AN APPLICATION OF LINEAR PROGRAMMING TO THE SCHEDULING OF TOLL COLLECTORS

by

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SUMMARY

The thesis is concerned with the practical problem of rostering manpower for toll facilities. The demand for labour is determined by dividing the day into periods of suitable length and estimating the number of collectors required for each period. The schedule is then required to supply the necessary number of collectors in each period while maintaining idle time at the lowest possible level.

The introduction to the thesis reviews the existing publications in the field of manpower scheduling and describes the general nature of the toll problem comparing it, in particular, to the scheduling of bus crews. Unlike the latter, the toll problem can be represented as a linear program of manageable proportions. Because a schedule must conform to the regulations contained in the collectors' award, a large number of additional considerations, related to the provision of rest breaks, are involved in the generation of an acceptable schedule. Although this suggests the use of a model of considerable dimensions, the thesis illustrates how a substantial reduction in the dimensions may be made.

The method used to solve the problem is based on the Revised Simplex Method but as the problem is an integer programming problem, the primary effort is directed towards satisfying the integer requirement as the size of the matrix is beyond the capabilities of available integer algorithms. The procedure used is dynamic and earlier phases establish efficient general bounds which locate a solution in the vicinity of the required integer solution. The method then proceeds by constraining individual variables by the use of a specially devised technique.

As the algorithm has been subjected to a considerable amount of testing, the thesis concludes with a summary of results together with a general discussion of the steps necessary to ensure economy in scheduling. The wider implications of the techniques are also referred to in the concluding chapter.

SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made, in the text of the thesis.

J.L. Byrne

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CHAPTER 1

INTRODUCTION

1.1 Background

The rostering of manpower in transportation systems has been the subject of considerable research and development in recent years and there is now a considerable literature of research papers on the subject, particularly in the field of public transport. As a result, manual methods of rostering are now being replaced by automatic computer methods which, to a varying degree, have been based upon sophisticated mathematical analysis.

A time consuming task, manual scheduling involves complexities which make it difficult to obtain economical solutions while it is practically impossible to control additional factors such as shift starting times or to evaluate the effects of changes to working regulations. Delays inherent in the manual process prohibit the preparation of new schedules with the necessary frequency and in consequence affect the standard of service to patrons. The resulting schedules are therefore less than satisfactory to employer, employee and patron alike. Under these circumstances, the use of a satisfactory computer method is necessary not so much to reduce the cost of manual scheduling, but to improve the effective management of the total budget allocation by achieving closer conformity between demand and service throughout the year and by providing a means of evaluating any changes proposed with the intention of rationalising operations generally.

For bus operators, the scheduling problem essentially involves the provision of bus crews to match a bus timetable which is prepared to meet passenger demand at an agreed level of service and the majority of methods devised for the solution of this problem rely either on simulation or partitioning of the problem. Elias [8],[9] describes simulation methods used by American transit companies and in a third paper [10] discusses the difficulties encountered in attempting to solve the complete problem as a linear program. This latter paper also contains a description of a heuristic run cutting technique which obtains a set of shifts by making a selection from a relatively small subset of the total alternatives available. Bennett and Potts [5] partition the problem and apply mathematical techniques to one section while the remainder is completed by a simulation method [17].

Although the amount of literature related to the scheduling of bus crews has become quite substantial, that specifically applicable to the toll problem, which is the concern of this thesis, is almost entirely restricted to the efforts made by the <u>Port of New York Authority</u> (P.N.Y.A.) to develop a satisfactory method of scheduling full time and part time collectors.* Edie [7] and Ashe [1] deal with the problem of establishing and maintaining satisfactory levels of service at toll facilities and Ashe [2] examines the current situation at P.N.Y.A. facilities, particularly with regard to scheduling collectors. The simulation method devised for the P.N.Y.A. by the Illinois Institute of Technology is described in [19] and a programmed version of the method is contained in [16]. Foote [11] establishes the necessity to computerise scheduling procedures and refers to the difficulties encountered with the simulation method.

1.2 Basic Features of the Problem

Basically, the problem at toll facilities is that of scheduling the least possible quantity of labour so that the necessary tolls are collected with a minimum of delay and inconvenience to the motorist using the facility. "Minimum" is used in the sense which implies the shortest delay consistent with economical collection.

The nature of the problem and the subsequent solution is determined largely by the demand pattern and by the framework of rules and regulations which define the working conditions for collectors.

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* A glossary containing abbreviations is included immediately before the bibliography.

The problem, as stated above, may be divided into two sub-problems:

(a) The determination of the delay time acceptable to the administering authority and the motorist, the calculation of the collection rate to be used and the calculation of the number of collectors needed to maintain satisfactory service throughout the day.

(b) The determination of the numbers and types of shifts necessary to meet the demand, together with their starting times and the times for the accompanying rest breaks.

The estimation of demand [2] requires that the total time interval under consideration be divided into a number of discrete periods whose length depends on the circumstances prevailing at the particular facility. The required number of collectors is then determined for each period throughout the day so that demand may be represented by a histogram as shown in figure (18). At facilities where automatic devices are installed, the contribution made by the latter must first be taken into account before the estimate of collectors is made. At this stage it suffices to say that the periods should be of small enough duration so that the number of collectors provided in each period can accurately reflect the changing traffic situation throughout the day.

Delay and the determination of demand has been the subject of intensive study by the P.N.Y.A. [1],[2],[7] and no attempt will be made to expand on this. It is important to recognise, however, that since demand is based upon predictions of future traffic activity, some degree of day by day variation must be expected. Further, demand depends to some extent on the collection rate used and this rate may be varied within certain limits. Whereas it is desirable to set an acceptable standard of service and adhere to this as closely as possible, the demand, as initially calculated, is not necessarily completely final and inflexible.

Since a collector cannot be expected to remain on duty continuously for the entire duration of his shift, certain intervals must be set aside for meals and rest purposes. In a shift of 8 hours a collector may have two or three rest breaks of specified durations taken at times within limits specified by the working regulations. The set of rest breaks contained in a shift (Fig.1a) will be referred to as the <u>relief</u> for that particular shift and since working regulations contain some latitude as to the location of the various breaks, a number of possible alternative reliefs will be available for each shift. The provision of these reliefs further complicates the situation.

While the problem of scheduling toll collectors is similar to that of scheduling bus crews, closer examination reveals that this similarity is to some extent superficial. The demand patterns for both are similar in that each is bi-modal with the modes corresponding to the morning and evening traffic peaks (Fig.18). Each problem contains the features which make a solution potentially wasteful in terms of manhours used and similarities exist in the working regulations for both problems. The differences are such however, that an entirely different method of solution is necessary.

Because he remains in the same locality, a toll collector may cease duty at any suitable time. A bus driver, on the other hand, must remain with his vehicle at least until it passes the next relief point where an exchange of drivers may take place. Demand in the bus problem therefore tends to be in terms of indivisible pieces of work of varying lengths and this factor has introduced a degree of inflexibility into the bus problem which does not appear in the toll problem. In the latter, the basic units of demand are periods of the same length rather than miscellaneous pieces of work of varying lengths and this important feature facilitates the representation of the latter problem as a linear program of manageable dimensions.

Relief requirements for the two problems also differ. While both problems require that periodic breaks be inserted into the shifts, all breaks for bus drivers, except one meal break, occur naturally at the end of each trip. All reliefs required for toll collectors must be provided by manipulation of the shifts themselves so that while the bus problem is mainly concerned with the minimization of overtime, the success of a toll schedule depends largely on the skill displayed in the provision of a comparatively large number of breaks.

Examination of the demand histogram reveals several features which contribute to the difficulty in producing economical rosters the most prominent of which are the two peaks enclosing a substantial dip in demand in the middle of the day (Fig.18). The accommodation of these irregularities alone implies a considerable wastage of manpower since demands at the peaks must be met and shifts used to satisfy demands in this region must oversupply labour at other times if they extend much beyond the region of the peaks. This difficulty becomes more pronounced as the difference between peak demand and mid-day demand increases. Further, the histogram shows that a large proportion of the demand is spread over an interval which is too long to be spanned by one shift of approximately eight hours and too short to warrant the use of two such shifts. Unfortunately, overmanning with a system of eight hour shifts is unavoidable simply because the system is less flexible than necessary and a schedule of full time shifts cannot be adapted to fit the problem with any great degree of efficiency. This fact is illustrated by a P.N.Y.A. study of the situation at the Lincoln Tunnel [2]. While estimates of demand were found to be accurate for 69% of the total periods in a test week, including Saturday and Sunday, subsequent rostering difficulties were such that the numbers of booths actually used were correct for only 28% of the total periods. For weekdays the percentages were respectively 75% and 30%.

These difficulties are common to both the bus problem and the toll problem and various techniques are used to overcome them. Bus operators usually resort to the use of broken shifts to reduce labour costs to a more reasonable level [5]. Broken shifts consist of two parts, corresponding to the peak periods, which are separated by a break of several hours corresponding to the fall in demand in the middle of the day. In general, broken shifts have the effect of spreading the working day while the break maintains actual working time in the vicinity of eight hours.

At least one toll authority, the Department of Main Roads of New South Wales (D.M.R.), has attempted to improve economy by the use of broken shifts on the <u>Sydney Harbour Bridge</u> but their value in this situation is limited. A very small proportion of broken shifts usually appeared in each roster and their use has now been discontinued completely.

Of greater interest is a system proposed by the P.N.Y.A. [2],[16] and lately adopted by D.M.R. where the usual <u>full time shifts</u> (F.T.S.) of approximately 8 hours are supplemented by <u>part time shifts</u> (P.T.S.) of lesser duration, thereby increasing substantially the flexibility of the labour supply by increasing the options available in schedule preparation.

1.3 Working Regulations

The working conditions for collectors at a facility are specified by a set of regulations which establish primarily the duration of the shifts used and the latitude permitted in providing the necessary rest breaks. Although the regulations vary from one facility to the next, there tends to be a close similarity in the broad principles applied to each case. As an example, the following sample contains the essential features of many awards and, with the exception of rule 4, resembles closely the regulations applied to full time collectors by D.M.R.

(1) A F.T.S. will be of 8 hours duration.

(2) A F.T.S. will contain three rest breaks, the

first and third of 20 minutes and the second of 40 minutes duration.

(3) The breaks in a F.T.S. must be located so that no collector is rostered for an interval of continuous duty in excess of 2 hours.

(4) No rest break may be rostered in the first or last hours of a shift.

In this situation the 8 hour shift is divided into four segments by the three breaks stipulated in rule 2. The duration of each segment is limited to a maximum of 2 hours by rule 3 which, together with rule 4, establishes control over the locations of breaks to ensure that they serve the purpose for which they are intended (Fig.1a). D.M.R. regulations specify 30 minutes for the first and third breaks as an extra 10 minutes is added to each for the purpose of counting and depositing receipts.

As a further example, the following six regulations are basically those used to establish the working conditions for full time collectors employed by the P.N.Y.A.

(a) A F.T.S. will be of 7 hours and 50 minutes duration.

(b) A F.T.S. must contain one personal break of 30 minutes duration and one meal break of 45 minutes duration.

(c) No segment of a shift may be less than 1 hour's duration nor more than 3 hours' duration.

(d) No shift may commence or finish between 1:00 a.m. and 6:00 a.m. .

(e) Collectors whose shifts commence between 6:00 a.m. and 10:30 a.m. must receive their meal breaks between 9:45 a.m. and 1:45 p.m. . Those commencing their shifts between 12:45 p.m. and 5:10 p.m. must receive their meal breaks between 4:45 p.m. and 8:45 p.m. .

(f) Collectors whose shifts commence between 10:30 a.m. and 12:45 p.m. may receive their meal break in either the noon or the evening meal period.

In this case a two part relief divides the shift into three segments and the maximum length of the segment of continuous duty is three hours. Although the latter implies much greater flexibility, the additional regulations, particularly that restricting the location of the meal break, greatly reduce the available options.

Although it is not possible to anticipate the regulations concerning P.T.S. by reference to current practice, experience indicates that the duration of the shift is not merely a matter of choice and should be determined by the requirements of the particular facility. The proposal by the P.N.Y.A. and the intentions of D.M.R. suggest that a one break relief will be adequate so that a P.T.S. will consist of two segments of continuous duty. Some extension of the maximum permissible segment may also be considered for P.T.S. since the shift is of limited duration.

While the use of P.T.S. makes an economical solution a possibility, a satisfactory method of automatically producing schedules is necessary to take full advantage of the opportunities offered. The purpose of this thesis is to describe an algorithm which may be used to produce efficient rosters involving shift work and which, in this case, is specifically applied to the scheduling of full time and part time labour for toll facilities. Chapter 2 introduces three possible formulations of the problem as a linear program and discusses the methods used to reduce the dimensions of the resulting matrices. Chapter 3 describes the techniques used to obtain a solution with particular reference to those concerned with the integer requirement. Chapter 4 contains an evaluation of the algorithm and deals with the additional features required in a satisfactory schedule. Chapter 4 also illustrates the use of the algorithm as an efficient device for the evaluation of various possible combinations of working regulations and the thesis is completed with a short discussion in Chapter 5.

CHAPTER 2

THE MATHEMATICAL MODEL

2.1 The Unit Period

The mathematical models have been based on a unit period which is also adopted for the estimation of demand and the elements of the schedule, that is the shifts and rest breaks, must be expressed in terms of this unit period. For example, if the unit period is of 20 minutes duration and a schedule is to be prepared for the interval 6 a.m. to 12 midnight, then the interval will be 54 unit periods in duration. Similarly, with a 20 minute unit period, an 8 hour shift would be 24 unit periods in duration. If the regulations call for 3 rest breaks, the first and third of 20 minutes duration and the second of 40 minutes duration, this would constitute a 1-2-1 relief in terms of 20 minute periods. Since the shift is of 24 unit periods duration and because 4 unit periods are allocated to rest breaks, a net 20 periods are actually contributed by the shift. The example is contained in figure 1a.

The adoption of the unit period also requires that a shift commences at the beginning of a period and concludes at the end of a period. In the example given above, a shift could commence at the beginning of period 1 and conclude at the end of period 24. Alternatively, a shift could commence at the beginning of period 2 and conclude at the end of period 25 and so on. Any shift must therefore be scheduled in one of a finite number of locations (Fig.1b). As well, the commencement and conclusion of a rest break must coincide with the beginning and end of a period so that associated with each shift, there exists a finite number of alternative reliefs. For brevity, reliefs for F.T.S. will be referred to by the initials F.T.R. and those for P.T.S. by P.T.R. Figure 2 illustrates the alternatives F.T.R. existing under rules 1 to 4 of chapter 1 and with a 20 minute unit period. Time is therefore expressed in terms of the unit period adopted and the notation to be used is as follows:

Let τ = the duration of the unit period in minutes.

T = the number of unit periods in the interval for which the schedule is to be prepared.

d_f = the duration in unit periods of a F.T.S.

d_p = the duration in unit periods of a P.T.S.

C_f = the net contribution in unit periods of a F.T.S.

C_p = the net contribution in unit periods of a P.T.S.

L_f = the number of alternative locations for F.T.S.

L_p = the number of alternative locations for P.T.S.

R_f = the number of alternative reliefs associated with each F.T.S.

R_p = the number of alternative reliefs associated with each P.T.S.

For example, in figure 1a, $\tau = 20$, $d_f = 24$ and $C_f = 20$. In figure 1b, $\tau = 60$, $T = 18$, $d_f = 8$, $d_p = 4$,

$L_f = 11$ and $L_p = 6$. In figure 2, $\tau = 20$ and $R_f = 33$.

2.2 Formulation 1

The obvious difficulty in applying linear programming to the problem concerns the anticipated size of the model which seeks to include the large number of relief alternatives available, particularly in view of the fact that as fractions of shifts and reliefs cannot be permitted in this situation, all variables must take integer values. Under these circumstances, partitioning of the problem seems indicated and a formulation exists which will avoid the difficulty referred to.

Ignoring the relief requirements temporarily, it is possible to provide an optimal quantity of labour to meet the initial demand by the use of enumeration and the assignment algorithm.

While demand may be represented as a histogram (Fig.3a), it may also be represented as a series of straight lines (Fig.3b) similar to those used in the preparation of schedules for bus crews. Let there be n such lines and let each line have a starting point S_i (i=1...n) and a finishing point F_j (j=1...n) where S_i and F_j represent the time elapsed from midnight. As in bus scheduling, the problem becomes one of cutting and joining the lines so that eventually the result coincides as closely as possible with a set of available shifts. This result may also be obtained by reassigning the pairs of starting and finishing points.

A point S_i may be joined to any point F_j, if $S_i < F_j$, and the resulting interval spanned by a shift or combination of shifts which equals or exceeds the interval in length. The shift or combination of shifts may then be used to satisfy the demand represented by the interval. As there may be alternatives, a cost c_{ij}, representing the cheapest alternative, is assigned to the possible linkage S_i to F_j. For example in figure 3c, if S_2 is $6\frac{1}{2}$ and F_4 is $19\frac{1}{2}$ and 8 hour F.T.S. and 5 hour P.T.S. are used, the best combination available to span the 13 hour interval involves the use of one F.T.S. and one P.T.S. and the corresponding cost, c_{ij}, is the cost of the F.T.S. plus the cost of the P.T.S. Since the number of alternatives is strictly limited and since each starting point S_i may be joined to any one of the finishing points F_j, enumeration may be used to generate $n \times n$ possibilities together with an associated cost c_{ij}. Where $S_i > F_j$, c_{ij} will be a prohibitive cost. The selection of the n optimal pairings is then performed by the following linear program.

$$\text{Minimise} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \qquad (1)$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j=1...n) \qquad (2)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i=1...n) \qquad (3)$$

and $$x_{ij} = \begin{cases} 1 & \text{if starting point } S_i \text{ is paired with finishing point } F_j. \\ 0 & \text{otherwise.} \end{cases} \qquad (4)$$

Because a shift or combination of shifts is associated with each of the n optimal linkages, the output from the assignment algorithm represents a set of shifts which matches the demand in an optimal manner. Since reliefs have not been included, a second step using simulation would be required to supply them.

This two step method resembles the procedure used by the P.N.Y.A. which, by simulation, also satisfies the initial demand before providing the reliefs. Unfortunately, as the reliefs represent a comparatively large additional demand, the problem solved in step 1 may have only a tenuous relationship to the complete problem and any advantage gained by an optimal solution to the initial demand is largely dissipated in the attempt to provide the necessary reliefs.

Under these circumstances it is obvious that the reliefs cannot be treated as an afterthought since the provision of reliefs is a major part of the overall task. As they represent a vital and integrated facet of the problem, a successful model must therefore cope with shifts and reliefs at the same time.

2.3 Formulation 2

A model incorporating reliefs consists of four main partitions containing respectively α variables corresponding to the α locations for F.T.S. and P.T.S., ϕ variables corresponding to the total alternative F.T.R., ψ variables corresponding to the total alternative P.T.R. and T variables corresponding to the surplus in each of the T periods.

Since α corresponds to the total number of locations for shifts of both categories then

$$\alpha = L_f + L_p . \tag{5}$$

If there are R_f alternative F.T.R. for each F.T.S., then ϕ is given by

$$\phi = L_f . R_f \tag{6}$$

as shifts in the same locations share the same relief possibilities.

Similarly ψ is given by

$$\psi = L_p . R_p . \tag{7}$$

For example in figure 1b, $L_f = 11$, $L_p = 6$ and α is therefore 17. In the same example, if $R_f = 4$ then $\phi = 44$ and if R_p is 2, ψ is 12. In this case T is 18.

Further, β is defined by

$$\beta = \alpha + \emptyset \tag{8}$$

and ω is defined by

$$\omega = \beta + \psi. \tag{9}$$

β therefore corresponds to the total number of locations for F.T.S. and P.T.S. plus the total number of F.T.R. and in the example, $\beta = 17 + 44 = 61$. Similarly ω corresponds to the total number of locations for F.T.S. and P.T.S. plus the total number of both F.T.R. and P.T.R. In the example given, $\omega = 61 + 12 = 73$.

If x_j, $0<j\leq L_f$ is the number of F.T.S. in the jth location for F.T.S., (10)

$x_{L_f+j}, 0<j\leq L_p$ is the number of P.T.S. in the jth location for P.T.S., (11)

$x_{\alpha+j}, 0<j\leq\emptyset$ is the number of the jth alternative F.T.R. included in the schedule, (12)

$x_{\beta+j}, 0<j\leq\psi$ is the number of the jth alternative P.T.R. included in the schedule, (13)

$x_{\omega+j}, 0<j\leq T$ is the number of surplus collectors in period j, (14)

and if C_f = the net contribution made by a F.T.S., (15)

C_p = the net contribution made by a P.T.S., (16)

b_i = the number of collectors required in period i, (17)

$$a_{ij} = \begin{cases} 1 & \text{if the } j\text{th location for F.T.S. includes the } i\text{th period,} \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

$$d_{ij} = \begin{cases} 1 & \text{if the } j\text{th location for P.T.S. includes the } i\text{th period,} \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$

$$e_{ij} = \begin{cases} 1 & \text{if the jth F.T.R. includes the ith period,} \\ 0 & \text{otherwise,} \end{cases} \qquad (20)$$

$$f_{ij} = \begin{cases} 1 & \text{if the jth P.T.R. includes the ith period,} \\ 0 & \text{otherwise,} \end{cases} \qquad (21)$$

$$p_{ij} = \begin{cases} 1 & \text{if the jth F.T.R. is an alternative relief for the ith location for F.T.S.,} \\ 0 & \text{otherwise,} \end{cases} \qquad (22)$$

$$q_{ij} = \begin{cases} 1 & \text{if the jth P.T.R. is an alternative relief for the ith location for P.T.S.,} \\ 0 & \text{otherwise,} \end{cases} \qquad (23)$$

then the model takes the following form:

$$\text{Minimise} \sum_{j=1}^{L_f} C_f x_j + \sum_{j=1}^{L_p} C_p \, x_{L_f+j} \qquad (24)$$

subject to

$$\sum_{j=1}^{L_f} a_{ij}x_j + \sum_{j=1}^{L_p} d_{ij}x_{L_f+j} - \sum_{j=1}^{\phi} e_{ij}x_{\alpha+j} - \sum_{j=1}^{\psi} f_{ij}x_{\beta+j} \qquad (25)$$

$$- \sum_{j=1}^{T} \delta_{ij}x_{\omega+j} = b_i$$

for $i=1\ldots T$,

and

$$x_i - \sum_{j=1}^{\phi} p_{ij}x_{\alpha+j} = 0 \quad \text{for} \quad i=1\ldots L_f, \qquad (26)$$

and

$$x_{L_f+i} - \sum_{j=1}^{\psi} q_{ij}x_{\beta+j} = 0 \quad \text{for} \quad i=1\ldots L_p, \qquad (27)$$

$$x_j \geqslant 0, \; x_j \; \text{integer.} \qquad (28)$$

δ_{ij} is the usual kronecker delta symbol.

The set of constraints (25) states that the shifts supplied must meet the requirements of the demand histogram plus the extra requirements needed for reliefs. The sets of constraints (26) and (27) ensure that for every shift used, a relief is selected from the set of possible reliefs for that shift and included in the solution. The model therefore caters for both aspects of the problem namely, meeting the demands of the histogram and supplying the necessary reliefs.

2.4 Formulation 3.

Formulation 2, due to the sets of constraints (26) and (27), involves a high degree of initial degeneracy. This can be avoided in an alternative formulation which requires an upper bound to be placed on the number of shifts to be scheduled in each alternative location. If the bound is K then the purpose of K is merely to permit a reformulation and K must not be permitted to interfere with the eventual solution. In other words, K must be such that the solutions obtained by formulation 2 and formulation 3 are the same. K must therefore exceed the greatest number of shifts required in any particular location.

Let the total demand, corresponding to the area under the histogram, be D unit periods. Since the demand is D unit periods of collection time, the total number of shifts supplied in the resulting schedule cannot exceed D since otherwise there would be at least one completely redundant shift rostered which contributes nothing to the reduction of demand. Since the total number of shifts cannot exceed D, the number of shifts of either category scheduled in a particular location cannot exceed D. An upper bound $K \geqslant D+1$, is therefore selected and K will always be adequate for the requirements of the solution. As an example K for the histogram in figure 18 is such that $K \geqslant 465 + 1 = 466$.

Let a_{ij}, d_{ij}, e_{ij}, f_{ij}, p_{ij} and q_{ij} be given by (18) to (23) respectively.

Let c_j be given by

$$c_j = -C_f \quad 0<j\leqslant L_f \tag{29}$$

$$c_j = -C_p \quad L_f<j\leqslant\alpha \tag{30}$$

$$c_j = 0 \quad j>\alpha. \tag{31}$$

If f_j is the number of F.T.S. in location j and if p_j is the number of P.T.S. in the jth location for P.T.S., then let x_j be defined by

$$f_j = K-x_j \quad 0<j\leqslant L_f \tag{32}$$

$$p_j = K-x_j \quad L_f<j\leqslant\alpha \tag{33}$$

and x_j, $j>\alpha$ as for formulation 2 and defined by (12), (13) and (14).

The x_j, $j=1\ldots L_f$ are such that x_j corresponds to the jth location for F.T.S. and location j commences in period j and concludes in period $j+d_f-1$.

The $x_{L_f+j}, j=1\ldots L_p$ are such that x_{L_f+j} is associated with the jth location for P.T.S. commencing in period j' and concluding in period $j'+d_p-1$ and

$$j < \underline{j} \Rightarrow j' < \underline{j}'. \tag{34}$$

Further, b_i' is defined by

$$b_i' = \sum_{j=1}^{L_f} a_{ij}K + \sum_{j=1}^{L_p} d_{ij}K - b_i \quad (i=1\ldots T). \tag{35}$$

The problem is now reformulated as follows:

$$\text{Minimise} \quad \sum_{j=1}^{L_f} c_j x_j + \sum_{j=1}^{L_p} c_{L_f+j} x_{L_f+j} \tag{36}$$

subject to

$$\sum_{j=1}^{L_f} a_{ij}x_j + \sum_{j=1}^{L_p} d_{ij}x_{L_f+j} + \sum_{j=1}^{\phi} e_{ij}x_{\alpha+j}$$

$$+ \sum_{j=1}^{\psi} f_{ij}x_{\beta+j} + \sum_{j=1}^{T} \delta_{ij}x_{\omega+j} = b_i' \tag{37}$$

for $i=1\ldots T$,

$$x_i + \sum_{j=1}^{\phi} p_{ij}x_{\alpha+j} = K \quad i=1\ldots L_f, \tag{38}$$

$$x_{L_f+i} + \sum_{j=1}^{\psi} q_{ij}x_{\beta+j} = K \quad i=1\ldots L_p, \tag{39}$$

$$x_j \geq 0, \quad x_j \quad \text{integer}. \tag{40}$$

The right hand sides of (38) and (39) are now non-zero and it will be shown that the use of the bound reduces the number of iterations and leads to further economy by effectively reducing the number of variables.

