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CENTRES, FIXED POINTS AND INVARIANT INTEGRATION

by

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SUMMARY

In this thesis we are interested in fixed point Theorems and their relation to the existence of right invariant integrals on compact and locally compact groups and semigroups.

We construct two types of centre for non-empty convex compact sets C and use these for fixed point Theorems.

For the first of these, the radius defined centre, we consider a normed vector space, endowed with a locally convex Hausdorff topology $\mathcal{T}$ such that the norm is lower semicontinuous with respect to $\mathcal{T}$. The centre is shown to be non-empty. The norm is then assumed to be locally uniformly convex and the centre shown to consist of one point. These properties of the centre are used to show that any set $\mathcal{H}$ of non-expansive mappings of C onto C has a common fixed point in C and any left reversible semigroup of continuous (with respect to $\mathcal{T}$) non-expansive affine mappings of C into C has a common fixed point in C. By the construction of a special norm, the first of these Theorems is applied to show the existence of a right invariant integral on a compact semigroup. A counter example where the centre equals C is also constructed.

For the second, the quotient defined centre, we consider a vector space X with a locally convex metrizable

topology $\mathcal{J}$. When C contains an internal point the centre is shown to exist and be a non-empty proper subset of C. C is then assumed to have quotient structure and it is shown that any set $\mathcal{H}$ of 1:1 affine mappings of C onto C has a common fixed point in C and any left reversible semigroup of continuous 1:1 affine mappings of C into C has a common fixed point in C. Both these Theorems are applied to show that any non-empty convex compact subset C of the n-dimensional Euclidean space R^n contains a common fixed point for any set of affine mappings of C onto C and, also, any left reversible semigroup of continuous affine mappings of C into C. Furthermore it is shown that any strictly convex subset of X, the set of end points of lines through which is closed, and, when X also has a norm, any uniformly convex subset of X contain' common fixed points for the mappings of $\mathcal{H}$.

Sneperman's fixed point Theorem is used to show the existence, on a locally compact metrizable space X, of an integral which is right invariant under a left reversible semigroup of bounded continuous mappings of X into X, which satisfy certain compactness conditions. For a locally compact topological group G with a countable basis, a non-empty convex weakly compact subset Γ of positive linear functionals on $C_0(G)$, the space of all real valued continuous functions on G with compact support, is constructed and shown to be right invariant under the group operations. No fixed point Theorem giving a common fixed point in Γ for the group operations has been found, but an interesting comparison is made with Γ', a subset of bounded members of Γ.

SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, this thesis contains no material previously published or written by any other person, except when due reference is made in the text of this thesis.

(T.J. Cooper)

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CHAPTER I

INTRODUCTION

For a locally compact semigroup G, $C_0(G)$ will denote the linear space of continuous real valued functions f on G, with compact support $spt(f)$ and with norm defined by

$$|f| = \sup_{x \in G} |f(x)|.$$

The algebraic dual Λ consists of all linear functionals λ on $C_0(G)$ and is assumed to be endowed with the weak topology, the basis of whose open sets at the origin is the collection of sets of the type

$$U(0,A,\varepsilon) = \{\mu \in \Lambda / |\mu(f)| < \varepsilon \text{ for all } f \in A\},$$

where ε is any positive real number and A is any finite subset of $C_0(G)$. Each of the positive λ in Λ can be considered as an integral on G.

For each $a \in G$ and $f \in C_0(G)$, a mapping f_a, which belongs to $C_0(G)$ if there are suitable compactness conditions on G, can be defined by

$$f_a(x) = f(xa)$$

for all $x \in G$. Let $\mathcal{H}$ denote the semigroup of all transformations T_a of Λ into itself, where T_a is defined, for each $a \in G$, by putting

$$T_a\lambda(f) = \lambda(f_a)$$

for all $\lambda \in \Lambda$. If λ is positive and such that

$$T_a\lambda = \lambda$$

for all $a \in G$, then λ is considered to be a right invariant integral on G. In a similar manner a left invariant integral can be defined. Furthermore, if an integral is both left and right invariant, it is called invariant.

If $\mathcal{L}$ is a set of mappings of a subset C, of a space X, into C, then a point x in C is said to be a common fixed point, in C, for the mappings of $\mathcal{L}$, if

$$Lx = x$$

for all $L \in \mathcal{L}$. Fixed point theory is concerned with the weakening of the conditions on $\mathcal{L}$, C and X necessary to prove the existence of the common fixed point. Usually $\mathcal{L}$ is, at least, a semigroup of affine maps, C is a nonempty convex compact subset and X is, at least, linear, Hausdorff and locally convex.

The existence of invariant integrals and the existence of fixed points are related.

$\mathcal{H}$ is a set of mappings of Λ into itself. Therefore, if a nonempty convex compact subset Γ of positive members of Λ can be constructed such that

$$T_a\Gamma \subseteq \Gamma$$

for all $a \in G$, and the conditions on $\mathcal{H}$, Γ and Λ

satisfy the requirements of a fixed point Theorem, then there is in Γ a common fixed point for the mappings of $\mathcal{H}$. This common fixed point is a right invariant integral on G. In many cases too, the existence of invariant integrals leads to the existence of common fixed points.

The theory for the existence of invariant integrals on locally compact semigroups is fairly complete. The work of J.H. Michael [17], P.S. Mostert [21] and L.N. Argabright [1] has shown that a locally compact semigroup G, satisfying a condition to ensure that f_a belongs to $C_0(G)$ for each $f \in C_0(G)$, admits a right invariant integral if G contains a unique minimal left ideal (which is necessarily closed).

The theory for the existence of common fixed points is not so complete. The most general Theorems, Sneperman ([24]), C. Ryll-Nardzewski ([23]) and R.D. Holmes and A.T. Lau ([14], and [15]), require conditions of equicontinuity, non contractibility (0 does not belong to the closure of the set $\{Lx-Ly/L \in \mathcal{L}\}$ for any x and $y \in C$ such that $x \neq y$) and asymptotic non expansiveness (there exists a left ideal J in $\mathcal{L}$ such that

$$p(Lx-Ly) \leq p(x-y),$$

for all x and $y \in C$, $L \in J$ and $p \in Q$, the collection of seminorms generating the topology of X). Although the first of these theorems has been used to show the existence of a right invariant integral on a compact semi-group, it does not seem possible to use any of them to show the existence of a right invariant integral on a locally compact semigroup or a locally compact group. To this extent, fixed point theory has not caught up to the knowledge of invariant integrals.

It should be noted that many of the most recent papers on fixed point theory consider the "action" of a semigroup on C, but because the interest in this thesis lies with Theorems that can be applied to show right invariant integrals on semigroups, the concept of "action" will not be introduced. Those interested may see Holmes and Lau [14].

The relation between fixed point Theorems and invariant integrals has received a great deal of attention and has stimulated the production of more general fixed point Theorems. The work of M.M. Day [5] and [6], T. Mitchell [19] and A.T. Lau [16] gives a good introduction to this work.

The above is a brief outline of the present state of fixed point theory and invariant integration. This thesis interests itself in fixed point Theorems and their relation to the existence of a right invariant integral on a compact or locally compact semigroup.

The three papers that have given direction to this thesis are those of Sneperman [24] and M. Edelstein [10] and [11]. Sneperman generalized Kakutani's fixed point Theorem (see page 457 of [9]) to make it applicable to compact semigroups (see the beginning of Chapter 4 for a statement of this Theorem). He then constructed a non-empty convex compact subset Γ in Λ which he showed was invariant under $\mathfrak{H}$. $\mathfrak{H}$ was shown to be weakly equi-continuous and the fixed point Theorem used to show the existence of a right invariant integral on the compact semi-group. Edelstein in the first paper [10] constructed an asymptotic centre for a bounded sequence of points $\{u_n\}$ in a closed convex subset C of a uniformly convex Banach space X. For each positive integer m, he defined

$$r_m(x) = \sup\{|u_n - x| / n \geq m\}$$

for each $x \in C$ and let a_m be the unique point in C such that

$$r_m(a_m) = \inf\{r_m(x) | x \in C\}.$$

The sequence $\{a_m\}$ was shown to converge to a point a in C, this point being the asymptotic centre. He then showed that for a function f of C into itself, the asymptotic centre of the sequence $\{f^n(x)\}$ in C, for some $x \in C$, is a fixed point of f, if f also satisfied a special nonexpansive property which takes account of the x chosen. In the second paper [11], he has further generalized this Theorem and used it to show that a commutative family of mappings, which satisfies a special non expansive condition, has a common fixed point. We have, in this thesis, extended the concept of centre to find fixed point Theorems and used a method of application similar to Sneperman's.

One type of centre is discussed in chapter 2 of this thesis. We consider a normed vector space X, which is also endowed with a locally convex Hausdorff topology such that the norm is lower semicontinuous with respect to $\mathcal{J}$, and a nonempty convex subset C of X, which is bounded with respect to the norm and compact with respect to $\mathcal{J}$. Then a concept of radius is used to define a centre of C, which is shown to be invariant under any set of non expansive mappings of C onto C. When the norm is locally uniformly convex, the centre has only one point. This yields two fixed point Theorems, the first of which is sufficient to show the existence of a right invariant integral on a compact semigroup, using a special norm on Λ. Hence in this application, the Theorem is as good as the best of the others presently known. The second fixed point Theorem is similar to the first and also is applicable to show the existence of a right invariant integral on a compact semigroup.

A second type of centre is discussed in Chapter 3. We consider a vector space X with a locally convex metrizable topology and a nonempty convex compact subset C of X, which contains at least one internal point. A quotient concept for lines through C is used to define a centre of C, which is independent of any norm or other topology on X. This centre is shown to be invariant under any set of 1:1 affine maps of C onto C, but only to exist when C contains an internal point. This also yields two fixed point Theorems, when the set C satisfies a condition called quotient structure (for something similar see normal structure in Brodskii and Milman [3]). These Theorems are applied to the n-dimensional Euclidean space R^n to show that any nonempty convex compact subset C of R^n contains a common fixed point for any set of affine mappings of C onto C or any left reversible semigroup of continuous affine mappings of C into C. Furthermore, in the general theory, when the set C satisfies certain convexity properties, it is shown to contain at least one internal point and the centre is shown to consist of only one point which yields two further fixed point Theorems. The interest in this chapter is that the fixed point Theorems result from conditions on the set C, not on the mappings and that we seem to be following a new direction in which the fixed point is constructed independently of the topology.

