frac{w}{c}, \quad u' = \frac{u}{c}$$

$$G' = \frac{G}{d}, \quad \Psi' = \frac{\Psi}{cd^2}, \quad t' = \frac{ct}{d}, \quad p' = \frac{p}{\rho c^2}$$

$$\varepsilon = \frac{A}{d}, \quad \alpha = \frac{2\pi d}{\lambda}$$

$$\mathrm{Re} = \frac{c^{2-n} d^n}{m} \rho$$

$$\theta' = \left(\frac{d^2}{c^2}\right)^{\frac{n-1}{2}} \theta$$

$$\left(\frac{\partial p}{\partial z}\right)_0' = \frac{d}{\rho c^2}\left(\frac{\partial p}{\partial z}\right)_0$$


### Acknowledgements

The authors wish to thank Matthew Russack and Bill Ellis for their assistance in the outcome of this research.

The authors are also grateful to the referees for their useful and constructive suggestions, most of which have been taken into account while revising this paper. Their valuable comments will certainly help us in future work.


### References


(1) Mazumdar, J. N. *Biofluid Mechanics* World Scientific Publishing, (1992)

(3.17)

(2) Y. C. Fung ,*Biomechanics Mechanical Properties of Living Tissues*, Springer-Verlag, New York (1981)

(3) Raju, K. K & Devanathan, R *Peristaltic motion of a non-Newtonian Fluid.* Rheol. Acta 11, 170-178 (1972)

(4) Liron, N *A new look at peristalsis and its functions.* Horizons in Biochemistry and Biophysics V5, pp161-182. (1978)

(5) Fung, Y. C & Yih, C. S
*Peristaltic Transport.*
J. Applied Mechanics, Trans. of the ASME
V35, pp669-675 (1968)

(6) Yin, F & Fung, Y. C
*Peristaltic waves in circular cylindrical tubes.*
J. Applied Mechanics, Trans. of the ASME
V36,pp579-587 (1969)

(7) A.H.Shapiro, M.Y.Jaffrin and
S.L.Weinberg
*Peristaltic pumping with long wavelengths at Low Reynolds number.*
J.Fluid Mech.V37,pp799-825 (1969)

(8) L.M.Srivastava and V.P.Srivastava
*Peristaltic transport of a non-Newtonian fluid: Applications to the vas deferens and small intestine.*
Annals of Biomedical Eng.V13,pp137-153 (1985)

(9) R.Devanathan and S.Parvathamma
*Long wavelength peristaltic transport of non-Newtonian fluids*
Indian J.Pure Appl. Math,V11(4)pp507-517 (1979)

(10) D.J.Griffiths
*Flow of urine through the ureter: A collapsible muscular tube undergoing peristalsis*
J. Biomedical Eng. Trans of the ASME
V111pp206-211 (1989)

(11) J.C. Burns and T. Parkes
*Peristaltic motion*, J. Fluid Mech.V29 (4) pp731-743 (1967)

(12) P.S.Lykoudis and R. Roos
*The fluid mechanics of the ureter from a lubrication theory point of view.*
J. Fluid Mech. V43(4), pp661-674, (1970)

(13) Miller-Keane
*Encyclopedia & Dictionary of Medicine, Nursing and Applied Health*, 5th Edition (1992)

(14) K.K. Raju
*Flow of a non-Newtonian Fluid in a tube with Sinusoidal Deformation* J.Phys,Soc. of Japan
V33, No.1, pp225-231 (1972)

Also, for the non-Newtonian case n = 0.5, k= 1, the approximating function for $\hat{F}_0(r)$ is given by

$$\hat{F}_0(r) = 0.25\left(3r^5 - 7r^2\right) \qquad (4.6)$$

A comparison between values obtained from (4.6) and exact values as recorded in (3) is given in Table 3.5, where error term is approximate.

| r | $\hat{F}_0(r)$ n=0.5 | $F_0(r)$ n=0.6 | $E_{F_0(r)}$ |
|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.017 | 0.015 | 0.002 |
| 0.2 | 0.070 | 0.062 | 0.018 |
| 0.3 | 0.156 | 0.140 | 0.016 |
| 0.4 | 0.272 | 0.249 | 0.023 |
| 0.5 | 0.414 | 0.385 | 0.029 |
| 0.6 | 0.572 | 0.542 | 0.030 |
| 0.7 | 0.731 | 0.707 | 0.024 |
| 0.8 | 0.874 | 0.860 | 0.014 |
| 0.9 | 0.975 | 0.972 | 0.003 |
| 1.0 | 1.000 | 1.000 | 0.000 |

**Table 3.5,** *n=0.5, n=0.6, k=1, where n is flow behaviour index, showing error term*

It is worth mentioning again that a graphical plot of the closeness of solutions for $F_0(r)$ is graphically depicted in Figs. 3.1 - 3.4 for behavior index (n)=$n_i$ and k=1

From these tables and Figs. it can be seen that the approximate solutions are very accurate for n = 1.0, 1.2 & 1.4 and less accurate for n = 0.8 & 0.5, but still good enough to be used to find approximate solutions for the stream function and pressure gradient for those values of n.

From (3.1a) it is clear that for $\varepsilon$ very small the stream function, can be approximated as

$$\Psi = \Psi_0 + \varepsilon\Psi_1 \qquad (4.7)$$

Thus, defining, $\Psi^N$ to be $\Psi$ in the case n = N and defining $\hat{F}^N(r)$ as the approximate solution for *F(r)* in the case n = N, (4.7) gives the approximation to the stream function, along with (3.2) and (3.8) as

$$\Psi^N \approx \left(\frac{k}{2}\right)^{\frac{1}{N}} \frac{N}{N+1}\left(\frac{r^2}{2} - \frac{Nr^{\frac{3N+1}{N}}}{3N+1}\right) + \varepsilon\left[\hat{F}^N(r)\cos\alpha(z-t) + \hat{G}^N(r)\sin\alpha(z-t)\right] \qquad (4.8)$$

where $\hat{G}^N(r)$, is the approximate function to *G(r)* in the case n = N.

From (3.9) it can be seen that

$$\varepsilon\left[\hat{F}^N(r)\cos\alpha(z-t) + \hat{G}^N(r)\sin\alpha(z-t)\right] =$$

$$\cos\alpha(z-t)\left[\varepsilon F_0^N(r) + G\,\mathrm{Re}^2\right] + \sin\alpha(z-t)\left[\varepsilon\,\mathrm{Re}\,G^N(r) + O\left(\varepsilon\,\mathrm{Re}^3\right)\right] \qquad (4.9)$$

under the assumption, $\varepsilon$, & Re small and defining

$$\alpha(z-t) = \bar{z} \qquad (4.10)$$

(4.8) , (4.9)&(4.10) give an approximation to the stream function as,

$$\Psi^N \approx \left(\frac{k}{2}\right)^{\frac{1}{N}} \frac{N}{N+1}\left(\frac{r^2}{2} - \frac{Nr^{\frac{3N+1}{N}}}{3N+1}\right) + \varepsilon\left[\hat{F}_0^N(r)\cos(\bar{z})\right] \qquad (4.11)$$

Streamlines for , n = N = 0.5, 1.0, 1.2 are given in Figs. 3.5, 3.6 and 3.7 respectively, where k = 1.0 and $\varepsilon$ =0.01.

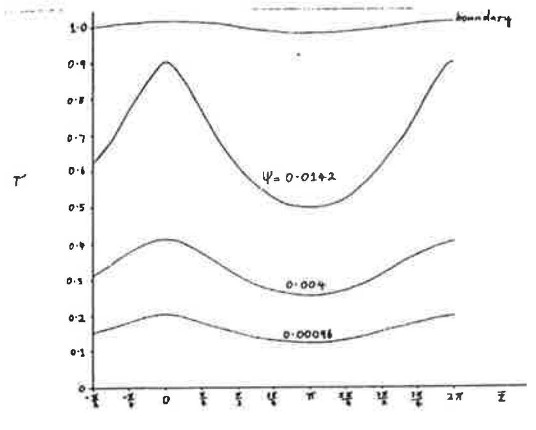


**Figure 3.5**,*Streamline Ψ for n=0.5, k=1,where n is flow behaviour index*

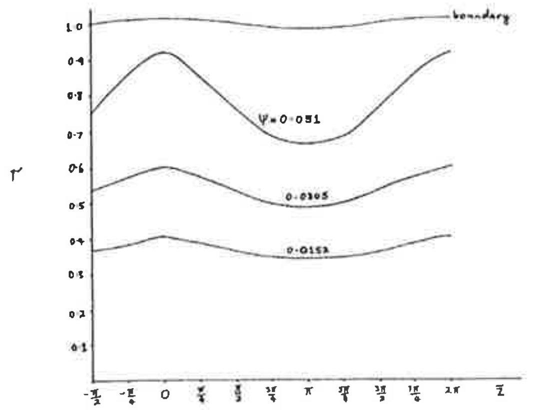


**Figure 3.6**,*Streamline Ψ for n=1, k=1,where n is flow behaviour index*

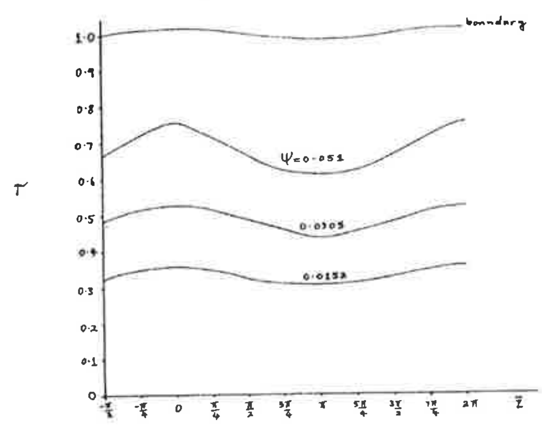


**Figure 3.7**,*Streamline Ψ for n=1.2, k=1, where n is flow behaviour index*

The approximating functions for $F_0(r)$ in the case k=0.01 and n=0.5, 0.8, 1.0 and 1.2 are found in the same manner as the approximating function $F_0(r)$k=1, and are given by

$$\hat{F}_0^{0.5}(r) = 0.666675r^5 - 1.666675r^2 \qquad (4.12)$$

$$\hat{F}_0^{0.8}(r) = 1.267376r^{4.25} - 2.267376r^{2.375} \qquad (4.13)$$

$$\hat{F}_0^{1.0}(r) = 1.0025r^4 - 2.0025r^2 \qquad (4.14)$$

$$\hat{F}_0^{1.2}(r) = 1.097504r^{\frac{23}{6}} - 2.097504r^2 \qquad (4.15)$$

Streamlines for n = 0.5, 0.8, 1.0 & 1.2 in the case k = 0.01 are plotted in Figs. 3.8 - 3.11 respectively where $\varepsilon$ =0.01; and are derived from (4.11) using (4.12) - (4.15).

It can be seen that the streamlines plotted in (Fig 3.5-3.7 )( high pressure gradient ) and Fig. (3.8 - 3.11) (low pressure gradient ) are similar to those established in Raju and Devanathan(3).

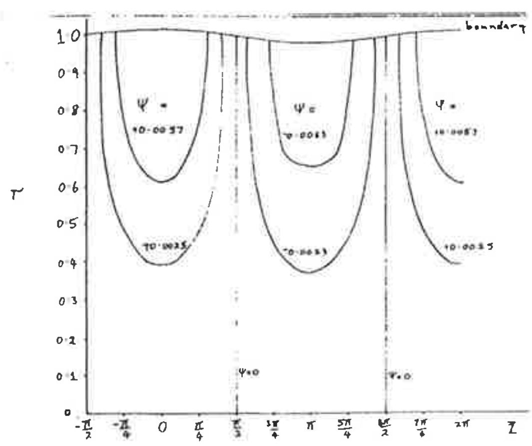


**Figure 3.8**,*Streamline Ψ for n=0.5, k=0.01, where n is flow behaviour index*

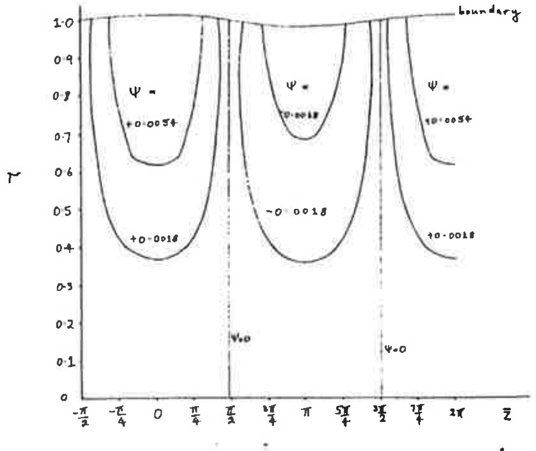


**Figure 3.9,** *Streamline* $\Psi$ *for n=0.8, k=0.01, where n is flow behaviour index*

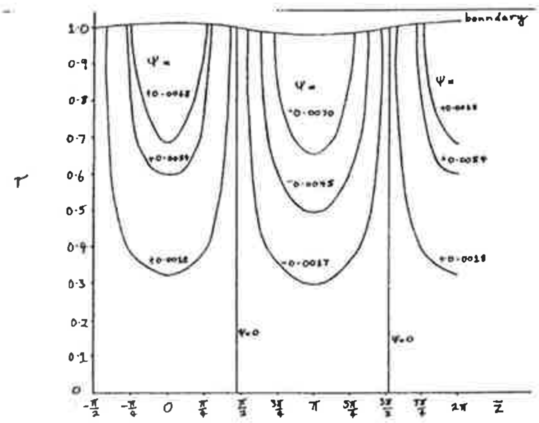


**Figure 3.10,** *Streamline* $\Psi$ *for n=1.0, k=0.01, where n is flow behaviour index*

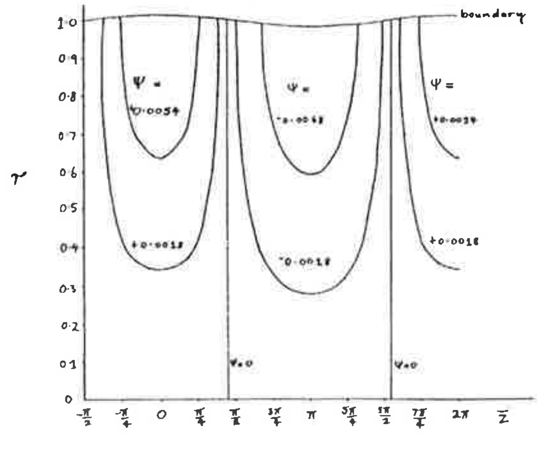


**Figure 3.11,** *Streamline* $\Psi$ *for n=1.2, k=0.01, where n is flow behaviour index*

The difference between streamlines for high and low pressure is apparent due to the boundary of the tube has a more significant impact with regard to low pressure gradient.

All approximate results have been obtained by assuming that the terms of the order $\alpha^2$ and higher are negligible and the parameters and Re are small.

More accurate results could be obtained by neglecting terms $\alpha^3$ and higher but keeping order $\alpha^2$ and $\alpha$ as well as $\alpha^0$.

## Conclusion

The results obtained in this study are consistent with those established by Raju and Devanathan(3) in their study of peristaltic motion of a non-Newtonian fluid.

It was seen that the equation of motion for the non-Newtonian (power law) model of peristaltic motion reduced to the governing equations for the Newtonian model in the case of flow behaviour index, n = 1.

Approximate results for the stream function $\Psi$ were obtained and is seen for n >= 1 are close to the exact results given in the literature. For the case n <= 1, the results were less accurate, but nevertheless to an acceptable level, if we consider the magnitude of the errors.

The streamlines obtained from the approximate results were found to be very similar to those found in Raju and Devanathan(3), thus demonstrating the effect of pressure gradient and boundary structure. In fact, it is found that for low pressure gradient the streamlines form closed loops and for higher presure gradient the streamlines run parallel to the axis of the tube when considered near the axis , whereas a considerable deformation is noted near the boundary. An explanation for this is that the region may be considered to be consisting of two regions; one the central core region and the other near the boundary, the boundary layer region. Highlighting the validity of current modelling of this type. In the case of low pressure gradient, the effects of the wave travelling along the boundary of the tube are more dominant. But as the pressure gradient increases, we find that the streamlines in the central part of the region are more influenced by it, than the motion of the boundary, hence run approximately parallel to the axis. One conclusion reached by these studies is that peristalsis is an effective method to move fluid only if the fluid is transported in the form of a series of isolated boluses. If the amplitude of displacement of the wall is small compared to the tube radius, very little pressure gradient can be generated by the travelling wave. Pressure gradient increases significantly when the radius of minimum section approaches zero, (complete occlusion). It is thus understandable why peristalsis is a common phenomena in the lymphatics, intestine, ureter,and many other biological systems and peristaltic pumps (dialysis machines and heart-lung machines).

