INHOMOGENEOUS CONFORMAL COSMOLOGICAL MODELS

by

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APPENDIX 36

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SUMMARY

This thesis presents an investigation of some of the properties of cosmological models that possess the property of self-similarity. These models are inhomogeneous. In this respect the motivation is to find suitable models for study to try and answer some of the questions about the structure of the universe left unanswered by the standard homogeneous models.

The major proportion of this study concerns itself with a class of cosmological models which admit a three parameter group of conformal motions which act on the space-like hypersurfaces. The metric in these models is conformal to that of the well-known homogeneous models which admit a three parameter group of isometries. Homothetic models are included as a special case.

Initially we consider the group theoretic and Lie derivative properties of these models. This leads to a discussion about the nature of energy-momentum tensor admitted by these models and the kinematical behaviour of the fluid. It is noted that only a restricted class of conformal models will admit a perfect fluid and that in general these models with matter will be filled with non-zero acceleration and vorticity.

To investigate the existence and nature of diagonal models admitting a perfect fluid, we use Einstein's equations written in tetrad form. These solutions are found to be restricted to Bianchi types I, V and VII. However, the energy - momentum tensor in these models has either an unphysical equation of state or the fluid quantities are unrealistic in view of the currently accepted nature of the universe.

In the next two sections we consider two further aspects of these conformal models. At first we examine the problem of placing a conformal motion symmetry upon the initial data in the Cauchy hypersurface, and then find the conditions imposed on this initial data for the space-time to admit a local conformal motion. If the model admits a perfect fluid, it is noted that certain constraints must be satisfied. However,

these constraints vanish for the special case of homothetic motions. We then consider the applicability of the Hamiltonian methods developed by Ryan to the class of homothetic cosmological models. While only a restrictive class of model is allowed, the method is found to be admissable to the homothetic generalizations of some of the homogeneous Class B models previously disallowed.

In the latter part of this thesis we consider self-similar spherically symmetric universe models admitting the conformal vector $\underset{\sim}{\xi} = \alpha\partial t + \beta\partial r$. These models, while presenting an opportunity to study inhomogeneous models with a completely different global structure than the previous, also allow us to study the singularity structure in a simple inhomogeneous model. We find that this situation allows a singularity structure completely different to that found in the standard Friedmann models with the possible existence of a continuous big-bang presence and timelike singularities. A short discussion is also given on the redshift relations found in these models.

The thesis ends with a brief examination of the need for further work into the nature and existence of inhomogeneous models of the universe.

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NOTATION

Notation used is as follows: the metric tensor g_{AB} has signature (-+++). Covariant differentiation in the X^A direction is $\nabla_X = {}_{;A}X^A$; partial differentiation in the X^A direction is ${}_{,A}X^A$.

A vector is regarded as a directional derivative. A basis of vectors is $\partial/\partial x^A$, so $X = X^A\partial/\partial x^A$, where X^A are components of the vector with respect to the basis. Thus $X(f) = X^A\partial f/\partial x^A = f_{,A}X^A$. The commutator of the vectors X, Y is [X,Y] defined by $[X,Y]f = X(Yf) - Y(Xf) = (X^AY^B{}_{,A} - Y^AX^B{}_{,A})\partial f/\partial x^B = (L_XY)f$ where L_XY is the Lie derivative of Y with respect to X.

A set of vectors $\{E_a\}$ that are orthonormal at each point is called a tetrad. The notation ∂_a is used to emphasize the action of these vectors as directional derivatives: $\partial_a f = E_a f$.

The indices A,B,C... are coordinate indices and a,b,c... are tetrad indices. Both run from 0 to 3. The indices i,j,k,... are coordinate indices and $\alpha,\beta,\gamma,\ldots$ are tetrad indices and run from 1 to 3. These indices are also used to label the elements of the 3-dimensional groups C_3 and G_3. Round brackets denote symmetrized indices and square brackets denote skew-symmetrized indices.

In the following we shall be dealing with two metric tensors - the homogeneous metric with components g_{AB} and the conformally related metric with components $\hat{g}_{AB} = e^{2\sigma}g_{AB}$. To distinguish those conformal quantities written with respect to $\hat{g}_{AB}$ we write '^' over them. Also, the covariant derivative with respect to $\hat{g}_{AB}$ is written as a bar '|', while the covariant derivative with respect to g_{AB} is written as a semi-colon ';'.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university and to the best of my knowledge and belief, contains no material previously published or written by another person except when due reference is made in the text of the thesis.

Signed:

To my parents

- for their continued support and encouragement

CHAPTER 1

INHOMOGENEOUS COSMOLOGICAL MODELS

§1.1 Introduction

The primary focus of cosmological thought in the present century has been on interpreting the observations of the sample of the universe available to our telescopes in terms of a set of models based on various theories of gravitation, especially general relativity. That general relativity, in principle, provides a new insight into the properties of the world as a whole was first indicated by Einstein in 1919. Subsequent progress in relativistic cosmology was initially connected principally with the solution of Einstein's gravitation equations first obtained by Friedmann in 1922.

The improvements in our observational knowledge of the universe, coupled with the better understanding of the theory of matter over the past few decades, has opened up a veritable Pandora's box of relativistic models of the universe [e.g. see Narlikar [1] for a survey]. However, in face of the continued success of Einstein's theory of general relativity as the most aesthetically pleasing theory fitting the known experimental facts, it is widely held that the standard model of the universe, based on the Friedmann models, fits most features of the actual universe quite well [see Weinberg [2] for a thorough examination of this model]. Nevertheless, the assumption of homogeneity used in this model is to be regarded merely as a working hypothesis, suggested by the state of present observations.

§1.2 Standard Cosmological Models

As is well known, these solutions are based on the assumption that the distribution of matter in space is completely homogeneous and isotropic. The main property of these solutions, that the universe is not stationary, has found confirmation in the red-shift effect and this property must be basic to any modern description of the state of the universe.

Given the 'success' of the standard models, at the same time it is clear that the universal assumption that the universe is homogeneous can be satisfied at best only approximately. Belief in homogeneity is really the outcome of a continuing series of reverses for a geocentric point of view. Briefly, these were [3] a) Copernicus' 1543 proposal that the Earth is not the centre of the universe, b) Shapley's 1918 discovery that the Sun is not at the centre of our Galaxy, c) Hubble's 1924 confirmation that the 'island nebulae' were other galaxies, and d) Baade's 1952 revision of the distance scale showing that our galaxy is not the biggest in the universe. The consequence is a widely held belief, known as the Copernican principle, [4] that the Earth is in no special position in the universe. Thus if we see isotropy, everybody must see isotropy and we are lead to homogeneity in space.

The only attempts at direct testing of homogeneity use the distribution of galaxies, and since the galaxies appear to be clustered on scales which may be very large, the outcome of these tests is disputed. However, the problem faced here is one that confronts all theories of the universe, as outlined by Ellis in a paper entitled 'Cosmology and Verifiability', op.cit, [4]. Given that the subject of relativistic cosmology is the determination of the smoothed out metrical structure of the universe, Ellis argues that the problem of determining this structure is centred on the fact that there is only one universe to be observed, and that we can effectively only observe it from one-point in space-time. Because it is a unique object, we cannot infer its probable nature by comparing it with similar objects. This leads to very real limitations in our observational knowledge as to the actual nature of the universe and its contents.

Given this situation, we are unable to obtain a model of the universe without making some specifically cosmological assumptions which are completely unverifiable. (Although we can presumably make some extrapolations of the conditions observed in our immediate neighbourhood to greater distances, we have no real justification for assuming that the whole universe has the same properties.) Thus we see that any theory we have of the universe will be heavily influenced by the assumptions we make.

Because the universe is so complex, it is immediately obvious that if we are to have a workable model of the universe, we must commence by simplifying it and discarding what hopefully we believe to be the irrelevant aspects. We start by discarding the planets and usually, in an all or nothing spirit, follow by dismissing the entire range of stellar and galactic structure. Everything is smeared into a uniform fluid and we are left with an idealized universe that is virtually little more than the 'grin on the Cheshire cat'. Just before all structure is dissolved away we hold on to some of the grosser rudiments of the universe - this is usually achieved by having the model satisfy some symmetry constraint. However, we must interpret these statements as meaning that the actual inhomogeneity and anisotropy at a point, are only small statistical deviations from the underlying symmetry.

In the standard models, the homogeneity working hypothesis is justified on the ground that on a sufficiently large scale the universe does appear to be roughly homogeneous. If it were not, then large clumps of matter would produce anisotropies in the microwave radiation in excess of those observed. Also, from a mathematical point of view, the study of homogeneous model universes has enormous advantages. Whereas, in general, non-homogeneous model universes involve us in global questions, the beauty of homogeneous models is that they can be studied mainly locally; any part is representative of the whole. Also, the field equations become more tractable (reducing to ordinary differential equations) and from the general class of spatially homogeneous models, a Bianchi-Behr classification scheme has been devised that indicates which are the most general [5].

We have thus created a smooth and featureless model of the universe. The next step is to show that irregularities grow and in the course of time the unstructured universe becomes the structured universe we observe today. On this point however, progress has been slow and the structure in the universe has become difficult to relate to models based on smoothing postulates. This has usually resulted in separate theoretical approaches to the origin of the various structures in the universe, and while these approaches have met with some success, they are usually inadequately related to one another and to cosmological theories.

§1.3 Homogeneity vs. Inhomogeneity

In this situation we are forced to reconsider our initial generalizations about isotropy and homogeneity. This view is also reinforced by the discovery that the standard model [6]

a) may not be sufficiently general for problems where generality conditions are of prime importance e.g. in the study of singularities.

b) seems incapable of explaining such phenomena as the homogenization and isotropization of the universe (see e.g. Misner's programme of chaotic cosmology, [7]).

c) do not provide a suitable background for the formation of galaxies from small random fluctuations.

Also, with the homogeneous models that are generally favoured for giving the best fit to the observations, the observational tests for discriminating between the various models are usually difficult to carry out and at best only marginal. It is unfortunate that the large amount of information contained in the various sub-structures of the universe cannot be used in testing these models.

Thus, we see that there is a need for more complete and descriptive cosmological models and in order to study questions as the above it will be necessary to consider inhomogeneous cosmological models. The homogeneity question should be formulated not as 'Is the universe homogeneous?' but rather as 'To what degree is the universe homogeneous?'.

In any case, careful study of other models can advance our understanding of relativity (see [8]). We must always keep an open mind as to changes and improvements which could make a better or more extended theory possible.

§1.4 Inhomogeneous Cosmological Models

Many exact solutions of Einstein's field equations are known and the qualitative behaviour of certain general classes has been studied. However, most of these solutions in one way or another are ruled out as cosmological models, and even less qualify as inhomogeneous cosmological models. In this section we briefly outline the work which has been done in this area.

Apart from the mathematical difficulties, the generally accepted cosmological principle, which together with the isotropy of space-time about a single observer leads unambiguously to the Friedmann models, has lead, until recently, to a lack of work on inhomogeneous models. As has been noted already, the standard models cannot explain fully a number of phenomena and Alfven [9] claims that some observations actually disagree with the theoretical predictions and can be bought into apparent agreement only by a number of ad hoc assumptions. Also, Ellis et al. [10] claim that while isotropy is directly observable, homogeneity (on a cosmological scale) is not. Thus, if the assumption that the universe is homogeneous is discarded, the situation is not so clear.

Given this situation of the observable isotropy about us, one of the first studies of inhomogeneous models was by Omer in 1949, which used the spherically symmetric Tolman-Bondi solution [12]. Although mathematical models for such Earth-centred cosmologies have occasionally been investigated, they have not been taken seriously as they are philosophically unattractive since it is believed to be unreasonable that we should be near the centre of the universe [see §1.2]. It is usually argued that it is unlikely that certain isotropies are peculiar to the observer's location, so that the observer infers that the symmetries he perceives are not fortuituous but that probably they exist everywhere. Thus aimed with this principle the observer usually infers widespread symmetry [13]. However in a recent paper by Ellis et al. [10], some intriging aspects of such universes are presented. It would certainly be consistent with the present observations that we were near the centre of such a universe and Varshni [14] argues that the distribution of quasars infact implies such a case. These statements have also been supported by Ellis [4].

It has been stated [6] that the main stumbling block in the development of inhomogeneous cosmologies has been the need to impose in a covariant way, symmetries which are sufficiently strong to render the field equations tractable, while being not so strong that they require spatial homogeneity. One way to introduce inhomogeneity is to still require the space-time to admit a group of isometries but that the orbits of the group are not three-dimensional. In this case we obtain the locally-rotationally symmetric (L.R.S.) models of Ellis & Stewart [15] where the space-time is invariant under a spatial rotation about a spacelike axis of symmetry at each point. Further, fairly symmetric cases occur when the space-time admits a two parameter group of motions. These include the Gowdy universes [16] and the stiff equation of state models of Wainwright et al. [17].

Also, one can impose less restrictive symmetries upon the space-time. Eardley [18], following the idea of Cahill and Taub [19] to consider 'self-similar' cosmological models, considered those space-times that admitted a 3-parameter group of homothetic motions that acted simply transitive on the spacelike hypersurfaces, and was able to generalize the Bianchi-Behr classification to include new and more general inhomogeneous models. Similarly motivated, Wesson [20] devised a new dimensional cosmological principle [21] that lead him to new 'self-similar' models, which being non-Friedmann, he hoped would solve some of the puzzling problems that face Friedmann models.

Another inhomogeneous exact model that has been used for cosmological purposes is one in which spherical regions in a Friedmann model are removed and replaced by part of the well-known spherically symmetric Schwarzschild solution. This model is often called the 'swiss-cheese' model [22]. This model has been used to discuss the influence of concentrations of matter on the propagation of light. An alternate approach has been to treat, approximately, perturbed spatially homogeneous models. This method was used by Lifschitz and Khalatnikov [23] in a classic discussion, and has subsequently been explored by many other authors.

More recently, Szekeres [24] discovered a useful and fairly wide set of dust cosmologies by assuming that the metric have a particular form. These models generalize the previously known Tolman-Bondi and Kantowski-Sachs [25] models. These models were extended by Szafron [26] to include pressure but the matter content is in general an unusual one, if not unphysical, in that although we have for the density $\rho = \rho(t,x,y,z)$, we have for the pressure $p = p(t)$ only. Although these spacetimes do not in general admit any Killing vectors [27] they do admit a preferred two-parameter family of two-surfaces of constant curvature. Also, each comoving space slice t = const. is conformally flat [28].

The study of the Szekeres solutions has suggested a different approach to the study of inhomogeneous models when the group theoretic techniques may no longer be applicable. The assumption of conformally flat space sections may provide the symmetry necessary to solve the Einstein equations when the condition of spatial homogeneity is relaxed. Spero and Szafron [29] have shown that any family of inhomogeneous solutions with conformally flat space-sections can only contain homogeneous solutions invariant under groups of Bianchi type I, VII_o, V, IX, VII_h or VI_{-1} or be of the Kantowski-Sachs form. The Szekeres solutions extend a subset of the above listed spaces to a family of inhomogeneous perfect fluid solutions with the following properties:

a) irrotational, geodesic flow and an expansion tensor with two equal eigenvalues.

b) conformally flat, comoving hypersurfaces whose Ricci tensor has two equal eigenvalues (Petrov type D).

c) a nonbaratropic equation of state.

Their results indicate that, if one wishes to generalize that subclass of perfect fluid solutions satisfying (b), then some portion of (a) must be discarded. If one does this, then the extension must include other spatially inhomogeneous models.

Most recently, Collins and Szafron [6] have suggested another alternative to space-time symmetries. Not unlike the previous problem, they impose restrictions on certain sub-manifolds instead of considering the full space-time. The problem can be regarded as one of classifying three-dimensional Riemannian geometries (i.e. the intrinsic geometry of the hypersurfaces) and of classifying normal time-like congruences. The properties of the normal time-like congruences relates to the way in which the hypersurfaces are imbedded in space-time (i.e. their extrinsic geometry). This classification is thus purely geometrical and is thus independent of any field equations. Their classification is based on the Ricci tensor of the metric induced on the hypersurfaces and on the shear tensor of the normal congruence. An investigation in which the normal congruence is geodesic and the hypersurfaces conformally flat led to a characterization of the Szekeres solutions.

This approach is being extended by Wainwright [30] in order to set up a classification scheme which is sufficiently general to distinguish the various known inhomogeneous solutions. Some of these exact solutions - such as the Szekeres solutions, the type-N perfect solutions of Oleson [31] and a class of algebraically special solutions found by Wainwright [32] - do not admit any Killing vectors in general.

§1.5 Aim of Thesis

Motivated by the foregoing work on inhomogeneous cosmological models, in the following we shall extend to work of Eardley [13] in considering models that admit a three-parameter group of conformal motions that act simply transitively on space-like hypersurfaces. These models will be inhomogeneous. In this respect it is hoped to find suitable models for study to try and answer some of the questions about the structure of the universe left unanswered by the standard homogeneous models.

In Chapter 2 we define the mathematical nature of a conformal motion and obtain the metric governing these models. In Chapter 3 we elucidate the Lie derivative of certain geometrical objects used in describing these models. From this we consider the nature of the energy-momentum tensor admitted by these models and examine the kinematical properties of the fluid. To help us we introduce a tetrad basis. Using this basis, the Einstein field equations governing these models are written down, and the existence and nature of diagonal perfect fluid solutions is examined in Chapter 4. In the next two chapters we consider two further aspects of these models. First we consider the initial value problem in relation to the preservation of a conformal symmetry in the initial data. Secondly, we investigate the applicability of the Hamiltonian methods developed by Ryan for giving a qualitative description of these models.

In Chapter 7, we break with the foregoing work and consider spherically symmetric models admitting the conformal vector $\underset{\sim}{\xi} = \alpha\partial t + \beta\partial r$. These models, while presenting an opportunity to study inhomogeneous universe models with a completely different global structure than the previous, also gives rise to the study of the singularity structure in a simple inhomogeneous model. The thesis ends with a brief examination of further work needed in the study of inhomogeneous models of the universe.

CHAPTER 2

CONFORMALLY HOMOGENEOUS MODELS

§2.1 Inhomogeneity as a Similarity Construct

As we have seen in Chapter 1, if one is to pursue the study of galaxies and similar structures, then we obviously need a cosmological structure which acts as an environment that allows for aggolomerations of matter. On this ground, one would exclude the homogeneous models - in view of the fundamental equivalence of geometry and physics stated by Einstein's equations, physical inhomogeneities such as the aggolomeration of matter into galaxies etc. imply spatial geometric inhomogeneities. Edelen and Wilson [33], in their study of discretization in astronomy argue that the simplest manner of introducing such inhomogeneities, and one that is consistent with the idea that the inhomogeneities can be viewed as 'bumps' on a homogeneous substratum, is that in which the spatial geometry of the inhomogeneities is similar to the geometry of the homogeneous substratum. They reasoned that the factor of proportionality obtained in stating the similarity as an equality would then describe the inhomogeneities through the variation in the values of the proportionality factor from point to point.

Edelen and Wilson also argued that the spatial inhomogeneities would in turn imply inhomogeneity in the energy-density and proper time rates. Hence, in considering both the spatial and temporal inhomogeneities, they constructed cosmological models that were conformal to the classical Friedmann models.

In this thesis we will be considering the wider class of models which are conformal to the well-known spatially homogeneous models which admit a three-parameter group of isometries [34]. In particular, we will be considering those metric spaces which admit a three-parameter group of conformal motions (which includes the class of homothetic motions as a special case) on their space-sections. Geometrically, this means that lengths are not necessarily preserved upon transformation from point to point upon the spacelike orbits, but may be multiplied by a conformal factor.

In constructing these models, our work follows closely Eardley's [18] paper on spatially homothetic cosmological models, where the conformal factor in the transformation mentioned above is a constant. Eardley called these models 'self-similar' space-times and cited the use of such solutions in classical hydrodynamics where the physical systems involved have no intrinsic scale of length. This led him to use the notion of self-similarity as meaning invariance under scale transformations. Such a scale free property could be a desirable property of a cosmological model which had 'forgotten' its initial conditions and had become scale invariant e.g. the expansion of the universe.

Finally, above and beyond the arguments of both mathematical and physical simplicity, conformally equivalent models possess a unique property that singles them out from all other possible inhomogeneous models [33]. It is known that the complete curvature tensor of a metric space can be decomposed uniquely into the sum of two tensors, one of which is the Weyl conformal tensor and the other which is uniquely determined by the metric tensor and the energy-momentum tensor. Now, the Weyl tensor is that part of the curvature tensor that is not determined locally by the matter and may thus be viewed as representing the 'free gravitational field'; further, the Weyl tensor is the unique curvature invariant under conformal changes in the metric tensor. It thus follows that conformally homogeneous models describe inhomogeneous distributions of matter whose free gravitational fields are identical with the free gravitational fields of the corresponding homogeneous models.

§2.2 Spatially Homogeneous Cosmological Models

We collect here those results from the spatially homogeneous models that are required for the purposes of reference and comparison [for a recent review, see MacCallum [3]].

Let E denote the class of Einstein-Riemannian spaces admissible in spatially homogeneous models (see below) and let $g_{AB}(\underline{x})$ denote the components of the metric tensor of an element of E. We then have

$$ds^2(g) = g_{AB}dx^A dx^B = -dt^2 + \nu_{\alpha\beta}(t)w^\alpha \otimes w^\beta \tag{2.1a}$$

$$\stackrel{*}{=} -dt^2 + \nu_{\alpha\beta}(t)w^\alpha{}_i w^\beta{}_j dx^i dx^j \tag{2.1b}$$

where $\stackrel{*}{=}$ denotes evaluation in a coordinate system in which the matter in the models is at rest (i.e. comoving coordinates).

Spatial homogeneity is specified mathematically in that the space-times (2.1) admit a three-parameter group of motions (isometries) simply transitive on the spacelike hypersurfaces. [Note that this definition doesn't include the Gödel universe, which while being homogeneous, rotating and shear-free, does not possess spatial surfaces of homogeneity. Also, if we weaken the above definition of homogeneity so that space-time need only be locally invariant under a group of isometries whose surfaces of transitively are spacelike surfaces, then it can be shown [35] that the general group Gr will possess a subgroup G_3 which satisfies the definition of spatial homogeneity except in the case of the Kantowski-Sachs type I universes [25].]

By definition, the spaces E admit an isometry if there exists a vector field ξ such that

$$\xi_{(A;B)} \equiv L_\xi g_{AB} = 0 \tag{2.2}$$

where L_ξ denotes the Lie derivative along the vector field ξ [36]. This is known as Killing's equation and ξ is called a Killing vector. Any Killing vector field generates isometries, and the set of all Killing vector fields forms a Lie algebra, which is the Lie algebra of the group of isometries. Physically, this corresponds to a transformation that maps the metric g_{AB} at some point p on the spacelike hypersurface to the same metric at another point q on the hypersurface (i.e. it preserves all length measurements). If we choose a basis $\{\xi_i\}$ for the group G_3, then the Lie algebra $\hat{G}_3$ will be specified by the commutation relations

$$[\xi_i, \xi_j] = C^k{}_{ij}\xi_k \tag{2.3}$$

where the structure constants $C^k{}_{ij}$ satisfy the antisymmetry condition and Jacobi identities.

It is possible to systematically list all real three-dimensional Lie algebras which are non-isomorphic [5]. For the three-dimensional Lie algebras above, this was first done by Bianchi, and it is this classification that is relevant to spatially homogeneous models. The classification examines the commutators. These themselves form a Lie algebra, which is a subalgebra of $\hat{G}$, called the derived algebra. Bianchi's method was to consider first the dimension of the derived algebra and then to enumerate all possibilities. This gave him nine inequivalent types of which type I is abelian, and has zero-dimensional Lie algebra, types II and III have a 1-dimensional Lie-algebra, types IV, V, VI and VII are 2-dimensional and types VIII and IX are 3-dimensional. Types VI and VII, in fact, are one-parameter families of algebras, where certain values of the parameters are excluded because they yield types III and V instead. Bianchi's method has been modified in recent years and the present method is as follows [34].

Take any (positive definite) scalar product on $\hat{G}$, and suppose its components in the basis $\{\xi_i\}$ are g_{ij}. Then write

$$C^i{}_{jk} = \varepsilon_{\ell jk} n^{i\ell} + 2\delta^i{}_{[k} a_{j]} \tag{2.4a}$$

This defines the vector a_j (on $\hat{G}$) uniquely, since

$$a_j = \tfrac{1}{2} C^i{}_{ji} \tag{2.4b}$$

and it defines n^{ij}, which is symmetric, up to an overall scalar factor. The Jacobi identity is equivalent to

$$n^{ij} a_j = 0 \tag{2.5}$$

The classification now gives two broad classes; Class A where $a_i = 0$, and class B ($a_i \neq 0$), each divided into several types according to the rank and signature of n^{ij}. When $a_i \neq 0$, there is a further invariant h, which can be defined by

$$(1 + h)C^i{}_{ji}C^k{}_{\ell k} = - 2hC^i{}_{kj}C^k{}_{i\ell} \quad . \tag{2.6}$$

This is the parameter required to subclassify types VI and VII.

By rotating the basis $\{\xi_i\}$ we can diagonalize the matrix n^{ij} so that $a_i = (0,0,a)$ and $n^{ij} = \mathrm{diag}(n_1,n_2,n_3)$ and then by scaling the basis we can set the non-zero entries in n_{ij} to +1 or -1. In types IV and V we can also scale $a = 1$. In general, $h = a^2/n_1n_2$ so the scaling gives $a = \sqrt{|h|}$. The resulting classification and canonical forms are shown in Table 1.

Table 1

Class	A						B				
Type	I	II	VI_o	VII_o	VIII	IX	V	IV	III	VI_h	VII_h
Rank n^{ij}	0	1	2	2	3	3	0	1	2	2	2
a	0	0	0	0	0	0	1	1	1	$(-h)^{1/2}$	$h^{1/2}$
n_1	0	1	1	1	1	1	0	1	1	1	1
n_2	0	0	-1	1	1	1	0	0	-1	-1	1
n_3	0	0	0	0	-1	1	0	0	0	0	0

The spatial basis vectors in (2.1), $\{X_i\}$, can be found in several ways. Essentially, these all make use of the result that in the case of a simply-transitive group with a basis of infinitesimal transformations $\{\xi_i\}$, one can find a set of three vector fields $\{X_i\}$ spanning the tangent space at every orbit such that [37],

$$[X_i,\xi_j] = 0 \qquad i,j = 1,2,3 \tag{2.7}$$

Moreover, since we can choose arbitrarily both the initial values of $\{\xi_i\}$ and $\{X_i\}$, we can ensure that at one point we have $X_i = -\xi_i$ for all i. Hence, defining functions $x^i{}_j$ by $X_j = x^i{}_j\xi_i$ (so that at the origin $x^i{}_j = -\delta^i{}_j$) and defining $D^i{}_{jk}$ by

$$[X_i,X_j] = D^k{}_{ij}X_k \tag{2.8}$$

, we find by substitution for X_i in (2.7) and (2.8) that

$$x^i{}_j x^\ell{}_k C^m{}_{i\ell}\xi_m = D^i{}_{jk}x_i .$$

Thus that at the origin

$$C^i{}_{jk} = D^i{}_{jk} \tag{2.9}$$

Hence, the algebraic structure of the Lie algebra of the $\{X_i\}$ at a point is the same as that of the algebra of infinitesimal transformations. In addition, substituting (2.7) and (2.8) in the Jacobi identities for $\{\xi_i,X_j,X_k\}$ shows that the $D^i{}_{jk}$ are constants in the orbit.

Let us denote the duals of the $\{X_i\}$ by $\{w^i\}$. Then to express the metric in the form (2.1) we consider the dual set of 1-forms $\{w^i\}$ defined by

$$w^i{}_A\xi^A{}_j = \delta^i{}_j \quad . \tag{2.10}$$

These invariant 1-forms satisfy the curl relations

$$dw^i = -\tfrac{1}{2}C^i{}_{jk}w^j \wedge w^k \quad . \tag{2.11}$$

Since the hypersurfaces being considered here admit a simply transitive group of motions, they are diffeomorphic to the group itself. Using the basis of vector fields w_i, so that $g = g_{ij}w^i w^j$ we find

$$\mathcal{L}_\xi g_{ij} = 0$$

and so the g_{ij} are constants. Also, since the group orbits are hypersurfaces, we can prove that the (unit) normals of these hypersurfaces are geodesic. Then taking t to be the affine parameter along these geodesics we obtain, if $\underline{n}$ is non-null

$$ds^2 = -dt^2 + g_{ij}w^i \otimes w^j$$

where as we have seen the g_{ij} are constant within each orbit.

For each type in Table 1, we can calculate the infinitesimal generators ξ_i in a canonical basis $\{\partial_i \equiv \partial/\partial x_i\}$ and hence obtain the explicit forms of the three invariant 1-forms dual to the ξ_i (see for example, [38]). For completeness they are listed in Appendix A.

We now see that the elements of E are metrics having the form (2.1) and which are solutions of Einstein's field equations

$$G_{AB} = R_{AB} - \tfrac{1}{2}Rg_{AB} = T_{AB} \tag{2.12}$$

in which the matter tensor takes the form of a perfect fluid

$$T_{AB} = (\rho + p)u_A u_B + p g_{AB} \tag{2.13}$$

where u_A, $(u_A u^A = -1)$ is the velocity vector and ρ and p are the density and pressure respectively. We shall also assume the energy condition

$$\rho + p > 0 \tag{2.14}$$

The simplest spatially homogeneous models are the isotropic Friedmann models while the simplest spatially homogeneous anisotropic models are the Bianchi type I models.

§2.3 Conformal Space-Times

Let $\mathcal{C}$ denote the class of Einstein-Riemannian spaces that are conformal to the elements of $\mathcal{E}$, and let $\hat{g}_{AB}(x^C)$ denote the components of the metric tensor of such spaces. We then have

$$\hat{g}_{AB}(x^C) = e^{2\sigma(x^C)} g_{AB}(x^C) \tag{3.1}$$

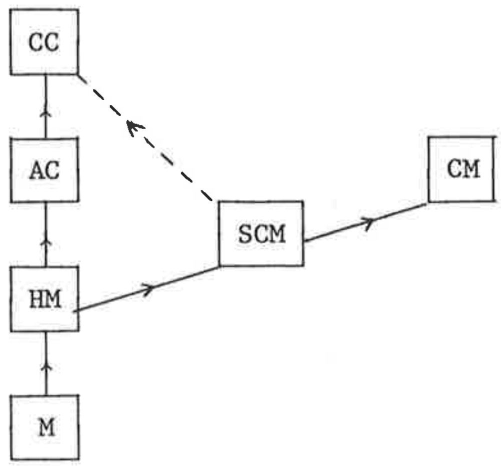
where
$$ds^2(\hat{g}) = \hat{g}_{AB} dx^A dx^B = e^{2\sigma} ds^2(g) \tag{3.2}$$

is the fundamental metric form on elements of $\mathcal{C}$ and $\sigma(x^C)$ is the conformal coefficient. We assume that $\mathcal{E}$ and $\mathcal{C}$ have the same coordinate patches and coordinate functions [i.e. $\{dx^A\}$ in $\mathcal{E}$ are the same as $\{dx^A\}$ in $\mathcal{C}$, and the respective points of $\mathcal{E}$ and $\mathcal{C}$ have the same coordinates with respect to the coordinate system in which the $\{dx^A\}$ are computed].

We now wish to restrict our attention to those models, having a metric tensor of the form (3.1), which admit transformations that map the metric tensor $\hat{g}_{AB}$ at some point p on a spacelike hypersurface to some tensor $\hat{f}_{AB}$ at another point q on the hypersurface, such that $\hat{f}_{AB}$ is a multiple of the metric $\hat{g}_{AB}$ at q i.e. $\hat{f}_{AB} = \phi(x^C)\hat{g}_{AB}$ for some scalar function ϕ. Such a transformation is called a conformal motion, and scales up all length measurements but preserves angles [see Yano, [36]]. If ϕ is a constant ($\phi \neq 1$) then the motion is called a homothetic motion and if the transformation leaves the metric invariant ($\phi = 1$), then it is an isometry.

The diagram below, taken from Oliver and Davis [39] shows how the various symmetries are related.

Fig. 1



M - Isometry $L_\xi g_{AB} = 0$

HM - Homothetic Motion $L_\xi g_{AB} = 2\psi g_{AB}$ ψ = constant

S.C.M. - Special Conformal Motion $L_\xi g_{AB} = 2\psi g_{AB}$ $\nabla_A\nabla_B = 0$

CM - Conformal Motion $L_\xi g_{AB} = 2\psi g_{AB}$

AC - Affine Connection $L_\xi \Gamma^A{}_{BC} = 0$

CC - Curvature Collineation $L_\xi R^A{}_{BCD} = 0$.

It is seen here that mathematically a homothetic motion is a special case of a more general class of symmetries known as curvature collineations. Collinson and French [40] have shown that for non-flat vacuum space-times not of Petrov type N with hypersurface orthogonal geodesic rays, the more general collineations reduce to a homothetic motion.

When the desired transformation is infinitesimal,

$$x^{A'} = f^{A}(x^{i},t) = x^{A} + \xi^{A}dt \tag{3.3}$$

the condition for a conformal motion becomes

$$L_{\xi}\hat{g}_{AB} = 2\phi\hat{g}_{AB} \tag{3.4}$$

The vector ξ^{A} is called a conformal Killing vector (C.K.F.) and is said to generate consometries. From (3.4) we find

$$L_{\xi}\hat{g} = 2\phi \qquad \hat{g} = |\det \hat{g}_{AB}|$$

and so by eliminating ϕ we have the result

$$L_{\xi}G_{AB} = 0 \qquad G_{AB} = \hat{g}^{-\frac{1}{n}}\hat{g}_{AB} \quad . \tag{3.5}$$

Hence G_{AB} is an invariant geometric object under the group of conformal transformations.

Now the system (3.4) does not depend on the choice of coordinate systems and so by choosing a coordinate system in a suitable neighbourhood of a regular point of ξ^{A} such that in this neighbourhood

$$\xi^{A} = \delta^{A}_{1}$$

then the infinitesimal point transformation generates the transformation

$$\hat{g}_{AB,1} - 2\phi\hat{g}_{AB} = 0 \quad .$$

This can be integrated to give

$$\hat{g}_{AB} = \exp\left\{\int 2\phi dx^1\right\} g_{AB}(x^2,..x^n) \quad . \tag{3.6}$$

Also, we see that $\hat{g}_{AB}$ can be rescaled to yield a g_{AB} for which ξ^A is a Killing vector. Conversely, if ξ^A is a Killing vector for some g_{AB},

$$g_{AB,C}\xi^C + g_{AC}\xi^C{}_{,B} + g_{BC}\xi^C{}_{,A} = 0$$

then for a $\hat{g}_{AB}$ conformal to g_{AB}, $\hat{g}_{AB} = e^{2\sigma}g_{AB}$, we regain (3.4) with

$$\xi^C\sigma_{,C} = \phi \quad . \tag{3.7}$$

Generalizing this result, we obtain the result first proven by Yano [36].