In chapter 4 we apply Sneperman's fixed point Theorem to show the existence, on a locally compact metrizable space X, of an integral which is right invariant under a left reversible semigroup of bounded continuous mappings of X into X which satisfies certain compactness conditions. In chapter 5, we construct a nonempty convex weakly compact subset Γ of positive linear functionals on $C_o(G)$, where G is a locally compact topological group with a countable basis, and show that Γ is invariant under $\aleph$, the group of transformations. A fixed point Theorem which would give a common fixed point in Γ for the mappings of $\aleph$ has not been found, but Γ', a subset of bounded members of Γ, is shown to be nonempty, convex, weakly compact and invariant under $\aleph$, yet not to contain the right invariant integral (i.e. the common fixed point for the mappings of $\aleph$). This demonstrates the difficulty of finding such a fixed point Theorem and is interesting for further research, because it shows that the fixed point Theorem applicable to the locally compact group must include Γ but not Γ'.

CHAPTER 2

THE RADIUS DEFINED CENTRE

The work in this chapter was done jointly with J.H. Michael.

Throughout this chapter, X will denote a normed vector space which is also endowed with a locally convex Hausdorff topology $\mathcal{T}$, such that

(i) the vector space operations are continuous with respect to $\mathcal{T}$ in the usual way, and

(ii) the norm is lower semicontinuous with respect to $\mathcal{T}$ in the following sense:

for every $x \in X$ and $\varepsilon > 0$, there exists a $U \in \mathcal{T}$, such that $x \in U$ and

$$|y| > |x| - \varepsilon$$

for all $y \in U$.

C is a non-empty convex subset of X, which is bounded with respect to the norm and compact with respect to $\mathcal{T}$. For each $x \in C$, define

$$r(x) = \sup_{y \in C} |x-y|. \tag{1}$$

Put
$$\alpha(C) = \inf_{x \in C} r(x) \tag{2}$$

and let $\gamma(C)$ denote the set

$$\{x \in C \,/\, r(x) = \alpha(C)\}. \tag{3}$$

$\gamma(C)$ is called <u>the radius defined centre of C</u> or, when no confusion is likely to arise, <u>the centre of C.</u>

(It could also be referred to as the Chebyshev centre of

C in C.)

The concept of centre has been previously used to show the existence of fixed points by M. Edelstein [10] and [11] and L.P. Belluce and W.A. Kirke [2]. The concept of centre has also been used by M.S. Brodskii and D.P. Mil'man [3].

It will be shown that $\gamma(C)$ is non-empty, convex and compact with respect to $\mathcal{J}$. It will also be shown that every mapping of C onto C, which is nonexpansive with respect to the norm, takes $\gamma(C)$ into $\gamma(C)$ and that when the norm satisfies a special convexity condition, $\gamma(C)$ has exactly one point. In this case $\gamma(C)$ is therefore a common fixed point for all non-expansive mappings of C onto C.

The existence of a common fixed point is then used to prove the existence of an invariant integral on a compact metrizable semigroup. The existence of such an integral, has of course been known since 1956, when it was established by Rosen in [22]. It is given here as an application of the fixed point theorem. In [24] Sneperman has given a similar application using a somewhat different fixed point theorem.

The existence of the common fixed point for onto mappings is also used to show the existence of a common fixed point for nonexpansive mappings of $\gamma(C)$ into $\gamma(C)$ when the mappings satisfy the special intersection property of left reversibility (equivalent in the application, to right reversibility of the semigroup).

2:1 Theorem:

If $\mathcal{J}'$ is the restriction of the topology $\mathcal{J}$ to the set C, then r is lower semicontinuous on C with respect to $\mathcal{J}'$.

Proof:

Let $x \in C$ and $\varepsilon > 0$. There exists $z \in C$ such that

$$|x-z| > r(x) - \varepsilon/2 \qquad (4)$$

Since the norm is lower semicontinuous with respect to $\mathcal{J}$, then there exists a $V \in \mathcal{J}$ such that $x-z \in V$ and

$$|w| > |x-z| - \varepsilon/2 \qquad (5)$$

for all $w \in V$. Put $U = (V+z) \cap C$. Then $U \in \mathcal{J}'$, $x \in U$ and for all $y \in U$, $y-z \in V$, so that by (4) and (5),

$$r(y) \geq |y-z| > r(x) - \varepsilon$$

for all $y \in U$. #

2:2 Theorem:

$\gamma(C)$ is a nonempty convex subset of C which is bounded with respect to the norm and compact with respect to $\mathcal{J}$.

Proof:

For each n, let

$$\gamma_n(C) = \{x \in C \,/\, r(x) \leq \alpha(C) + \tfrac{1}{n}\}.$$

Since r is lower semicontinuous and C is compact, $\gamma_n(C)$ is compact with respect to $\mathcal{J}$ for all n. From (2), $\gamma_n(C)$ is nonempty for all n. Therefore

$$\gamma(C) = \bigcap_{n=1}^{\infty} \gamma_n(C)$$

is a nonempty compact subset of C. From (1), (2) and (3), the boundedness of $\gamma(C)$ is obvious and the convexity straightforward. #

We define a mapping T of C into C to be non-expansive if, for all x and $y \in C$,

$$|T(x) - T(y)| \leq |x-y|. \qquad (6)$$

2:3 Theorem:

Every non-expansive mapping T of C onto C maps $\gamma(C)$ into $\gamma(C)$.

Proof:

Consider any $x \in \gamma(C)$ and $y \in C$. There exists $\xi \in C$ such that $T(\xi) = y$. Now

$$|x-\xi| \leq \alpha(C).$$

Hence from (6),

$$|T(x) - T(\xi)| \leq \alpha(C),$$

i.e.

$$|T(x) - y| \leq \alpha(C).$$

Since this holds for all $y \in C$, then $r(T(x)) = \alpha(C)$, $T(x) \in \gamma(C)$ and T maps $\gamma(C)$ into $\gamma(C)$. #

The norm $\|$ is said to be locally uniformly convex if for every x and $y \in X$, with $x \neq y$, and every $D \geq \frac{1}{2}|x-y|$,

$$\delta(x,y,D) = \inf\{D-\|\tfrac{1}{2}(x+y)-\xi\| / \xi \in X, |x-\xi| \leq D, |y-\xi| \leq D\}$$

is a positive number.

2:4 Theorem:

If the norm is locally uniformly convex, then $\gamma(C)$ contains only one point.

Remark: The norm need only be locally uniformly convex on C.

Proof:

Suppose $\gamma(C)$ contains two distinct points x and y. Consider any $\xi \in C$. Then $|x-\xi| \leq \alpha(C)$, $|y-\xi| \leq \alpha(C)$ and

$$\delta(x,y,\alpha(C)) \leq \inf\{\alpha(C)-|\tfrac{1}{2}(x+y)-z| / z \in C, |x-z| \leq \alpha(C), |y-z| \leq \alpha(C)\}$$
$$\leq \alpha(C) - |\tfrac{1}{2}(x+y) - \xi|.$$

Hence

$$|\tfrac{1}{2}(x+y) - \xi| \leq \alpha(C) - \delta(x,y,\alpha(C)).$$

This holds for all $\xi \in C$ and therefore

$$r(\tfrac{1}{2}(x+y)) \leq \alpha(C) - \delta(x,y,\alpha(C)).$$

But since $\alpha(C) \geq \frac{1}{2}|x-y|$, then $\delta(x,y,\alpha(C))$ is a positive number and so $r(\frac{1}{2}(x+y)) < \alpha(C)$, a contradiction.

#

2:5 Theorem:

If the norm is locally uniformly convex and if $\mathcal{H}$ is any set of non-expansive mappings of C onto C, then the mappings of $\mathcal{H}$ have a common fixed point in C.

Remark: The members of $\mathcal{H}$ do not have to be linear.

Proof:

If C contains only one point, we have nothing to prove. Assume that C contains more than one point. Then, from 2:3, all members of $\mathcal{H}$ map $\gamma(C)$ into $\gamma(C)$ and, from 2:4, $\gamma(C)$ contains only one member. This member is the required fixed point. #

2:6 An application of the fixed point theorem 2:5 to show the existence of a right invariant integral on a compact metrizable semigroup:

G is a compact metrizable semigroup. Let d be a metric generating the topology of G. $C(G)$ is the Banach space of all real valued continuous functions on G with the supremum norm. $\{f^{(n)}\}$ is a sequence of members of $C(G)$ such that the linear manifold $\mathcal{M}$ spanned by them is dense in $C(G)$ and

$$|f^{(n)}| = 1 \tag{7}$$

for all n. (See page 246 of [12] for proof of separability of $C(G)$). For each $f \in C(G)$ and $a \in G$, f_a is the member of $C(G)$ defined by

$$f_a(x) = f(xa),$$

for all $x \in G$. Λ is the normed vector space of all bounded linear functionals λ on $C(G)$. For each $\lambda \in \Lambda$ and $a \in G$, define

$$\phi(\lambda,a) = [\sum_{n=1}^{\infty} 2^{-n}(\lambda(f_a^{(n)}))^2]^{\frac{1}{2}} . \tag{8}$$

Define a norm on Λ, by

$$|\lambda| = \sup_{a\in G} \phi(\lambda,a) . \tag{9}$$

2:6:1 Lemma:

If $f \in C(G)$ and $\{a_k\}$ is a sequence in G which converges to an element a of G, then

$$|f_{a_k} - f_a| \to 0,$$

as $k \to \infty$.

Proof:

Suppose $|f_{a_k} - f_a|$ does not $\to 0$ as $k \to \infty$. Then there exists an $\varepsilon' > 0$ and a subsequence $\{b_r\}$ of $\{a_k\}$, such that

$$|f_{b_r} - f_a| \geqslant \varepsilon'$$

for all r. For each r, let $x_r \in G$ be such that

$$|f_{b_r}(x_r) - f_a(x_r)| = |f_{b_r} - f_a|,$$

so that

$$|f(x_r b_r) - f(x_r a)| \geqslant \varepsilon'$$

for all r. Let $\{x_{r_s}\}$ be a subsequence of $\{x_r\}$ which converges to an element x' of G. Then

$$|f(x_{r_s} b_{r_s}) - f(x_{r_s} a)| \geqslant \varepsilon' \tag{10}$$

for all s. But $x_{r_s} b_{r_s} \to x' a$ and $x_{r_s} a \to x' a$ as $s \to \infty$. This contradicts (10). #

2:6:2 Theorem:

For all $\lambda \in \Lambda$ there exists an $a \in G$ such that

$$|\lambda| = \phi(\lambda, a). \tag{11}$$

Proof:

Consider any $\lambda \in \Lambda$. Choose a sequence $\{a_k\}$ in G such that

$$|\lambda| \geqslant \phi(\lambda, a_k) > |\lambda| - \frac{1}{k} \tag{12}$$

for all k. Since G is compact metric, there exists a subsequence $\{b_r\}$ of $\{a_k\}$ converging to an element a of G. By (12),

$$\phi(\lambda, b_r) \to (\lambda) \tag{13}$$

as $r \to \infty$. Let K be a constant such that

$$|\lambda(f)| \leqslant K|f|, \tag{14}$$

for all $f \in C(G)$. Therefore, by (7),

$$|\lambda(f_{b_r}^{(n)})| \leqslant K \tag{15}$$

for all r and n.