**Appendix A**

Solution to (3.10) and (3.11) may now be sought as

$$\hat{F}(r) = Ar^{m_1} + Br^{m_2} + Cr^{m_3} + Dr^{m_4} \quad \text{(A1)}$$

where $m_{ii}$, i = 1-4 are to be determined by making the substitution

$$\hat{F}(r) = r^{m} \quad \text{(A2)}$$

On substitution (A1) into (3.10) yields

$$r^{m}\left\{nm(m-1)(m-2)(m-3) - \frac{2}{n}\left(n^2 - n + 1\right)m(m-1)(m-2) + \frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}(m-1)m - \frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2}m\right\} = 0 \quad \text{(A3)}$$

This finally reduces to,

$$m(m-2)\left[nm^2 - \left(6n - 2 + \frac{2}{n}\right)m\right] + m(m-2)\left[9n - 6 + \frac{3}{n} + \frac{2}{n^2}\right] = 0 \quad \text{(A4)}$$

Where

$m_1 = 0$

$m_2 = 2$

The other two roots are evaluated by determining the determinate of (A4) to see if all the roots are real, hence

$$\Delta = \frac{4}{n^2}(2n-1)^2 \quad \text{(A5)}$$

$\Delta \geq 0$

$$so, \quad m_3 = 3 + \frac{1}{n}, \quad m_4 = 3 - \frac{3}{n} + \frac{2}{n^2}$$

our general solution to (3.10) is

$$\hat{F}_0(r) = A + Br^{2} + Cr^{3+\frac{1}{n}} + Dr^{\frac{3n^2-3n+2}{n^2}} \quad \text{(A6)}$$

There is now sufficient information to find $F_0(r)$ and $G_1(r)$

For the Newtonian case n = 1 (3.13) gives the general solution to equation (3.10) as

$$\hat{F}_0(r) = A + Br^2 + Cr^4 + Dr^2 \ln r \quad \text{(A7)}$$

where lnr is introduced because $m_2 = m_4 = 2$

From (A8)

$$\hat{F}_0''(r) = 2B + 3D + 2D \ln r \quad \text{(A8)}$$

keeping this second derivative bounded at r= 0 means,

D = 0

The boundary conditions (3.27) imply

A = 0

B + C = -1

$$2B + 4C = \frac{k}{2} \quad \text{(A9)}$$

For, n = k = 1 the homogeneous solution to (3.11) again keeping the second derivative bounded at r=0 is

$$G'_n(r) = A_1 + B_1 r^2 + C_1 r^4 \quad \text{(A10)}$$

A particular solution to (3.26) can be sought of the form,

$$\hat{G}_p(r) = D_1 r^6 + E_1 r^8 \quad \text{(A11)}$$

Therefore substitution of (A13) into (3.11) with n = 1 yields

$$192Dr^6 + 1152Er^8 = \frac{5\alpha}{2}\left[5r^6 - r^8\right]$$

Hence $D = \frac{25\alpha}{384}$ $\quad E = \frac{-5\alpha}{2304}$ (A12)

so from (A12) and (A13) the solution to (3.11) is

$$\hat{G}_1(r) = A_1 + B_1 r^2 + C_1 r^4 + \frac{25\alpha}{384} r^6 - \frac{5\alpha}{2304} r^8 \quad \text{(A13)}$$

The boundary conditions imply from (3.14)

$$A_1 = 0$$

$$B_1 + C_1 = \frac{5\alpha}{2304} - \frac{25\alpha}{384}$$

$$2B_1 + 4C_1 = \frac{5\alpha}{288} - \frac{25\alpha}{64}$$

$$\therefore 2C_1 = -\frac{95\alpha}{384}$$

$$\therefore B_1 = \frac{275\alpha}{2304} \quad \text{(A14)}$$

For the non-Newtonian case n = 0.8, k = 1

$$\hat{F}(r) = A + Br^2 + Cr^{4.25} + Dr^{2.375} \quad \text{(A15)}$$

therefore using the boundary conditions (3.14)

A = 0

B + C + D = 1

2B + 4.25C + 2.375D = 0.420448 (A16)

Hence another boundary condition is required, therefore using the condition of symmetry, ie, $\Psi_{1rr} = 0$, at r = 0

C = 1.490906, D = -2.490906
B=0 (A17)
therefore,

$$\hat{F}_0(r) = 1.490906r^{4.25} - 2.490906r^{2.375} \quad \text{(A18)}$$

**Appendix B**

w = 0 at r = d + G(z,t)

$u = \frac{\partial}{\partial t} G(z,t) = Ac\frac{2\pi}{\lambda}\sin\frac{2\pi}{\lambda}(z - ct)$
at r = d + G(z,t)

therefore using (1.11)

$-\frac{1}{r}\frac{\partial \Psi}{\partial z} = \frac{2\pi}{\lambda} A c \sin\frac{2\pi}{\lambda}(z - ct)$
at r = d +G(z,t)

and $\frac{1}{r}\frac{\partial \Psi}{\partial r} = 0$ at r = d + G(z,t)

(2.1B)

Because we are dealing with an infinite tube the end conditions are not specified; instead the pressure gradient in the longitudinal (z) direction is specified and is assumed to be of the form,

$$\frac{\partial p}{\partial z}=\left(\frac{\partial p}{\partial z}\right)_0+A\left(\frac{\partial p}{\partial z}\right)_1+A^2\left(\frac{\partial p}{\partial z}\right)_2+\text{K}$$

(2.2B)

$\left(\frac{\partial p}{\partial z}\right)_0$ =constant and $\left(\frac{\partial p}{\partial z}\right)_i=F(z,r,t)$

(2.3B)

where A is specified in Figure 2.

$$-\text{Re}\left(\frac{\partial p}{\partial z}\right)_0=$$

$$\frac{n}{r}\frac{d^3\Psi_0}{dr^3}\left\{\frac{1}{r}\frac{d^2\Psi_0}{dr^2}-\frac{1}{r^2}\frac{d\Psi_0}{dr}\right\}^{n-1}$$

$$-\frac{(2n-1)}{r}\left\{\frac{1}{r}\frac{d^2\Psi_0}{dr^2}-\frac{1}{r^2}\frac{d\Psi_0}{dr}\right\}^{n}$$

(3.1B)

if we now let

$$k=\text{Re}\left(\frac{\partial p}{\partial z}\right)_0 \qquad (3.2B)$$

(3.3)gives us the differential equation for, $\Psi_0(r)$, as

$$\Psi_0(r)=A\left\{\frac{nr^{3+\frac{1}{n}}}{3n+1}-\frac{r^2}{2}\right\}$$

(3.3B)

which satisfies the required boundary conditions,

$$\Psi_0(0)=0$$
$$\Psi_{0r}(1)=0$$

On substitution of (3.3B) into (3.1) yields

$$A=\left(\frac{k}{2}\right)^{\frac{1}{n}}\frac{n}{n+1} \qquad (3.3aB)$$

Collection of coefficents in $\varepsilon$ in equation (1.6) after substitution of (1.11) we obtain

$$\frac{1}{r}\Psi_{1zt}-\frac{1}{r^2}\Psi_{0r}\Psi_{1zz}=-\frac{\partial p}{\partial r}+$$

$$\frac{1}{\text{Re}}\left\{\left(\frac{1}{r}\Psi_{0rr}-\frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}}X$$

$$\left\{\frac{-n}{r}\Psi_{1zzz}+\left(\frac{n-2}{r}\right)\Psi_{1zrr}\right\}+$$

$$\frac{1}{\text{Re}}\left\{\left(\frac{1}{r}\Psi_{0rr}-\frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}}X$$

$$\left\{\left(\frac{2-n^2}{nr^2}\right)\Psi_{1zr}\right\}+$$

$$\frac{1}{\text{Re}}\left\{\left(\frac{1}{r}\Psi_{0rr}-\frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}}X$$

$$\left\{\left(\frac{2(n-1)}{nr^3}\right)\Psi_{1z}\right\}$$

(3.4B)

Similarly collection ofcoefficients in $\varepsilon$ in equation (1.7) after substitution of (1.11) we obtain,

$$\frac{1}{r}\Psi_{1rt}-\frac{1}{r^2}\Psi_{0r}\Psi_{1rz}+\frac{1}{r^2}\Psi_{0rr}\Psi_{1z}-$$

$$\frac{1}{r^3}\Psi_{0r}\Psi_{1z}=$$

$$-\frac{\partial p}{\partial z}+\frac{1}{\text{Re}}\left\{\left(\frac{1}{r}\Psi_{0rr}-\frac{1}{r^2}\Psi_{0r}\right)^2\right\}^{\frac{n-1}{2}}X$$

$$\frac{n}{r}\Psi_{1rrr}+\left(\frac{2-n}{r}\right)\Psi_{1zzr}+\left(\frac{3n-2-2n^2}{nr^2}\right)\Psi_{1rr}$$

$$+\frac{(n-1)(n-2)}{nr^2}\Psi_{1zz}+\left(\frac{2n^2+2-3n}{nr^3}\right)\Psi_{1r}$$

(3.5B)

Substituting for, $\Psi_0(r)$, as given in (3.2) and, $\Psi_1$, as given in (3.8) and collecting coefficients of, $\cos\alpha(z-t)$, on either side of the resulting equation after multiplying both sides by,

$$\mathrm{Re}\left(\frac{k_0}{2}\right)^{\frac{1-n}{n}} r^{\frac{1+4n}{n}}$$ yields

$$
\begin{aligned}
& nr^4F^{1V} - \frac{2}{n}\left(n^2-n+1\right)r^3F''' + \\
& \left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r^2 + (2n-4)\alpha^2r^4\right\}F'' - \\
& \left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r + \left(\frac{2n^2-2n-2}{n}\right)\alpha^2r^3\right\}F' \\
& +\left\{n\alpha^4r^4 + \left(\frac{n^3-3n+2}{n^2}\right)\alpha^2r^2\right\}F = \\
& -\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}} - r^{\frac{1+3n}{n}}\right)G''\right\} + \\
& \mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{1+2n}{n}} - r^{\frac{2+3n}{n}}\right)G'\right\} + \\
& \mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\left(\frac{n-1}{n}\right)r^{\frac{2+2n}{n}}G\right\} + \\
& \mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{1+3n}{n}} - r^{\frac{2+4n}{n}}\right)\alpha^2G\right\} + \\
& \mathrm{Re}\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{-r^{\frac{1+3n}{n}}G'' + r^{\frac{1+2n}{n}}G'\right\} + \\
& \mathrm{Re}\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{\alpha^2r^{\frac{1+3n}{n}}G\right\}
\end{aligned}
$$

(3.6B)

making the substitution as before for, $\Psi_0(r)$, as given in (3.2) and, $\Psi_1$ as given in (3.8) and collecting coefficients of, $\sin\alpha(z-t)$, after multiplying both sides by ,

$$\mathrm{Re}\left(\frac{k_0}{2}\right)^{\frac{1-n}{n}} r^{\frac{1+4n}{n}}$$ yields,

$$
\begin{aligned}
& nr^4G^{1V} - \frac{2}{n}\left(n^2-n+1\right)r^3G''' + \\
& \left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r^2\right\}G'' + \\
& \left\{(2n-4)\alpha^2r^4\right\}G'' \\
& -\left\{\left(\frac{4n^3-4n^2+n+2}{n^2}\right)r\right\}G' + \\
& \left\{+\left(\frac{2n^2-2n-2}{n}\right)\alpha^2r^3\right\}G' \\
& +\left\{n\alpha^4r^4 + \left(\frac{n^3-3n+2}{n^2}\right)\alpha^2r^2\right\}G = \\
& -\mathrm{Re}\left(\frac{k}{2}\right)^{\frac{2-n}{n}}\alpha\left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}} - r^{\frac{1+3n}{n}}\right)F'' + \right. \\
& \frac{n}{n+1}\left(r^{\frac{1+2n}{n}} - r^{\frac{3+2n}{n}}\right)F' + \frac{n-1}{n}r^{\frac{2+2n}{n}}F \\
& \left. +\frac{n}{n+1}\left(r^{\frac{1+3n}{n}} - r^{\frac{2+4n}{n}}\right)\alpha^2F\right\} + \\
& \mathrm{Re}\left(\frac{k}{2}\right)^{\frac{1-n}{n}}\alpha\left\{r^{\frac{1+3n}{n}}F'' - r^{\frac{1+2n}{n}}F' - \alpha^2r^{\frac{1+3n}{n}}F\right\}
\end{aligned}
$$

(3.7B)

$$n r^4 F_0^{IV} - \frac{2}{n}\left(n^2 - n + 1\right) r^3 F_0''' +$$

$$\left\{\frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2} r^2 + (2n-4)\alpha^2 r^4\right\} F_0''$$

$$-\left\{\frac{\left(4n^3 - 4n^2 + n + 2\right)}{n^2} r + \left(\frac{2n^2 - 2n - 2}{n}\right)\alpha^2 r^3\right\} F_0'$$

$$+\left\{n\alpha^4 r^4 + \left(\frac{n^3 - 3n + 2}{n^2}\right)\alpha^2 r^2\right\} F_0 = 0$$

(3.8B)

and

$$n r^4 G_1^{IV} - \frac{2}{n}\left(n^2 - n + 1\right) r^3 G_1''' +$$

$$\left\{\left(\frac{4n^3 - 4n^2 + n + 2}{n^2}\right) r^2 + (2n-4)\alpha^2 r^4\right\} G_1''$$

$$-\left\{\left(\frac{4n^3 - 4n^2 + n + 2}{n^2}\right) r + \left(\frac{2n^2 - 2n - 2}{n}\right)\alpha^2 r^3\right\} G_1'$$

$$+\left\{n\alpha^4 r^4 + \left(\frac{n^3 - 3n + 2}{n^2}\right)\alpha^2 r^2\right\} G_1 =$$

$$-\left(\frac{k}{2}\right)^{\frac{2-n}{n}} \alpha \left\{\frac{n}{n+1}\left(r^{\frac{2+4n}{n}} - r^{\frac{1+3n}{n}}\right) F_0''' +\right.$$

$$\frac{n}{n+1}\left(r^{\frac{1+2n}{n}} - r^{\frac{3+2n}{n}}\right) F_0' + \frac{n-1}{n} r^{\frac{2+2n}{n}} F_0$$

$$\left. + \frac{n}{n+1}\left(r^{\frac{1+3n}{n}} - r^{\frac{2+4n}{n}}\right)\alpha^2 F_0 \right\} +$$

$$\left(\frac{k}{2}\right)^{\frac{1-n}{n}} \alpha \left\{ r^{\frac{1+3n}{n}} F_0''' - r^{\frac{1+2n}{n}} F_0' - \alpha^2 r^{\frac{1+3n}{n}} F_0 \right\}$$

(3.9B)

## Appendix C

- Dimensionless variables and parameters are defined as follows

$$r' = \frac{r}{d}, \quad z' = \frac{z}{d}, \quad w' = \frac{w}{c}, \quad u' = \frac{u}{c}$$

$$G' = \frac{G}{d}, \quad \Psi' = \frac{\Psi}{cd^2}, \quad t' = \frac{ct}{d}, \quad p' = \frac{p}{\rho c^2}$$

$$\varepsilon = \frac{A}{d}, \quad \alpha = \frac{2\pi d}{\lambda}$$

$$\mathrm{Re} = \frac{c^{2-n} d^n}{m} \rho$$

$$\theta' = \left(\frac{d^2}{c^2}\right)^{\frac{n-1}{2}} \theta$$

$$\left(\frac{\partial p}{\partial z}\right)_0' = \frac{d}{\rho c^2}\left(\frac{\partial p}{\partial z}\right)_0$$


### Acknowledgements

The authors wish to thank Matthew Russack and Bill Ellis for their assistance in the outcome of this research.

The authors are also grateful to the referees for their useful and constructive suggestions, most of which have been taken into account while revising this paper. Their valuable comments will certainly help us in future work.


### References


(1) Mazumdar, J. N. *Biofluid Mechanics* World Scientific Publishing, (1992)

(3.17)

(2) Y. C. Fung ,*Biomechanics Mechanical Properties of Living Tissues*, Springer-Verlag, New York (1981)

(3) Raju, K. K & Devanathan, R *Peristaltic motion of a non-Newtonian Fluid.* Rheol. Acta 11, 170-178 (1972)

(4) Liron, N *A new look at peristalsis and its functions.* Horizons in Biochemistry and Biophysics V5, pp161-182. (1978)

(5) Fung, Y. C & Yih, C. S
*Peristaltic Transport.*
J. Applied Mechanics, Trans. of the ASME
V35, pp669-675 (1968)

(6) Yin, F & Fung, Y. C
*Peristaltic waves in circular cylindrical tubes.*
J. Applied Mechanics, Trans. of the ASME
V36,pp579-587 (1969)

(7) A.H.Shapiro, M.Y.Jaffrin and
S.L.Weinberg
*Peristaltic pumping with long wavelengths at Low Reynolds number.*
J.Fluid Mech.V37,pp799-825 (1969)

(8) L.M.Srivastava and V.P.Srivastava
*Peristaltic transport of a non-Newtonian fluid: Applications to the vas deferens and small intestine.*
Annals of Biomedical Eng.V13,pp137-153 (1985)

(9) R.Devanathan and S.Parvathamma
*Long wavelength peristaltic transport of non-Newtonian fluids*
Indian J.Pure Appl. Math,V11(4)pp507-517 (1979)

(10) D.J.Griffiths
*Flow of urine through the ureter: A collapsible muscular tube undergoing peristalsis*
J. Biomedical Eng. Trans of the ASME
V111pp206-211 (1989)

(11) J.C. Burns and T. Parkes
*Peristaltic motion*, J. Fluid Mech.V29 (4) pp731-743 (1967)

(12) P.S.Lykoudis and R. Roos
*The fluid mechanics of the ureter from a lubrication theory point of view.*
J. Fluid Mech. V43(4), pp661-674, (1970)

(13) Miller-Keane
*Encyclopedia & Dictionary of Medicine, Nursing and Applied Health*, 5th Edition (1992)

(14) K.K. Raju
*Flow of a non-Newtonian Fluid in a tube with Sinusoidal Deformation* J.Phys,Soc. of Japan
V33, No.1, pp225-231 (1972)

A.V. Mernone and J.N. Mazumdar (1998) Peristaltic flow of a non-Newtonian fluid and it’s biomedical applications. EMAC’98, 3rd Biennial Engineering Mathematics and Applications Conference. IEAust, pp. 367-370, 1998

NOTE: This publication is included in the print copy of the thesis held in the University of Adelaide Library.

# A Mathematical Study of Peristaltic Transport of a Casson Fluid.

A.V. Mernone[1], J. N. Mazumdar

Department of Applied Mathematics, The University of Adelaide, Australia 5005

S. K. Lucas

School of Mathematics, University of South Australia, Mawson Lakes, Australia 5095

**Abstract**

In this paper, the peristaltic flow of rheologically complex physiological fluids when modelled by a non-Newtonian Casson fluid in a two-dimensional channel is considered. A perturbation series method of solution of the stream function for zeroth and first order in amplitude ratio is sought. Of interest is the difference between peristaltic transport of Newtonian and non-Newtonian fluids. It is found that Newtonian fluid is an important sub-class of non-Newtonian fluids that may adequately represent some physiological phenomena. Analytical and numerical solutions are found for the zeroth and first order in stream function and compared to well-documented research in the literature. It is shown that for a Casson fluid, when certain approximations are made in the most generalised form of constitutive equation, the fluid may be adequately represented as an improvement of a Newtonian fluid.

**Keywords** Mathematical Modelling, Casson Fluid, Peristalsis, Perturbation Series Method

## INTRODUCTION

As mentioned in an earlier paper in this sequel Mernone and Mazumdar[1] peristalsis is the phenomenon in which a circumferential progressive wave of contraction or expansion (or both)

* Corresponding Author:A.Mernone
email: amernone@maths.adelaide.edu.au

propagates along a tube. If the tube is long enough, one might see several identical waves moving along the tube simultaneously. Peristalsis appears in many organisms and a variety of organs.

Peristalsis is now well known to physiologists to be one of the major mechanisms for fluid transport in many biological systems. In particular, peristaltic mechanisms may be involved in urine transport from the kidney to the bladder through the ureter, the movement of chyme in the gastrointestinal tract, the transport of spermatozoa in the ductus efferentes of the male reproductive tract and in the cervical canal, the movement of ova in the fallopian tubes, the transport of lymph in the lymphatic vessels and in the vascomotion in small blood vessels.

These flows also provide efficient means for sanitary fluid transport and are thus exploited in industrial peristaltic pumping and medical devices, for example, industrial applications of mechanical roller pumps using viscous fluids in the printing industry and the peristaltic transport of noxious fluid in the nuclear industry. In addition, peristaltic pumping occurs in many practical applications involving biomedical systems. Many modern medical devices have been designed on the principle of peristaltic pumping to transport fluids without internal moving parts, for example, the blood in the heart-lung machine.