In considering the use of formulation 3, several requirements emerge as being of primary importance.

(a) It must be possible to select suitable unit periods which accurately reflect the fluctuations in demand and which permit the use of acceptable scheduling arrangements.

(b) Given that suitable unit periods are available, the resulting matrix must be of such dimensions that storage requirements and computer run times are not prohibitive.

(c) The values of variables in the solution must be integer since fractions of shifts and fractions of reliefs are meaningless in practice and satisfactory solutions cannot be obtained by rounding the variables in a fractional solution.

2.5 The Determination of T.

For practical reasons it is necessary to restrict the size of the model as much as possible provided these restrictions do not erode the quality of the solution. An opportunity to remove a number of periods from the twenty-four hour day is available in an interval of low traffic activity in the early hours of the morning. Even at the larger installations such as those administered by the P.N.Y.A. and D.M.R., only three or four collectors are required over this period.

The provision of collectors for this interval is easily accomplished by manual means particularly in the case of the P.N.Y.A. where no collector may start or finish a shift between 1 a.m. and 6 a.m.. In D.M.R. regulations, this restriction is not specified but shifts starting or finishing in this interval are avoided. In the tests conducted, the practice has been to remove the interval entirely by manual means. Since this can be accomplished quickly and economically, there appears to be no justification for its inclusion in the mathematical treatment as the presence of this early morning interval merely increases the size of the problem for no apparent benefit. The means therefore exist to restrict the problem to an 18 or 19 hour period starting at approximately 6 a.m.

It is extremely unlikely that P.T.S. will be considered in the early hours of the morning so the requirements of this interval will be met by F.T.S. and for reasons which will be given later, these shifts should commence as early as possible, ideally at approximately 10 p.m. and concluding at approximately 6 a.m.

2.6 The Determination of τ

The size of a practical problem is very much dependent on the value of τ where τ is the duration of a unit period. If M represents the number of constraints, then M is largely dependent on T, the total number of unit periods in the interval under consideration, L_f, the number of alternative locations for F.T.S. and L_p, the number of alternative locations for P.T.S. Since $T \times \tau = \text{const.}$, T is inversely proportional to the length of the unit period. Inspection shows that if all available locations for F.T.S. are used, then $L_f = T - d_f + 1$. For example in figure (1b), T is 18 and d_f is 8. L_f is therefore equal to 11 as shown so that L_f is also dependent on τ. Similarly L_p is dependent on τ; however, it has been established that all alternative locations for P.T.S. need not be considered.

Intuition suggests that the function of the part time collectors is to assist in intervals of increased activity and to supplement full time collectors in situations where demand cannot be economically satisfied by the use of full time collectors alone. Logically, this implies a need for P.T.S. in the region of the peaks and in the early and late periods of the day where they may act as an extension of a F.T.S. Testing has supported this approach and in practice the available P.T.S. have been restricted to two intervals, generally speaking, in the neighbourhood of the two peaks. P.T.S. are therefore divided into two categories, namely, a.m. P.T.S. in the region of the a.m. peak and p.m. P.T.S. which are located in the neighbourhood of the p.m. peak or in the evening interval.

Since some extra constraints will be required, the need for which will be explained in the description of the algorithm, the total number of constraints M, is given by

$$M = T + L_f + L_p + B \qquad (41)$$

where B is the number of locations for P.T.S. on the a.m. peak, plus 3. As T, L_f and L_p are each dependent on τ and since B can be expected to increase as τ decreases, the number of constraints will increase significantly as τ becomes smaller.

If N is the total number of variables then N also depends on τ since

$$N \approx L_f + L_p + (L_f \times R_f) + (L_p \times R_p) + M. \qquad (42)$$

The total number of relief alternatives for F.T.S., $L_f \times R_f$, accounts for the majority of variables and both R_f and R_p increase as τ becomes smaller. The increase in the number of variables is, however, less important as the structure allows considerable economy in the handling of the variables.

In selecting a suitable value for τ then, three major points must be kept in mind:

(a) τ must be small enough to cater accurately for the changes in traffic volume as they occur throughout the day.

(b) The durations of shifts and the duration of each break in a relief must be a multiple of τ, so τ must be such that satisfactory shift lengths and relief patterns are possible.

(c) As the size of the matrix depends on τ, satisfactory award conditions must be sought which tend to increase τ.

Since different working regulations will cause the size of the matrix to vary, it is not possible to specify M and N exactly for various values of τ, however, Tables I and II give some indication of the dimensions to be expected. For the estimates appearing in Table I, regulations similar to those used by D.M.R. were adopted. The duration of a F.T.S. is 8 hours and the maximum duration of a segment of continuous duty is 2 hours.

The table illustrates how the situation changes from modest dimensions at τ equal to 1 hour, to approximately 160 constraints and over three thousand variables in an extreme example with τ at 15 minutes. In the latter case there is a large degree of repetition in the matrix and it will be shown that the number of variables can be effectively reduced to a small fraction of the apparent total, however, it should be appreciated that such dimensions will prove more expensive as regards computing time.

Although the regulations used by the P.N.Y.A. control the positions of breaks by specifying limits on the three segments of continuous duty, the additional regulation which restricts the position of the meal break leads to a variation in the number of alternative F.T.R. associated with each location. The estimates appearing in Table II have been derived using regulations of the P.N.Y.A. type with F.T.S. of 8 hours and rest breaks as shown in column 2.

Under these restrictions the number of relief alternatives varies for each F.T.S. and depends on the starting time for the shift. For example, Table II shows that a shift beginning at 8 a.m. may have approximately 50% more alternative reliefs than a shift commencing at 6 a.m. The total number of F.T.R. will, if the meal breaks are confined to specified periods during the day, depend upon the conditions imposed although regulations similar to those used by the P.N.Y.A., which are quite liberal, will produce dimensions well within practical limits for τ at 20 mins. It should be pointed out also that the number of multi-break reliefs depends also on the size of the breaks in the reliefs, tending to increase as the breaks increase in size.

For estimation purposes, hourly records of traffic flow are prepared and converted to half hourly estimates by interpolation by both the P.N.Y.A. and D.M.R. The result is adjusted for any anticipated variation and used to produce the estimates of collection requirements. It would therefore appear that an acceptable value for τ would be in the vicinity of 30 minutes and to satisfy all requirements, it is suggested that one of a number of available possibilities be chosen with τ in the neighbourhood of 30 minutes.

The award regulations for toll collectors may specify that shifts contain extra paid time for purposes other than the collection of tolls. Common examples of this include time to count and deposit receipts and time to exchange a uniform for street clothes. D.M.R. uses F.T.S. of a total duration of 8 hours and 40 minutes. This time includes three breaks, the first and third of 30 minutes duration and the second of 40 minutes duration. In addition, the collector is entitled to 11 minutes at the end of the shift to make the final pay-in and change to street clothes. The remaining 6 hours 49 minutes is the time actually spent collecting tolls and from the administrative point of view, the productive time. Any alternative arrangement should therefore produce at least this much working time since, although it is possible to make a small pay adjustment if the work content of a shift is increased slightly, a reduction in work content implies a reduction in pay which, in practice, introduces difficulties. A loss resulting from a reduction in productive time would therefore be borne entirely by the employer and, depending on the extent of the reduction, an alternative involving such a loss would tend to be unacceptable from the administrative viewpoint. The remaining alternatives present such a choice, however, that is it difficult to foresee any problem in reaching a compromise acceptable to all concerned.

The options available for D.M.R. are contained in Table III while those in Table IV are applicable to the P.N.Y.A. It can be seen that a number of these alternatives involve only minor changes to the existing conditions for example, the 4th.alternative in Table III and the 6th. alternative in Table IV. Alternative 5 in Table III is included since a loss of as little as 1 minute is unlikely to involve any extra expense in a practical situation. By considering fractions of a minute, even closer approximations can be obtained. For example, the alternative

1'. 20½-41-20½ 8.12½ 6.50½ pay 1½ mins. 5 mins. approx.

in Table III, or

5¼	$24\frac{3}{4}$-$49\frac{1}{2}$.$49\frac{1}{2}$-$24\frac{3}{4}$	7.50¼	6.30¼	pay ¼ min.	5 mins.approx.

in Table IV could be considered. However, there appears to be ample scope for agreement without the need to include such artificial possibilities.

The question of compatability with previously existing regulations does not arise for P.T.S. The selection of the shift length and the size and location of the break is, within the limitations imposed by the unit period and the needs of the collectors, determined by the particular demand situation.

The need for an adjustment may also exist if τ does not divide evenly into 24 hours. τ in number 3, Table III for example, is 22 minutes. Dividing 24 hours by 22 minutes leaves a remainder of 10 minutes so that a special 10 minute period would have to be inserted into the histogram. The most logical place for its inclusion would be in the interval of least activity, namely at approximately 2 a.m. or 3 a.m. where it will affect the least number of shifts. So that these shifts will start and finish at the beginning of a regular period and at the end of a regular period respectively, 10 minutes overtime may have to be attached to each shift spanning the extra 10 minute period. For D.M.R., three or four shifts would be involved so that an extra 24 minutes would be paid for with overtime at time and a half. The adjustment would cost less than 50c. a day at current rates and compared to the anticipated savings, is negligible.

Alternatively, the payment of overtime could be avoided altogether if a satisfactory adjustment could be made to the collection rate for a short interval at either end of the shifts concerned, the steps taken in this regard depending largely on the conditions existing at the particular facility.

With τ between 20 and 30 minutes, it appears that all requirements can be satisfied as demand can be satisfactorily represented, working regulations can be accommodated and the dimensions of the matrix are well within manageable proportions. Whereas τ could be reduced even further to produce further alternatives, it appears pointless to increase the size of the matrix if a more economical value is available.

2.7 Computer Storage

Although modest in comparison to models of similar problems, for example that formulated by Elias [10] for the bus problem, the anticipated size of the toll problem does raise the question of computer storage and the facility with which the matrix can be accommodated on a computer of moderate size depends on the sparseness and the special features which can be exploited to advantage.

Allowing for minor variations in the model, it is anticipated that the matrix will be approximately 95% sparce as only a limited number of periods will be involved in each relief. Under these circumstances the number of non-zero matrix locations is very modest considering the dimensions involved and since each non-zero element will be unity, only the location of the element and not its magnitude need be stored.

Of greater importance is the structure. The matrix is composed mainly of F.T.R. which exhibit a sequential character. The award regulations will specify a set of possible reliefs for a particular shift (Fig.2) and provided the award does not distinguish between shifts on the basis of starting times, the relief patterns for all F.T.S. will be the same differing only in their location. For example, if shift 2 commences one period later than shift 1, the pattern for 2 will be located one period later than the pattern for 1 but in all other respects the patterns will be the same. Awards which distinguish between shifts on the basis of starting times will also exhibit this sequential characteristic with the difference that some of the alternatives will be invalid.

This feature is important and the storage of all reliefs becomes unnecessary since the basic pattern will suffice for all shifts, an adjustment to the pattern being all that is required for each successive location. Using this technique, all information concerning the matrix may be stored in a few hundred words. This system requires that the matrix be regenerated for each iteration; but with further economies, this regeneration assumes little importance.

CHAPTER 3

THE ALGORITHM

3.1 Integer Programming Algorithms

In view of the integer requirement, the most obvious method of solution requires the use of an integer programming algorithm. The anticipated dimensions of the matrix are such, however, that economical run times are unlikely in practice since the capabilities of the available integer algorithms are extremely limited. Trauth and Woolsey [18] suggest that the maximum dimensions under favourable conditions are in the vicinity of 100×100 and because the problem under discussion will usually contain in excess of a thousand variables, the prospects for the successful use of integer programming methods are poor.

Gomory's first algorithm [12], aided by a number of special constraints, showed some promise in tests and solutions to comparatively small problems containing some 40 constraints and approximately 120 variables were obtained in a matter of a few seconds. This performance is due largely to the fact that it is possible to specify efficient a priori bounds leaving the algorithm merely to seek out from a limited number of alternatives a solution which satisfies the integer requirement.

Whereas it is always possible to obtain efficient general bounds, the latter are not necessarily immutable and where the situation requires a variation in these bounds, the integer programming approach tends to become much less efficient. Increased selectivity in the order of application of the cuts improves run times but performance becomes generally erratic as the size of the problem increases and this, together with the difficulties associated with increasing numbers of large Gomory constraints, rules out the approach as a practical proposition. Since no satisfactory integer algorithm is available, a method, specifically applicable to labour scheduling, has been developed which deals effectively with the integer requirement and achieves the required objectives.

3.2 The Method of Solution

Based on the Revised Simplex Algorithm, the procedure is dynamic and includes a number of phases, the action taken in each depending on the results in previous phases. Successive solutions are generated, each containing additional information which is used to constrain subsequent solutions but at no stage is the procedure isolated from the requirements of the overall problem.

Essentially the method may be divided into three main sections consisting of 8 phases:

(1) Iterate to an initial optimal solution in phase 1.

(2) Constrain general features of the solution in phases 2 to 4.

(3) Constrain particular features of the solution by

the manipulation of individual variables in phases 5 to 8.

Because of the excessive number of iterations required for the degenerate formulation 2, formulation 3 is used and the discussion of the techniques applied refers to this formulation only. The Revised Simplex Method was modified to take advantage of the economies in storage already described and the further substantial economies in computation offered by the matrix.

3.3 Economies in Computation

The sequential nature of the reliefs can also be exploited in the search for the incumbent variable as an overlap exists in the reliefs for adjacent shifts. A relief for one shift may, because of the latitude contained in the regulations, be a legitimate relief for an adjacent shift so that a collector may find that his breaks coincide with those of a collector who commences his shift earlier or later.

Let a_{ij}, $i=1...M$, $j=1...N$, be the coefficients in the original matrix, u_i, $i=1...M$, be the elements in the cost row of the inverse and c_j, $j=1...N$, the costs assigned to the variables in the objective function. In terms of the Revised Simplex method, calculations of the type

$$z_j = \sum_{i=1}^{M} a_{ij}u_i - c_j \tag{43}$$

are needed to identify the incumbent variable.

Since attention is temporarily restricted to those variables related to the F.T.R., (43) may be rewritten as

$$z_j = \sum_{i=1}^{T} a_{ij}u_i + \sum_{i=T+1}^{T+L_f} a_{ij}u_i - c_j. \qquad (44)$$

As the costs applicable to F.T.R. are zero, the cost, c_j, may be discarded. If variable x_j corresponds to a relief contained in the set for location d, (44) becomes

$$z_j = \sum_{i=1}^{T} a_{ij}u_i + u_{T+d} \qquad (45)$$

or

$$z_j = S_j + u_{T+d} \qquad (46)$$

since $a_{T+d,j}$ will equal 1 and a_{ij}, for all other values of i in the second term, will equal 0.

If z^* is the current minimum, it is necessary to determine

$$\text{Min.}(z^*, z_j)$$

or

$$\text{Min.}(z^* - u_{T+d}, S_j).$$

Let the variable x_p correspond to a relief contained in the set for location e, then

$$z_p = \sum_{i=1}^{T} a_{ip}u_i + u_{T+e}. \qquad (47)$$

If the breaks in the reliefs represented by x_j and x_p correspond, then

$$\sum_{i=1}^{T} a_{ip}u_i = \sum_{i=1}^{T} a_{ij}u_i = S_j \tag{48}$$

since $a_{ip} = a_{ij} = 1$ for the same values of i. For the pth variable it is necessary to determine

$$\text{Min}(z^*, z_p)$$

or
$$\text{Min.}(z^*-u_{T+e}, S_j).$$

Provided that the adjustment implied by z^*-u_{T+e} is made when the set of reliefs belonging to location e are encountered, it is not necessary to calculate z_p as the criterion required to determine whether x_p is the possible incumbent variable, namely S_j, is already available.

In cases where this repetition does not apply, substantial savings in computation may also be made if reliefs are grouped into subsets differing in say, only one break. With τ at 20 minutes and using a 1-2-1 relief with breaks excluded from the first and last hours of a shift, each F.T.S. has thirty three alternative reliefs if the maximum segment of continuous duty is 2 hours (Fig.2). z_j is calculated for all reliefs for the first location but for each subsequent location, 18 of the reliefs require no calculation at all and only 6 of the 33 need be calculated in full. In the example in figure 2 the alternatives have been reordered according to computational requirements.

As the relief patterns become larger, the repetitive element tends to increase with the result that the time taken to determine the next entrant to the basis is considerably reduced and run time is much less dependent on the number of variables. The time required to locate the incumbent may be further reduced as a_{ij} in (43) is always 1 or 0, so that the need to multiply is eliminated.

Additional savings follow from the use of the bound K. If K is large it can be expected that x_j, $j=1....\alpha$ will be much greater than zero after being initially introduced into the basis. Since it is unlikely that these variables would be removed from the basis at a later stage, except in phases 5 to 7, where they are removed permanently, they may be subsequently ignored as possible incumbents provided a simple check is made at each iteration. In practice therefore, the matrix is further reduced by α variables.

In all integer programming problems, efficiency is greatly enhanced if bounds can be identified which will locate an initial solution in the vicinity of the required integer solution. The failure of generalised integer algorithms to produce this bound and so effectively discard a large proportion of the unacceptable solutions, contributes to their disappointing performances. Fortunately, in practice, an efficient bound may sometimes be computed by the use of methods related to the particular problem and such a situation exists in the toll problem. The earlier phases therefore establish a number of general but necessary conditions which are, in no small way, responsible for the efficiency of the method.

3.4 Phase 1 - The Initial Solution

Initially the integer requirement is disregarded and an optimal solution is obtained where x_j, $j=1...N$, may take any non-negative values. The non-zero costs in phase 1 are those related to the shifts only and since the concept embodied in formulation 3 involves an initial oversupply of K shifts in each location, formulation 3 approaches the optimum by removing surplus shifts. The costs c_j, $j=1...\alpha$, therefore refer to the advantage to be gained in removing a shift rather than the actual cost incurred in its use, which is given by $-c_j$, and it is necessary to appreciate this when considering the effects of the cost changes made in subsequent phases.

Using the constraints (37), (38) and (39), phase 1 results in a solution containing the minimum paid time or alternatively, a solution containing the minimum number of spare periods. This solution is invariably fractional and merely provides a lower bound. Succeeding phases then impose the conditions necessary for an integer solution.

3.5 Phase 2 - The Determination of Idle Time

Although spare periods have been minimised in phase 1, this minimum level does not necessarily represent the minimum number of spare periods present in an optimal integer solution and, depending upon the circumstances, it may be necessary to increase the spare time content.

If an integer solution is to be obtained, some integer number of F.T.S. and some integer number of P.T.S., say F and P respectively, must be selected so that

$$D \leqslant C^* \leqslant F.C_f + P.C_p \qquad (49)$$

where D is the total demand in unit periods given by the demand histogram and C^* is the total contribution in unit periods in the initial solution in phase 1. C^* therefore provides a lower bound and if w represents the number of spare periods, then (49) may be rewritten as

$$D + w = C = F.C_f + P.C_p \qquad (50)$$

where $w \geqslant 0$ is as small as possible and $C \geqslant C^*$ represents the contribution in unit periods corresponding to a number F of F.T.S. and a number P of P.T.S. .

If H is the H.C.F. of C_f and C_p and

$$C_f = H.G \qquad (51)$$

and $$C_p = H.Q \qquad (52)$$

then from (50)

$$D + w = C = H(F.G + P.Q) \qquad (53)$$

$$= HI' \qquad I' \in I \qquad (54)$$

where I is the set of all integers.

The number of periods supplied must therefore be an integer multiple of H if an integer solution is to be obtained. As an example, suppose that D is 365 unit periods, C^* is 366 unit periods, C_f is 12 unit periods and C_p is 8 unit periods. The H.C.F. of 12 and 8 is 4, so that $C = D+w$ must be a multiple of 4; i.e.

$$365 + w = 4 \times I'.$$

The smallest possible w is selected and C is given by

$$C = 365 + 3 = 4 \times 92 = 368.$$

The number of units supplied must therefore be at least 368 unit periods and the integer solution must contain at least 3 spare periods.

The solution is constrained to include w spare periods by the use of the first of the additional constraints viz.

$$\sum_{j=1}^{T} x_{\omega+j} = w. \qquad (55)$$

In practice, this constraint must, until this stage, remain inoperative. Initially the right hand side of (55) is given a large value greater than D and the constraint is satisfied by the use of an extra slack available at zero cost. To introduce the required number of spare periods, the right hand side of (55) is changed to w and the extra slack is given a high prohibitive cost to expel it from the solution. Such action necessarily renders the current solution infeasible and the Simplex iterative process must be recommenced to restore feasibility and to produce a solution supplying C unit periods of labour and containing w spare periods.

3.6 The Determination of the Part Time Component

After introducing the minimum number of spare periods consistent with existing requirements, a further condition must be satisfied by the solution, namely that the sum of the F.T.S. and the sum of the P.T.S. in the schedule, must each be integer. Figure 6 shows that if the F.T.S. used are plotted against the P.T.S. used, then for any value of C, a line may be drawn representing the combinations of the two categories which yield C unit periods. Further, if C is a multiple of H, the line will intersect grid points representing integer combinations and one such point must be selected where a feasible solution may be obtained. It has already been established that at least one combination exists which permits a feasible solution and which contains a contribution of C^* unit periods. It becomes necessary to determine the effect of an increase in the contribution on the availability of feasible combinations.

<u>Theorem</u>

If there is one combination which yields a feasible solution supplying C^* unit periods, then there is an interval on the line of combinations corresponding to C which yields feasible solutions if $C > C^*$.

<u>Proof</u>

Suppose the feasible solution supplying C^* unit periods contains P' P.T.S. and F' F.T.S.

An increase in the contribution from C^* to C unit periods is required so that X F.T.S., each with a net contribution of C_f unit periods, are added to the solution containing C^* unit periods and X is defined by

$$X = (C - C^*)/C_f. \tag{56}$$

Because the additional shifts are attached to an already feasible solution, their entire contribution is redundant and they consist entirely of spare periods any of which may be assigned to relief breaks. The augmented solution is therefore feasible and contains C unit periods. If $F'' = F' + X$, the augmented solution corresponds to the point P', F'' in figure 4.

Similarly by adding P.T.S., an augmented solution may be obtained at P'', F'. Any point P, F on the line representing C such that $P' < P < P''$ and $F'' > F > F'$, merely represents the situation where the contribution is increased to C unit periods by the addition of both F.T.S. and P.T.S. and is also a feasible point so that a feasible interval exists (Fig.4).

By continuing this argument, it may be shown that as C is increased, the length of the feasible interval increases and hence the number of feasible combinations also increases.

It follows that if an integer solution containing C unit periods is to be obtained, there must exist on the corresponding line a feasible point P,F, where P and F are integers, or a feasible interval such that $P' \leq P \leq P''$ and $F'' \geq F \geq F'$, where the points P', F'' and P'', F' are the extremities of the feasible interval (Fig.5). If no such point exists then C must be increased by H unit periods and the integer points corresponding to the increased value of C examined.

The possibility also exists that while feasible points may be available, none represents an acceptable part time component. If such is the case, C must be increased as the range of feasibility must be increased to include the required ratio.

3.7 Integer Points and Working Regulations

The prevalence of integer points for a particular value of C also depends on H. Given any combination of P.T.S. and F.T.S., other combinations may be derived from it while C remains constant provided that part time and full time contributions are exchanged in fixed proportions. Because C_f/C_p P.T.S. may be exchanged for 1 F.T.S. with C constant, one integer point may be derived from another if the exchange involves an integer number of F.T.S., say J, such that

$$I^* = J.C_f/C_p \quad I^* \in I. \tag{57}$$

Since $$I^* = J.\frac{G.H}{Q.H} \tag{58}$$

by (51) and (52) and since the smallest J for which this holds is Q, it follows that Q F.T.S. may be exchanged for G P.T.S.

Because $Q = C_p/H$, the minimum number of F.T.S. permitted in an integer exchange will be large if H is small so that the length of the interval between neighbouring integer points on the line corresponding to C is inversely proportional to H.

The distribution of the integer points therefore depends on the H.C.F. of C_f and C_p which in turn is determined by the working regulations. If the H.C.F. is H, then, commencing at the line corresponding to C given by (50), every Hth. line will include integer points. If H is large then such lines will be encountered less frequently as C is increased but in this case the line will contain a comparatively large number of integer points (Fig.6). If H is small the lines will be encountered more frequently but each line will contain relatively fewer integer points (Fig.7).

While the dispersal pattern suggests that a larger number of spare periods may be necessary if H is large, in practice the same amount of wastage is usually encountered if H is small since there are relatively fewer points to choose from at a given value of C and a point representing the required level of part time participation may only be available if C is increased.

3.8 Phase 3 - The Initial Schedule

In phase 3, the objective in the initial run is to select an integer point which establishes an acceptable ratio of part time to full time work in toll facility schedules. Subsequent schedules must seek to minimise the variation in this ratio since any radical departure from it entails dismissal for collectors in one category and their replacement by collectors in the other. Whereas wastage and growth will provide opportunities for the gradual variation of the ratio, the initial establishment of the part time component will largely determine the composition of the work force if subsequent disruptions are to be avoided. Under the circumstances a careful appraisal of the whole situation is indicated and all feasible integer points may be required in the initial run, probably for a number of values of C. The procedure used initially will therefore vary from that used for subsequent schedules.