By 2:6:1, $|f_{b_r}^{(n)} - f_a^{(n)}| \to 0$ as $r \to \infty$, hence by (14) $\lambda(f_{b_r}^{(n)}) \to \lambda(f_a^{(n)})$ as $r \to \infty$. By (15), the series concerned is uniformly convergent and hence

$$\phi(\lambda, b_r) \to \phi(\lambda, a)$$

as $r \to \infty$, so that, by (13),

$$\phi(\lambda, a) = |\lambda|.$$ #

Define on Λ the functional β by

$$\beta(\lambda) = \inf_{a \in G} \phi(\lambda, a). \qquad (16)$$

2:6:3 Theorem:

For all $\lambda \in \Lambda$, there exists an $a \in G$ such that

$$\beta(\lambda) = \phi(\lambda, a).$$

Proof:

Similar to 2:6:2. #

2:6:4 Theorem:

If G is left simple, i.e. $Gx = G$ for all $x \in G$, then

$$\phi(\lambda, b) > 0 \qquad (17)$$

for all $\lambda \in \Lambda$, with $\lambda \neq 0$, and all $b \in G$. Hence $\beta(\lambda)$ is a positive number for all $\lambda \in \Lambda$, with $\lambda \neq 0$.

Proof:

Let $\lambda \in \Lambda$, with $\lambda \neq 0$ and let $b \in G$. Since G is compact it contains at least one idempotent ([27]). By Theorem 1-27 on page 38 of [4], G is a left group. Then the mapping χ of G onto G, defined by

$$\chi(x) = xb,$$

for all $x \in G$, is 1:1. Hence χ is a homeomorphism of G onto G. It follows that the linear manifold spanned by the set of functions

$$\{f_b^{(n)}/n = 1, 2, \ldots\}$$

is dense in $C(G)$. Then

$$\phi(\lambda,b) > 0. \qquad \#$$

2:6:5 Theorem:

If G is left simple, then the norm for Λ is locally uniformly convex.

Proof:

Consider any λ and $\mu \in \Lambda$ with $\lambda \neq \mu$, and any real number D, with $D \geq \frac{1}{2}|\lambda-\mu|$. We have to show that

$$\delta(\lambda,\mu,D) = \inf\{D-|\tfrac{1}{2}(\lambda+\mu)-\xi| / \xi \in \Lambda, |\lambda-\xi| \leq D, |\mu-\xi| \leq D\}$$

is a positive number. Consider any $\xi \in \Lambda$ with $|\lambda-\xi| \leq D$ and $|\mu-\xi| \leq D$.

There exists an element $b \in G$ such that

$$\begin{aligned} |\tfrac{1}{2}(\lambda+\mu)-\xi|^2 &= [\phi(\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu-\xi,b)]^2 \\ &= \tfrac{1}{2}[\phi(\lambda-\xi,b)]^2+\tfrac{1}{2}[\phi(\mu-\xi,b)]^2-\tfrac{1}{4}[\phi(\lambda-\mu,b)]^2 \\ &\leq \tfrac{1}{2}[\sup_{a\in G}\phi(\lambda-\xi,a)]^2+\tfrac{1}{2}[\sup_{a\in G}\phi(\mu-\xi,a)]^2-\tfrac{1}{4}[\inf_{a\in G}\phi(\lambda-\mu,a)]^2 \\ &= \tfrac{1}{2}|\lambda-\xi|^2+\tfrac{1}{2}|\mu-\xi|^2-\tfrac{1}{4}[\beta(\lambda-\mu)]^2 \\ &\leq D^2 - \tfrac{1}{4}[\beta(\lambda-\mu)]^2. \end{aligned}$$

Since $\lambda \neq \mu$, then, from 2:6:4, $\beta(\lambda-\mu)$ is a positive number. Hence there exists a positive number ∂ such that, for all $\xi \in \Lambda$, with $|\lambda-\xi| \leq D$ and $|\mu-\xi| \leq D$,

$$|\tfrac{1}{2}(\lambda+\mu)-\xi| \leq D-\partial, \qquad (18)$$

where ∂ is dependent only on λ,μ and D. Hence

$$\delta(\lambda,\mu,D) \geqslant \partial$$

and therefore it is a positive number. #

Let $\mathcal{J}$ denote the weak topology.

2:6:6 Theorem:

The norm for Λ is lower semicontinuous with respect to $\mathcal{J}$.

Proof:

Let $\lambda \in \Lambda$ and let $\varepsilon > 0$. We have to show that there exists $U \in \mathcal{J}$ such that $\lambda \in U$ and

$$|\mu| > |\lambda| - \varepsilon \tag{19}$$

for all $\mu \in U$. Let $b \in G$ be such that

$$|\lambda| = \phi(\lambda,b).$$

Hence there exists a positive integer N, such that

$$\Big[\sum_{n=1}^{N} 2^{-n}(\lambda(f_b^{(n)}))^2\Big]^{\frac{1}{2}} > |\lambda| - \tfrac{1}{2}\varepsilon. \tag{20}$$

Let ψ be a function defined on Λ by

$$\psi(\eta) = \Big[\sum_{n=1}^{N} 2^{-n}(\eta(f_b^{(n)}))^2\Big]^{\frac{1}{2}}, \tag{21}$$

for all $\eta \in \Lambda$. Let $\delta > 0$ be such that

$$N^{\frac{1}{2}}\delta < \tfrac{1}{2}\varepsilon. \tag{22}$$

Put

$$U = \{\mu \in \Lambda / |(\lambda-\mu)(f_b^{(n)})| < \delta \quad \text{for} \quad n=1,2,..,N\}. \tag{23}$$

Then $U \in \mathcal{J}$, $\lambda \in U$ and, for all $\mu \in U$,

$$\psi(\lambda) \leqslant \psi(\mu) + \psi(\lambda-\mu),$$

so that, by (20) and (23),

$$|\lambda| - \tfrac{1}{2}\varepsilon < \phi(\mu,b) + N^{\frac{1}{2}}\delta$$

and, by (22),

$$< |\mu| + \tfrac{1}{2}\varepsilon.$$

Thus

$$|\mu| > |\lambda| - \varepsilon$$

for all $\mu \in U$. Therefore (19) holds. #

Let Γ be the set of all positive linear functionals λ in Λ for which

$$\lambda(1) = 1 \tag{24}$$

and

$$|\lambda(f)| \leqslant 1, \tag{25}$$

for all $f \in C(G)$ with $|f| \leqslant 1$.

2:6:7 Theorem:

Γ is a non empty convex subset of Λ which is bounded with respect to the norm and weakly compact.

Proof:

The convexity and boundedness of Γ are straightforward. The existence of a positive linear functional with $\lambda(1) = 1$ and $|\lambda(f)| \leqslant |f|$, for all $f \in C(G)$ with $|f| \leqslant 1$, can be shown by letting $a \in G$ and defining

$$\lambda(f) = f(a)$$

for all $f \in C(G)$.

For weak compactness, we need only show that Γ is weakly closed (see Theorem 4-61-A, page 228 of [25]). The weak closure of Γ follows easily from its definition.

#

We note that Γ does not contain the zero functional.

Define a semigroup $\mathcal{H}$ of transformations T_a, for all $a \in G$, of Γ into Γ by

$$(T_a\lambda)(f) = \lambda(f_a). \qquad (26)$$

for all $f \in C(G)$ and $a \in G$.

From Theorem 2:6:4, G contains idempotent e and G is a left group. Hence

$$xe = x$$

for all $x \in G$. Hence

$$T_e\lambda(f) = \lambda(f) \qquad (27)$$

for all $\lambda \in \Lambda$ and $f \in C(G)$, since

$$f_e(x) = f(xe) = f(x)$$

for all $f \in C(G)$ and $x \in G$.

2:6:8 Theorem:

For all $a \in G$, T_a is a non-expansive map of Γ into Γ and, if G is left simple, Γ onto Γ.

Proof:

Consider any $\lambda \in \Gamma$ and any $a \in G$. Then

$$(T_a\lambda)(1) = \lambda(1_a) = \lambda(1) = 1.$$

Consider any $f \in C(G)$ with $|f| \leqslant 1$. Then

$$|f_a| \leqslant |f| \leqslant 1$$

and therefore

$$|(T_a\lambda)(f)| = |\lambda(f_a)| \leqslant 1.$$

Hence $T_a\lambda \in \Gamma$ and T_a maps Γ into Γ for all $a \in G$.

For non-expansive, consider any $a \in G$ and λ and $\mu \in \Gamma$. We have to show that $|T_a(\lambda-\mu)| \leqslant |\lambda-\mu|$. Now from 2:6:3 there exists an element $c \in G$ such that

$$|\lambda-\mu| = \phi(\lambda-\mu, c).$$

But, for all b and $d \in G$, $\xi \in \Gamma$ and $g \in C(G)$,

$$(g_b)_d(x) = g_b(xd) = g(xdb) = g_{db}(x)$$

for all $x \in G$, and, therefore,

$$(T_d\xi)(g_b) = \xi((g_b)_d) = \xi(g_{db}).$$

Hence

$$\begin{aligned} |T_a(\lambda-\mu)| &= \phi(\lambda-\mu, ac) \\ &\leqslant \sup_{x \in G} \phi(\lambda-\mu, x) \\ &= |\lambda-\mu|. \end{aligned}$$

For G left simple, consider any $a \in G$ and any $\lambda \in \Gamma$. Let e be the idempotent in G such that

$$xe = x$$

for all $x \in G$. Then there exists $b \in G$ such that $ba=e$.