The main motivation for any mathematical analysis of physiological fluid flows is to ultimately have a better understanding of the particular flow being modelled. If there is similarity between the results obtained from the analysis and experimental and clinical data, then the mechanism of flow can at least be explained. Because peristalsis is evident in many physiological flows, an accurate mathematical study can help explain the major contributing factors to many flows in the human body. When comparing results between the mathematical model and the experimental and clinical data it is desirable that the data obtained from experimental research be as close as possible to the actual physiological parameter being analysed. That is to say, it may be necessary to take into account the effect of the measuring instrument or device or procedure has on the data obtained.

The study of the mechanisms of peristalsis, in both mechanical and physiological situations, has become the subject of scientific research for quite some time. Since the first investigation of Latham[2]several theoretical and experimental attempts have been made to understand peristaltic action in different situations. Interest in peristaltic pumping has been quite recently stimulated by its relevance to ureteral function. As reliable and accurate urometric measurements became available through the work of Kiil[3]and Boyarsky[4] several hydrodynamic models of ureteral function invoking peristalsis were attempted. The earliest models by Shapiro[5], Fung[6] and Shapiro, et al[7] were idealised and represented the peristalsis by an infinite train of sinusoidal waves in a two-dimensional channel; thus they could pretend to only a qualitative relationship with the ureter. These models concerned themselves, in part, with offering an explanation of the biologically and medically important phenomenon of 'reflux'. One manifestation of this reflux is that bacteria sometimes travel from the bladder to the kidney against the mean urine flow. A similar phenomenon has been observed in the small bowel. These observations are puzzling because the travel times are too small to be explained by diffusion and also because retrograde peristaltic waves have not usually been observed. Later, Lykoudis[8]and Weinberg, et al [9] proposed models that represent ureteral waves more realistically. Fung[10] investigated the coupling between the forces of fluid-mechanical origin and the dynamics of the ureteral muscle. Some of these models showed that observed urometric pressure pulses and flow rates could be accounted for by assuming internal dimensions of the ureter which seem physiologically plausible. But ureteral physiology has not been the only motivation for the study of peristalsis.

Burns and Parkes[11] and Hanin[12] contributed to the theory of peristaltic pumping without reference to physiological applications. Barton and Raynor[13] made a calculation based on peristalsis theory of the time required for chyme to traverse the small intestine and found that this calculation compared favourably with observed values. In addition, Fung[10] studied peristaltic flow taking muscle action in the tube wall into account. Some new examples of peristalsis were given in Liron[14]. Considerable experimental investigations of peristaltic pumping have also been undertaken, for example, Latham[2], Mank[15], Shapiro and Latham[16], Eckstein[17[, Weinberg[18] Weinberg, et al[9], Yin and Fung[19] and Hung and Brown[20}. Most of the theoretical investigations have been carried out by assuming blood and other physiological fluids behave like a Newtonian fluid. Although this approach may provide a satisfactory understanding of the peristaltic mechanism in the ureter, it fails to provide a satisfactory model when the peristaltic mechanism is involved in small blood vessels, lymphatic vessels, intestine, ductus efferentes of the male reproductive transport and in the transport of spermatozoa in the cervical canal. It has now been accepted that most of the physiological fluids behave like non-Newtonian fluids. But it appears that no quantitative rigorous attempt has been made to understand the problem of a non-Newtonian fluid before the investigation of Raju and Devanathan[21] in the case of small wave amplitude. Subsequently, Srivastava and Srivastava[22] investigated the problem of peristaltic transport of blood assuming a single layered Casson fluid and ignoring the presence of a peripheral layer. Later on, Srivastava[23] considered the axisymmetric flow of a Casson fluid in a circular non uniform tube. More recently, Siddiqui,et al

[24] investigated peristaltic motion of a non-Newtonian fluid modelled with a constitutive equation for a second order fluid for the case of a planar channel. A perturbation series was used representing parameters such as curvature, inertia and the non-Newtonian character of the fluid. Tang and Rankin[25] proposed a mathematical model for peristaltic motion of a nonlinear viscous flow where they used an iterative method to solve a free boundary problem. Das and Batra[30] studied the fully developed, steady flow of a Casson fluid through a curved tube for small values of Dean number. A plug core formation region at the centre is considered where the shear stress in not sufficient to exceed the yield value. Elshehawey et al[31] consider the problem of peristaltic transport of a non-Newtonian (Carreau) fluid in a non-uniform channel under zero Reynolds number with long wavelength approximation. The problem is formulated using a perturbation expansion in terms of a variant of Weissenberg number. They find that pressure rise and friction force are smaller than the corresponding values in the case of uniform geometry. However, in the present paper we propose to study peristaltic transport of physiological fluids in a planar channel using the most generalised form of constitutive equation, for Casson fluid, as given by Fung[26]. The final analysis is done by using a perturbation method in the same way as was done in our previous paper, Mernone and Mazumdar[1]. To the author's knowledge the use of this generalised equation has not been considered previously in the literature.

## PROBLEM FORMULATION

### Dimensionless Variables in a Two dimensionalChannel

$$x' = \frac{x}{d} \quad y' = \frac{y}{d} \quad u' = \frac{u}{c} \quad v' = \frac{v}{c} \quad \psi' = \frac{\psi}{cd}$$

$$\tilde{\alpha} = \frac{2\pi d}{\lambda} \quad \varepsilon = \frac{A}{d} \quad t' = \frac{ct}{d} \quad G' = \frac{G}{d} \quad p' = \frac{p}{\rho c^2}$$

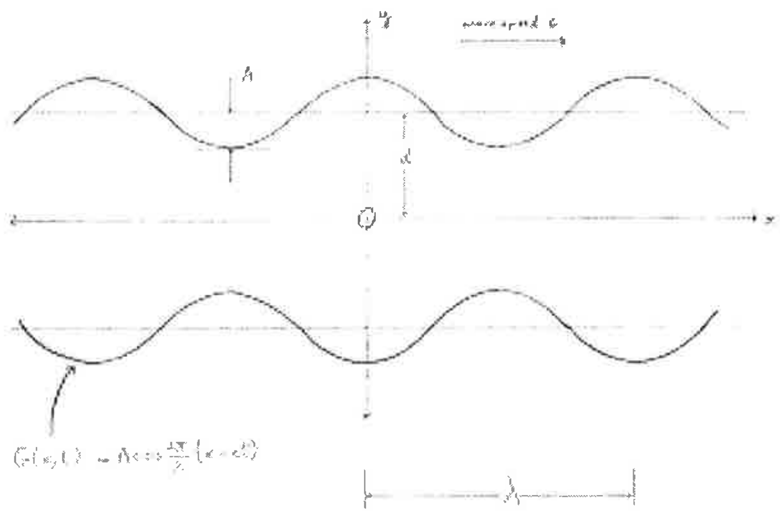

**Figure 1**:*Peristaltic flow in a Two-dimensional Channel*

**Statement of Problem**

Consider the peristaltic motion of a non-Newtonian fluid, modelled as a Casson fluid in a two-dimensinal channel, where, d, is the undeformed width of the channel and the channel is considered to be infinitely long; A, represents the amplitude of the sinusiodal waves travelling along the channel at velocity c; $\lambda$, is the wavelength (Figure.1). A rectangular co-ordinate system is chosen for the channel with x along the centre line and y normal to it. Let u and v be the longitudinal and transverse velocity components, respectively. It is assumed that an infinite train of sinusiodal waves progresses along the walls in the x direction. The vertical displacements for the upper and lower walls are G and -G for peristaltic flow at time t, where G is defined by,

$$G(x,t) = A\cos\frac{2\pi}{\lambda}(x - ct) \qquad \textbf{(1)}$$

We assume that there is no motion of the wall in the longitudinal direction(extensible or elastic wall).

For the case of peristaltic pumping of a Casson fluid in a planar channel the stress-strain relationship in tensor format is given by Fung[26] as

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(J_2)V_{ij} \tag{2}$$

where

$$\mu(J_2) = \left[(\eta^2 J_2)^{\frac{1}{4}} + 2^{-\frac{1}{2}}\tau_y^{\frac{1}{2}}\right]^2 J_2^{-\frac{1}{2}} = \left[\eta^{\frac{1}{2}} + 2^{-\frac{1}{2}}\tau_y^{\frac{1}{2}} J_2^{-\frac{1}{4}}\right]^2 = \left[\alpha + \beta J_2^{-\frac{1}{4}}\right]^2 = \mu(say) \tag{3}$$

Here we have denoted

$$\alpha = \eta^{\frac{1}{2}} \quad : \quad \beta = 2^{-\frac{1}{2}}\tau_y^{\frac{1}{2}} \tag{4}$$

*where* $\eta$ *is the Casson coefficient of viscosity, and* $\tau_y$ *is the yield stress*

Here,

$$V_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{5}$$

and

$$J_2 = \frac{1}{2}V_{ij}V_{ij} = \frac{1}{2}\left(V_{11}^{\,2} + V_{22}^{\,2} + 2V_{12}^{\,2}\right) \tag{6}$$

where

$$V_{11} = \frac{\partial u}{\partial x},\ V_{22} = \frac{\partial v}{\partial y},\ V_{12} = V_{21} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

**Mathematical Modelling of a Casson Fluid in a Two-Dimensional Channel**

Substituting equations (2-6) into the basic equations for continuity and momentum respectively given by

$$div\underline{q} = 0 \tag{7}$$

and

$$\rho \frac{Dq_i}{Dt} = \frac{\partial}{\partial x_j}\sigma_{ij} \tag{8}$$

we have

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + 2\mu_x\frac{\partial u}{\partial x} + \mu_y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

which, using continuity reduces to

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + 2\mu_x\frac{\partial u}{\partial x} + \mu_y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \mu\nabla^2 u \tag{9}$$

Similarly,

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + 2\mu_y\frac{\partial v}{\partial y} + \mu_x\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \mu\nabla^2 v \tag{10}$$

Defining a stream function as $u = \psi_y$ *and* $v = -\psi_x$ we obtain from equations (9) and (10) respectively

$$\rho\left(\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy}\right) = -\frac{\partial p}{\partial x} + 2\mu_x\psi_{xy} + \mu_y\left(\psi_{yy} - \psi_{xx}\right) + \mu\nabla^2\psi_y \tag{11}$$

and

$$\rho\left(-\psi_{xt} - \psi_y\psi_{xx} + \psi_x\psi_{xy}\right) = -\frac{\partial p}{\partial y} - 2\mu_y\psi_{xy} + \mu_x\left(\psi_{yy} - \psi_{xx}\right) - \mu\nabla^2\psi_x \tag{12}$$

where $\mu_x = \frac{\partial}{\partial x}\left[\mu\left(J_2\right)\right]$ *and* $\mu_y = \frac{\partial}{\partial y}\left[\mu\left(J_2\right)\right]$

**Solution Procedure( Zeroth Order Approximation)**

Expressing stream function, $\psi$, pressure, p, and $\mu$ as a series in terms of amplitude ratio $\varepsilon = \frac{A}{d}$, where A is the amplitude and ,d, is the undeformed width of the channel, (Figure 1), we have

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + O\left(\varepsilon^3\right) \tag{13}$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + O(\varepsilon^3) \quad \textbf{(14)}$$

$$\mu = \mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + O(\varepsilon^3) \quad \textbf{(15)}$$

where it is assumed that $\psi_0$ is a function of y only, ie, $\psi_0 = \psi_0(y)$, because of zeroth order axial pressure gradient. We finally obtain from equations (11 & 13-15) after collecting coefficents of $\varepsilon^0$.

$$\frac{\partial p_0}{\partial x} = 2\mu_{0x}\psi_{0yx} + \mu_{0y}\psi_{0yy} - \mu_{0y}\psi_{0xx} + \mu_0\psi_{0xxy} + \mu_0\psi_{0yyy}$$

that is, $$\frac{\partial p_0}{\partial x} = \mu_{0y}\psi_{0yy} + \mu_0\psi_{0yyy} \quad \textbf{(16)}$$

Therefore

$$\frac{\partial p_0}{\partial x} = \frac{\partial}{\partial y}\left(\mu_0\psi_{0yy}\right) \quad \textbf{(17)}$$

We now need to find the zeroth order expression for $\mu_0 = \mu(J_2)_0$

From equation (6) and expanding and substituting we have

$$J_2 = \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{2}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right\} = \frac{1}{2}\left\{\psi_{xy}^{\ 2} + \psi_{xy}^{\ 2} + \frac{1}{2}\left(\psi_{yy} - \psi_{xx}\right)^2\right\}$$

$$= \left\{\psi_{xy}^{\ 2} + \frac{1}{4}\left(\psi_{yy} - \psi_{xx}\right)^2\right\} \quad \textbf{(18)}$$

Therefore we have, after introducing equations (13, 15 and 18)

$$\mu(J_2) = \left\{\alpha + \beta\left[\frac{1}{4}\left(\psi_{0yy}^{\ 2} + 2\varepsilon\psi_{0yy}(\psi_{1yy} - \psi_{1xx})\right)\right]^{-\frac{1}{4}} + O(\varepsilon^2)\right\}^2 \quad \textbf{(19)}$$

Neglecting $O(\varepsilon^2)$ and higher in equation (19) and expanding we have

$$\mu(J_2) \approx \left\{\alpha + \beta\left(\frac{1}{4}\right)^{-\frac{1}{4}} \psi_{0yy}^{-\frac{1}{2}}\left(1 - \frac{\varepsilon}{2}\psi_{0yy}^{-1}(\psi_{1yy} - \psi_{1xx})\right)\right\}^2 \quad \textbf{(20)}$$

Which after further expansion and collecting terms in amplitude ratio for the first two terms and using equation(15) yield equations for $\mu_0 = \mu(J_2)_0$ and $\mu_1 = \mu(J_2)_1$ as

$$\mu_0 = \alpha^2 + 2\sqrt{2}\alpha\beta\psi_{0yy}^{-\frac{1}{2}} + 2\beta^2\psi_{0yy}^{-1} \quad \textbf{(21)}$$

$$\mu_1 = -\alpha\beta\sqrt{2}\psi_{0yy}^{-\frac{3}{2}}\left(\psi_{1yy} - \psi_{1xx}\right) - 2\beta^2\psi_{0yy}^{-2}\left(\psi_{1yy} - \psi_{1xx}\right) \quad \textbf{(22)}$$

Solving equation(15) by using equation(19) and applying the symmetry boundary condition $\psi_{0yy}(0) = 0$ we have

$$Ky + L = \mu_0\psi_{0yy}$$
$$where\ K = \rho\sqrt{c^3}d\frac{\partial p_0}{\partial x} \quad and\ L = 2\beta^2 \quad \textbf{(23)}$$

Our equation to solve for $\psi_0(y)$ then becomes

$$\alpha^2\psi_{0yy} + 2\sqrt{2}\alpha\beta\psi_{0yy}^{\frac{1}{2}} - Ky = 0 \quad \textbf{(24)}$$

If we set

$$\psi_{0yy} = W^2 \quad \textbf{(25)}$$

then equation(24) becomes a quadratic in W as

$$\alpha^2W^2 + 2\sqrt{2}\alpha\beta W - Ky = 0 \quad \textbf{(26)}$$

whose roots are given by

$$W = -\sqrt{2}\frac{\beta}{\alpha} \pm \frac{1}{\alpha}\sqrt{2\beta^2 + Ky} \quad \textbf{(27)}$$

Using equation(25 & 27) we obtain

$$\psi_{0yy} = \left(-\sqrt{2}\frac{\beta}{\alpha} \pm \frac{1}{\alpha}\sqrt{2\beta^2 + Ky}\right)^2 \quad \textbf{(28)}$$

But the symmetry boundary condition $\psi_{0yy}(0) = 0$ demands only the positive sign to be valid, therefore

$$\psi_{0yy} = \left(-\sqrt{2}\frac{\beta}{\alpha} + \frac{1}{\alpha}\sqrt{2\beta^2 + Ky}\right)^2 \quad \textbf{(29)}$$

Integrating equation(29) twice we obtain

$$\psi_0(y) = \frac{2\beta^2}{\alpha^2}y^2 + \frac{K}{6\alpha^2}y^3 - \frac{8\sqrt{2}\beta}{15K^2\alpha^2}\left(2\beta^2 + Ky\right)^{\frac{5}{2}} + Ay + B \quad \textbf{(30)}$$

Where A and B are constants of integration.

Using the boundary conditions $\psi_{0y}(1) = 0 \quad and \quad \psi_0(0) = 0$

we find

$$A = \frac{4\sqrt{2}\beta}{3\alpha^2 K}\left(2\beta^2 + K\right)^{\frac{3}{2}} - \frac{4\beta^2}{\alpha^2} - \frac{K}{2\alpha^2} \quad \textbf{(31)}$$

$$B = \frac{8\sqrt{2}\beta}{15K^2\alpha^2}\left(2\beta^2\right)^{\frac{5}{2}} \quad \textbf{(32)}$$

If we let $\beta \to 0 \quad that\ is\ \tau_y \to 0$ from equation(4), we obtain the Newtonian case in the form

$$\psi_0(y) = -\frac{K}{2\alpha^2}\left(y - \frac{y^3}{3}\right)$$

which coincides with the literature, Fung[6].

We now seek to determine the dimensionless pressure rise, $\Delta p_0$, where

$$\Delta p_0 = \int_0^1 \frac{\partial p_0}{\partial x}dx \quad \textbf{(33)}$$

The flow rate, q, is given by

$$q = \int_0^1 u dy = \int_0^1 \frac{\partial \psi}{\partial y} dy = \psi(1) - \psi(0) \tag{34}$$

therefore from equations(30-32)

$$\psi(1) - \psi(0) = -\frac{2\beta^2}{\alpha^2} - \frac{K}{3\alpha^2} + \frac{4\sqrt{2}}{3\alpha^2 K}\left(2\beta^2 + K\right)^{\frac{3}{2}} = q \tag{35}$$

Applying expansion gives

$$q = \frac{2\beta^2}{\alpha^2} - \frac{K}{3\alpha^2} + \frac{16\beta^4}{3\alpha^2 K} \tag{36}$$

Separating the pressure gradient after solving for quadratic in $K$, and using equation(23) gives

$$\frac{\partial p_0}{\partial x} = \frac{K}{\rho\sqrt{c^3 d}} = \frac{1}{\rho\sqrt{c^3 d}}\left\{-\frac{1}{2}\left(3\alpha^2 q - 6\beta^2\right) \pm \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2}\right\} \tag{37}$$

Hence using equation(33) pressure rise is

$$\begin{aligned}
\Delta p_0 &= \frac{1}{\rho\sqrt{c^3 d}} \int_0^1 \left\{-\frac{1}{2}\left(3\alpha^2 - 6\beta^2\right) \pm \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2}\right\} dx \\
&\textit{because} -\frac{1}{2}\left(3\alpha^2 - 6\beta^2\right) \pm \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2} = constant \\
\Delta p_0 &= \frac{1}{\rho\sqrt{c^3 d}} \left\{-\frac{1}{2}\left(3\alpha^2 - 6\beta^2\right) \pm \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2}\right\}
\end{aligned} \tag{38}$$

However, the condition $\beta = 0$, implies that only the negative sign of the quadratic to be valid,

therefore

$$\Delta p_0 = \frac{1}{\rho\sqrt{c^3 d}} \left\{-\frac{1}{2}\left(3\alpha^2 - 6\beta^2\right) - \frac{1}{2}\sqrt{9\alpha^4 q^2 + 100\beta^4 - 36\alpha^2 q\beta^2}\right\} \tag{39}$$

This is graphically dipicted in Figure(2).