- In an n-dimensional Riemannian or pseudo-Riemannian manifold (Mn,g), every r-dimensional Lie group Cr of local conformal transformations that is simply transitive is conformally isometric.

This means that if a space-time admits a group Cr of conformal motions with generators $\xi^A{}_\alpha$ ($\alpha = 1,\ldots,r$) acting on (Mn,g) [i.e. there exist functions ϕ_α on Mn such that $L_\xi g_{AB} = 2\phi_\alpha g_{AB}$], then one can rescale g_{AB} such that the resulting metric tensor, $e^{2\sigma}g_{AB}$, will admit the vectors $\xi^A{}_\alpha$ as Killing vector fields and

$$L_{\xi_\alpha}\sigma - \phi_\alpha = 0 \tag{3.8}$$

These results give us the desired models mentioned at the beginning of this section [see also Defrise-Carter, [41]]. In summary we have -

If a space-time admits a 3-parameter group of conformal motions which act simply-transitively on the spacelike hypersurface, then there exists a coordinate system with respect to which the metric $\hat{g}_{AB}$ of the space-time has the form $\hat{g}_{AB} = e^{2\sigma(x^C)} g_{AB}(x^C)$ where $g_{AB}(x^C)$ is the metric of a space-time which admits a 3-parameter group of Killing vectors and so is an element of E. Hence the original space-time is an element of C.

The significance of this result is that we can use the classification scheme previously outlined in the last section to classify the conformal space-times. As we shall see, with certain specifications on the conformal factor σ, we can obtain a Bianchi-Behr classification similar to that of the spatially homogeneous models.

To eludicate this behaviour we look at the structure of the group of transformations more closely. The 3-parameter group of conformal motions forms a continuous Lie group C_3 which acts on $(M,\hat{g})$ and let G_m $(m \leqslant 3)$ be the corresponding isometry group. If $\sigma = 0$, then the conformal motions are trivial $(\phi_\alpha = 0)$ and reduce to isometries and $G_m = C_3$. However, if C_3 is non-trivial $G_m \subset C_3$. The infinitesimal generators ξ_α of C_3 are vector fields on M and form a Lie algebra $\hat{C}_3$; each $\xi^A \in \hat{C}_3$ obeys

$$L_\xi \hat{g}_{AB} = 2\langle\phi,\xi_\alpha\rangle \hat{g}_{AB} \tag{3.9}$$

Here $\langle\phi,\xi_\alpha\rangle \equiv \phi_\alpha$ is a scalar function depending on the choice of ξ_α, i.e.

$\langle\phi,\rangle$ is a linear functional on $\hat{C}_3$. In particular, each $\xi_\beta \in G_m$ is a Killing vector field satisfying

$$L_\xi \hat{g}_{AB} = 0 \qquad\qquad \beta = 1,..m$$

Thus
$$G_m = \text{Kernel}\langle\phi,\rangle$$

and ϕ_α can be interpreted as an element of the dual space to the Lie algebra $\hat{C}_3$. Hence m = 2 if C_3 is non-trivial since $\hat{G}_m$ is the subspace orthogonal to the covector ϕ_α corresponding to the non-trivial conformal motion ξ_α i.e. $(M,\hat{g})$ admits at most one independent, non-trivial conformal motion [see Eardley, [18] for spatially homothetic case].

Consider now the commutator of $\xi_\alpha, \xi_\beta \in C_3$

$$\begin{aligned}
L_{[\xi_\alpha,\xi_\beta]}\hat{g}_{AB} &= [L_{\xi_\alpha}, L_{\xi_\beta}]\hat{g}_{AB} \\
&= L_{\xi_\alpha}(2\phi_\beta \hat{g}_{AB}) - L_{\xi_\beta}(2\phi_\alpha \hat{g}_{AB}) \\
&= 2(L_{\xi_\alpha} L_{\xi_\beta}\sigma)\hat{g}_{AB} + 4\phi_\beta\phi_\alpha \hat{g}_{AB} \\
&\quad - 2(L_{\xi_\beta} L_{\xi_\alpha}\sigma)\hat{g}_{AB} - 4\phi_\alpha\phi_\beta \hat{g}_{AB} \\
&= +\,2(L_{[\xi_\alpha,\xi_\beta]}\sigma)\hat{g}_{AB} \\
&= 2\phi_{\alpha\beta}\hat{g}_{AB} \qquad \phi_{\alpha\beta} = L_{[\xi_\alpha,\xi_\beta]}\sigma
\end{aligned}$$

where we have used the result from (3.8) that $\phi_\alpha = L_{\xi_\alpha}\sigma$. Thus in general the commutator is an element of C_3. However, the commutator can be shown to be an isometry i.e. $L_{[\xi_\alpha,\xi_\beta]}\sigma = 0$

This follows from the result that if H is a normal Lie subgroup of a Lie group G, then the corresponding Lie algebras $\hat{H}$ and $\hat{G}$ satisfy [84]

$$[\hat{G},\hat{H}] \subset \hat{H} \tag{3.11}$$

Now since $(M,\hat{g})$ admits at most one independent, non-trivial conformal motion and G_m = Kernel $\langle\phi,\rangle$ (and is thus a normal subgroup of C_3), then the commutator $[\xi_\alpha,\xi_\beta]$ where $\alpha \neq \beta$ takes the form of (3.11) and so is a killing vector i.e.

$$[\hat{C}_3,\hat{G}_2] \subseteq \hat{G}_2$$

Hence we have the result

$$[\xi_\alpha,\xi_\beta]\sigma = C^\nu{}_{\alpha\beta}\xi_\nu\sigma = 0 \Rightarrow C^\nu{}_{\alpha\beta}\phi_\nu = 0$$

where the structure constants correspond to one of the canonical forms listed in section 2.

Using the decomposition (2.4) this gives the restriction

$$(n^{\alpha\beta} + a_\gamma\varepsilon^{\alpha\gamma\beta})\phi_\beta = 0 \quad . \tag{3.12}$$

Using the normalized basis of section 2, with $a_\alpha = a_3\delta^3{}_\alpha$, one can now use (3.12) to refine the classification of Table 1 in terms of the sets $\{n^{\alpha\beta}, a_\alpha, \phi_\alpha\}$ that are non-equivalent under a change of basis $\{\xi_\alpha\}$. This classification was first obtained by Eardley [18]. We obtain four classes

Class A	:	$a_\alpha = 0$	$\phi_\alpha = 0$
Class B	:	$a_\alpha \neq 0$	$\phi_\alpha = 0$
Class C	:	$a_\alpha = 0$	$\phi_\alpha \neq 0$
Class D	:	$a_\alpha \neq 0$	$\phi_\alpha \neq 0$

Obviously the two classes A and B correspond to the spatially homogeneous models of section 2 while classes C and D are generalizations of the previous two classes respectively.

To find a canonical form for ϕ_α, consider equation (3.8) while it can be written in the form

$$\xi^A_\alpha \frac{\partial\sigma}{\partial x^A} = \phi_\alpha \tag{3.13a}$$

Introducing the dual invariant 1-forms $\{w^\alpha\}$ defined by (2.10) and (2.11), we find that σ is defined by

$$d\sigma = \phi_\alpha w^\alpha \tag{3.13b}$$

Hence, the spatial derivative of the conformal factor lies in the vector space of left-invariant 1-forms on C_3 and it follows that σ is independent of t, the group invariant scalar field which specifies each spacelike hypersurface s(t). Now, for a given type there will exist a non-trivial consometry if we can find a linearly independent combination $\phi_\alpha w^\alpha$ that is locally a total differential. By inspection of the explicit form of the w^α for the various types, as listed in Appendix A, we find that we can reduce ϕ_α to the canonical form

$$\phi_\alpha = \phi\delta^3{}_\alpha \tag{3.14}$$

where $\qquad \phi = F(x^3)$ for type I

and $\qquad \phi = F(x^1)$ for types II - VII

where F is any function of the indicated variable. We see immediately that for types VIII and IX, although a G_3 of isometries exists, there does not exist a non trivial conformal motion. The reason for this is that the existence of the spatial derivative of the conformal factor in the vector space of left-invariant 1-forms implies that C_3 is not semi-simple and hence the semi-simple Bianchi types VIII and IX does not allow conformal extensions. To see this more clearly, we have, putting $a_\alpha = 0$ in (3.12) the equation

$$n^{\alpha\beta}\phi_\alpha = 0 \quad .$$

Hence, from (3.14) and the canonical basis for $n^{\alpha\beta}$ we have $n_3\phi = 0$ and thus if we require $\phi \neq 0$ then $n_3 = 0$ and this excludes Types VIII and IX. This is unfortunate as we do not have a generalization of the closed Friedmann model (k = 1), which is a special case of the Type IX models.

In Table 2 we now list those types, together with the appropriate canonical forms, which are allowed in Classes C and D.

Table 2

Class	C				D			
Type	I	II	VI_o	VII_o	V	IV	VI_h	VII_h
a	0	0	0	0	1	1	$(-h)^{\frac{1}{2}}$	$h^{\frac{1}{2}}$
n_1	0	1	1	1	0	1	1	1
n_2	0	0	-1	1	0	0	-1	1
n_3	0	0	0	0	0	0	0	0
ϕ	$F(x^3)$	$F(x^1)$	$F(x^1)$	$F(x^1)$	$F(x^1)$	$F(x^1)$	$-hF(x^1)$	$hF(x^1)$

Note: We have written $\phi = F(x^1)a$ for latter convenience.

Type III = Type VI_{-1}.

2.4 Remarks

It was stated at the beginning that the Weyl tensor is the unique curvature invariant under conformal changes in the metric tensor. Thus, from the group theoretic point of view, the conformal motions preserve the conformal group structure of the homogeneous models while destroying the metric motions that give rise to spatial homogeneity. This first point is demonstrated by the fact that physically the light geometry of the conformal models is the same as that of the spatially homogeneous models. Null curves remain null curves because the square of their tangent vector remains zero, i.e.

$$0 = \underline{\ell}.\underline{\ell} = g_{ab}\ell^a\ell^b \Rightarrow e^{2\sigma}g_{ab}\ell^a\ell^b = 0$$

Also, it is the second point above that hopefully will afford us the freedom to model the actual agglomerations of matter in the forms of galaxies and clusters of galaxies that are precluded by the assumptions of homogeneity that underlie the standard models.

Thus, the conformally related models developed here may be expected to preserve most of the agreement between the predictions of the standard theories and the cosmological observables as well as acting as an environment that allows for agglomerations of matter.

CHAPTER 3

LIE DERIVATIVE AND KINEMATICAL PROPERTIES

§3.1 Introduction

In the last chapter we studied the metrical structure and group classification of the conformally related models. Here we shall study some of the more salient physical properties of these space-times. We also introduce a tetrad coordinate system which will be used in subsequent chapters.

In the first section we investigate the Lie derivative of certain geometrical objects which we use to describe the conformal models and we obtain the relations with the corresponding homogeneous quantities. While these relations are of a simple nature for homothetic motions, they are more complicated for the conformal case. Also, while we will be restricting our attention to those models which admit a perfect-fluid energy-momentum tensor, it is seen that in general the conformal models will demand an energy-momentum tensor with anisotropic stress terms.

We then investigate the kinematical structure of the fluid in the models. In the investigations of homogeneous anisotropic models, filled with a perfect fluid, the relation between the 4-normal to the hypersurfaces s(t) and the 4-velocity of the fluid plays an essential role. If they are collinear, then the world lines of matter are geodesic, even in the case of non-zero pressure. However, if $\hat{\underline{n}}$ and $\hat{\underline{u}}$ are not collinear, then in general the 4-acceleration is not zero and hydrodynamical effects become apparent. Relations are obtained for the various kinematical quantities used in describing the fluid and it is seen that in general the fluid will have non-zero acceleration and vorticity.

To examine the space-times in accordance with the group classification a tetrad coordinate system is introduced. This is again used in the next chapter to investigate the existence of perfect fluid solutions.

§3.2 The Lie Derivative And Conformal Motions

In this section we wish to list some properties of the Lie derivative and some results concerning conformal motions. Firstly, the following expressions for the Lie derivative with respect to a vector field $\underline{\xi}$ of a scalar field A, a vector field A^A, and a 2-rank tensor field A_{AB} are used extensively [36].

$$L_\xi A = A_{,B}\xi^B$$

$$L_\xi A^B = A^B{}_{,C}\xi^C - A^C\xi^B{}_{,C}$$

$$L_\xi A_{BC} = A_{BC,D}\xi^D + A_{DC}\xi^D{}_{,B} + A_{BD}\xi^D{}_{,C}$$

In these formulas, partial derivatives, A can be replaced by covariant derivatives $;_A$ throughout.

We now let the vector field ξ generate on the space-time a 1-parameter group of conformal transformations which satisfy the relations

$$L_\xi \hat{g}_{AB} = 2\phi\hat{g}_{AB} \Leftrightarrow \xi_{A|B} + \xi_{B|A} = 2\phi\hat{g}_{AB} \tag{2.1}$$

where $$\phi = \phi(x^A) \quad \text{and} \quad \phi_A = \nabla_A\phi\ . \tag{2.2}$$

We wish to find out how the Riemann curvature tensor $R^{A}{}_{BCD}$ transforms. To do this we will need to find the Lie derivative of both sides of the Ricci identity

$$\hat{R}^{A}{}_{BCD}X^{B} = [\hat{\nabla}_C\hat{\nabla}_D - \hat{\nabla}_D\hat{\nabla}_C]X^{A} \quad . \tag{2.3}$$

Consider the following expressions

$$\hat{\nabla}_A[L_\xi Y^B] = \hat{\nabla}_A[Y^B{}_{|C}\xi^C - Y^C\xi^B{}_{|C}]$$

$$= (\hat{\nabla}_A\hat{\nabla}_C Y^B)\xi^C + \hat{\nabla}_C Y^B\hat{\nabla}_A\xi^C - \hat{\nabla}_A Y^C\hat{\nabla}_C\xi^B - Y^C\hat{\nabla}_A\hat{\nabla}_C\xi^B$$

and $$L_\xi[\hat{\nabla}_A Y^B] = \hat{\nabla}_C(\hat{\nabla}_A Y^B)\xi^C - \hat{\nabla}_A Y^C\xi^B{}_{|C} + \hat{\nabla}_C Y^B\hat{\nabla}_A\xi^C$$

$$= (\hat{\nabla}_C\hat{\nabla}_A Y^B)\xi^C - \hat{\nabla}_A Y^C\hat{\nabla}_C\xi^B + \hat{\nabla}_C Y^B\hat{\nabla}_A\xi^C$$

$$\Rightarrow \quad \hat{\nabla}_A[L_\xi Y^B] - L_\xi[\hat{\nabla}_A Y^B]$$

$$= - Y^B\phi_A + Y_A\phi_B - \delta^B{}_A(Y^C\phi_C) \tag{2.4a}$$

and $\hat{\nabla}_A[L_\xi Y_B] - L_\xi[\hat{\nabla}_A Y_B]$

$$= Y_B\phi_A + Y_A\phi_B - g_{AB}(Y^C\phi_C) \tag{2.4b}$$

and combining the above two expressions we obtain

$$\hat{\nabla}_A[L_\xi Y^D Y_B] - L_\xi[\hat{\nabla}_A(Y^D Y_B)]$$

$$= Y_A Y_B\phi^D + Y_A Y^D\phi_B - \delta^D{}_A Y_B Y^C\phi_C - g_{AB}Y^D Y^C\phi_C$$

and from this we obtain for a mixed second order tensor $T^{D}{}_{B}$ the result

$$\hat{\nabla}_A[L_\xi T^D{}_B] - L_\xi[\hat{\nabla}_A T^D{}_B]$$

$$= T_{AB}\phi^D + T^D{}_A\phi_B - T^D{}_C g_{AB}\phi^C - T^C{}_B(\phi_C\delta^D{}_A) \quad .$$

Let $T^D{}_B = \hat{\nabla}_B X^D$, then

$$L_\xi[\hat{\nabla}_A\hat{\nabla}_B X^D] = \hat{\nabla}_A[L_\xi\hat{\nabla}_B X^D] - \hat{\nabla}_B X_A\phi^D - \hat{\nabla}_A X^D\phi_B$$

$$+ (\hat{\nabla}_C X^D)\phi^C g_{AB} + (\hat{\nabla}_B X^C)\phi_C\delta^D{}_A$$

Interchanging A and B and subtracting

$$L_\xi[(\hat{\nabla}_A\hat{\nabla}_B - \hat{\nabla}_B\hat{\nabla}_A)X^D] = \hat{\nabla}_A[L_\xi\hat{\nabla}_B X^D] - \hat{\nabla}_B[L_\xi\hat{\nabla}_A X^D]$$

$$- \hat{\nabla}_B X^A\phi^D + \hat{\nabla}_A X_B\phi^D - \hat{\nabla}_A X^D\phi_B + \hat{\nabla}_B X^D\phi_A$$

$$+ (\hat{\nabla}_B X^C)\phi_C\delta^D{}_A - (\hat{\nabla}_A X^C)\phi_C\delta^D{}_B$$

$$\Rightarrow \qquad L_\xi[2\hat{\nabla}_{[A}\hat{\nabla}_{B]}X^D] = 2\hat{\nabla}_{[A}[L_\xi\hat{\nabla}_{B]}X^D] + 2\hat{\nabla}_{[A}X_{B]}\phi^D$$

$$- 2\hat{\nabla}_{[A}X^D\phi_{B]} + 2\delta^D{}_{[A}\hat{\nabla}_{B]}X^C\phi_C \qquad (2.5)$$

Now, from (2.4) we have

$$L_\xi[\hat{\nabla}_B Y^D] = \hat{\nabla}_B[L_\xi X^D] + X^D\phi_B - X_B\phi^D + \delta^D{}_B(X^C\phi_C)$$

Substituting this into (2.5) we obtain, after rearranging

$$L_\xi[2\hat{\nabla}_{[A}\hat{\nabla}_{B]}X^D] = 2\hat{\nabla}_{[A}\hat{\nabla}_{B]}[L_\xi X^D] - 2X^D\hat{\nabla}_{[B}\hat{\nabla}_{A]}\phi$$

$$+ 2X_{[A}\hat{\nabla}_{B]}\phi^D + 2\delta^D{}_{[B}\hat{\nabla}_{A]}\phi_C X^C$$

Now, from (2.3) we have

$$2\hat{\nabla}_{[A}\hat{\nabla}_{B]}[L_\xi X^D] = \hat{R}^D{}_{CAB}[L_\xi X^C]$$

and substituting back into the previous expression

$$L_\xi[\hat{R}^D{}_{CAB}X^C] = \hat{R}^D{}_{CAB}[L_\xi X^C] + 2X^D\hat{\nabla}_{[A}\phi_{B]}$$

$$+ 2X_{[A}\hat{\nabla}_{B]}\phi^D - 2\delta^D{}_{[A}\hat{\nabla}_{B]}\phi_C X^C$$

and expanding the left-hand side we obtain the desired result

$$[L_\xi\hat{R}^D{}_{CAB}]X^C = 2X^D\hat{\nabla}_{[A}\phi_{B]} + 2X_{[A}\hat{\nabla}_{B]}\phi^D - 2\delta^D{}_{[A}\hat{\nabla}_{B]}\phi_C X^C$$

$$= 2X^C\delta^D{}_C\hat{\nabla}_{[A}\phi_{B]} + 2X^C g_{C[A}\hat{\nabla}_{B]}\phi^D - 2X^C\delta^D{}_{[A}\hat{\nabla}_{B]}\phi_C$$

$$\Rightarrow \quad [L_\xi\hat{R}^D{}_{CAB}] = 2g_{C[A}\hat{\nabla}_{B]}\phi^D - 2\delta^D{}_{[A}\hat{\nabla}_{B]}\phi_C + 2\delta^D{}_C\hat{\nabla}_{[A}\phi_{B]} \qquad (2.7)$$

Since $\hat{R}_{CB} = \hat{R}^A{}_{CAB}$ we also find

$$L_\xi\hat{R}_{CB} = -2\hat{\nabla}_{(C}\phi_{B)} - \hat{g}_{CB}\hat{\nabla}_D\phi^D \qquad (2.8)$$

and also, since $\hat{R} = \hat{g}^{CB}\hat{R}_{CB}$ we have

$$L_\xi\hat{R} = -2\phi\hat{R} - 6\hat{\nabla}_C\phi^C \qquad (2.9)$$

By putting ϕ = constant we obtain the results for a homothetic motion

$$\left.\begin{aligned} \mathcal{L}_\xi \hat{R}^D{}_{CAB} &= 0 \\ \mathcal{L}_\xi \hat{R}_{CD} &= 0 \\ \text{and} \qquad \mathcal{L}_\xi \hat{R} &= -\,2\phi\hat{R} \end{aligned}\right\} \tag{2.10}$$

as obtained by Wainwright and Yaremovicz [42].

In the study of spatially homogeneous models where it is assumed that the space-time metric admits one or more Killing vector fields $\underline{\xi}$ i.e. ($\phi = 0$), which generate a group of isometries, it has been shown that when the energy-momentum tensor has the form of a perfect fluid that the source quantities are invariant under the group, in the sense that

$$\mathcal{L}_\xi u_A = 0 \qquad \mathcal{L}_\xi \rho = 0 \qquad \mathcal{L}_\xi p = 0$$

where u_A, ρ and p are respectively the fluid velocity vector, density and pressure. In this study we shall again assume that the energy-momentum associated with the conformal models is a perfect fluid, as this assumption appears essential for the physical interpretations in the corresponding standard models. Thus

$$\hat{T}_{AB} = (\hat{\rho} + \hat{p})\hat{u}_A\hat{u}_B + \hat{p}\hat{g}_{AB} \tag{2.11}$$

and using the results (2.8) and (2.9) we can study the Lie derivatives of the conformal source quantities under the conformal group.

From (2.8), (2.9) and Einstein's field equations

$$\mathcal{L}_\xi \hat{T}_{AB} = -\,2\phi_{(A|B)} + 2\hat{g}_{AB}\phi^C{}_{|C} \tag{2.12}$$

Now the perfect fluid $\hat{T}_{AB}$ is so structured that it has a simple eigenvalue yielding a timelike eigenvector (the velocity vector $\hat{u}$) and an eigenvector yielding an eigenvalue of multiplicity three. Hence using the relation

$$\hat{T}_{AB}\hat{u}^B = -\hat{\rho}\hat{g}_{AB}\hat{u}^B$$

we find

$$L_\xi\hat{\rho} = -2\phi\hat{\rho} - 2[\phi_{(A|B)} - \hat{g}_{AB}\phi^C{}_{|C}]\hat{u}^A\hat{u}^B \tag{2.13a}$$

$$L_\xi\hat{u}_A = \phi\hat{u}_A + 2(\hat{\rho} + \hat{p})^{-1}[\phi_{(C|D)}\hat{u}^C\hat{u}^D g_{AB} + \phi_{(A|B)}]\hat{u}^B \tag{2.13b}$$

and

$$L_\xi\hat{p} = -2\phi\hat{p} - \frac{2}{3}[\phi_{(A|B)} - \hat{g}_{AB}\phi^C{}_{|C}]\hat{h}^{AB} \tag{2.13c}$$

where

$$\hat{h}_{AB} = \hat{g}_{AB} + \hat{u}_A\hat{u}_B \ .$$

Thus we can see from these expressions that in general the geometric objects in the conformal spaces depend in a complicated way upon the conformal factor ϕ and its derivative.

From equations (2.7) - (2.10) and (2.13) we have the following conclusion. We see that each physical geometric object Λ is invariant under a group of Killing vectors ($\phi = 0$) and has a dimension q such that under a homothetic motion, Λ transforms like

$$L_\xi\Lambda = q\langle\phi,\xi\rangle\Lambda \ . \tag{2.14a}$$

Thus a spatially homothetic field $\hat{\Lambda}$ of dimension q is related to a spatially homogeneous field $\Lambda(t)$ by

$$\hat{\Lambda} = e^{q\sigma}\Lambda(t) \tag{2.14b}$$

Since the covariant metric has dimension 2, the dimension depends on the positioning of the indices [18]. From (2.13) we thus have

$$\hat{p} = e^{-2\sigma}p'(t) \qquad \hat{\rho} = e^{-2\sigma}\rho'(t) \qquad \hat{u}_A = e^{\sigma}u'_A(t) \tag{2.15}$$

Note that ρ', p' and u'_A may depend upon the homothic constant ϕ but will reduce to the homogeneous quantities of the models of Class E when $\phi = 0$. When the space-time admits a more general group of conformal motions, the relationship between $\hat{\Lambda}$ and $\Lambda'(t)$ is more complicated, but can be found by using the respective expressions for the geometric objects and substituting the relation

$$\hat{g}_{AB} = e^{2\sigma}g_{AB}(x^C). \tag{2.16}$$

We find

$$\hat{\Gamma}^A{}_{BC} = \tfrac{1}{2}\hat{g}^{AD}[\hat{g}_{DB,C} + \hat{g}_{DC,B} - \hat{g}_{BC,D}]$$

$$= \Gamma^A{}_{BC}(g) + (\delta^A{}_B\partial_C\sigma + \delta^A{}_C\partial_B\sigma - g^{AD}g_{BC}\partial_D\sigma) \tag{2.17a}$$

$$\hat{R}_{AB} = \partial_C\hat{\Gamma}^C{}_{BA} - \partial_B\hat{\Gamma}^C{}_{CA} + \hat{\Gamma}^C{}_{CD}\hat{\Gamma}^D{}_{BA} - \hat{\Gamma}^C{}_{DA}\hat{\Gamma}^D{}_{CB}$$

$$= R_{AB}(g) - 2\nabla_B\nabla_A\sigma + 2\partial_A\sigma\partial_B\sigma - g_{AB}(\nabla_C\nabla^C\sigma + 2\nabla_C\sigma\nabla^C\sigma) \tag{2.17b}$$

and contraction gives

$$\hat{R} = e^{-2\sigma}[R(g) - 6\nabla^A\nabla_A\sigma - 6\partial_A\sigma\partial^A\sigma] \tag{2.17c}$$

It can be shown, using the relations $L_\xi(\nabla_B\partial_A\sigma) = \nabla_B(L_\xi\partial_A\sigma)$ and $L_\xi(\partial_A\sigma) = \phi_A$, that the above equations are solutions of equations (2.8) and (2.9).

From these equations, we note that a spatially conformal field $\hat{\Lambda}$ of dimension q is related to another field $\Lambda'(x^A)$ where

$$\hat{\Lambda} = e^{q\sigma}\Lambda'(x^A) \tag{2.18}$$

This form will be used in the rest of this chapter, where appropriate.

§3.3 Energy-Momentum Tensor

We are now in a position to be able to study some of the properties of the energy-momentum tensor in relation to the conformal models. From Einstein's equations we have

$$\hat{T}_{AB} = \hat{R}_{AB} - \tfrac{1}{2}\hat{g}_{AB}\hat{R} = \hat{G}_{AB} \tag{3.1}$$

where $\hat{G}_{AB}$ is the Einstein tensor. Substituting in (2.17 b,c) we have

$$\begin{aligned}\hat{T}_{AB} &= T_{AB} - 2\nabla_A\nabla_B\sigma + 2\partial_A\sigma\partial_B\sigma \\ &\quad + g_{AB}[2\nabla^C\partial_C\sigma + \partial^C\sigma\partial_C\sigma]\end{aligned} \tag{3.2}$$

Assuming that the spatially homogeneous models admit a perfect fluid energy-momentum tensor we have

$$\begin{aligned}\hat{T}_{AB} &= (\rho + p)u_A u_B - 2\nabla_A\nabla_B\sigma + 2\partial_A\sigma\partial_B\sigma \\ &\quad + g_{AB}[2\nabla^C\partial_C\sigma + \partial^C\sigma\partial_C\sigma + p]\end{aligned} \tag{3.3}$$

However, we see from this equation that, in general, the terms $\nabla_A \nabla_B \sigma$ and $\partial_A \sigma \partial_B \sigma$ will preclude the eigenvalue and eigenvector structure of a perfect-fluid energy-momentum tensor [see last section]. Thus, $\hat{T}_{AB}$ will, in general take the more general form

$$\hat{T}_{AB} = (\hat{\rho} + \hat{p})\hat{u}_A \hat{u}_B + \hat{p}\hat{g}_{AB} + \hat{q}_{AB} \tag{3.4}$$

where $\hat{q}_{AB}$ constitute the components of a symmetric tensor which describes anisotropy [since $\hat{q}_{AB} \neq 0$ precludes the isotropic 3-space of eigenvectors of $\hat{T}_{AB}$]. Conformally homogeneous models thus, in general, demand more general dynamical processes than those exhibited by a perfect fluid [see next section]. Only a restricted class of conformal models will admit a perfect fluid [see Chapter 4]. In these models we require

$$- 2\nabla_A \nabla_B \sigma + 2\partial_A \sigma \partial_B \sigma = Au_A u_B + Bg_{AB} \ .$$

In (3.4), if $\hat{\rho}$ is to be the eigenvalue corresponding to the eigenvalue $\hat{u}_A$, we require $\hat{q}_{AB}\hat{u}^B = 0$ and we can also normalize $\hat{\underline{u}}$. Under these conditions $\hat{\rho}$ and $\hat{u}_A$ can be identified with the total rest energy and the velocity vector field of fluid elements of C, respectively.

Following Edelen and Wilson [33] we write

$$\hat{u}_A = \lambda u_A + v_A \qquad v^A u_A = 0 \tag{3.5}$$

where λ and v_A are to be determined. $\underline{V}$ has been interpreted as the dispersion vector of the conformal models with respect to the corresponding velocity vector field of the homogeneous model. In this way we can make sense of the conformal models even though there are, in general, no isotropic comoving coordinate systems on elements of C (see next section).

§3.4 Kinematical Properties

The immediate geometrical objects defined in a space-time in which a spacelike group C_3 of conformal motions acts are the surfaces in which the group acts and defines, s(t) and the fluid flow vector $\hat{u}^A$. The surfaces s(t) then determine a unique-future directed normal field $\hat{n}^A$, $(n^A n_A = -1)$ which, by definition, is rotation-free.

In the investigations of homogeneous anisotropic models filled with a perfect fluid, the relation between the 4-normal $\hat{n}^A$ and the 4-velocity $\hat{u}^A$ plays an essential role. If they are collinear, then the world lines of the matter are geodesic, even in the case of non-zero pressure. However, if $\hat{n}^A$ and $\hat{u}^A$ are not collinear, then in general the 4-acceleration is not zero and hydrodynamical effects become apparent. McIntosh [43] has already shown that, in general, homothetic models are tilted and we can expect a similar result for conformal models.

Considering firstly the normal field $\hat{n}^A$, we find that unlike the homogeneous models, it is not geodesic. To see this consider a family of conformal vector fields $\xi^A{}_\alpha$ which are linearly independent at each point. Then we have

$$\xi_{\alpha A}\hat{n}^A = 0 \qquad \alpha = 1,\ldots,3$$

$$\Rightarrow \qquad \xi_{\alpha A|B}\hat{n}^A + \xi_{\alpha A}\hat{n}^A{}_{|B} = 0 \; ^*$$

*Covariant differentiation w.r.t. $\hat{g}_{AB}$ is denoted by a bar '|'

Multiply by $\hat{n}^B \Rightarrow \tfrac{1}{2}\xi_{\alpha(A|B)}\hat{n}^A\hat{n}^B + \xi_{\alpha A}\hat{n}^A{}_{|B}\hat{n}^B = 0$

and since $$\xi_{\alpha(A|B)} = 2\phi_\alpha g_{AB}$$

we have $$\xi_{\alpha A}\hat{n}^A{}_{|B}\hat{n}^B = -\phi_\alpha g_{AB}\hat{n}^A\hat{n}^B = \phi_\alpha = \xi^A{}_\alpha \partial_A\sigma$$

$$\Rightarrow \qquad \hat{n}_{A|B}\hat{n}^B = \partial_A\sigma \ . \tag{4.1}$$

Now, since the normal congruence is rotation-free and thus satisfies the condition

$$\hat{n}_{[A}\hat{n}_{B|C]} = 0$$

, then by standard theorems it can be shown [see Ellis in [37]) that $\hat{n}_A$ is proportional to a gradient i.e.

$$w = 0 \Leftrightarrow \exists \text{ locally functions } \ell(x^A),\ t(x^A) : \hat{n}_A = -\ell t_{,A} \tag{4.2}$$

Since $t_{,A}$ is a vector normal to the surfaces $\{t = \text{constant}\}$ this is the condition that $\hat{n}_A$ be orthogonal to these surfaces. Thus the spaces defined at each point by the spatial projection tensor $\hat{h}_{AB} = g_{AB} + \hat{n}_A\hat{n}_B$ mesh together in this case to form spacelike surfaces $\{t = \text{constant}\}$ orthogonal to $\hat{n}^A$ (i.e. $\hat{h}_{AB}\hat{n}^B = 0$). Thus the function $t(x^A)$ may be thought of as a cosmological 'time' coordinate defined by the normal congruence. However, since these future-directed normal vector fields are not geodesic, the function $t(x^A)$ does not measure proper time along the world lines. Thus the hypersurface normals are the tangent vector fields of a non-geodesic hypersurface orthogonal congruence. A similar result was noted by Eardley [18].

Traditionally, the problems of relativistic cosmology are attacked by choosing a coordinate system in which all fundamental particles are at rest. This result shows that the fact that such a coordinate system exists is by no means obvious and is usually closely related to the principle of homogeneity [44].

The motion of the cosmological fluid is described by a congruence of timelike curves tangent to the unit vector field $\hat{u}^A [\hat{u}^A \hat{u}_A = -1]$, where the fluid flow vector is uniquely defined as the future directed timelike eigenvector of the Ricci tensor. In general the surfaces S(t) will not be orthogonal to $\underline{\hat{u}}$. This is the case whenever the vorticity of the fluid congruence is different from zero. Thus the rest spaces H orthogonal to $\underline{\hat{u}}$, defined by the projection operator $\hat{m}_{AB} = \hat{g}_{AB} + \hat{u}_A \hat{u}_B$ $(\hat{m}_{AB}\hat{u}^B = 0)$, are in general tilted with respect to the surfaces s(t). Note however that tilt does not necessarily imply non-zero vorticity. The geometry is displayed in fig. 1.