But for all $\lambda \in \Gamma$ and $f \in C(G)$,

$$(T_{ba}\lambda)(f) = \lambda(f_{ba}) = \lambda((f_a)_b) = (T_b\lambda)(f_a) = (T_aT_b\lambda)(f)$$

and, from above, $T_b\lambda \in \Gamma$. Hence, by (27),

$$\lambda = T_e\lambda = T_{ba}\lambda$$
$$= T_a(T_b\lambda)$$

and therefore T_a maps Γ onto Γ. #

2:6:9 Theorem:

If G is left simple, then there exists a positive right invariant integral on G.

Proof:

Λ with the above norm and the weak topology of the dual, $\mathcal{H}$ and Γ satisfy the conditions of Theorem 2:5. Therefore the mappings of $\mathcal{H}$ have a common fixed point, λ_0 say, in Γ. Hence, for all $a \in G$ and $f \in C(G)$,

$$\lambda_0(f_a) = \lambda_0(f).$$

This λ_0 is the required positive right invariant integral on G. #

We now assume that G contains a unique minimal left ideal. One of the results of W.G. Rosen [22] is that this is a necessary and sufficient condition for the existence of a right invariant integral. Sneperman has also shown, in [24], that the right reversibility of G

(i.e. $Gx \cap Gy$ is non empty for all x and $y \in G$) is equivalent to the statement that the minimal two sided ideal of A is a subsemigroup of A with left inverses; i.e. to the statement that the minimal two sided ideal of A is a subsemigroup of A which does not possess a proper left ideal; i.e., since the union of all minimal left ideals of a semigroup is a minimal two sided ideal, to the existence of a unique minimal left ideal. Granirer has shown that every left amenable semigroup (i.e. a semigroup which admits a left invariant mean) is left reversible ([13]).

Let this unique minimal left ideal be K. Then it can be shown that K is a compact metrizable left simple semigroup (see J.H. Michael [17] Theorems 2:1 and 5:1).

Therefore from 2:6:9, there exists a positive right invariant integral, λ_0 say, on K.

For each $f \in C(G)$, let f^* denote the restriction of f to K and define positive linear functional $\lambda_1 \in \Lambda$ by

$$\lambda_1(f) = \lambda_0(f^*) \qquad (28)$$

for all $f \in C(G)$. Since for every real valued continuous function g on K there exists a $f \in C(G)$ with $f^*=g$, λ_1 is non trivial. We shall now show that λ_1 is a right invariant integral on G, i.e. for all $a \in G$ and $f \in C(G)$,

$$\lambda_1(f_a) = \lambda_1(f). \qquad (29)$$

Now K has the properties

$$Kx \supseteq K \tag{30}$$

for all $x \in S$ and

$$Kx = K \tag{31}$$

for all $x \in K$. Let $a \in K$ and $f \in C(G)$. Then, from (31),

$$f_a^*(x) = f_a(x) = f(xa) = f^*(xa) = (f^*)_a(x) \tag{32}$$

for all $x \in K$. Hence, from (28) and (32),

$$\begin{aligned} \lambda_1(f_a) &= \lambda_0(f_a^*) \\ &= \lambda_0((f^*)_a) \\ &= \lambda_0(f^*) \\ &= \lambda_1(f). \end{aligned} \tag{33}$$

Let $a \in G \sim K$ and $f \in C(G)$. Let $b \in K$. Then, from (30), there exists $c \in K$ such that $ca=b$. Hence, from (33),

$$\begin{aligned} \lambda_1(f) &= \lambda_1(f_b) \\ &= \lambda_1((f_a)_c) \\ &= \lambda_1(f_a). \end{aligned}$$

Therefore (29) holds and we have constructed a right invariant integral on G.

Note: In a similar manner Theorem 2:5 could be applied to show the existence of a fixed point in a weakly closed bounded convex subset C of a Hilbert space under any set of non-expansive mappings of C onto C.

This completes the discussion of the applications. The general fixed point theory will now be continued.

2:7 Theorem:

If the norm is locally uniformly convex and if $\mathcal{H}$ is any semigroup of continuous (with respect to $\mathcal{J}$) non-expansive affine mappings of C into C such that $\mathcal{H}$ is left reversible, i.e.

$$T\mathcal{H} \cap T'\mathcal{H} \text{ is non empty} \qquad (34)$$

for all T and $T' \in \mathcal{H}$, then the mappings of $\mathcal{H}$ have a common fixed point in C.

Proof:

(This is similar to Sneperman's Proof).

If C contains only one point, there is nothing to prove. Therefore assume C contains more than one point. Let $\mathcal{K}$ be the collection of all subsets K of C which are non-empty convex and compact, with respect to $\mathcal{J}$, and for which $\mathcal{H}K$ is a subset of K. Order $\mathcal{K}$ by inclusion. Then $(\mathcal{K}, \subseteq)$ is a pre-ordering. Consider a chain $\mathcal{K}_0$ in $\mathcal{K}$. Then, since any two members of $\mathcal{K}_0$ are related, $A_1 \cap A_2 \cap \ldots \cap A_r$ is nonempty for any finite sequence $A_1, A_2, \ldots, A_r$ in $\mathcal{K}_0$. Hence

$$K_1 = \cap\{A/A \in \mathcal{K}_0\}$$

is nonempty. It is straightforward to show that K_1 is convex and compact, with respect to $\mathcal{J}$, and contains $\mathcal{H}K_1$. Hence K_1 is a member of $\mathcal{K}$. It is obviously also the lower bound of the chain $\mathcal{K}_0$. Therefore by Zorns Lemma, $\mathcal{K}$ has a minimal element, K_0 say. If K_0 contains only one point there is nothing further to prove, therefore assume K_0 contains more than one point.

We now show that $TK_0 = K_0$ for all $T \in \mathcal{H}$. We first show that, for all n and $T_1, T_2, .., T_n \in \mathcal{H}$, there exists $H_1, H_2, ..., H_n \in \mathcal{H}$ such that

$$T_1H_1 = T_2H_2 = ... = T_nH_n . \qquad (35)$$

From (34), there exists H_1 and H_2 such that (35) holds for $n=2$. Suppose there exists $H_1, H_2, ..., H_{k-1} \in \mathcal{H}$ such that (35) holds for $n=k-1$. Then, from (34), there exists H and $H_k \in \mathcal{H}$ such that $T_1H_1H = T_kH_k$. Hence

$$T_1H_1H = T_2H_2H = ... = T_{k-1}H_{k-1}H = T_kH_k$$

and so (35) holds for $n=k$. By Mathematical Induction, (35) holds for all n.

Let $x \in K_0$. Then (35) gives for all n

$$T_1H_1(x) = T_2H_2(x) = .. = T_nH_n(x)$$

i.e. for all n and all $T_1, T_2, .., T_n \in \mathcal{H}$, there exists $x_1, x_2, .., x_n \in K_0$ such that

$$T_1x_1 = T_2x_2 = .. = T_nx_n.$$

Now it is clear that for every finite sequence $T_1, T_2, .., T_n$ in $\mathcal{H}$

$$\bigcap_{i=1}^{n} T_i(K_0) \quad \text{is non empty.}$$

Since K_0 is compact, with respect to $\mathcal{J}$, and for each

$T \in \mathcal{H}$, $T(K_o)$ is closed, it follows that

$$K_o' = \bigcap_{T \in \mathcal{H}} T(K_o) \qquad (36)$$

is non empty and compact, with respect to $\mathcal{J}$. Since each T is affine, K_o' is convex.

If now T_o and T_1 are arbitrary transformations in $\mathcal{H}$, then from (34) there exists H_o and H_1 such that $T_o H_o = T_1 H_1$ and therefore, since $K_o' \subseteq H_o(K_o)$, it follows that,

$$\begin{aligned} T_o(K_o') &\subseteq T_o[H_o(K_o)] \\ &= T_1[H_1(K_o)] \\ &\subseteq T_1(K_o) \ . \end{aligned}$$

Hence $T_o(K_o') \subseteq T(K_o)$, for all $T \in \mathcal{H}$, and so, from (36),

$$T_o(K_o') \subseteq K_o'.$$

Therefore $K_o' = K_o$, since K_o is minimal. Hence by (36) and the minimality of K_o, $T(K_o) = K_o$ for all $T \in \mathcal{H}$.

Then K_o and $\mathcal{H}$ satisfy the conditions of Theorem 2:5 and therefore K_o contains a fixed point under $\mathcal{H}$. This is also a fixed point in C. #

If the norm topology and $\mathcal{J}$ are the same and X is a Banach space, then from lemma 1 of [7], if the diameter of $C(\mathrm{diam}(C))$ is positive, there exists an element $u \in C$ such that

$$r(u) < \text{diam}(C).$$

It can be easily shown that $\gamma(C) = C$ iff

$$r(x) = \text{diam }(C)$$

for all $x \in C$. Therefore in the above case $\gamma(C)$ is a proper subset of C if C contains more than one point. Hence the existence of a fixed point in C, under a left reversible semigroup of non-expansive affine mappings of C into C, can be shown by using a Zorn's lemma argument similar to the proof of Kakutani's fixed point theorem as proved on page 457 of [9]. This will be valid without the norm property of local uniform convexity. ~~In fact, since Theorem 2:4 is not needed and Theorem 2:3 does not use the convexity of C, the existence of a fixed point in C can be shown for maps which are not affine.~~ This is a slightly less general ~~the same~~ version of the fixed point theorem ~~as that~~ shown by Mitchell in [8].

If the norm and the topology $\mathcal{T}$ are not the same, but C has normal structure or the stronger condition of completely normal structure (see Brodskii and Milman [3] and Belluce and Kirke [2]), then $\gamma(K)$ can be shown to be a proper subset of any convex subset K of C which is compact with respect to $\mathcal{T}$ and contains more than one point. Similarly to the above, a Zorn's Lemma argument shows the existence of a fixed point in C under a left reversible semigroup of non-expansive affine mappings of C into C. Local uniform convexity of the norm ~~and affineness of the~~

mappings is again not required. This is similar to the fixed point Theorem of Belluce and Kirke [2], the paper other than those of Edelstein's that uses a centre. It weakens many of their assumptions, but, of course, it requires the norm to be lower semicontinuous with respect to the topology $\mathcal{T}$ and the mappings to be affine.

But in general this can not be done because there are C for which $\gamma(C) = C$, as shown in the following.

2:8 Counter Example

Let $\mathcal{M}$ be the space of all bounded real sequences $\{a_n\}_{n=1}^{\infty}$. $\mathcal{T}$ is the Tychonoff product topology. $|\ |$ is the supremum norm;

$$|\{a_n\}| = \sup_{n\in N} |a_n|.$$

This norm is lower semicontinuous with respect to $\mathcal{T}$.