$\Delta p_0$

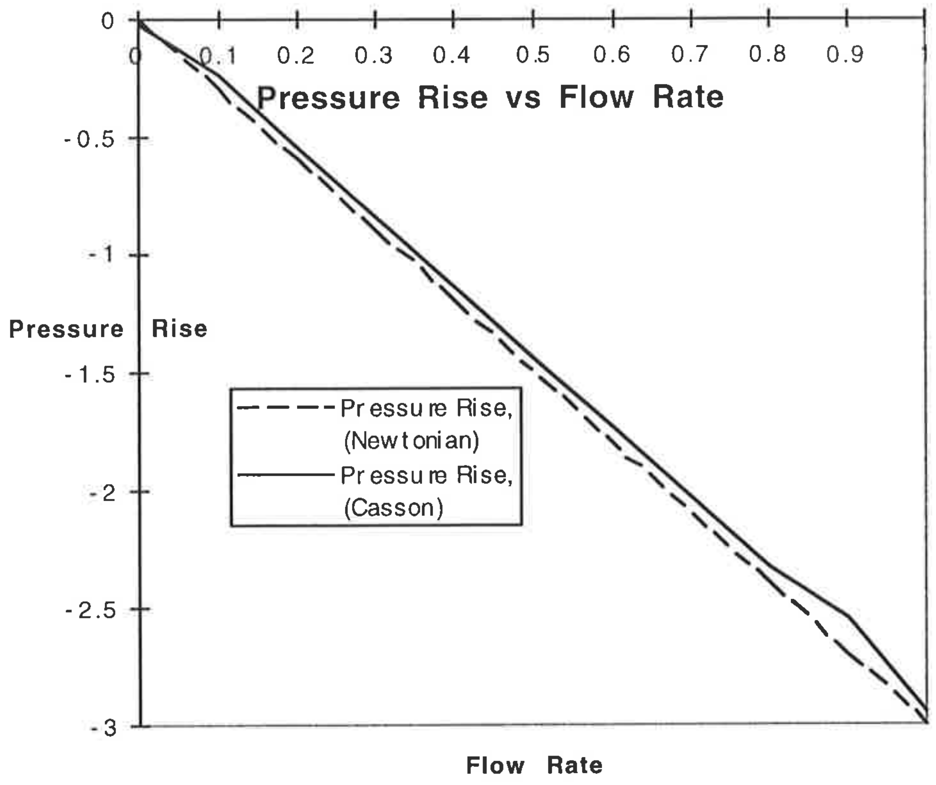


**Figure 2:** *Pressure Rise vs Flow Rate*

**Solution Procedure( First Order Approximation )**

We now look at the procedure for determining $\psi_1(x,y,t)$.

The boundary conditions for $\psi_1(x,y,t)$ are derived as follows; Assuming that there is no horizontal displacement of the tube walls during the peristaltic motion, the boundary conditions at the walls are

$$\begin{aligned} &(a)\ \ no-slip\ condition : u = 0\ at\ y = \pm[d + G(x,t)] \\ &(b)\ \ impermeable\ condition : v = \pm\frac{\partial}{\partial t}G(x,t)\ at\ y = \pm[d + G(x,t)] \end{aligned} \qquad \textbf{(40a\&b)}$$

Using $G(x,t) = A\cos\frac{2\pi}{\lambda}(x - ct)$ and equation (13) and non-dimensionalising as defined above we obtain

$$\psi_y = 0 \; at \; y = \pm\left[1 + \varepsilon\cos\tilde{\alpha}(x-t)\right]$$
$$\psi_x = \mp\frac{2\pi Ac}{\lambda}\sin\frac{2\pi}{\lambda}(x-ct) = \mp\tilde{\alpha}\,\varepsilon\sin\tilde{\alpha}(x-t) \; at \; y = \pm\left[1+\varepsilon\cos\tilde{\alpha}(x-t)\right] \qquad \textbf{(41a\&b)}$$

The boundary conditions (41) can be written, using Taylor series expansions about $y = \pm(1+G)$ *where here* $G = \varepsilon\cos\tilde{\alpha}(x-t)$ as, after equating terms of the same order in $\varepsilon$, on either side of the equations, which gives

$$\psi_y(\pm1) \pm G\psi_{yy}(\pm1) + \frac{G^2}{2}\psi_{yyy}(\pm1) \pm O\left(G^3\right) = 0$$
$$\psi_x(\pm1) \pm G\psi_{xy}(\pm1) + \frac{G^2}{2}\psi_{xyy}(\pm1) \pm O\left(G^3\right) = \mp\tilde{\alpha}\,\varepsilon\sin\tilde{\alpha}(x-t) \qquad \textbf{(42a\&b)}$$

Substituting equation (13) into equation (42) and collecting terms of the same order in $\varepsilon$, gives

$$\psi_{0y}(\pm1) = 0$$
$$\psi_{1y}(\pm1) \pm \psi_{0yy}(\pm1)\cos\tilde{\alpha}(x-t) = 0$$
$$\psi_{0x}(\pm1) = 0 \qquad \textbf{(43)}$$
$$\psi_{1x}(\pm1) \pm \psi_{0xy}(\pm1)\cos\tilde{\alpha}(x-t) = \mp\tilde{\alpha}\sin\tilde{\alpha}(x-t)$$

*and so on for higher order terms in* $\varepsilon$.

Taking the positive sign of the boundary conditions as given in equation (43) yields the boundary conditions as

$$\psi_{1y}(1) = -\psi_{0yy}(1)\cos\tilde{\alpha}(x-t)$$
$$\psi_{1x}(1) = -\tilde{\alpha}\sin\tilde{\alpha}(x-t) \qquad \textbf{(44a\&b)}$$

From these boundary conditions, it can be assumed that $\psi_1(x,y,t)$ can be obtained in the form

$$\psi_1(x,y,t) = f(y)\cos\tilde{\alpha}(x-t) + g(y)\sin\tilde{\alpha}(x-t) \qquad \textbf{(45)}$$

Eliminating the pressure terms in equations (11) and (12) by cross-differentiation and subtraction, the following equation is obtained:

$$\rho(\nabla^2\psi_t + \psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y) = 4\mu_{xy}\psi_{yx} + 2\mu_x\psi_{yyx} + \mu_{yy}(\psi_{yy} - \psi_{xx}) + \mu_y(\psi_{yyy} - \psi_{xxy}) + \mu_y\nabla^2\psi_y + \mu\nabla^2\psi_{yy} + 2\mu_y\psi_{xxy} - \mu_{xx}(\psi_{yy} - \psi_{xx}) - \mu_x(\psi_{yyx} - \psi_{xxx}) + \mu_x\nabla^2\psi_x + \mu\nabla^2\psi_{xx} \quad \textbf{(46)}$$

By substituting equation (45) for $\psi_1(x,y,t)$ and equation (29) for $\psi_0(y)$ into equation (45), and collecting coefficients of $\cos\tilde{\alpha}(x-t)$ *and* $\sin\tilde{\alpha}(x-t)$ on either side of the resulting equation, two differential equations for $f(y)$ *and* $g(y)$ are obtained.

Due to the length and complexity of these equations approximate solutions are obtained by assuming that the parameter, $\tilde{\alpha}$ , which is $\frac{2\pi d}{\lambda}$ is small. As a first approximation, the terms of order $\tilde{\alpha}^2$ and higher can be neglected; as a second approximation, the terms of order $\tilde{\alpha}^3$ and higher can be neglected and so on.

Hence the following equation is obtained from equation(45) by expanding in a pertubations series as indicated in equations(13-15) after collecting terms of the first order

in amplitude ratio, $\varepsilon$

$$\rho\sqrt{c^3 d}\left(\psi_{1yyt} + \psi_{0y}\psi_{1yyx} - \psi_{1x}\psi_{0yyy}\right) = \mu_{0yy}\psi_{1yy} + \mu_{1yy}\psi_{0yy} + 2\mu_{0y}\psi_{1yyy} + 2\mu_{1y}\psi_{0yyy} + \mu_0\psi_{1yyyy} + \mu_1\psi_{0yyyy} \quad \textbf{(47)}$$

where $\mu_0, \mu_{0y}, \mu_{0yy}$ *and* $\mu_1, \mu_{1y}, \mu_{1yy}$ are extensive and complicated equations and are obtained from equation (21) and equation (22) respectively, as follows,

$$\mu_{0y} = -\sqrt{2}\alpha\beta\psi_{0yy}^{-\frac{3}{2}}\psi_{0yyy} - 2\beta^2\psi_{0yy}^{-2}\psi_{0yyy} \quad \textbf{(48)}$$

$$\mu_{0yy} = \frac{3\alpha\beta}{\sqrt{2}}\psi_{0yy}^{-\frac{5}{2}}\psi_{0yyy}^{2} - \sqrt{2}\alpha\beta\psi_{0yy}^{-\frac{3}{2}}\psi_{0yyyy} + 4\beta^2\psi_{0yy}^{-3}\psi_{0yyy}^{2} - 2\beta^2\psi_{0yy}^{-2}\psi_{0yyyy} \quad \textbf{(49)}$$

$$\mu_{1y} = \frac{3\alpha\beta}{\sqrt{2}}\psi_{0yy}^{-\frac{5}{2}}\psi_{0yyy}\psi_{1yy} - \sqrt{2}\alpha\beta\psi_{0yy}^{-\frac{3}{2}}\psi_{1yyy} + 4\beta^2\psi_{0yy}^{-3}\psi_{0yyy}\psi_{1yy} - 2\beta^2\psi_{0yy}^{-2}\psi_{1yyy} \quad \textbf{(50)}$$

$$\begin{aligned}\mu_{1y} = &-\frac{15\alpha\beta}{2\sqrt{2}}\psi_{0yy}^{-\frac{7}{2}}\psi_{0yyy}^{2}\psi_{1yy} + \frac{3\alpha\beta}{\sqrt{2}}\psi_{0yy}^{-\frac{5}{2}}\psi_{0yyyy}\psi_{1yy} + \frac{6\alpha\beta}{\sqrt{2}}\psi_{0yy}^{-\frac{5}{2}}\psi_{0yyy}\psi_{1yyy} - \sqrt{2}\alpha\beta\psi_{0yy}^{-\frac{3}{2}}\psi_{1yyyy} \\ &- 12\beta^2\psi_{0yy}^{-4}\psi_{0yyy}^{2}\psi_{1yy} + 4\beta^2\psi_{0yy}^{-3}\psi_{0yyyy}\psi_{1yy} + 8\beta^2\psi_{0yy}^{-3}\psi_{0yyy}\psi_{1yyy} - 2\beta^2\psi_{0yy}^{-2}\psi_{1yyyy}\end{aligned} \quad \textbf{(51)}$$

After substituting for the various terms in equation(47) and collecting terms and remembering the approximation made on terms in the parameter, $\bar{\alpha}$, the following ordinary differential equation is

$$\rho\sqrt{c^3 d}\left\{\begin{array}{l}\bar{\alpha} f''\sin\bar{\alpha}(x-t) - \bar{\alpha} g''\cos\bar{\alpha}(x-t) + \\ \left[\frac{4\beta^2}{\alpha^2}y + \frac{k}{2\alpha^2}y^2 - \frac{4\beta\sqrt{2}}{3k\alpha^2}\left(2\beta^2+ky\right)^{\frac{3}{2}} + A\right]\left(-\bar{\alpha} f''\sin\bar{\alpha}(x-t) + \bar{\alpha} g''\cos\bar{\alpha}(x-t)\right) \\ -\left(-\bar{\alpha} f\sin\bar{\alpha}(x-t) + \bar{\alpha} g\cos\bar{\alpha}(x-t)\right)\frac{k^2}{4\beta^2\alpha^2}y\end{array}\right\} =$$

$$\left(\frac{k^2 y}{4\beta^2\alpha^2}\right)^2\left(f''\cos\bar{\alpha}(x-t) + g''\sin\bar{\alpha}(x-t)\right)\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}} -$$

$$\left(\frac{ky}{4\beta^2\alpha^2}\right)\left(f'''\cos\bar{\alpha}(x-t) + g'''\sin\bar{\alpha}(x-t)\right)\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} +$$

$$\alpha^2\left(f^{iv}\cos\bar{\alpha}(x-t) + g^{iv}\sin\bar{\alpha}(x-t)\right) -$$

$$\left(\frac{k^2}{4\alpha^2\beta^2} - \frac{3k^3}{16\beta^4\alpha^2}y\right)\frac{\alpha\beta}{\sqrt{2}}\left(f''\cos\bar{\alpha}(x-t) + g''\sin\bar{\alpha}(x-t)\right)\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} +$$

$$\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(f^{iv}\cos\bar{\alpha}(x-t) + g^{iv}\sin\bar{\alpha}(x-t)\right) \quad \textbf{(52)}$$

where the constant A is given in equation(31).

Collecting coefficients of $\cos\tilde{\alpha}(x-t)$ in equation(52) gives

$$\rho\sqrt{c^3}d\left\{-\tilde{\alpha}g''+\left[\frac{4\beta^2}{\alpha^2}y+\frac{k}{2\alpha^2}y^2-\frac{4\beta\sqrt{2}}{3k\alpha^2}\left(2\beta^2+ky\right)^{\frac{3}{2}}+A\right]\left(\tilde{\alpha}g''\right)-\left(\tilde{\alpha}g\right)\frac{k^2}{4\beta^2\alpha^2}y\right\}=$$

$$\left(\frac{k^2y}{4\beta^2\alpha^2}\right)^2(f'')\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}}-$$

$$\left(\frac{ky}{4\beta^2\alpha^2}\right)(f''')\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}}+\alpha^2\left(f^{iv}\right)- \tag{53}$$

$$\left(\frac{k^2}{4\alpha^2\beta^2}-\frac{3k^3}{16\beta^4\alpha^2}y\right)\frac{\alpha\beta}{\sqrt{2}}(f'')\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}}+$$

$$\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(f^{iv}\right)$$

Collecting coefficients of $\sin\tilde{\alpha}(x-t)$ in equation(52) gives

$$\rho\sqrt{c^3}d\left\{\tilde{\alpha}f''-\left[\frac{4\beta^2}{\alpha^2}y+\frac{k}{2\alpha^2}y^2-\frac{4\beta\sqrt{2}}{3k\alpha^2}\left(2\beta^2+ky\right)^{\frac{3}{2}}+A\right]\left(\tilde{\alpha}f''\right)+\left(\tilde{\alpha}f\right)\frac{k^2}{4\beta^2\alpha^2}y\right\}=$$

$$\left(\frac{k^2y}{4\beta^2\alpha^2}\right)^2(g'')\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}}-$$

$$\left(\frac{ky}{4\beta^2\alpha^2}\right)(g''')\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}}+\alpha^2\left(g^{iv}\right)- \tag{54}$$

$$\left(\frac{k^2}{4\alpha^2\beta^2}-\frac{3k^3}{16\beta^4\alpha^2}y\right)\frac{\alpha\beta}{\sqrt{2}}(g'')\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}}+$$

$$\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2}+\frac{k}{\alpha^2}y-\frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2+ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(g^{iv}\right)$$

The equations for f(y) and g(y) can be simplified by assuming that the Reynolds Number associated with the present model is small, where the associated Reynolds Number is given as

$$\text{Re} = \rho\sqrt{c^3 d} \tag{55}$$

Therefore,

$$\begin{aligned} f(y) &= f_0(y) + \text{Re}^2 f_2(y) + \textit{higher order terms in } \text{Re} \\ g(y) &= \text{Re}\, g_1(y) + \text{Re}^3 g_3(y) + \textit{higher order terms in } \text{Re} \end{aligned} \tag{56}$$

Hence, evaluating equation (53) and equation(54) with equation(56) and equating equal terms in Reynolds Number the following ordinary differential equations are obtained for $f_0(y)$ *and* $g_1(y)$ respectively,

$$\begin{aligned} &\left(\frac{k^2 y}{4\beta^2\alpha^2}\right)^2 \left(f_0''\right)\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}} - \\ &\left(\frac{ky}{4\beta^2\alpha^2}\right)\left(f_0''\right)\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} + \alpha^2\left(f_0^{iv}\right) - \\ &\left(\frac{k^2}{4\alpha^2\beta^2} - \frac{3k^3}{16\beta^4\alpha^2}y\right)\frac{\alpha\beta}{\sqrt{2}}\left(f_0''\right)\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} + \\ &\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2}y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(f_0^{iv}\right) = 0 \end{aligned} \tag{57}$$

$$\left\{\tilde{\alpha} f_0'' - \left[\frac{4\beta^2}{\alpha^2} y + \frac{k}{2\alpha^2} y^2 - \frac{4\beta\sqrt{2}}{3k\alpha^2}\left(2\beta^2 + ky\right)^{\frac{3}{2}} + A\right]\left(\tilde{\alpha} f_0''\right) + \left(\tilde{\alpha} f_0\right)\frac{k^2}{4\beta^2\alpha^2} y\right\} =$$

$$\left(\frac{k^2 y}{4\beta^2\alpha^2}\right)^2 \left(g_1''\right)\frac{3\alpha\beta}{2\sqrt{2}}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2} y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{5}{2}} -$$

$$\left(\frac{ky}{4\beta^2\alpha^2}\right)\left(g_1'''\right)\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2} y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} + \alpha^2\left(g_1^{iv}\right) - \tag{58}$$

$$\left(\frac{k^2}{4\alpha^2\beta^2} - \frac{3k^3}{16\beta^4\alpha^2} y\right)\frac{\alpha\beta}{\sqrt{2}}\left(g_1''\right)\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2} y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}} +$$

$$\alpha\beta\sqrt{2}\left[\frac{4\beta^2}{\alpha^2} + \frac{k}{\alpha^2} y - \frac{2\beta\sqrt{2}}{\alpha^2}\left(2\beta^2 + ky\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left(g_1^{iv}\right)$$

From equation(44) the boundary conditions for $f_0(y)$ and $g_1(y)$ are given as

$$\begin{aligned} &f_0(0) = 0 \quad f_0'(0) = 0 \quad f_0(1) = 1 \quad f_0'(1) = -\psi_{0yy} \\ &g_1(0) = g_1'(0) = g_1(1) = g_1'(1) = 0 \end{aligned} \tag{59}$$

The analytical solution to equation(57) is found by using the text[38].