To locate the available integer points, all integer points associated with C and their positions relative to the feasible interval are determined.

$$\text{If} \quad F^* = \left[\frac{C}{C_f}\right] \tag{59}$$

where the brackets indicate "integer part of", then it is impossible to obtain an integer solution supplying C unit periods and containing more than F^* F.T.S. To calculate the required integer combinations, all combinations containing i F.T.S., where i is integer and $0 \leq i \leq F^*$, are derived using

$$i' = (C - iC_f)/C_p \tag{60}$$

where i' represents the number of P.T.S. associated with i F.T.S. and a total contribution of C unit periods. Where an integer value for i produces an integer value for i' in equation (60), i' and i represent an integer combination containing i'P.T.S. and i F.T.S. and if there are q such points $P_r, F_r, (r=1...q)$, those lying in the feasible interval remain to be identified.

The extremities of the feasible interval are the points representing feasible solutions containing respectively the maximum and the minimum P.T.S. and contributing a total of C unit periods. These extremes are therefore determined by varying the costs associated with the P.T.S.

and generating each of the required solutions.

In phase 1, the costs assigned to the two categories of shifts are based solely on the net contributions made by each so that no preference is established. If costs c''_j are now assigned to each of the variables associated with P.T.S. so that

$$c''_j = c_j + u \quad j = L_f+1 \ldots . L_f+L_p \tag{61}$$

where the c_j are the previous costs and u is a small positive constant, then the balance is upset and a preference for P.T.S. is indicated since the cost per unit period for part time labour is now less than that for full time labour. By recommencing the iterative process a solution will be obtained containing P'' P.T.S., the maximum part time component consistent with feasibility and a total contribution of C unit periods. If the number of F.T.S. contained in the solution is F', then the point P'',F' represents one extremity of the feasible interval.

Similarly, by making a cost change of the form

$$c'_j = c_j - u \quad j = L_f+1 \ldots . L_f+L_p \tag{62}$$

a solution may be obtained containing P' P.T.S., the minimum part time component.

The points P_r, F_r such that $P' \leq P_r \leq P''$, are then the required feasible integer points and if the need to consider further possibilities arises, then C must be increased by H and the process repeated. Figure 8 illustrates the path taken by a solution where $H=4$, $C_f=12$ and $C_p=8$. The feasible integer points have been examined for two values of C, namely 84 and 88.

From the alternatives produced, a choice is made and a further constraint must be introduced to fix part time participation at the chosen level. If P represents the number of P.T.S. required in the solution, the constraint necessary is

$$\sum_{j=1}^{L_p} x_{L_f+j} = (L_p.K) - P. \qquad (63)$$

The procedure used to incorporate this constraint is the same as that used for constraint (55) and by re-starting the iterative process, a feasible solution will be obtained incorporating the selected ratio.

Since

$$C = F.C_f + P.C_p \qquad (64)$$

and since C and P are now constrained, F, the number of F.T.S. in the solution, is also effectively constrained although no constraint is specifically assigned this function. A feasible solution will therefore be available containing a fixed level of spare time and a fixed proportion of full time and part time work and on the application of the remaining phases, will lead to the required solution. If stored, the solution obtained at phase 3 may serve an additional purpose since it provides an initial solution for the subsequent schedule which will be required to meet an altered demand at some time in the future.

3.9 Phase 3 - Subsequent Schedules

To maintain the ratio of part time to full time shifts, a subsequent schedule must utilize one of a number of alternatives represented by integer points in close proximity to P,F, the current point. For example, if an increase in demand must be catered for, the three closest points may entail respectively:

(1) The addition of a P.T.S.

(2) The replacement of a P.T.S. by a F.T.S.

(3) The addition of a F.T.S.

Combinations of these alternatives will produce further possibilities so that a list of priorities may be established and investigated which supply the required increase in labour for the new schedule and which are associated with integer points in the neighbourhood of P,F. Whereas it is not possible to provide a useful preferred solution for the first schedule, the subsequent run may use the solution at P,F produced by phase 3 of the initial run. In this case, computation must be commenced by altering the right hand sides of the relevant constraints in (37) to accommodate the changes in demand in the preferred solution. The first priority is then selected and w and P, the levels of spare periods and P.T.S. respectively, are adjusted by modifying the right hand sides of constraints (55) and (63). An attempt is made to generate a feasible solution at the selected point but if none is available, then by suitably manipulating constraints (55) and (63), an ordered investigation of the integer points in the priority list is then conducted until a feasible solution is obtained. For example with $C_f=12$, $C_p=8$ and $H=4$ unit periods, figure 9 shows the alternative points in the vicinity of (2,6) which appears in figure 8. By increasing C by 4 and exchanging a F.T.S. for a P.T.S., the point (1,7) may be reached. By adding one P.T.S., the point (3,6) may be reached, representing an increase of 8 unit periods. Similarly, the point (2,7) represents the addition of one F.T.S. and the addition of 12 unit periods.

After the first schedule has been produced, the procedure may then bypass phases 1 and 2 through the use of the preferred solution available from the previous run. The steps to be followed in each case are shown in figure 10.

3.10 Phase 4 - The Distribution of P.T.S.

One further general condition concerning the approximate positioning of the P.T.S. remains to be imposed. As previously described, P.T.S. must belong to one of two categories, namely a.m.P.T.S. or p.m.P.T.S. Since an integer solution must contain an integer number of each category, a further constraint is required to accomplish this and phase 4 limits the participation of the a.m.P.T.S to some acceptable integer number. As in phase 3, alternatives normally exist and at least one extremity of the range of alternatives must first be determined. For present purposes it is assumed that the number of a.m.P.T.S. is to be minimised and a cost change is used to determine A*, the minimum integer consistent with feasibility.

If the first $\underline{a}$ of the L_p locations for P.T.S. are related to the a.m.P.T.S., the altered costs are given by

$$c'_j = c_j - u \quad j=L_f+1 \ldots L_f+\underline{a} \tag{65}$$

where c'_j is the new cost, c_j is the previous cost and u is a small positive constant. The result of the cost change is that the cost of the a.m.P.T.S. is now greater than that for p.m.P.T.S. and by recommencing the Simplex iterations, a solution will be obtained containing the minimum number of a.m.P.T.S. If A' is the required minimum and if $A' \in I$, then $A^*=A'$. If $A' \notin I$, then A^* is given by

$$A^* = [A'] + 1 \tag{66}$$

and the required constraint is therefore

$$\sum_{j=1}^{a} x_{L_f+j} = (\underline{a}K) - A^*. \qquad (67)$$

Because the total number of P.T.S. is constrained to be integer and because the sum of the a.m.P.T.S. is integer, the sum of the p.m.P.T.S. will also be integer.

3.11 Weighted Costs

The restrictions imposed prior to and during phase 4, while necessary, are not sufficient to produce an integer solution and a large number of alternative solutions may exist from which an integer solution must be extracted. Since, at this stage, all useful general conditions have been specified, further restrictions must necessarily be applied to individual variables. Before the latter may be constrained however, some prior knowledge of their likely integer values must be obtained by observing the response to certain cost changes.

Suppose that a change is applied to the cost of a variable x_i, $i \leqslant L_f$, with the object of maximising, subject to the existing conditions, the participation of F.T.S. in the corresponding location. Let it be assumed also that the maximum participation in this location is given by

$$f_i = V^*, \quad V^* \notin I. \qquad (68)$$

Since V^* is a maximum and since it is fractional, then, provided the existing conditions remain constant, it can be assumed that the value of f_i in the eventual integer solution will be such that

$$f_i = V' \leqslant [V^*] \quad V' \in I. \tag{69}$$

As it is possible that V' may take any integer value from 0 to $[V^*]$, the situation suggests the use of an enumerative procedure where all possible values for all variables may be investigated. Enumeration implies an unrealistic level of computation and although the method used is related to this concept, the structure of the problem is such that a high degree of interdependence exists between the variables with the result that the testing of numerous alternative values is quite unnecessary. To further simplify the process, weighting may be applied to the costs of variables in groups so that in practice, the required integer values may be determined with relatively little computation.

The approach is best illustrated by the use of the following example. Suppose a weighting is applied to the costs so that the cost of using a F.T.S. increases as i increases where i represents the location. Suppose also that if f_t represents the number of F.T.S. in location t in the resulting solution, f_t is given by

$$f_t = K - x_t = X_t', \quad X_t' \notin I \tag{70}$$

and $f_i \in I$ for $i<t$.

Since the value of f_t indicates the use of a fractional shift in location t, the current solution is unacceptable in practice and after constraining the $f_i, i<t$, to their current integer values, some integer alternative for f_t must be selected. As the cost structure induces the earliest possible commencement, it is assumed that there is no alternative feasible solution containing $f_t > X_t^{\prime}$, so that the integer value chosen for f_t is $[X_t^{\prime}]$. f_t is constrained to $[X_t^{\prime}]$, an alternative feasible solution complying with the additional constraint is then sought by the use of the Simplex Algorithm and the f_i, $i>t$, are examined for further fractions.

The desired practical result of the application of the weighted costs is that a solution will be obtained containing an integer allocation of F.T.S. in the earliest possible locations. The result of the weighting is, however, that the integer objective is exceeded since although it may not be possible for a shift to advance from location i to location i-1 say, it may be possible for a part of the shift to do so. As there is no way of controlling this through the costs, subsequent action must be taken to correct the excesses. The nature of the bias makes it possible to anticipate the corrective action necessary and by constraining the biased variables in order, fractional shifts and reliefs are excluded from their appointed locations. By prohibiting their presence in successive locations, the fractional parts of shifts and reliefs are therefore forced to coincide to form integer shifts and reliefs. In the example given, the earliest locations are constrained first and any fractional F.T.S. are relegated to later locations. If fractional shifts are encountered, the number of F.T.S. permitted in the particular location is reduced to the nearest lower integer and although the fractional part is driven from the location, no attempt is made to specify an alternative location. The Simplex procedure merely seeks out any alternative solution which complies with the additional constraint. In this example therefore, all locations possible for fractional F.T.S. are successively eliminated and by applying the technique to the various elements of the schedule in turn, an integer solution is obtained.

3.12 Implied Constraints

Whereas the necessity to constrain individual variables suggests an unrealistic increase in the dimensions of the problem, no such difficulty exists in practice. The effect of scheduling a F.T.S. shift in location $\underline{i}$ for example, is that demand is reduced by 1 in each of the periods i such that $\underline{i} \leq i \leq \underline{i}+d_r-1$. On the other hand, reliefs may be regarded as additional demands and the inclusion of a particular relief in a solution may be represented by increasing the demand by 1 in each of the relevant periods corresponding to the rest breaks. A variable may be removed from the solution therefore, and its subsequent presence implied by altering the initial demand. By using this technique, it is possible to constrain a variable to any desired value by excluding it entirely from the basis and implying its presence at the required level. Such constraints will be referred to as Implied Constraints and provided that the object is to restrict a single variable to a selected value, no formal constraint is then required.

As an example, suppose a variable x_j, corresponding to a P.T.R. is such that

$$x_j = X_j' \qquad X_j' \notin I. \tag{71}$$

Let it also be assumed that it is necessary to constrain x_j so that

$$x_j = X_j = [X_j']. \tag{72}$$

Let

$$I_j = \{i : f_{ij} = 1, \quad i \leq T\} \tag{73}$$

where the f_{ij} are given by (37).

Since the presence of a relief is to be implied, the initial demand is increased. The alterations to the right hand sides of the first T constraints are given by

$$b_i'' = \sum_{j=1}^{L_f} a_{ij}K + \sum_{j=1}^{L_p} d_{ij}K - b_i - X_j \qquad i \in I_j \quad (74)$$

or
$$b_i'' = b_i' - X_j \qquad i \in I_j \quad (75)$$

where b_i'' is the new value of the right hand side for constraint i.

In addition, since the presence of X_j reliefs will be implied and since these reliefs will not appear in the basis, a corresponding adjustment must be made. For example, if the reliefs are associated with the P.T.S. in location r, then the constraint maintaining the relationship between location r and its corresponding relief alternatives must be relaxed so that the rth. constraint in (39) viz.

$$x_{L_f+r} + \sum_{y=1}^{R_p} x_{\sigma+y} = K \quad (76)$$

becomes

$$x_{L_f+r} + \sum_{y=1}^{R_p} x_{\sigma+y} = K - X_j \quad (77)$$

where $\sigma = \beta + R_p(r-1)$ and β is given by (8).

If the variable x_j is expelled from the basis, subsequent solutions will contain exactly X_j of the corresponding reliefs.

Similar techniques may be used to constrain the shifts and the situation applicable to the F.T.S. is taken as an example. Assume that x_r, corresponding to the shifts in location r is such that

$$x_r = K - f_r = X_r + \rho \quad X_r \in I, \; 0<\rho<1 \tag{78}$$

where, since formulation 3 applies, f_r, the number of shifts actually used, is given by the difference between the bound K and x_r.

Since K is integer, by (78)

$$f_r = X_r' + \rho' \quad X_r' \in I, \; 0<\rho'<1 \tag{79}$$

so that if f_r is to be constrained to the integer X_r', then x_r, by (78) and (79) must be such that

$$x_r = K - X_r' = X_r + \rho + \rho' \tag{80}$$

and since from (78) and (79), $\rho + \rho' = 1$, then

$$x_r = X_r + 1. \tag{81}$$

In other words, if X_r' F.T.S. are required in location r, the variable x_r must be constrained to the next highest integer.

Let

$$I_r = \{i : a_{ir} = 1, \; i \leq T\} \tag{82}$$

where the a_{ir} are given by (37).

Since x_r and not f_r is constrained in practice, the necessary alterations involve a reduction to the right hand sides given by

$$b_i'' = \sum_{j=1}^{L_f} a_{ij}K + \sum_{j=1}^{L_p} d_{ij}K - b_i - (X_r+1) \quad i \in I_r \qquad (83)$$

or $$b_i'' = b_i' - (X_r+1) \quad i \in I_r. \qquad (84)$$

Again the relationship between the shifts and their reliefs must be relaxed and the rth. constraint in (38) viz.

$$x_r + \sum_{y=1}^{R_f} x_{\sigma+y} = K \qquad (85)$$

becomes

$$x_r + \sum_{y=1}^{R_f} x_{\sigma+y} = K - (X_r+1) \qquad (86)$$

where $\sigma = \alpha + (r-1)R_f$ and α is given by (5).

Where shifts, for example the a.m.P.T.S., are constrained by any of the auxiliary constraints, these also must be relaxed. To maintain its implied value at X_r+1, x_r is excluded from the basis and X_r' F.T.S. in location r will be contained in the schedule.

Since spare periods may also be constrained in this manner, the means exist to constrain any variable x_j (j=1...N), to any desired integer value and phases 5 to 7 proceed to constrain all variables corresponding to the shifts and all variables related to the a.m.P.T.R. Constraining the latter is advisable since the sharing of the integer time allocation by a.m.P.T.R. and F.T.R. is prevented and earlier termination may result. Figures 16a to 16f illustrate by means of Gantt type charts the gradual approach to an integer solution, commencing with the solution obtained at phase 4 and concluding with an integer solution at phase 8.

3.13 Phase 5

The first of the weighted costs are applied to a.m.P.T.R. and it is assumed that the latter consist of 1 break of 1 or more unit periods.

If

$$J^i = \{j : x_j \text{ corresponds to a possible relief for the P.T.S. in location } i\} \quad (87)$$

and if

$$J_i = \{j : x_j \text{ corresponds to a P.T.R. commencing in period } i\} \quad (88)$$

then the costs c_j, are given by

$$c_j = \sum_{i=1}^{b} (u - iv) + av \quad j \in J^a \cap J_b \quad (89)$$

where the relief is associated with the P.T.S. in location a, and b is the starting period for the relief. u and v are small positive constants selected such that $u > (a+b)v$, for all a and b so that $c_j < c_{j'}$, for all $j \in J_b$ and all $j' \in J_{b+1}$.

The purpose of this cost function is to ensure that all a.m.P.T.R. and, by implication, all a.m.P.T.S., commence as early as possible and that any time available for P.T.R. is allocated to the earliest P.T.S. not already containing a relief. The subsequent procedures are therefore designed to relegate fractional P.T.S. and their reliefs to later periods. Further, since the possibility arises that if a relief is advanced from period j to period $j-1$, a later relief may be displaced from period j' to period $j'+1$, where $j < j'$, the costs are generated such that

$$|c_j - c_{j'}| > |c_j - c_{j''}| \quad j' \in J_{b-1},\ j \in J_b,\ j'' \in J_{b+1}. \qquad (90)$$

The number of alternative locations for a.m.P.T.S., namely $\underline{a}$, will vary with the circumstances but experience has shown that if a' is the period of highest demand on the a.m. peak, then locations for P.T.S. commencing in each period t should be considered where $t < a'$, so that generally $\underline{a} = a'$. For the first $\underline{a}$ periods then, the commencement of both categories of shifts must be considered and the techniques described in phase 5 are applied successively to the periods 1 to $\underline{a}$. The first $\underline{a}$ locations for F.T.S. represented by x_j, $j=1\ldots\underline{a}$ and the first $\underline{a}$ locations for P.T.S. represented by x_j, $j=L_f+1\ldots L_f+\underline{a}$, are therefore involved. The procedure is divided into three steps.

1. Because of the possible flexibility remaining in the solution initially, greater precautions are required in the earlier periods and auxiliary constraints are used to initiate the process for each period by constraining the sum of the shifts commencing in the period to some suitable integer. If p_i and f_i represent the numbers of P.T.S. and F.T.S. commencing in period i, then let

$$p_i + f_i = s_i = X_i' . \qquad (91)$$

This sum is first constrained to $X_i = [X_i']$ by the use of the constraint

$$x_i + x_{L+i} = 2K - X_i . \qquad (92)$$

If $X_i \neq X_i'$, subsequent use of the iterative process will produce a solution complying with the extra constraint, but if $X_i = X_i'$, no Simplex iterations will be necessary in step 1.

2. If, following step 1, p_i is such that $p_i = X_p'$, the number of P.T.S. must be constrained to $X_p = [X_p']$ and since only one variable is involved, an implied constraint is used. If $X_p \neq X_p'$, the iterative process must be recommenced before proceeding to step 3. Because s_i and p_i are now constrained to integer values, f_i, the number of F.T.S. commencing in i, must also assume an integer value.

3. To exclude the possibility that fractional P.T.R. may commence in period i, these also are examined. If

$$r < s \Rightarrow j < j' \qquad j \in J^r,\ j' \in J^s \tag{93}$$

the first relief to be investigated in period i will be that associated with the earliest shift and is represented by the variable x_r where

$$r = \text{Min.}(j : j \in J_i). \tag{94}$$

If $x_r = X_r'$ and if $X_r' = [X_r']$, then x_r is implicitly constrained to X_r' and the variable x_p such that

$$p = \text{Min.}(j : j \in J_i, j > r) \tag{95}$$

is then examined, the process continuing until either all $x_j, j \in J_i$, have been constrained, in which case the procedure is applied to period $i+1$, or a fractional relief is encountered.

If $X_r' \neq [X_r']$, x_r is constrained to $X_r = [X_r']$ and all x_j, $j \in J_i, j > r$, are constrained to 0 since

$$X_r' \neq [X_r'] \Rightarrow x_j = 0 \qquad j \in J_i, j > r. \tag{96}$$

Again the Simplex iterative procedure is used only if $X_r' \neq [X_r']$.

The result of phase 5 is that all P.T.S. and F.T.S. together with all P.T.R. commencing prior to period $\underline{t}$ where $\underline{t} = \underline{a} + 1$, may only exist in the solution as integers (Fig.16c). Since the supply of labour reduces demand and since a part of this supply is now fixed, phase 5 has effectively reduced the time span of the problem with the result that subsequent phases are in fact dealing with a reduced problem requiring less sophisticated treatment. As experience has shown that there is no further need for additional formal constraints, none are used and all remaining variables are implicitly constrained, commencing in phase 6 with those related to the remaining F.T.S. . The procedure for phases 4 and 5 is summarised in figure 11.

3.14 Phase 6.

As before, phase 6 involves the application of weighted costs, in this case to the remaining F.T.S. contained in the locations represented by x_j, $j=\underline{t}\ldots L_f$, and as before, subsequent action is needed to correct the effects of an oversatisfied objective. As well, any remaining a.m. P.T.R. are also constrained.

If

$$c'_{\underline{t}} = c_{\underline{t}} \tag{97}$$

where $c_{\underline{t}}$ is the previous cost of $x_{\underline{t}}$, then the costs applicable in phase 6 are given by

$$c'_j = c'_{j-1} - (g-j)e \qquad j = \underline{t}+1\ldots L_f \tag{98}$$

where c'_j is the altered cost of x_j, e is a small positive constant and $g = L_f+1$. Again the costs are generated so that $|c_j - c_{j-1}| > |c_j - c_{j+1}|$. Since the actual cost of a shift is $-c_j$, it can be seen that the weighting is such that F.T.S. commence in the earliest possible locations. Under the circumstances, the corrective procedure for phase 6 begins at period $\underline{t}$ and continues period by period to L_f, constraining all x_j, $j=\underline{t}\ldots L_f$ and relegating fractional shifts where necessary to later locations. Let $f_i, i = \underline{t}\ldots L_f$, be the number of F.T.S. commencing in period i and let t^* represent the latest period in which the latest a.m.P.T.R. may commence.

If $f_i = X_i'$, then f_i is implicitly constrained to $X_i = [X_i']$. Again, if $X_i \neq X_i'$, Simplex iterations are necessary to produce a new solution.

If $i \leq t^*$, then any P.T.R. commencing in i are also constrained as in phase 5. The situation at the conclusion to phase 6 is illustrated in figure 16d.

3.15 Phase 7

The shifts remaining to be constrained are the p.m.P.T.S. and the procedure followed in phase 7 to constrain them is similar to that used in phase 6.

If $q = L_f + \underline{t}$ and since $\alpha = L_f + L_p$, then all variables x_j where $j = q\ldots\alpha$, will be constrained in phase 7. The procedure commences by changing the corresponding costs and if

$$c_q' = c_q \qquad (99)$$

where c_q is the previous cost of x_q, the remaining altered costs may be derived from

$$c'_j = c'_{j-1} + je \qquad j = q+1 \ldots \alpha \qquad (100)$$

where c'_j represents the new cost of x_j and e is a small positive constant.

In this case, the latest locations are favoured so that the corrective procedure commences by constraining the variable x_α and progresses to the variable related to the earliest location, namely, x_q.

The situation at the conclusion to phase 7 is represented by figure 16e. All shifts are now constrained and although the a.m.P.T.R. are constrained, a further phase is necessary to ensure that the variables related to the remaining reliefs are also integer. The procedure for phases 6 and 7 is illustrated by figure 12.

3.16 Infeasibility

The ability to produce an integer solution at the level of efficiency specified in phase 3 depends on the ability of the linear program to produce, after each restart, a new solution complying with the additional restrictions. While the integer points are reliable indicators, it is possible that in some circumstances to be discussed later, a stage may be reached subsequent to phase 3 where an infeasible solution is encountered denoting conflict between the existing constraints. In this situation, the need arises for some procedure to restore feasibility and short of abandoning all constraints imposed after phase 1, such a procedure may involve one or more of the following possibilities:

(a) A variation to current restrictions involving no increase in the labour supply.

(b) A relaxation of certain constraints to increase the labour supply.

(c) A reduction in demand.

Tests using (a) above have revealed that there is little prospect of restoring feasibility by altering previously established constraints and in no case was the use of this technique successful, suggesting that no integer solution was available while the solution remained constrained to the current integer point. Moreover, the investigation of numerous alternatives is expensive in run time so that (a) above, may be discarded for all practical purposes. Useful alternatives are offered by (b) and (c) however, and each maintains a high standard of efficiency.

3.17 The Addition of Labour

In every case the imposition of an additional restriction involves the expulsion of a variable from the basis either by the use of a special objective function or by the application of a prohibitive cost. For convenience, this latter variable will be referred to as a prohibited variable. If infeasibility is encountered, it may be manifested by the failure to exclude the prohibited variable from the basis or, if suitable slack variables are available, by the failure of the solution to remain at the chosen integer point indicating that a variation in the labour supply has automatically been made to compensate for a deficiency disclosed by the additional constraint.

If (b) above is to be utilized to restore feasibility, then some indication is necessary that an increase in labour will, in fact, be sufficient as in isolated circumstances it may prove necessary to alter the most recently established constraint.

To illustrate, suppose that variables x_s and $x_{s'}$ are high cost slack variables inserted into constraints (55) and (63) at the conclusion to phase 3, so that constraint (55) becomes

$$\sum_{j=1}^{T} x_{\omega+j} - x_s = W \qquad (101)$$

and (63) becomes

$$\sum_{j=1}^{L_p} x_{L_f+j} + x_{s'} = (L_p K) - P \qquad (102)$$

and the corresponding costs c_s and $c_{s'}$ are chosen so that a variation in C will not be permitted to interfere with the objectives of phases 5 to 7. After weighting is applied to the relevant costs, the greatest cost reduction is achieved in each of phases 5,6 and 7 if the elements of the solution, that is the shifts and reliefs, cease to be associated with the least favoured locations and become associated with the most favoured locations as determined by the weighted costs. For each element for which weighted costs apply, the greatest cost variation is less than ut^* in phase 5 and equals $(c_{\underline{t}} - c_{L_f})$ and $(c_\alpha - c_{L_f+\underline{t}})$ in phases 6 and 7. Since there are at most D elements involved in each phase, a standard cost k_c is adopted such that

$$k_c > D \text{ Max.}(ut^*, c_{\underline{t}} - c_{L_f}, c_\alpha - c_{L_f+\underline{t}}). \qquad (103)$$

The costs c_s and $c_{s'}$ are chosen so that

$$c_s > k_c \quad \text{and} \quad C_p c_s = c_{s'} >> 0. \qquad (104)$$

Since the cost of increasing the labour supply by one unit exceeds the greatest possible advantage to be gained in the resulting variations to the weighted variables in phases 5,6 or 7, x_s and $x_{s'}$ will enter the basis only when absolutely necessary, that is, when the need arises to compensate for a deficiency in the labour contribution C.