Let C be the weakly closed convex hull of

$$\begin{array}{ll} 2,2,2,\ldots & \{a_n^{(0)}\} \\ 1,2,2,\ldots & \{a_n^{(1)}\} \\ 2,1,2,\ldots & \{a_n^{(2)}\} \\ 2,2,1,\ldots & \{a_n^{(3)}\} \\ \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot & \end{array}$$

For this we shall show $\gamma(C) = C$.

Obviously C is convex and nonempty and every member of C is a weak limit of members of the convex hull of $\{\{a_n^{(0)}\}, \{a_n^{(1)}\}, \{a_n^{(2)}\},\ldots\}$. For any $\{b_n\} \in C$, $1 \leqslant b_n \leqslant 2$, for all $n \in N$, and so C is bounded with respect to the norm. By using a diagonalising argument one can find, for any sequence in C, a subsequence which converges weakly to an element in C, so C is sequentially compact with respect to $\mathcal{J}$. Since $\mathcal{J}$ is metrizable, C is compact with respect to $\mathcal{J}$. The diameter of C can be shown to be 1 and, by considering the convex hull of $\{\{a_n^{(0)}\}, \{a_n^{(1)}\}, \{a_n^{(2)}\},\ldots\}$ and weak limits, $|\{b_n\}|$ can be shown to be 2 for all $\{b_n\}$ in C.

But then if we consider any $\{b_n\} \in C$, by looking at $|\{b_n\} - \{a_n^{(k)}\}|$, for all k, it can be shown that

$$r(\{b_n\}) = 1.$$

Therefore $\alpha(C) = 1$ and hence $\gamma(C) = C$.

So we have constructed a nonempty convex bounded weakly compact set C whose radius defined centre is itself.

CHAPTER 3

THE QUOTIENT DEFINED CENTRE

X is a vector space endowed with a locally convex metrizable topology $\mathcal{J}$ such that the vector space operations are continuous, with respect to $\mathcal{J}$, in the usual manner. C is a nonempty convex compact subset of X.

We call p and internal point of C if there exists a positive δ such that, for all $z \in C$ and all t, with $|t| \leq \delta$,

$$p+t(z-p) \in C. \qquad (1)$$

I(C) is the set of all internal points of C.

For any p and $q \in X$, define

$$[p,q] = \{\lambda_p + (1-\lambda)q \in X/0 \leq \lambda \leq 1\}.$$

For any x and $y \in X$, $[x,y]$ is defined to be a line through C if $x \neq y$ and $1 = \sup\{\lambda \in R/\lambda x+(1-\lambda)y \in C\}$ and $0 = \inf\{\lambda \in R/\lambda x+(1-\lambda)y \in C\}$. For any x,y,p and $q \in X$, $[x,y]$ is defined to be parallel to $[p,q]$ iff $(x-y)$ is a non negative multiple of $(p-q)$. If p and $q \in C$, $[x,y]$ is a line through C with $p \in [x,y]$ and $[p,q]$ is parallel to $[x,y]$, then $q \in [x,y]$.

L(C) is the set of all lines through C and E(C) is the set of all end points of lines through C. (The compactness of C ensures that these end points belong to C). For every $p \in C$, $L_p(C)$ is the set of all lines containing p, through C and $E_p(C)$ is the set of all end points of all lines, containing p, through C.

We assume that C contains at least one internal point.

For any p, x and $y \in X$, such that $p \in [x,y]$, define $\lambda_p(x,y)$ to be the unique real number such that

$$0 \leqslant \lambda_p(x,y) \leqslant 1 \tag{2}$$

and

$$p = \lambda_p(x,y)x + (1-\lambda_p(x,y))y. \tag{3}$$

Obviously

$$\lambda_p(x,y) = 1 - \lambda_p(y,x). \tag{4}$$

If x,y,z and $p \in X$ and are such that $z \in [x,y]$ and $p \in [x,z]$, then

$$\lambda_p(x,y) \geqslant \lambda_p(x,z). \tag{5}$$

If $z \neq y$, then

$$\lambda_p(x,y) > \lambda_p(x,z). \tag{6}$$

If x,y,z and $p \in X$ and are such that $p \in [x,y]$ and $z \in [x,p]$, then

$$\lambda_p(x,y) \leqslant \lambda_p(z,y). \tag{7}$$

If $z \neq x$,

$$\lambda_p(x,y) < \lambda_p(z,y).$$

For any p, x and $y \in X$ with $p \in [x,y]$, $p \neq x$ and $p \neq y$, define

$$\alpha_p(x,y) = \frac{1 - \lambda_p(x,y)}{\lambda_p(x,y)}. \tag{8}$$

Then, from (4),

$$\alpha_p(x,y) = \frac{1}{\alpha_p(y,x)} \tag{9}$$

and

$$\lambda_p(x,y) = \frac{1}{1+\alpha_p(x,y)} . \tag{10}$$

We also note that, from (3),

$$\lambda_p(x,y)(p-x) = (1-\lambda_p(x,y))(y-p)$$

and, therefore, if X is also endowed with a norm,

$$\alpha_p(x,y) = \frac{1-\lambda_p(x,y)}{\lambda_p(x,y)} = \frac{|p-x|}{|p-y|}. \tag{11}$$

But it should be recognised that the definition of $\alpha_p(x,y)$ is independent of any concept of distance and dependent only on the vector space properties of X.

3:1 Theorem:

p is an internal point of C iff there exists a positive constant K such that

$$\alpha_p(x,y) \leqslant K$$

for all $[x,y] \in L_p(C)$.

Note: It is obvious that if $p \in I(C)$ and $[x,y] \in L_p(C)$, then $p \neq x$ and $p \neq y$.

Proof:

Only if: Let p be an internal point of C. Then, from (1), there exists positive real number δ such that for all $z \in C$ and for all t, such that $|t| \leqslant \delta$,

$p+t(z-p) \in C$. Put

$$K = \frac{1}{\delta}.$$

Consider any $[x,y] \in L_p(C)$. Then $p-\delta(x-p) \in C$ and, obviously, $\in [x,y]$. Hence, from (5),

$$\lambda_p(x,y) \geqslant \lambda_p(x), \ p-\delta(x-p)) = \frac{\delta}{\delta+1}.$$

Hence, from (8),

$$\alpha_p(x,y) \leqslant \frac{1}{\delta} = K.$$

If: Let K be such that $\alpha_p(x,y) \leqslant K$ for all $[x,y] \in L_p(C)$. Let $z \in C$ and be such that $z \neq p$. Extend $[z,p]$ to $[x,y] \in L_p(C)$. Since $\alpha_p(x,y) \leqslant K$ and $\alpha_p(x,p - \frac{1}{K}(x-p)) = K$,

$$[x,p - \tfrac{1}{K}(x-p)] \subseteq [x,y].$$

Therefore, since $z \in [x,p]$,

$$[z,p - \tfrac{1}{K}(z-p)] \subseteq [x,y].$$

Put

$$\delta = \min\left(\frac{1}{K},1\right).$$

Then

$$[p+\delta(z-p),p-\delta(z-p)] \subseteq [z,p - \tfrac{1}{K}(z-p)] \subseteq C.$$

When $z=p$, then $p+t(z-p) = p$ and $\in C$, for all t. Hence for all z and for all t, such that $|t| < \delta$, $p+t(z-p) \in C$; i.e. p is an internal point of C. #

3:2 Corollary:

The intersection of I(C) and E(C) is empty.

For any $p \in I(C)$, define

$$\alpha_p(C) = \sup_{[x,y] \in L_p(C)} \alpha_p(x,y). \tag{12}$$

Define

$$\alpha(C) = \inf_{p \in I(C)} \alpha_p(C). \tag{13}$$

If $p \notin I(C)$, then the supremum in (12) is undefined. In fact, if $p \in E(C)$, then there exists $[x,y] \in L_p(C)$ with $y=p$, which means $\alpha_p(x,y)$ is undefined for this $[x,y]$. Therefore $\alpha(C)$ is equal to the infimum of $\alpha_p(C)$ taken over all p in C. If $\alpha(C) < 1$, then there is an internal point p such that $\alpha_p(C) < 1$. Therefore for any $[x,y] \in L_p(C)$, $\alpha_p(x,y) < 1$. But, from (9), $\alpha_p(y\ x) > 1$ and, therefore, from (12), $\alpha_p(C) > 1$, a contradiction. Hence

$$\alpha(C) \geq 1.$$

Define

$$\gamma(C) = \{p \in I(C)/\alpha_p(C) = \alpha(C)\}. \tag{14}$$

This is called the <u>quotient defined centre</u> of C, or when no confusion is likely to arise, the <u>centre</u> of C. We note here that this centre, unlike the radius defined centre, is independent of the topology on X. It depends only on the shape of the non empty convex compact set C. Obviously $\gamma(C)$ is a <u>proper subset</u> of C.

It will be shown that $\gamma(C)$ is nonempty convex and compact. It will also be shown that every mapping of C onto C which is 1:1 and affine, maps $\gamma(C)$ into $\gamma(C)$. This, with the property that $\gamma(C)$ is a proper subset of C, will be used to show that C, if it has quotient structure, contains a common fixed point for all 1:1 affine mappings of C onto C and a common fixed point for a left reversible semigroup of continuous 1:1 affine mappings of C into C.

The existence of these common fixed points is then applied to finite dimensional spaces to show the existence of common fixed points, in nonempty convex compact subsets of these spaces, for affine mappings of C onto C and left reversible semigroups of continuous affine mappings.

After this application, sets for which the quotient defined centre consists of only one point are discussed. Sets of this type will contain common fixed points if they contain at least one internal point and do not require quotient structure. In particular it is shown, when C is either uniformly convex or strictly convex with $E(C)$ closed, that C contains at least one internal point and that $\gamma(C)$ consists of only one point. In these cases $\gamma(C)$ is therefore shown to be a common fixed point for 1:1 affine mappings of C onto C or left reversible semigroups of continuous 1:1 affine mappings of C into C.

3.3 Theorem:

If $p \in C$ and $p \notin E(C)$ and $y \in E(C)$, then the intersection of the half line

$$\{\alpha p + (1-\alpha)y \in X/\alpha \leqslant 1\}$$

and $E(C)$ consists only of y.