Reducing equation(57) to a second order equation and then integrating twice the solution is found. Comparing our reduced second order equation to and using part 28 on page 134 of text[38] with their notations

$$\alpha = \frac{A}{E},\ \beta = \frac{B}{E},\ \gamma = -\frac{C}{E}$$

where, $A = \dfrac{k^4 3\alpha\beta\left(\dfrac{4\beta^2}{\alpha^2}\right)^{-\frac{5}{2}}\dfrac{14}{4}}{2\sqrt{2}\left(4\beta^2\alpha^2\right)^2}$ $\quad B = \dfrac{3k^3\alpha\beta\left(\dfrac{4\beta^2}{\alpha^2}\right)^{-\frac{3}{2}}\dfrac{10}{4}}{16\beta^4\alpha^2 2\sqrt{2}}$ $\quad C = \dfrac{k^2\alpha\beta\left(\dfrac{4\beta^2}{\alpha^2}\right)^{-\frac{3}{2}}\dfrac{10}{4}}{4\sqrt{2}\beta^2\alpha^2}$

$$D = \frac{k\sqrt{2}\alpha\beta\left(\dfrac{4\beta^2}{\alpha^2}\right)^{-\frac{3}{2}}\dfrac{10}{4}}{4\beta^2\alpha^2} \qquad E = \alpha^2 + \sqrt{2}\alpha\beta\left(\frac{4\beta^2}{\alpha^2}\right)^{-\frac{1}{2}}\frac{6}{4} \tag{60}$$

Consequently, with $w_0 = f_0''$ equation(57) reduces to

$$w_0'' - \frac{D}{E} y w_0' + \frac{1}{E}\left\{Ay^2 + By - C\right\} w_0 = 0 \qquad \textbf{(61)}$$

Then following through the analysis they describe where

$$s = \frac{D}{4E} \pm \frac{1}{2}\sqrt{\frac{D^2}{4E^2} - \frac{A}{E}} \qquad \textbf{(62)}$$

is the root of the quadratic(see page134 of text[38])

$$4s^2 + 2as + \alpha = 0\,,\; a = -\frac{D}{E}\,,\; b = 0 \qquad \textbf{(63)}$$

It is found that if we consider the first two terms of the series,

$$w_0 = f_0'' = \exp(hy)\exp(sy^2)z(\xi) \quad \textit{where} \quad z(\xi)\ \textit{is found in Table2.2 page 143 of text[38]}$$

$$= \exp(hy)\exp(sy^2)\left\{C_1\left[1 + \sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n}\frac{\left(k'\zeta^2\right)^n}{n!}\right] + C_2 y^{\frac{1}{2}}\left[1 + \sum_{n=1}^{\infty}\frac{\left(a+\frac{1}{2}\right)_n}{(b)_n}\frac{\left(k'\zeta^2\right)^n}{n!}\right]\right\} \qquad \textbf{(64)}$$

where

$$z(\zeta) = \xi(a, \tfrac{1}{2}, k'\zeta^2)$$
$$\zeta = \frac{y-\mu}{\lambda}\,,\; \mu = -\frac{2b_2 h + b_1}{a_1}\,,\; \lambda = 1\,, \qquad \textbf{(65)}$$

$\xi\left(a, \frac{1}{2}, k'\zeta^2\right)$ is the degenerate hypergeometric solution and is found on page143, part 103 and page137, part 65 of [38].

Subsequently, the solution to equation(57) and hence equation(64 & 65), after applying symbolic integration twice using MATLAB v5.3 is very intricate and given in Appendix A. Numerical solutions of equations (57) for $f_0(y)$ and (58) for $g_1(y)$ result in the plot in Figures(4-7). Figure (4) shows a comparison of $f_0(y)$ with other models[21]. Figure (4) shows the curves for

$f_0(y)$ and $f_0'(y)$ with varying values of yield stress. That is, $\beta$ is gradually varied between zero and unity. Figures (5-7) show curves for $g_1(y)$ and $g_1'(y)$ with varying values of yield stress and various values of wave number. Figure(8) gives a plot of the function, $\psi$ vs $\bar{x}$, where $\bar{x} = x - t$ as derived in this paper from equations(13), (30-32), (45) and (56), which are very similar to plots given in [21].

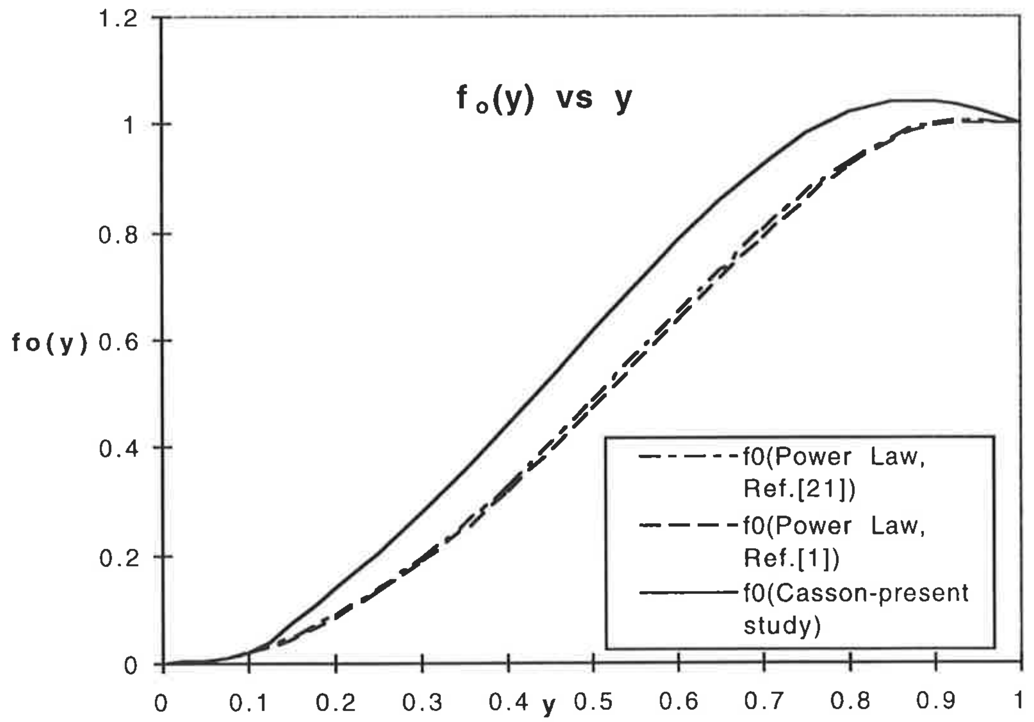


**Figure 3:** *Comparing $f_0$ vs y with [21]*

When comparing the values of our Casson model in Figure 3, obtained from numerical integration, of the first order in stream function with those of the power-law model of Mernone & Mazumdar[1] and Raju & Devanathan[21] the results are similar but noticably different. The Casson model indicating the effects of the yield stress and Casson viscosity on the stream function.

However, there are a similarities between the two models in form. Initially the two models coincide then diversify with as increasing values.

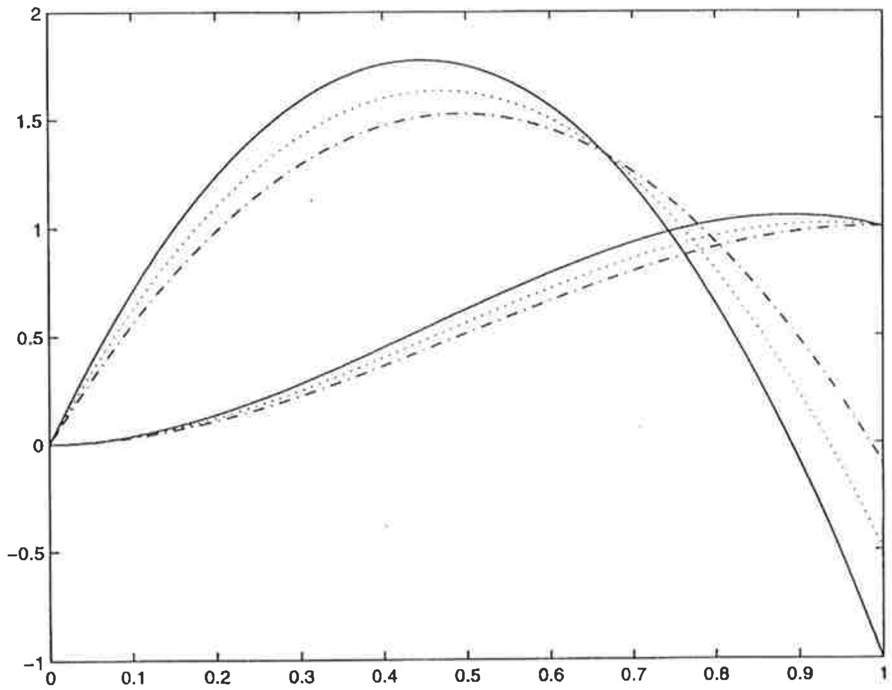


**Figure 4:** $f_0(y)$ & $f_0'(y)$ *with*, ____ $f_0'(1) = -1(\beta = 0)$, ......... $f_0'(1) = -0.5$, −.−.− $f_0'(1) = -0.1(\beta = 1)$

When considering the $f_0(y)$ and $f_0'(y)$in Figure 4 it is found that as the yield stress $\beta$ is gradually varied between zero and unity; the effects on both $f_0(y)$ and $f_0'(y)$are noticable and significant. It appears that the maximum value for $f_0'(y)$ is shifted slightly to the right.

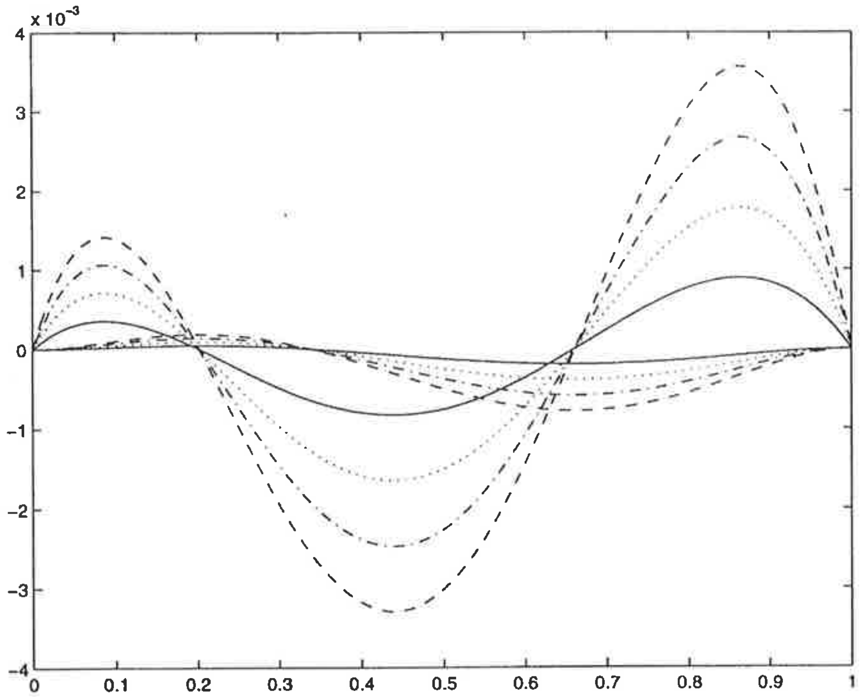


**Figure 5:** $g_1(y)$ & $g_1'(y)$ *with* $f_0'(1) = -1$ : ____ $\tilde{\alpha} = 0.2$, ....... $\tilde{\alpha} = 0.4$, -.- $\tilde{\alpha} = 0.6$, - - - $\tilde{\alpha} = 0.8$

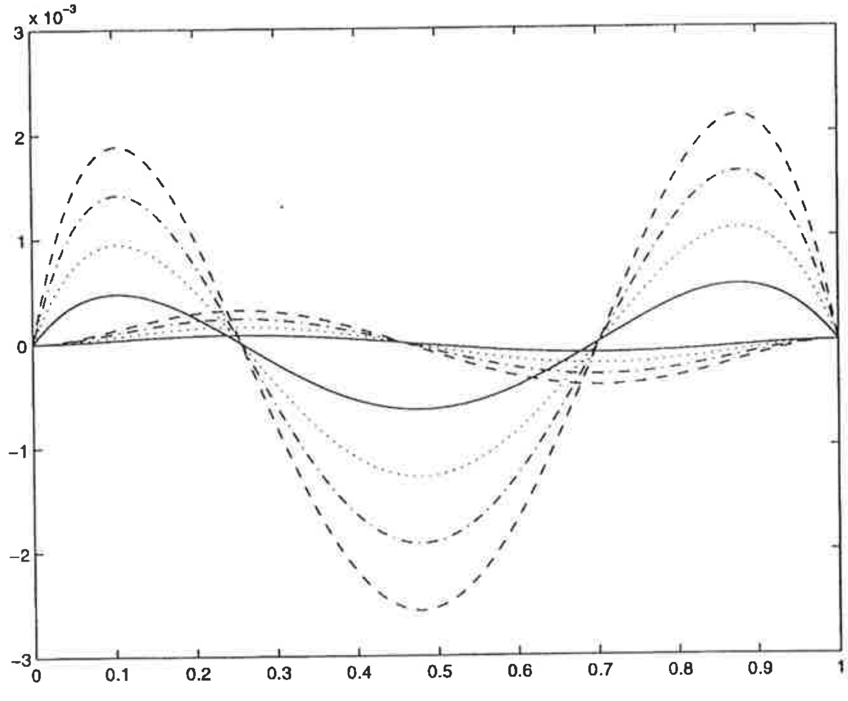


**Figure 6:** $g_1(y)$ & $g_1'(y)$ *with* $f_0'(1) = -0.5$ : ____ $\tilde{\alpha} = 0.2$, ....... $\tilde{\alpha} = 0.4$, -.- $\tilde{\alpha} = 0.6$, - - - $\tilde{\alpha} = 0.8$

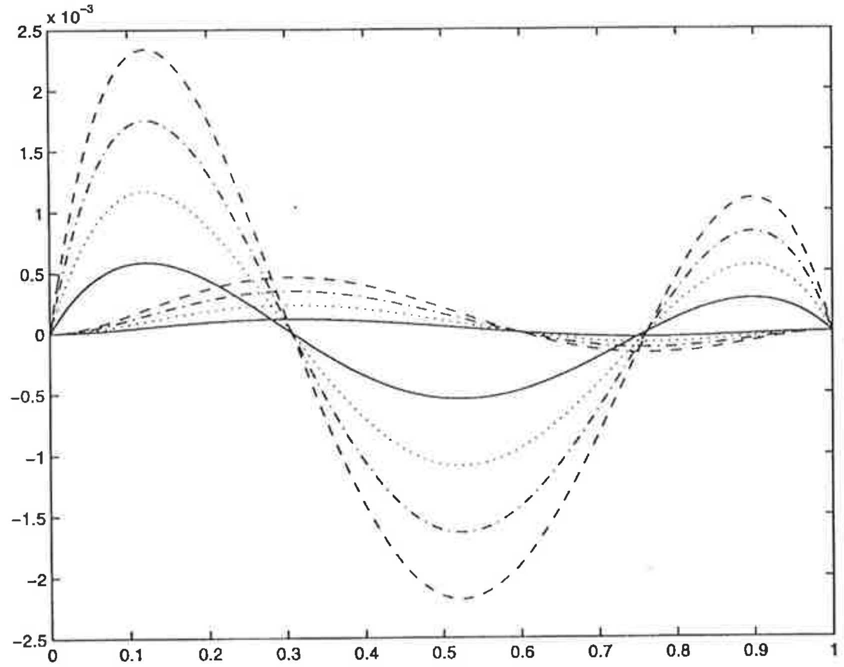


**Figure 7:** $g_1(y)$ & $g_1'(y)$ *with* $f_0'(1) = -0.1$ : ____ $\tilde{\alpha} = 0.2$, ....... $\tilde{\alpha} = 0.4$, -.- $\tilde{\alpha} = 0.6$, - - - $\tilde{\alpha} = 0.8$

Similarly, in Figures 5-7 when considering the functions $g_1(y)$ and $g_1'(y)$ we find that the wave number $\tilde{\alpha}$ has considerable affect on the curves. It appears that as the yield stress $\beta$ is gradually varied between zero and unity, and therefore the value of $f_0'(y) = -\psi_{0yy}(y)$, there is a shift in the size and shape of the left side and right side in the curve representing $g_1'(y)$. There seems to be a reversal in the location of peaks between the right side and left side. It is of interest to note that the points of inflection occur in exactly the same location when considering each of the respective graph of $g_1(y)$ and $g_1'(y)$. As the yield stress $\beta$ is gradually varied between zero and unity the points of inflection are shifted slightly to the right. The numerical values obtained for $f_0(y)$ and $f_0'(y)$, and $g_1(y)$ and $g_1'(y)$ are indicative of the validity of the perturbation analysis used throughout this research as indicated in equation (45). It is seen that the order in magnitude of $f_0(y)$ is very much greater than that of $g_1(y)$ as is suggested by the perturbation method. From the numerical calculations we find that the change in behaviour of the streamfunction occur depending on many parameters, including $K,\ \tilde{\alpha}, \alpha, \beta, R_e, and\ \varepsilon$. Just for the sake of understanding peristaltics, we have taken some basic values of the parameters, with $\varepsilon = 0.01$.