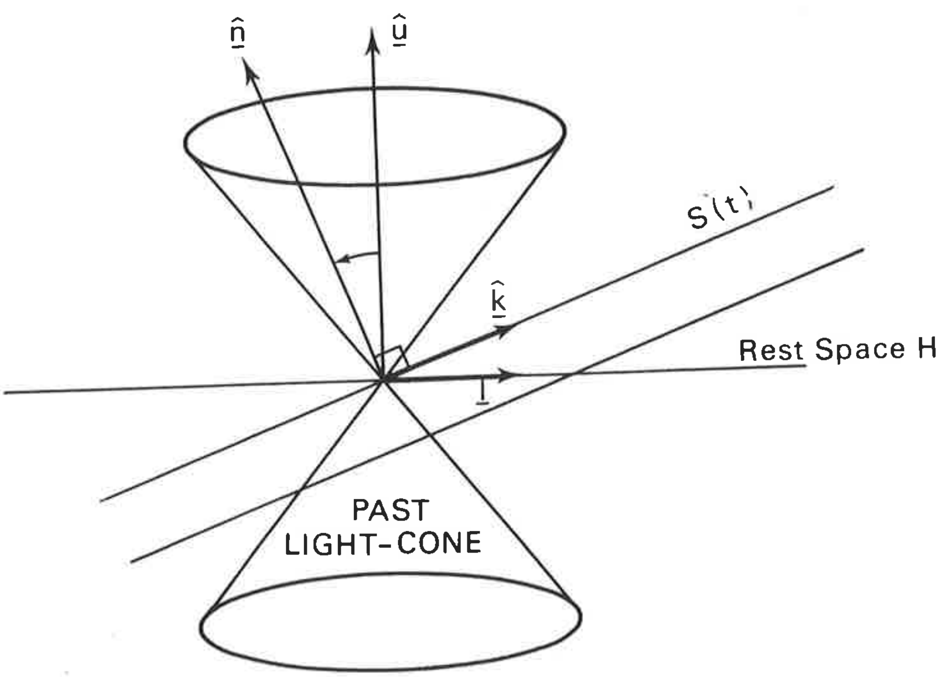


fig 1

Following the notation of Ellis and King [35] we define the relation between $\hat{u}^A$ and $\hat{n}^A$ by

(a) the hyperbolic angle of tilt β, where

$$\cosh\beta = -\hat{u}^A\hat{n}_A \qquad \beta \geqslant 0 \tag{4.3}$$

and the direction of tilt, specified either by

(b) the direction $\hat{k}^A$ of the projection of $\hat{u}^A$ in s(t)

$$\hat{h}^A{}_B\hat{u}^B = \sinh\beta\hat{k}^A \Rightarrow \hat{k}_A\hat{n}^A = 0,\ \hat{k}_A\hat{k}^A = 1 \tag{4.4}$$

or by

(c) the direction $\hat{\ell}^A$ of $\hat{n}^A$ perpendicular to $\hat{u}^A$

$$\hat{m}^A{}_B\hat{n}^B = -\sinh\beta\hat{\ell}^A \Rightarrow \hat{\ell}_A\hat{u}^A = 0,\ \hat{\ell}_A\hat{\ell}^A = 1 \tag{4.5}$$

Then one has the relations

$$\hat{u}^A = \cosh\beta\hat{n}^A + \sinh\beta\hat{k}^A \tag{4.6a}$$

$$\hat{n}^A = \cosh\beta\hat{u}^A - \sinh\beta\hat{\ell}^A \ .$$

When $\beta \neq 0$, $\hat{k}^A$ and $\hat{\ell}^A$ are uniquely defined by (4.4), (4.5) and (4.6) expresses the way in which $\hat{u}^A$ is tilted with respect to the surfaces s(t) [see fig. 1].

We now wish to relate the various quantities introduced above to their respective counterparts which apply in the spatially homogeneous models. We note that from the form of the conformal metric (4.2) that the normal vectors of the two models are related as follows;

$$\hat{n}_A = e^{\sigma} n_A \tag{4.7}$$

Since in the spatially homogeneous models the normal is geodesic and rotation-free, we have

$$n_A = -\, t_{,A} \tag{4.8}$$

where t measures proper time. Comparing (4.7) and (4.8) with (4.2) we find

$$\ell(x^A) = e^{\sigma}$$

$$\Rightarrow \quad \hat{n}_A = -\, e^{\sigma} t_{,A} \tag{4.9}$$

Similarly, one can define a direction k_A' such that

$$\hat{k}_A = e^{\sigma} k_A'$$

and so we may write

$$\hat{u}_A = e^{\sigma}[\cosh\beta n_A + \sinh\beta k_A'] \tag{4.10}$$

The corresponding velocity vector in the spatially homogeneous models can similarly be written as

$$u_A = \cosh\theta n_A + \sinh\theta k_A \tag{4.11}$$

where the angle and direction of tilt, θ and k_A, are defined as before. Solving for n_A from (4.11) and substituting into (4.10) we have the expression

$$\hat{u}_A = \left(\frac{e^{\sigma}\cosh\beta}{\cosh\theta}\right) u_A + e^{\sigma}(\sinh\beta k_A' - \tanh\theta\cosh\beta k_A)$$

and this has the form (3.5). We see that the dispersion vector is so called since it is the direction vector corresponding to the difference between the two directions of tilt*.

We can now use the above relations to give a kinematical description of the fluid as seen by the local inertial observer, one whose frame is Fermi propagated along the world lines of the matter. In the usual decomposition,

$$\hat{u}_{A|B} = \hat{w}_{AB} + \hat{\sigma}_{AB} + \frac{1}{3}\hat{\theta}\hat{m}_{AB} - \hat{\dot{u}}_A\hat{u}_B \tag{4.12}$$

, where the acceleration $\hat{\dot{u}}_A$, volume expansion $\hat{\theta}$, shear tensor $\hat{\sigma}_{AB}$, and vorticity tensor $\hat{w}_{AB}$ are defined by

$$\left.\begin{aligned} \hat{\dot{u}}_A &= \hat{u}_{A|B}\hat{u}^B \\ \hat{\theta} &= \hat{u}_{A|B}\hat{m}^{AB} \\ \hat{\sigma}_{AB} &= \hat{u}_{(C|D)}\hat{m}^C{}_A\hat{m}^D{}_B - \frac{1}{3}\hat{\theta}\hat{m}_{AB} \\ \hat{w}_{AB} &= \hat{u}_{[C|D]}\hat{m}^C{}_A\hat{m}^D{}_B \end{aligned}\right\} \tag{4.13}$$

Substituting (4.10) into these expressions we have:

$$\hat{u}_{A|B} = \hat{u}_A\partial_B\sigma + e^{\sigma}[\cosh\beta(n_{A|B} + \partial_B\beta k_A{}') + \sinh\beta(k'{}_{A|B} + \partial_B\beta n_A)] \tag{4.14}$$

where, using (2.17a) we may write

*In the homothetic case, a spatial homothetic field ϕ' of dimension q is always related to a spatially homogeneous field by $\phi' = e^{q\sigma}\phi$. The velocity vector u^A has $q = -1$ and so u_A has $q = 1$. Hence we obtain $\beta = \beta(t)$.

$$n_{A|B} = \partial_B n_A - \hat{\Gamma}^C{}_{AB} n_C$$

$$= \partial_B n_A - [\Gamma^C{}_{AB} + \delta^C{}_A \partial_B \sigma + \delta^C{}_B \partial_A \sigma - g_{AB} \partial^C \sigma] n_C$$

$$= n_{A;B} - n_A \partial_B \sigma - n_B \partial_A \sigma + g_{AB} \partial^C \sigma n_C$$

$$= \theta_{AB} - \dot{n}_A n_B + w_{AB} - n_A \partial_B \sigma - n_B \partial_A \sigma + g_{AB} \partial_C \sigma$$

$$= \theta_{AB} - n_A \partial_B \sigma - n_B \partial_A \sigma$$

where we have employed a similar decomposition of $n_{A;B}$ as used in (4.12), where in this case, the acceleration and vorticity are zero. Note that if the homogeneous models are non-tilted (i.e. $\theta = 0$ in (4.11)) then $n_A = u_A$ and θ_{AB} then becomes the expansion tensor of the fluid in these models. Again, as noted earlier, the acceleration and vorticity in the non-tilted homogeneous models is zero.

Similarly, we have

$$k'_{A|B} = k'_{A;B} - k_A' \partial_B \sigma - k'_B \partial_A \sigma + g_{AB} \partial^C \sigma k_C'$$

Hence

$$\hat{u}_{A|B} = e^{\sigma}[\ell_A' \partial_B \beta + \cosh\beta(\theta_{AB} - n_B \partial_A \sigma) + \sinh\beta(k'_{A;B} - k'_B \partial_A \sigma + g_{AB} \partial^C \sigma k'_C)] \tag{4.15}$$

where $$\ell_A' \equiv e^{-\sigma} \hat{\ell}_A$$

Substituting (4.15) in (4.13) we obtain expressions for the fluid quantities in terms of the respective homogeneous quantities with respect to the normal congruence, the angle of tilt and the conformal factor. The acceleration is

$$\hat{\dot{u}}_A = \cosh\beta\ell_A{}'\partial_B\beta n^B + \sinh\beta\ell'_A\partial_B\beta k'^B + \partial_A\sigma$$

$$+ \cosh\beta\sinh\beta(\theta_{AB}k'^B + k'_{A;B}n^B + n_Ak'_B\partial^B\sigma) \qquad (4.16)$$

$$+ \sinh^2\beta(k'_{A;B}k'^B + k'_Ak'_B\partial^B\sigma)$$

$$= \dot{u}_A + u_A\partial_B\sigma u^B + \partial_A\sigma$$

; the expansion is

$$\hat{\theta} = e^{-\sigma}[\ell'_A\partial^A\beta + \theta\cosh\beta + \sinh\beta(k^A{}_{;A} + 3k_A\partial^A\sigma)] \qquad (4.17)$$

$$= e^{-\sigma}(\theta_{(h)} + 3\partial_A\sigma u^A)$$

; the vorticity vector $\hat{w}^A = \frac{1}{2}\eta^{ABCD}\hat{u}_B\hat{u}_{D;C}$ is

$$\hat{w}^A = \tfrac{1}{2}e^{2\sigma}\eta^{ABCD}[\cosh\beta\sinh\beta n_Bk'_{C;D} + \sinh^2\beta k'_Bk'_{C;D}]$$

$$= e^{2\sigma}w^A{}_{(h)} \qquad (4.18)$$

and while the expressions for the expansion, shear and rotation tensors are somewhat more complex, we have

$$\hat{\sigma}_{AB} = e^{\sigma}\sigma_{AB(h)} \quad \text{and} \quad \hat{w}_{AB} = e^{\sigma}w_{AB(h)}$$

In these expressions, $\theta_{(h)}$, $w_{(h)}$, $\sigma_{AB(h)}$ and $w_{AB(h)}$ refer to the corresponding terms in the homogeneous models which one regains by putting $\sigma = 0$ [see Ellis and King [35]. Since the shear tensor, vorticity vector and vorticity tensor in the conformal models differ from their homogeneous counterparts only by a multiplicative conformal factor, they have the same physical significance in both models. Further, unlike the standard models, the acceleration and vorticity will be non-zero in these models.

From (4.18), the projection of $\hat{w}^A$ in the direction of $\hat{k}^A$ is

$$\hat{k}^A\hat{w}_A = -\cosh\beta n^A\hat{w}_A$$

Thus one sees that the vector $\hat{w}^A$ lies in the surface $s(t)$ iff it is perpendicular to the tilt directions $\hat{k}^A$ and $\hat{\ell}^A$. To examine the behaviour of particular group types it will be convenient to introduce the group classification of Chapter 2. This is achieved in the next section where we write the above equations in terms of a tetrad basis.

Finally, consider the energy-momentum conservation equations $\hat{T}^{AB}{}_{|B} = 0$ for the perfect fluid

$$\hat{T}_{AB} = (\hat{\rho} + \hat{p})\hat{u}_A\hat{u}_B + \hat{g}_{AB}\hat{p} \tag{4.19}$$

The components $\hat{u}_A\hat{T}^{AB}{}_{|B} = 0$ and $\hat{m}^C{}_A\hat{T}^{AB}{}_{|B} = 0$ become

$$\hat{\rho}_{,A}\hat{u}^A + (\hat{\rho} + \hat{p})\hat{\theta} = 0$$

and

$$\hat{m}_A{}^B\hat{p}_{,B} + (\hat{\rho} + \hat{p})\hat{u}_{A|B}\hat{u}^B = 0$$

Substituting in the forms $\hat{\rho} = e^{-2\sigma}\rho'(x^A)$ and $\hat{p} = e^{-2\sigma}p'(x^A)$ these equations become

$$\rho'_{,A}\hat{u}^A + (\rho' + p')\hat{\theta} = 2\rho'\partial_A\sigma\hat{u}^A \tag{4.20a}$$

$$\hat{m}_A{}^B p'_{,B} + (\rho' + p')(\hat{u}_A)^{\bullet} = 2p'\partial_B\sigma\hat{m}_A{}^B \tag{4.20b}$$

From (4.20b) we see that for dust solutions (i.e. $p' = 0$) the acceleration $(\hat{u}_A)^{\bullet}$ is zero. However, from (4.1) we have $(\hat{u}_A)^{\bullet} = \partial_A\sigma$ if $\beta = 0$ i.e. if the conformal model is non-tilted. Hence we have the result that there are no non-tilted dust conformal models.

For homothetic 'self-similar' models $\rho' = \rho'(t)$ and $p' = p'(t)$ and so we may define the functions [34]

$$w(t) \equiv \exp\int_{t_0}^{t}\frac{d\rho'}{\rho' + p'} \qquad r(t) \equiv \exp\int_{t_0}^{t}\frac{dp'}{\rho' + p'} \tag{4.21}$$

and substituting into equations (4.20) we have

$$e^{-\sigma}\cosh\beta d(\log w)/dt + \hat{\theta} = \frac{2\rho'\hat{\partial}_A\sigma\hat{u}^A}{(\rho' + p')} \qquad (4.22a)$$

$$\sinh\beta d(\log r)/dt\hat{\ell}_A + (\hat{u}_A)^{\bullet} = \frac{2p'\hat{\partial}_B\sigma\hat{m}^B{}_A}{(\rho' + p')} \qquad (4.22b)$$

Substituting (4.16) into (4.22b) we have after contracting with k^A

$$\underline{d}\log(r\sinh\beta)/dz + \hat{\theta}_{AB}\hat{k}^A\hat{k}^B = \frac{(p' - \rho')}{p' + \rho'}\,\frac{\cosh\beta}{\sinh\beta}\,\partial_A\sigma\hat{k}^A \qquad (4.23)$$

The form of this equation shows that if $\hat{\rho} = \hat{p}$, then β is either zero or non-zero for all t i.e. a tilted model stays tilted. Hence, combining this result with the result first proven by McIntosh [43] that all homothetic models with perfect fluids are titled if $\hat{\rho} \neq \hat{p}$, we find that in fact all homothetic models are either tilted or non-tilted for all t.

3.5 Tetrad Description

So far, when components of tensors have been written, it has been with respect to a coordinate basis. However, any set of linearly independent vectors will do as a basis at each point and it is convenient to write the equations in terms of an orthonormal tetrad $\{e_a\}$ [34]. We denote the derivative of any function F in the basis vector directions by $\hat{\partial}_a F$, so if $\hat{e}_a{}^A$ are the components of the vectors $\hat{e}_a$ in a local coordinate system $\hat{\partial}_a F = \hat{e}_a{}^A\partial_A F$. The inverse matrix will be denoted by $\hat{e}^a{}_A$, so $\hat{e}^a{}_A\hat{e}^B{}_a = \delta_A{}^B$. Then, any tensor with coordinate components $T^{A\ldots B}{}_{C\ldots D}$ has tetrad components $T^{a..b}{}_{c..d}$ defined by

$$T^{a\ldots b}{}_{c\ldots d} = \hat{e}^{a}{}_{A..}\hat{e}^{b}{}_{B}\hat{e}^{c}{}_{C..}\hat{e}^{D}{}_{d}T^{A..B}{}_{C..D}$$

The metric tensor components $\hat{g}_{ab}$ are defined by

$$\hat{g}_{ab} = \hat{e}_{a}\cdot\hat{e}_{b} = \hat{e}_{a}{}^{A}\hat{e}_{b}{}^{B}g_{AB} \tag{5.1a}$$

and

$$\hat{g}^{ab} = \hat{e}^{a}{}_{A}\hat{e}^{b}{}_{B}g^{AB} \tag{5.1b}$$

The differential properties of the basis may be characterized by the rotation coefficients $\hat{\Gamma}_{abc}$ or $\hat{\Gamma}^{a}{}_{bc}$ where

$$\hat{\Gamma}_{abc} = \hat{e}_{a}{}^{A}\hat{e}_{cA|B}E_{b}{}^{B} \qquad \hat{\Gamma}^{a}{}_{bc} = \hat{g}^{ad}\hat{\Gamma}_{dbc} \tag{5.2}$$

Alternatively, one may consider the basis vector commutators. We define the commutation functions $\hat{\nu}^{a}{}_{bc}$, $\hat{\nu}_{abc}$ by

$$[\hat{e}_{a},\hat{e}_{b}] = \hat{\nu}^{c}{}_{ab}\hat{e}_{c} \qquad \hat{\nu}_{cab} = \hat{g}_{cd}\hat{\nu}^{d}{}_{ab} \tag{5.3}$$

and it follows that

$$\hat{\nu}^{a}{}_{bc} = \hat{\Gamma}^{a}{}_{bc} - \hat{\Gamma}^{a}{}_{cb} \Leftrightarrow \hat{\nu}_{abc} = \hat{\Gamma}_{abc} - \hat{\Gamma}_{acb} \tag{5.4}$$

and

$$\hat{\Gamma}_{abc} = \tfrac{1}{2}[\hat{\partial}_{b}\hat{g}_{ca} + \hat{\partial}_{c}\hat{g}_{ab} - \hat{\partial}_{a}\hat{g}_{bc} + \hat{\nu}_{abc} + \hat{\nu}_{cab} - \hat{\nu}_{bca}] \tag{5.5}$$

Taking the tetrad components of the curvature tensor and contracting, one obtains the field equations in tetrad form

$$\hat{\partial}_{d}\hat{\Gamma}^{c}{}_{cb} - \hat{\partial}_{c}\hat{\Gamma}^{c}{}_{db} - \hat{\Gamma}^{c}{}_{cs}\hat{\Gamma}^{s}{}_{db} + \hat{\Gamma}^{s}{}_{cb}\hat{\Gamma}^{c}{}_{sd} = \hat{T}_{db} - \tfrac{1}{2}\hat{T}\hat{g}_{bd} \tag{5.6a}$$

The identities $\hat{R}^{a}{}_{[bcd]} = 0$ are equivalent to the Jacobi identities $[\hat{e}_b, [\hat{e}_c, \hat{e}_d]] + [\hat{e}_d, [\hat{e}_b, \hat{e}_c]] + [\hat{e}_c, [\hat{e}_d, \hat{e}_b]] = 0$

$$\Leftrightarrow \quad \hat{\partial}_{[d}\hat{\nu}^{f}{}_{bc]} + \hat{\nu}^{s}{}_{[db}\hat{\nu}^{f}{}_{c]s} = 0 \tag{5.6b}$$

In a similar way we can set up an orthonormal basis of vectors $\{e_a\}$ on the homogeneous spaces. Equations (5.1) - (5.6) again hold for this basis. The homogeneity of these models is expressed by the restrictions [34]

$$g_{ab} = g_{ab}(t) \qquad \nu^{c}{}_{ab} = \nu^{c}{}_{ab}(t) \qquad \Gamma^{c}{}_{ab} = \Gamma^{c}{}_{ab}(t) \quad . \tag{5.7}$$

The group of motions which act in these models is classified by considering the triad of vectors $\{e_\nu\}$ which span the surfaces of homogeneity s(t) at each point and which is invariant under this group. From (5.3) we have $[e_\alpha, e_\beta] = \nu^{\delta}{}_{\alpha\beta}e_\delta$ and as in Chapter 2 we decompose the $\nu^{\delta}{}_{\alpha\beta}$ into a symmetric tensor $n^{\alpha\beta}$ and a relative tensor a_α;

$$\nu^{\delta}{}_{\alpha\beta} = \varepsilon_{\alpha\beta\mu}n^{\mu\delta} - \delta^{\delta}{}_{\beta}a_\alpha - \delta^{\delta}{}_{\alpha}a_\beta \tag{5.8}$$

The Jacobi identities for the vectors $\{e_\alpha\}$ give

$$n^{\alpha\beta}a_\alpha = 0 \tag{5.9}$$

and as before the solutions of this equation give the Bianchi classification of group types.

To complete the triads $\{\hat{e}_\mu\}$ and $\{e_\mu\}$ to obtain a complete set of basis vectors we add the normal vectors making the normal bases

$\{\hat{e}_a\} = \{\hat{\underline{n}},e_\mu\}$ and $\{e_a\} = \{\underline{n},e_\mu\}$. While for non-tilted models there is no other compelling choice, two alternatives are more closely related to the fluid properties. However the tilted basis $\{\hat{e}_a\} = \{\hat{\underline{u}},e_\alpha\}$ is not orthogonal and the fluid basis $\{\tilde{e}_a\} = \{\hat{\underline{u}},\tilde{e}_\alpha\}$, where the vector $\{\tilde{e}_\alpha\}$ are orthogonal to $\underline{u}$, does not span the surfaces s(t). The normal bases are those most closely related to the symmetry properties of the space-time. As we have restricted the bases to be orthonormal it follows that the metric components have the form

$$\hat{g}_{ab} = \mathrm{diag}(-1,1,1,1) \quad \text{and} \quad g_{ab} = \mathrm{diag}(-1,1,1,1) \tag{5.10}$$

To establish the relationship between the two sets of tetrad components we note that

$$\hat{g}_{AB} = \hat{e}^a{}_A \hat{e}^b{}_B \hat{g}_{ab} = e^{2\sigma} g_{AB} = e^{2\sigma} e^a{}_A e^b{}_B g_{ab}$$

$$\Rightarrow \quad \hat{e}^a{}_A = e^\sigma e^a{}_A \tag{5.11}$$

From this, it follows directly that

$$\hat{\nu}^c{}_{ab} = e^{-\sigma}[\nu^c{}_{ab}(t) + \delta^c{}_a \partial_b \sigma - \delta^c{}_b \partial_a \sigma] \tag{5.12}$$

and
$$\hat{\Gamma}^a{}_{bc} = e^{-\sigma}[\Gamma^a{}_{bc}(t) + \delta^a{}_b \partial_c \sigma - g_{bc} \partial^a \sigma] \tag{5.13}$$

Also, the choice of normalized basis $\{e_a\} = \{\underline{n},e_\alpha\}$ where $n^A = \delta^A{}_0$, and the decomposition

$$n_{a;b} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + w_{ab} - \dot{n}_a n_b \tag{5.14}$$

implies
$$\nu^{o}{}_{o\alpha} = \dot{n}_{\alpha} \qquad \nu^{\alpha}{}_{o\alpha} = -\theta_{\alpha} \text{ (no sum)}$$
$$\nu^{o}{}_{\alpha\beta} = -2\varepsilon_{\alpha\beta\delta}\omega^{\delta}$$
$$\nu^{\alpha}{}_{o\beta} = -\varepsilon_{\alpha\beta\delta}\Omega^{\delta} - \sigma_{\alpha\beta} - \varepsilon_{\alpha\beta\delta}\omega^{\delta} \qquad (5.15)$$
$$\nu^{\delta}{}_{\alpha\beta} = \varepsilon_{\alpha\beta\mu}n^{\mu\delta} + \delta^{\delta}{}_{\beta}a_{\alpha} - \delta^{\delta}{}_{\alpha}a_{\beta}$$

where $\Omega^{a} = \frac{1}{2}\eta^{abcd}n_{b}\dot{e}_{c}.e_{d}$ gives the angular velocity, of the triad $\{e_{\alpha}\}$ with respect to a set of Fermi-propaged axes. Then from (5.6) it follows that

$$\Gamma^{o}{}_{o\beta} = \dot{n}_{\beta} \qquad \Gamma^{o}{}_{\beta o} = 0 \qquad \Gamma^{\beta}{}_{\beta o} = \theta \qquad \Gamma^{\beta}{}_{o\beta} = 0$$
$$\Gamma^{c}{}_{oo} = \dot{n}^{c} \qquad \Gamma^{o}{}_{\beta\alpha} = \varepsilon_{\alpha\beta\delta}\omega^{\delta} + \theta_{\alpha\beta}$$
$$\Gamma^{\beta}{}_{\alpha o} = \varepsilon^{\beta}{}_{\alpha\delta}\omega^{\delta} + \theta^{\beta}{}_{\alpha} \qquad \Gamma^{\beta}{}_{o\alpha} = -\varepsilon^{\beta}{}_{\alpha\delta}\omega^{\delta} \qquad (5.16)$$
$$\Gamma^{\sigma}{}_{\beta\alpha} = \frac{1}{2}[\varepsilon_{\beta\alpha\delta}n^{\delta\sigma} + 2n^{\nu}{}_{(\alpha}\varepsilon^{\sigma}{}_{\beta)\nu} - 2\delta^{\sigma}{}_{\beta}a_{\alpha} + 2g_{\alpha\beta}a^{\sigma}]$$
$$\Gamma^{\alpha}{}_{\alpha\beta} = -2a_{\beta} \qquad \Gamma^{\alpha}{}_{\beta\alpha} = 0$$

We can now write out the kinematical quantities of the conformal fluid in terms of the above tetrad quantities. For example, the components of the vorticity vector (4.18) in the normal frame are

$$w^{o} = \frac{1}{2}e^{2\sigma}\sinh^{2}\beta k'_{\alpha}k'_{\beta}n^{\alpha\beta} \qquad (5.17a)$$

$$w^{\alpha} = \frac{1}{2}e^{2\sigma}\tanh\beta(n^{\alpha\beta}k'_{\beta} + \varepsilon^{\alpha\beta\delta}k'_{\beta}a_{\delta} + \sinh^{2}\beta k'^{\alpha}k'_{\beta}n^{\beta\delta}k'_{\delta}) \qquad (5.17b)$$

From the Jacobi identity $n^{\alpha\beta} a_\beta = 0$, it can be seen that if the tilt vector k'_α is parallel to a_α, then the vorticity vector is zero. It also follows from the time preservation of the Jacobi identity that if the above is true at any time, the same is true at all times. Thus w^a is either zero or non-zero at all times (i.e. it is an invariant relation quantity [35]).

When $n^{\alpha\beta} = 0$, the vorticity vector becomes

$$w^\alpha = \tfrac{1}{2} e^{2\sigma} \tanh\beta \ \epsilon^{\alpha\beta\delta} k'_\beta a_\delta$$

and so Type I models have zero vorticity. Similarly, Type V models have non-zero vorticity unless k'_α is parallel to a_β ; then w_α and k'_α are orthogonal. Further, the vanishing of vorticity in other class C models (i.e. Types ${}_1\mathrm{II}$, ${}_f\mathrm{VI}_0$, and ${}_f\mathrm{VII}_0$) corresponds to the vanishing of $n^{\alpha\beta}k'_\beta$ and from (5.17b) we see that in these cases k_α' cannot be perpendicular to w^a since $k_\alpha' w^\alpha = \tfrac{1}{2}\sinh\beta\cosh\beta k'_\alpha k'_\beta n^{\alpha\beta} \neq 0$.

The above comments hold for all conformal extensions of homogeneous models (M,g). However, if one restricts oneself to homothetic extensions then $\partial_\alpha \sigma$ is a constant b_α . In these cases much of the above still holds, as work recently published by Chao [81] demonstrates. However, in an attempt to consider whether any simple conformal extensions actually exist, we must solve Einstein's field equations. These field equations, using the above tetrad have been written out in Appendix B.

CHAPTER 4

PERFECT FLUID CONFORMAL MODELS

§4.1 Introduction

In this chapter we shall investigate the existence of perfect fluid solutions of class C which are conformal to homogeneous perfect fluid solutions of class E. To do this we shall use the tetrad form of Einstein's field equations derived in the last chapter for conformally homogeneous models.

In the first section we consider the form of the metric and for later ease in calculations we restrict ourselves to diagonal models. Next we consider the form and nature of the governing Einstein equations in both a coordinate and tetrad frame. To aid in solving these equations we at first restrict ourselves to considering those models which are conformal extensions of non-tilted homogeneous models. We then consider conformal extensions which are non-tilted. To illustrate the form of the fluid quantities in these models, Bianchi type I models are examined using known general solutions.

For homothetic models we again establish the result first found by McIntosh that non-tilted homothetic models only admit a hard equation of state (i.e. $\rho = p$).

§4.2 The Metric

As we have seen, the metric for spatially homogeneous models has the form

$$ds^2 = -dt^2 + \nu_{ij}(t)w^i \otimes w^j \tag{2.1}$$

where the three invariant 1-forms have the following structure [see Appendix A]

$$w^1 = (sx^3 + rx^2)dx^1 + dx^2 = Adx^1 + dx^2$$

$$w^2 = (rx^3 + qx^2)dx^1 + dx^3 = Bdx^1 + dx^3$$

$$w^3 = - dx^1$$

where we have put

$$A = (sx^3 + rx^2) \qquad B = (rx^3 + qx^2)$$

,except for Bianchi Type I where $w^i = dx^i$. The values of the constants s, r, and q for each type are given in Table 3.

In the coordinate base $\{dt, dx^i\}$ we can rewrite the metric (2.1) in the form

$$ds^2 = - dt^2 + g_{ij}dx^i dx^j$$

where $g_{11} = \nu_{11}A^2 + 2\nu_{12}AB + \nu_{33} - 2\nu_{13}A + \nu_{22}B^2 - 2\nu_{23}B$

$$g_{12} = \nu_{11}A + \nu_{12}B - \nu_{13}$$

$$g_{13} = \nu_{12}A + \nu_{22}B - \nu_{23}$$

$$g_{22} = \nu_{11} \qquad g_{23} = \nu_{12} \qquad g_{33} = \nu_{22}$$

In the following we shall restrict ourselves to diagonal metrics where

$$\nu_{12} = \nu_{23} = \nu_{13} = 0 \qquad (2.2)$$

TABLE 3

TYPE	S	r	q
II	1	0	0
VI_o	1	0	1
VII_o	1	0	-1
V	0	-1	0
IV	1	-1	0
VI_h	1	-a	1
VII_h	1	-a	-1
$III = VI_{-1}$	1	-1	1

We then have

$$g_{11} = \nu_{11}A^2 + \nu_{22}B^2 + \nu_{33} \qquad g_{22} = \nu_{11}$$

$$g_{12} = \nu_{11}A \qquad g_{23} = 0$$

$$g_{13} = \nu_{22}B \qquad g_{33} = \nu_{22}$$

The conformally homogeneous metrics thus have the form

$$ds^2 = e^{2\sigma}[- dt^2 + (\nu_{11}A^2 + \nu_{22}B^2 + \nu_{33})(dx^1)^2 + 2\nu_{11}Adx^1dx^2$$

$$+ 2\nu_{22}Bdx^1dx^3 + \nu_{11}(dx^2)^2 + \nu_{22}(dx^3)^2] \qquad (2.3)$$

where from (2.3.13) and (2.3.14) we have the conformal coefficient

$\sigma = -\int \phi dx^1$ for Bianchi types II - VII and $\sigma = \int \phi dx^3$ for Bianchi type I.

§4.3 The Governing Equations

In the next section we shall investigate the existence of perfect fluid solutions of class C [i.e. having the metric form (2.3)] which are conformal to homogeneous perfect fluid solutions of class E. To do this we shall use the tetrad form of Einstein's field equations introduced in the last chapter. However, we first consider the form of these equations in a coordinate frame and the tetrad frame.

i) Coordinate Frame

We obtain the governing equations by eliminating $\hat{T}_{AB}$ between equations (3.3.3) and (3.3.4) and substituting in (3.3.5), putting $\hat{q}_{AB} = 0$. This gives the equation

$$(\hat{\rho} + \hat{p})e^{4\sigma}[\lambda^2 u_A u_B + v_A v_B + \lambda(u_A v_B + u_B v_A)] + \hat{p}e^{2\sigma} g_{AB}$$

$$= (\rho + p)u_A u_B - 2\nabla_A \nabla_B \sigma + 2\partial_A \sigma \partial_B \sigma + g_{AB}(2g^{AB}\nabla_A \nabla_B \sigma + g^{AB}\partial_A \sigma \partial_B \sigma + p) \tag{3.1}$$

Note that the metric tensor g_{AB} is assumed to be known from the homogeneous models and hence its ten independent components are not to be counted as unknown variables in the above system. The same holds for the source quantities ρ and p.

Multiplication of (3.1) by $u^A u^B$ and summing gives,

$$\hat{\rho} = e^{-2\sigma}\rho + V^2(\rho + p) - e^{-2\sigma}g^{AB}(2\nabla_A\nabla_B\sigma + \partial_A\sigma\partial_B\sigma)$$

$$- u^Au^B[2\nabla_A\nabla_B\sigma - 2\partial_A\sigma\partial_B\sigma] \tag{3.2}$$

where $V^2 = g^{AB}v_Av_B$. Also, multiplication of (3.1) by g^{AB} and summing gives,

$$(3\hat{p} - \hat{\rho})e^{2\sigma} = (3p - \rho) + 6g^{AB}(\nabla_A\nabla_B\sigma + \partial_A\sigma\partial_B\sigma) \tag{3.3}$$

for which, using (3.2), we obtain

$$3\hat{p} = 3e^{-2\sigma}p + V^2(\rho + p) + e^{-2\sigma}[4\nabla_A\nabla_B\sigma + \partial_A\sigma\partial_B\sigma]g^{AB}$$

$$- u^Au^B[2\nabla_A\nabla_B\sigma - 2\partial_A\sigma\partial_B\sigma] \tag{3.4}$$

The above system of equations gives the relations between the salient physical quantities of the conformally homogeneous models, the conformal coefficient and its derivatives and the known quantities of the corresponding homogeneous models. We can hence solve these equations to find the unknowns $\hat{\rho}$, $\hat{p}$ and $\{v_A\}$. This will involve solving the constraint equations (3.1) in conjunction with the full system of homogeneous field equations. Since these equations are of second order in a coordinate frame, it is easier to work with an orthonormal frame where the differential equations are of first order only. However, in this latter frame, the time evolution of any constraint equation needs to be found to ensure consistency. This procedure will give rise to further constraint equations which must themselves be conserved in time giving rise to yet more constraints. Either these sets of constraints will lead to inconsistencies or there will be a stage when the new constraints found are merely identities by virtue of the previous equations and we have a consistent solution.