Suppose that x_r is the variable corresponding to the rth. location for F.T.S., commencing in period r, and

$$x_r = X'_r, \quad X'_r \notin I$$

and $$f_r = K - X'_r = X_r.$$

The procedure requires that x_r be implicitly constrained to $[X'_r] + 1$ so that f_r, the number of F.T.S. in location r, is reduced to $[X_r]$. The implied constraint also requires that x_r be regarded as a prohibited variable and removed from the solution. If priority is given to the removal of x_r and if an infeasible solution results, the non-feasible status will normally be indicated by the presence in the basis of either x_s or $x_{s'}$ or, if infeasibility occurs in phase 7, by both x_s and $x_{s'}$, indicating that the chosen integer point is no longer suitable. Under these circumstances feasibility may be restored and the integer requirement satisfied if the solution is constrained to a new integer point related to a sufficiently higher labour contribution. In some circumstances it may prove impossible to remove x_r, indicating that $f_r = [X_r]$ is totally incompatible with existing constraints. In this example, such a situation can only be caused by a deficiency in period i, where $r \leqslant i < r+d_f$, which cannot be reconciled unless f_r is assigned a higher value. In this case, before increasing C to restore feasibility, the value of x_r is reduced to $[X'_r]$ thereby increasing f_r to $[X_r] + 1$. Since $f_r = X_r$ implies an adequate contribution in all periods i, $r \leq i < r+d_f$, so also must $f_r = [X_r] + 1$.

Because infeasibility denotes a deficiency in a period or periods, priority in the removal of the prohibited variable is ensured if the associated cost is greater than the largest possible cost of compensating for the greatest possible deficiency. Since the deficiency can never exceed D unit periods and because at worst, D shifts would be required to compensate for this deficiency, c_r, the cost applied to the prohibited variables will be sufficiently large if

$$c_r > D \, \text{Max}(C_p c_s + c_{s'}, C_f c_s). \qquad (105)$$

Alternatively, since all prohibited variables become, in effect, artificial variables, they may be driven from the basis by the use of a special objective function.

As any automatic increase in C, corresponding to the entry into the basis of the special slack variables, is usually a fractional increase, any addition of labour must be controlled by altering constraints (101) and where necessary (102), so that the solution will be constrained to a new integer point. Since it is necessary to maintain approximately the existing proportions in the work force, the point will be selected from a list of priorities as previously described and will be the first in the list which restores feasibility. If procedure (b) alone is used to restore feasibility, three constraints at most will be involved, namely, the most recently imposed constraint if a prohibited variable is contained in the basis, constraint (101) where an increased integer value of w will apply and constraint (102) where the part time component may be suitably varied. Because the majority of constraints remain unchanged, the increase is therefore excluded from already satisfied areas of the problem and restricted to the integer addition of labour in locations not as yet constrained. The procedure appears in figure 14.

3.18 Reducing Demand

Experience suggests that if an addition to C is required subsequent to phase 3, then this increase will usually be necessary only once in the course of computing a schedule. The first alternative integer point in the list of priorities will normally involve the smallest possible increase in labour and it can be expected that this will be sufficient to restore feasibility. Testing has also shown that infeasibility is usually related to minimum shortages in one or two periods only. Under these circumstances a whole shift may be used to compensate for a single deficiency in a single period. If, however, a limited reduction in demand is permitted, expensive additions to C may be avoided altogether.

As an example, the P.N.Y.A. has established a level of service which requires a collection rate in the vicinity of 325 vehicles per hour. Although this rate has been adopted with service in mind, it appears that it is significantly lower than the limit of a collector's capacity since collectors employed by D.M.R., due to limitations on the number of booths possible, are sometimes expected to process up to 700 vehicles per hour in intervals of greatest activity. It would seem, therefore, that at some facilities, there is a possibility of increasing the collection rate above the ideal determined by service requirements if a significant advantage can be gained. Whereas a rate near the maximum for extended intervals is unfair to collectors and patrons, a small increase over a few periods may lead to economies which more than compensate for the resulting limited reduction in the level of service.

To illustrate, consider the case where 15 collectors are specified for a certain 20 minute period to process 1625 vehicles at a rate of 325 vehicles per hour. If the number of collectors is reduced by 1, the collection rate for the period will increase to approximately 349 vehicles per hour for each of the remaining 14 collectors and it is suggested that such an effect on service could be tolerated if it resulted in significant economy.

If a reduction in demand is to be used to maintain feasibility, experience indicates that a limited reduction in a few isolated periods, at most, will be involved. Provided this facility is not used to excess, for example to make substantial variations in the part time component, no widespread tampering with the established level of service is envisaged. Accordingly, a reduction in demand will be considered only after phase 3 and it will be necessary to specify those periods where a reduction may be made and the extent of the permitted reductions. The use of this feature will also require certain fundamental changes to the model and to the procedure used.

If, corresponding to every period, there is an r_j, j=1...T, where r_j represents the maximum permissible reduction in period j, then a set $\underline{J}$ exists such that

$$\underline{J} = \{j:\ j \leq T,\ r_j > 0\}. \qquad (106)$$

Since $\underline{J}$ is the set of all periods in which a reduction is permitted, the possibility of reducing demand may be considered in the original model by including a sixth term in (37) viz.

$$\sum_{j=1}^{L_f} a_{ij}x_j + \sum_{j=1}^{L_p} d_{ij}x_{L_f+j} + \sum_{j=1}^{\phi} e_{ij}x_{\alpha+j} + \sum_{j=1}^{\psi} f_{ij}x_{\beta+j} + \sum_{j=1}^{T} \delta_{ij}x_{\omega+j} - \sum_{j\in\underline{J}} \delta_{ij}x_{\gamma+j} = b_i \qquad (107)$$

where $\gamma = \omega + T$ (108)

and where $x_{\gamma+j}$, $j \in \underline{J}$ denotes the number by which demand is reduced in period j. Further, since C=D+w, the number of spare periods, w, must increase if D is decreased and C remains constant or if C is increased and D remains constant. The entry of the slack x_s into the basis may now be due either to an increase in C or to a reduction in D and the cause may not be immediately obvious if constraint (101) is used. It becomes more convenient therefore to exert a more direct control over the contribution of labour than that afforded by constraints (101) and (102), so that in this situation, (101) is discarded at phase 3 to be replaced by

$$\sum_{j=1}^{L_f} x_j + x_{s''} = (L_f K) - F \qquad (109)$$

where $x_{s''}$ is a high cost slack variable with L_f and K as before.

Because (109) specifies F, the number of F.T.S. contained in the schedule, (102) and (109) are sufficient to constrain the solution to the chosen integer point with the result that (101) becomes redundant and may be returned to its original inoperative status. The presence in the basis of either $x_{s'}$ or $x_{s''}$ is then used to immediately indicate that C has been increased.

The corresponding costs are chosen so that

$$c_{\gamma+j} = k_c, \; j \in \underline{J} \tag{110}$$

while $$c_{s''} > c_{s'} > C_f k_c. \tag{111}$$

Again, c_r is greater than the greatest cost of compensating for D deficient periods and in this case

$$c_r > Dc_{s''}. \tag{112}$$

Because it is necessary to maintain control over the reduction in demand in each period, successive solutions are examined to ensure that $x_{\gamma+j} \leq r_j$. Provided the $x_{\gamma+j}$, $j \in \underline{J}$ remain within the prescribed limits, the procedure may continue as before but if any exceed the limits, these variables must also be constrained.

To illustrate the procedure used, assume that in step 1 a variable x_r, $r \leq \alpha$, has been constrained and suppose that the resulting solution contains the variable $x_{\gamma+j''} > r_{j''}$. It is necessary in step 2 to reduce the value of $x_{\gamma+j''}$ at least to the permitted maximum and $x_{\gamma+j''}$ is constrained to $r_{j''}$. Step 2 then continues by constraining any further $x_{\gamma+j}, j \in \underline{J}$ until all such variables are within the limits prescribed or an infeasible solution is encountered. In the latter case, the implication is that there is no solution complying with existing conditions such that $x_{\gamma+j} \leq r_j$, $j \in \underline{J}$ and two possible courses of action must be considered. If a prohibited variable is present in the basis then it denotes that the constraining of x_r has initiated a succession of solutions which can never lead to the required feasible integer solution. The constraint is then removed from each $x_{\gamma+j}$ constrained in step 2 and the value of x_r is reset to the alternative. On the other hand, if no prohibited variable is present and $x_{s'}$ or $x_{s''}$ indicate a need to increase C, then the constraint is removed from each $x_{\gamma+j}$ constrained in step 2 and the procedure needed to increase C is invoked. In this case, therefore, the procedure attempts to produce a solution containing the level of wastage specified at phase 3,but if this is not possible, the algorithm proceeds by maintaining C at the current level and reducing demand. An increase in C is made only if it proves impossible to maintain the reduction in demand within the preset limits. A detailed description of the process is contained in figure 15.

The procedure prior to and including phase 7 may now be summarised as follows. Initially, a number of a priori conditions are specified which apply to the solution as a whole and which restrict the more general features of the solution. Subsequently, phases 5 to 7 establish within the general framework of the initial restrictions, a localised control over the provision of labour and the method employed to constrain the relevant variables is equivalent to a gradual emaciation of the demand so that each successive solution is related to a problem of lessening dimensions. If, at any stage, the imposition of a constraint causes conflict between the various requirements of the solution, the constraint must be altered if it prohibits the eventual generation of the required solution, or the conflict is resolved either by artificially reducing the requirements of the problem or by increasing the possibilities permitted by the initial general restrictions.

3.19 Manual Completion

At the conclusion to phase 7, all shifts have been specified as integers and in this labour supply provision has been made for all reliefs, including those not yet constrained. If feasibility has been restored, where necessary, by increasing C and if allowance is made for the fact that a.m.P.T.R. have already been constrained, the time provided for the remaining reliefs can be determined simply by subtracting demand from supply in each period 1 to T. This surplus may then be allocated manually to the individual shifts requiring reliefs. While the overall problem is complex and time consuming, the reduced problem remaining after phase 7 is not and the manual allocation of the remaining reliefs has been accomplished in the largest problems in less than 15 minutes. The relative facility with which the schedule can be completed is due to the fact that the locations for breaks over a large portion of the problem are unique if trivial alternatives, such as the swapping of reliefs by shifts in the same locations are ignored. If a reduction in demand has been employed, manual allocation may take slightly longer.

3.20 L.P. Completion

Because manual methods of completion amount to little more than a clerical task, no attempt has been made to constrain the majority of reliefs prior to phase 8 and the procedure adopted in this phase is in the nature of a compromise to be used more for convenience than necessity.

At the conclusion to phase 7 the basis will contain those non-zero variables corresponding to the F.T.R., the p.m.P.T.R. and the spare periods. If a reduction in demand has been made, the corresponding variables may also be present. The elements selected for attention are the F.T.R. and phase 8 involves a stage by stage process to test and constrain the corresponding variables. It has been determined in this case that it is more economical to weight individual variables so that phase 8 consists of successive stages, one variable being constrained at each stage. If

$$J' = \{j : x_j \text{ corresponds to a F.T.R.}\}$$

then all x_j in the basis such that

$$x_j = X_j + \rho_j \qquad j \in J', \; X_j \in I, \; 0<\rho_j<1 \qquad (113)$$

are examined at stage i and x_r corresponding to $\rho_r = \text{Max.}(\rho_j)$, is selected to be constrained. It must be determined whether a solution exists in which x_r assumes an integer value and accordingly a cost $\underline{c}$ is assigned to x_r so that

$$\underline{c} < 0 \quad \text{and} \quad |\underline{c}| << k_c. \qquad (114)$$

The Simplex iterative process is recommenced and the value of x_r in the resulting solution is given by $x_r = X_r'$, which represents the maximum possible value for x_r subject to the existing conditions. As before, x_r is constrained to the maximum integer value indicated, namely $V(i) = [X_r']$, and x_r is removed from the solution. If necessary a further stage, i+1, is commenced and the procedure continues until all $x_j, j \in J'$ are integer or an infeasible solution is encountered. In the latter case, the need may arise to change the value of a variable constrained at a previous stage.

As an example, let the tableau

i	1	2	3	4	5
r	269	796	842	105	364
V(i)	1	2	1	0	0

represent the position after the completion of 5 stages. If at stage 5, after constraining x_{364} to 0, an infeasible solution is encountered, conflict exists between the constraints imposed in stages 1 to 4 and that imposed in stage 5. In this situation alternative values must be investigated for the variables constrained in stages 1 to 4 and an alternative exists if there is a $V(i)$ such that

$$V(i) > 0 \quad \text{at stage} \quad i < 5.$$

In other words, it remains to be determined whether infeasibility could have been avoided if the variables constrained at stages $i<5$ had assumed values less than the maximum possible.

If an exhaustive investigation of all possibilities is to be undertaken, a tree search involving backtracking may be used. Originally, all x_j $j \in J'$ are contained in U, the set of unconstrained variables so that the procedure at each stage selects and constrains some x_j such that $j \in J' \cap U$. If infeasibility occurs at stage i, the procedure backtracks to stage i' where i' is the maximum for which $V(i') > 0$ and $i' < i$. $V(i')$ is then reduced by 1 and all variables constrained at stages $\underline{i}$ such that $i' < \underline{i} \leq i$, are returned to U and the associated costs increased to zero. The procedure then recommences at stage $i'+1$. In the example above, i' is 3, $V(3)$ is reduced from 1 to 0 and the procedure returns again to stage 4. If, on the other hand, a non-feasible solution is encountered at stage i and there is no $i' < i$ such that $V(i') > 0$, then there is no integer solution available.

All possibilities may be examined in this manner but because the restrictions imposed prior to phase 8 have such a binding effect, the number of options to be considered in practice is extremely limited with the result that back-tracking is seldom necessary.

One further point requires consideration and this concerns the possibility that integer values for all $x_j, j \in J'$ may not be available unless either C is increased or D is reduced. Figure 17 contains a diagram illustrating how a number of 1-2-1 reliefs may become involved in a fractional deadlock. In this case the procedure has converged on a solution containing six fractional reliefs and the situation may be corrected by the provision of one extra collector in any one of five alternative periods or by a corresponding reduction in demand. Such an occurrence is possible only with certain working regulations and will involve the allocation of a small number of reliefs as simple fractions, however, the conditions to be satisfied in this situation are such that a deadlock will occur only on rare occasions, if at all. Whereas procedures similar to those used in phases 4 to 7 are available to correct the corresponding deficiency, their use cannot be justified on computational grounds as the necessary steps involved in manual correction entail little difficulty.

3.21 Phase 8 - A Shortened Procedure

Experience has shown that phase 8 will be used only on a limited scale since, in more than 75% of the examples tested, the required integer solutions were produced prior to this phase. Where phase 8 was utilized, the integer solution was normally produced after two or three stages at most and in no case was there any need for further constraints for the p.m.P.T.R. or the spare periods. A deficiency was encountered only once and backtracking was used on this occasion and in one other example.

The situation subsequent to phase 7 therefore, is that only a minimum of computation may be justified and then only for convenience. Further, in all but exceptional cases, only a minimum is necessary. If any situation arises where significant run time is used in phase 8, this will be due to the repeated necessity to backtrack and because there will be little if any benefit to be derived, a suitable termination point must be adopted. Experience suggests that backtracking should be dispensed with and the tree search discarded and that computation should be terminated either when $j \in J' \Rightarrow x_j \in I$ or, in isolated cases, when an infeasible solution is encountered in phase 8. In this latter case it will be necessary to release the constraint leading to infeasibility and to complete the schedule manually. Since the shortened procedure has demonstrated the ability to quickly suppress any remaining fractions or at worst, isolate any minor complications remaining, economy in computation is assured. The procedure appears in figure 13.

CHAPTER 4

PRACTICAL EXPERIENCE

4.1 Optimal Solutions

At the conclusion to phase 3, a suitable level of labour has been specified which is sufficient to meet the static demands of the histogram and includes sufficient extra time for the provision of the necessary reliefs. Subsequent to phase 3, the imposition of the additional restrictions necessary to produce a feasible integer solution may, or may not require an increase in C. If, in phase 4, a feasible solution containing an integer number of a.m.P.T.S. cannot be obtained, then obviously there is no integer solution available at the current integer point and appropriate measures must be taken. Subsequent to phase 4, the situation is less obvious.

If y is the number of periods of collectors' time by which C is increased subsequent to phase 4, and if $y = 0$, the solution is a proven optimal integer solution since it contains the minimum wastage consistent with all requirements. If $y > 0$, it is not possible to prove that the solution obtained is an optimal integer solution as it is not possible to show that the restrictions imposed after phase 4 warrant the increase in C. Testing undertaken to evaluate the algorithm under varying conditions has shown that proven optimal integer solutions can be obtained over a wide range of practical situations and the results strongly suggest that even where $y > 0$, the solutions obtained are also optimal integer solutions, as $y > 0$ is invariably minimal and always directly related to more stringent or less suitable working conditions.

4.2 Additional Objectives

Whereas minimising idle time is undoubtedly of prime importance, other factors must be considered if an acceptable solution is to be obtained. Important practical considerations must be taken into account when determining the ratio of full time to part time work. A maximum part time component is not beneficial to the administering authority, nor to the full time collectors. For the authority, a large casual work force leads to greater administrative costs, a greater likelihood of absenteeism and less control over staff. In addition, there is a possibility of a loading for casual work as the part time collector may be paid slightly more per hour than his full time counterpart. It can be assumed also that the collectors' union would be in favour of using as few part time collectors as possible as an increase in the number of part time collectors causes a corresponding decrease in the number of full time collectors. It appears desirable therefore to minimise both the overall part time contribution and the number of part time collectors. In addition, an attempt must be made to obtain a satisfactory spread of starting times for F.T.S. so that a collector may expect to work a reasonable proportion of early shifts and late shifts.

Given that the number of P.T.S. has been set at some acceptable figure, there is usually a considerable difference between the maximum number of P.T.S. which can be located in the a.m. region and the minimum, the former being sometimes twice the latter. This variation has a pronounced effect on the appearance of the ultimate schedule. Maximising the number of a.m.P.T.S. will minimise full time participation in this region so that the F.T.S. are relegated to later periods, a situation which would be unpopular with full time collectors who would then be required to work a preponderance of late shifts. Minimising the number of a.m.P.T.S. has the opposite effect and requires F.T.S. to commence in earlier periods. The locations of the F.T.S. can therefore be controlled by this means. The effect of minimising the a.m.P.T.S. varies with the problem but values in the vicinity of the minimum tend to produce more satisfactory distributions of starting times for F.T.S.

Since the above requirements are of the utmost practical importance, it is desirable that the regulations applicable should be sufficiently flexible to allow a solution meeting all objectives. In practice, the regulations likely to be encountered provide the necessary flexibility so that a number of alternative solutions will normally be available at a given value of C and the most suitable alternatives tend to contain limited part time participation and a limited number of a.m.P.T.S.

Assuming that some choice will be available in selecting a suitable part time contribution, it must be appreciated that the result of reducing the part time component is that the solution is forced to approach a full time solution where wastage is at a maximum. As can be expected, examples have been encountered where the selection of the feasible integer point related to the lowest level of part time labour, has led to increased idle time. The point in the feasible interval associated with the minimum part time contribution is therefore more likely to lead to an integer solution containing $y > 0$ although this appears to rely on the overall situation rather than on the extent of part time participation alone. Although, in practice, it is desirable to reduce the part time component, the accompanying loss in flexibility must also be considered when the proportion of part time work is established.

Some degree of caution is also warranted in choosing the number of a.m.P.T.S. since, although an estimate of the minimum is obtained in phase 4, it is not absolutely certain that the minimum indicated is also suitable for an integer solution. For this reason it appears prudent to choose a value in the vicinity of the minimum rather than the minimum itself.

4.3 Test Results

Practical problems can be expected to vary in two respects, namely, in the size and shape of demand and in the working regulations applicable. The algorithm was programmed in Fortran for a CDC6400 computer and tests involving in excess of 70 different problems were conducted to evaluate reliability in these two broad areas. In the first series of tests, working regulations were stabilised and solutions were obtained using a variety of histograms. Further tests were then conducted by selecting particular histograms and using a variety of working regulations. To clarify the subsequent discussion and to standardise the measure of efficiency, it is assumed that no reduction in initial demand is permitted and compensation for any deficiency encountered is made by way of an increase in C.

Conclusions

The results show that a high degree of reliability can be expected from the algorithm. No evidence has, in fact, been found which suggests that the efficiency of a schedule has been affected by shortcomings in the method used to produce it and any wastage appears to depend solely on the restrictions implied by the various working regulations. In practice, unless the regulations adopted are extremely unsuitable, a level of wastage in phase 2 greater than the minimum possible, is unusual. Additional wastage in phase 3 depends on the particular circumstances but especially on the H.C.F. of C_f and C_p and the latitude permitted by the regulations. Increases in wastage subsequent to phase 3 appear to depend entirely on the working regulations used. Whereas y can, in most circumstances, be maintained at zero, the selection of an integer point does not provide an absolute guarantee that there is a corresponding integer solution since all conditions necessary for an integer solution have not been imposed at phase 3. However, if $y > 0$ is necessary, it can be expected that the minimum possible increase will be required only once in the course of computation as extremely severe restrictions can be expected to result in a higher initial bound in phases 1 to 4. Although the feasible integer point does not provide a guarantee of an associated integer solution, experience suggests that it does guarantee that at least there will be an integer solution associated with a point in the immediate vicinity. Testing has therefore established that in practical situations, the highest level of efficiency, or at worst, a very high level of efficiency can be anticipated.

In addition to evaluating the efficiency of the algorithm, the tests provided valuable information as to the desirability of the various working regulations and experience indicates that maximum efficiency will be obtained if the regulations adopted are suitably related to the requirements of the particular facility. It has been determined that the principal factors on which efficiency depends are:

(a) The shape of the histogram and the length of the P.T.S.

(b) The locations available for P.T.S.

(c) The latitude permitted in providing reliefs.

Whereas each of the above has some effect on efficiency, none could be described as critical and an increase in wastage is usually due to a combination of restrictive factors rather than to one alone.

4.4 The Determination of d_p

Basically, three different demand histograms were considered. The first was calculated from traffic figures obtained from D.M.R. for the Sydney Harbour Bridge (Fig.18). The second represents the requirements for toll collection at the Lincoln Tunnel (Fig.19) and is contained in the P.N.Y.A. publication [2]. The third was derived from traffic estimates for the new Lower Yarra Crossing in

Melbourne (Fig.20). These three examples were chosen as each differs in important characteristics from the others. The D.M.R. histogram is comparatively large reflecting the heavy volume of traffic using the bridge. The a.m. and p.m. peaks are high compared to the requirements for the rest of the day but are of shorter duration than those in histogram for the Lincoln Tunnel. Peak demand for the latter tends to remain constant for some hours and there is also a lesser tendency for demand to decrease in the middle of the day so that the difference between demand at the peaks and demand at midday is not as pronounced as in the D.M.R. histogram. A further important difference concerns demand in the late p.m. interval where the requirements tend to be relatively higher at the Lincoln Tunnel than on the Sydney Harbour Bridge. The histogram for the Lower Yarra Crossing bears some resemblance to the histogram for the Lincoln Tunnel in that it tends to be a much smoother profile than that for the Sydney Harbour Bridge, the peaks being flatter and the drop in demand at midday less pronounced. Demand, however, is about half that for the Lincoln Tunnel and late p.m. demand tends to be small compared to the demand at the peaks. The three initial histograms therefore represent significantly differing situations and by varying the size, the shape and the value of τ, a substantial number of additional histograms were also derived for testing purposes.

Although the magnitude of demand made little appreciable difference, it is quite obvious that the shape of the histogram is important. Smoother demand patterns were easier to satisfy then those of the D.M.R. type; however, optimal solutions to the latter can be obtained with little difficulty. Satisfactory solutions to the smoother patterns were obtained using P.T.S. up to a length of 5 hours but as the peakedness increased, flexibility was reduced and the ability to satisfy additional requirements diminished. While solutions containing a minimum of wastage could be obtained to problems slightly less extreme than the D.M.R. problem, a higher level of wastage was invariably contained in solutions to the latter where five hour P.T.S. were used. The shape of the histogram therefore does impose some limits and this can be appreciated since F.T.S. and five hour P.T.S. must be used to meet the demand of a high narrow peak. As these shifts extend well beyond the peak, it is inevitable that there will be a surplus if demand decreases significantly in the neighbourhood of the peak. Further, although it may be possible to avoid increased wastage under these conditions, the ability to obtain a solution complying with all requirements is severely curtailed as the number of alternatives involving minimum wastage is greatly reduced. This loss of flexibility is due to the fact that as the length of the P.T.S. is increased, the solution must again approach a full time solution and the prospect of greater wastage is increased. The failure to match the length of the P.T.S. to the requirements appears, in practice, to be the factor most likely to cause an increased level of wastage in phases 1 and 2. It is undesirable therefore to adopt in the first instance, a system which is basically unsuitable or which could interfere with efficiency in the future, especially when the means of circumventing these difficulties are available. In this case, the situation requires the use of shorter P.T.S. which readily produce satisfactory results. Although P.T.S. of up to 4½ hours could be used in some circumstances by D.M.R., tests show that shifts of about three to three and a half hours are most suitable. Generally, although it is better to limit the number of collectors to as few as possible so that longer P.T.S. of about 5 hours would be indicated, greater flexibility is obtained with the use of shorter shifts with less likelihood of increased wastage. In addition, as they allow more flexibility, shorter P.T.S. assist in reducing the overall part time component in the solution although the latter is largely determined by the peaks. Higher, more pronounced peaks will require a relatively greater part time contribution so that for example, the proportion of part time work on the Sydney Harbour Bridge would be greater than that at the Lincoln Tunnel, given similar regulations.

4.5 The Determination of L_p

A further important factor affecting economy concerns the number of optional locations permitted for P.T.S.. In every problem tested, every possible location for F.T.S. was considered. This was not the case for P.T.S. where only selected options were used. It has not been established that P.T.S. will be permitted at any time of the day but it is clear that P.T.S. will be required at certain times only. Intuitively, P.T.S. are required to cater for the extra work at the peaks and to supplement F.T.S. by satisfying demand in situations where the use of F.T.S. is uneconomic. Alternatively, P.T.S. can be regarded as reducing the demand in such a way that F.T.S. can cope economically with the residue. Sufficient optional locations must therefore be available to enable the P.T.S. to fulfil this function. Again, an approach to a full time solution is implied by restrictions on the locations of P.T.S. but again, this factor is not necessarily critical. Flexibility merely deteriorates gradually as the number of options is reduced and in most practical situations it will be possible to obtain economical solutions with available locations well below any anticipated limits. For example, in some situations with τ at 30 minutes, solutions containing the least possible waste were obtained to the Lower Yarra Crossing problem with only six alternative locations for P.T.S. over the whole problem. There is, however, the increasing prospect that as the number of alternative locations is reduced, the ability to reach all objectives will be reduced.