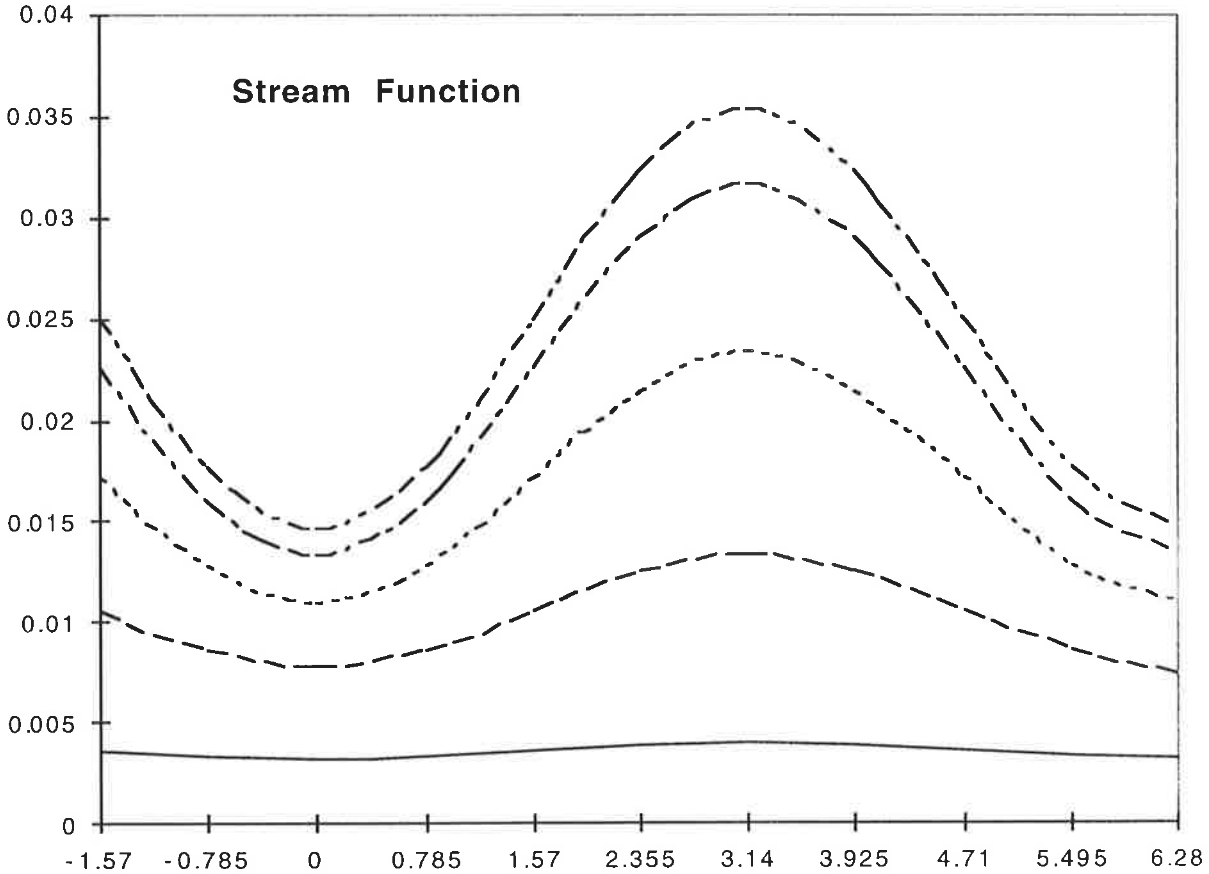


**Figure 8:** Plot of function $\psi$ vs $\bar{x}$ as given by equations (30-32, 45 & 56)

When we consider Figure (8), which is a plot of the function $\psi$ given by equations(13 and 30) and equations(45 and 54) and selecting $\varepsilon = 0.01$; for the case of high pressure gradient, with ___ representing $\psi_{0.1}$(y=0.1), ---- representing $\psi_{0.3}$, ..... representing $\psi_{0.5}$, -.-.-. representing $\psi_{0.7}$ and -..-..-representing $\psi_{0.9}$(y=0.9); it is found that the curves for streamfunction $\psi$ run parallel to the axis of the channel when considered near the axis(y=0.1), whereas considerable deformation is observed when they are considered near the boundary(y=0.9). Perhaps a possible explanation for this sort of behaviour of the streamlines can be given by considering the region as consisting of two parts - a central core and a boundary layer region. As the pressure gradient increases, we find that the streamfunction $\psi$ in the central region are more influenced by it, than by the motion of the boundary and hence the values for the streamfunction run parallel to the axis, while in the region near the boundary the flow is influenced by both the wave and the pressure gradient.

## CONCLUSION

In this research it is found that for the Casson model the governing partial differential equations are indeed extensive and complicated. If however we use the fact that the zeroth order perturbation in stream function is a function of the axial coordinate only, because the zeroth order axial pressure gradient is constant, we find that the Casson model may be quantitatively expressed as a Newtonian model (Figure 2 ).

It is found that in the zeroth order approximation in stream function that there is a dependence on the Casson coefficient of viscosity, yield stress, the density of the fluid, the wave speed and the dimensions of the channel.

When considering this approximation in ther zeroth order stream function, results show the difference between Newtonian(dashed line) and non-Newtonian(bold line) in Figure 2 seems to be slightly significant, and consistent with that of a Newtonian model, with slight anomalies at very low and very hiogh flow rates.

However, we see that for the first order in stream function the differential equation to be solved is complex, and the analytical solution derived from symbolic integration is more so.

The values for the first order in streamfunction are indicative of the perturbation method used and results in Figures(4-7) are consistent with that given the literature.

This modelling is appropriate as it may allow insight into the validity of the reduction of the complexity of modelling some non-Newtonian fluids like flow of urine in the ureter and blood flow in the blood vessels under certain physiological conditions.

**Ackowlegements**

We would like to express sincere thanks to Drs. Peter Gill, Hilary Booth and Paul McCann for their assistance during the preparation of this manuscript.

Also, sincere gratitude goes to the reviewers for their valuable advice most of which has been taken into consideration in the revised version of this paper.

**APPENDIX A**

$$f_0 = \frac{C_1\left(1-\frac{2B^2Ea_1}{\Omega b_1}\right)}{2}\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)\frac{1}{s}\left(\left(s^{\frac{1}{2}}y-\frac{h}{2\sqrt{s}}\right)erf\left(s^{\frac{1}{2}}y-\frac{h}{2\sqrt{s}}\right)+\frac{\exp\left(-\left(s^{\frac{1}{2}}y-\frac{h}{2s^{\frac{1}{2}}}\right)^2\right)}{\sqrt{\pi}}\right)$$

$$-\frac{C_1}{2}\frac{a_1\Omega^3}{2Eb_1 s}\left\{-\frac{\left(-\frac{\exp\left(-sy^2+hy\right)}{2s}+\frac{h\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)erf(\%1)}{4s^{\frac{3}{2}}}\right)}{2s}+\frac{1}{2hs}\left(-\frac{\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)erf(\%1)}{4s^{\frac{3}{2}}}+\frac{h\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)\left(\%1.erf(\%1)+\frac{\exp(\%1^2)}{\sqrt{\pi}}\right)}{4s^2}\right)+\frac{\sqrt{\pi}\exp\left(\frac{h^2}{4s}\right)\left(\%1.erf(\%1)+\frac{\exp(\%1^2)}{\sqrt{\pi}}\right)}{4s^2}\right\}$$

$$+\frac{C_1a_1B\sqrt{\pi}}{b_1}\exp\left(\frac{h^2}{4s}\right)\frac{1}{s^{\frac{3}{2}}}\left(\frac{erf(\%1)\%1^2}{2}+\frac{erf(\%1)h\%1}{2s^{\frac{1}{2}}}-2\frac{-\frac{\%1}{4\exp(\%1^2)}+\frac{\sqrt{\pi}erf(\%1)}{8}-\frac{h}{4\sqrt{s}\exp(\%1^2)}}{\sqrt{\pi}}\right)$$

$$-\frac{C_1a_1B\sqrt{\pi}}{b_1}\exp\left(\frac{h^2}{4s}\right)\frac{1}{s^{\frac{3}{2}}}\left(\frac{erf(\%1)\%1^2}{2}-\frac{-\frac{\%1}{2\exp(\%1^2)}+\frac{\sqrt{\pi}erf(\%)}{4}}{\sqrt{\pi}}+\frac{erf(\%1)}{2\sqrt{s}}\right)$$

$$+C_2\left(1-\frac{2B^2E(a+\frac{1}{2})}{\Omega b_1}\right)\sum_{k_2=0}^{\infty}\left(\begin{array}{l}\frac{1}{3}\exp\left(\frac{h^2}{4s}\right)(-1)^{k_2}2^{(-1-2k_2)}\left(-\frac{h}{\sqrt{s}}\right)^{(-1+2k_2)}y^{\frac{3}{2}}\frac{{}_2F_1([3/2,-1-2k_2],[5/2],2\frac{ys}{h})}{(\sqrt{s}(1/2+k_2)\Gamma(1+k_2))}\\+\frac{1}{3}2^{(-1-2k_2)}\exp\left(\frac{h^2}{4s}\right)h(-1)^{k_2}\left(-\frac{h}{\sqrt{s}}\right)^{(2k_2)}y^{\frac{3}{2}}\frac{{}_2F_1([1/2,-1-2k_2],[5/2],2\frac{ys}{h})}{(s(1/2+k_2)\Gamma(1+k_2))}\end{array}\right)$$

$$+C_2\left(\frac{\Omega^3(a+\frac{1}{2})}{2Eb_1}\right)\sum_{k_2=0}^{\infty}\left(\begin{array}{l}\frac{1}{7}\exp\left(\frac{h^2}{4s}\right)(-1)^{k_2}2^{(-1-2k_2)}\left(-\frac{h}{\sqrt{s}}\right)^{(-1+2k_2}y^{\frac{7}{2}}\frac{{}_2F_1([7/2,-1-2k_2],[9/2],2\frac{ys}{h})}{(\sqrt{s}(1/2+k_2)\Gamma(1+k_2))}\\+\frac{1}{7}2^{(-1-2k_2)}\exp\left(\frac{h^2}{4s}\right)h(-1)^{k_2}\left(-\frac{h}{\sqrt{s}}\right)^{(2k_2)}y^{\frac{7}{2}}\frac{{}_2F_1([5/2,-1-2k_2],[9/2],2\frac{ys}{h})}{(s(1/2+k_2)\Gamma(1+k_2))}\end{array}\right)$$

$$+C_2\left(\frac{2B(a+\frac{1}{2})}{b_1}\right)\sum_{k_2=0}^{\infty}\left(\begin{array}{l}\frac{1}{5}\exp\left(\frac{h^2}{4s}\right)(-1)^{k_2}2^{(-1-2k_2)}\left(-\frac{h}{\sqrt{s}}\right)^{(-1+2k_2}y^{\frac{5}{2}}\frac{{}_2F_1([5/2,-1-2k_2],[7/2],2\frac{ys}{h})}{(\sqrt{s}(1/2+k_2)\Gamma(1+k_2))}\\+\frac{1}{5}2^{(-1-2k_2)}\exp\left(\frac{h^2}{4s}\right)h(-1)^{k_2}\left(-\frac{h}{\sqrt{s}}\right)^{(2k_2)}y^{\frac{7}{2}}\frac{{}_2F_1([3/2,-1-2k_2],[7/2],2\frac{ys}{h})}{(s(1/2+k_2)\Gamma(1+k_2))}\end{array}\right)$$

$$+C_3+C_4$$

where $C_1, C_2, C_3, C_4$ are constants of integration determined by the boundary conditions in equation(66).

where $\Omega = D - 4sE$ and $\%1 = y\sqrt{s} - \frac{h}{2\sqrt{s}}$

## REFERENCES

1. A.V.Mernone and J.Mazumdar, Mathematical Modelling of peristaltic Transport of a Non-Newtonian Fluid, J.Australasian Physical and engineering Sciences in Medicine,21,3,126-140 (1998)

2. T.W. Latham, Fluid motion in a peristaltic pump, M.S Thesis M.I.T., Cambridge(1966).

3. F. Kiil, The function of the ureter and the renal pelvis, Philadelphia Saunders(1957)

4. S. Boyarsky, Surgical physiology of the renal pelvis, Monogr. Surg. Sci 1:173-213(1964)

5. A.H. Shapiro, Pumping and retrograde diffusion in peristaltic waves, Proc. Workshop Ureteral Reflux Children, Nat Acad. Sci. Wash.,D.C(1967)

6. Y.C. Fung and C.S. Yih, Peristaltic Transport, J. Appl. Mech. 35:669-75(1968)

7. A.H. Shapiro, M.Y. Jaffrin and S.L. Weinberg, Peristaltic pumping with long wavelengths at low Reynolds number. J. Fluid Mech. 37:799-825(1969)

8. P.S. Lykoudis, Peristaltic pumping; a bioengineering model. Proc. Workshop Hydrodynam. Upper Urinary Tract, Nat Acad. Sci. Wash.,D.C(1971)

9. S.L. Weinberg, M.Y. Jaffrin and A.H. Shapiro, A hydrodynamical model of ureteral function, Proc. Workshop Hydrodynam. Upper Urinary Tract, Nat Acad. Sci. Wash.,D.C(1971)

10. Y. C. Fung, Peristaltic pumping; a bioengineering model, Proc. Workshop Hydrodynam. Upper Urinary Tract, Nat Acad. Sci. Wash.,D.C(1971)

11. J.C Burns and T. Parkes, Peristaltic Motion, J. Fluid Mech. 29:731-43(1968)

12. M. Hanin, The flow through a channel due to traversely oscillating walls, Israel J. Technol. 6:67-71(1968)

13. C. Barton and S. Raynor, Peristaltic flow in tubes, Bull. Math. Biophys.30:663-80(1968)

14. N. Liron, A new look at peristalsis and its functions, Horizons Biochem. Biophys.5:161-82(1978)

15. M.G. Mank, Berechung der peristalticchen flussigkeits Forderong mit Method der Finiten Element. Dissertation, Hanover(1976)

16. A.H. Shapiro and T.W. Latham, On peristaltic pumping(abstact), Proc. of Annual Conference on Engineering in Medicine and Biology, Holden Day, San Francisco, 8:147(1966)

17. E.C. Eckstein, Experimental and theoretical pressure studies of peristaltic pumping. S.M Thesis, Dept. Mech. Eng. M.I.T., Cambridge(1970)

18. S.L. Weinberg, Theoretical and experimental treatment of peristaltic pumping and its relation to ureteral function. PhD Thesis M.I.T. Cambridge(1970)

19. Y.C. Fung and F.C.P. Yin, Comparison of theory and experiment in peristaltic transport, J.Fluid Mech. 47:93-112(1971)

20. T.K. Hung and T.D. Brown, Solid-particle motion in two-dimensional peristaltic flows, J.Fluid Mech. 73:77-96(1976)

21. K.K. Raju and R. Devanathan, Peristaltic motion of a non-Newtonian fluid, Rheol. Acta, 170-179(1972)

22. L.M.Srivastava and V.P. Srivastava, Peristaltic transport of Blood:Casson Model II, J. Biomechanics 17:No.11, 821-829(1984)

23. L.M. Srivastava, Peristaltic transport of a Casson fluid, Nig.J.Sci.Res. 1:71-82(1987)

24. A.M. Siddiqui, A. Provost and W.H. Schwarz, Peristaltic pumping of a second order fluid in a planar channel, Rheol. Acta 30:249-262(1991)

25. D. Tang and S. Rankin, Numerical and asymptotic solutions for peristaltic motion of non-linear viscous flows with elastic boundaries, SIAM J.Sci.Compt. 14:No.6,1300-1319(1993)

26. Y.C. Fung, Biomechanics, Mechanical Properties of Living Tissues, Springer-Verlag. N.Y(1981)

27. Y.C. Fung, Biomechanics, Motion, Flow, Stress and Growth, Properties of Living Tissue, Springer-Verlag. N.Y(1990)

28. Y.C. Fung, Biomechanics, Circulation, Springer-Verlag. N.Y(1984)

29. R.L. Batra and B. Jena, Flow of a Casson fluid in a slightly curved tube, Int. J.Eng.Sci 29:No.10,1245-1258(1991)

30. B.Das and R.L. Batra, Secondary flow of a Casson fluid in a slightly curved tube. Int. J. Non-linear Mechanics, 28,5, 567-577 (1993)

31. J. Elshehawey, et al, Peristaltic motion of Generalised Newtonian fluid in a non-uniform channel, J. Phys. Soc. Japan 67, 434-440, (1998)

32. W. P. Walawender et al, An approximate Casson Fluid model for tube flow of blood. Biorheology, 12, 111-119 (1975)

33. E. Kreyszig, Advanced Engineering Mathematics, 1979, Fourth Edition, Wiley & Sons

34. A.M. El Misery, E.F. Elshehawey and A.A. Hakeem, Peristaltic Motion of an Incompressible Generalised Newtonian Fluid in a Planar Channel, J. Phys. Soc. Japan, 65, 11 pp3524-3529 (1996)

35. L.M. Srivastava, V.P. Srivastava and S.N. Sinha, Peristaltic Transport of a Physiological Fluid Part I Flow in non-Uniform Geometry, Biorheology, 29 153-166 (1983)

36. L.M. Srivastava and V.P. Srivastava, Peristaltic transport of a power-law fluid: application to ductus efferentes of the reproductive tract, Rheol. Acta: 27. 428-433, (1988)

37. L. Liethold, The Calculus with Analytic Geometry,3rd Edition

38. A.D. Polyanin & V.F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations

# Biomathematical Modelling of Physiological Fluids using a Casson Fluid with emphasis to Peristalsis.

A.V. Mernone[1], J.N Mazumdar

Department of Applied Mathematics, Adelaide University, Australia 5005

**Abstract**

In this paper, the peristaltic flow of rheologically complex physiological fluids when modelled by a non-Newtonian Casson fluid in a two-dimensional channel is considered. Of interest is the difference between peristaltic transport of Newtonian and non-Newtonian fluids. A perturbation series method of solution of the stream function in amplitude ratio is sought. It is found that Newtonian fluid is an important sub-class of non-Newtonian fluids that may adequately represent some physiological phenomena. It is shown that for a Casson fluid, when certain simplifications and approximations are made in the most generalised form of constitutive equation, the fluid may be adequately represented as an improvement of a Newtonian fluid.

---

* Corresponding Author: A.Mernone
email: amernone@maths.adelaide.edu.au

**Keywords** Mathematical Modelling, Casson Fluid, Peristalsis, Perturbation Series Method

## INTRODUCTION

As mentioned in Mernone and Mazumdar[33] peristalsis is the phenomenon in which a circumferential progressive wave of contraction or expansion (or both) propagates along a tube. If the

tube is long enough, one might see several identical waves moving along the tube simultaneously. Peristalsis appears in many organisms and a variety of organs.

Peristalsis is now well known to physiologists to be one of the major mechanisms for fluid transport in many biological systems. In particular, peristaltic mechanisms may be involved in urine transport from the kidney to the bladder through the ureter, the movement of chyme in the gastrointestinal tract, the transport of spermatozoa in the ductus efferentes of the male reproductive tract and in the cervical canal, the movement of ova in the fallopian tubes, the transport of lymph in the lymphatic vessels and in the vascomotion in small blood vessels.

These flows also provide efficient means for sanitary fluid transport

and are thus exploited in industrial peristaltic pumping and medical devices, for example, industrial applications are mechanical roller pumps using viscous fluids in the printing industry and the peristaltic transport of noxious fluid in the nuclear industry. In addition, peristaltic pumping occurs in many practical applications involving biomedical systems. Many modern medical devices have been designed on the principle of peristaltic pumping to transport fluids without internal moving parts, for example, the blood in the heart-lung machine.