The solution of the coordinate system of equations (3.1) - (3.3) together with the homogeneous field equations in coordinate form is outlined for the interested reader in Appendix C.

ii) Orthonormal Frame

The construction of Einstein's field equations in an orthonormal frame has been outlined in Appendix B. For a tilted conformal model it was found that the field equations reduced to the following system of equations;

$$- \dot{\theta} - \theta_{\alpha\beta}\theta^{\alpha\beta} + \partial_{\alpha}\partial^{\alpha}\sigma + 2\partial_{\alpha}\sigma\partial^{\alpha}\sigma - 2a_{\alpha}\partial^{\alpha}\sigma$$

$$= \tfrac{1}{2}\rho'(1 + 2\sinh^2\beta) + \frac{3}{2}(1 + \frac{2}{3}\sinh^2\beta)p' \tag{3.5a}$$

$$\varepsilon_{\alpha\beta\delta}n^{\delta\mu}\sigma^{\beta}{}_{\mu} - 3\sigma^{\beta}{}_{\alpha}a_{\beta} + \frac{2}{3}\theta\partial_{\alpha}\sigma + 2\sigma^{\beta}{}_{\alpha}\partial_{\beta}\sigma$$

$$= - (\rho' + p')\sinh\beta\cosh\beta k'_{\alpha} \tag{3.5b}$$

$$\partial_0\sigma_{\alpha\beta} + \theta\sigma_{\alpha\beta} + 2\sigma^{\nu}{}_{(\alpha}\varepsilon_{\beta)\delta\nu}\Omega^{\delta} + 2n^{\delta}{}_{(\alpha}n_{\beta)\delta} - 2\varepsilon_{\nu\delta(\alpha}n^{\nu}{}_{\beta)}a^{\delta} - nn_{\alpha\beta} - 2\partial_{\beta}\partial_{\alpha}\sigma$$

$$- \varepsilon_{\alpha\beta\delta}n^{\delta\nu}\partial_{\nu}\sigma - 2a_{\alpha}\partial_{\beta}\sigma + 2\partial_{\alpha}\sigma\partial_{\beta}\sigma + 2\varepsilon_{\nu\delta(\alpha}n_{\beta)}{}^{\nu}\partial^{\delta}\sigma$$

$$+ \frac{1}{3}\delta_{\alpha\beta}(n^2 - 2n^{\mu\nu}n_{\mu\nu} - 2\partial^{\nu}\sigma\partial_{\nu}\sigma + 2\partial^{\nu}\partial_{\nu}\sigma + 2a_{\nu}\partial^{\nu}\sigma)$$

$$= (\rho' + p')\sinh^2\beta(k'_{\alpha}k'_{\beta} - \frac{1}{3}\delta_{\alpha\beta}) \tag{3.5c}$$

$$\frac{1}{3}\theta^2 - \tfrac{1}{2}\sigma^2 - 3a^\alpha a_\alpha + \tfrac{1}{2}(\tfrac{1}{2}n^2 - n^{\alpha\beta}n_{\alpha\beta}) - 2\partial^\alpha\partial_\alpha\sigma - \partial^\alpha\sigma\partial_\alpha\sigma + 4a_\alpha\partial^\alpha\sigma$$

$$= \cosh^2\beta\rho' + \sinh^2\beta p' \tag{3.5d}$$

where the various terms refer to the homogeneous quantities describing the normal congruence (and hence the fluid quantities in a non-tilted homogeneous model), the angle and direction of tilt and the conformal factor and its derivatives.

Since these models are conformal to homogeneous models, the various homogeneous terms will thus satisfy the field equations for a homogeneous model. If the tilt in these homogeneous models is described by equation (3.4.11), then the corresponding field equations are [35],

$$\partial_0\theta + \frac{1}{3}\theta^2 + \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \tfrac{1}{2}\rho(1 + 2\sinh^2\theta) + \frac{2}{3}p(1 + \frac{2}{3}\sinh^2\theta) = 0 \tag{3.6a}$$

$$3a_\beta\sigma^\beta{}_\alpha - \varepsilon_{\alpha\beta\nu}n^{\nu\mu}\sigma^\beta{}_\mu = (\rho + p)\sinh\theta\cosh\theta k_\alpha \tag{3.6b}$$

$$\partial_0\sigma_{\alpha\beta} + \theta\sigma_{\alpha\beta} + 2\sigma^\nu{}_{(\alpha}\varepsilon_{\beta)\mu\nu}\Omega^\mu - 2\varepsilon_{\nu\mu(\alpha}n_{\beta)}{}^\nu a^\mu + 2n_{\alpha\mu}n^\mu{}_\beta$$

$$- nn_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}(2n^{\mu\nu}n_{\mu\nu} - n^2)$$

$$= (\rho + p)\sinh^2\theta(k_\alpha k_\beta - \frac{1}{3}\delta_{\alpha\beta}) \tag{3.6c}$$

$$\frac{1}{3}\theta^2 - \tfrac{1}{2}\sigma^2 - 3a_\alpha a^\alpha + \tfrac{1}{2}(\tfrac{1}{2}n^2 - n^{\alpha\beta}n_{\alpha\beta})$$

$$= \rho\cosh^2\theta + p\sinh^2\theta \tag{3.6d}$$

Substituting these equations into the set (3.5) we obtain the following constraints.

$$\partial_\alpha \partial^\alpha \sigma + 2\partial_\alpha \sigma \partial^\alpha \sigma - 2a_\alpha \partial^\alpha \sigma$$

$$= \tfrac{1}{2}(\rho' - \rho) + \frac{3}{2}(p' - p) + (\rho' + p')\sin^2\beta - (\rho + p)\sinh^2\theta \tag{3.7a}$$

$$2\theta_\alpha{}^\beta \partial_\beta \sigma = (\rho + p)\sinh\theta\cosh\theta k_\alpha - (\rho' + p')\sinh\beta\cosh\beta k'_\alpha \tag{3.7b}$$

$$- 2\partial_\beta \partial_\alpha \sigma - \varepsilon_{\alpha\beta\mu} n^{\mu\nu} \partial_\nu \sigma - 2a_\alpha \partial_\beta \sigma + 2\partial_\alpha \sigma \partial_\beta \sigma$$

$$+ 2\varepsilon_{\mu\nu(\alpha} n_{\beta)}{}^\mu \partial^\nu \sigma + \frac{1}{3}\delta_{\alpha\beta}(2a_\nu \partial^\nu \sigma + 2\partial^\nu \partial_\nu \sigma - 2\partial^\nu \sigma \partial_\nu \sigma)$$

$$= (\rho' + p')\sinh^2\beta(k'_\alpha k'_\beta - \frac{1}{3}\delta_{\alpha\beta}) - (\rho + p)\sinh^2\theta(k_\alpha k_\beta - \frac{1}{3}\delta_{\alpha\beta}) \tag{3.7c}$$

$$4a_\alpha \partial^\alpha \sigma - 2\partial^\alpha \partial_\alpha \sigma - \partial^\alpha \sigma \partial_\alpha \sigma = \rho' \cosh^2\beta - \rho\cosh^2\theta + p'\sinh^2\beta - p\sinh^2\theta \tag{3.7d}$$

To write out these equations in detail we use the basis $\{e_\mu\}$ introduced in Appendix B such that

$$a_\alpha = (0,0,a) \qquad n_{\alpha\beta} = \mathrm{diag}(n_1, n_2, n_3) \tag{3.8}$$

Using this basis, and the results from Appendix D that

$$\partial_\alpha \sigma = f\delta^3{}_\alpha \qquad\qquad f = \nu_{33}{}^{-\frac{1}{2}}F(x^1) \tag{3.9a}$$

$$\partial_\beta\partial_\alpha \sigma = \partial f\delta^3{}_\alpha\delta^3{}_\beta \qquad \partial f = -\,\nu_{33}{}^{-1}\partial F(x^1) \tag{3.9b}$$

,except for Bianchi type I where $\partial f = \nu_{33}{}^{-1}\partial F$ and $F = F(x^3)$, then equations (3.7) can be written in the form,

$$\partial f + 2f^2 - 2af = \tfrac{1}{2}(\rho' - \rho) + \frac{3}{2}(p' - p) + (\rho' + p')\sinh^2\beta - (\rho + p)\sinh^2\theta \tag{3.10}$$

$$2\theta_{13}f = (\rho + p)\sinh\theta\cosh\theta k_1 - (\rho' + p')\sinh\beta\cosh\beta k_1' \tag{3.11a}$$

$$2\theta_{23}f = (\rho + p)\sinh\theta\cosh\theta k_2 - (\rho' + p')\sinh\beta\cosh\beta k_2' \tag{3.11b}$$

$$2\theta_{33}f = (\rho + p)\sinh\theta\cosh\theta k_3 - (\rho' + p')\sinh\beta\cosh\beta k_3' \tag{3.11c}$$

$$2af + 2\partial f - 2f^2 = 3(\rho' + p')\sinh^2\beta(k_1'^2 - \frac{1}{3}) - 3(\rho + p)\sinh^2\theta(k_1^2 - \frac{1}{3}) \tag{3.12a}$$

$$(n_2 - n_1 - n_3)f = (\rho' + p')\sinh^2\beta k_1'k_2' - (\rho + p)\sinh^2\theta k_1k_2 \tag{3.12b}$$

$$0 = (\rho' + p')\sinh^2\beta k_1'k_3' - (\rho + p)\sinh^2\theta k_1k_3 \tag{3.12c}$$

$$2af + 2\partial f - 2f^2 = 3(\rho' + p')\sinh^2\beta(k_2'^2 - \frac{1}{3}) - 3(\rho + p)\sinh^2\theta(k_2^2 - \frac{1}{3}) \tag{3.12d}$$

$$0 = (\rho' + p')\sinh^2\beta k_2'k_3' - (\rho + p)\sinh^2\theta k_2k_3 \tag{3.13e}$$

$$4f^2 - 4af - 4\partial f = 3(\rho' + p')\sinh^2\beta(k_3'^2 - \frac{1}{3}) - 3(\rho + p)\sinh^2\theta(k_3^2 - \frac{1}{3}) \tag{3.12f}$$

$$4af - 2\partial f - f^2 = \rho'\cosh^2\beta + p'\sinh^2\beta - \rho\cosh^2\theta - p\sinh^2\theta \qquad (3.13)$$

We can now solve the above constraint equations and use the conditions thus found in solving the system of homogeneous field equations for allowable models.

§4.4 Conformal Extensions

i) Non-tilted homogeneous models ($\theta = 0$, $k_\alpha = 0$): We consider first conformal extensions from non-tilted homogeneous models. From (3.12a) and (3.12d) we have

$$2af + 2\partial f - 2f^2 = 3(\rho' + p')\sinh^2\beta(k_1'^2 - \tfrac{1}{3})$$

$$2af + 2\partial f - 2f^2 = 3(\rho' + p')\sinh^2\beta(k_2'^2 - \tfrac{1}{3})$$

$$\Rightarrow \quad k_1' = k_2'$$

Then (3.12c) and (3.12e) give the equation

$$0 = (\rho' + p')\sinh^2\beta k_1' k_3'$$

Now, if $k_3' = 0$, then from (3.11c) we have, since $f \neq 0$, $\theta_3 = 0$. Hence using equation (3.4.15) one can show that $\theta_3' = 0$. However, since observation shows that the universe is expanding, we discard this result as being unrealistic. Therefore we have $k_1' = 0$ and the result

$$k_1' = k_2' = 0 \quad k_3' = 1 \qquad (4.1)$$

Thus, the conformal models will always be tilted if the homogeneous model is not. We see immediately from (3.11a) and (3.11b) that

$$\sigma_{13} = \sigma_{23} = 0 \tag{4.2}$$

and from (3.12b) that

$$(n_2 - n_1 - n_3)f = 0 \tag{4.3}$$

Since $n_3 = 0$ for all conformal models, we have the result that $n_1 = n_2$ and hence the only allowable Bianchi types are types I, V and VII. Also, Ellis and MacCallum [34] have shown that a non-tilted homogeneous model of class B has $n_1 = n_2$ on an open neighbourhood iff there is a group of Type V. Hence, for class B models we need only consider Type V models.

Using these results (3.12a), (3.12d) and (3.12f) give

$$2af + 2\partial f - 2f^2 = -(\rho' + p')\sinh^2\beta \tag{4.4}$$

Substituting this into (3.13) we obtain

$$\rho' = \rho + 3f(2a - f) \tag{4.5a}$$

and then (3.10) gives

$$p' = p - f(2a - f) + 2\partial f \tag{4.5b}$$

Noting the values for f given by equations (3.9), these equations are identical to those obtained by the coordinate method in Appendix C.

Finally, equation (3.11c) becomes the constraint equation

$$2\theta_3 f = -\ (\rho' + p')\sinh\beta\cosh\beta$$

Squaring, and substituting for $(\rho' + p')$ from (4.5) and $\sinh^2\beta$ from (4.4), we obtain

$$2\theta_3^2 f^2 = (f^2 - af - \partial f)(\rho + p + 2af) \qquad (4.6)$$

Again, by substituting in for the values of f given in (3.9) and noting the additional coordinate transformation $a = -\ \nu_{33}^{-\frac{1}{2}} r$ found in Appendix D, then (4.6) can be written in the coordinate form:

$$2\theta_3^2 F^2 = (F^2 + rF + \partial F)(\rho + p - 2\nu_{33}^{-1} rF) \qquad (4.7a)$$

for Bianchi types II - VII, and

$$2\theta_3^2 F^2 = (F^2 - \partial F)(\rho + p) \qquad (4.7b)$$

for Bianchi type I $(r = 0)$.

We are now in the position of using the above results in conjunction with the homogeneous field equations (3.6) in obtaining the properties of those homogeneous solutions allowing a conformal extension. We have seen that we need restrict our attention to Bianchi types I, VII_o and V only. It has been shown that for such models [34] that there exists an orthonormal tetrad such that

$$\Omega_\alpha = 0 \qquad \sigma_{\alpha\beta} = 0\ (\alpha \neq \beta) \qquad (4.8a)$$

and $$\theta_{\alpha\beta} = \mathrm{diag}(\theta_1,\theta_2,\theta_3) \tag{4.8b}$$

Substituting (4.8) into the system of homogeneous equations (B.14) of Appendix B leaves

$$\partial_0 a + \theta_3 a = 0 \tag{4.9}$$

$$\partial_0 n_1 + (\theta_2 + \theta_3 - \theta_1)n_1 = 0 \tag{4.10a}$$

$$\partial_0 n_2 + (\theta_1 + \theta_3 - \theta_2)n_2 = 0 \tag{4.10b}$$

$$\partial_0 \theta + \theta_1^2 + \theta_2^2 + \theta_3^2 + \tfrac{1}{2}\rho + \tfrac{3}{2}p = 0 \tag{4.11}$$

$$a(2\theta_3 - \theta_1 - \theta_2) = 0 \tag{4.12}$$

$$\partial_0 \theta_1 = -\theta\theta_1 + 2a^2 + \tfrac{1}{2}(\rho - p) \tag{4.13a}$$

$$\partial_0 \theta_2 = -\theta\theta_2 + 2a^2 + \tfrac{1}{2}(\rho - p) \tag{4.13b}$$

$$\partial_0 \theta_3 = -\theta\theta_3 + 2a^2 + \tfrac{1}{2}(\rho - p) \tag{4.13c}$$

In solving these equations we shall consider Class B models first (i.e. type V). We shall then consider the simpler Class A models.

ii) Type V Models ($a \neq 0$ or $r \neq 0$). Take the time derivative of the conformal constraint (4.7a), first noting that from (4.9)

$$\partial_0 a = -\partial_0(\nu_{33}^{-\frac{1}{2}} r) = \tfrac{1}{2}\nu_{33}^{-\frac{3}{2}}\dot{\nu}_{33} r = -\tfrac{1}{2}\nu_{33}^{-1}\dot{\nu}_{33} a = -\theta_3 a$$

$$\Rightarrow \quad \theta_3 = \tfrac{1}{2}\nu_{33}^{-1}\dot{\nu}_{33} \quad \text{and} \quad \dot{\nu}_{33} = 2\nu_{33}\theta_3 \tag{4.14}$$

Also, writing the equation of state in the form $p = \nu\rho$ and using equation (3.4.19a), which in the homogeneous case becomes

$$\partial_0\rho = -(1+\nu)\theta\rho \tag{4.15}$$

we obtain the result

$$-4\theta_3^2\theta F^2 + 8r^2\theta_3 F^2\nu_{33}^{-1} + 2\theta_3(1-\nu)\rho F^2$$

$$= (F^2 + rF + \partial F)[-(1+\nu)^2\theta\rho + 4\nu_{33}^{-1}\theta_3 rF]$$

Substituting in the original constraint (4.7a), we obtain

$$\{2\nu_{33}^{-1}r[8r^2\theta_3\nu_{33}^{-1} - 4\theta_3^2\theta + 2\theta_3(1-\nu)\rho] + 8\theta_3^3\nu_{33}^{-1}r\}F$$

$$= \rho(1+\nu)[8r^2\theta_3\nu_{33}^{-1} - 4\theta_3^2\theta + 2\theta_3(1-\nu)\rho + 2\theta_3^2\theta(1+\nu)]$$

Inspection of this equation shows that it has the following form

$$A(t)F(x) = B(t)$$

Thus, this equation can be consistent only if $F(x)$ = constant unless $A(t) = B(t) = 0$, in which case $F(x)$ is arbitrary.

ii-a) $F(x) \neq$ constant - Conformal. In this case we have

$$8r^2\theta_3\nu_{33}^{-1} - 4\theta_3^2\theta + 2\theta_3(1-\nu)\rho + 4\theta_3^3 = 0 \tag{4.16}$$

and $\quad 4\theta_3^2 - 8r^2\theta_3\nu_{33}^{-1} - 2\theta_3(1-\nu)\rho - 2\theta_3^2\theta(1+\nu) = 0$

Adding these equations we obtain

$$4\theta_3^3 = 2\theta_3^2\theta(1 + \nu)$$

and since we require θ_3 to be non-zero (otherwise $\hat{\theta}_3 = 0$), we have

$$2\theta_3 = \theta(1 + \nu) \tag{4.17}$$

Also, from (4.12) we have since $a \neq 0$

$$2\theta_3 = \theta_1 + \theta_2 \tag{4.18a}$$

$$\Rightarrow \qquad \theta = 3\theta_3 \tag{4.18b}$$

Substituting this into (4.17) gives

$$(1 + \nu) = \frac{2}{3} \qquad \nu = -\frac{1}{3}$$

We thus have an unphysical equation of state for the homogeneous models, where from (4.16) we have

$$\rho = 3\theta_3^2 - 3r^2\nu_{33}^{-1} \tag{4.19}$$

We now consider the equation of state in the conformal model. Substituting (4.18) and (4.19) into the constraint equation (4.7a) we have

$$2\theta_3^2F^2 = (F^2 + rF + \partial F)(2\theta_3^2 - 2r^2\nu_{33}^{-1} - 2\nu_{33}^{-1}rF) \tag{4.20}$$

and since $F(x) \neq$ constant, this equation is consistent only if $\theta_3 = \beta\nu_{33}^{-\frac{1}{2}}$ where $\beta \in \mathbb{R}$. In this case, (4.20) becomes

$$\beta^2 F^2 = (F^2 + rF + \partial F)(\beta^2 - r^2 - rF) \tag{4.21}$$

Also, from (4.5) and since $p = -\frac{1}{3}\rho$ we have the result

$$p' = -\frac{1}{3}\rho' - 2\nu_{33}{}^{-1}\partial F \tag{4.22}$$

Hence, this is an unrealistic fluid since it will be impossible to satisfy the conditions $\rho' > 0$ and $p' \geqslant 0$ for all space-time points. Thus, we conclude that there are no Class B homogeneous models admitting a conformal extension which has a realistic perfect fluid. We now consider the other alternative where $F(x)$ = constant.

ii-b) $F(x)$ = constant: Homothetic. Here the constraint (4.7a) reduces to

$$2\theta_3^2 F = (F + r)(\rho + p - 2\nu_{33}{}^{-1} rF) \tag{4.23}$$

and the time derivative gives, using (4.17)

$$12\theta_3^2 F - 8r^2 F\nu_{33}{}^{-1} - 2(1 - \nu)\rho F = (F + r)[3(1 + \nu)^2\rho - 4\nu_{33}{}^{-1} rF]$$

Eliminating θ_3 by substituting in (4.23) gives

$$8rF(F + 2r)\nu_{33}{}^{-1} = (1 - \nu)[F + 3\nu F + 3r + 3\nu r]\rho \tag{4.24}$$

If $\rho \neq 0$, this constraint will be satisfied if $F = -2r$ and $\nu = 1$ or $\nu = \frac{1}{3}$. If, however, none of these conditions hold, then time evolution of (4.24) leads to the condition $\nu = -\frac{1}{3}$ and from (4.22) we have $p' = -\frac{1}{3}\rho'$ and we have an unrealistic equation of state. Thus we consider the case where $F = -2r$.

Substituting this condition into the homothetic constraint (4.23) gives

$$4\theta_3^2 = (1 + \nu)\rho + 4\nu_{33}^{-1}r^2 = (1 + \nu)\rho + 4a^2 \tag{4.25}$$

Also, since $\theta = 3\theta_3$ and using equations (4.11) and (4.13) we have

$$\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 = 3a^2 + \rho$$

and since $(\theta_1 + \theta_2) = 2\theta_3$, and using (4.25) we obtain

$$\theta_1\theta_2 = a^2 + \tfrac{1}{2}(1 - \nu)\rho \tag{4.26}$$

We now consider the two cases $\nu = 1$, $\nu = \frac{1}{3}$ separately.

Case I: $(\nu = \frac{1}{3})$. Here (4.25) and (4.26) become

$$\tfrac{1}{3}\rho = \theta_3^2 - a^2$$

and

$$\tfrac{1}{3}\rho = \theta_1\theta_2 - a^2$$

Subtracting we obtain $\theta_3^2 = \theta_1\theta_2$ and together with (4.18) we have the result $\theta_1 = \theta_2 = \theta_3$. All constraint equations are satisfied and we thus have the isotropic Friedmann model of type V.

Case II: $(\nu = 1)$. Here (4.25) and (4.26) become

$$\rho = 2\theta_3^2 - 2a^2$$

and

$$a^2 = \theta_1\theta_2$$

Both these equations are consistent with the remaining homogeneous field equations and thus represent a solution. Note that in general these models will be non-isotropic; otherwise $\rho = 0$.

Now, given the condition $F = -2r$, which is equivalent to $f = 2a$, we have from equations (4.5)

$$\rho' = \rho \quad \text{and} \quad p' = p$$

Hence, the density and pressure in the conformal models is given by the expressions

$$\hat{\rho} = e^{-2\sigma}\rho \quad \text{and} \quad \hat{p} = e^{-2\sigma}p \tag{4.27}$$

where $\sigma = -\int F dx^1 = 2r\int dx^1 = 2rx^1$. However, referring to Table 4.1, for type V models $r = -1$ and so, although the density and pressure satisfy the requirements of always being positive, they become unbounded as x_1 approaches infinity. This would seem to be unrealistic, however, in an open universe.

The above theory holds for Class B (type V) models where $a \neq 0$. We shall next consider Class A models: in particular Bianchi type I.

iii) Bianchi Type I models: Equation (4.7b) gave the conformal constraint applicable for type I models:

$$2\theta_3^2F^2 = (F^2 - \partial F)(\rho + p)$$

or

$$2\theta_3^2F^2 = (F^2 - \partial F)(1 + \nu)\rho \tag{4.28}$$

where $p = \nu\rho$. This equation has some immediate consequences. Since ρ and θ_3 are homogeneous functions of t, while F is a function of x^3, we have that

$$1 - \partial F\, F^{-2} = \text{constant} = \beta^2 \qquad (4.29)$$

Note that since $(1 + \nu)\rho$ and $\theta_3{}^2$ are positive ($\rho > 0$), so is the constant β^2. Rearranging (4.29) we obtain

$$(\beta^2 - 1)F^2 + \partial F = 0 \qquad (4.30)$$

If $\beta^2 = 1$, then $\partial F = 0$ and F = constant. This corresponds to the homothetic case. Hence, for conformal models we require $\beta^2 \neq 1$.

As an example of a solution of the differential equation (4.30) consider $F = cx_3{}^n$. Substitution gives the result $n = -1$, $c = (\beta^2 - 1)^{-1}$. Hence a solution of (4.30) is

$$F = \frac{1}{(\beta^2 - 1)x_3}, \quad \beta^2 \neq 1 \qquad (4.31)$$

and from this we have

$$\sigma(x_3) = \int F dx_3 = \frac{\ell n x_3}{(\beta^2 - 1)}, \quad \beta^2 \neq 1 \qquad (4.32)$$

Now, substituting (4.29) back into (4.28) we have

$$2\theta_3^2 = \beta^2(1 + \nu)\rho \qquad (4.33)$$

As in the previous case, we take the time evolution of this equation which gives

$$(1 - \nu)\theta_3\theta = (1 - \nu)\rho \tag{4.34}$$

If $\nu \neq 1$, we obtain the condition

$$\rho = \theta_3\theta \tag{4.35}$$

Substituting this into (4.33) we obtain

$$2\theta_3 = \beta^2(1 + \nu)\theta \tag{4.36}$$

which gives $\beta^2(1 + \nu) = \frac{2}{3}$ upon taking the time derivative. Substituting this result back in (4.36) gives $\theta = 3\theta_3$. From (4.35) we have $\rho = 3\theta_3^2$ and from the homogeneous field equations we obtain the condition $\theta_1 = \theta_2 = \theta_3$. Hence, the homogeneous models with $\nu \neq 1$ must be isotropic.

When $\nu = 1$, the conformal constraint (4.33) becomes $\theta_3^2 = \beta^2\rho$ and from (4.34) we saw that its time evolution was immediately satisfied. Thus, there are no constraints on the homogeneous models as the above equation defines the value of β for each case. For example, if one considers an isotropic homogeneous model then from the field equations we obtain the condition $\rho = 3\theta_3^2$. Comparing this with the above condition we find $\beta^2 = \frac{1}{3}$.

Finally, when $\beta^2 = 1$, we again reduce to the homothetic case and the conformal constraint reduces to

$$2\theta_3^2 = (1 + \nu)\rho$$

Comparing this with equation (4.36) and the condition obtained from its time evolution, we see that if $\nu \neq 1$, we have the result $\nu = -\frac{1}{3}$. Using equations (4.5) then gives

$$\rho' = \rho - 3\nu_{33}{}^{-1}F^2$$

$$p' = p + \nu_{33}{}^{-1}F^2 = -\tfrac{1}{3}(\rho - 3\nu_{33}{}^{-1}F^2)$$

and hence the equation of state for the homothetic models is unrealistic.

§4.5 Bianchi - Type I Models

In this section we shall investigate the form and properties of Bianchi - type I models allowing conformal extensions, by using known analytic type I solutions. Perfect fluid type I models have been investigated by Jacobs [45] and we use his closed form solutions for dust and hard equation of state ($\nu = 1$). We at first show that these solutions satisfy the conformal constraint, reaffirming some of the results of the previous section, then obtain explicit expressions for the fluid quantities.

Using equations (4.14) and (4.33) we write the conformal constraint in the form

$$2(\rho + p)\beta^2 = (\nu_{33}{}^{-1}\dot{\nu}_{33})^2 \tag{5.1}$$

From the homogeneous field equations we can find expressions for ρ and p in terms of the metric coefficients and substitute back into (5.1). This gives

$$\beta^2[(\nu_{33}{}^{-1}\dot{\nu}_{33})^2 - 2\nu_{33}{}^{-1}\ddot{\nu}_{33} + \nu_{11}{}^{-1}\dot{\nu}_{11}\nu_{22}{}^{-1}\dot{\nu}_{22}] = (\nu_{33}{}^{-1}\dot{\nu}_{33})^2 \tag{5.2}$$

We can now take the known analytic type I solutions and see whether this constraint is satisfied.

i) Dust Solutions:

From Jacobs [45] we have the solution

$$\nu_{ii} \equiv \nu_i = [x_D(x_D + |\varepsilon_D|)]^{\frac{2}{3}}[(x_D + |\varepsilon_D|)/x_D]^{\frac{4X_i}{3}} \tag{5.3a}$$

where
$$X \equiv \sin\left(\psi, \psi + \frac{2\pi}{3}, \psi + \frac{4\pi}{3}\right) \tag{5.3b}$$

and
$$x_D = \text{normalized time} = (t + t_D^*)/\tau_D$$

$$\tau_D = \text{time scale} = (6\pi\rho_0)^{-\frac{1}{2}}$$

Substituting (5.3) into (5.2) and equating coefficients of x_D^2 we find $\beta^2 = \frac{2}{3}$. Equating coefficients of x_D we have, if $\varepsilon_D \neq 0$, $X_3 = 0$ and from the x_D^0 equation $X_1^2 = \frac{3}{8}$. However, these last two conditions do not satisfy (5.3b) and hence we require $\varepsilon_D = 0$ which satisfies these equations. Thus, isotropic dust solutions will give a perfect fluid conformal extension where $\beta^2 = \frac{2}{3}$. This result is in agreement with the discussion after equation (4.3b) in the previous section.

Substituting $\varepsilon_D = 0$ into (5.3) we find

$$\nu_{33} = x_D^{4/3} \quad \text{and} \quad \rho = \rho_0 x_D^{-2}$$

Hence, from equations (4.5), the density and pressure in the conformal model are

$$\hat{\rho} = e^{-2\sigma}\left\{\frac{\rho_0}{x_D^2} - \frac{3F^2}{x_D^{4/3}}\right\} \tag{5.4a}$$

$$\hat{p} = \frac{e^{-2\sigma}}{x_D^{4/3}}[2\partial F + F^2] = \frac{5e^{-2\sigma}}{3x_D^{4/3}} F^2 \qquad (5.4b)$$

where we have used (4.30) in eliminating ∂F in (5.4b). Note that while the pressure is always positive, this need not always be the case with the density. Also, depending upon the choice of solution of (4.30) for F, the behaviour of the source terms could be unrealistic. As an example, consider the solution (4.31) with $\beta^2 = \frac{2}{3}$ giving $F = -3x_3^{-1}$.

$$\hat{\rho} = x_3^6\left\{\frac{\rho_0}{x_D^2} - \frac{27}{x_3^2 x_D^{4/3}}\right\}$$

$$\hat{p} = \frac{15x_3^4}{x_D^{4/3}}$$

As with the homothetic Type V models found in the last section we see that these quantities increase quite rapidly in one direction.

ii) Hard Equation of State Solutions ($\nu = 1$)

From Jacobs [45] we have the solution

$$\nu_i = \nu_{io} x_Z^{2/3} x_Z^{\frac{4\delta X_i}{3}}, \quad 0 \leqslant \delta < 1 \qquad (5.5)$$

where $\quad x_Z = \text{normalized time} = (t + t_Z^*)/\tau_Z$

$$\tau_Z = \text{time scale} = [(1 - \delta^2)/24\rho_0]^{\frac{1}{2}}$$

and X_i is as above. Substituting (5.5) into the constraint (5.2) gives the single equation

$$(3\beta^2 - 1) + 4\beta^2 X_1 X_2 \delta^2 - 4(\beta^2 + 1)\delta^2 X_3^2 - 4\delta X_3 = 0 \tag{5.6}$$

With the free parameters at our disposal, this equation can be satisfied. For example, putting $\delta = 0$ we obtain an isotropic homogeneous model and $\beta^2 = \frac{1}{3}$. Again this agrees with the discussion in the last section. Expressions for the fluid quantities for the conformal models may be obtained and similar comments as above hold.

Putting $\beta^2 = 1$ into (5.6) we obtain the homothetic case, and the equation

$$1 + 2\delta^2 X_1 X_2 - 4\delta^2 X_3^2 - 2\delta X_3 = 0$$

From this equation we note that the homogeneous models must be anisotropic, as putting $\delta = 0$ leads to a contradiction. Using expression (5.5) and the fact that $\rho = \rho_0 x_Z^{-2}$, we find the homothetic fluid quantities to be

$$\hat{\rho} = e^{-2\sigma}\left(\frac{\rho_0}{x_Z^2} - \frac{3F^2}{\nu_{33}}\right) \tag{5.7a}$$

$$\hat{p} = e^{-2\sigma}\left(\frac{\rho_0}{x_Z^2} + \frac{F^2}{\nu_{33}}\right) \tag{5.7b}$$

where

$$\sigma = \int F dx^3 = Fx^3 \quad \text{since } F \text{ is a constant.}$$

Since the denominator of the second term in (5.7a) increases less rapidly with time than the first term, a time will come when the density becomes negative. However, if $F > 0$, then both quantities approach zero as x_3 increases without bound.

§4.6 Non-Tilted Conformal Extensions

In this section we now consider tilted homogeneous models allowing non-tilted conformal extensions. We use the tetrad frame and the governing equations are obtained by putting $\beta = 0$, $k'_\alpha = 0$ in equations (3.10) - (3.13). In this manner, equation (3.11c) has an immediate consequence. Since all terms except the function f are homogeneous quantities, then we must have $\partial f = 0$. Thus only homothetic extensions are allowed.