Optional locations for P.T.S. are necessary, broadly speaking, in the region of the two peaks. Although the requirements must be determined for each particular situation, testing shows that with τ at 30 minutes, about five locations should be considered for a.m.P.T.S. and about seven are required with τ equal to 20 minutes. As a general rule, the optional locations for a.m.P.T.S. should be such that P.T.S. may commence in any period between the first period and the period of maximum demand on the a.m. peak.

The remaining options are located in the region of the p.m. peak so that ideally, a P.T.S. may commence in any period between the period of minimum demand, at about midday, and the period of maximum demand on the p.m. peak. In some cases, a complication may occur if demand after the p.m. peak remains high. This is extreme in the demand pattern for the Lincoln Tunnel where demand remains steady until midnight at about half that for the p.m. peak. If P.T.S. are not available in the evening periods, a large number of F.T.S. will be required in this interval, concluding at about midnight. If the starting times for these late shifts coincide with the p.m. peak, a difficulty may result the nature of which may be appreciated if the time supplied by these late p.m.F.T.S. is used to reduce the initial demand to leave a residue. Under the conditions described, the characteristics of the p.m. peak may be changed so that a high narrow peak remains making it difficult to complete the solution without incurring a penalty. An example in figure 21 illustrates the effect on the demand histogram of scheduling 7 late evening F.T.S. Although in this example, reliefs are not considered, the diagram shows clearly that the residue contains a comparatively high narrow p.m. peak. While the shape of the initial demand suggests that satisfactory solutions may be readily obtained, the additional regulation prohibiting late evening P.T.S. presents a difficulty and may regularly lead to a higher level of wastage. In this case, late evening P.T.S. can be effectively used and the effect of a high late evening demand can be further diminished if the F.T.S. rostered for the early a.m. interval commence as early as possible, ideally at about 10 p.m.

It is important therefore not to place extra limitations on the solution by inadvertently excluding P.T.S. from locations where their presence may be required. Testing has shown that if any limitations are placed on the use of part time collectors, then such limitations should be related to the particular circumstances. Although a restriction on the locations of P.T.S. will not unduly effect solutions in some circumstances, in others it may affect economy and lead to less acceptable solutions from the collectors' point of view. In such a situation, a restriction on the number of P.T.S. may prove more acceptable.

4.6 The determination of R_f and R_p

The number of relief alternatives available may affect economy and for a given value of τ, this number will depend on the maximum interval of continuous duty, the minimum interval of duty permitted and any further regulations which limit the locations of the various breaks. These latter regulations, to a large extent, will be determined by the particular circumstances. For example, D.M.R. are reluctant to allow any collector to work longer than 2 hours without a break because of the high collection rate and for reasons of security. Although this appears to be much more restrictive than the corresponding P.N.Y.A. regulation, the tests indicate that a two hour maximum for full time collectors is quite satisfactory under most circumstances even where a minimum of 1 hour is also included.

As all indications suggest a single break for a P.T.S., the limitations implied by the use of a maximum two hour sequence for part time collectors, depend mainly on the length of the P.T.S. If, for example, 5 hour shifts are used and a maximum sequence of two hours is imposed, the relief is effectively fixed at about the centre of the shift and such rigidity would be quite unsatisfactory. If, on the other hand, a two hour limit is applied to a three hour shift, the relief may be taken at any time in a comparatively large interval so that the latter system leads to much more flexibility. Relief requirements may therefore determine the most suitable length for P.T.S. as flexibility is greatly effected by the relationship between these two features.

Again, experience indicates that this factor alone is not critical unless the regulations applied are extreme and much less flexible than those commonly in use. It has been clearly demonstrated that economy will depend on the ability to establish a satisfactory relationship between the various regulations in the award and the suitability of such regulations may be measured in each particular situation.

4.7 Examples

The foregoing remarks are illustrated by the schedules contained in figures 22 to 26, all of which represent solutions to the least accommodating demand pattern, namely, that for the Sydney Harbour Bridge which is contained in figure 18 and for which τ is 30 minutes. The solutions in figures 22, 23 and 24 contain the minimum possible idle time and are therefore optimal in this respect. Those in figures 25 and 26 contain $y > 0$ so that the status of the latter two is uncertain; however, in each case, $y > 0$ is a minimum.

Initially the regulations applied were as follows:

1. F.T.S. will be of 8 hours duration.

2. A F.T.S. will contain three rest breaks, the first and third of 30 minutes duration and the second of 60 minutes duration.

3. No full time collector may remain on duty continuously for longer than 2 hours.

4. No break may be scheduled in the first or last hours of a F.T.S.

5. P.T.S. will be of $3\frac{1}{2}$ hours duration.

6. A P.T.S. will contain one break of 30 minutes duration.

7. No part time collector may remain on duty continuously for longer than $2\frac{1}{2}$ hours.

8. P.T.S. may commence between 6 a.m. and 8.00 a.m. and between 12 noon and 8.30 p.m.

Schedule 1, figure 22, has been computed to comply with the above regulations and represents an optimal solution containing only 1 spare period. If the measure of efficiency is given by

$$\text{Efficiency} = \frac{C-w}{C} \times \frac{100}{1}$$

then efficiency in this case exceeds 99%; however no attempt has been made to satisfy other requirements. A satisfactory spread of starting times has not been achieved for F.T.S. and the schedule, including more than 50% of part time work, contains far more P.T.S. than is necessary or desirable. For these reasons Schedule 1 must be considered unsatisfactory.

In Schedule 2, figure 23, the part time component has been reduced to approximately 40% and the number of a.m.P.T.S. has been adjusted to obtain a more suitable spread of starting times for F.T.S. In addition, the number of possible locations for P.T.S. has been reduced so that in Schedule 2, no p.m.P.T.S. are permitted to commence before 3.30 p.m. In spite of the additional limitations, all objectives have been reached and wastage remains at the minimum of 1 period.

In Schedule 3, figure 24, the same conditions apply as in Schedule 2 except that regulation 7 above, is altered to reduce maximum continuous duty for part time collectors to 2 hours. This causes a significant reduction in flexibility as the break in a P.T.S. is now limited to only three alternative locations. Efficiency is maintained at approximately $99\frac{3}{4}\%$ as the schedule still contains only 1 spare period; however, some difficulty is experienced in obtaining the desired spread of starting times for F.T.S. as the reduced flexibility requires a much higher part time contribution in the vicinity of the a.m. peak and a consequent reduction in full time participation in this region. The loss of flexibility therefore suggests that a shorter P.T.S. would be more suitable if part time collectors are limited to 2 hours continuous duty.

Schedule 4, figure 25, complies with the regulations for Schedule 2 with one important addition. As well as the maximum 2 hour limit on continuous duty for full time collectors, the interval between breaks is limited to a minimum of 1 hour. This requirement prohibits a large number of relief possibilities and the consequent loss in flexibility causes a drop in efficiency to $98\frac{1}{2}\%$ as the number of spare periods increases to 7. The algorithm has been unable to produce a schedule at the previous level of efficiency and an extra P.T.S. has been included to overcome the difficulty. Schedules at maximum efficiency and containing this restriction have, nevertheless, been obtained by increasing the part time component.

In Schedule 5, figure 26, the basic regulations applied are similar to those for Schedule 2, with the difference that the part time component has been reduced to approximately 30% and all P.T.S. have been prohibited from the late evening interval. In this schedule, starting times for p.m. P.T.S. are confined to the interval between 3.30 p.m. and 5 p.m. Again, flexibility is lost and a penalty is involved as 7 spare periods are necessary and efficiency is once more reduced to 98½%. Inspection of Schedule 5 suggests that a further attempt to reduce the part time component will lead to an accelerated loss of efficiency as the P.T.S. contained in the schedule are necessary to cater for the increased work load at the peaks. It is indicated therefore that the minimum part time contribution consistent with economy, for this highly peaked demand, is in the vicinity of 30%.

Schedule 6, figure 28, contains a further example of a possible schedule for the Sydney Harbour Bridge problem, τ in this case being reduced to 20 minutes. The associated demand histogram in figure 27 has been derived from the initial figures by means of interpolation. The regulations applied are basically similar to those used above but in this case, the schedule represents an attempt to impose the most stringent regulations possible while maintaining minimum idle time. A maximum of 2 hours continuous duty is applied to both F.T.S. and P.T.S. To preserve flexibility, the length of the P.T.S. has been reduced from 3½ hours to 3 hours. In addition, a minimum limit of 1 hour applies to the interval between breaks in F.T.S. No late evening P.T.S. are permitted and starting times for p.m.P.T.S. have been restricted to the interval between 2 p.m. and 6 p.m.

The regulations applying in Schedule 6 may be summarised as follows:

1. F.T.S. will be of 8 hours duration.

2. A F.T.S. must contain three rest breaks, the first and third of 20 minutes duration and the second of 40 minutes duration.

3. No collector, full time or part time may remain on duty for longer than 2 hours continuously.

4. Breaks in a F.T.S. must be separated by at least 1 hour of duty.

5. No break in a F.T.S. may be scheduled in the first or last hours of a shift.

6. P.T.S. will be of 3 hours duration.

7. A P.T.S. must contain one rest break of 20 minutes duration.

8. P.T.S. may commence between 6 a.m. and 8 a.m. and between 2 p.m. and 6 p.m.

Schedule 6 contains the minimum amount of idle time, in this case, 3 spare periods. The part time component constitutes 34% of the total requirement and a satisfactory spread of starting times has been obtained. Although the regulations are far more restrictive than would be anticipated in practice, the algorithm has produced a schedule which achieves all objectives. However, because of the excessively restrictive nature of the regulations, there is an ever present possibility that subsequent schedules seeking to cater for an altered demand pattern, will contain a higher level of idle time. In practice then, the aim should be to select a set of regulations which allows sufficient flexibility to cater for the variations which will inevitably occur in the demand pattern. A summary of all example schedules is contained in Table VII.

Ashe [2] has pointed out that due to periodic variations in traffic, schedules should be revised four times per year. During the lifetime of a particular schedule, a gradual change in demand must be expected therefore. Since some day by day variation in demand is also involved, wastage, in practice, will also vary and will depend particularly on the attitude adopted in estimating demand.

4.8 Computation Time

C.P.U. time depends on a number of factors, the most important of which are:

(a) The value of τ.

(b) The number of solutions generated.

(c) The number of iterations required for each solution.

Since the number of variables and the number of constraints is directly related to τ, run times may be expected to increase as τ decreases. For example, there is a considerable difference in run times for τ at 60 minutes and τ at 30 minutes. As a general rule, the number of iterations will increase as τ decreases but there are many other factors influencing the number of iterations and schedules with τ at 20 minutes frequently require fewer iterations than similar schedules with τ at 30 minutes, in spite of the considerable difference in the number of constraints. The value of τ appears to have a more direct influence on the time per iteration. Although the increase in the latter is significant as τ is reduced, greater economies are possible as the matrix becomes larger, for example, the overlap between variables becomes more pronounced, so that the increase in the time per iteration is less than may be expected. As a comparison of Schedules 4 and 6 in Table V indicates, the time per iteration with τ at 20 minutes is less than twice the time per iteration with τ at 30 minutes.

An inspection of Table V reveals that there may be a considerable variation in the number of solutions produced by the algorithm before termination is reached. In Schedules 2 and 3, termination occurred in phase 5 and only 6 solutions were generated before the integer solution was obtained. In Schedules 4 and 5, on the other hand, 20 and 21 solutions respectively were produced. Where the need arises to constrain the solution to an alternative integer point, some delay may be experienced as the algorithm exhausts all possibilities before increasing C and this is reflected in the increased number of solutions in Schedules 4 and 5. Although termination may be expected to occur when all non-integer solutions have been discarded, integer solutions may be obtained long before this stage so that with the exception of cases where deficiencies are encountered, the number of solutions produced depends on the number of phases completed and this latter is largely unpredictable.

In addition to the variation in the number of solutions produced, the number of iterations required to produce each solution may vary as Table VI illustrates. In some cases, as few as 5 iterations are required to generate the subsequent solution and in others, as many as

100 may be required. Again, an increase in C usually leads to extra iterations as the inclusion of extra time may lead to a major reorganisation of the unconstrained elements of the partly completed schedule. Although such generalisations may be made, the number of iterations required for each successive solution is also unpredictable and depends on the reaction to the particular constraint imposed.

Whereas run times depend to some extent on the problem, they may be expected to vary between modest and predictable limits. For example, Schedule 5 in Table V is in the vicinity of the upper limit with τ at 30 minutes. Mean run time with τ at 30 minutes and with a preferred solution, was 1 minute 20 seconds while the mean run time with τ at 20 minutes and with a preferred solution, was 3 minutes and 30 seconds. It would appear therefore that problems with τ equal to 20 minutes are well within reasonable limits and although it depends upon the computing facilities available to the user, there appears to be considerable scope to reduce τ even further if necessary.

CHAPTER 5.

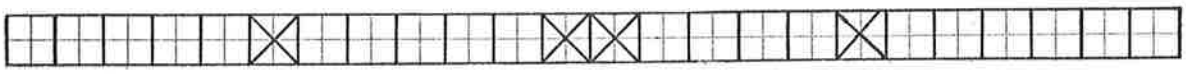
DISCUSSION

Since the success of schedules depends on the working regulations used, basic safeguards must be adopted to preserve economy. It is apparent that even the most restrictive of current regulations can be accommodated provided that suitable conditions for part time collectors are carefully selected. However, since the introduction of part time work, itself, constitutes a major and basic change to the system of rostering, the opportunity exists for a thorough re-examination of all regulations. In some cases, it appears that regulations and procedures have been devised in an attempt to overcome the rigidity resulting from the use of F.T.S. alone. For example, it seems in some instances that comparatively longer breaks have been used to extend the overall duration of shifts so that particular shifts may then participate to a greater extent in both peak periods. The increased flexibility introduced by the use of P.T.S. disposes of many of the problems encountered in full time rosters and the possibility arises that conditions may be revised to the benefit of all concerned.

Considered in this context and because the options available in any case involve small alterations to existing conditions, potential difficulties in relating existing regulations to the requirements of the mathematical model assume less importance. For example, D.M.R. officers who have been associated with the project have indicated that the adoption of one of the available alternatives in Table III represents no practical obstacle whatever. Since satisfactory solutions can be obtained under widely varying conditions, considerable latitude exists in achieving the desired degree of compatibility and since the algorithm provides an efficient means of measurement, there appears to be no reason why a very close relationship between the requirements of employer and employee cannot be established.

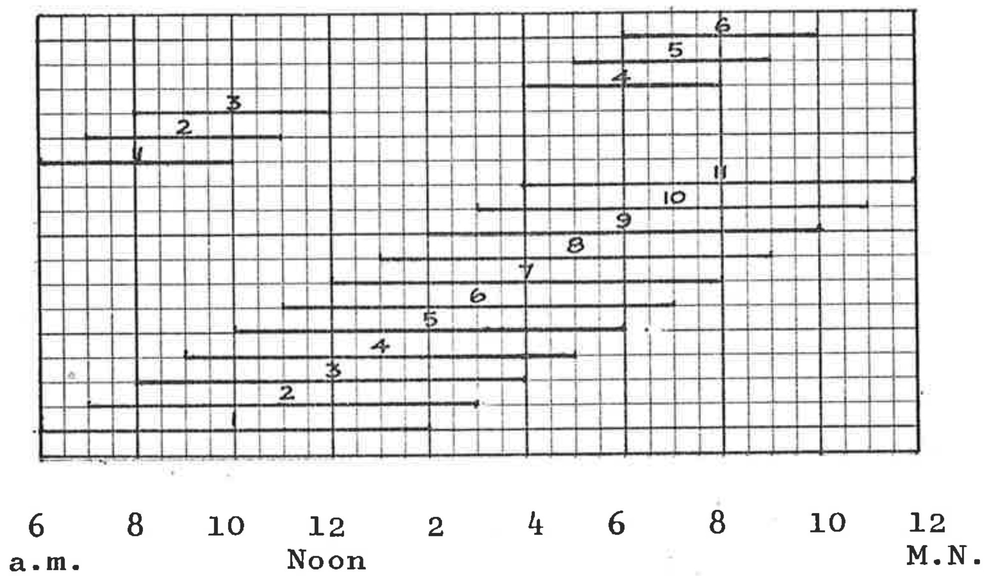
It is envisaged that the algorithm will constitute the most important segment in a complete computer system concerned with the problems of staffing, from the initial stage of estimating demand to a final stage of producing time sheets and preparing payrolls. Such a system will assist greatly in speeding up the allocation of manpower with the result that economy in operation and service to the public will be closely correlated. In view of the tests conducted, efficiency in the vicinity of 100% can be anticipated, especially if minor variations to the standard of service are permitted. The algorithm therefore provides all features required of it as it represents a rapid and convenient method of rostering part time and full time collectors for any toll facility at a high level of efficiency and provides a reliable means of measuring overall effectiveness in operations. The problem, as described in Chapter 1, has therefore been satisfactorily solved.

Although this thesis has dealt specifically with the problem of toll collection, the techniques described are not restricted to this application alone, nor are they restricted to situations where full time and part time labour is used. The algorithm may be used as effectively in other spheres where the same broad principles of demand and supply prevail. An interesting comparison exists in the scheduling of telephone operators in large trunk exchanges. Although an extra evening peak exists in the latter problem, this does not present any additional difficulty and since there is a close similarity in all other respects, the techniques described may be applied to this problem with equal success. In addition, various features of the algorithm suggest interesting possibilities for future experimentation concerning integer programming generally and zero-one programming in particular. The algorithm may therefore be regarded as being of more general significance the extent of which will be determined by further research.



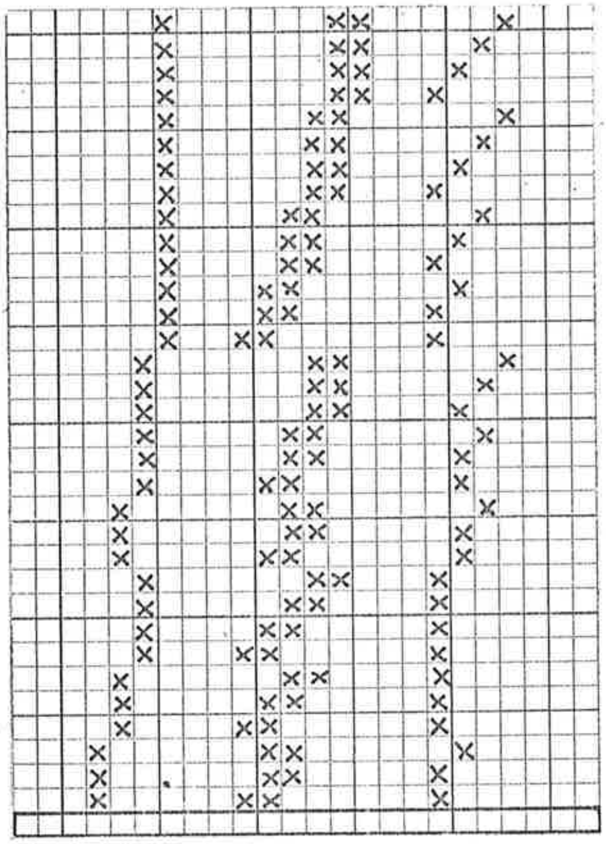
An 8 hour shift containing 3 rest breaks dividing the shift into 4 segments. τ is 2o minutes, d_f is 24 unit periods and C is 20 unit periods. The relief is a 1-2-1 relief with breaks of 20, 40 and 20 minutes duration respectively.

Figure 1a



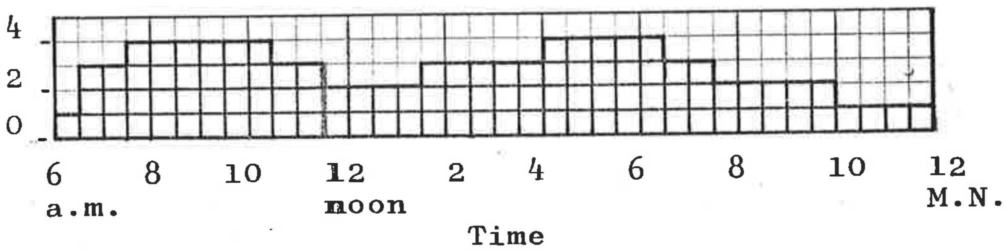
With τ =60 minutes, T=18 unit periods and d_f =8 unit periods, L_f =11. With d_p =4 unit periods and L_p =6, only a proportion of the available locations for P.T.S. are considered.

Figure 1b



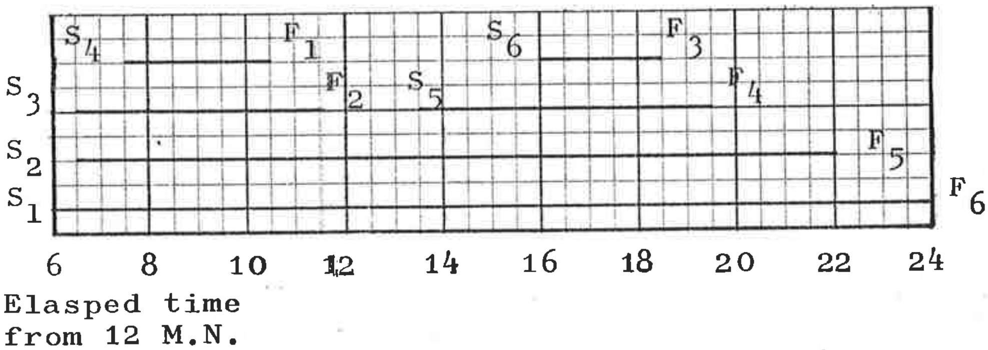
The diagram represents an 8 hour shift with τ at 20 minutes together with 33 alternative reliefs. Each relief consists of 3 breaks of 20, 40 and 20 minutes respectively. No breaks are permitted in the first or last hours and the maximum segment of continuous duty is 2 hours.

Figure 2



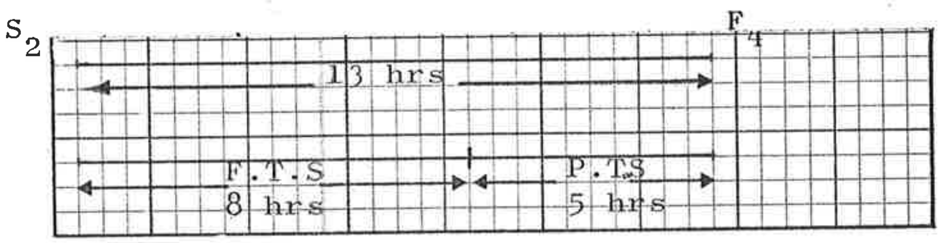
Demand represented by a histogram.

Figure 3 a



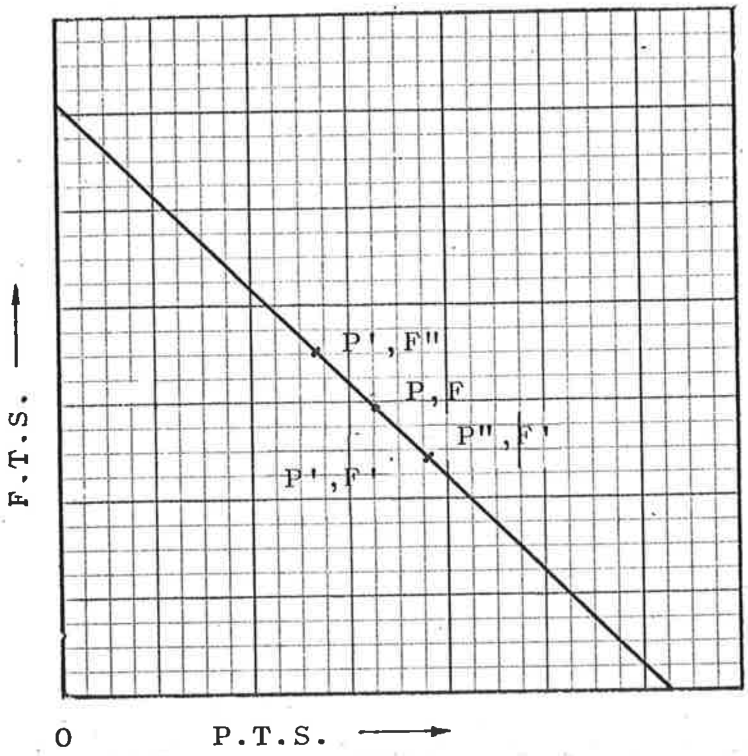
Demand represented as a line graph.