The main motivation for any mathematical analysis of physiological fluid flows is to ultimately have a better understanding of the particular flow being modelled. If there is similarity between the results obtained from the analysis and experimental and clinical data, then the mechanism of flow can at least be explained. Because peristalsis is evident in many physiological flows, an accurate mathematical study can help explain the major contributing factors to many flows in the human body. When comparing results between the mathematical model and the experimental and clinical data it is desirable that the data obtained from experimental research be as close as possible to the actual physiological parameter being analysed. That is to say, it may be necessary to take into account the effect the measuring instrument or device or procedure has on the data obtained.

The study of the mechanisms of peristalsis, in both mechanical and physiological situations, has recently become the subject of scientific research. Since the first investigation of Latham [1] several theoretical and experimental attempts have been made to understand peristaltic action in different situations. Interest in peristaltic pumping has been quite recently stimulated by its relevance to ureteral function. As reliable and accurate urometric measurements became available through the work of Kiil[2] and Boyarsky[3] several hydrodynamic models of ureteral function invoking peristalsis were attempted. The earliest models, Shapiro[4], Fung[5] and Shapiro, Jaffrin and Weinberg[6] were idealised and represented the peristalsis by an infinite train of sinusoidal waves in a two-dimensional channel; thus they could pretend to only a qualitative relationship with the ureter. These models concerned themselves, in part, with offering an explanation of the biologically and medically important phenomenon of 'reflux'. One manifestation of this reflux is that bacteria sometimes travel from the bladder to the kidney against the mean urine flow. A similar phenomenon has been observed in the small bowel. These observations are puzzling because the travel times are too small to be explained by diffusion and also because retrograde peristaltic waves have not usually been observed. Later, Lykoudis[7]and Weinberg, Jaffrin and Shapiro[8] proposed models that represent ureteral waves more realistically. Fung[9] investigated the coupling between the forces of fluid-mechanical origin and the dynamics of the ureteral muscle. Some of these models showed that observed urometric pressure pulses and flow rates could be accounted for by assuming internal dimensions of the ureter which seem physiologically plausible. But ureteral physiology has not been the only motivation for the study of peristalsis.

Burns and Parkes[10] and Hanin[11] contributed to the theory of peristaltic pumping without reference to physiological applications. Barton and Raynor[12] made a calculation based on peristalsis theory of the time required for chyme to traverse the small intestine and found that this calculation compared favourably with observed values. In addition, Fung[9] studied peristaltic flow taking muscle action in the tube wall into account. Some new examples of peristalsis were given in Liron[13]. Considerable experimental investigations of peristaltic pumping have also been undertaken, for example, Latham[1], Mank[14], Shapiro & Latham[15], Eckstein[16], Weinberg[17] Weinberg[8] et al, Yin & Fung[18] Hung & Brown[19]. Most of the theoretical investigations have been carried out by assuming blood and other physiological fluids behave like a Newtonian fluid. Although this approach may provide a satisfactory understanding of the peristaltic mechanism in the ureter, it fails to provide a satisfactory model when the peristaltic mechanism is involved in small blood vessels, lymphatic vessels, intestine, ductus efferentes of the male reproductive transport and in the transport of spermatozoa in the cervical canal. It has now been accepted that most of the physiological fluids behave like non-Newtonian fluids. But it appears that no quantitative rigorous attempt has been made to understand the problem of a non-Newtonian fluid before the investigation of Raju & Devanathan[20] in the case of small wave amplitude. Subsequently, Srivastava & Srivastava[21] investigated the problem of peristaltic transport of blood assuming a single layered Casson fluid and ignoring the presence of a peripheral layer. Later on, Srivastava[22] considered the axisymmetric flow of a Casson fluid in a circular non uniform tube. More recently, Siddiqui, Provost & Schwarz[23] investigated peristaltic motion of a non-Newtonian fluid modelled with a constitutive equation for a second order fluid for the case of a planar channel. A perturbation series was used representing parameters such as curvature, inertia and the non-Newtonian character of the fluid. Tang and Mankin[24] proposed a mathematical model for peristaltic motion of a nonlinear viscous flow where they used an iterative method to solve a free boundary problem. Das & Batra[29] studied the fully developed, steady flow of a Casson fluid through a curved tube for small values of Dean number. A plug core formation region at the centre is considered where the shear stress in not sufficient to exceed the yield value. Elshehawey[30]et al consider the problem of peristaltic transport of a non-Newtonian (Carreau) fluid in a non-uniform channel under zero Reynolds number with long wavelength approximation. The problem is formulated using a perturbation expansion in terms of a variant of Weissenberg number. They find that pressure rise and friction force are smaller than the corresponding values in the case of uniform geometry. However, in the present paper we propose to study peristaltic transport of physiological fluids in a planar channel using the most generalised form of constitutive equation, for Casson fluid, as given by Fung[25]. The final analysis is done by using a perturbation method in the same way as was done in our previous paper, Mernone & Mazumdar[33]. To the author's knowledge the use of this generalised equation has not been considered previously in the literature.

## PROBLEM FORMULATION

**Dimensionless Variables in a Two dimensionalChannel**

$$x' = \frac{x}{d} \quad y' = \frac{y}{d} \quad u' = \frac{u}{c} \quad v' = \frac{v}{c}$$

$$R_e = \frac{cd}{\nu} \quad \nu = \frac{\mu}{\rho} \quad \psi' = \frac{\psi}{cd} \quad \alpha = \frac{2\pi d}{\lambda}$$

$$\varepsilon = \frac{A}{d} \quad t' = \frac{ct}{d} \quad G' = \frac{G}{d} \quad p' = \frac{p}{\rho c^2}$$

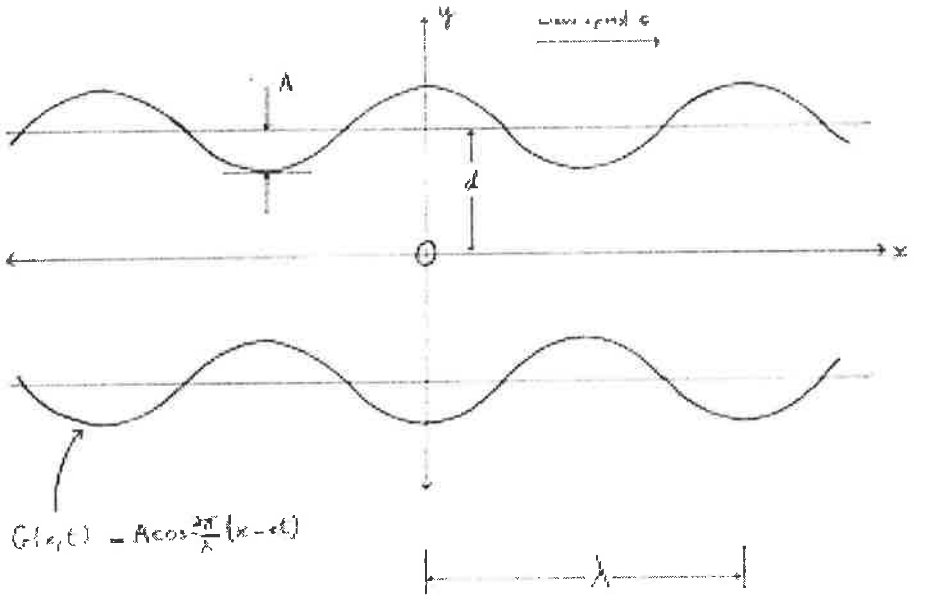


**Figure 1**: *Peristaltic flow in a Two-dimensional Channel*

**Statement of Problem**

Consider the peristaltic motion of a non-Newtonian fluid, modelled as a Casson fluid, which is viscous and incompressible in a two-dimensinal channel, where, d, is the undeformed width of the channel and the channel is considered infinitely long. A, represents the amplitude of the sinusiodal waves travelling along the channel, $\lambda$, is the wavelength and they are travelling at velocity, c (Figure.1). A rectangular co-ordinate system is chosen for the channel with x along the centre line and y normal to it. Let u and v be the longitudinal and transverse velocity components, respectively. It is assumed that an infinite train of sinusiodal waves progresses along the walls in the x direction. The height of the wall for peristaltic flow at time t is defined by,

$$G(x,t) = A\cos\frac{2\pi}{\lambda}(x-ct) \qquad (1)$$

We assume that there is no motion of the wall in the longitudinal direction(extensible or elastic wall).

For the case of peristaltic pumping of a Casson fluid in a planar channel the stress-strain relationship in tensor format is given by Fung[25] as

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(J_2)V_{ij} \qquad (2)$$

where

$$\mu(J_2) = \left[(\eta^2 J_2)^{\frac{1}{4}} + 2^{-\frac{1}{2}}\tau_y^{\frac{1}{2}}\right]^2 J_2^{-\frac{1}{2}}$$

$$= \left[\eta^{\frac{1}{2}} + 2^{-\frac{1}{2}}\tau_y^{\frac{1}{2}} J_2^{-\frac{1}{4}}\right]^2 \qquad (3)$$

$$= \left[\alpha + \beta J_2^{-\frac{1}{4}}\right]^2 = \mu(say)$$

and

$$\alpha = \eta^{\frac{1}{2}} \quad : \quad \beta = 2^{-\frac{1}{2}}\tau_y^{\frac{1}{2}} \qquad (4)$$

*where* $\eta$ *is the Casson coefficient of viscosity,*
$\tau_y$ *is the yield stress*

Here,

$$V_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (5)$$

where

$$J_2 = \frac{1}{2} V_{ij} V_{ij} = \frac{1}{2}\left(V_{11}{}^2 + V_{22}{}^2 + 2V_{12}{}^2\right) \quad (6)$$

$$V_{11} = \frac{\partial u}{\partial x},\ V_{22} = \frac{\partial v}{\partial y},\ V_{12} = V_{21} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

**Mathematical Modelling of a Casson Fluid in a Two-Dimensional Channel**

If we use the basic equations for continuity and momentum we have

$$div\underline{q} = 0 \quad \textit{continuity equation} \quad (7)$$

$$\rho \frac{Dq_i}{Dt} = \frac{\partial}{\partial x_j}\sigma_{ij} \ \textit{momentum equation} \quad (8)$$

Substituting equations (2-6) into equation (8) our momentum equations become, after defining a stream function, as $u = \psi_y$ *and* $v = -\psi_x$

$$\rho\left(\psi_{yt} + \psi_y \psi_{xy} - \psi_x \psi_{yy}\right) =$$

$$-\frac{\partial p}{\partial x} + 2\mu_x \psi_{xy} + \mu_y\left(\psi_{yy} - \psi_{xx}\right) + \mu \nabla^2 \psi_y$$

and (9)

$$\rho\left(-\psi_{xt} - \psi_y \psi_{xx} + \psi_x \psi_{xy}\right) =$$

$$-\frac{\partial p}{\partial y} - 2\mu_y \psi_{xy} + \mu_x\left(\psi_{yy} - \psi_{xx}\right) - \mu \nabla^2 \psi_x$$

(10)

where $\mu_x = \frac{\partial}{\partial x}\left[\mu\left(J_2\right)\right]$ *and* $\mu_y = \frac{\partial}{\partial y}\left[\mu\left(J_2\right)\right]$

**A Method of Solution of a Casson Fluid in a Two-Dimensional Channel**

Expressing stream function, $\psi$, pressure, p, and $\mu$ expressed as a series in terms of amplitude ratio $\varepsilon = \frac{A}{d}$, where A is the amplitude and ,d, is the undeformed width of the channel, (Figure 1), we have

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots \tag{11}$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots \tag{12}$$

$$\mu = \mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \dots \tag{13}$$

Where it is assumed that $\psi_0$ is a function of y only, ie, $\psi_0 = \psi_0(y)$, because of zeroth order axial pressure gradient, we finally obtain from equations (9 & 11-13) after collecting coefficents of $\varepsilon^0$.

$$\frac{\partial p_0}{\partial x} = 2\mu_{0x}\psi_{0yx} + \mu_{0y}\psi_{0yy} -$$
$$\mu_{0y}\psi_{0xx} + \mu_0\psi_{0xxy} + \mu_0\psi_{0yyy}$$

$$\frac{\partial p_0}{\partial x} = \mu_{0y}\psi_{0yy} + \mu_0\psi_{0yyy} \tag{14}$$

Therefore

$$\frac{\partial p_0}{\partial x} = \frac{\partial}{\partial y}\left(\mu_0\psi_{0yy}\right) \tag{15}$$

We now need to find the zeroth order expression for $\mu_0 = \mu\left(J_2\right)_0$

From equation (6) and expanding and substituting we have

$$J_2 = \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{2}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\psi_{xy}{}^{2} + \psi_{xy}{}^{2} + \frac{1}{2}\left(\psi_{yy} - \psi_{xx}\right)^{2}\right\}$$

$$= \left\{\psi_{xy}{}^{2} + \frac{1}{4}\left(\psi_{yy} - \psi_{xx}\right)^{2}\right\} \qquad \textbf{(16)}$$

Therefore we have, after introducing equations (11, 13 and 16)

$$\mu\left(J_2\right) =$$

$$\left\{\alpha + \beta\begin{bmatrix}\frac{1}{4}\left(\psi_{0yy}{}^{2} + 2\varepsilon\psi_{0yy}(\psi_{1yy} - \psi_{1xx}\right) + \\ O(\varepsilon^2)\end{bmatrix}^{-\frac{1}{4}}\right\}^{2} \qquad \textbf{(17)}$$

Neglecting $O(\varepsilon^2)$ and higher in equation (17) and expanding we have

$$\mu\left(J_2\right) \approx$$

$$\left\{\alpha + \beta\left(\frac{1}{4}\right)^{-\frac{1}{4}}\psi_{0yy}{}^{-\frac{1}{2}}\left(1 - \frac{\varepsilon}{2}\psi_{0yy}{}^{-1}(\psi_{1yy} - \psi_{1xx})\right)\right\}^{2} \qquad \textbf{(18)}$$

Expanding equation(18) and collecting terms in amplitude ratio for the first two terms and using eqaution(13) we obtain equations for $\mu_0 = \mu\left(J_2\right)_0$ and $\mu_1 = \mu\left(J_2\right)_1$ as

$$\mu_0 = \alpha^2 + 2\sqrt{2}\alpha\beta\psi_{0yy}{}^{-\frac{1}{2}} + 2\beta^2\psi_{0yy}{}^{-1} \qquad \textbf{(19)}$$

$$\mu_1 = -\alpha\beta\sqrt{2}\psi_{0yy}{}^{-\frac{3}{2}}\left(\psi_{1yy} - \psi_{1xx}\right) - $$
$$2\beta^2\psi_{0yy}{}^{-2}\left(\psi_{1yy} - \psi_{1xx}\right) \qquad \textbf{(20)}$$

Solving equation(15) by using equation(19) and applying the symmetry boundary condition

$\psi_{0yy}(0) = 0$ we have

$$Ky + A = \mu_0 \psi_{0yy}$$

*where*

$$K = \rho\sqrt{c^3 d}\,\frac{\partial p_0}{\partial x} \qquad \textbf{(21)}$$

*and* $A = constant\ of\ integration = 2\beta^2$

Our equation to solve for $\psi_0(y)$ then becomes

$$\alpha^2 \psi_{0yy} + 2\sqrt{2}\alpha\beta\psi_{0yy}^{\frac{1}{2}} - Ky = 0 \qquad \textbf{(22)}$$

If we set

$$\psi_{0yy} = W^2 \quad ,ie, \quad \psi_{0yy}^{\frac{1}{2}} = W \qquad \textbf{(23)}$$

Equation(22) becomes a quadratic in W as

$$\alpha^2 W^2 + 2\sqrt{2}\alpha\beta W - Ky = 0 \qquad \textbf{(24)}$$

whose solution is given by

$$W = -\sqrt{2}\frac{\beta}{\alpha} \pm \frac{1}{\alpha}\sqrt{2\beta^2 + Ky} \qquad \textbf{(25)}$$

Using equation(23) and equation(25)

$$\psi_{0yy} = \left(-\sqrt{2}\frac{\beta}{\alpha} \pm \frac{1}{\alpha}\sqrt{2\beta^2 + Ky}\right)^2 \qquad \textbf{(26)}$$

But the symmetry boundary condition $\psi_{0yy}(0) = 0$ demands only the positive sign to be valid,

therefore

$$\psi_{0yy} = \left(-\sqrt{2}\frac{\beta}{\alpha} + \frac{1}{\alpha}\sqrt{2\beta^2 + Ky}\right)^2 \qquad \textbf{(27)}$$

Integrating equation(27) twice we obtain

$$\psi_0(y) = \frac{2\beta^2}{\alpha^2}y^2 + \frac{K}{6\alpha^2}y^3 - \frac{8\sqrt{2}\beta}{15K^2\alpha^2}\left(2\beta^2 + Ky\right)^{\frac{5}{2}} + Ay + B \qquad \textbf{(28)}$$

Where A and B are constants of integration.