As before, solving these equations gives the results:

$$k_1 = k_2 = 0 \quad k_3 = 1 \tag{6.1}$$

$$\rho' = \rho + 3f(2a - f) \tag{6.2a}$$

$$p' = p + f(f - 2a) \tag{6.2b}$$

and as before the only admissible Bianchi types are those where $n_1 = n_2$ i.e. types I, VII_o and V. Again it is possible to find a basis such that

$$\Omega_\alpha = 0, \quad \sigma_{\alpha\beta} = 0 \quad (\alpha \neq \beta), \quad \theta_{\alpha\beta} = \mathrm{diag}(\theta_1, \theta_2, \theta_3) \tag{6.3}$$

and we obtain the following constraint

$$2\theta_3^2 f = (a - f)(\rho + p + 2af - 2f^2) \tag{6.4}$$

Substituting the above results into the homogeneous field equations (B.14) of Appendix B leaves

$$\partial_0 a + \theta_3 a = 0 \tag{6.5}$$

$$\partial_0 n_1 + (\theta_2 + \theta_3 - \theta_1)n_1 = 0 \tag{6.6a}$$

$$\partial_0 n_2 + (\theta_1 + \theta_3 - \theta_2)n_2 = 0 \tag{6.6b}$$

$$\partial_0\theta + \theta_1^2 + \theta_2^2 + \theta_3^2 + \tfrac{1}{2}(1 + 2\sinh^2\theta)\rho + \frac{3}{2}(1 + \frac{2}{3}\sinh^2\theta)p = 0 \tag{6.7}$$

$$a(2\theta_3 - \theta_1 - \theta_2) = (\rho + p)\sinh\theta\cosh\theta \tag{6.8}$$

$$\partial_0\theta_1 = -\theta\theta_1 + 2a^2 + \tfrac{1}{2}(\rho - p) \tag{6.9a}$$

$$\partial_0\theta_2 = -\theta\theta_2 + 2a^2 + \tfrac{1}{2}(\rho - p) \tag{6.9b}$$

$$\partial_0\theta_3 = -\theta\theta_3 + 2a^2 + \tfrac{1}{2}(\rho - p) + (\rho + p)\sinh^2\theta \tag{6.9c}$$

From (6.8) we note that since the angle θ is non-zero, and since we want $(\rho + p) \neq 0$, [otherwise $p = \rho = 0$ and from (6.2) $p' = -\frac{1}{3}\rho'$ which is unrealistic], then $a(2\theta_3 - \theta_1 - \theta_2) \neq 0$ and hence $a \neq 0$. Thus there are no Class A homogeneous perfect-fluid models giving non-tilted homothetic extensions. Thus we need only consider Type V models.

Combining (3.11c) and (6.8) gives

$$2\theta_3 f = a(2\theta_3 - \theta_1 - \theta_2)$$

or

$$2\theta_3(F + r) = r(\theta_1 + \theta_2) \tag{6.10}$$

Taking the time derivative and using (3.12a) we obtain

$$(\rho - p)\nu_{33} = 8rF + 4F^2 \qquad (6.11)$$

However, from (6.2) we also have

$$\rho - p = \rho' - p' + (8rF + 4F^2)\nu_{33}{}^{-1} \qquad (6.12)$$

Hence, combining (6.11) and (6.12) gives the condition

$$\rho' - p' = 0$$

Thus the only non-tilted homothetic models allowed have a hard equation of state. This result was first obtained by McIntosh [43].

Now taking the time derivative of (6.11), if $\nu \neq 1$, gives

$$(1 + \nu)\theta = 2\theta_3 \qquad (6.13)$$

and similarly, the time evolution of this equation gives

$$2(1 + \nu)F = - r(1 + 3\nu) \qquad (6.14)$$

which is also obtained by substituting (6.13) into (6.10). Also, substituting (6.14) and (6.11) into (6.2) we obtain

$$\rho' = p' = - \frac{F^2(3 + \nu)}{(1 - \nu)} \nu_{33}{}^{-1}$$

and hence $\rho' > 0$ only when $\nu > 1$ or $\nu < -3$. However, if $\nu \neq 1$, then the time evolution of the homothetic constraint (6.4) admits the solutions $\nu = -3$ or $\nu = -\frac{3}{5}$ only. Hence, if $\nu \neq 1$, the homothetic extensions have a negative pressure and density or are a vacuum.

When $\nu = 1$, then from (6.11) we have $F = - 2r$, or $f = 2a$, and substitution into (6.2) gives the fluid quantities as $\rho' = p' = \rho$. However, from (3.11f) we have $- 4a^2 = 2\rho\sinh^2\theta$ and so $\rho < 0$, and hence the homothetic extensions again have a negative density.

§4.7 Discussion

We have considered the restrictions implied by the condition that a perfect fluid model be a conformal extension of a perfect fluid homogeneous model. We have, however, restricted our attention to models where the metric has the following form

$$ds^2 = - dt^2 + \nu_{11}(w^1)^2 + \nu_{22}(w^2)^2 + \nu_{33}(w^3)^2$$

In general it has been found that the conformal models are tilted i.e. the fluid 4-velocity is not the normal direction to the surfaces upon which the group acts. In fact, the velocity vector of the conformal models is always tilted with respect to the velocity vector of the homogeneous model, for if one puts $k'_a = k_a$ in equations (3.10) - (3.13) we obtain the result $f = 0$.

In the models investigated above, either the conformal or homogeneous model was constrained to be non-tilted. In these cases the only perfect fluid models allowed were of Bianchi types I, VII_o or V. Unfortunately, it was usually found in all cases that the conformal equation of state was unphysical or the fluid quantities were unrealistic in view of the currently accepted observational data. The form of these fluid quantities, being inhomogeneous in one direction, was due to the fact that only one non-trivial conformal motion is allowed by these models. This will be the form of even the most general solutions not considered here.

Some of the fluid kinematical quantities can also be obtained using the expressions in §3.5. For example, for the tilted models it is seen that the direction of tilt k'_α is parallel to the vector a_α and so from the discussion at the end of §3.5, these models have zero vorticity. This is unfortunate as not many models with non-zero rotation are known with closed form solutions.

The restrictions we have considered here have been both dynamical and kinematical. However, we have not considered the nature of the dynamical evolution of these models. One aspect which is of particular interest, the initial value problem, we consider in the next chapter.

CHAPTER 5

CONFORMAL MOTIONS AND THE INITIAL VALUE PROBLEM

§5.1 Introduction

We have seen that the application of various symmetries to space-time has proved to be useful in finding exact solutions and in classifying space-times. Also, to consider a solution of Einstein's equations as the time evolution of an initial spacelike hypersurface has proven successful in various applications. We consider here the problem of placing a conformal motion symmetry upon the initial data in the Cauchy hypersurface, and then finding the conditions imposed on this initial data for the space-time to possess a local conformal motion, i.e. given initial data admits a conformal motion, does a solution to the evolution equations exist which also admits a conformal motion?

We shall follow closely the work of B. Berger [46] who obtained the appropriate equations and constraints for the case of vacuum space-times. In the following we extend this work to include a non-zero energy-momentum tensor.

§5.2 The Initial Value Problem

Consider the spacetime M with metric g_{ab} which satisfies Einstein's field equations

$$R_{ab} - \tfrac{1}{2}g_{ab}R = T_{ab} \tag{2.1}$$

where T_{ab} is an arbitrary stress-energy tensor

$$T_{ab} = \rho n_a n_b + p h_{ab} + 2q_{(a}n_{b)} + \pi_{ab} \tag{2.2}$$

Consider a spacelike hypersurface S, defined as having constant coordinate time t, embedded in the space-time. Let n^a denote the unit normal to this hypersurface and let λn^a be the connecting vector from each surface to nearby surfaces $[n_a = - \lambda \nabla_a t,\ \lambda = [- \nabla_a t \nabla^a t]^{-\frac{1}{2}}]$. We wish to discuss the geometry within the surfaces S(t) in terms of the intrinsic tensor field $h_{ab} = g_{ab} + n_a n_b$ and the extrinsic curvature $K_{ab} = h^c{}_a h^d{}_b \nabla_c n_d$. These tensor fields on S describe the intrinsic geometry of S and the embedding of S in M, respectively and constitute the initial data for Einstein's equations.

The initial value problem has the usual structure. Einstein's equations (2.1) can be written down as four constraint equations and twelve evolution equations for h_{ab} and K_{ab}. Roughly speaking, one sets initial data $\{h_{ab}(x^a),\ K_{ab}(x^a)\}$ at some initial time on a space-slice S(o) satisfying the constraint equations

$$G^o{}_o = T^o{}_o \tag{2.3a}$$

$$G^o{}_a = T^o{}_a \tag{2.3b}$$

and then uses the evolution equations

$$G^\alpha{}_\alpha = T^\alpha{}_\alpha \tag{2.4a}$$

$$G^\alpha{}_\beta - \frac{1}{3}\delta^\alpha{}_\beta G^\gamma{}_\gamma = T^\alpha{}_\beta - \frac{1}{3}\delta^\alpha{}_\beta T^\gamma{}_\gamma \tag{2.4b}$$

, which give the change in the data from one instant to the next, to see whether or not there exists a unique solution which preserves the constraints over the space-time region of interest.

To write out equations (2.3) and (2.4) in such a way that all tensors and tensor operations on S involve tensors and operations in M we follow Geroch [47] and use the intrinsic covariant derivative defined by

$$D_e T^{a\ldots c}{}_{b\ldots d} = h_e{}^r h^a{}_m \ldots h^b{}_n h^p{}_c \ldots h_d{}^q \nabla_r T^{m\ldots n}{}_{p\ldots q}$$

and such that $D_a h_{bc} = 0$. Using this Geroch has shown that the four constraint equations (2.3) can be written in the form [extended here to include non-zero T_{ab}]

$$\mathcal{R} - K^{ab}K_{ab} + K^2 = 2T_{ab}n^a n^b \tag{2.5}$$

$$D_b(K^{ab} - Kh^{ab}) = h^{ab}n^m T_{bm} \tag{2.6}$$

where $\mathcal{R}$ is the curvature scalar formed from h_{ab} using D, and the twelve evolution equations (2.4) give

$$\dot{h}_{ab} = \mathcal{L}_{\lambda n}h_{ab} = -\,2\lambda K_{ab} \tag{2.7}$$

and

$$\dot{K}_{ab} = \mathcal{L}_{\lambda n}K_{ab} = -\,2\lambda K_{am}K^m{}_b + \lambda K K_{ab} + \lambda \mathcal{R}_{ab}$$

$$- D_a D_b \lambda - \lambda(h^m{}_a h^n{}_b T_{mn} - \tfrac{1}{2}h_{ab}T) \tag{2.8}$$

where $\mathcal{R}_{ab}$ is the Ricci tensor formed from D and h_{ab}.
The function λ in (2.7) and (2.8) allows the evolution to proceed at

different rates at different points of S and even into the future at certain points of S (where $\lambda > 0$) and into the past at others ($\lambda < 0$).

In the following we first derive the constraints imposed on the initial data in the hypersurface due to the presence of a conformal motion and then consider whether these constraints are preserved by Einstein's equations.

§5.3 The Conformal Constraints

The spacetime is assumed to possess a conformal motion generated by the vector field ξ^a such that

$$L_\xi g_{ab} = 2\phi g_{ab} \tag{3.1}$$

where ϕ is an arbitrary function. We now find the components of equation (3.1) with respect to the hypersurface S restricting our attention to spacelike conformal Killing vectors i.e. $n^a \xi_a = 0$. The normal component gives

$$L_\xi \lambda = \phi\lambda \tag{3.2}$$

; the mixed component

$$L_{\lambda n} \xi^a = 0 \Rightarrow L_n \xi^a = \phi n^a \tag{3.3}$$

and the spatial component

$$L_\xi h_{ab} = 2\phi h_{ab} \quad . \tag{3.4}$$

To obtain an additional constraint on S, operate on (3.4) with $L_{\lambda n}$ and use the relation

$$\mathcal{L}_v \mathcal{L}_u h_{ab} = \mathcal{L}_u \mathcal{L}_v h_{ab} + \mathcal{L}_{\mathcal{L}_v \mu} h_{ab}$$

together with (2.7), (2.8), (3.2) and (3.3). We obtain

$$\mathcal{L}_\xi K_{ab} = \phi K_{ab} - h_{ab} \mathcal{L}_n \phi$$

or, using the result that $\mathcal{L}_\xi K_{ab} = {}^3\mathcal{L}_\xi K_{ab}$ we have

$$ {}^3\mathcal{L}_\xi K_{ab} = \phi K_{ab} - h_{ab} \mathcal{L}_n \phi \ . \tag{3.5}$$

Equations (3.4) and (3.5) are constraints which must be satisfied by the intrinsic metric and extrinsic curvature on S in the presence of a spacelike conformal Killing vector (CKV). All the above equations correspond to similar equations found by Berger [46].

§5.4 Evolution and Results

It is now possible to see whether the constraints (3.4) and (3.5) are preserved under evolution using Einstein's equations. This will involve acting on the above constraints with the operator $\mathcal{L}_{\lambda n}$ and seeing whether or not an identity results. The evolution of (13.4) yields (13.5), so we need only consider

$$\mathcal{L}_{\lambda n} \mathcal{L}_\xi K_{ab} = \mathcal{L}_{\lambda n}(\phi K_{ab}) - \mathcal{L}_{\lambda n}(h_{ab} \mathcal{L}_n \phi)$$

A term by term calculation of the right hand side gives

$$L_{\lambda n} L_\xi K_{ab} = \phi L_{\lambda n} K_{ab} + 3\lambda K_{ab} L_n \phi - \lambda h_{ab} L_n L_n \phi \tag{4.1}$$

For the calculation of the left hand side the reader is referred to Appendix A of Berger, [46], modified here to include non-zero energy-momentum tensor. We find

$$\begin{aligned} L_{\lambda n} L_\xi K_{ab} = {} & \phi L_{\lambda n} K_{ab} + \lambda K_{ab} L_n \phi - 2\lambda D_a D_b \phi \\ & - \lambda h_{ab} D_c D^c \phi - \lambda h_{ab} K L_n \phi \\ & - h_{ab} D_c \lambda D^c \phi - \lambda [L_\xi (h^m{}_a h^n{}_b T_{mn}) - \tfrac{1}{2} L_\xi (h_{ab} T)] \end{aligned} \tag{4.2}$$

Comparing (4.1) with (4.2) we obtain the following identity

$$\begin{aligned} & - 2D_a D_b \phi - 2K_{ab} L_n \phi + h_{ab} [L_n L_n \phi - \lambda^{-1} D^c \lambda D_c \phi - D_c D^c \phi - K L_n \phi] \\ & = L_\xi (h^m{}_a h^n{}_b T_{mn}) - \tfrac{1}{2} L_\xi (h_{ab} T) \\ & = (L_\xi T_{mn})(h^m{}_a h^n{}_b - \tfrac{1}{2} h_{ab} g^{mn}) \end{aligned} \tag{4.3}$$

Thus for a given energy-momentum tensor, this equation is a restriction on ϕ. If ϕ does not satisfy (4.3) then the conformal constraint (3.5) is not preserved by Einstein's evolution equations, which in fact restrict the conformal motions compatible with solutions of Einstein's equations.

Additional restrictions in the hypersurface S also arise from requiring the Lie derivative along the conformal Killing vector of the Einstein constraints (2.5) and (2.6) to be zero i.e. $$\mathcal{L}_\xi(K_{ab}K^{ab} - K^2 - R) = - 2\mathcal{L}_\xi\rho \tag{4.4a}$$

and $$\mathcal{L}_\xi(D_b(K^{ab} - Kh^{ab})) = - \mathcal{L}_\xi q^b \tag{4.4b}$$

Direct calculation yields the restrictions

$$D_cD^c\phi + K\mathcal{L}_n\phi = - \mathcal{L}_\xi\rho - 2\phi\rho \tag{4.5a}$$

and $$D_b\mathcal{L}_n\phi + K_{ab}D^a\phi = \mathcal{L}_\xi q_b - \phi q_b \tag{4.5b}$$

These equations give the correct extensions to equations (49) and (51) of Berger's paper for non-zero energy-momentum tensor. That this is the case can be seen by examining the case for homothetic motions. As was mentioned in Chapter 2 under a homothetic motion any geometric object with dimension $(\text{length})^q$ transforms with a factor $\exp(q\phi)$. Thus we have

$$\mathcal{L}_\xi\rho = - 2\phi\rho \quad \text{and} \quad \mathcal{L}_\xi q_b = \phi q_b$$

and so we see from equations (4.5) that these constraints are satisfied for ϕ constant, as pointed out by Berger. Also, we obtain from (4.3) the result shown by Berger that the conformal constraints are conserved if $\mathcal{L}_\xi(T_{ab}) = 0$.

Now, from equation (3.2.12) we have the result that for a general energy-momentum tensor, such as (2.2)

$$\tfrac{1}{2}\mathcal{L}_\xi(T_{ab}) = g_{ab}\nabla^c\nabla_c\phi - \nabla_a\nabla_b\phi \tag{4.6}$$

Using this result it is easy to show that the constraints (4.3) and (4.5a,b) are satisfied. Thus, in general, if in a spacelike hypersurface S, a set of initial data satisfy the conformal constraints (3.4) and (3.5), [in addition to the Einstein constraints], for a conformal Killing vector ξ^a, then the space-time development of S contains a conformal motion with generator ξ^a. However, in general this development will be compatible only with an imperfect fluid stress-energy tensor, as we noted in Chapter 3. Hence, we wish to find out whether, given the initial conditions of S(o), the space-time development still admits a conformal motion when we constrain the energy-momentum tensor to be that of a perfect fluid

$$T_{ab} = \rho n_a n_b + p h_{ab} \tag{4.7}$$

Using (4.6), (3.3) and (3.4) we have

$$\tfrac{1}{2}\mathcal{L}_\xi \rho = - D^c D_c \phi - K L_n \phi - \phi\rho \tag{4.8a}$$

and
$$\tfrac{1}{2}\mathcal{L}_\xi p = \tfrac{2}{3} D^c D_c \phi + \tfrac{2}{3} K L_n \phi + \lambda^{-1} D^c \lambda D_c \phi - L_n L_n \phi - \phi p \tag{4.8b}$$

These expressions also follow from equations (3.2.13a) and (3.2.13c) when written out in terms of tensors and tensor operations on S. Thus we have for the perfect fluid (47)

$$\tfrac{1}{2}\mathcal{L}_\xi T_{ab} = (\tfrac{2}{3} h_{ab} - n_a n_b)(K L_n \phi + D^c D_c \phi)$$

$$+ h_{ab}(\lambda^{-1} D^c \lambda D_c \phi - L_n L_n \phi) \tag{4.9}$$

and substituting this result into the constraint (4.3) gives the following restriction upon ϕ

$$\frac{1}{3}h_{ab}(D_c D^c \phi + K L_n \phi) - D_a D_b \phi - K_{ab} L_n \phi = 0 \tag{4.10}$$

Also, substituting (4.8a) into (4.5a) gives an identity while (4.5b) gives the constraint

$$D_b L_n \phi + K_{ab} D^a \phi = 0 \tag{4.11}$$

It is immediately seen that equations (4.10) and (4.11) are not automatically satisfied for ϕ nonconstant. These restrictions serve, in general, to prevent a conformal motion in spacelike initial data from being a spacetime conformal motion.

Putting ϕ = constant, one immediately sees that all the constraints are satisfied. In fact, from (4.6) $\mathcal{L}_\xi(T_{ab}) = 0$ whenever ϕ = constant and so the Einstein and homothetic constraints on the initial hypersurface cannot spoil the compatibility of Einstein's equations and a homothetic motion. Thus we see that when ϕ is a constant, each spacelike point effectively evolves in a similar manner and so the symmetry property of the space-time is conserved. However, this is not the case when ϕ is an arbitrary function of the space variables.

Finally, if ϕ is independent of the time coordinate, then constraints (4.10) and (4.11) reduce to

$$D_a D_b \phi - \frac{1}{3}h_{ab} D_c D^c \phi = 0 \tag{4.12a}$$

$$D^a \phi K_{ab} = 0 \tag{4.12b}$$

We see that if $\phi = F(x^3)$, as for the models considered in the previous chapter, then we have from (4.12a) $D_c D^c \phi = 0$ or $F' = 0$ and thus we have the result that no conformal solutions having a non-tilted velocity vector admit a perfect fluid. This agrees with the result found in §4.4.

CHAPTER 6

HAMILTONIAN COSMOLOGY AND SPATIALLY HOMOTHETIC MODELS

§6.1 Introduction

The value of an action principle as a vehicle for intuition in the study of homogeneous cosmologies was first demonstrated by Misner in his programme of Chaotic cosmology [48]. Since then, Hamiltonian cosmology, the study of cosmological models by means of equations of motion in Hamiltonian form, has received considerable attention, especially due to the work of Ryan [49]. The cosmological models which have received detailed examination are the Kantowski-Sachs models, the spatially homogeneous models and Eardley has initiated work on spatially homothetic models [18].

Apart from Eardley's work, Hamiltonian cosmology has not been applied to inhomogeneous cosmological models. Nevertheless, studies have been made of non-cosmological metrics which have inhomogeneous space sections. Kuchar [50] has studied the Einstein-Rosen cylindrical wave metric in the ADM formulation; Berger et al. [51] have applied this formalism to the study of spherically symmetric gravitational fields,while Lund [52] has considered the complete vacuum Schwarzschild solution. Ryan [49] sees the most outstanding problem that will arise in considering inhomogeneous three spaces is that the Hamiltonian will become a Hamiltonian density and so we must deal with a field theory instead of a particle problem.

In the following we reconsider and extend Eardley's work on homothetic models. Conformal models are ruled out since ϕ_α is not a constant and so a necessary spatial integration cannot be carried out.

In section 6.2 we briefly review the A.D.M. procedure but the reader is referred to the review by Ryan [49] and the relevant chapters in the book 'Relativistic Homogeneous Cosmologies (Princeton U.P.) by Ryan and Shepley. In section (6.3) we consider the applicability of this method to homothetic models and show that only models where $\phi_\alpha = 2a_\alpha$ will give valid field equations from variation of the action. Then in section (6.4) we apply the qualitative methods of Ryan to these models.

§6.2 A.D.M. Formalism

The first step in the Hamiltonian formulation is the identification of the field variables with the metric. However, the general coordinate invariance of the theory creates problems, introducing redundant variables to insure the correct transformation properties. Thus it is necessary to separate the metric into the parts carrying the true dynamical information and those parts characterizing the coordinate system. When in canonical form, the Hamiltonian will involve the minimal number of variables specifying the state of the system.

The usual action integral for general relativity is

$$I = \int d^4x\mathcal{L} = \int d^4x\sqrt{-g}\;R \tag{2.1}$$

One obtains Einstein's equations upon variation in the metric. The three dimensional quantities appropriate for the Einstein field are

$$g_{ij} \equiv {}^4g_{ij} \qquad N \equiv (-\,{}^4g^{00})^{-\frac{1}{2}} \qquad N_i \equiv {}^4g_{oi} \tag{2.2a}$$

$$\pi^{ij} \equiv \sqrt{-{}^4g}\,({}^4\Gamma^o{}_{pq} - g_{pq}\,{}^4\Gamma^o{}_{rs}g^{rs})g^{ip}g^{jq} \tag{2.2b}$$

Here, and subsequently, we mark every 4-dimensional quantity with the prefix ^{4}, so that all unmarked quantities are understood as 3-dimensional.

In terms of the basic quantities (2.2), the Lagrangian becomes

$$L = \sqrt{-{}^4g}\ {}^4R = - g_{ij}\pi^{ij}{}_{,o} - NC^o - N_iC^i$$
$$- 2(\pi^{ij}N_j - \tfrac{1}{2}\pi N^i + N^{|i}\sqrt{g})_{,i} \qquad (2.3)$$

where

$$C^o \equiv \sqrt{-g}({}^3R + g^{-1}(\tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}) \qquad (2.4a)$$

$$C^i \equiv - 2\pi^{ij}{}_{|j} \qquad (2.4b)$$

The quantity ${}^3R$ to the curvature scalar formed from the spatial metric g_{ij}; $|$ indicates the covariant derivative using this metric, and the spatial indices are raised and lowered using g^{ij} and g_{ij}.

The use of the Palatini Lagrangian - writing L linear in the time derivatives - and the 3 + 1 dimensional notation does not impair the general covariance of the theory under arbitrary coordinate transformations and hence the action is analogous to the parametrized form of mechanics in which the Hamiltonian and the time derivative are introduced as a conjugate pair of variables. Consider the example [58] of a system with M degrees of freedom. Its action may be written

$$I = \int_{t_2}^{t_1} L dt = \int_{t_2}^{t_1} \left(\sum_{i=1}^{m} p_i \dot{q}_i - H(p,q) \right) dt \qquad \dot{q}_i = \frac{dq_i}{dt}$$

where L is linear in time variables. The action may be cast into parametrized form in which the time is regarded as a function q_{m+1} of an arbitrary parameter τ

$$I = \int_{\tau_2}^{\tau_1} L_\tau d\tau \equiv \int_{\tau_2}^{\tau_1} d\tau \left\{ \sum_{i=1}^{m+1} p_i q_i' \right\} \qquad q_i' \equiv \frac{dq_i}{d\tau}$$

and the constraint equation $p_{m+1} + H(p,q) = 0$ holds. One may equally replace this constraint by an additional term in the action

$$I = \int_{\tau_2}^{\tau_1} d\tau \left\{ \sum_{i=1}^{m+1} p_i q_i' - NR \right\} \tag{2.5}$$

where $N(\tau)$ is a Lagrange multiplier. Its variation yields the constraint $R(p_{m+1},p,q) = 0$ having solution $p_{m+1} = -H(p,q)$. The theory, as cast in form (2.5), is now generally covariant with respect to arbitrary coordinate transformations $\bar{\tau} = \bar{\tau}(\tau)$. The price of achieving this has been not only the introduction of the $(m + 1)$st degree of freedom, but loss of canonical form. Also, the Hamiltonian $H' = NR$ now vanishes due to the constraint.

In changing from a particle case to a field theory, the coordinates now appear as four new field variables $q^{m+\mu} = x^\mu(\tau^\alpha)$ and there are four extra momenta. Four constraints are now required and four Lagrange multipliers. From (2.3), we see N and N_i are the Lagrange multipliers corresponding to the constraints C_o and C^i. Variation with respect to π^{ij}, g_{ij}, N and N_i gives the field equations as

$$g_{ij,o} = 2Ng^{-\frac{1}{2}}(\pi_{ij} - \tfrac{1}{2}g_{ij}\pi) + 2N_{(i|j)} \tag{2.6a}$$

$$\pi^{ij}{}_{,o} = - Ng^{\frac{1}{2}}({}^3R^{ij} - \tfrac{1}{2}g^{ij}\,{}^3R) + \tfrac{1}{2}Ng^{-\frac{1}{2}}g^{ij}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}\pi^2)$$

$$- 2Ng^{-\frac{1}{2}}(\pi^{im}\pi_m{}^j - \tfrac{1}{2}\pi\pi^{ij}) + g^{\frac{1}{2}}(N^{|ij} - g^{ij}N_{|m}{}^{|m})$$

$$+ (\pi^{ij}N^m)_{|m} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi} \qquad (2.6b)$$

$$C^o = 0 \qquad (2.6c)$$

$$C^i = 0 \qquad (2.6d)$$

Note that the spatial divergence in the integrand plays no part in the variational principle and may be neglected.

To reduce the Lagrangian (2.3) to canonical form, one inserts the solution of the constraint equations and then imposes coordinate conditions (equivalent to introducing intrinsic coordinates). Only after this will the true non-vanishing Hamiltonian of the theory arise. The canonical formalism necessarily destroys the space-time covariance of the theory by cutting space-time into slices and investigating their geometrical properties. In the ADM approach, a definite slicing of space-time and a definite coordinatization of the slices are picked out by the coordinate conditions.

In applying the ADM procedure to cosmological models, the basic method is to freeze all but a few degrees of the infinitely many degrees of freedom of the gravitational field by putting a number of the canonical coordinates and their momenta zero. The pioneer of this approach was DeWitt who first applied the Dirac method to the Friedman universes [53]. The second model, treated by the ADM method, was Misner's mixmaster universe. However, because of the high symmetry of these models, a privileged slicing of space-time exists, such that the intrinsic geometry of the slices is homogeneous. The symmetry thus provides a unique 1-parameter family of spacelike hypersurfaces on which the further formalism is based. This goes against the usual properties of the ADM method, which picks out a 1-parameter family of slices by coordinate conditions, rather than by symmetry requirements.

Also, the homogeneity of these models was responsible for the major reduction in the number of gravitational variables. In a typical field theory we can expect to find several degrees of freedom at each point of space and we can study the interaction between the degrees of freedom at neighbouring points. However, the requirement of homogeneity ties the corresponding degrees of freedom at different points rigidly together and so the field aspect of gravity thus almost completely disappears from the model.

The methods used by Misner and Ryan in the study of homogeneous cosmologies involved assuming a metric of the form

$$g_{ij} = g_{ab}(\tau)\sigma^a{}_i\sigma^b{}_j \tag{2.7}$$

(see (2.2.1)) where the σ^a are three time-independent 1-forms. With this assumption used in the variational principle, the new generalized coordinates and momenta (g_{ab},π^{ab}) are now discrete variables. However, as first noted by Hawking [54] the resultant variational principle does not always give the correct field equations. This difficulty was first investigated by MacCallum and Taub [55] and later by Ryan [56], but it was not until the work of Sneddon [57] that the situation was clarified. It was shown that whereas the variational principle works for non-coordinate frames, the requirement of spatial homogeneity prevents a boundary term being set to zero.

In the following we first discuss a similar problem when we require spatial homogeneity (first studied by Eardley) and then apply the qualitative methods of Ryan to the applicable cases.

§6.3 ADM Approach and Spatially Homothetic Cosmologies

We now apply the action (2.1) with Lagrangian given by (2.3) to homothetic Bianchi metrics. In so doing we closely follow Sneddon's work [57] on homogeneous metrics. We work in the non-coordinate frame given by the following transformation

$$\omega^{\alpha} = \sigma^{\alpha}{}_{i}dx^{i} \tag{3.1}$$

so that the components of the 3-metric have the form

$$\hat{g}_{ij} = \hat{g}_{\alpha\beta}\sigma^{\alpha}{}_{i}\sigma^{\beta}{}_{j} \tag{3.2}$$

From Chapter 2 we note that the invariant 1-forms ω^{α} are independent of the 'time' parameter τ and satisfy

$$d\omega^{\alpha} = -\tfrac{1}{2}C^{\alpha}{}_{\beta\nu}\omega^{\beta}\wedge\omega^{\nu}$$

where the $C^{\alpha}{}_{\beta\nu}$ are the structure constants and have one of the nine canonical forms found by Bianchi.

The conjugate momenta may be written in the form

$$\hat{\pi}^{ij} = (\det\sigma^{\alpha}{}_{i})\hat{\pi}^{\alpha\beta}\sigma_{\alpha}{}^{i}\sigma_{\beta}{}^{j} \qquad (3.3)$$

where $\sigma^{\alpha}{}_{i}\sigma^{i}{}_{\beta} = \delta^{\alpha}{}_{\beta}$. The first factor on the right-hand-side of (3.3) appears because π^{ij} is a tensor density. Because of this we must be careful in transforming from one frame to another. However, one can see from (2.6) that there are no problems when $\hat{N}_i = 0$ because only 'time' derivatives of π^{ij} appear and the frame we are transforming to has $\sigma^{o} = d\tau$. The coordinate condition $\hat{N}_i = 0$ (i.e. ${}^4g_{oi} = 0$) is usual in cosmology and applies here.