Figure 3 b



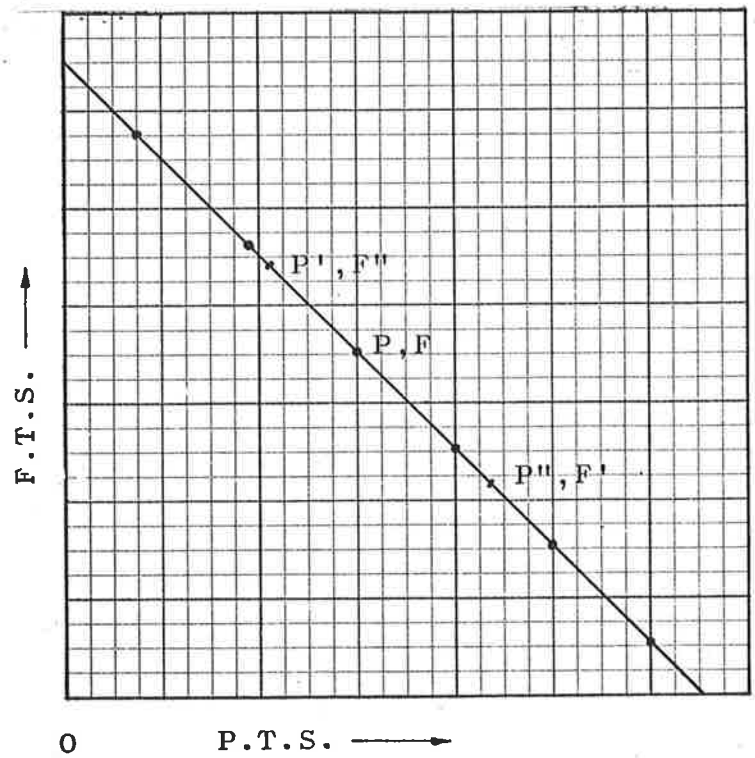
A F.T.S. and a P.T.S. span the 13 hour interval S_2 to F_4

Figure 3 c



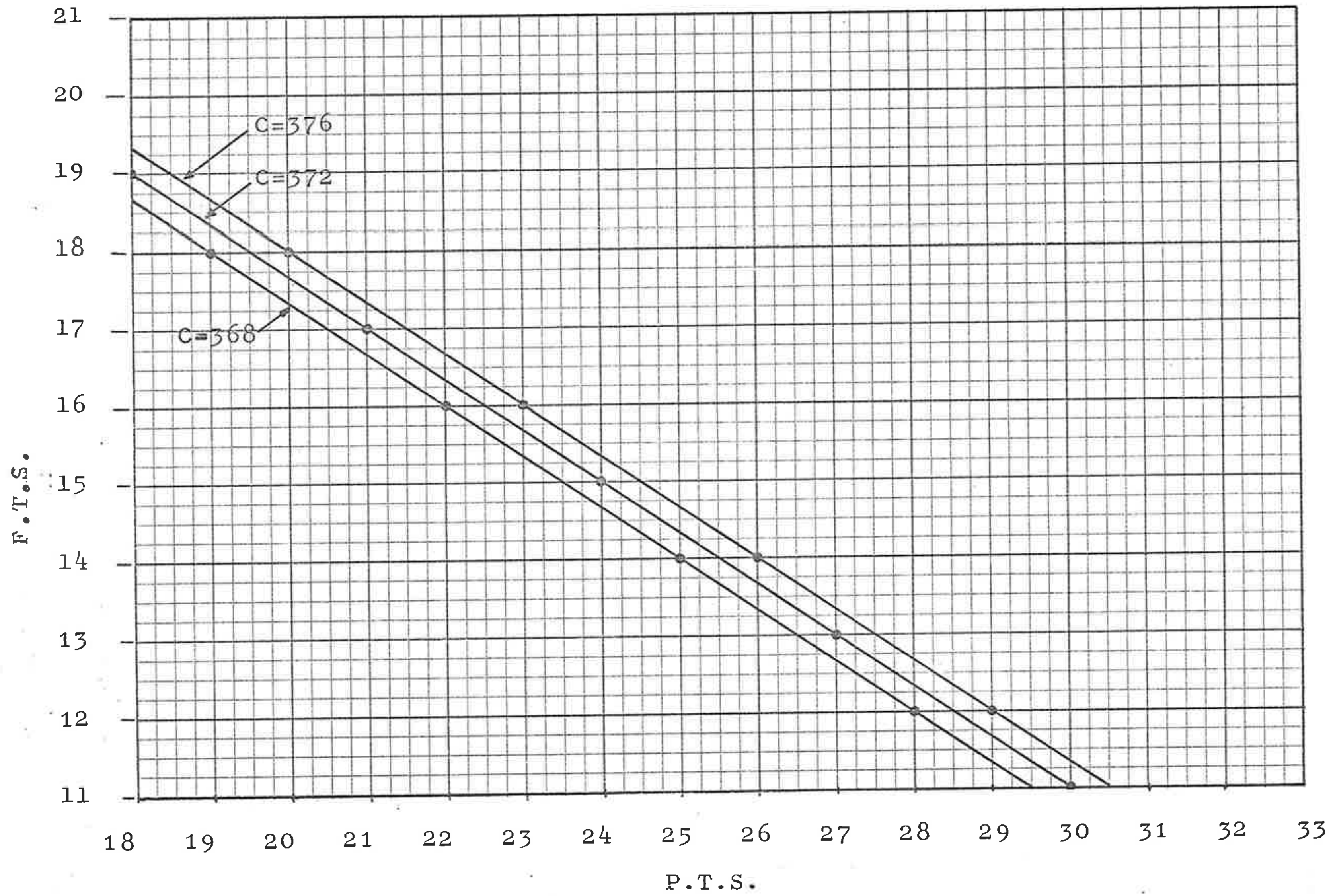
Commencing at the feasible point P',F' the feasible point P',F" is obtained by increasing the number of F.T.S. By increasing the number of P.T.S. the feasible point P", F' is obtained. By increasing both F.T.S. and P.T.S. the feasible point P,F is obtained.

Figure 4



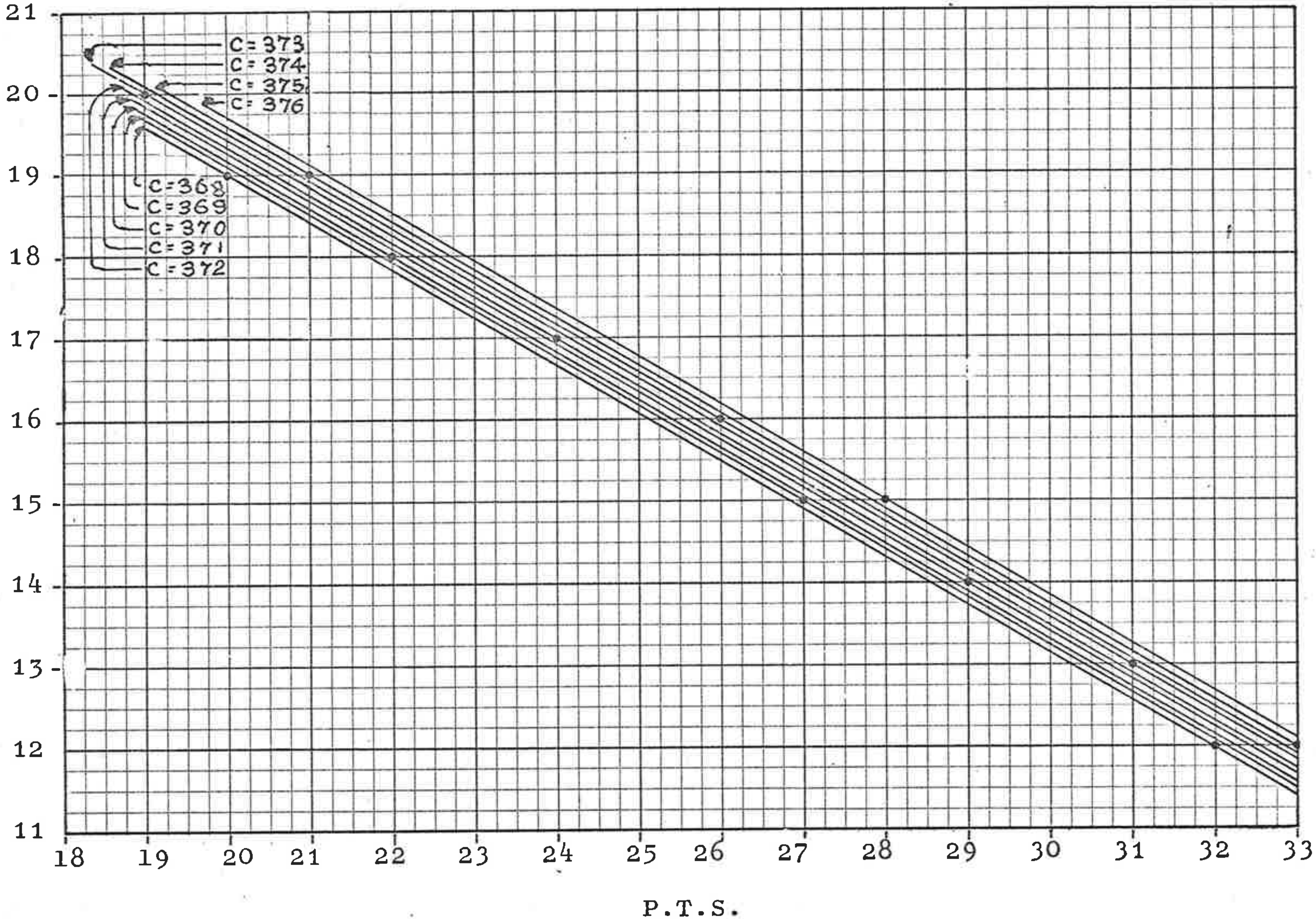
The extremes of the feasible interval are the points P',F" and P",F'. Two feasible integer points lie on this interval, one of which is the point P,F.

Figure 5



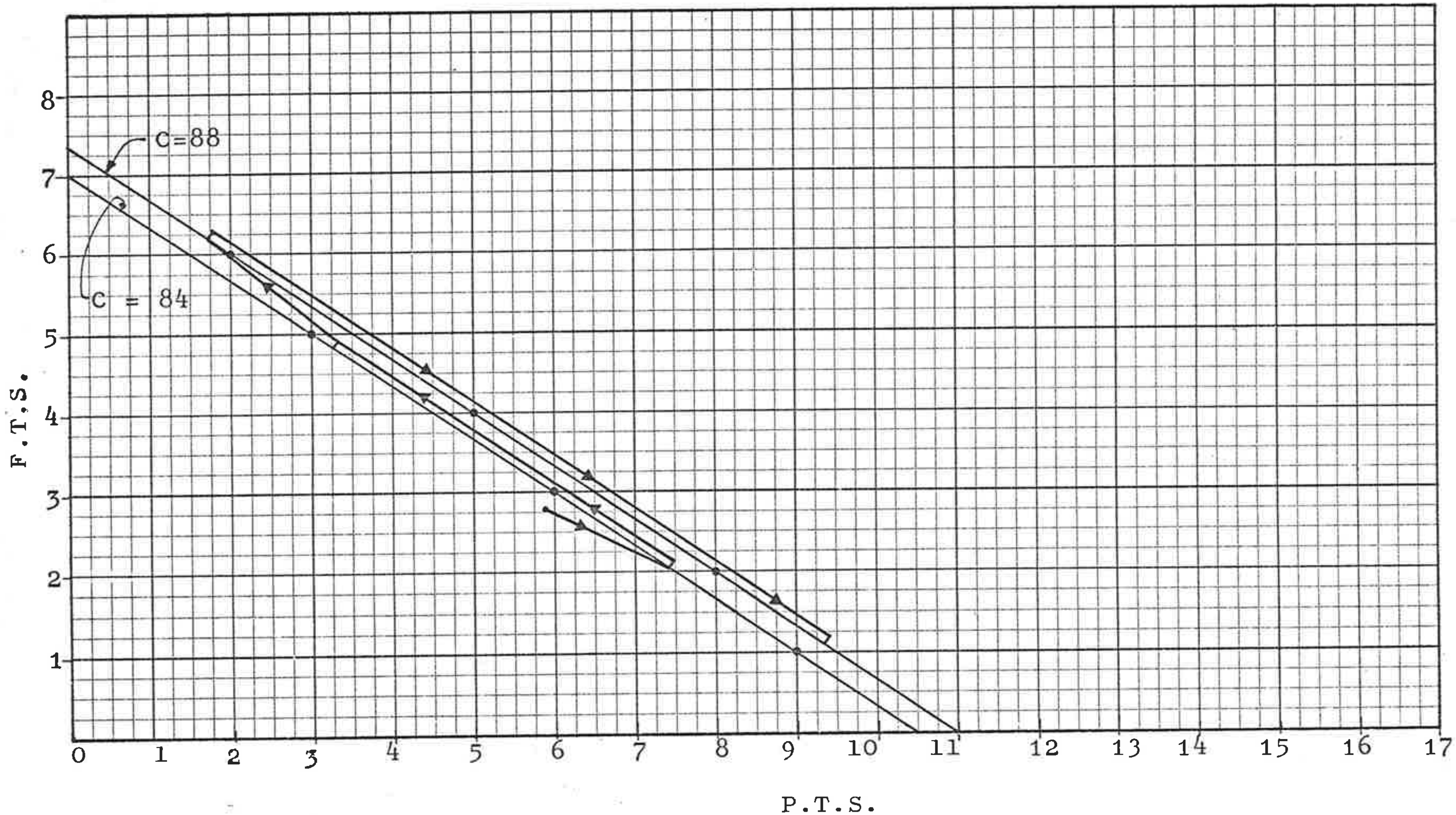
The distribution of points with C_f = 12, C_p = 8 and H = 4.

Figure 6



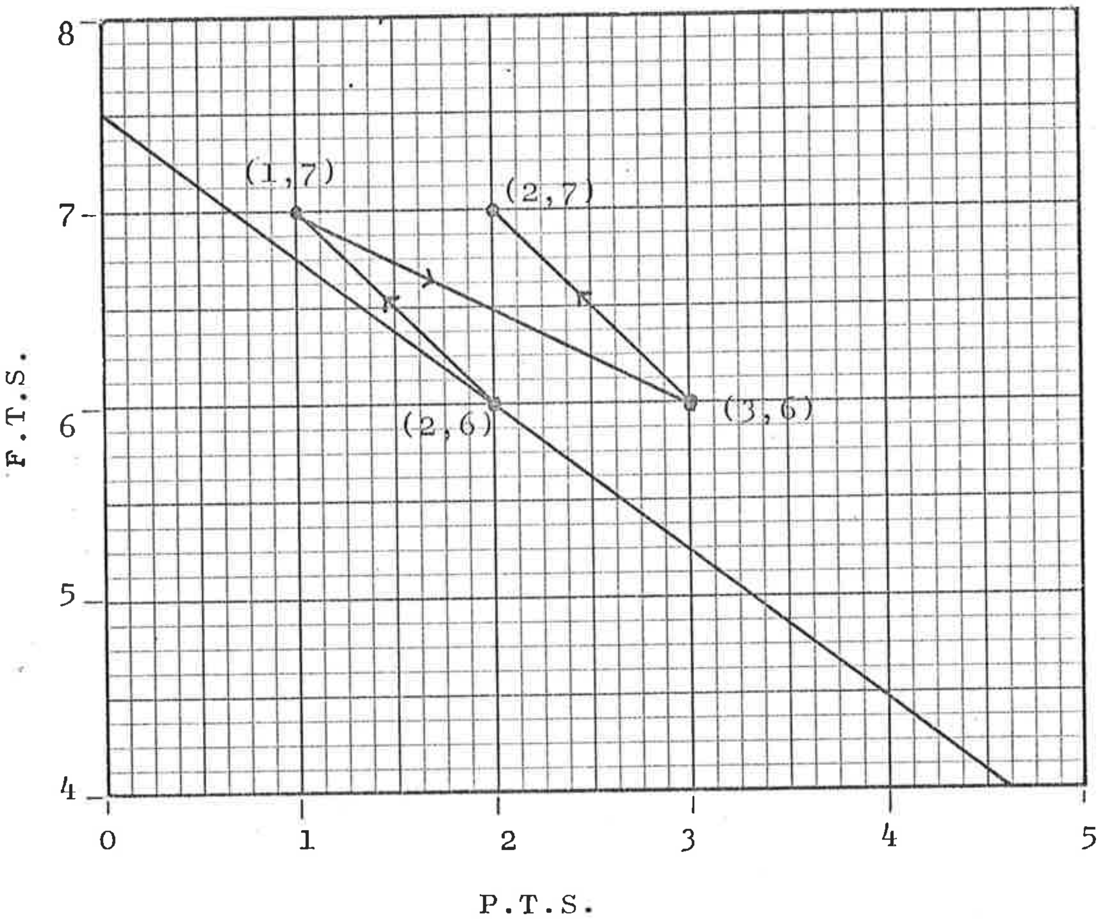
The distribution of points with $C_f = 12$, $C_p = 7$ and $H = 1$

Figure 7



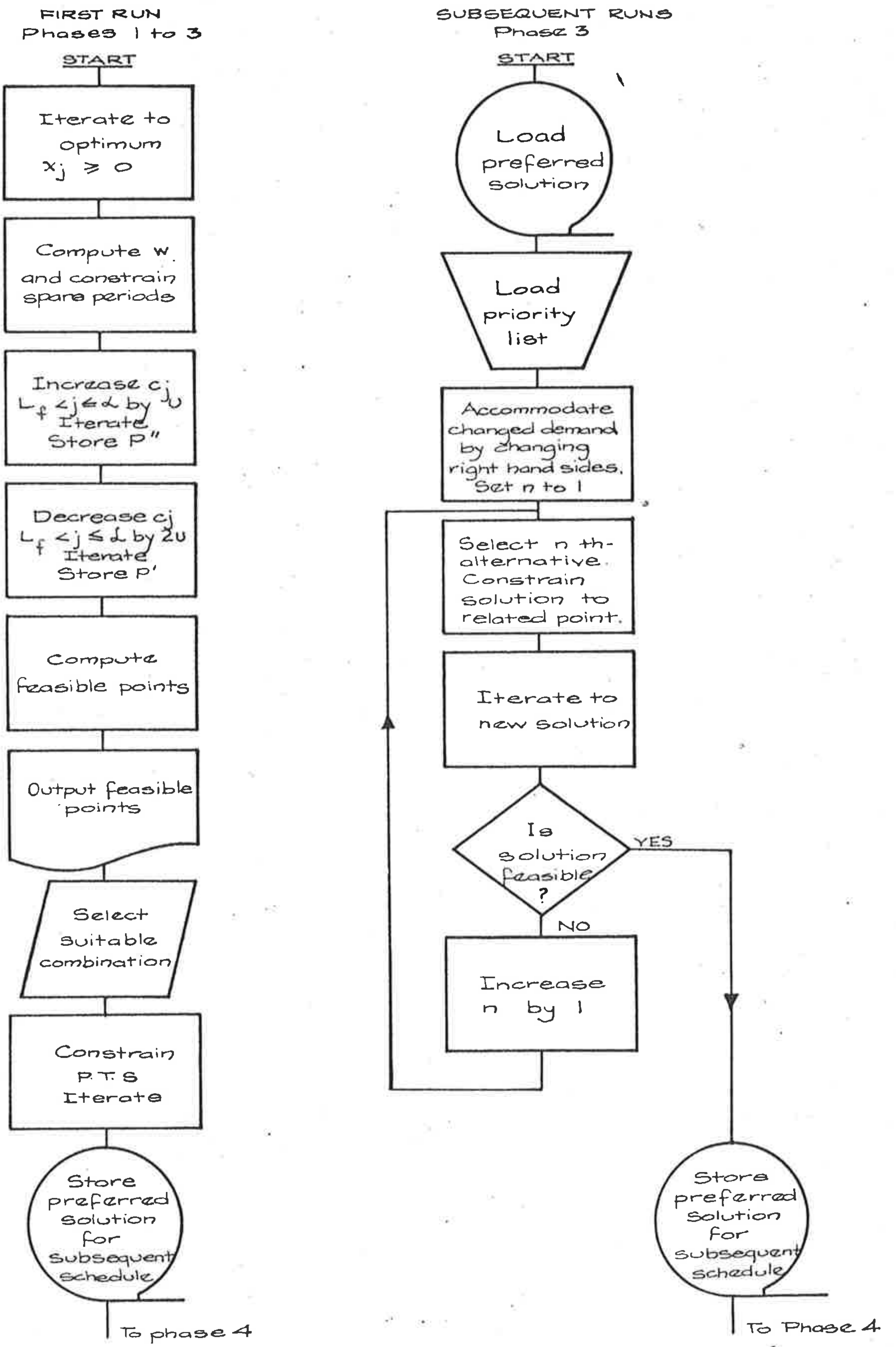
The Solution in phase 1 contains 5 7/8 P.T.S. and 2 5/6 F.T.S. with C^{*} = 81 periods
The integer point is chosen from those available with C at 84 and 88 unit periods.

Figure 8



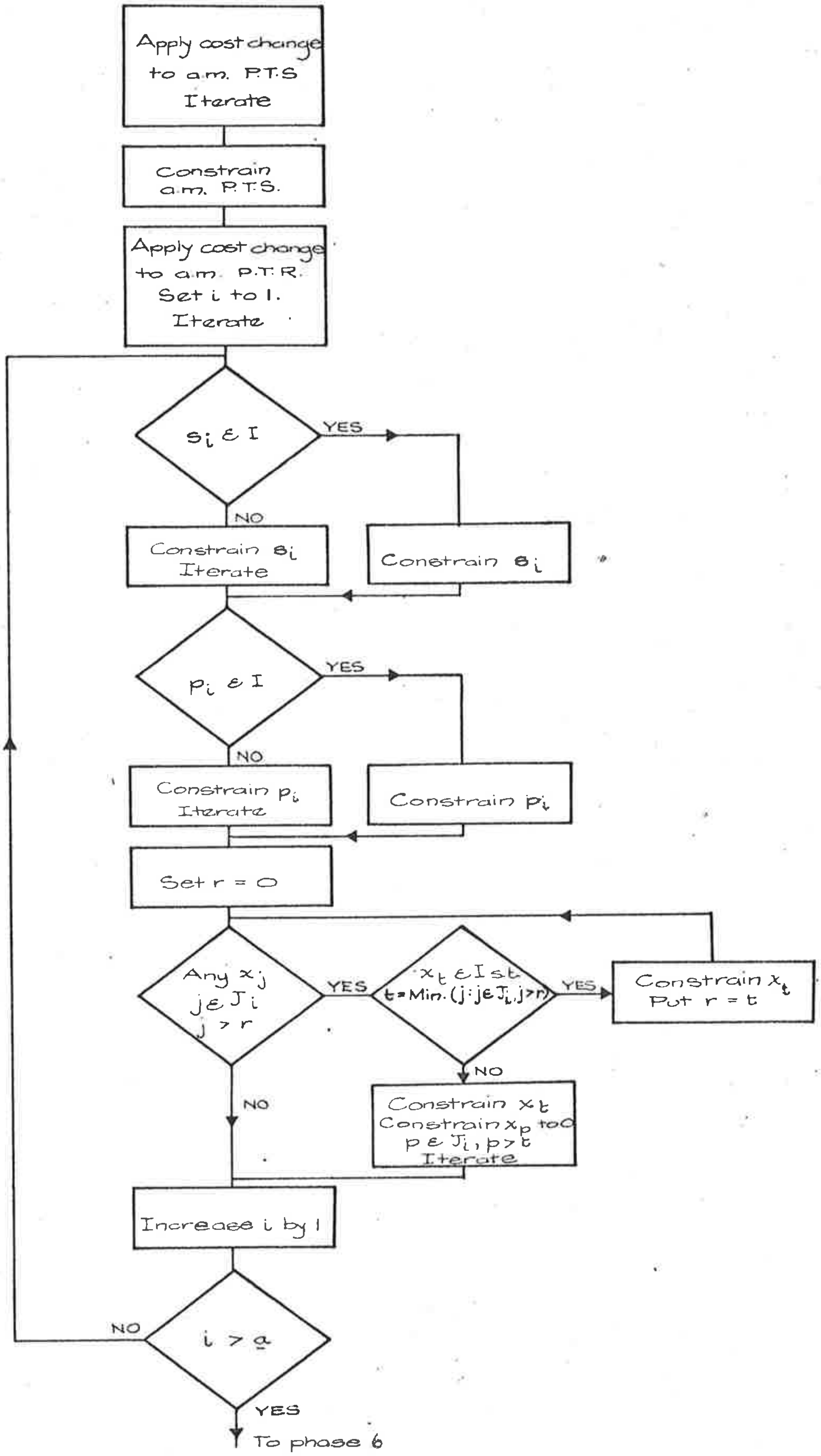
The alternative points in the vicinity of (2,6). The path indicates the order of investigation of the available points if idle time is to be minimised.