Using the following boundary conditions $\psi_{0y}(1) = 0 \quad and \quad \psi_0(0) = 0$

we find

$$A = \frac{4\sqrt{2}\beta}{3\alpha^2 K}\left(2\beta^2 + K\right)^{\frac{3}{2}} - \frac{4\beta^2}{\alpha^2} - \frac{K}{2\alpha^2} \qquad \textbf{(29)}$$

$$B = \frac{8\sqrt{2}\beta}{15K^2\alpha^2}\left(2\beta^2\right)^{\frac{5}{2}} \qquad \textbf{(30)}$$

If we set $\beta \to 0 \Rightarrow \tau_y \to 0$ from equation(4) we obtain the Newtonian case in the form

$$\psi_0(y) = -\frac{K}{2\alpha^2}\left(y - \frac{y^3}{3}\right)$$

which is as recorded in the literature, Fung[6]. We now seek to determine the dimensionless pressure rise, $\Delta p_0$, where

$$\Delta p_0 = \int_0^1 \frac{\partial p_0}{\partial x}dx \qquad \textbf{(31)}$$

The flow rate, q, is given by

$$q = \int_0^G u\,dy = \int_0^G \frac{\partial \psi}{\partial y}dy \qquad \textbf{(32)}$$

$$where \quad G = \varepsilon\cos\alpha(x - t)$$

hence from equations (21, 28,29, 30, 31 and 32)

$$\Delta p_0 = \frac{1}{\rho\sqrt{c^3 d}}\int_0^1 \frac{6q\alpha^2 - 12\beta^2\varepsilon^3\cos^3 x}{\varepsilon^3\cos^3 x - 3\varepsilon\cos x}dx \qquad \textbf{(33)}$$

The solution to equation(33) is found by making the substitution

$$z = \tan\frac{1}{2}x$$

*therefore* **(34)**

$$\cos x = \frac{1-z^2}{1+z^2} \quad \sin x = \frac{2z}{1+z^2} \quad dx = \frac{2}{1+z^2}dz$$

Seperating integrands and using the method of partial fractions we obtain three integrals to be solved; they are

$$\frac{12q\alpha^2}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546}\frac{A_1 z + B_1}{\left(z^2 - (\theta+\gamma)\right)}dz\right] + \frac{12q\alpha^2}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546}\frac{C_1 z + D_1}{\left(z^2 - (\theta-\gamma)\right)}dz\right] + \frac{12q\alpha^2}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546}\frac{E_1 z + F_1}{\left(1 - z^2\right)}dz\right] \qquad \textbf{(35)}$$

$$-\frac{12\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546}\frac{A_2 z + B_2}{\left(z^2 - (\theta+\gamma)\right)}dz\right] - \frac{12\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546}\frac{C_2 z + D_2}{\left(z^2 - (\theta-\gamma)\right)}dz\right] \qquad \textbf{(36)}$$

$$+\frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546}\frac{A_3 z+B_3}{\left(z^2-(\theta+\gamma)\right)}dz\right]$$

$$+\frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546}\frac{C_3 z+D_3}{\left(z^2-(\theta-\gamma)\right)}dz\right] \quad \textbf{(37)}$$

$$+\frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3 d}}\left[\int_0^{0.546}\frac{E_3 z+F_3}{\left(1+z^2\right)}dz\right]$$

where

$$\theta=\frac{b}{a} \quad \gamma=\sqrt{\frac{b^2}{a^2}-1} \quad \textbf{(38)}$$

$$a=\varepsilon^3-3\varepsilon \quad b=\varepsilon^3+3\varepsilon$$

Clearly, in our case, $a$ *and* $\theta$ are always negative and $b$ *and* $\gamma$ are always positive.

The coefficients $A_i, B_i, C_i, D_i, E_i, F_i, i=1,2,3$ are found by equating coefficients in equations (35-37) after substituting $\cos^2 x = 1-\sin^2 x$ in equation(33) and applying Gauss-Jordan elimination, and are found to be

$$A_1=C_1=E_1=0$$

$$F_1=-\frac{\{(-\theta+\gamma+1)-2\theta(2\gamma+2)(-\theta+\gamma-1)\}}{\{(-1+\theta+\gamma)(\theta-\gamma)+2\theta(-(\theta+\gamma)(\theta-\gamma-1)+2\theta)(2\gamma+2)\}} \quad \textbf{(39)}$$

$$D_1=-\frac{(-\theta+\gamma+1)}{2\theta}+\frac{(-1+\theta+\gamma)(\theta-\gamma)}{2\theta}F_1$$

$$B_1=-1-D_1+F_1$$

$$A_2=C_2=0$$

$$D_2=-\frac{(1+\theta-\gamma)}{2\gamma} \quad \textbf{(40)}$$

$$B_2=\frac{(\theta+\gamma+1)}{2\gamma}$$

$$A_3 = C_3 = E_3 = 0$$
$$F_3 = -\frac{1}{(1+\theta+\gamma)(1-(\theta-\gamma))}$$
$$D_3 = \frac{1}{2\gamma}\left(-1+\frac{1}{(1-(\theta-\gamma))}\right) \qquad \textbf{(41)}$$
$$B_3 = -D_3 - F_3$$

The solution to equation(33) after considering equation(35-37) takes the form,

$$\Delta p_0 = \frac{12q\alpha^2}{\rho\sqrt{c^3}d}\begin{bmatrix} \frac{B_1}{2\sqrt{\theta+\gamma}}\ln\left(\frac{z-\sqrt{\theta+\gamma}}{z+\sqrt{\theta+\gamma}}\right)+ \\ \frac{D_1}{2\sqrt{\theta-\gamma}}\ln\left(\frac{z-\sqrt{\theta-\gamma}}{z+\sqrt{\theta-\gamma}}\right)+ \\ \frac{F_1}{2}tahn^{-1}z \end{bmatrix} - \frac{12\beta^2\varepsilon^3}{\rho\sqrt{c^3}d}\begin{bmatrix} \frac{B_2}{2\sqrt{\theta+\gamma}}\ln\left(\frac{z-\sqrt{\theta+\gamma}}{z+\sqrt{\theta+\gamma}}\right)+ \\ \frac{D_2}{2\sqrt{\theta-\gamma}}\ln\left(\frac{z-\sqrt{\theta-\gamma}}{z+\sqrt{\theta-\gamma}}\right) \end{bmatrix} + \frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3}d}\begin{bmatrix} \frac{B_3}{2\sqrt{\theta+\gamma}}\ln\left(\frac{z-\sqrt{\theta+\gamma}}{z+\sqrt{\theta+\gamma}}\right)+ \\ \frac{D_3}{2\sqrt{\theta-\gamma}}\ln\left(\frac{z-\sqrt{\theta-\gamma}}{z+\sqrt{\theta-\gamma}}\right)+ \\ \frac{F_3}{2}\tan^{-1}z \end{bmatrix} \qquad \textbf{(42)}$$

if we assume $\sqrt{\theta-\gamma} = i\delta_1 \quad \sqrt{\theta+\gamma} = i\delta_2$

since $\theta-\gamma$ *and* $\theta+\gamma$ s negative, then we have

$$\ln\left|\frac{z-i\delta_j}{z+i\delta_j}\right| = -2i\tan^{-1}\left(\frac{\delta_j}{z}\right)$$

*where* $i=\sqrt{-1}$ $j=1,2$ **(43)**

$$\Delta p_0 = -\frac{12q\alpha^2}{\rho\sqrt{c^3}d}\left[\begin{array}{l}\frac{B_1}{\delta_2}\tan^{-1}\left(\frac{\delta_2}{z}\right)+\\ \frac{D_1}{\delta_1}\tan^{-1}\left(\frac{\delta_1}{z}\right)-\\ \frac{F_1}{2}\tanh^{-1}z\end{array}\right]$$
$$+\frac{12\beta^2\varepsilon^3}{\rho\sqrt{c^3}d}\left[\frac{B_2}{\delta_2}\tan^{-1}\left(\frac{\delta_2}{z}\right)+\frac{D_2}{\delta_1}\tan^{-1}\left(\frac{\delta_1}{z}\right)\right] \quad \textbf{(44)}$$
$$-\frac{96\beta^2\varepsilon^3}{\rho\sqrt{c^3}d}\left[\begin{array}{l}\frac{B_3}{\delta_2}\tan^{-1}\left(\frac{\delta_2}{z}\right)+\\ \frac{D_3}{\delta_1}\tan^{-1}\left(\frac{\delta_1}{z}\right)-\\ \frac{F_3}{2}\tan^{-1}z\end{array}\right]$$

Representation of pressure rise vs flow rate, Figure(2) and Figure(3) are obtained for varying values of amplitude ratio.

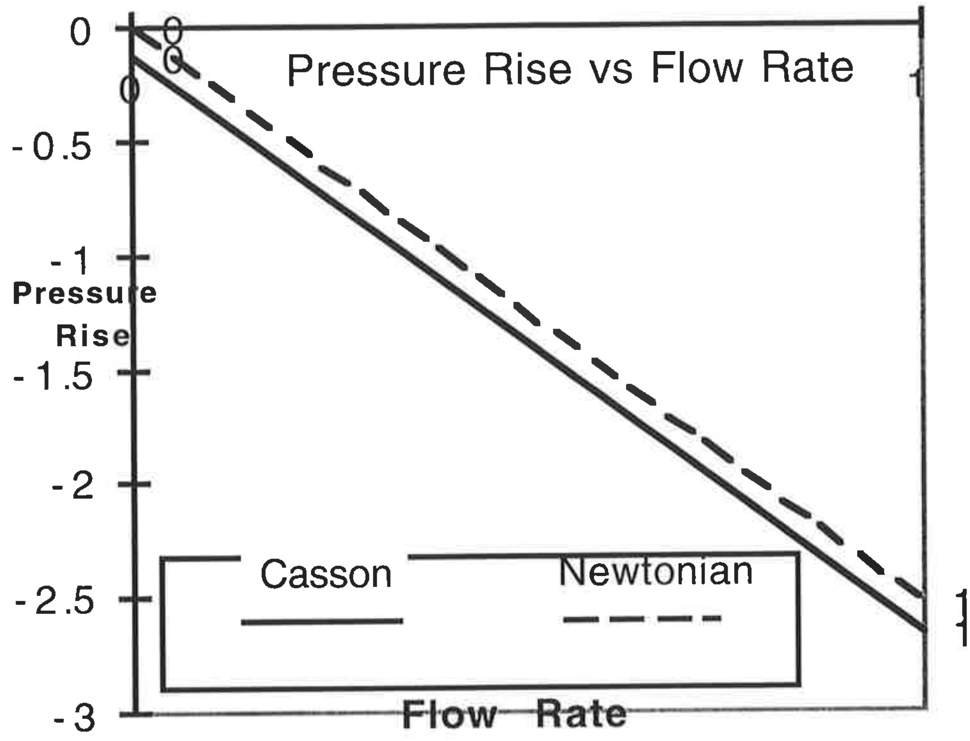


**Figure 2**:*Pressure Rise vs Flow Rate,* $\varepsilon=0.2$

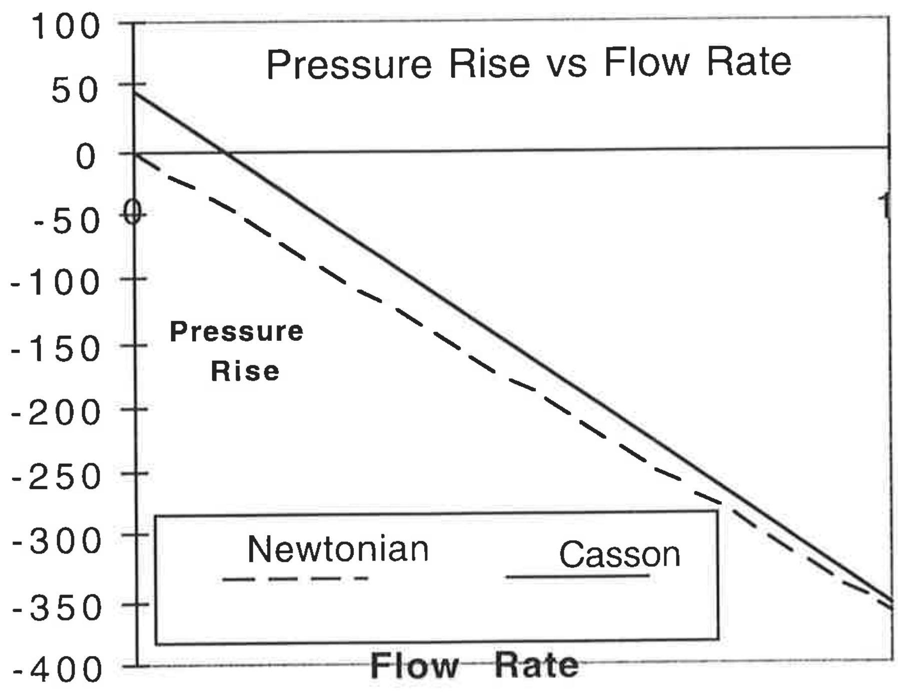


**Figure 3**:*Pressure Rise vs Flow Rate,* $\varepsilon=0.8$

## CONCLUSION

It should be emphasised that this research represents a fluid dynamical biomathematical model of the phenomena of peristalsis whereby a non-Newtonian Casson fluid is travelling through a channel with sinusoidally varying waves travelling along the upper and lower boundary of the channel. It is found that for the Casson model the governing partial differential equations are indeed extensive and complicated. If however the fact that the zeroth order perturbation in stream function is a function of the axial coordinate only, because the zeroth order axial pressure gradient is constant, we find that the Casson model may be quantitatively expressed as a Newtonian model.

Figures 2 and 3 are graphs of Pressure Rise vs Flow Rate for the case of amplitude ratio, $\varepsilon$=0.2 and 0.8, respectively.

It is found that in the zeroth order approximation in stream function that there is a dependence on the Casson coefficient of viscosity, yield stress, the density of the fluid, the wave speed and the dimensions of the channel.

When considering this approximation in stream function, results show for lower values of amplitude ratio the difference between Newtonian(dashed line) and non-Newtonian(bold line) seems to be slightly significant. However, for higher values of amplitude ratio, the pressure gradient is noticeably affected by the non-Newtonian character of the fluid. The effect appears to increase as the occlusion gets larger.

This modelling is appropriate as it may allow insight into the validity of the reduction of the complexity of modelling some non-Newtonian fluids like flow of urine in the ureter and blood flow in the blood vessels under certain physiological conditions.

The analysis here is restricted to the zeroth order approximation. The first order approximation using the most generalised form of constitutive equation for a Casson fluid is under active study.

**Acknowledgements**

The authors wish to express gratitude to the reviewers for their constructive and valuable comments.

**REFERENCES**

1. T.W. Latham, Fluid motion in a peristaltic pump, M.S Thesis M.I.T., Cambridge(1966).
2. F. Kiil, The function of the ureter and the renal pelvis, Philadelphia Saunders(1957)
3. S. Boyarsky, Surgical physiology of the renal pelvis, Monogr. Surg. Sci 1:173-213(1964)
4. A.H. Shapiro, Pumping and retrograde diffusion in peristaltic waves, Proc. Workshop Ureteral Reflux Children, Nat Acad. Sci. Wash.,D.C(1967)
5. Y.C. Fung and C.S. Yih, Peristaltic Transport, J. Appl. Mech. 35:669-75(1968)
6. A.H. Shapiro, M.Y. Jaffrin and S.L. Weinberg, Peristaltic pumping with long wavelengths at low Reynolds number. J. Fluid Mech. 37:799-825(1969)
7. P.S. Lykoudis, Peristaltic pumping; a bioengineering model. Proc. Workshop Hydrodynam. Upper Urinary Tract, Nat Acad. Sci. Wash.,D.C(1971)
8. S.L. Weinberg, M.Y. Jaffrin and A.H. Shapiro, A hydrodynamical model of ureteral function, Proc. Workshop Hydrodynam. Upper Urinary Tract, Nat Acad. Sci. Wash.,D.C(1971)
9. Y. C. Fung, Peristaltic pumping; a bioengineering model, Proc. Workshop Hydrodynam. Upper Urinary Tract, Nat Acad. Sci. Wash.,D.C(1971)
10. J.C Burns and T. Parkes, Peristaltic Motion, J. Fluid Mech. 29:731-43(1968)

11. M. Hanin, The flow through a channel due to traversely oscillating walls, Israel J. Technol. 6:67-71(1968)

12. C. Barton and S. Raynor, Peristaltic flow in tubes, Bull. Math. Biophys.30:663-80(1968)

13. N. Liron, A new look at peristalsis and its functions, Horizons Biochem. Biophys.5:161-82(1978)

14. M.G. Mank, Berechung der peristalticchen flussigkeits Forderong mit Method der Finiten Element. Dissertation, Hanover(1976)

15. A.H. Shapiro and T.W. Latham, On peristaltic pumping(abstact), Proc. of Annual Conference on Engineering in Medicine and Biology, Holden Day, San Francisco, 8:147(1966)

16. E.C. Eckstein, Experimental and theoretical pressure studies of peristaltic pumping. S.M Thesis, Dept. Mech. Eng. M.I.T., Cambridge(1970)

17. S.L. Weinberg, Theoretical and experimental treatment of peristaltic pumping and its relation to ureteral function. PhD Thesis M.I.T. Cambridge(1970)

18. Y.C. Fung and F.C.P. Yin, Comparison of theory and experiment in peristaltic transport, J.Fluid Mech. 47:93-112(1971)

19. T.K. Hung and T.D. Brown, Solid-particle motion in two-dimensional peristaltic flows, J.Fluid Mech. 73:77-96(1976)

20. K.K. Raju and R. Devanathan, Peristaltic motion of a non-Newtonian fluid, Rheol. Acta, 170-179(1972)

21. L.M.Srivastava and V.P. Srivastava, Peristaltic transport of Blood:Casson Model II, J. Biomechanics 17:No.11, 821-829(1984)

22. L.M. Srivastava, Peristaltic transport of a Casson fluid, Nig.J.Sci.Res. 1:71-82(1987)

23. A.M. Siddiqui, A. Provost and W.H. Schwarz, Peristaltic pumping of a second order fluid in a planar channel, Rheol. Acta 30:249-262(1991)

24. D. Tang and S. Rankin, Numerical and asymptotic solutions for peristaltic motion of non-linear viscous flows with elastic boundaries, SIAM J.Sci.Compt. 14:No.6,1300-1319(1993)

25. Y.C. Fung, Biomechanics, Mechanical Properties of Living Tissues, Springer-Verlag. N.Y(1981)

26. Y.C. Fung, Biomechanics, Motion, Flow, Stress and Growth, Properties of Living Tissue, Springer-Verlag. N.Y(1990)

27. Y.C. Fung, Biomechanics, Circulation, Springer-Verlag. N.Y(1984)

28. R.L. Batra and B. Jena, Flow of a Casson fluid in a slightly curved tube, Int. J.Eng.Sci 29:No.10,1245-1258(1991)

29. B.Das and R.L. Batra, Secondary flow of a Casson fluid in a slightly curved tube. Int. J. Non-linear Mechanics, 28,5, 567-577 (1993)

30. J. Elshehawey, et al, Peristaltic motion of Generalised Newtonian fluid in a non-uniform channel, J. Phys. Soc. Japan 67, 434-440, (1998)

31. W. P. Walawender et al, An approximate Casson Fluid model for tube flow of blood. Biorheology, 12, 111-119 (1975)

32. E. Kreyszig, Advanced Engineering Mathematics, 1979, Fourth Edition, Wiley & Sons

33. A.V.Mernone and J.Mazumdar, Mathematical Modelling of peristaltic Transport of a Non-Newtonian Fluid, J.Australasian Physical and engineering Sciences in Medicine,21,3,126-140 (1998)

34. A.M. El Misery, E.F. Elshehawey and A.A. Hakeem, Peristaltic Motion of an Incompressible Generalised Newtonian Fluid in a Planar Channel, J. Phys. Soc. Japan, 65, 11 pp3524-3529 (1996)

35. L.M. Srivastava, V.P. Srivastava and S.N. Sinha, Peristaltic Transport of a Physiological Fluid Part I Flow in non-Uniform Geometry, Biorheology, 29 153-166 (1983)

36. L.M. Srivastava and V.P. Srivastava, Peristaltic transport of a power-law fluid: application to ductus efferentes of the reproductive tract, Rheol. Acta: 27. 428-433, (1988)

37. L. Liethold, The Calculus with Analytic Geometry,3rd Edition