Using the above results, the action (2.1) with $N_i = 0$ can be written as

$$\delta \int \Big\{- \hat{g}_{\alpha\beta}\sigma^{\alpha}{}_{i}\sigma^{\beta}{}_{j}\, \frac{\partial}{\partial\tau}\{(\det\sigma^{\alpha}{}_{i})\hat{\pi}^{\nu\delta}\sigma_{\nu}{}^{i}\sigma_{\delta}{}^{j}\}$$

$$- \hat{N}\Big\{(\det\sigma^{\alpha}{}_{i})^{-1}\hat{g}^{-\frac{1}{2}}\Big[(\det\sigma^{\alpha}{}_{i})^{2}\hat{\pi}^{\alpha\beta}\sigma_{\alpha}{}^{i}\sigma_{\beta}{}^{j}\hat{\pi}_{\gamma\delta}\sigma^{\gamma}{}_{i}\sigma^{\delta}{}_{j}$$

$$- \tfrac{1}{2}(\det\sigma^{\alpha}{}_{i})^{2}\hat{\pi}^{2}\Big] - (\det\sigma^{\alpha}{}_{i})\hat{g}^{\frac{1}{2}}({}^{3}\hat{R})\Big\}\Big\}d\tau d^{3}x = 0$$

$$\Rightarrow \delta \int (\det\sigma^{\alpha}{}_{i})\Big\{- \hat{g}_{\alpha\beta}\hat{\pi}^{\alpha\beta}{}_{,\tau} - \hat{N}\Big[\hat{g}^{-\frac{1}{2}}(\hat{\pi}^{\alpha\beta}\hat{\pi}_{\alpha\beta} - \tfrac{1}{2}\hat{\pi}^{2}) - \hat{g}^{\frac{1}{2}}{}^{3}\hat{R}\Big]\Big\}d\tau d^{3}x = 0 \qquad (3.4)$$

and equations (2.6) become

$$\hat{g}_{\alpha\beta,o} = 2\hat{N}\hat{g}^{-\frac{1}{2}}(\hat{\pi}_{\alpha\beta} - \tfrac{1}{2}\hat{g}_{\alpha\beta}\hat{\pi}) \qquad (3.5a)$$

$$\hat{\pi}^{\alpha\beta}{}_{,o} = - \hat{N}\hat{g}^{\frac{1}{2}}({}^{3}\hat{R}^{\alpha\beta} - \tfrac{1}{2}\hat{g}^{\alpha\beta}\,{}^{3}\hat{R}) + \tfrac{1}{2}\hat{N}\hat{g}^{-\frac{1}{2}}(\hat{\pi}^{\gamma\delta}\hat{\pi}_{\gamma\delta} - \tfrac{1}{2}\hat{\pi}^{2})\hat{g}^{\alpha\beta}$$

$$- 2\hat{N}\hat{g}^{-\frac{1}{2}}(\hat{\pi}^{\alpha\gamma}\hat{\pi}_{\gamma}{}^{\beta} - \tfrac{1}{2}\hat{\pi}\hat{\pi}^{\alpha\beta}) \qquad (3.5b)$$

$$H = \hat{g}^{-\frac{1}{2}}(\hat{\pi}^{\alpha\beta}\hat{\pi}_{\alpha\beta} - \tfrac{1}{2}\hat{\pi}^2) - \hat{g}^{\frac{1}{2}}({}^3\hat{R}) = 0 \tag{3.5c}$$

We can now consider the specific form of the homothetic models under construction. We have from previous results

$$\hat{g}_{\alpha\beta} = e^{2\sigma}g_{\alpha\beta}(t) \tag{3.6}$$

where σ is independent of t. Hence, we need to substitute (3.6) into the action (3.4) and see if the subsequent variations give the field equations with (3.6) substituted. From (3.6) we have the following transformations

$$\left.\begin{array}{ll} \hat{\pi}_{\alpha\beta} = e^{4\sigma}\pi_{\alpha\beta}(t) & \hat{\pi} = e^{2\sigma}\pi \\ \hat{N} = e^{\sigma}N \quad \text{and} & {}^3\hat{g} = e^{6\sigma}\,{}^3g \end{array}\right\} \tag{3.7}$$

To see how ${}^3\hat{R}^{\alpha\beta}$ and ${}^3\hat{R}$ transform consider the following. In the non-coordinate basis we are dealing with, we have the relations

$$\hat{R}_{\alpha\beta} = \hat{\Gamma}^{\gamma}{}_{\alpha\beta,\gamma} - \hat{\Gamma}^{\gamma}{}_{\alpha\gamma,\beta} + \hat{\Gamma}^{\gamma}{}_{\alpha\beta}\hat{\Gamma}^{\delta}{}_{\gamma\delta} - \hat{\Gamma}^{\gamma}{}_{\alpha\delta}\hat{\Gamma}^{\delta}{}_{\beta\gamma} \tag{3.8}$$

where

$$\hat{\Gamma}^{\gamma}{}_{\alpha\beta} = \tfrac{1}{2}g^{\gamma\delta}[\hat{C}_{\delta\beta\alpha} + \hat{C}_{\alpha\delta\beta} - \hat{C}_{\beta\alpha\delta} + \hat{g}_{\delta\beta,\alpha} + \hat{g}_{\alpha\delta,\beta} - \hat{g}_{\beta\alpha,\delta}] \tag{3.9}$$

and

$$\hat{C}^{\delta}{}_{\alpha\beta} = g^{\delta\gamma}\hat{C}_{\gamma\alpha\beta} \; .$$

Also, from section (2.3) we have

$$d\sigma = \frac{\partial\sigma}{\partial x^i} dx^i = \frac{\partial\sigma}{\partial x^\alpha} \omega^\alpha = \phi_\alpha \omega^\alpha$$

$$\Rightarrow \qquad \frac{\partial\sigma}{\partial x^\alpha} = \phi_\alpha = \text{constant} \quad . \tag{3.10}$$

Since the structure constants depend upon the basis vectors which are unaffected by the transformation (3.6) we have

$$\hat{C}^\alpha{}_{\beta\delta} = C^\alpha{}_{\beta\delta} \tag{3.11}$$

Also, substituting (3.11) and (3.6) into (3.9) gives

$$\hat{\Gamma}^\delta{}_{\beta\alpha} = \Gamma^\delta{}_{\beta\alpha} + [\delta^\delta{}_\alpha\phi_\beta + \delta^\delta{}_\beta\phi_\alpha - g_{\alpha\beta}\phi^\delta] \tag{3.12}$$

and using this result in (3.8) gives

$$\begin{aligned} {}^3\hat{R}_{\alpha\beta} &= {}^3R_{\alpha\beta} + \phi_\alpha\phi_\beta - g_{\alpha\beta}(\phi_\delta\phi^\delta + \Gamma^\delta{}_{\gamma\delta}\phi^\nu) + \phi_\delta C^\delta{}_{\beta\alpha} + \phi_\delta\Gamma^\delta{}_{\alpha\beta} + 2\phi^\delta\Gamma_{(\alpha\beta)\delta} \\ &= {}^3R_{\alpha\beta} + \phi_\alpha\phi_\beta + (2a_\delta - \phi_\delta)\phi^\delta g_{\alpha\beta} - \phi^\delta C_{(\alpha\beta)\delta} \end{aligned} \tag{3.13}$$

since $2a_\alpha = C^\beta{}_{\alpha\beta}$ and from equation (2.3.11) we have the condition $\phi_\delta C^\delta{}_{\alpha\beta} = 0$. Contracting (3.13) then gives

$${}^3\hat{R} = e^{-2\sigma}[{}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha] \tag{3.14}$$

Substituting (3.7), (3.13) and (3.14) into the action (3.4) now gives

$$\begin{aligned} \delta I = \delta \int & e^{2\sigma}(\det\sigma^\alpha{}_i)\Big\{- g_{\alpha\beta}(t)\pi^{\alpha\beta}(t)_{,0} \\ & - N\Big\{g^{-\frac{1}{2}}(\pi^{\alpha\beta}(t)\pi_{\alpha\beta}(t) - \tfrac{1}{2}\pi^2(t)) - g^{\frac{1}{2}}({}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha)\Big\}\Big\}dt\,d^3X \\ = 0 & \end{aligned}$$

The spatial integration can now be performed and the variational principle becomes

$$\delta I = \delta \int \left\{- g_{\alpha\beta}\pi^{\alpha\beta}{}_{,o} - N\left\{g^{-\frac{1}{2}}(\pi^{\alpha\beta}\pi_{\alpha\beta} - \tfrac{1}{2}\pi^2) - g^{\frac{1}{2}}({}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha)\right\}\right\}dt$$

$$= 0 \tag{3.15}$$

As several authors have noticed for the homogeneous case, trouble arises when the variation of ${}^3R\sqrt{g}$ is taken with respect to $g_{\alpha\beta}$. Thus we consider the term

$$\delta \int ({}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha)\sqrt{g}\ dt \tag{3.16}$$

$$= - \int (R^{\alpha\beta} + (8a^\alpha - 2\phi^\alpha)\phi^\beta)\delta g_{\alpha\beta}\sqrt{g}\ dt$$

$$+ \tfrac{1}{2} \int g^{\alpha\beta}({}^3R + (8a_\nu - 2\phi_\nu)\phi^\nu)\sqrt{g}\ \delta g_{\alpha\beta} dt + \int g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{g}\ dt$$

and following the work of Sneddon, one can show that

$$\int g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{g}\ dt = \int (4a^\alpha a^\beta + 2a^\nu C^{(\alpha\beta)}{}_\nu)\sqrt{g}\ \delta g_{\alpha\beta}\ dt$$

and so equation (3.16) becomes

$$\delta \int ({}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha)\sqrt{g}\ dt$$

$$= \int \left\{{}^3R^{\alpha\beta} + (8a^\alpha - 2\phi^\alpha)\phi^\beta - 4a^\alpha a^\beta - 2a^\gamma C^{(\alpha\beta)}{}_\gamma\right.$$

$$\left. - \tfrac{1}{2}g^{\alpha\beta}({}^3R + (8a_\nu - 2\phi_\nu)\phi^\nu\right\}\sqrt{g}\ \delta g_{\alpha\beta}\ dt \tag{3.17}$$

Substituting (3.17) back into (3.15) and taking the variation with respect to $g_{\alpha\beta}$ we obtain the equation

$$\pi^{\alpha\beta}{}_{,o} = -Ng^{\frac{1}{2}}\Big[{}^3R^{\alpha\beta} + (8a^{\alpha} - 2\phi^{\alpha})\phi^{\beta} - 4a^{\alpha}a^{\beta} - 2a^{\gamma}C^{(\alpha\beta)}{}_{\gamma}$$

$$- \tfrac{1}{2}g^{\alpha\beta}({}^3R + (8a_{\gamma} - 2\phi_{\nu})\phi^{\nu})\Big]$$

$$+ \tfrac{1}{2}Ng^{-\frac{1}{2}}(\pi^{\gamma\delta}\pi_{\gamma\delta} - \tfrac{1}{2}\pi^2)g^{\alpha\beta} - 2Ng^{-\frac{1}{2}}(\pi^{\alpha\nu}\pi_{\nu}{}^{\beta} - \tfrac{1}{2}\pi\pi^{\alpha\beta}) \quad (3.18)$$

Also, substituting (3.6), (3.7), (3.11), (3.13) and (3.14) into the field equation (3.5b) we also have the equation

$$\pi^{\alpha\beta}{}_{,o} = -Ng^{\frac{1}{2}}\Big[{}^3R^{\alpha\beta} + (2a_{\nu} - \phi_{\nu})\phi^{\nu}g^{\alpha\beta} + \phi^{\alpha}\phi^{\beta} - \phi^{\nu}C^{(\alpha\beta)}{}_{\nu}$$

$$- \tfrac{1}{2}g^{\alpha\beta}({}^3R + (8a_{\nu} - 2\phi_{\nu})\phi^{\nu})\Big]$$

$$+ \tfrac{1}{2}Ng^{-\frac{1}{2}}(\pi^{\gamma\delta}\pi_{\gamma\delta} - \tfrac{1}{2}\pi^2)g^{\alpha\beta} - 2Ng^{-\frac{1}{2}}(\pi^{\alpha\delta}\pi_{\delta}{}^{\beta} - \tfrac{1}{2}\pi\pi^{\alpha\beta}) \quad (3.19)$$

Comparing equations (3.18) and (3.19) we see that they are equivalent only when we apply the additional constraint

$$\phi_{\alpha} = 2a_{\alpha}$$

One can also check this result by first taking the variation of (3.4) with respect to $\hat{g}_{\alpha\beta}$ and then substituting in the homothetic condition (3.6). As before we have

$$\delta\int {}^3\hat{R}\sqrt{\hat{g}}\,dV = \int (-\,{}^3\hat{R}^{\alpha\beta} + \tfrac{1}{2}\hat{g}^{\alpha\beta}\,{}^3\hat{R})\sqrt{\hat{g}}\;\delta\hat{g}_{\alpha\beta}dV + \int \delta\hat{R}_{\alpha\beta}\;\hat{g}^{\alpha\beta}dV \quad (3.21)$$

and from the last term we have

$$\hat{g}^{\alpha\beta}\delta\hat{R}_{\alpha\beta} = \hat{\omega}^{\alpha}{}_{|\alpha}$$

where $$\hat{\omega}^{\alpha} = e^{-2\sigma}(g^{\alpha\nu}g^{\delta\beta} - g^{\alpha\beta}g^{\delta\gamma})((\delta g_{\gamma\delta})_{;\beta} + \delta g_{\gamma\delta}\phi_{\beta}) \qquad (3.22)$$

Now provided $\delta g_{\alpha\beta}$ and $(\delta g_{\alpha\beta})_{,\gamma}$ can be made to vanish on the boundary, the last term in equation (3.21) will vanish and the variation will give the usual result. However, as pointed out by Sneddon (op.cit), if $g_{\alpha\beta}$ and $\delta g_{\alpha\beta}$ are to be constants (or functions of time only) these conditions cannot be satisfied without $\delta g_{\alpha\beta}$ vanishing everywhere. Thus

$$\int \hat{\omega}^{\alpha}{}_{|\alpha}\sqrt{\hat{g}}\, dV = \int (\phi_{\alpha} - 2a_{\alpha})\hat{\omega}^{\alpha}\sqrt{\hat{g}}\, dV$$

will not vanish unless $\phi_{\alpha} = 2a_{\alpha}$ as before. Setting $\phi_{\alpha} = 0$ we obtain the homogeneous result $a_{\alpha} = 0$.

As in the homogeneous case, this result means that whenever the Einstein variational principle is used, care should be taken to ensure that the correct field equations are obtained. The usual variational principle breaks down in a number of places, as before. By neglecting some terms in the Lagrangian, some of these difficulties can be overcome, but as we have seen incorrect terms still arise from a surface integral that does not vanish. The exceptions are spaces of Class A where $a_{\alpha} = \phi_{\alpha} = 0$ and the subspace of Class D spaces for which $\phi_{\alpha} = 2a_{\alpha}$. It can also be noted that this result coincides with the result found in Chapter 4 that Class B homothetic extensions admitting a perfect fluid were possibly only when F = 2a. Also in these cases the fluid sources in the homothetic models were described by the expressions

$$\hat{\rho} = e^{-2\sigma}\rho \qquad \hat{p} = e^{-2\sigma}p$$

and so it can be seen that if the homogeneous models are vacuum models then so are the homothetic models and this agrees with the conclusion that the above variational principle gives the Einstein vacuum field equations.

Note that Eardley [18] found the condition $a_\alpha = \phi_\alpha$ for the variational principle to give the correct field equations. However, he assumed that the last term in (3.21) had the form $\hat{\omega}^\alpha{}_{|\alpha} = (e^{2\sigma}\omega^\alpha(t))_{|\alpha}$; from (3.22) we see that it in fact has the form $\hat{\omega}^\alpha{}_{|\alpha} = (e^{-2\sigma}\omega^\alpha(t))_{|\alpha}$.

§6.4 Qualitative Description of Homothetic Models

We are now in a position to use the fruitful mathematical techniques developed by Misner and Ryan to study the qualitative aspects of spatially homothetic cosmological models. Eardley briefly considered Type VII models and we extend this work to all admissible types.

From the previous section we have seen that the action can be written in the form

$$I = \int \pi^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial t} d^4X \tag{4.1}$$

subject to the constraints

$$C^0 = -\, g^{\frac{1}{2}}\left[{}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha + g^{-1}(\tfrac{1}{2}(\pi^k{}_k)^2 - \pi^{\gamma\delta}\pi_{\gamma\delta})\right] = 0 \quad (4.2)$$

$$C^\alpha = -\, 2\pi^{\alpha\beta}{}_{;\beta} - 4\pi^{\alpha\beta}\phi_\beta + 2\phi^\alpha\pi^\beta{}_\beta = 0 \quad (4.3)$$

It is seen that the scalar σ (or its derivatives ϕ_α) does not appear in the action (4.1), and it is thus in a form identical to the action found in the homogeneous case. We can hence follow the work of Ryan [49] in reducing (4.1) to canonical form. However, the constraint equations have extra terms involving ϕ_α and we shall see that this leads to a modified Hamiltonian. In reducing the action to canonical form we follow the procedure outlined in section (6.2) of choosing four of the twelve $g_{\alpha\beta}$ and $\pi^{\alpha\beta}$ as intrinsic coordinates and by solving the four constraint equations to eliminate four more.

To begin with we parametrize $g_{\alpha\beta}$ by means of Misner's parametrization, $g_{\alpha\beta} = R_o{}^2 e^{-2\Omega} e^{2B}{}_{\alpha\beta}$ where $\Omega(t)$ is a scalar and $B(t)$ is a 3×3 symmetric traceless matrix. R_o is a constant included for convenience in choosing units. Inserting this into (4.1) we have

$$I = \frac{1}{16\pi} \int 2\left\{- \pi^\delta{}_\delta + (e^B \pi e^{-B})_{\alpha\beta} \frac{\mathfrak{d}B_{\alpha\beta}}{d\Omega}\right\} d^4x$$

where $\dfrac{\mathfrak{d}B}{d\Omega} = \tfrac{1}{2}\left\{e^{-B}\dfrac{de^B}{d\Omega} + \dfrac{de^B}{d\Omega}e^{-B}\right\}$. Integrating over the space variables one has

$$I = (2\pi) \int [(e^B \pi e^{-B})_{\alpha\beta}\mathfrak{d}B_{\alpha\beta} - \pi^\alpha{}_\alpha d\Omega] \quad (4.4)$$

Now, whenever Ω is a monotonic function of t we can choose Ω as our 'time' coordinate. This choice represents the first step in the ADM procedure ; that is, choosing a function of the $g_{\alpha\beta}$ and $\pi^{\alpha\beta}$ as a coordinate. In this case we have

$$\Omega = -\frac{1}{6}\,\ell n\,[\det(g_{\alpha\beta})] \qquad (4.5)$$

We now see that the action (4.4) is in canonical form if we define the Hamiltonian $H = (2\pi)\pi^{\alpha}{}_{\alpha}$ and if we can find a matrix $p_{\alpha\beta}$ such that $2\,(e^{B}\pi e^{-B})_{\alpha\beta}dB_{\alpha\beta} = p_{\alpha\beta}dB_{\alpha\beta} = p_A dB_A$ where B_A are the parameters which determine the B-matrix and may number from two to six [49]. Finally, to give the metric completely we need only specify N. For our choice of Ω as time, it has been shown that $dt = -\,Nd\Omega$ where

$$N = H^{-1}e^{-3\Omega}(12\pi R_0^3) \qquad (4.6)$$

§ 6.5 DIAGONAL SPACE-TIMES

Follwing Ryan [49], we define the matrix $p_{\alpha\beta}$ as

$$p_{\alpha\beta} = 2\pi(e^{B}\,\pi^{\delta}{}_{\delta}\,e^{-B})_{\alpha\beta} - \frac{2\pi}{3}\,\delta_{\alpha\beta}\pi^{\delta}{}_{\delta} \qquad (5.1)$$

and proceed to parametrize $B_{\alpha\beta}$ and $p_{\alpha\beta}$ in order to reduce the first term in (4.4) to form $p_A dB_A$. We can then obtain H as a function of the canonical variables using the constraints. To begin with we shall consider the simplest case ; that for when $B_{\alpha\beta}$ is diagonal. In this case we use the parametrization

$$B_{\alpha\beta} = B_d = \mathrm{diag}(B_+ + \sqrt{3}B_-\,,\; B_+ - \sqrt{3}B_-,\; -\,2B_+)$$

and

$$6p_{\alpha\beta} = R_0^{-1}\,\mathrm{diag}\,(p_+ + \sqrt{3}p_-\,,\; p_+ - \sqrt{3}p_-,\; -2p_+)$$

In this case the action reduces to

$$I = \int p_+dB_+ + p_-dB_- - Hd\Omega$$

which is subject to constraints (4.2) and (4.3).

From the constraint $C^0 = 0$ we have

$$H^2 = 6p_{\alpha\beta}p^{\alpha\beta} - 24\pi^2 g({}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha) \qquad (5.2)$$

$$= p_+^{\ 2} + p_-^{\ 2} - 24\pi^2 g({}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha)$$

and as in the homogeneous models this equation gives a Hamiltonian corresponding to a particle moving in two dimensions on the B_+, B_- plane where $g({}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha)$ acts as a time-dependent potential. Defining V by the equation

$$g({}^3R + (8a_\alpha - 2\phi_\alpha)\phi^\alpha) = -\frac{3}{2} R_0^{\ 4} e^{-4\Omega} (V-1)$$

and using the expressions

$${}^3R_{\alpha\beta} = -C^{(\delta\nu)}{}_\beta C_{\delta\nu\alpha} + 2a^\delta C_{(\alpha\beta)\delta} + \frac{1}{4} C_\beta{}^{\delta\nu} C_{\alpha\delta\nu}$$

and $$C^\alpha{}_{\beta\delta} = \epsilon_{\beta\delta\nu} n^{\nu\alpha} + \delta^\alpha{}_\delta a_\beta - \delta^\alpha{}_\beta a_\delta \qquad (5.3)$$

we have

$$V = 1 + \frac{1}{3}\Big\{2e^{2B}{}_{\alpha\beta}\, e^{2B}{}_{\nu\delta}\, n^{\alpha\nu} n^{\beta\delta} - (n^{\alpha\beta} e^{2B}{}_{\alpha\beta})^2 + (12a_\alpha a_\beta - 16a_\alpha\phi_\beta$$

$$+ 4\phi_\alpha\phi_\beta) e^{-2B}{}_{\alpha\beta}\Big\} \qquad (5.4)$$

Putting $\phi_\alpha = fa_\alpha$, and using the classification scheme outlined in Table 2(p.28) together with the parametrization given above for diagonal models we have the following expressions for the potential in each of these models.

$$\text{TYPE IV} \qquad V = 1 + \frac{e^{4B_+}}{3}\Big\{12 - 16f + 4f^2 + e^{4\sqrt{3}B_-}\Big\}$$

$$\text{TYPE V} \qquad V = 1 + \frac{e^{4B_+}}{3}\Big\{12 - 16f + 4f^2\Big\} \qquad (5.5)$$

$$\text{TYPE VI}_h \qquad V = 1 + \frac{2e^{4B_+}}{3}\Big\{\cosh(4\sqrt{3}B_-) - h(6 - 8f + 2f^2)\Big\}$$

$$\text{TYPE VII}_h \qquad V = 1 + \frac{2e^{4B_+}}{3}\Big\{\cosh(4\sqrt{3}B_-) + h(6 - 8f + 2f^2)\Big\}$$

When $f = 0$ we recover the potentials appropriate for the vacuum homogeneous universes. Note that some of these expressions differ from the potentials for homogeneous models given in Table 11.1 in Ryan and Shepley [82] because of differences in the classification scheme used by them.

From the space constraint $C^{\alpha} = 0$ we also have

$$g^{\alpha\nu} C^{\delta}_{\nu\beta} \pi^{\beta}{}_{\delta} + 2(\phi_{\beta} - a_{\beta})\pi^{\alpha\beta} - \phi^{\alpha}\pi^{\beta}{}_{\beta} = 0$$

and using equation (5.3) we find

$$\epsilon^{\alpha}{}_{\beta\delta} n^{\delta\mu}\pi^{\beta}{}_{\mu} + (2f - 3)a_{\beta}\pi^{\alpha\beta} + (1 - f)a^{\alpha}\pi^{\beta}{}_{\beta} = 0 \qquad (5.6)$$

where we have put $\phi_{\alpha} = fa_{\alpha}$. For diagonal models we see from (5.1) that the matrix $\pi_{\alpha\beta}$ is diagonal.

In this instance the first term in (5.4) is identically zero since for our Bianchi classification scheme $n^{\alpha\beta}$ is also diagonal. Thus this constraint reduces to the same expression for each Bianchi type :

$$C_1 = C_2 = 0 \qquad (5.7a)$$

and

$$C^3 = -\frac{g^{33}}{3\pi}\left[p_{+}(2f - 3) + fH \right\} = 0 \qquad (5.7b)$$

$$\Rightarrow \quad (2f - 3)p_{+} + fH = 0$$

We see that this constraint contains the Hamiltonian when $f \neq 0$. Indeed, we can use this equation, instead of (5.2) to define H.

The usual methods of exegesis [see Ryan] may now be employed to discover the qualitative behaviour of these models. As the potentials are exponenial, they are replaced by walls in the first approximation. From the previous sections we noted that the ADM method was valid for homogeneous models only when $a_{\alpha} = 0$ (i.e. Class A). Thus extensive work has been done on studying the Bianchi type I and IX models, since they generalize the open and closed Friedmann models with $k = 0$ and $k = 1$ respectively, but little work has been done on Type V models, which mimic the $k = -1$ Friedmann models. Nevertheless, the ADM method is valid for a subset of class D homothetic models and these can be considered as inhomogenous generalizations of the homogeneous class B models. Thus this allows us to study universe models of Bianchi type V which it has been argued, using present observational evidence, give the best representation of the real universe.

Putting $f = 2$ in equation (5.5) we have

$$V = 1 - \frac{4e^{4B_+}}{3} \tag{5.8}$$

and so we see that the potential associated with vacuum type V models falls away exponentially as B_+ increases. This is illustrated in figure 1a. From the constraint (5.7) we have upon putting $f = 2$,

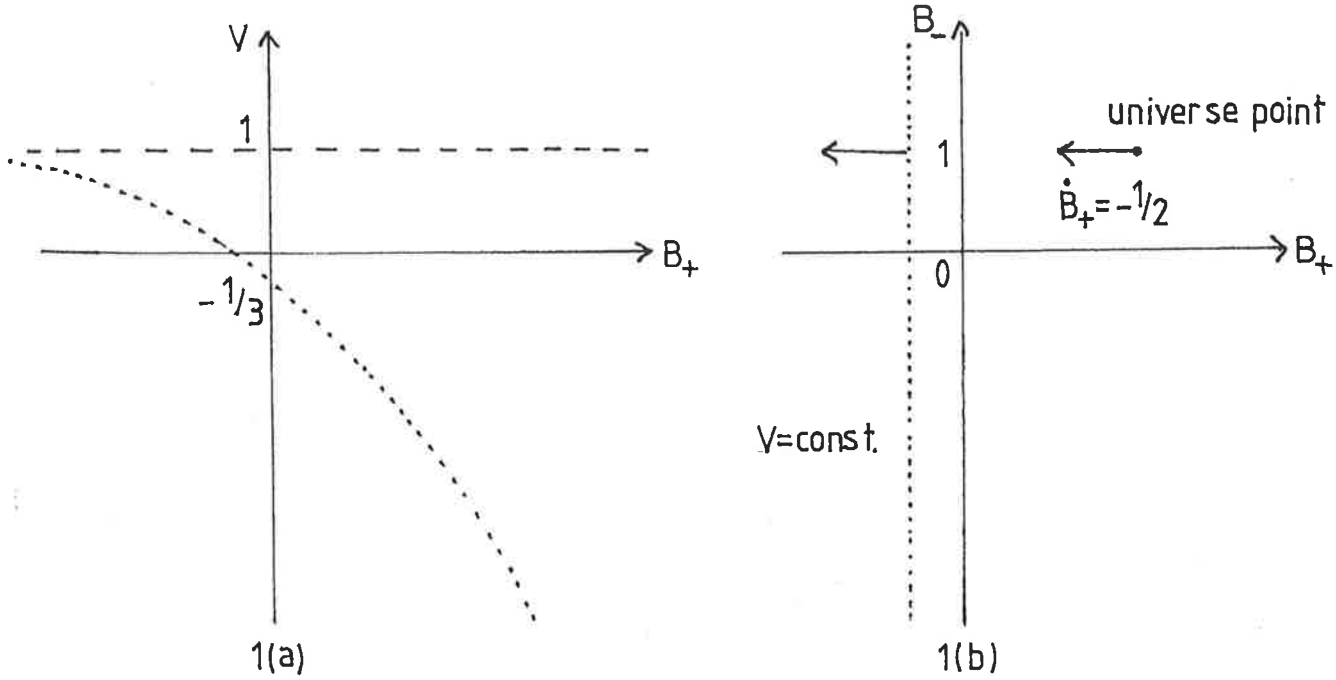


Figure 1 : Potential diagram for vacuum type V model

$$H = -\frac{1}{2} P_+ \tag{5.9}$$

and Hamilton's equations imply that p_+, p_-, B_- and H are constants of the motion. The equation of motion for B_+ is given by $\dot{B}_+ = \partial H/\partial p_+ = -\frac{1}{2}$. Thus we see that the universe point moves with a velocity of one half in the direction of decreasing B_+.

In closed universe models, the position of the wall is usually defined where $\underset{\sim}{p} = 0$ i.e. this gives the position of the particle when a 'bounce' occurs. However, putting $\underset{\sim}{p} = 0$ into equation (5.2) gives H^2 negative. Since this is not allowed we conclude that the universe point never catches up to the potential wall. Thus whereas the potential was zero in type I universe models, and exponentially steep in type IX models where the universe point bounces off the walls, in this model the universe point approaches, but never reaches the potential wall. Indeed, from figure 1b we see that if one moves on the line $\beta_- = 1$, then the universe point approaches a rising potential that flattens out as $B_+ \to -\infty$. This seems to correspond to the deceleration and infinite expansion of the $k = -1$ Friedmann models.

§6.6 MATTER IN THE A.D.M. FORMALISM

In certain universe models (especially those with non-diagonal $g_{\alpha\beta}(t)$) the postulated form of the metric is inconsistent with a vacuum solution. For this reason, and because it is customary to consider non-empty universe models in any case we shall outline the necessary changes in the foregoing formalism when a non-zero energy-momentum tensor is to be included. This follows the work of Ryan [49].

In order to add matter to the Einstein equations, it is necessary to modify the action (2.1) to read

$$I = \int (R\sqrt{-g} + L_m) d^4x$$

where the Lagrangian density for matter L_m satisfies

$$\delta \int L_m d^4x = -8\pi \int T_{\mu\nu} (-g)^{\frac{1}{2}} \delta g^{\mu\nu} d^4x. \tag{6.1}$$

Once such a modified action is obtained it is necessary to break up L_m into terms such as $p_i q_i$ and $NL^0{}_m$ and $N_i L^i{}_m$ (c.f. equation (2.5)). The first of these introduces new independent coordinates and second two quantities modify the constraints which now read

$$C^{0'} = C^0 + L^0{}_m = 0 \tag{6.2a}$$

$$C^{\alpha'} = C^\alpha + L^\alpha{}_m = 0 \tag{6.2b}$$

For homogeneous models of Bianchi class A with a perfect fluid $[T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + pg_{\alpha\beta}]$ and having an equation of state $p = (\nu - 1)\rho$, $1 \leqslant \nu \leqslant 2$, such a Lagrangian has been found by Ryan :

$$L_m = -16\pi\rho(u^0)^{\nu} N^{\nu} R_0{}^3 e^{-3\Omega} \{\nu N(1 + R_0^{-2} e^{2\Omega} e^{-B}{}_{\alpha\beta} u_\alpha u_\beta)^{(1 - \nu/2)}$$

$$- N(\nu - 1)(1 + R_0^{-2}\, e^{2\Omega} e^{-2\beta}{}_{\alpha\beta} u_\alpha u_\beta)^{-\nu/2} \tag{6.3}$$

$$+ N_i{}^{\nu}(1 + R_0^{-2}\, e^{2\Omega} e^{-2\beta}{}_{\alpha\beta} u_\alpha u_\beta)^{\frac{(1-\nu)}{2}} u_\alpha g^{\alpha\beta} \}$$

The density ρ is then eliminated from this expression by solving the conservation equation $u_\alpha T^{\alpha\beta}{}_{;\beta} = 0$.

For homothetic models we have from equations (2.2.15)

$$\hat{\rho} = e^{-2\sigma}\rho(t) \qquad \hat{p} = e^{-2\sigma}p(t) \qquad \hat{u}_\alpha = e^{\sigma}u_\alpha(t) \tag{6.4}$$

and upon substituting into (6.1) and performing a spatial integration, the variational principle for $\hat{L}_m$ becomes

$$\delta \int \hat{L}_m\, d^4x = - \int T_{\alpha\beta}(-{}^4g)\,\delta g^{\alpha\beta} dt.$$

Hence $\hat{L}_m$ satisfies the same equation as L_m and so has the form (6.3). In order to eliminate ρ from this equation we consider $\hat{u}_\alpha \hat{T}^{\alpha\beta}{}_{|\beta} = 0$. Using the equation of state $\hat{p} = (\nu - 1)\hat{\rho}$ this equation becomes

$$(\hat{\rho}^{\,1/\nu}\, \hat{u}^\alpha)_{|\alpha} = 0 \tag{6.5}$$

Now using the identity

$$A^\alpha{}_{|\alpha} = \frac{1}{g^{1/2}}(g^{1/2} A^\alpha)_{,\alpha} + C^\alpha{}_{\alpha\beta} A^\beta$$

together with expressions (6.4), the equation (6.5) becomes

$$(NR_0^3 e^{-3\Omega} \rho^{1/\nu} u^0)_{,0} = - NR_0^3\, e^{-3\Omega} \rho^{1/\nu}\{\phi^\alpha u_\alpha(3 - \frac{2}{\nu}) + u^\beta C^\alpha{}_{\alpha\beta}\} \tag{6.6}$$

In order to carry out the integration we consider

$$\phi^\alpha u_\alpha(3 - \frac{2}{\nu}) + u^\beta C^\alpha{}_{\alpha\beta} = 0$$

Since $C^{\alpha}{}_{\alpha\beta} = -2a_{\beta}$ and putting $\phi^{\alpha} = 2a^{\alpha}$ we have

$$a^{\alpha}u_{\alpha}\left(1 - \frac{1}{\nu}\right) = 0$$

If $a^{\alpha}u_{\alpha} \neq 0$, then $\nu = 1$; otherwise $a^{\alpha}u_{\alpha} = 0$ and since $a_{\alpha} = a\delta^{3}{}_{\alpha}$ we require $u^{3} = 0$. We thus have the result that unless $u^{3} = 0$, the only models allowing equation (6.6) to be integrated are dust models. In either case we find

$$\rho = \mu N^{-\nu}(u^{0})^{-\nu} R_{0}^{-3} e^{3\nu\Omega} \quad (\mu = \text{constant}) \tag{6.7}$$

With this we can complete the Hamiltonian formalism for homothetic models.

Upon substituting (6.7) into (6.3) we notice that the Langrangian has the form $L_{m} = NL^{0}{}_{m} + N_{\alpha}L^{\alpha}{}_{m}$, so the addition of matter in the allowed models leaves (4.4) unchanged (i.e. no new independent coordinates) and the constraints (6.2) now give

$$H^{2} = H^{2}{}_{vac} - 24\pi^{2}g^{1/2}L^{0}{}_{m} \tag{6.8}$$

and

$$C^{\alpha} = -L^{\alpha}{}_{m} \tag{6.9}$$

where

$$L^{0}{}_{m} = -16\pi\mu R_{0}{}^{3(1-\nu)}e^{3(\nu-1)\Omega}\Big\{\nu(1 + R_{0}{}^{-2}e^{2\Omega}e^{-2B}{}_{\alpha\beta}u_{\alpha}u_{\beta})^{(1-\frac{\nu}{2})}$$

$$-(\nu - 1)(1 + R_{0}{}^{-2}e^{2\Omega}e^{-2B}{}_{\alpha\beta}u_{\alpha}u_{\beta})^{-\frac{\nu}{2}}\Big\}.$$

$$L^{\alpha}{}_{m} = -16\pi\mu R_{0}{}^{3(1-\nu)}e^{3(\nu-1)\Omega}\nu(1 + R_{0}{}^{-2}e^{2\Omega}e^{-2B}{}_{\alpha\beta}u_{\alpha}u_{\beta})^{\frac{(1-\nu)}{2}}u^{\alpha}$$

In these equations u^{α} are the space components of the fluid velocity and are solved for as functions of Ω by use of the auxiliary geodesic equations.

For diagonal universe models considered in the previous section we found $C^1 = C^2 = 0$. Hence from equation (6.9) we find $u^1 = u^2 = 0$ in these models. Using a previous result that non-tilted homothetic models admit a perfect fluid only when it has a hard equation of state $(\nu = 2)$, we conclude that the above formalism is only valid for the following two cases.