Figure 9



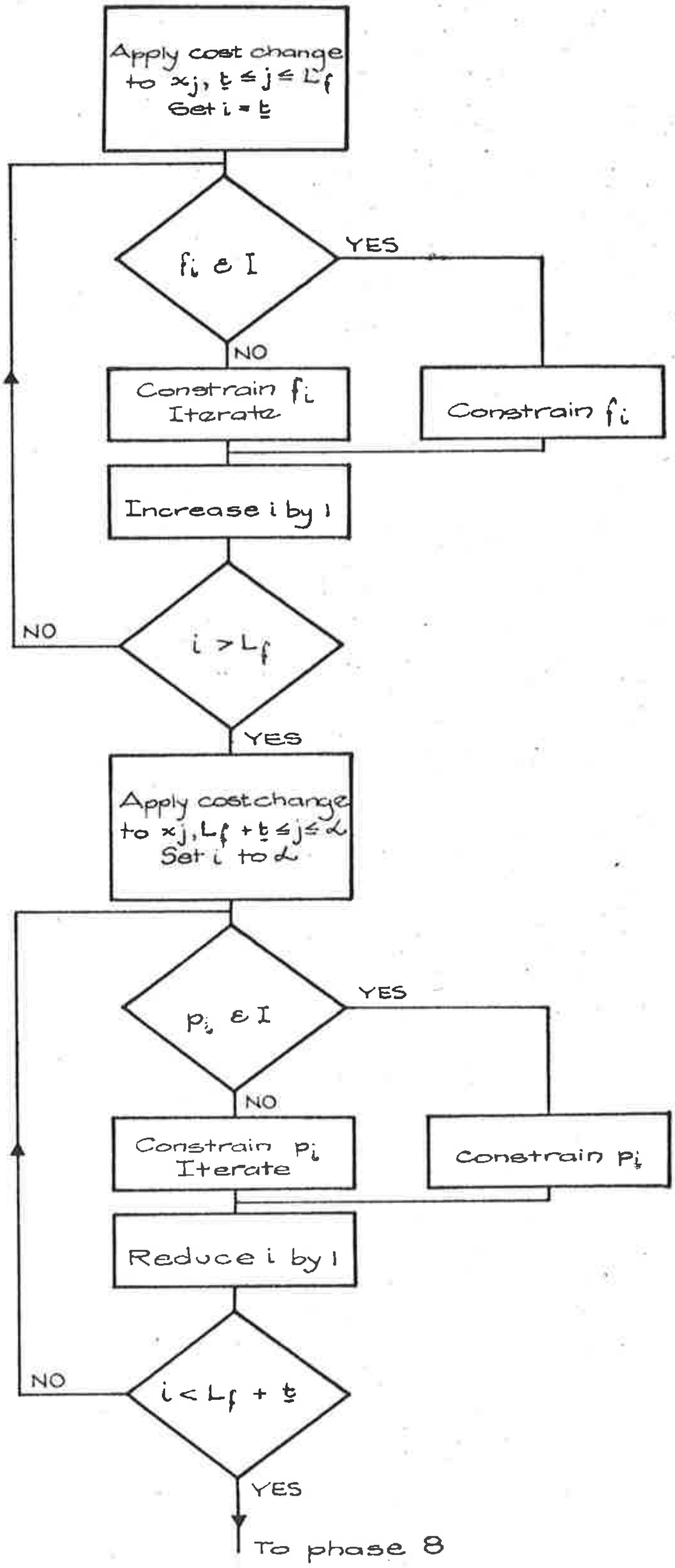
INITIAL PROCEDURES

Figure 10



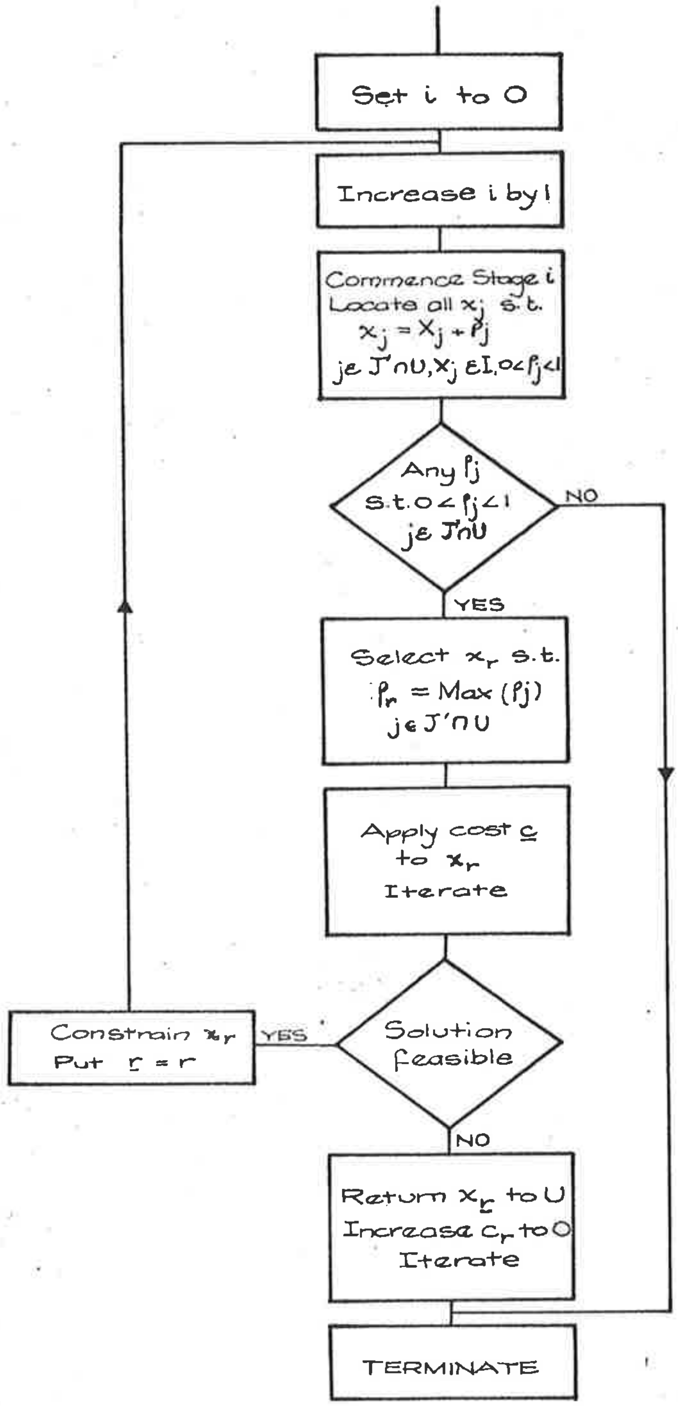
Apply cost change to a.m. P.T.S Iterate
Constrain a.m. P.T.S.
Apply cost change to a.m. P.T.R. Set i to 1. Iterate
$s_i \in I$
YES
NO
Constrain s_i Iterate
Constrain s_i
$p_i \in I$
YES
NO
Constrain p_i Iterate
Constrain p_i
Set r = 0
Any x_j $j \in J_i$ $j > r$
YES
$x_t \in I$ s.t. $t = \text{Min.}(j : j \in J_i, j > r)$
YES
Constrain x_t Put r = t
NO
Constrain x_t Constrain x_p to 0 $p \in J_i, p > t$ Iterate
NO
Increase i by 1
NO
$i > a$
YES
To phase 6

PHASES 4 & 5



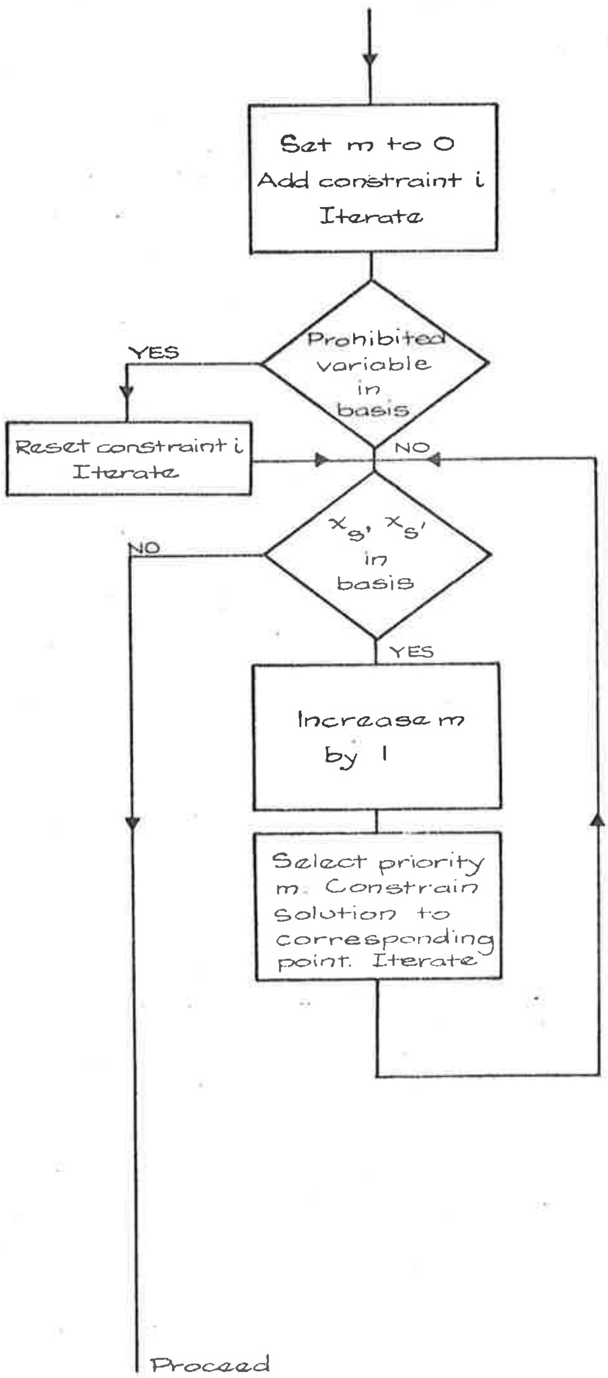
PHASES 6 & 7

FIGURE 12



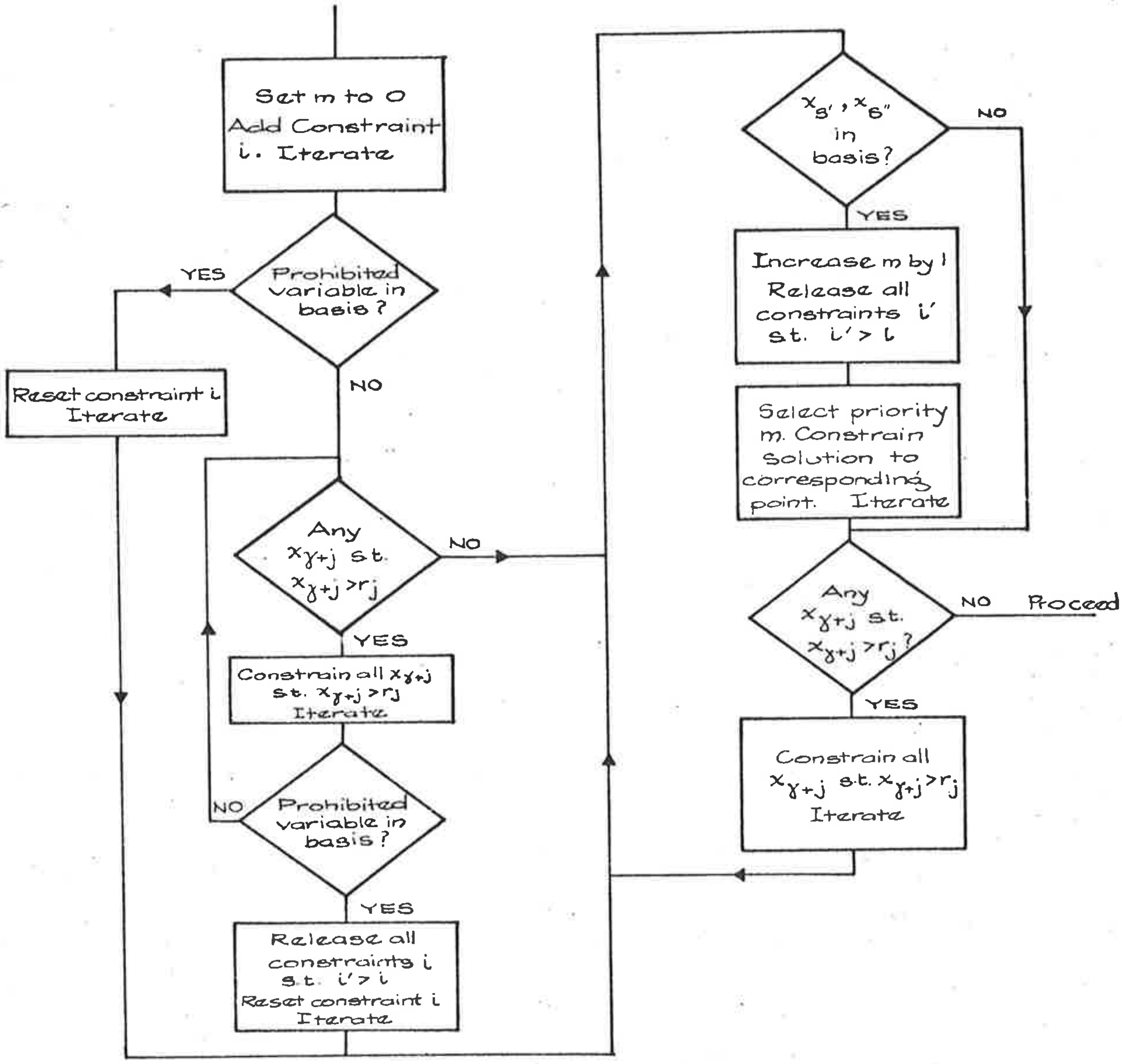
PHASE 8

FIGURE 13



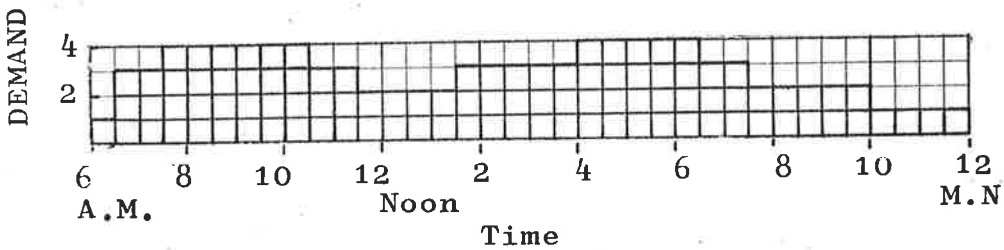
INCREASING C - THE LABOUR CONTRIBUTION

Figure 14



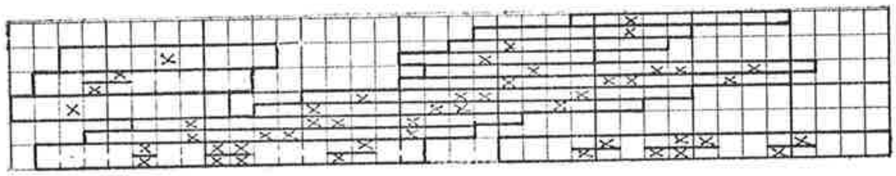
REDUCING D - THE DEMAND

Figure 15



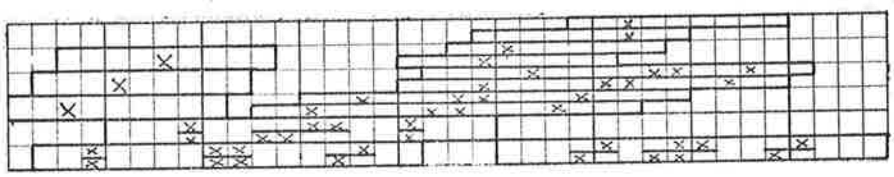
Demand Histogram

Figure 16a



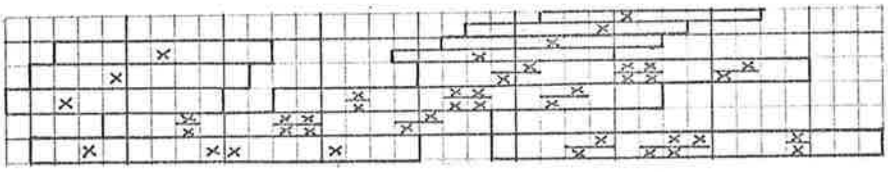
At the conclusion to phase 4, a large number of shifts and reliefs are allocated in halves

Figure 16b



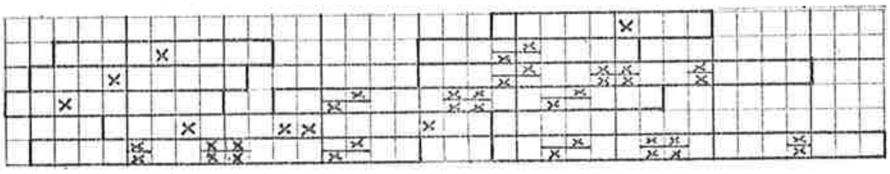
At the conclusion to phase 5, a.m. P.T.S. and early F.T.S. together with a.m. P.T.R. are integer,

Figure 16c



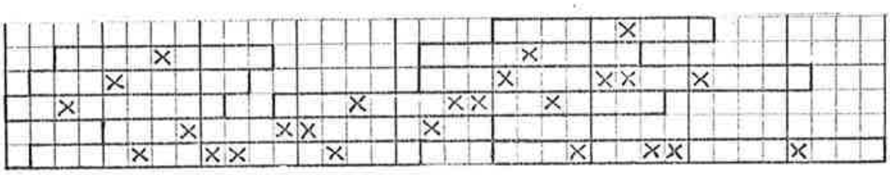
At the conclusion to phase 6 all F.T.S. are integer.

Figure 16d



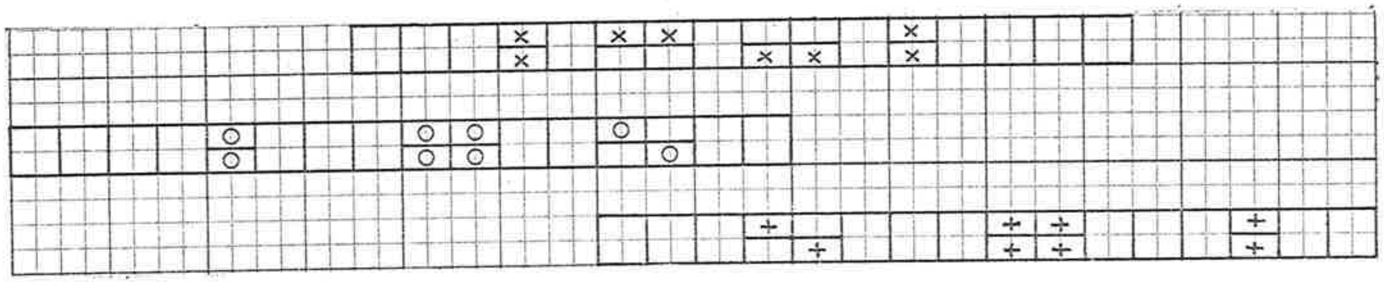
At the conclusion to phase 7 all shifts are allocated as integers.

Figure 16e



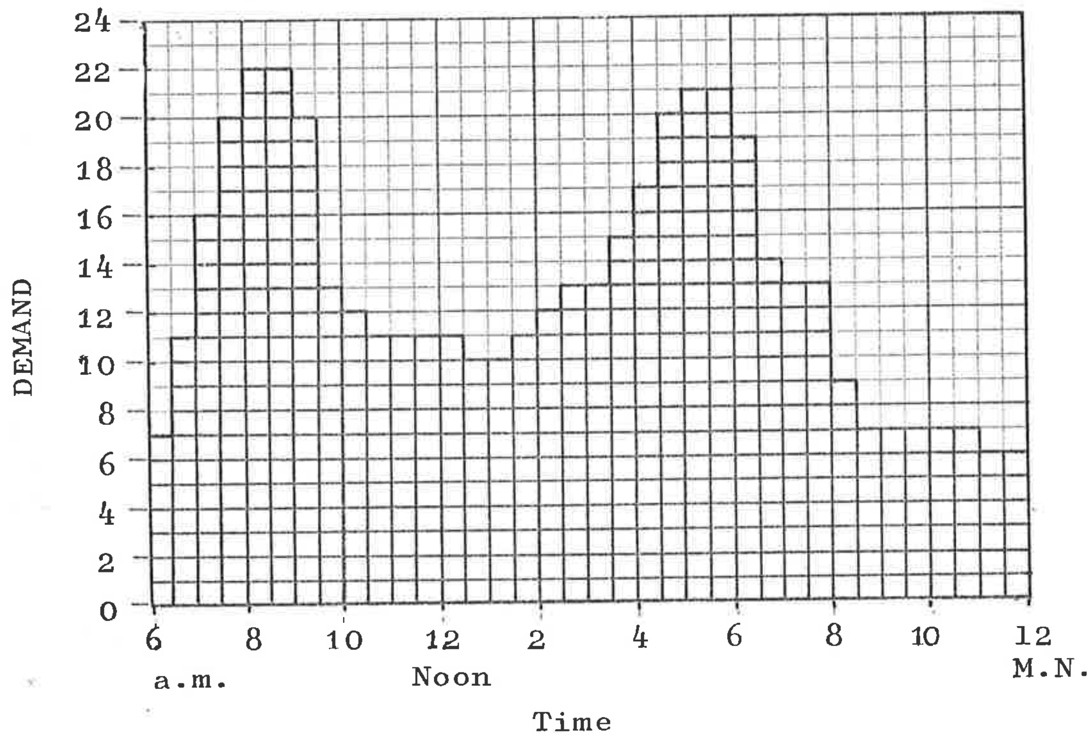
In phase 8 any remaining reliefs are constrained and the accompanying diagram represents a completed solution.

Figure 16f



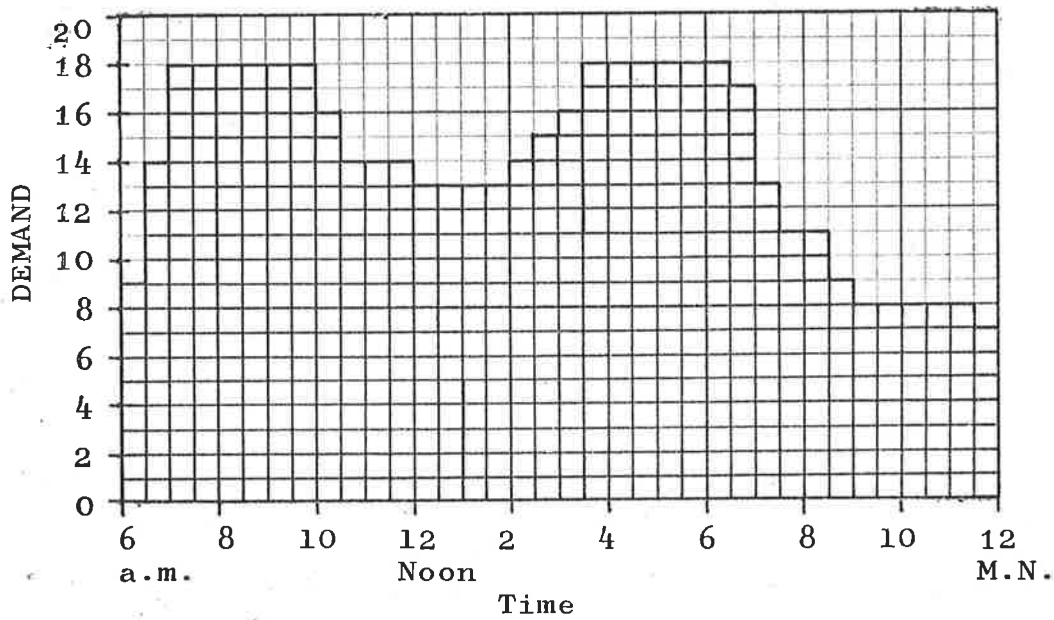
Three 8 hour shifts are represented with τ at 30 minutes. The breaks are of 30, 60 and 30 minutes duration respectively. If the maximum segment of continuous duty is 2 hours, it is not possible to allocate integer reliefs under the existing conditions.

Figure 17



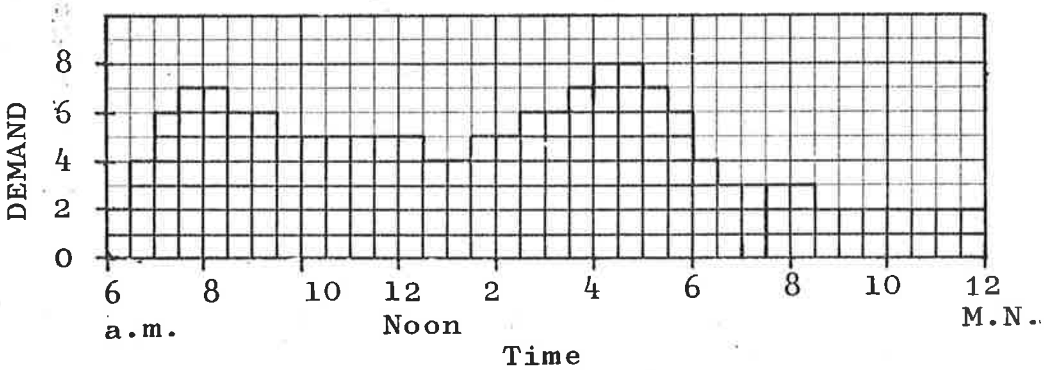
Demand histogram for the Sydney Harbour Bridge with τ at 30 minutes.

Figure 18



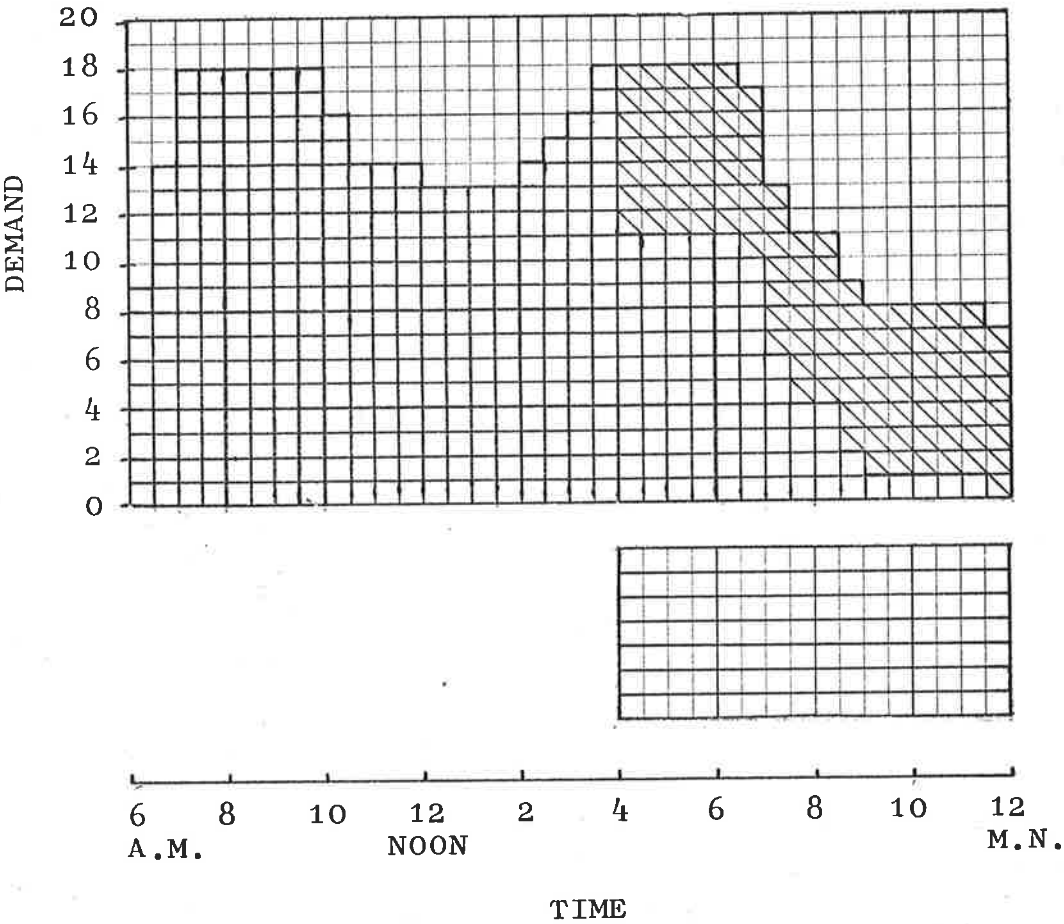
Demand histogram for the Lincoln Tunnel with τ at 30 minutes - Spring 1967.