Proof:

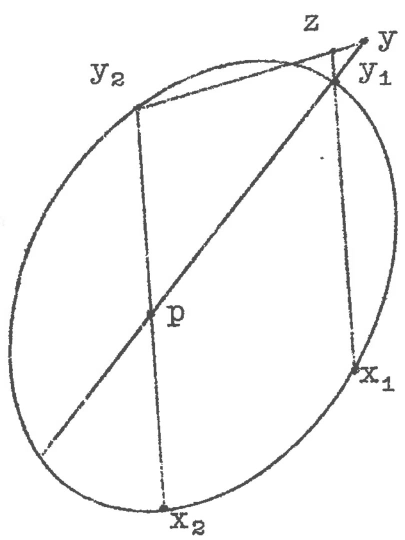
Let p be any member of C, not in $E(C)$, and y be any member of $E(C)$. Let

$$\ell = \{\alpha p+(1-\alpha)y \in X/\alpha \leqslant 1\}.$$

Assume there exists in the intersection of ℓ and $E(C)$ an element $y_1 \in E(C)$, such that $y_1 \neq y$. Then $y_1 \in [p,y]$ or $y \in [p,y_1]$. If $y_1 \in [p,y]$, then

$$y_1 = \lambda p + (1-\lambda)y$$

for some real number λ such that $0 < \lambda < 1$. Now $y_1 \in E(C)$ means that there exists $x_1 \in E(C)$ such that $[x_1,y_1] \in L(C)$. $p \notin [x_1,y_1]$ otherwise $y_1 = y$. Therefore there exists $[x_2,y_2] \in L(C)$ and parallel to $[x_1,y_1]$. Let



$$z = \lambda y_2 + (1-\lambda)y.$$

Then $z \in C$ and

$$z-y_1 = \lambda y_2 + (1-\lambda)y - \lambda p - (1-\lambda)y = \lambda(y_2-p).$$

Hence $[z,y_1]$ is parallel to $[x,y_1]$. Therefore, since

$p \notin E(C)$, z is on the indefinite extension of $[x_1,y_1]$, but does not belong to $[x_1,y_1]$. This is a contradiction of $z \in C$. If $y \in [p,y_1]$, we arrive at a similar contradiction in the above way with the positions of y and y_1 interchanged. Hence the intersection of ℓ and $E(C)$ consists only of y.

#

3:4 Corollary:

If $p \in C$ and $p \notin E(C)$ and $[x,y] \in L_p(C)$, then the intersection of the indefinite extension of $[x,y]$ and $E(C)$ consists only of x and y.

3:5 Theorem:

If $p \in C$ and $p \notin E(C)$, then $E_p(C) = E(C)$.

Proof:

Let p be any member of C not belonging to $E(C)$. Obviously $E_p(C) \leq E(C)$. Consider any $x \in E(C)$. Then $[x,p]$ can be extended to $[x_1,y_1] \in L_p(C)$. From theorem 3:3, $x_1=x$. Hence $x \in E_p(C)$. Therefore $E(C) = E_p(C)$.

#

3:6 Theorem:

For all p_1 and $p_2 \in C \sim E(C)$ and for all $0 < \lambda < 1$, $\lambda p_1+(1-\lambda)p_2 \notin E(C)$. Hence $C \sim E(C)$ is a convex set.

Proof:

Let p_1 and p_2 be any members of C, not in $E(C)$, and λ be any real number strictly between 0 and 1. Then $\lambda p_1 + (1-\lambda)p_2 \in [p_1,p_2]$ and $[p_1,p_2]$ can be extended to $[x,y] \in L(C)$. From Corollary 3:4, $\lambda p_1+(1-\lambda)p_2 \notin E(C)$.

#

3:7 Lemma:

Let $\{p_n\}$ be a sequence in C with limit p, $\{y_n\}$ be a sequence in $E(C)$ and x and $y \in E(C)$, such that $x \neq p$. Suppose that

$$[x,y_n] \in L_{p_n}(C),$$

for all n, and $[x,y] \in L_p(C)$. Then there exists a subsequence $\{y_{m_r}\}$ of $\{y_n\}$ with limit y_o in C and such that $p \in [x,y_o]$,

$$\lambda_p(x,y_o) \leq \lambda_p(x,y)$$

and $\lambda_p(x,y_o)$ is the limit of the sequence $\{\lambda_{p_{m_r}}(x,y_{m_r})\}$ as $r \to \infty$.

Proof:

Let $\{p_n\}$ be any sequence in C with limit p. Let $\{y_n\}$ be any sequence in $E(C)$ and x and y be any members of $E(C)$, such that $x \neq p$,

$$[x,y_n] \in L_{p_n}(C),$$

for all n, and $[x,y] \in L_p(C)$. From (2) and the properties of real numbers, there is a subsequence $\{\lambda_{p_{n_q}}(x,y_{n_q})\}$ of $\{\lambda_{p_n}(x,y_n)\}$ with limit λ_o, a real number such that $0 \leq \lambda_o \leq 1$. From the compactness of C, there exists a

subsequence $\{y_{m_r}\}$ of $\{y_{n_q}\}$ with limit y_0 in C.
Hence by taking limits as $r \to \infty$ of both sides of

$$p_{m_r} = \lambda_{p_{m_r}}(x,y_{m_r})x + (1-\lambda_{p_{m_r}}(x,y_{m_r}))y_{m_r},$$

we have

$$p = \lambda_0 x + (1-\lambda_0)y_0.$$

Therefore $p \in [x,y_0]$, $[x,y_0]$ is a subset of $[x,y]$, since $p \neq x$, and $\lambda_0 = \lambda_p(x,y_0)$. Hence, from (5),

$$\lambda_p(x,y_0) \leqslant \lambda_p(x,y).$$

#

3:8 Theorem:

If $\{p_n\}$ is a sequence in $I(C)$ with limit p in C and β is a positive real number such that

$$\alpha_{p_n}(C) \leqslant \beta$$

for all n; then $p \notin E(C)$ and $\alpha_p(C) \leqslant \beta$. (Hence from Theorem 3:1, $p \in I(C)$.)

Proof:

Let $\{p_n\}$ be any sequwnce in $I(C)$ with limit p in C. Let β be any positive real number such that

$$\alpha_{p_n}(C) \leqslant \beta \tag{15}$$

for all n.

Consider $p \in E(C)$. Then there exists $x \in E(C)$, such that $x \neq p$ and $[x,p] \in L(C)$, and a sequence $\{y_n\}$ in $E(C)$, such that $[x,y_n] \in L_{p_n}(C)$ for all n. Then, by Lemma 3:7, there exists a subsequence $\{y_{n_q}\}$ of $\{y_n\}$ with limit y_0 in C such that $p \in [x,y_0]$. Hence, from Corollary 3:4, $y_0=p$. Hence $\lambda_p(x,y_0) = 0$.

But, from Lemma 3:7, this means that

$$\lambda_{p_{n_q}}(x,y_{n_q}) \to 0$$

as $q \to \infty$. Therefore, from (8), the sequence $\{\alpha_{p_{n_q}}(x,y_{n_q})\}$ is unbounded. Hence, from (12), the sequence $\{\alpha_{p_{n_q}}(C)\}$ is unbounded. This contradicts (15). Therefore $p \notin E(C)$.

Let $[x,y]$ be any line through C containing p. We will show that $\alpha_p(x,y) \leq \beta$. Since $p \notin E(C)$, $x \neq p$. Let a sequence $\{z_n\}$ in E(C) be such that $[x,z_n] \in L_{p_n}(C)$. Lemma 3:7 shows that there exists a subsequence $\{z_{n_q}\}$ of $\{z_n\}$ with limit z_0 in C such that $p \in [x,z_0]$ and $\lambda_p(x,z_0)$ is the limit of the sequence $\{\lambda_{p_{n_q}}(x,z_{n_q})\}$. From (12),

$$\alpha_{p_{n_q}}(x,z_{n_q}) \leq \beta$$

for all q. Therefore, from (10),

$$\lambda_{p_{n_q}}(x,z_{n_q}) \geq \frac{1}{1+\beta}$$

for all q. Taking limits,

$$\lambda_p(x,z_0) \geq \frac{1}{1+\beta} \,.$$

But, from Lemma 3:7, $\lambda_p(x,z_0) \leq \lambda_p(x,y)$. Hence

$$\lambda_p(x,y) \geq \frac{1}{1+\beta} .$$

Hence, from (10),

$$\alpha_p(x,y) \leq \beta.$$

#

3:9 Theorem:

$\gamma(C)$ is a nonempty convex compact proper subset of C.

Proof:

For each k, define

$$\gamma_k(C) = \{p \in I(C)/\alpha_p(C) \leq \alpha(C) + \frac{1}{k}\}.$$

From (13), the assumption that C contains at least one internal point and Theorem 3:1, $\gamma_k(C)$ is nonempty for all k. From Theorem 3:8, for any k, if $\{p_n\}$ is any sequence in $\gamma_k(C)$ with limit p in C, then $p \in \gamma_k(C)$. Hence, for any k, $\gamma_k(C)$ is closed and hence compact. Therefore we have a decreasing sequence $\{\gamma_k(C)\}$ of non-empty compact sets. Therefore

$$\gamma(C) = \bigcap_{k=1}^{\infty} \gamma_k(C)$$

is nonempty and compact.

For the convexity of $\gamma(C)$, let p and q be any members of $\gamma(C)$ and let λ be any real number strictly between 0 and 1. Let

$$s = \lambda p + (1-\lambda)q.$$

Now from Theorem 3:6, $s \notin E(C)$. We shall show that $s \in \gamma(C)$.

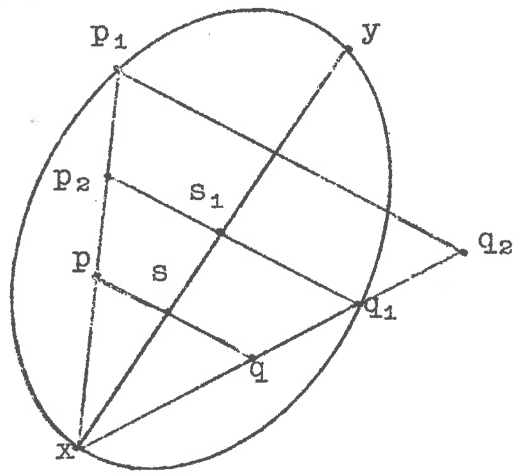
Assume $s \notin \gamma(C)$. Then $\alpha_s(C) > \alpha(C)$ or the set

$$\{\alpha_s(x,y)/[x,y] \in L_s(C)\}$$

is unbounded. In either case (from (12) in the first case) there exists $[x,y] \in L_s(C)$ such that

$$\alpha_s(x,y) > \alpha(C). \tag{16}$$

Let p_1 and $q_1 \in E(C)$ and be such that $[x,p_1] \in L_p(C)$ and $[x,q_1] \in L_q(C)$.