<u>Case 1</u> Dust Models with tilt $(u_1 = u_2 = 0,\ u_3 \neq 0)$, for which

$$L^0{}_m = -16\pi\mu\ (1 + R_0^{-2} e^{2\Omega} e^{-2B_{33}} u_3 u_3)^{1/2}$$

$$L^1{}_m = L^2{}_m = 0 \qquad L^3{}_m = -16\pi\mu u^3$$

<u>Case 2</u> Non-tilted models $(u^\alpha = 0)$ with hard equation of state, for which

$$L^0{}_m = -16\pi\mu R_0^{-3} e^{3\Omega} \qquad L^\alpha{}_m = 0$$

To illustrate the behaviour of the universe point in this latter case, we consider the type model of the previous section. Thus for a non-tilted type V model with a hard equation of state the constraint equations give

$$H^2 = p_+^2 + p_-^2 - 48\pi^2 R_0^4 e^{-4\Omega + 4B_+} + 384\pi^3\mu$$

$$H = -\frac{1}{2} p_+ .$$

It is easy to see that the only change from the vacuum case is the addition of the constant $384\mu\pi^3$ to the first expression. As this is equivalent to the addition of this constant to the potential, the dynamics of the motion is not changed. From (6.7) the density in this model is given by $\rho = \mu R_0^{-6} e^{6\Omega}$. Using equation (4.6) together with $dt = -N d\Omega$ and the fact that H is a constant we find $t = (4\pi R_0^3/H)e^{-3\Omega}$. We thus have

$$\rho = \frac{16\pi^2\mu}{H^2 t^2}$$

and we see that there is a singularity at $t = 0$.

§6.7 FURTHER MODELS

In the above examples the dynamics of the models have been constrained by the space constraint $C^3 = L^3_m$ which has given $H = -\frac{1}{2}p_+$. In order to achieve more general motion we can consider either the tilted dust models mentioned above or non-diagonal models.

For dust models, the non-zero space constraint gives

$$H = -\frac{1}{2}p_+ - 24\pi^2\mu u_3$$

To find expressions for u^3 (and u^0) we need to consider the geodesic equation $\hat{u}_{\alpha|\beta}\hat{u}^{\beta} = 0$. When $u_1 = u_2 = 0$ we find

$$u_0 = -N\cosh f(\Omega)$$

$$u_3 = R_0 e^{-\Omega} e^{-2B_+} \sinh f(\Omega)$$

where $f(\Omega)$ satisfies

$$\frac{df(\Omega)}{d\Omega} = -N\left\{g_{33}^{-1/2}b + \frac{1}{2Ng_{33}}\frac{dg_{33}}{d\Omega}\tanh f(\Omega)\right\}$$

However, this last equation is difficult to solve when H is not constant. We can expect this difficulty in non-diagonal dust models as well so one is left to use numerical methods of analysis.

For non-tilted non-diagonal models we can parametrize the matrix $B_{\alpha\beta}$ so that there are five independent coordinates. In the so called 'symmetric' case one introduces the off-diagonal coordinate ϕ and the corresponding momentum coordinate p_ϕ [see Ryan and Shepley [81]]. The corresponding constraint (6.8) now gives

$$H^2 = p_+^2 + p_-^2 + \frac{3(P\phi)^2}{\sinh(2\sqrt{3}\beta-)} - 48\pi^2 R_0^4 e^{-4\Omega + 4B_+} + 384\pi^3\mu.$$

The new term in the Hamiltonian, being proportional to $(P\phi)^2$ is called the centrifugal potential V_c, because it is the analogue of the centrifugal potential in the Kepler problem of Newtonian mechanics [49]. We see that this term, and thus the Hamiltonian is singular at $\beta_- = 0$. According to Eardley [18] this locus of points is associated with a Cauchy horizon, and arises due to the assumption that the hypersurfaces $S(z)$ of transitivity of the homothetic group H_3 were spacelike. However, Eardley argues that this is not a necessary restriction on the global causal structure of the $S(z)$, and that some of the $S(z)$ may be timelike. Further, if one assumes analyticity, it should be possible to extend these models through the Cauchy horizon (as it is possible for certain homogeneous cosmologies).

To help examine the causal and singularity structure in an inhomogeneous class of universe models, in the next chapter we shall briefly investigate such behaviour in the special case of spherically symmetric self-similar space-times.

CHAPTER 7

SPHERICALLY SYMMETRIC SELF-SIMILAR MODELS

§ 7.1 SELF SIMILARITY AND SPHERICAL SYMMETRY

In this chapter we shall consider another class of space-times which possess both the properties of self-similarity and spherical symmetry.

A space-time is spherically symmetric if it admits the orthogonal group $O(3,R)$ as a symmetry group of two dimensional space-like trajectories. Further, spherically symmetric self-similarity solutions of Einstein's field equations have been defined by Cahill and Taub [19] as those for which under the transformation

$$\bar{t} = at, \; \bar{r} = ar \qquad \bar{\theta} = \theta, \; \bar{\phi} = \phi \tag{7.1}$$

where a is a constant

$$\bar{g}_{\mu\nu}(\bar{r},\bar{t}) = g_{\sigma\tau} \frac{\partial x^{\sigma}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\nu}} = a^{-2} g_{\mu\nu}(\bar{r},\bar{t})$$

Such a solution thus gives a space-time which admits the transformation (7.1) as a homothetic transformation. The requirement that the barred coordinate system be comoving is also made.

These two requirements may be given a more general and invariant formulation. We shall define a similarity solution of the field equations as one for which the resulting space-time admits the conformal Killing vector field ξ^{μ} satisfying

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 2\phi(r,t) g_{\mu\nu} \tag{7.2}$$

where $\phi(r,t)$ is an arbitrary function of r and t.
The transformation (7.1) is thus a special case of the conditions (7.2) and (7.3) where $\phi(r,t) = \text{constant}$. As a consequence of this condition, the four-velocity vector u^{μ} satisfies

$$u^{\mu}{}_{;\nu}\xi^{\nu} - \xi^{\mu}{}_{;\nu}u^{\nu} = -\phi(r,t)u^{\mu} \tag{7.3}$$

For the spherically symmetric models to be considered here, we have

$$\xi^{\mu} = \alpha(r,t)\delta^{\mu}_{r} + \beta(r,t)\delta^{\mu}_{t} \tag{7.4}$$

Physically, Barenblatt and Zel'dovich [59] have shown that spherically-symmetric space-times (with $\Lambda = 0$) contain no fundamental scales, dimensional constraints or (at least for cosmologies) dimensional boundary conditions, and so they admit no preferred scale in space or time.

§7.2 THE MODEL

We consider spherically-symmetric Tolman-Bondi models having the metric of the form

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + Y^2(r,t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{7.5}$$

where $-\infty < t < \infty$, r is a comoving radial coordinate and θ,ϕ are the ususal spherical coordinates. The circumference of an azimuthal circle in the model is $2\pi Y$.

The central worldline, denoted by C, is at $r = 0$. Because one wants this to be a regular centre of the space-time, one requires

$$Y(r,t) \to 0, \qquad \frac{1}{X(r,t)}\frac{\partial Y}{\partial r}(r,t) \to 1 \quad \text{as } r \to 0 \tag{7.6}$$

The space-time is then spherically symmetric about the world-line C.

In our coordinates, the velocity vector takes the form

$$u^{\nu} = (1,0,0,0)$$

and we take the energy-momentum tensor to be that of a perfect fluid.

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}$$

However, it can be shown that for the metric (7.5), the pressure can have no radial dependence i.e. $p = p(t)$ only [26]. The energy density can still have a radial and time dependence, and so in general no equation of state of the form $p = p(\rho)$ can be imposed.

Since this dependence of the pressure on time only seems rather unrealistic, we shall consider a pressureless dust solution. This should closely approximate the present universe, but not necessarily the early universe. In this case, the Einstein equations for the metric (7.5), with $\Lambda = 0$, have been solved exactly. Following Szekeres [24], the simplest case is where the dust particles are marginally bound (the space-section t = constant are flat) and the metric has the form

$$ds^2 = -dt^2 + \frac{(t - t_1(r))^2}{(t - t_0(r))^{2/3}} dr^2 + r^2(t - t_0(r))^{4/3} d\Omega^2 \qquad (7.7a)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

$$t_1(r) = t_0(r) + \frac{2r}{3}\frac{dt_0}{dr}(r) \qquad (7.7b)$$

and $t_0(r)$ is an arbitrary function. Further

$$\rho = \frac{4}{3(t - t_0(r))(t - t_1(r))} \qquad (7.8)$$

The case t_0 = constant implies $t_0 = t_1$ by (7.7b) and the solution reduces to the Einstein-de Sitter model.

For this model to admit the conformal Killing vector (7.4) then equations (7.2) and (7.3) must be satisfied. The first of these reduce to [19]

$$\frac{\alpha}{Y}\frac{\partial r}{\partial r} + \frac{\beta}{Y}\frac{\partial Y}{\partial t} = \phi \qquad (7.9a)$$

$$\frac{\alpha}{X}\frac{\partial X}{\partial r} + \frac{\beta}{X}\frac{\partial X}{\partial t} + \frac{\partial \alpha}{\partial r} = \phi \qquad (7.9b)$$

$$\frac{\partial \beta}{\partial t} = \phi \qquad (7.9c)$$

$$\frac{-X\partial\alpha}{\partial t} + \frac{\partial \beta}{\partial r} = 0 \qquad (7.9d)$$

Equations (7.3) reduce to two equations, one being the third in the above set, the other being

$$\frac{\partial \alpha}{\partial t} = 0 \qquad (7.9e)$$

From (7.9d) we then have $\partial\beta/\partial r = 0$ ie $\beta = \beta(t)$ and from (7.9e) $\alpha = \alpha(r)$. Equation (7.9c) then requires $\phi = \phi(t)$ only, and from (7.9b) we see that, in fact, $\partial\alpha/\partial r$ and ϕ must be constant. Hence, the metric (7.7) only allows self-similarity transformations which are homothetic i.e. of the form (7.1)

From the remaining equations (7.9a,b) it is readily found that $\alpha = r/3$, $\beta = t$ and $t_0(r) = ar^3$, giving from (7.7b) $t_1(r) = 3ar^3$. We thus have the spherically symmetric self-similar space-time with the metric

$$ds^2 = -dt^2 + \frac{(t-3ar^3)^2}{(t-ar^3)^{2/3}}\, dr^2 - r^2(t - ar^3)^{4/3} d\Omega^2 \qquad (7.10)$$

admitting the homothetic Killing vector

$$\underset{\sim}{\xi} = \frac{1}{3} r\partial t + t\partial r$$

and having the energy density

$$\rho = \frac{4}{3(t-ar^3)(t-3ar^3)} \qquad (7.11)$$

This solution can be shown to be identical to that found by Henrikson and Wesson [60] by use of the coordinate transformation

$$r = R^{1/3} \qquad t = \frac{3T}{2} \qquad a = \pm\frac{3\alpha}{2} s$$

This model represents an inhomogeneous, spherically symmetric expanding space-time. This solution has also been studied by Dyer [83].

§7.3 PHILOSOPHICAL CONSIDERATIONS

Although mathematical models possessing spherical symmetry have occassionally been investigated [see Omer [11], Bonn r [61], and Wesson [62] for example], they are usually not taken seriously because it is believed to be unreasonable that we should be near the centre of the universe As we have seen, the isotropy of the microwave background radiation is usually used as evidence for the isoptrophy of the universe about the observer. Thus by use of the location principle (i.e. the belief that we don't occupy a privileged position in either space or time) one is lead to infer wide-spread homogeneity.

The limited attention given to spherically symmetric universes can be traced back to Einstein who tried to show that in contrast to

Newton's reasoning, there could be no centre of the universe. However Ellis, Maartens and Nel [10] argue that while it is certainly unreasonable to imply that the universe has been centred on our presence, there is no need for this attitude. Instead, they ask : given a universe of this type, where is one likely to find life as we know it? The situation would then not be that the universe had been created in an anthrocentric way, but rather that, the universe being in existence, our life would have evolved in the most probable region for life. This is the spatial analogue of Carter's statement that life only occurs at favourable times in the history of the universe [63]. For the static spherically symmetric universe model considered by Ellis et al, this principle was found to be satisfied by siting our Galaxy near the cool centre ; this being surrounded by a hot singularity. Indeed, in separate paper Ellis [4] has stated that it would certainly be consistent with the present observations if we were near the centre of the universe, and that, for example, radio sources were distributed spherically symmetrically about us in shells characterised by increasing source density and brightness as their distance from us increases. Varshni [14] argues that in fact the distribution of quasars implies such a case.

The investigations by Ellis et al suggested that while exactly static inhomogeneous models may not be viable, certain interesting features of such models may remain in expanding inhomogeneous models ; in particular the singularity structure in such models is completely different from that in a FRW model. More recently, Wesson [62,20] has studied a particular spherically symmetric self-similar model in the hope of resolving some of the long-standing problems encountered in Friedmann cosmologies. For example, by allowing $t \to \infty$ in (7.10), the metric becomes

$$ds^2 = -dt^2 + t^{4/3}(dr^2 + r^2 d\Omega^2)$$

Thus the model evolves into the homogeneous Einstein-de Sitter model implying that the present isotropy and homogeneity can be the product of evolution from conditions different from those exceedingly special ones required for FRW models.

§7.4 THE SINGULARITY

As mentioned above, the singularity structure of the static models studied by Ellis et al [10] displayed a behaviour very different from that of the Friedmann models. From equation (7.11) we see that the density has in general two singularities ; at $t = t_0 = ar^3$ and $t = t_1 = 3ar^3$. However, we define our time coordinate such that $t > \mathrm{Max}(t_0, t_1)$. Thus only one of the singularities needs be considered.

We see immediately, that in contrast to the FRW models, the spatial inhomogeneity of these models opens up the possibility that the big-bang does not, in any particularly natural sense, go off all at once. This possibility has been mentioned previously in connection with various astrophysical phenomena [see Miller [64] and references cited within]. Further, the behaviour of this singularity varies as to whether the arbitrary constant, a, in the above model is positive or negative.

Case I : $a < 0$

In this case the hypersurface $\Sigma(r) = \mathrm{Max}(t_0, t_1) = ar^3$ represents the 'big-bang' singularity, on which any co-moving observer's world-lines originate. Since such an observer with coordinate r emerges from the singularity at time $t = ar^3$, then the inequality $dt_0(r)/dr = 3ar^2 < 0$ implies that comoving observers with larger values of r enter the universe at earlier times. Further, since $t_0(0) = 0$, for all observers with $r > 0$, the big-bang time is negative [in the coordinate t used here]. The big-bang therefore acts like an implosion, the behaviour of which is illustrated in figure 2.

Backward radial null geodesics satisfy

$$\frac{dt}{dr} = -\frac{(t - t_1(r))}{(t - t_0(r))^{1/3}} = -\frac{(t - 3ar^3)}{(t - ar^3)^{1/3}} \qquad (7.12)$$

and thus except near the the big-bang hypersurface, $t = ar^3$, don't differ much from the curve $r = -t$. Those null geodesics arising at $t = ar^3$ are no longer initially horizontal as in FRW models, but are initially vertical and so the 'big-bang' hypersurface is inaccessible to future-directed

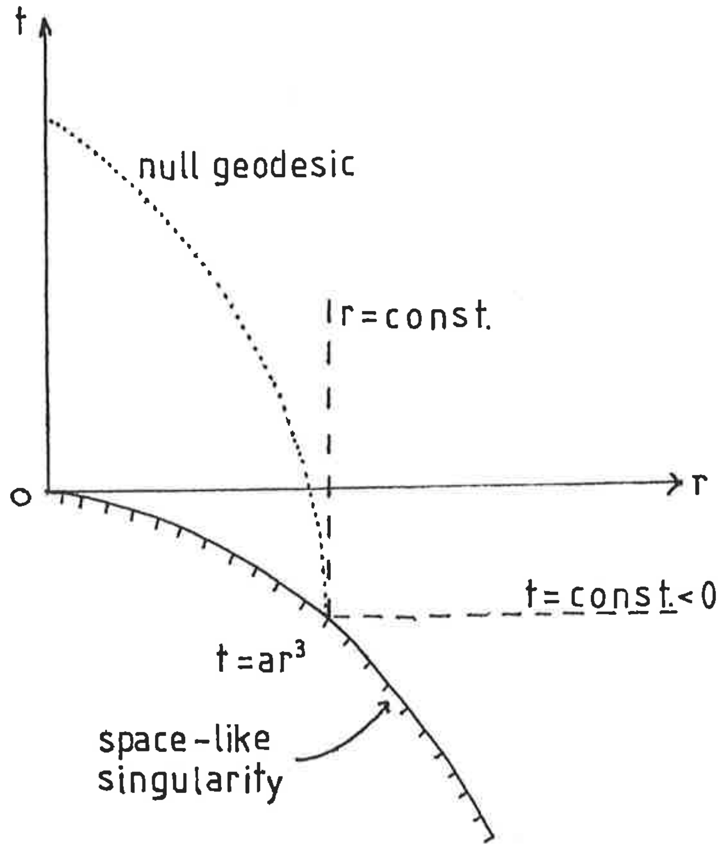


Figure 2. Singularity behaviour for the case $a < 0$.

causal curves. Such a singularity is called past space-like meaning that it cannot be influenced by observers within the spacetime - it can only be observed [65].

In this model the density on any hypersurface t = constant increases as one moves radially inwards. For $t > 0$, the central line C is characterised by a finite maximum for the density. However, for $t < 0$, as one moves radially inwards, eventually the singularity is reached at $r = (t/a)^{1/3}$ for which $\rho \to \infty$. This singularity disappears from the space-time at time $t = 0$, after which the observer at $r = 0$ rests in a thoroughly well-behaved region of space-time. However, in a universe of infinite extent $t_0(r) \to -\infty$ as $R \to \infty$ and so it is reasonable to ask whether this point in time is ever reached? Further, in his investigations of such spacelike hypersurfaces that contain 'lagging cores' of the big-bang, Miller [64] has shown that the field equations allow the masses of these lagging cores to become negative - more generally, that they allow the spacelike singularities to 'evolve into' timelike ones. Such behaviour is normally not allowed in any realistic spacetime model since it disobeys the strong form of the cosmic censorship hypothesis which states that no timelike singularities - whether primeval or having formed from initially non-singular circumstances are present in space-time [65].

If one assumes that the time $t = 0$ is never reached in these models, then one finds that the 'big-bang' is still going off. In this case the future of any spacelike hypersurface t = constant is only partially determined, at least from the point of view of observers on this hypersurface. However in such models the position of our Galaxy must be at a radius $r \gg (t/a)^{1/3}$ [given the low density and temperature in our neighbourhood] and this will have implications for the isotrophy of any observations made at such a position. This point will be considered in the next section.

Finally, one can avoid many of the difficulties in the singularity behaviour just described by simply requiring $t > 0$ in such a model. Then we have an initial big-bang at $r = 0$, as in the FRW models.

Case II : $a > 0$

In this case, the hypersurface $\Sigma(r) = \mathrm{Max}(t_0, t_1) = 3ar^3$ represents the big-bang singularity. As before $t_1(0) = 0$, so the comoving observer at $r = 0$ leaves the singularity at $t = 0$, but now, since $dt_1(r)/dr = 9ar^2 > 0$, for all other observers with $r > 0$, the big-bang time is positive. Thus one can now speak of an initial beginning to the universe, with comoving observers having larger values of r entering the universe at later times. Hence one can speak of an 'expanding-shell' of the big-bang in contrast to the 'lagging-core' just described. The behaviour of this big-bang is illustrated in figure 3.

Backwards radial null geodesics again satisfy equation (7.12) and, as in the FRW models, are initially horizontal at the big-bang hypersurface $t_1 = 3ar^3$. However, unlike the FRW models where only null and past timelike geodesics intersect the singularity, in this model all space-like geodesics intersect it as well. Thus the singularity both surrounds the central line C and bounds it past. In the former aspect, this model represents the expanding counterpart of the static spherically symmetric models considered by Ellis et al [10] , which like the model here is spatial finite and bound. This can be seen from the fact that at

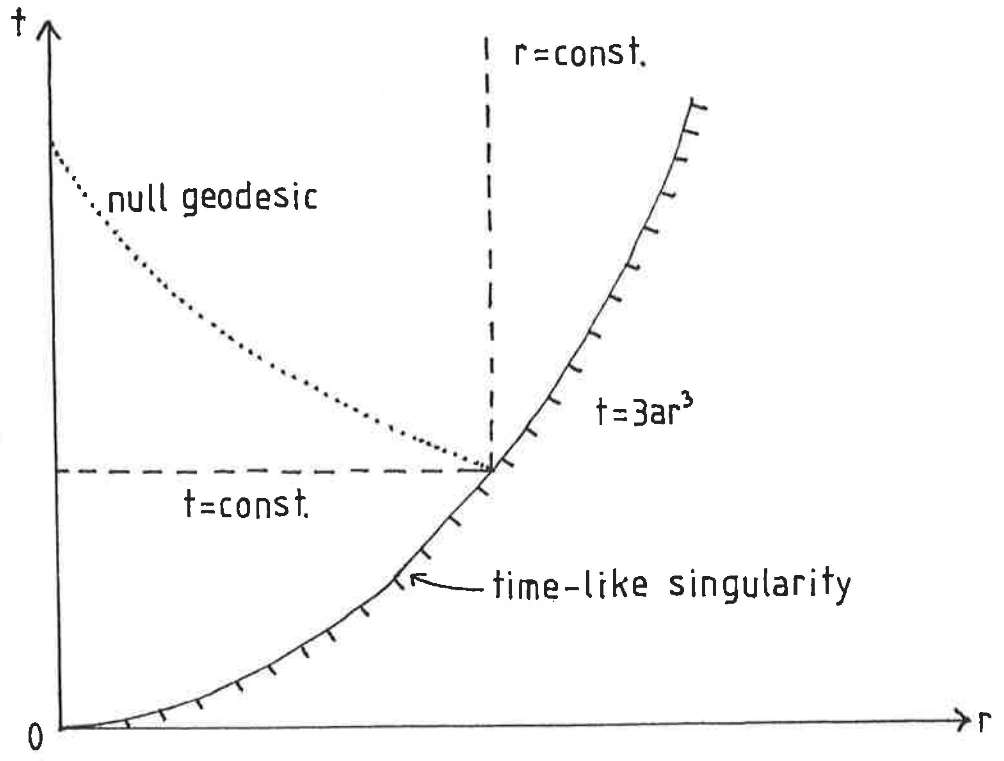


Figure 3 : Singularity behaviour for the case $a > 0$

any time t, the proper distance from C to the singularity is

$$\int_0^{R_s} X(r,t)dr = Y(R_s,t) = R_s(t - aR_s)^{2/3}$$

which is finite for all finite t.

The singularity in these space-times, as in the previous case where $a < 0$ and $t < 0$, can be considered as 'sitting over there'. This is unlike the FRW models where the singularity is hidden away inaccessibily in the past. Further, like the previous case again, we see that there are no global Cauchy surfaces and thus the singularity can influence the universe continually. This continuous interaction, according to Ellis et al, might be envisaged as a process which keeps the universe running i.e. one would in these models 'have the 'thermal history' of the universe taking place in a spatial rather than a time direction, with element formation taking place continuously in the hot fire-ball, pair production taking place continually, and soon'. However, unlike the previous case, those models where $a > 0$ possess a timelike singularity. Thus a radial null geodesic emitted from the singularity, goes through C and is eventually re-absorbed by the singularity. Hence the singularity is in effect both a source and a sink of information (and possibly matter) for the space-time.

As we have seen, this possibility of the singularity being influenced by the universe itself violates the cosmic censorship hypothesis.

7.5 REDSHIFTS

While the singularity structures of the spherically symmetric self-similar spacetime with metric (7.10) have some unfamiliar features, one must consider whether these models can adequately incorporate the observational relations supplied by astronomers. Of, these, perhaps the most important is the redshift - distance relation. In the static model considered by Ellis et al, while many of the features of a FRW model could be adquately described they were unable to fit the current (m,z) observations to their model.

Following Bondi [12], the redshift z of a source at $r = r_e$ as measured by an observer at $r = r_0$ is given by

$$\log(1 + z) = \int_{r_0}^{r_e} \left(\frac{\partial X}{\partial t}\right)_{r,T(r)} dr \qquad (7.13)$$

where $T(r)$ is the equation of the ray of light travelling radially inwards. From (7.13) we can write

$$1 + Z = \frac{f(r_e)}{f(r_0)} \quad \text{where} \quad f(r) = \exp\left\{\int\left(\frac{\partial X}{\partial t}\right)_{r,T(r)} dr\right\} \qquad (7.14)$$

Because of the spherical symmetry of the metric and the matter distribution, together with the centrality conditions, cosmological observations made at C will be exactly isotropic. By continuity, observations made by an observer near C will be nearly isotropic, except for small redshifts, where the proper motion of the objects will have an appreciable effect. Evidence for such anisotrophy in the Hubble parameter has been put forward by Fennelly [66]. While this result is difficult to produce in a FRW model, Fennelly shows that in the context of our expanding spherically symmetric model one can reproduce the desired angular gradient of H by placing our galaxy 132 Mpc from the centre of such models.

While analytic expressions for the redshift - distance relation (7.14) cannot be found, we can make the following comments. From (7.13) we have

$$\frac{dz}{dr} = \pm (1 + z)\left(\frac{\partial x}{\partial t}\right)_{r,T(r)} = \pm (1 + z) \left.\frac{2t}{3(t - ar^3)^{4/3}}\right|_{r,T(r)}$$

where the positive sign signifies outwards travelling radial null geodesics and the negative sign those travelling inwards. For the case $a > 0$, we find this equation results in a finite limiting blueshhift at the singularity. This result is inconsistent if one accepts the current belief that the background radiation has cooled from an infinite temperature to its current temperature.

When $a < 0$, we found that a singularity existed in these models when $t \leqslant 0$. In this instance, we find $dz/dr \to \infty$ on the singularity, thus giving the required infinite redshifting for the reason just mentioned. Further the numerical size of z is larger for sources seen in the direction of C (i.e. towards the singularity) than for those seen in the opposite direction, so there is a kind of redshift pile-up. This anisotrophy may be unecessarily large to accomodate present observations. The model with history $t > 0$ has been investigated by Wesson [20] in some detail. In this case there is a maximum observable radial redshift which is proportional to $(t_0/ar_0^3)^{2/3}$. Data on $m(z)$ and $n(z)$ indicate a value for this expression of about 50.

§7.6 REMARKS

The above model throws up various points of interest in relation to inhomogeneous cosmological models. First we have seen that there exist singularity structures in expanding but inhomogeneous models which are completely different in nature from those in the FRW universes, but which can give similar observational predictions. Thus FRW models may be quite restrictive in requiring that the 'big-bang' goes off simultaneously in the past of each matter world line.

Secondly, any off-centre observer, while in general measuring anisotropic galactic redshifts, will still observe isotropic background blackbody radiation. This follows from equation (7.14) since the temperature of the background radiation (emitted at temperature T_e at coordinate value r_e) measured by an observer at r will be given by the expression

$$T(r) = T_e(1 + Z)^{-1} = T_e f(r)/f(r_e)$$

and is thus independent of the direction of observation. This result is in line with the conjecture put forward by Ellis et. al [10] for such a model. As we have discussed, it is the isotropy of this background radiation, together with the location principle, which usually leads one to infer global homogeneity. Thus this model presents a challenge to the FRW models which are usually preferred on philosophical grounds rather than observational from other models which can give quite a reasonable picture of the universe. Thus the relative merits of models such as that outlined here should perhaps be more seriously explored before discarding them. We need to at least assess the assumption of homogeneity more fully relative to some of the alternatives. Indeed, recent observations of a quadrupole moment in the background radiation hint at such a revision of this assumption [67].

While spherically symmetric models present an alternative to FRW universe models, like these latter models they are based on special initial conditions, and so together these models are very implausible within the set of all possible universe models. There is thus an on-going need to study more general inhomogeneous models if the above task is to be more fully carried out.

CHAPTER 8 CONCLUSIONS

§8.1 THE STUDY OF INHOMOGENEOUS MODELS.

The basis upon which much of modern theoretical cosmology is founded is the spatially homogeneous and isotropic FRW models. While it is not believed that they truely represent the universe (they arè too simplistic to do that), it is believed that these models ,in some sense, are good global approximations of the present universe - indeed, it is often claimed that the isotropy and homogeneity of the universe can be partially justified observationally. However, as this thesis has attempted to show, the much used assumption of homogeneity should be regarded merely as a working hypothesis, suggested by the state of these present observations. Thus, it would be subject to modification or even dropped if more powerful telescopes (such as the proposed space telescope to be launched later this decade) should reveal a systematic lack of uniformity in different parts of the universe. Indeed, recent observations of galaxies with large red-shifts have shown that there are large regions of the universe (of the order of 10^6 Mpc) practically devoid of galaxies [68]. This evidence tends to support the line of reasoning put forward by de Vaucouleurs [69] who has pointed out that over the last three centuries we have repeatedly discovered ever larger inhomogeneities in the distribution of matter : stars, stellar clusters, galaxies, groups of galaxies, clusters of groups, clusters of clusters.

Following up this claim Oldershaw [70] has more recently argued that there is still no unambiguous evidence for cosmological homogeneity. In support of his case he presents evidence based on recent observations of the distribution of faint galaxies, in the distribution of radio sources, in the Hubble expansion and in the isotropy of the background radiations. Summing up his evidence he writes,

> 'Homogeneity on cosmological scales is most certainly not a fact ; it is still a reasonable approximation, but several lines of evidence gathered over the last decade now suggest that inhomogeneities may persist from the smallest to the largest observational scales'.

Motivated by this line of thinking, this study was an attempt to gain some understanding of kinematical and dynamical effects of inhomogeneities by carrying out an analysis on some of the more simple models - in this case those models which possess the character of self-similarity. First we examined solutions of Einstein's field equations for a perfect fluid which admit a three-parameter group of conformal motions simply transitive on the spatial sections. Unfortunately, the perfect fluid models found generally had an unrealistic equation of state. Secondly we examined spherically symmetric self-similar solutions for dust. While these models presented some difficulties in adequately modelling the present universe the singularity structures contained by them represented a dramatic departure from those in the standard FRW models. From these studies, some ideas as to the nature of the problems to be confronted by future studies of more general inhomogeneous models can be inferred.

Firstly, the condition that conformal models admit a perfect fluid has led to many restrictions. Also, these models have a tendency to be tilted. Hence, it is felt that if one is to obtain realistic inhomogeneous models then a more general energy - momentum tensor is required. This would, by necessity, include terms representing viscosity, taking into account dissipative processes. Thus, together with the non-zero acceleration and rotation likely to be met with in more general models, some explanation as to the origin and nature of the inhomogeneities currently observed in the universe might more readily be found.

Secondly, the singularities occurring in inhomogeneous models may be completely different from those occurring in homogeneous models. As we have seen, these models present one with the situation of an 'on-going' singularity which may act as a continuous creation of matter in the universe. Indeed, it has been suggested by Ne'eman [71] and Novikov [72] that when we observe quasars, we are actually observing matter which has only recently emerged from a 'lagging core' of the big-bang. Further, the particle horizons which limit communications in the standard models could be modified or even non-existent, and so the usual belief in the consequences of the existence of these horizons - together with timelike singularities - may need revision.

§8.2 OTHER MODELS

In order to be able to study the problems just mentioned, suitable inhomogeneous models will be required. In the past, cosmology has generally proceeded by a suitable symmetry being imposed upon Einstein's field equations. However, usually this is very restrictive. For example, the requirement that the space-time metric admits a conformal motion leads to a restricted class of Bianchi - type models. Thus the nature of imposing symmetries on the field equations in order to find suitable models needs to be examined. On this matter, Collins and Szafron [6] have recently suggested imposing restrictions on certain submanifolds instead of placing conditions on the full space-time manifold.

More ideally we should discard metrics of the form where spatial and temporal parts are separated since it is impossible to observe at any time a complete spatial section in such models. As an alternative we should write the metric down in terms of the light cone structure and then according to our observations place conditions on a backward null cone. This leads to mathematical difficulties however.

Another class of model that has received little attention are the hierarchical models proposed by de Vaucouleurs [69] and based upon an idea originally proposed by Charlier in 1908 [73] . It is a commonplace observation that nature loves hierarchies. Most of the complex systems that occurr in nature find their place in one or more of four intertwined hierarchial sequences. For example, analysis of chemical substances discloses sets of component molecules, within which are found atoms, then nuclei and electrons and finally (?) elementary particles. Further our experiences with many different types of complex systems, both natural and artificial, indicate that as systems grow in size and complexity, they reach a limit where a new level of hierarchical control is necessary if the system is to be efficient and reliable. As a result, hierarchies evolve much more rapidly from elementary constituents than non-hierarchic systems, containing the same number of elements [74]. Hence, almost all the very large systems we observe in nature have a hierarchic organisation. There are thus heuristic grounds for suspecting that the global design of nature might also involve such organization [75]. Indeed, Wesson claims that recent observations of global inhomogeneity are in fact quite close to that predicted by de Vaucouleur's hierarchical paradigm [20].

Finally, recent insights by Prigogine [76] into irreversible thermodynamic processes have lead to the development of a theory of natural self-organisation to explain the processes leading to the formation of structure in the universe. Described as 'order through fluctuations', this theory is concerned with systems that are initially in a state of randomness or homogeneity and affected by fluctuations. However, rather than being controlled or damped, as fluctuations tend to be in stable systems, they are 'amplified', and it is this amplification that gives rise to what are called 'dissipative structures'. This theory has been recently used by Zimmerman [77] to help describe the creation of structure in the lepton era of the early universe.

While many other approaches to studying inhomogeneous models of the universe have been proposed [see a review by Mac Callum [78]], general, the study of these models will involve us in global questions as to the nature of the universe as against the situation in homogeneous models where any part is representative of the whole.

§ 8.3 WHAT THE FUTURE HOLDS

As mentioned at the beginning of this thesis, there are various problems associated with the FRW models. Recent developments, however, with grand unified field theories may suggest avenues for the retention of the standard model. Thus the possibility of a phase transition occurring at about 10^{-35}S after the big bang could generate density fluctuations which, in turn, might give rise to the observed inhomogeneities on galactic and cosmic scales. Many of these new ideas have very recently been reviewed by Linde [79].

This aside, the dilemma that faces cosmologists is one quite familiar to those with an appreciation of the history of science [see Kuhn [80]]. On the one hand, we have a well-established paradigm (the standard model) which has served as an able guide to a generation of researchers and through its merits has gained widespread acceptance. On the other hand, there is a growing recognition of the fact that the major observational evidence that once provided the empirical foundation for this paradigm, is now providing insights into its inevitable limitations. The outcome of this present situation is eagerly awaited.

APPENDIX A

BIANCHI TYPES I - IX : VECTORS AND FORMS

We list here the canonical form of the (conformal) Killing vectors and the invariant basis for each Bianchi type [38].