Figure 19



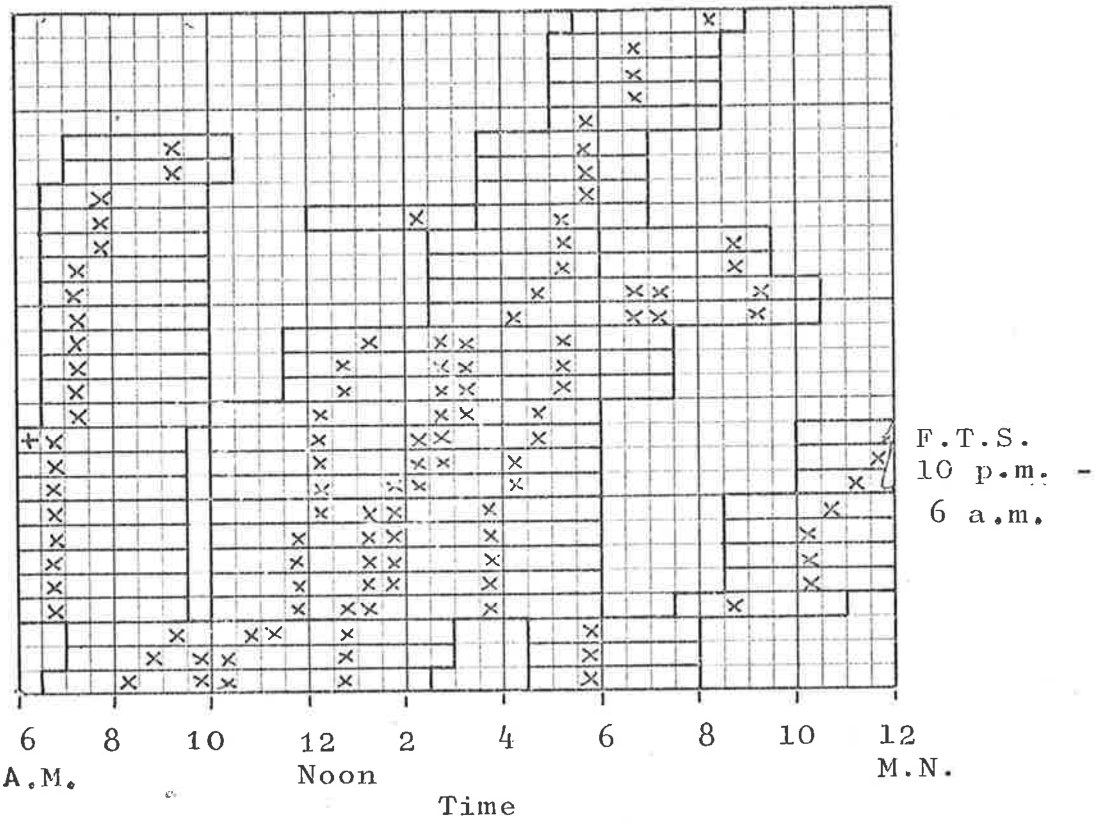
Demand histogram for the Lower Yarra Crossing with τ at 30 minutes.

Figure 20



7 late evening F.T.S., commencing at 4 p.m. and concluding at 12 midnight are scheduled. The residue contains an irregular peak between 2 p.m. and 4 p.m.

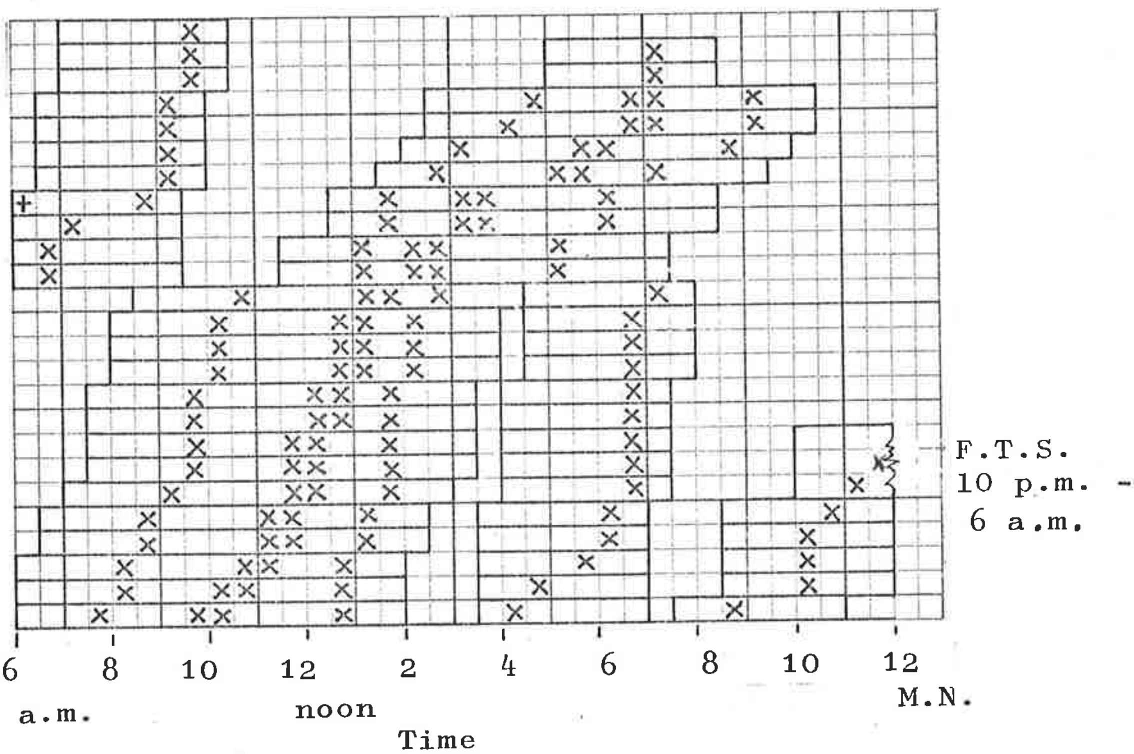
Figure 21



Schedule 1

Relief periods are represented by x and spare periods by +. τ is 30 minutes and the schedule contains 1 spare period. The part time component is in the vicinity of 50%.

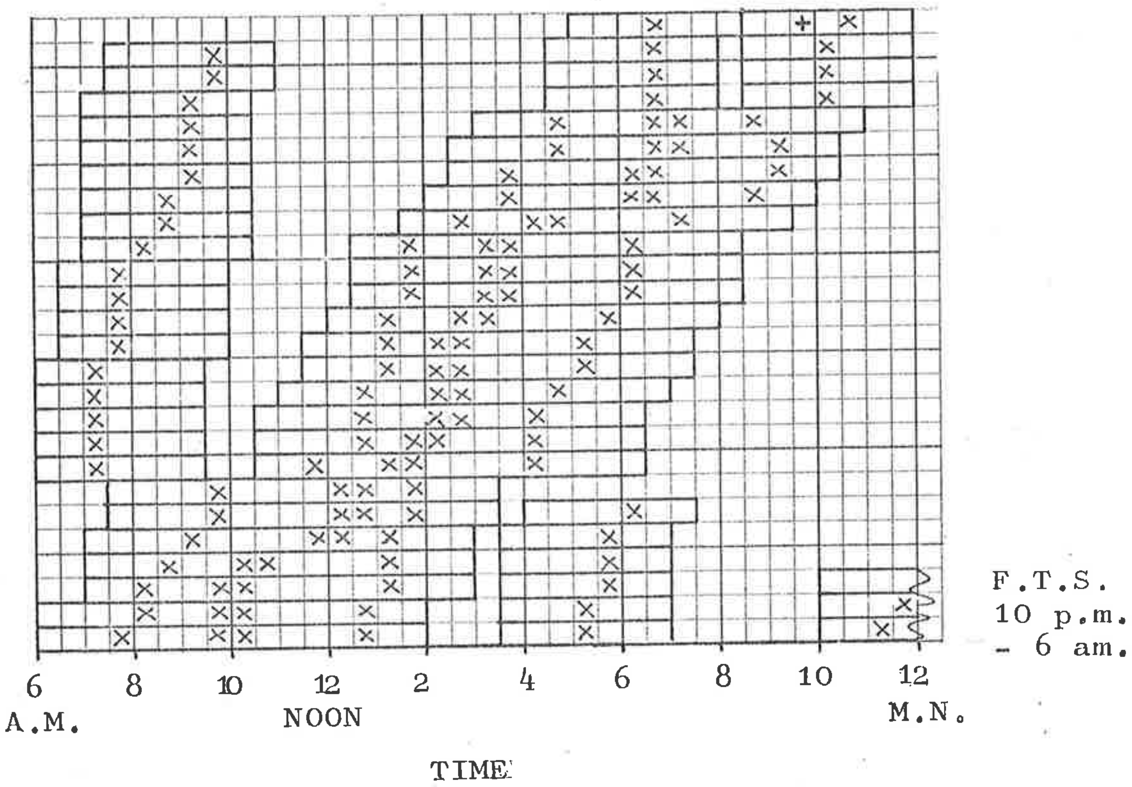
Figure 22



Schedule 2

Relief periods are represented by x and spare periods by +. The schedule contains a part time component of approximately 40% and 1 spare period.

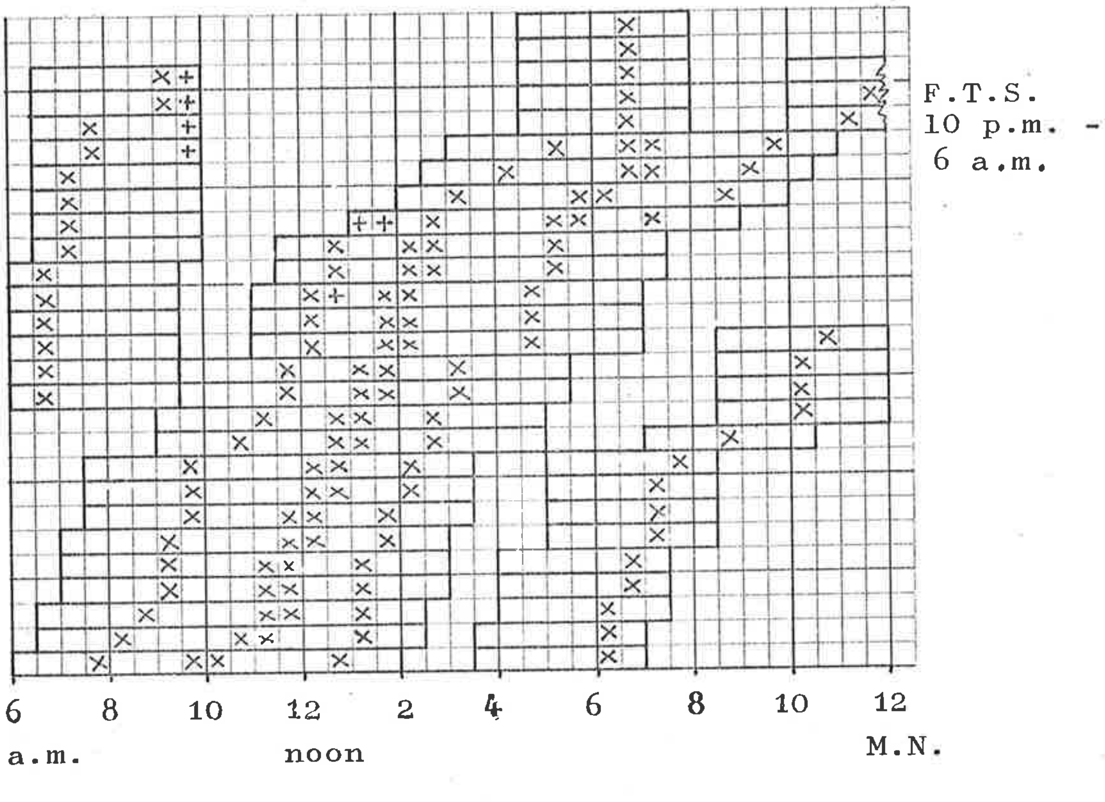
Figure 23



Schedule 3

Relief periods are represented by x and spare periods by +. The schedule contains 1 spare period and a part time component of approximately 40%, including an increased level of part time participation in the a.m. region.

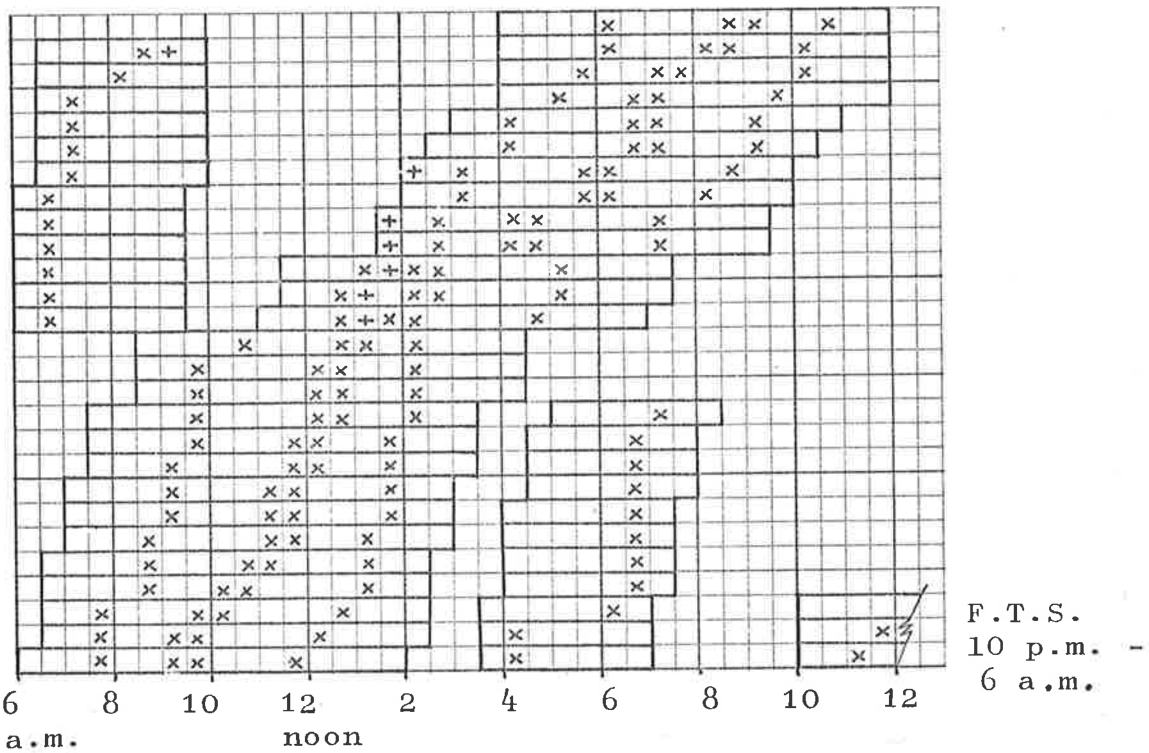
Figure 24



Schedule 4

Relief periods are represented by x and spare periods by +. The schedule contains a part time component of approximately 40% and 7 spare periods. The increased level of idle time is due to the additional restriction applying to the F.T.R.

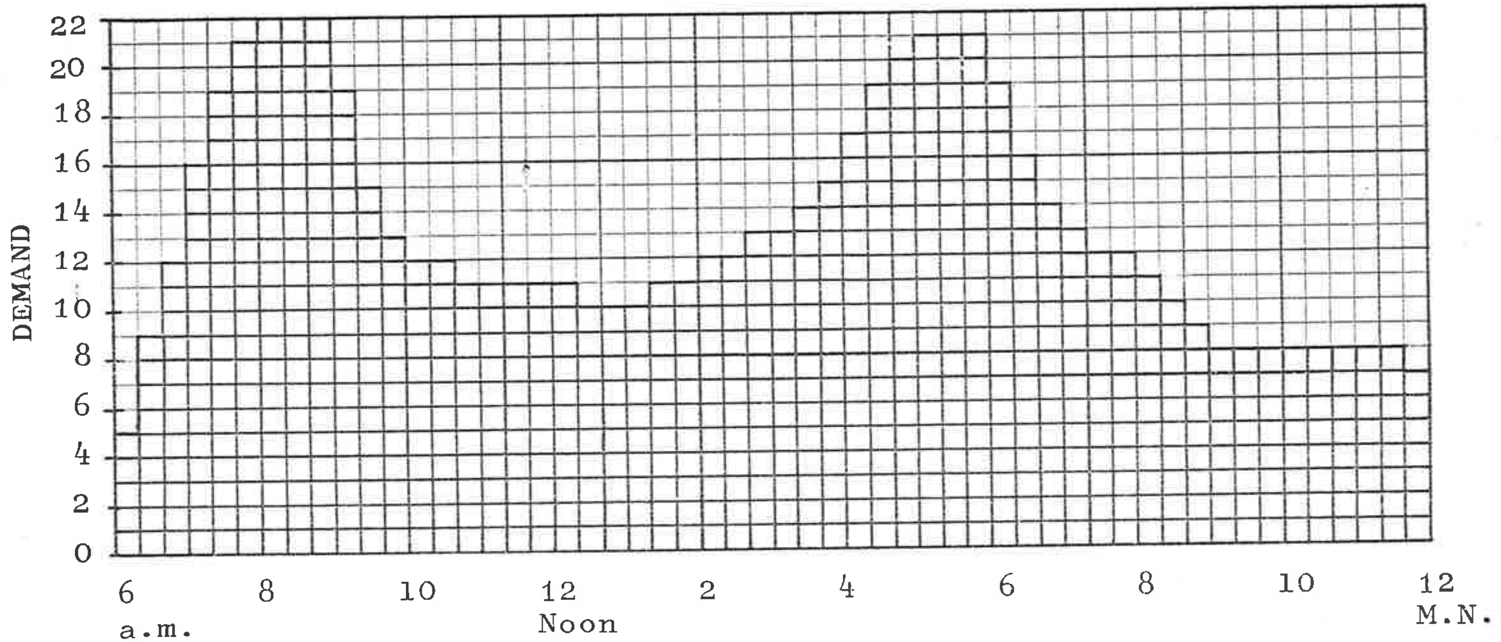
Figure 25



Schedule 5

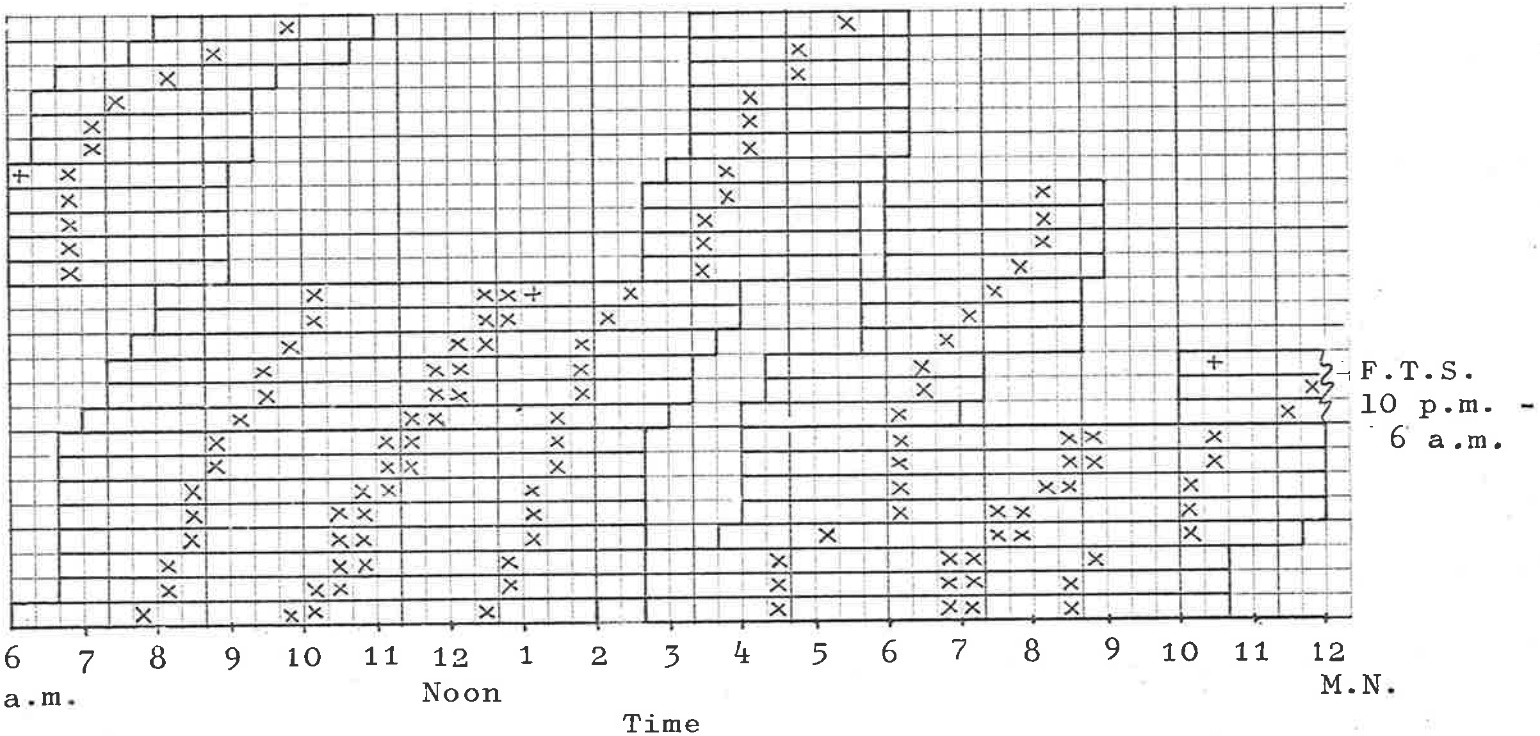
Relief periods are represented by x and spare periods by +. The schedule contains a part time component of approximately 30% and 7 spare periods.

Figure 26



The Sydney Harbour Bridge demand histogram with τ at 20 minutes.

Figure 27



Schedule 6.
Relief periods are represented by x and spare periods by +.
⊣ is 20 minutes, the part time component is approximately 34%
and there are 3 spare periods.

Figure 28.

τ	Relief Pattern	No. of F.T.S. Locations L_f	Relief options/ F.T.S. R_f	Total Constraints (Approx.) M	Total F.T.R. ϕ
60 mins	60-60-60	11	4	39	44
30 mins	30-60-30	21	20	80	420
20 mins	20-40-20	31	33	120	1023
15 mins	15-45-15	41	76	160	3116

TABLE I

τ	Relief Pattern	Commencement time for shift	Relief options/ shift
20 mins	(20-40)(40-20)	6 a.m.	28
20 mins	(20-40)(40-20)	8 a.m.	40
20 mins	(20-40)(40-20)	9.30 a.m.	28
15 mins	(15-45)(45-15)	6 a.m.	45
15 mins	(15-45)(45-15)	8 a.m.	69
15 mins	(15-45)(45-15)	9.30 a.m.	45

TABLE II

No.	τ mins.	Relief	Shift Length	Net. Working Time	Shift Adjustment Reqd.	Adjustment to Histogram
current		30-40-30	8.29	6.49	current	current
1.	20	20-40-20	8.20	7.00	pay 11 mins.	Nil
2.	21	21-42-21	8.24	7.00	pay 11 mins.	12 mins.
3.	22	22-44-22	8.26	6.58	pay 9 mins.	10 mins.
4.	23	23-46-23	8.26	6.54	pay 5 mins.	14 mins.
5.	24	24-48-24	8.24	6.48	down 1 min.	Nil.
6.	24	24-48-24	8.48	7.12	pay 23 mins.	Nil.
7.	25	25-50-25	8.45	7.05	pay 16 mins.	15 mins.
8.	26	26-52-26	8.40	6.56	pay 7 mins.	10 mins.
9.	30	30-30-30	8.30	7.00	pay 11 mins.	Nil.
10.	35	35-35-35	8.45	7.00	pay 11 mins.	5 mins.

Currently shifts include an 11 min. allowance at the end of the shift making a total shift length of 8.40. This allowance can be disregarded in the mathematical treatment.

TABLE III

No.	τ mins.	Relief	Shift Length	Net. Working Time	Shift Adjustment Reqd.	Adjustment to Histogram
current		30-45.45-30	7.50	6.35	current	current
1.	20	20-40.40-20	7.40	6.40	pay 5 mins.	Nil.
2.	20	40-40	8.00	6.40	pay 5 mins.	Nil.
3.	21	21-42.42-21	7.48	6.45	pay 10 mins.	12 mins.
4.	22	22-44.44-22	7.42	6.36	pay 1 min.	10 mins.
5.	24	24-48.48-24	8.00	6.48	pay 13 mins.	Nil.
6.	25	25-50.50-25	7.55	6.40	pay 5 mins.	15 mins.

TABLE IV

Schedule	Total Run Time No Preferred Solution	Approx.Run Time Preferred Solution	No. of Solns.	No. of Iterations	Termination Phase
1	2 mins. 1 sec.	1 min.	8	533	7
2	1 min. 20 secs.	40 secs.	6	452	5
3	54 secs.	20 secs.	6	286	5
4	2 mins. 29 secs.	1 min. 50 secs.	20	841	7
5	3 mins. 20 secs.	2 mins.30 secs.	21	1166	8
6	4 mins. 30 secs.	3 mins.	15	960	8

TABLE V

Phase	No. of Solns. Generated	No. of Iterations in Phase	Cumulative Iteration Count
1	1	302	302
2	1	55	357
3	1	65	422
4	1	53	475
5	6	160	635
6	1	35	670
7	3	201	871
8	1	89	960

Schedule 6.

TABLE VI.

Schedule	τ mins.	No. of P.T.S.	% of Part Time Work Approx.	No. of Spares	Efficiency % Approx.	Status
1	30	42	53	1	99	Optimal
2	30	32	40	1	99	Optimal
3	30	32	40	1	99	Optimal
4	30	33	41	7	98½	Not known
5	30	23	29	7	98½	Not known
6	20	32	34	3	99	Optimal

TABLE VII.

GLOSSARY

F.T.S.	Full time shift(s).
F.T.R.	Relief(s) for full time shifts.
P.T.S.	Part time shift(s).
a.m.P.T.S.	P.T.S. in the region of the a.m. peak.
p.m.P.T.S.	P.T.S. in the region of the p.m. peak or in the evening.
P.T.R.	Relief(s) for part time shifts.
a.m.P.T.R.	Relief(s) for part time shifts in the a.m. region.
p.m.P.T.R.	Relief(s) for part time shifts in the p.m. region.
D.M.R.	Department of Main Roads of New South Wales.
P.N.Y.A.	Port of New York Authority.

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