Consider lines through p_1 and q_1 parallel to $[p,q]$ cutting $[x,q_1]$ extended in q_2 and $[x,p_1]$ extended in p_2. Either $p_2 \in [x,p_1]$ or $q_2 \in [x,q_1]$. Say $p_2 \in [x,p_1]$.

Let $s_1 = \lambda p_2 + (1-\lambda)q_1$. Then $s_1 \in [s,y]$ and, from (5),

$$\lambda_s(x,y) \geq \lambda_s(x,s_1),$$

and, therefore, from (8),

$$\alpha_s(x,y) \leq \alpha_s(x,s_1). \tag{17}$$

Since $[p_2,q_1]$ is parallel to $[p,q]$, then

$$\alpha_q(x,q_1) = \alpha_s(x,s_1). \tag{18}$$

But $q \in \gamma(C)$. Therefore, from (12), $\alpha_q(x,q_1) \leq \alpha(C)$. Hence, from (16), (17) and (18), $\alpha(C) < \alpha(C)$, a contradiction. Therefore $s \in \gamma(C)$.

#

Let T be a map of C onto C which is 1:1 and affine. The inverse T^{-1} of T is also a 1:1 affine map of C onto C.

3:10 Lemma:

If T is a 1:1 affine map of C onto C, then T maps $E(C)$ onto $E(C)$.

Proof:

Let T be any 1:1 affine map of C onto C. Let x be any member of $E(C)$. Then there exists $y \in E(C)$ such that $[x,y] \in L(C)$. Since T is affine,

$$T([x,y]) = [Tx,Ty].$$

Extend $[Tx,Ty]$ to $[x_1,y_1] \in L(C)$. But then

$$T^{-1}([x_1,y_1]) = [T^{-1}x_1,T^{-1}y_1]$$

and

$$[x,y] = T^{-1}[Tx,Ty] \subseteq [T^{-1}x_1,T^{-1}y_1].$$

From Corollary 3:4, $T^{-1}x_1 = x$; i.e. $Tx = x_1 \in E(C)$. Hence T is a map of $E(C)$ into $E(C)$. Since T is 1:1 and maps C onto C, T is a map from $E(C)$ onto $E(C)$.

#

3:11 Theorem:

If T is a 1:1 affine map of C onto C, then T maps $\gamma(C)$ onto $\gamma(C)$.

Proof:

Let T be any 1:1 affine map of C onto C. Let p be any member of $\gamma(C)$. Since T is 1:1, from Lemma 3:10, Tp does not belong to $E(C)$. Consider any $[x,y] \in L_{Tp}(C)$. Since T and T^{-1} are affine and from Lemma 3:10, $[T^{-1}x,T^{-1}y] \in L_p(C)$ and

$$\lambda_p(T^{-1}x,T^{-1}y) = \lambda_{Tp}(x,y).$$

Hence, from (8), (12) and (14),

$$\alpha_{Tp}(x,y) = \alpha_p(T^{-1}x,T^{-1}y) \leqslant \alpha(C).$$

Therefore $\alpha_{Tp}(C) \leqslant \alpha(C)$. Hence $Tp \in \gamma(C)$. Therefore T is a map from $\gamma(C)$ into $\gamma(C)$ and since T is a 1:1 affine map from C onto C, T maps $\gamma(C)$ onto $\gamma(C)$.

#

For future brevity we shall define a nonempty convex compact set C to have <u>quotient structure</u> iff every convex compact subset of C which contains at least two members, contains at least one internal point.

3:12 Theorem:

If C has quotient structure and $\mathcal{H}$ is any set of 1:1 affine maps of C onto C; then the mappings of $\mathcal{H}$ have a common fixed point in C.

Proof:

If C contains only one point, there is nothing further to prove. Therefore assume C contains more than one point. Let $\mathcal{K}$ be the collection of all nonempty convex compact subsets K of C, for which $T(K) = K$ for all $T \in \mathcal{H}$. In a similar manner as in the proof of Theorem 2:7, a Zorn's Lemma argument gives a minimal member of $\mathcal{K}, K_o$ say.

If K_o contains only one element, there is nothing further to prove. Therefore assume it contains more than one element. Since C has quotient structure, K_o contains at least one internal point. Therefore we can construct $\gamma(K_o)$ which, by Theorem 3:9, is a nonempty convex compact proper subset of K_o. From Theorem 3:11,

$$T(\gamma(K_o)) = \gamma(K_o)$$

for all $T \in \mathcal{H}$. Hence $\gamma(K_o)$ contradicts the minimality of K_o.

#

3:13 Theorem:

If C has quotient structure and $\mathcal{H}$ is any left reversible semigroup of continuous 1:1 affine maps of C into C, then the mappings of $\mathcal{H}$ have a common fixed point in C.

Proof:

If C contains only one point, there is nothing further to prove. Therefore assume C contains more than one point. Let $\mathcal{K}$ be the collection of all nonempty convex compact subsets K of C, for which $T(K) = K$ for all $T \in \mathcal{H}$. Similar to Theorem 3:12, a Zorn's Lemma argument gives a minimal member of $\mathcal{K}$, K_o say.

If K_o contains only one element, there is nothing further to prove. Therefore we assume K_o contains more than one point. Similar to Theorem 3:12 we can construct $\gamma(K_o)$, a nonempty convex compact proper subset of K_o. Similarly to the proof of Theorem 2:7, we can use the left reversibility of $\mathcal{H}$ and the continuity of the members of $\mathcal{H}$ to show, with an induction argument, that $T(K_o) = K_o$, for all $T \in \mathcal{H}$. Then, from Theorem 3:11,

$$T(\gamma(K_o)) = \gamma(K_o)$$

for all $T \in \mathcal{H}$, and hence $\gamma(K_o) \in \mathcal{K}$, a contradiction of the minimality of K_o.

#

3:14 An application of the fixed point Theorems 3:12 and 3:13 to show the existence of fixed points in nonempty convex compact subsets of the finite dimensional Euclidean vector spaces under sets of affine maps:

R^n is the n-dimensional Euclidean vector space. The topology $\mathcal{J}$ is the usual norm topology. C is a nonempty convex compact subset of R^n.

Let a be any element of C. Consider $C-a$. This is a nonempty convex compact subset of R^n containing 0. Let $\{u_1,\ldots,u_k\}$ be a maximal set of linearly independent vectors in $C-a$. Construct the open geometric k-simplex

$$S_a = \langle a, a+u_1, \ldots, a+u_k \rangle . \tag{19}$$

Then S_a is a subset of C. Let

$$b = \frac{1}{k+1}\left(a+(a+u_1)+\ldots+(a+u_k)\right). \tag{20}$$

b is the barycentre of S_a. The $k+1$ elements $a, a+u_1, \ldots, a+u_k$ are the vertices of S_a. Obviously $b \in C$.

We shall show that b is an internal point of C and hence that C has quotient structure.

Let M be the k-dimensional Euclidean vector subspace of R^n spanned by the set of vectors $\{u_1,\ldots,u_k\}$ and endowed with a norm topology induced from R. Obviously $C-a$ is a subset of M and S_a-a is an open k-simplex in M with vertices $0, u_1, \ldots, u_k$ and barycentre $b-a$.

3:14:1 Lemma:

In M, $b-a$ is an interior point of $C-a$.

Proof:

Since C is convex, $C-a$ is convex and, therefore, the convex hull of the vertices of S_a-a is contained in $C-a$. The open k-simplex S_a-a is the interior of this convex hull (see the bottom of page 171 of [8]) in M. Hence S_a-a is contained, with the interior of $C-a$, in M. $b-a \in S_a-a$, hence, in M, $b-a$ is an interior point of $C-a$.

#

3:14:2 Theorem:

b is an internal point of C.

Proof:

To show that b is an internal point of C, we need only show that b-a is an internal point of C-a.

Let $\partial(C-a)$ be the boundary, in M, of C-a. Since each line through C-a is contained in M, the end points of these lines are contained in $\partial(C-a)$,

$$\text{i.e.}\quad E(C-a) = \partial(C-a). \tag{21}$$

From Lemma 3:14:1, b-a is an interior point of C-a and, therefore, there exists a positive real number ε such that

$$|(b-a) - z| > \varepsilon$$

for all $z \in \partial(C-a)$. Therefore, from (21),

$$|(b-a) - x| > \varepsilon \tag{22}$$

for all $x \in E(C-a)$. Since C is compact, C-a is compact and, therefore, the diameter of C-a, diam(C-a), is finite. Put

$$\delta = \frac{\varepsilon}{\operatorname{diam}(C-a)} .$$

Then for all $y \in C-a$ and for all t, such that $|t| \leqslant \delta$,

$$\begin{aligned} |(b-a) + t(y-(b-a)) - (b-a)| &= |t|\,|y-(b-a)| \\ &\leqslant \delta.\operatorname{diam}(C-a) \\ &= \varepsilon. \end{aligned}$$

Therefore, from (22), $(b-a) + t(y-(b-a)) \in C-a$; i.e. $b-a$ is an internal point of $C-a$.

#

For any nonempty convex compact subset of R^n, we can construct an internal point in the above way. Hence,

3:14:3 Corollary:

C has quotient structure.

This property of C leads directly to the following.

3:14:4 Fixed points for 1:1 affine maps:

If $\mathcal{H}$ is any set of 1:1 affine maps of C onto C, $\mathcal{H}$ and C satisfy the conditions of Theorem 3:12 and, therefore, the mappings of $\mathcal{H}$ have a common fixed point in C. If $\mathcal{H}_1$ is any left reversible semigroup of continuous 1:1 affine maps of C into C, $\mathcal{H}_1$ and C satisfy the conditions of Theorem 3:13 and, therefore, in this case too, the mappings of $\mathcal{H}_1$ have a common fixed point in C.

But these conditions on $\mathcal{H}$ and $\mathcal{H}_1$ may be further weakened.

3:14:5 Lemma:

For all $y \in C$, the maximal number of linearly independent points in $C-y$ is invariant (and equals k).

Proof:

Assume that there exists y and $z \in C$ such that $y \neq z$, the maximal number of linearly independent points in $C-y$ is n and the maximal number of linearly independent points in $C-z$ is m, with $m > n$. Let $u_1,\ldots,u_n$ be a maximal set of linearly independent points in $C-y$ and $v_1,\ldots,v_m$ be a maximal set of linearly independent points in $C-z$, with $v_1,\ldots,v_m$ so chosen that, if $\lambda_1,\ldots,\lambda_m \in R$ and are such that

$$y-z = \lambda_1 v_1+\ldots+\lambda_m v_m, \tag{23}$$

then $\lambda_1+\ldots+\lambda_m \neq 1$.