Class A and Class C

Types I and ${}_1$I : $C^A{}_{BC} = 0$

$$\xi_1 = \partial_1 \qquad \omega^1 = dx^1$$
$$\xi_2 = \partial_2 \qquad \omega^2 = dx^2$$
$$\xi_3 = \partial_3 \qquad \omega^3 = dx^3$$

Types II and ${}_1$II : $C^1{}_{23} = -\,C^1{}_{32} = 1$, rest zero

$$\xi_1 = \partial_2 \qquad \omega^1 = x^3dx^1 + dx^2$$
$$\xi_2 = \partial_3 \qquad \omega^2 = dx^3$$
$$\xi_3 = -\,\partial_1 + x^3\partial_2 \qquad \omega^3 = -\,dx^1$$

Types VI$_o$ and ${}_f$VI$_o$: $C^1{}_{23} = -\,C^1{}_{32} = 1,\ C^2{}_{13} = -\,C^2{}_{31} = 1$

$$\xi_1 = \partial_2 \qquad \omega^1 = x^3dx^1 + dx^2$$
$$\xi_2 = \partial_3 \qquad \omega^2 = x^2dx^1 + dx^3$$
$$\xi_3 = -\,\partial_1 + x^3\partial_2 + x^2\partial_3 \qquad \omega^3 = -\,dx^1$$

Types VII$_o$ and ${}_f$VII$_o$: $C^1{}_{23} = -\,C^1{}_{32} = 1,\ C^2{}_{13} = -\,C^2{}_{31} = -1$

$$\xi_1 = \partial_2 \qquad \omega^1 = x^3dx^1 + dx^2$$
$$\xi_2 = \partial_3 \qquad \omega^2 = -\,x^2dx^1 + dx^3$$
$$\xi_3 = -\,\partial_1 + x^3\partial_2 - x^2\partial_3 \qquad \omega^3 = -\,dx^1$$

Type VIII : $C^1{}_{23} = - C^1{}_{32} = 1,\ C^2{}_{31} = - C^2{}_{13} = 1,\ C^3{}_{12} = - C^3{}_{21} = -1$

$$\xi_1 = \tfrac{1}{2}e^{-x^3}\partial_1 - \tfrac{1}{2}[e^{x^3} + (x^2)^2 e^{-x^3}]\partial_2 - x^2 e^{-x^3}\partial_3$$

$$\xi_2 = \partial_3$$

$$\xi_3 = - \tfrac{1}{2}e^{-x^3}\partial_1 - \tfrac{1}{2}[e^{x^3} - (x^2)^2 e^{-x^3}]\partial_2 + x^2 e^{-x^3}\partial_3$$

$$\omega^1 = [e^{x^3} - (x^2)^2 e^{-x^3}]dx^1 - e^{-x^3}dx^2$$

$$\omega^2 = 2x^2 dx^1 + dx^3$$

$$\omega^3 = - [e^{x^3} + (x^2)^2 e^{-x^3}]dx^1 - e^{-x^3}dx^2$$

Type IX : $C^i{}_{jk} = \varepsilon_{ijk}$

$$\xi_1 = \partial_2$$

$$\xi_2 = \cos x^2 \partial_1 - \operatorname{cotan} x^1 \sin x^2 \partial_2 + \frac{\sin x^2}{\sin x^1}\partial_3$$

$$\xi_3 = - \sin x^2 \partial_1 - \operatorname{cotan} x^1 \cos x^2 \partial_2 + \frac{\cos x^2}{\sin x^1}\partial_3$$

$$\omega^1 = dx^2 + \cos x^1 dx^2$$

$$\omega^2 = \cos x^2 dx^1 + \sin x^2 \sin x^1 dx^3$$

$$\omega^3 = - \sin x^2 dx^1 + \cos x^2 \sin x^1 dx^3$$

Class B and Class D

Types V and Types ${}_fV$: $C^1{}_{13} = - C^1{}_{31} = -1,\ C^2{}_{23} = - C^2{}_{32} = -1$

$\xi_1 = \partial_2$ $\qquad \omega^1 = - x^2 dx^1 + dx^2$

$\xi_2 = \partial_3$ $\qquad \omega^2 = - x^3 dx^1 + dx^3$

$\xi_3 = - \partial_1 - x^2\partial_2 - x^3\partial_3$ $\qquad \omega^3 = - dx^1$

Types IV and ${}_f\mathrm{IV}$: $C^1{}_{13} = -\, C^1{}_{31} = -1,\ C^1{}_{23} = -\, C^1{}_{32} = 1,\ C^2{}_{23} = -\, C^2{}_{32} = -1.$

$$\xi_1 = \partial_2 \qquad\qquad \omega^1 = (x^3 - x^2)dx^1 + dx^2$$
$$\xi_2 = \partial_3 \qquad\qquad \omega^2 = -\, x^3 dx^1 + dx^3$$
$$\xi_3 = -\,\partial_1 + (x^3 - x^2)\partial_2 - x^3\partial_3 \qquad \omega^3 = -\, dx^1$$

Types VI_h and ${}_f\mathrm{VI}_h$: $C^1{}_{23} = -\, C^1{}_{32} = 1,\ C^2{}_{13} = -\, C^2{}_{31} = 1,\ C^1{}_{13} = -\, C^1{}_{31} = -(-h)^{\frac{1}{2}}$

$C^2{}_{23} = -\, C^2{}_{32} = -\,(-h)^{\frac{1}{2}}\ (h < 0)$

$$\xi_1 = \partial_2 \qquad\qquad \omega^1 = (x^3 - ax^2)dx^1 + dx^2$$
$$\xi_2 = \partial_3 \qquad\qquad \omega^2 = (x^2 - ax^3)dx^1 + dx^3$$
$$\xi_3 = -\,\partial_1 + (x^3 - ax^2)\partial_2 + (x^2 - ax^3)\partial_3 \qquad \omega^3 = -\, dx^1$$

Types $\mathrm{III} = \mathrm{VI}_{-1}$, ${}_f\mathrm{III} = {}_f\mathrm{VI}_{-1}$ and ${}_1^*\mathrm{III}$: $C^1{}_{23} = -\, C^1{}_{32} = 1,\ C^2{}_{13} = -\, C^2{}_{31} = 1$

$C^1{}_{13} = -\, C^1{}_{31} = -1,\ C^2{}_{23} = -C^2{}_{32} = -1$

$$\xi_1 = \partial_2 \qquad\qquad \omega^1 = (x^3 - x^2)dx^1 + dx^2$$
$$\xi_2 = \partial_3 \qquad\qquad \omega^2 = (x^2 - x^3)dx^1 + dx^3$$
$$\xi_3 = -\,\partial_1 + (x^3 - x^2)\partial_2 + (x^2 - x^3)\partial_3 \qquad \omega^3 = -\, dx^1$$

Types VII_h and ${}_f\mathrm{VII}_h$: $C^1{}_{23} = -\, C^1{}_{32} = 1,\ C^2{}_{13} = -\, C^2{}_{31} = -1,$

$C^1{}_{13} = -\, C^1{}_{31} = -\, h^{\frac{1}{2}},\ C^2{}_{23} = -\, C^2{}_{32} = -\, h^{\frac{1}{2}}\ (h > 0)$

$$\xi_1 = \partial_2 \qquad\qquad \omega^1 = (x^3 - ax^2)dx^1 + dx^2$$
$$\xi_2 = \partial_3 \qquad\qquad \omega^2 = -\,(x^2 + ax^3)dx^1 + dx^3$$
$$\xi_3 = -\,\partial_1 + (x^3 - ax^2)\partial_2 - (x^2 + ax^3)\partial_3 \qquad \omega^3 = -\, dx^1$$

APPENDIX B

TETRAD FORM OF EINSTEIN'S FIELD EQUATIONS

We consider here Einstein's field equations written out in the orthonormal frame introduced in Chapter 3. The virtue of the orthonormal tetrad approach is that the field equations are differential equations of only first order in the variables $\gamma^a{}_{bc}$ (or Γ_{abc}). The drawback is that, as compared with calculating from the coordinate form of the metric, we have more variables and more equations, since in addition to the field equations we must satisfy the Jacobi identities.

In the tetrad frame we have, using the Ricci identities

$$v_{a|bc} - v_{a|cb} = - R_{aebc}v^e \qquad \text{(B.1)}$$

and upon choosing $\underline{v}_a$ as the basis vector $\hat{e}_a$, the result

$$R^f{}_{bcd} = \partial_d\Gamma^f{}_{cb} - \partial_c\Gamma^f{}_{db} - \Gamma^f{}_{cg}\Gamma^g{}_{db} + \Gamma^f{}_{dg}\Gamma^g{}_{cb} - \Gamma^f{}_{gb}\gamma^g{}_{dc};$$

and upon contraction one obtains

$$R_{bd} = \partial_d\Gamma^c{}_{cb} - \partial_c\Gamma^c{}_{db} - \Gamma^c{}_{cs}\Gamma^s{}_{db} + \Gamma^s{}_{cb}\Gamma^c{}_{sd} . \qquad \text{(B.2)}$$

To write out the conformal components $\hat{R}_{bd}$ we now substitute equations (3.5.13) into (B.2) making use of the set of equations (3.5.16). One obtains equations (B.3):-

$$\hat{R}_{oo} = e^{-2\sigma}[R_{oo} - 3\partial_o^2\sigma - \theta\partial_o\sigma + \partial_\alpha\partial^\alpha\sigma + 3\dot{n}^\alpha\partial_\alpha\sigma$$

$$+ 2\partial_\alpha\sigma\partial^\alpha\sigma - 2a_\alpha\partial^\alpha\sigma]$$

$$\hat{R}_{o\alpha} = e^{-2\sigma}[R_{o\alpha} - 2\partial_\alpha\partial_o\sigma + \frac{2}{3}\partial_\alpha\sigma\theta + 2\partial_\alpha\sigma\partial_o\sigma$$

$$+ 2\sigma^\beta{}_\alpha\partial_\beta\sigma - 2\varepsilon_{\alpha\beta\delta}\omega^\delta\partial^\beta\sigma]$$

$$\hat{R}^*_{\alpha\beta} = e^{-2\sigma}[R^*_{\alpha\beta} - 2\partial_\beta\partial_\alpha\sigma - \varepsilon_{\alpha\beta\delta}n^{\delta\gamma}\partial_\nu\sigma - 2\partial_\beta\sigma a_\alpha$$

$$+ 2\partial_\alpha\sigma\partial_\beta\sigma + 2\varepsilon_{\nu\delta(\alpha}n^\nu{}_{\beta)}\partial^\delta\sigma$$

$$+ \frac{1}{3}\delta_{\alpha\beta}(2\partial_\alpha\partial^\alpha\sigma - 2\partial^\gamma\sigma\partial_\nu\sigma + 2a_\nu\partial^\nu\sigma)]$$

$$\hat{R}^* = e^{-2\sigma}[R^* - 5\partial_\alpha\partial^\alpha\sigma - 4\partial_\alpha\sigma\partial^\alpha\sigma + 10\partial^\alpha\sigma a_\alpha] \qquad \text{(B.3)}$$

where $\hat{R}^*$ is the trace of $\hat{R}_{\alpha\beta}$, $\hat{R}^*_{\alpha\beta}$ is the trace-free part of $\hat{R}_{\alpha\beta}$ and R_{ab} correspond to the non-conformal components and are given by equations (B.4):-

$$R_{oo} = - \dot{\theta} - \theta_{\alpha\beta}\theta^{\alpha\beta} + 2\omega^2 + \partial_\alpha\dot{n}^\alpha + \dot{n}_\alpha\dot{n}^\alpha - 2a_\alpha\dot{n}^\alpha$$

$$R_{o\alpha} = - \frac{2}{3}\partial_\alpha\theta + \partial_\beta\sigma^\beta{}_\alpha - \partial_\beta(\varepsilon^\beta{}_{\alpha\delta}\omega^\delta) - 3\sigma^\beta{}_\alpha a_\beta + n_{\alpha\beta}\omega^\beta$$

$$+ \varepsilon_{\alpha\beta\delta}\omega^\delta(a^\beta - 2\dot{n}^\beta) + \varepsilon_{\alpha\beta\delta}n^{\delta\mu}\sigma^\beta{}_\mu$$

$$R^*_{\alpha\beta} = \partial_0\sigma_{\alpha\beta} - \partial_{(\alpha}\dot{n}_{\beta)} + \frac{1}{3}g_{\alpha\beta}\partial_\mu\dot{n}^\mu - \dot{n}_\alpha\dot{n}_\beta + \frac{1}{3}g_{\alpha\beta}\dot{n}_\mu\dot{n}^\mu$$

$$- \dot{n}_{(\alpha}a_{\beta)} + n^\nu{}_{(\alpha}\varepsilon_{\beta)\nu\delta}\dot{n}^\delta + \theta\sigma_{\alpha\beta} + 2\sigma^\nu{}_{(\alpha}\varepsilon_{\beta)\delta\nu}\Omega^\delta$$

$$+ 2\Omega_{(\alpha}\omega_{\beta)} + \partial_{(\alpha}a_{\beta)} - \varepsilon_{\delta\nu(\alpha}\partial^\delta n^\nu{}_{\beta)} - 2\varepsilon_{\nu\delta(\alpha}n^\nu{}_{\beta)}a^\delta$$

$$+ 2n^\delta{}_{(\alpha}n_{\beta)\delta} - nn_{\alpha\beta}$$

$$+ \frac{1}{3}\delta_{\alpha\beta}(a_\mu\dot{n}^\mu - 2\omega_\mu\Omega^\mu - \partial_\mu a^\mu + n^2 - 2n^{\mu\nu}n_{\mu\nu})$$

$$R^* = \partial_0\theta - \partial_\alpha\dot{n}^\alpha - \dot{n}_\alpha\dot{n}^\alpha + 2\dot{n}_\alpha a^\alpha - 4\omega_\alpha\Omega^\alpha + 4\partial_\alpha a^\alpha$$

$$- n^{\alpha\beta}n_{\alpha\beta} + \tfrac{1}{2}n^2 - 6a_\alpha a^\alpha \tag{B.4}$$

The field equations can be written in the form

$$R_{ab} = T_{ab} - \tfrac{1}{2}Tg_{ab} \tag{B.5}$$

where $T_{ab} = (\rho + p)u_a u_b + pg_{ab}$ is the perfect-fluid energy-momentum tensor. Substituting in $u_a = \cosh\theta n_a + \sinh\theta k_a$, the stress tensor takes the form

$$T_{ab} = (\rho\cosh^2\theta + p\sinh^2\theta)n_a n_b + (p + \frac{1}{3}(\rho + p)\sinh^2\theta)h_{ab}$$

$$+ 2(\rho + p)\sinh\theta\cosh\theta k_{(a}n_{b)} + (\rho + p)\sinh^2\theta(k_a k_b - \frac{1}{3}h_{ab}) \tag{B.6}$$

when decomposed with respect to the vector n^a [35]. Hence, from (B.5) we obtain the components

$$\left.\begin{aligned}
R_{oo} &= \tfrac{1}{2}\rho(1 + 2\sinh^2\theta) + \frac{3}{2}p(1 + \frac{2}{3}\sinh^2\theta) \\
R_{o\alpha} &= -\ (\rho + p)\sinh\theta\cosh\theta k_\alpha \\
R^*_{\alpha\beta} &= (\rho + p)\sinh^2\theta(k_\alpha k_\beta - \frac{1}{3}\delta_{\alpha\beta}) \\
R^* &= \frac{3}{2}[(\rho - p) + \frac{2}{3}(\rho + p)\sinh^2\theta] \ .
\end{aligned}\right\} \qquad (B.7)$$

Equating equations (B.4) and (B.7) one now obtains the field equations. Notice however that one can substitute the R_{oo} component into the R^* equation, eliminating $\partial_0\theta$ and obtaining

$$\frac{1}{3}\theta^2 - \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}\omega^2 - 2\omega_\alpha\Omega^\alpha + 2\partial_\alpha a^\alpha - 3a_\alpha a^\alpha + \tfrac{1}{2}(\tfrac{1}{2}n^2 - n^{\alpha\beta}n_{\alpha\beta})$$

$$= \rho\cosh^2\theta + p\sinh^2\theta \qquad (B.8)$$

Similarly, one can write out the equations for the conformal models, where the perfect fluid is now given by

$$\hat{T}_{ab} = (\hat{\rho} + \hat{p})\hat{u}_a\hat{u}_b + \hat{p}\hat{g}_{ab}$$

where from (3.4.10) one has

$$\hat{u}_a = \cosh\beta n_a + \sinh\beta k'_a \ .$$

Writing $\hat{\rho} = e^{-2\sigma}\rho'$ and $\hat{p} = e^{-2\sigma}p'$ we obtain the equations (B.9):-

$$\hat{R}_{oo} = e^{-2\sigma}[\tfrac{1}{2}\rho'(1 + 2\sinh^2\beta) + \tfrac{3}{2}p'(1 + \tfrac{2}{3}\sinh^2\beta)]$$

$$\hat{R}_{o\alpha} = - e^{-2\sigma}[(\rho' + p')\sinh\beta\cosh\beta k'_\alpha]$$

$$\hat{R}^*_{\alpha\beta} = e^{-2\sigma}[(\rho' + p')\sinh^2\beta(k'_\alpha k'_\beta - \tfrac{1}{3}\delta_{\alpha\beta})]$$

$$\hat{R}^* = e^{-2\sigma}[\tfrac{3}{2}(\rho' - p') + (\rho' + p')\sinh^2\beta] \tag{B.9}$$

Equating equations (B.9) and (B.3) now gives the desired field equations. Also, as above, we substitute the $\hat{R}_{oo}$ equation into the $\hat{R}^*$ equation to give

$$\begin{aligned} &\tfrac{1}{3}\theta^2 - \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}\omega^2 - 2\omega_\alpha\Omega^\alpha + 2\partial_\alpha a^\alpha - 3a_\alpha a^\alpha + \tfrac{1}{2}(\tfrac{1}{2}n^2 - n^{\alpha\beta}n_{\alpha\beta}) \\ &- 2\partial^\alpha\partial_\alpha\sigma - \partial^\alpha\sigma\partial_\alpha\sigma + 4a_\alpha\partial^\alpha\sigma \\ &= \cosh^2\beta\rho' + p'\sinh^2\beta \end{aligned} \tag{B.10}$$

As mentioned at the beginning, we must also specify the Jacobi identities $R^a{}_{[bcd]} = 0$ which may be written in the form

$$\partial_{[d}\gamma^f{}_{bc]} + \gamma^s{}_{[db}\gamma^f{}_{c]s} = 0 \tag{B.11}$$

Substituting in equations (3.5.15) we may write these in the form (B.12):-

$$2\partial_0\omega^\alpha + \varepsilon^{\alpha\mu\nu}\partial_\mu\dot{n}_\nu - \dot{n}_\mu n^{\alpha\mu} - \varepsilon^{\alpha\mu\nu}a_\mu\dot{n}_\nu + 2\theta\omega^\alpha$$

$$- 2\varepsilon^{\alpha\mu\nu}\omega_\mu\Omega_\nu - 2\theta^\alpha{}_\mu\omega^\mu = 0$$

$$\partial_\alpha\omega^\alpha = \omega^\alpha(\dot{n}_\alpha + 2a_\alpha)$$

$$2\partial_0 a_\alpha - \partial_\mu\theta^\mu{}_\alpha + \partial_\alpha\theta + \varepsilon_\alpha{}^{\mu\nu}\partial_\mu(\Omega_\nu + \omega_\nu) + \theta\dot{n}_\alpha$$

$$+ \theta^\mu{}_\alpha(2a_\mu - \dot{n}_\mu) - \varepsilon_{\alpha\mu\nu}(2a^\mu - \dot{n}^\mu)(\Omega^\nu + \omega^\nu) = 0$$

$$\partial_\alpha n^{\alpha\mu} + \varepsilon^{\mu\alpha\nu}\partial_\alpha a_\nu - 2\theta^\mu{}_\alpha\omega^\alpha - 2n^\mu{}_\alpha a^\alpha - 2\varepsilon^{\mu\alpha\nu}\omega_\alpha\Omega_\nu = 0$$

$$\partial_0 n^{\alpha\beta} + \partial^{(\alpha}(\Omega^{\beta)} + \omega^{\beta)}) - 2n^{\nu(\alpha}\varepsilon^{\beta)}{}_{\nu\delta}(\Omega^\delta + \omega^\delta) +$$

$$+ \dot{n}^{(\alpha}(\omega^{\beta)} + \Omega^{\beta)}) - 2n_\nu{}^{(\alpha}\theta^{\beta)\gamma} + n^{\alpha\beta}\theta - \delta^{\alpha\beta}(\partial_\nu\Omega^\nu + 2\omega^\nu\dot{n}_\nu)$$

$$+ 2\omega^\nu a_\nu + \dot{n}^\nu\Omega_\nu) - \varepsilon^{\delta\nu(\alpha}\partial_\nu\theta^{\beta)}{}_\delta + \theta_\nu{}^{(\alpha}\varepsilon^{\beta)\delta\nu}\dot{n}_\delta = 0 \qquad \text{(B.12)}$$

Similarly, using equation (3.5.12) one can show that the Jacobi identities in the conformal case are identical to the above expressions.

To write out these equations in detail, one usually specifies the triad of basis vectors $\{e_\mu\}$ further. One way of doing this is to fit the triad $\{e_\mu\}$ to the tensor $n^{\alpha\beta}$ and the vector a_α such that

$$n_{\alpha\beta} = \mathrm{diag}(n_1,n_2,n_3) \qquad a_\alpha = (0,0,a) \qquad \text{(B.13)}$$

Writing out the field equations (B.7) and (B.8) and Jacobi identities (B.12) for homogeneous models where $\dot{n}_\alpha = \omega_\alpha = 0$ and all quantities are functions of time alone we have equations (B.14) [34].

$$a(\Omega_2 + \sigma_{13}) = 0$$

$$a(\Omega_1 + \sigma_{23}) = 0$$

$$\partial_0 a + \theta_3 a = 0$$

$$\partial_0 n_1 + (\theta_2 + \theta_3 - \theta_1)n_1 = 0$$

$$(n_1 - n_2)\Omega_3 - (n_1 + n_2)\sigma_{12} = 0$$

$$(n_3 - n_1)\Omega_2 - (n_1 + n_3)\sigma_{13} = 0$$

$$\partial_0 n_2 + (\theta_1 + \theta_3 - \theta_2)n_2 = 0$$

$$(n_2 - n_3)\Omega_1 - (n_2 + n_3)\sigma_{23} = 0$$

$$\partial_0 n_3 + (\theta_1 + \theta_2 - \theta_3)n_3 = 0$$

$$\partial_0 \theta + \theta_1{}^2 + \theta_2{}^2 + \theta_3{}^2 + 2\sigma_{12}^2 + 2\sigma_{13}^2 + 2\sigma_{23}^2$$

$$+ \tfrac{1}{2}(1 + 2\sinh^2\theta)\rho + \frac{3}{2}(1 + \frac{2}{3}\sinh^2\theta)p = 0$$

$$3a\sigma_{13} + \sigma_{23}(n_2 - n_3) = (\rho + p)\sinh\theta\cosh\theta k_1$$

$$3a\sigma_{23} + \sigma_{23}(n_3 - n_1) = (\rho + p)\sinh\theta\cosh\theta k_2$$

$$a(2\theta_3 - \theta_1 - \theta_2) + \sigma_{12}(n_1 - n_2) = (\rho + p)\sinh\theta\cosh\theta k_3$$

$$\partial_0\sigma_{12} = -\,\theta\sigma_{12} + \sigma_{13}\Omega_1 - \sigma_{23}\Omega_2 + (\theta_2 - \theta_1)\Omega_3$$

$$+ (n_2 - n_1)a + (\rho + p)\sinh^2\theta k_1k_2$$

$$\partial_0\sigma_{13} = -\,\theta\sigma_{13} + \sigma_{23}\Omega_3 - \sigma_{12}\Omega_1 + (\theta_1 - \theta_3)\Omega_2$$

$$+ (\rho + p)\sinh^2\theta k_1k_3$$

$$\partial_0\sigma_{23} = -\,\theta\sigma_{23} + \sigma_{12}\Omega_2 - \sigma_{13}\Omega_3 + (\theta_3 - \theta_2)\Omega_1$$

$$+ (\rho + p)\sinh^2\theta k_2k_3$$

$$\partial_0\theta_1 = -\,\theta\theta_1 + 2a^2 - \tfrac{1}{2}n_1{}^2 + \tfrac{1}{2}(n_2 - n_3)^2 + 2\sigma_{12}\Omega_3$$

$$- 2\sigma_{13}\Omega_2 + \tfrac{1}{2}(\rho - p) + (\rho + p)\sinh^2\theta k_1{}^2$$

$$\partial_0\theta_2 = -\,\theta\theta_2 + 2a^2 - \tfrac{1}{2}n_2{}^2 + \tfrac{1}{2}(n_1 - n_3)^2 + 2\sigma_{23}\Omega_1$$

$$- 2\sigma_{12}\Omega_3 + \tfrac{1}{2}(\rho - p) + (\rho + p)\sinh^2\theta k_2{}^2$$

$$\partial_0\theta_3 = -\,\theta\theta_3 + 2a^2 - \tfrac{1}{2}n_3{}^2 + \tfrac{1}{2}(n_1 - n_2)^2 + 2\sigma_{13}\Omega_2$$

$$- 2\sigma_{23}\Omega_1 + \tfrac{1}{2}(\rho - p) + (\rho + p)\sinh^2\theta k_3{}^2$$

$$\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 = \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 + 3a^2 + p\sinh^2\theta$$

$$+ \tfrac{1}{4}(n_1{}^2 + n_2{}^2 + n_3{}^2 - 2n_1n_2 - 2n_1n_3 - 2n_2n_3) + \rho\cosh^2\theta$$

APPENDIX C

SOLUTION OF COORDINATE EQUATIONS (4.3.1)

We outline here the solution of the coordinate equations (4.3.1). We consider only Bianchi types II - VII where $\sigma = \sigma(x^1) = -\int F(x^1)dx^1$. The simpler type I models follow the same procedure. These calculations provide a useful check on some of the results of §4.4.

For non-tilted homogeneous models we have

$$u^A = \delta^A_o \qquad u_A = -\delta^o_A \tag{C.1}$$

Since from (3.3.5), $\hat{u}^A = \lambda u^A + v^A$ where $u^A v_A = 0$, then

$$v^o = 0, \quad \hat{u}^o = \lambda \quad \text{and} \quad \hat{u}^i = v^i \tag{C.2}$$

Substituting conditions (C.1) and (C.2) into equations (4.3.1) we have the following system of equations:

i) (oo) component

$$(\hat{\rho} + \hat{p})e^{4\sigma}\lambda^2 - \hat{p}e^{2\sigma} - \rho = -g^{AB}(2\nabla_A\nabla_B\sigma + \partial_A\sigma\partial_B\sigma) \tag{C.3}$$

ii) (oi) components

$$(\hat{\rho} + \hat{p})e^{4\sigma}\lambda v_i = 2\nabla_o\nabla_i\sigma \tag{C.4}$$

iii) (ij) components

$$(\hat{\rho} + \hat{p})e^{4\sigma}v_i v_j + (\hat{p}e^{2\sigma} - p)g_{ij}$$

$$= - 2\nabla_i\nabla_j\sigma + 2\partial_i\sigma\partial_j\sigma + g_{ij}[2\nabla_A\nabla_B\sigma + \partial_A\sigma\partial_B\sigma]g^{AB} \qquad (C.5)$$

Putting $\sigma = 0$ gives the trivial solution $\hat{\rho} = \rho$, $\hat{p} = p$ and $\lambda = 1$, $v_i = 0$. From (C.4) we can see that the conformal models will in general be tilted i.e. $v_i \neq 0$.

Use the metric form (4.2.3) and calculating the Christoffel symbols, it can be shown that

$$g^{AB}\nabla_A\nabla_B\sigma = - \gamma_{33}{}^{-1}\partial F + 2r\gamma_{33}{}^{-1}F$$

$$g^{AB}\partial_A\sigma\partial_B\sigma = \gamma_{33}{}^{-1}F^2$$

and

$$\Gamma^1_{oi} = \tfrac{1}{2}\gamma_{33}{}^{-1}\gamma_{33}\delta^1_i$$

Substituting these results into the equations (C.3) - (C.5) yields

$$(\hat{\rho} + \hat{p})e^{4\sigma}\lambda^2 - \hat{p}e^{2\sigma} - \rho = - \gamma_{33}{}^{-1}F(4r + F) + 2\gamma_{33}{}^{-1}\partial F \qquad (C.6)$$

$$(\hat{\rho} + \hat{p})e^{4\sigma}\lambda v_i = \gamma_{33}{}^{-1}\gamma_{33}F\delta^1_i \qquad (C.7)$$

$$(\hat{\rho} + \hat{p})e^{4\sigma}v_i v_j + (p'e^{2\sigma} - p)g_{ij}$$

$$= (2\partial F + 2F^2)\delta^1_i\delta^1_j - 2\Gamma^1_{ij}F + g_{ij}\gamma_{33}{}^{-1}F(4r + F) - 2\gamma_{33}{}^{-1}\partial F \qquad (C.8)$$

From (C.6) we have $v_2 = v_3 = 0$ and from the (2 3) component of equation of (C.8) we find

$$\Gamma^1_{23} = 0 \Rightarrow \gamma_{11}s + \gamma_{22}q = 0 \qquad \text{(C.9)}$$

This constraint puts restrictions upon the allowed Bianchi types. From Table 3 we find that only types V and VII are admitted.

To solve equations (C.8) further we calculate the Christoffel symbols Γ^1_{ij}. A series of calculations give

$$\Gamma^1_{ij} = \gamma_{33}{}^{-1}g_{ij}r - r\delta^1_i\delta^1_j$$

Solving equations (C.6) - (C.8) now gives the solutions

$$\hat{\rho}e^{2\sigma} = \rho - 3\gamma_{33}{}^{-1}(2rF + F^2) \qquad \text{(C.9)}$$

$$\hat{p}e^{2\sigma} = p + \gamma_{33}{}^{-1}(2rF - \partial F + F^2) \qquad \text{(C.10)}$$

$$v_1 = \frac{\gamma_{33}{}^{-1}\dot{\gamma}_{33}F}{(\hat{\rho} + \hat{p})e^{4\sigma}\lambda}$$

and we have the constraint

$$[\rho + p - 2\gamma_{33}{}^{-1}rF][2rF + 2F^2 + 2\partial F] = (\gamma_{33}{}^{-1}\dot{\gamma}_{33})^2F^2 \qquad \text{(C.11)}$$

Equations (C.9), (C.10) and (C.11) correspond to the equations (4.5a), (4.5b) and (4.7a) respectively of §4.4.

APPENDIX D

TRANSFORMATION PROPERTIES

In the coordinate frame, for Bianchi Types II - VII we have $\sigma = -\int F(x^1)dx^1$. Hence

$$\partial_i\sigma = -F\delta^1_i$$

and
$$\partial_j\partial_i\sigma = -\partial F\ \delta^1_i\delta^1_j \quad \text{where } \partial F = dF/dx^1 .$$

Now consider transforming to the tetrad frame where the spatial part of the metric is written in the form

$$ds^2 = g_{\alpha\beta}\omega^\alpha\omega^\beta = g_{\alpha\beta}\omega^\alpha{}_i\omega^\beta{}_j dx^i dx^j$$

where $g_{\alpha\beta} = \mathrm{diag}(1,1,1)$. From equation (2.2.1) and Appendix A we have

$$\left.\begin{array}{l} e_1{}^2 = \gamma_{11}{}^{-\frac{1}{2}} \qquad e_2{}^3 = \gamma_{22}{}^{-\frac{1}{2}} \qquad e_3{}^1 = -\gamma_{33}{}^{-\frac{1}{2}} \\ e_3{}^2 = \gamma_{33}{}^{-\frac{1}{2}}(sx^3 + px^2)^{-1} \qquad e_3{}^3 = \gamma_{33}{}^{-\frac{1}{2}}(px^3 + qx^2)^{-1} \end{array}\right\} \qquad \text{(C.1)}$$

where
$$\omega^\alpha{}_i e_\beta{}^i = \delta^\alpha{}_\beta .$$

A covariant vector transforms according to

$$v_\alpha = e_\alpha{}^i v_i$$

Thus, in the tetrad frame we will have

$$\partial_\alpha \sigma = e_\alpha{}^i \partial_i \sigma = - e_\alpha{}^i F \delta_i^1 = - F e_\alpha^1$$

and $$\partial_\beta \partial_\alpha \sigma = e_\beta{}^i \partial_i (- F e_\alpha^1) = - e_\beta{}^i [\partial F \delta_i^1 e_\alpha^1 + F \partial_i e_\alpha^1]$$

$$= - [e_\beta^1 e_\alpha^1 \partial F + F e_\beta{}^i \partial_i e_\alpha^1]$$

Substituting in equations (C.1) we find

$$\partial_\alpha \sigma = \gamma_{33}{}^{-\frac{1}{2}} F \delta_\alpha^3 \tag{C.2a}$$

and $$\partial_\beta \partial_\alpha \sigma = - \gamma_{33}{}^{-1} \partial F \delta_\alpha^3 \delta_\beta^3 \tag{C.2b}$$

In the Bianchi Type I case $F = F(x^3)$ and it is trivial to show that

$$\partial_\alpha \sigma = \gamma_{33}{}^{-\frac{1}{2}} F \delta_\alpha^3$$

$$\partial_\beta \partial_\alpha \sigma = \gamma_{33}{}^{-1} \partial F \delta_\alpha^3 \delta_\beta^3$$

To note how the relative vector $\underset{\sim}{a}$ transforms from the invariant basis of Chapter 2 to the triad of orthonormal vectors, consider the following change in the basis vectors

$$e_1{}^i = \gamma_{11}{}^{-\frac{1}{2}} \xi_1{}^i \qquad e_2{}^i = \gamma_{22}{}^{-\frac{1}{2}} \xi_2{}^i \qquad e_3{}^i = \gamma_{33}{}^{-\frac{1}{2}} e_3{}^i$$

It then follows that

$$\gamma^\delta{}_{\alpha\beta} = \gamma_\delta{}^{\frac{1}{2}} \gamma_\alpha{}^{-\frac{1}{2}} \gamma_\beta{}^{-\frac{1}{2}} C^\delta{}_{\alpha\beta} \quad \text{(no sum)}$$

Hence, since $a_\alpha = \frac{1}{2}\gamma^\beta{}_{\alpha\beta}$ and $a_i = \frac{1}{2}C^j{}_{ij} = -r\delta^3_i$ we have

$$a_\alpha = -\gamma_{33}{}^{-\frac{1}{2}}\delta^3_\alpha r \quad .$$

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