Now, for $i = 1,2,\ldots,m$, $v_i + (z-y) \in C-y$.

Since

$$C-y \in Sp\{u_1,\ldots,u_n\}$$

and, since $m > n$, then $v_1 + (z-y),\ldots,v_m + (z-y)$ are linearly dependent points in $C-y$. Therefore there exists $\mu_1,\ldots,\mu_m \in R$ and not all zero, such that

$$\mu_1(v_1 + (z-y))+\ldots+\mu_m(v_m - (z-y)) = 0;$$

i.e., if we let $\eta = \mu_1+\ldots+\mu_m$,

$$\mu_1 v_1+\ldots+\mu_m v_m = \eta(y-z).$$

If $\eta \neq 0$, then

$$y-z = \frac{\mu_1}{\eta} v_1+\ldots+ \frac{\mu_m}{\eta} v_m,$$

which contradicts the choice of $v_1,\ldots,v_m$ in (23). If

$\eta=0$, then $\mu_1 v_1+\dots+\mu_m v_m = 0$, which contradicts the linear independence of $v_1,\dots,v_m$. #

3:14:6 Lemma:

If T is a linear map of R^n into R^n which takes C onto C, then the restriction of T to C is 1:1.

Proof:

Let T be any linear map of R^n into R^n which takes C onto C. Assume that the restriction of T to C is not 1:1; i.e. that there exists p and $q \in C$, such that $p \neq q$ and $Tp=Tq$. We will show that this gives rise to a contradiction of Lemma 3:14:5.

Let x be any member of C. Let $\{v_1,\dots,v_k\}$ be a maximal set of linearly independent points in C-x and let

$$S_x = \langle x,x+v_1,\dots,x+v_k\rangle.$$

Then

$$p-x = \lambda_1 v_1+\dots+\lambda_k v_k$$

and

$$q-x = \mu_1 v_1+\dots+\mu_k v_k$$

for some $\lambda_1,\dots,\lambda_k,\mu_1,\dots,\mu_k \in R$ and such that there exists K such that $1 \leq K \leq k$ and

$$\lambda_K \neq \mu_K \tag{24}$$

(or else $p=q$). Since $Tp = Tq$ and T is linear,

$$T(p-x) = T(q-x);$$

i.e. $\lambda_1 Tv_1 + \ldots + \lambda_k Tv_k = \mu_1 Tv_1 + \ldots + \mu_k Tv_k$

i.e. $(\lambda_1 - \mu_1)Tv_1 + \ldots + (\lambda_k - \mu_k)Tv_k = 0.$

From (24), this means that $Tv_1, \ldots, Tv_k$ are linearly dependent.

Now let w be any member of $C-x$. There exists $\xi_1, \ldots, \xi_k \in R$ and such that

$$w = \xi_1 v_1 + \ldots + \xi_k v_k.$$

Since T is linear,

$$Tw = \xi_1 Tv_1 + \ldots + \xi_k Tv_k.$$

Hence $T[C-x]$ is a subset of $Sp\{Tv_1, \ldots, Tv_k\}$. It is straightforward to show that

$$T[C-x] = C - Tx.$$

Hence $C-Tx$ is a subset of $Sp\{Tv_1, \ldots, Tv_k\}$, and since $Tv_1, \ldots, Tv_k$ are linearly dependent, then the maximal set of linearly independent points in $C-Tx$ is less than k, which contradicts Lemma 3:14:5. #

3:14:7 Theorem:

Every affine map of C onto C is 1:1.

Proof:

Let T be any affine map of C onto C. Let L be the 1:1 linear map of R^n onto R^n defined by

$$L(x) = x+a$$

for all $x \in R^n$. Then $L^{-1}TL$ is an affine map of C−a onto C−a. Extend $L^{-1}TL$ to map H of R^n into R^n by

$$\begin{aligned} H(x) &= \alpha L^{-1}TL(y); \text{ where there exists scalar } \alpha \text{ such that } x=\alpha y \text{ for some } y \in C-a, \\ &= 0; \text{ otherwise.} \end{aligned}$$

Then it is straightforward to show that H is a linear map of R^n into R^n which takes C−a onto C−a and for which

$$H(x) = L^{-1}TL(x) \tag{25}$$

for all $x \in C-a$. Let

$$T' = LHL^{-1}.$$

Then T' is a linear map of R^n into R^n which takes C onto C. From Lemma 3:14:6, the restriction of T' to C is 1:1. But, from (25),

$$\begin{aligned} T'(x) &= LL^{-1}TL^{-1}L(x) \\ &= T(x) \end{aligned}$$

for all $x \in C$. Hence T is 1:1. #

This Theorem leads on to the following.

3:14:8 Fixed points for affine maps.

If $\mathcal{H}$ is any set of affine maps of C onto C, $\mathcal{H}$ and C, from Theorem 3:14:7, satisfy the conditions of Theorem 3:12 and, therefore, the mappings of $\mathcal{H}$ have a common fixed point in C. If $\mathcal{H}_1$ is any left reversible semigroup of continuous affine maps of C into C, then $\mathcal{H}_1$ and C, even with Theorem 3:14:7, do not exactly satisfy the conditions of Theorem 3:13. But if we use Zorn's Lemma, similarly to the proof of Theorem 3:13, to find a minimal nonempty convex compact subset K_0 of C and if we use the left reversibility of $\mathcal{H}_1$ and the continuity of the maps of $\mathcal{H}_1$ to show that $T(K_0) = K_0$ for all $T \in \mathcal{H}_1$; then, from Theorem 3:14:7, $\mathcal{H}_1$ and K_0 satisfy the conditions of Theorem 3:12. Therefore the mappings of $\mathcal{H}_1$ have a common fixed point in K_0 and, therefore, in C.

This ends the discussion of finite dimensional applications. We now resume the study of the quotient defined centre and the fixed point theory.

If $\gamma(C)$ contains only one point, then, from Theorem 3:12, this point in C is a fixed point for 1:1 affine mappings of C onto C. In this case we would not require C to have quotient structure but only to contain an internal point. We now consider sets of this type.

3:15 Convexity:

For this section we assume that X is also endowed with a norm.

Remark: There need be no relationship between the norm and $\mathcal{J}$. The norm is only required for the study of uniform convexity.

C is defined to be strictly convex iff for all p_1 and $p_2 \in C$, such that $p_1 \neq p_2$, and for all $q \in C$ there exists an $\varepsilon > 0$ such that

$$(1-t)q + \frac{t}{2}(p_1+p_2) \in C \qquad (26)$$

for all $t \in [0,1+\varepsilon]$. C is defined to be locally uniformly convex iff for all distinct p_1 and $p_2 \in C$ there exists an $\varepsilon > 0$ such that (26) holds for all $q \in C$ and for all $t \in [0,1+\varepsilon]$. C is defined to be uniformly convex iff for all positive constants D there exists an $\varepsilon > 0$ such that (26) holds for all p_1, p_2 and $q \in C$, such that $|p_1-p_2| \geq D$, and for all $t \in [0,1+\varepsilon]$.

(These definitions were suggested by the previous concepts, in this thesis, of internal point and local uniform convexity of a norm and the concepts of strict and uniform convexity of a norm.)

Note: If U is the unit ball in X defined by the norm, and if the norm is strictly, locally uniformly or uniformly convex, then U has the same convexity as the norm.

Obviously uniform convexity implies local uniform convexity which implies strict convexity.

It can be easily shown that C is locally uniformly convex iff $\frac{1}{2}(p_1+p_2)$ is an internal point of C for all distinct p_1 and $p_2 \in C$. Hence if C is uniformly or locally uniformly convex it contains internal points. If C is strictly convex it may not contain internal points due to the fact that in the definition ε is dependent on q. But if C is strictly convex, it does contain internal points if $E(C)$ is also closed. In fact,

3:15:1 Lemma:

If $E(C)$ is closed, with respect to $\mathcal{J}$, and C is strictly convex, then C is locally uniformly convex; i.e. for all distinct p_1 and $p_2 \in C$, $\frac{1}{2}(p_1+p_2) \in I(C)$.

Proof:

Let p_1 and p_2 be any members of C. From the definition of strictly convex, $\frac{1}{2}(p_1+p_2)$ does not belong to $E(C)$. Therefore, if we let $z = \frac{1}{2}(p_1+p_2)$ and $[x,y]$ be any member of $L_z(C)$, $z \neq x$, $z \neq y$ and

$$0 < \lambda_z(x,y) < 1. \tag{27}$$

We can show C is locally uniformly convex if we show $z \in I(C)$ and this is true, from Theorem 3:1, if there exists a constant K such that

$$\alpha_z(x,y) \leqslant K$$

for all $[x,y] \in L_z(C)$.

Assume this is not so. Therefore, since $\mathcal{J}$ is metrizable, we can choose sequences $\{x_n\}$ and $\{y_n\}$ in $E(C)$ such that $[x_n,y_n] \in L_z(C)$, for all n, and

$$\alpha_z(x_n,y_n) \to \infty \tag{28}$$

as $n \to \infty$.

Since $E(C)$ is closed with respect to $\mathcal{J}$, it can be shown that there exists subsequences $\{x_{n_q}\}$ and $\{y_{n_q}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, with limits of x_0 and y_0 in $E(C)$. By an argument similar to Lemma 3:7, it can be shown that $[x_0,y_0] \in L_z(C)$ and

$$\lambda_z(x_{n_q},y_{n_q}) \to \lambda_z(x_0,y_0)$$

as $q \to \infty$. From (27), $0 < \lambda_z(x_0,y_0) < 1$ and therefore, from (8), it can be easily seen that $\alpha_z(x_0,y_0) < \infty$ and

$$\alpha_z(x_{n_q},y_{n_q}) \to \alpha_z(x_0,y_0)$$

as $q \to \infty$. Hence (28) does not occur. #

Note: It can be shown, with a similar argument to that used in the proof of Lemma 3:15:1, that, if C is strictly convex and $E(C)$ is closed with respect to $\mathcal{J}$, then for all $p \in I(C)$ there exists $[x,y] \in L_p(C)$ such that

$$\alpha_p(x,y) = \alpha_p(C). \tag{29}$$

This property is the result of the closure of the set $E(C)$ of end points.

3:15:2 Theorem:

If C is strictly convex and $E(C)$ is closed with respect to $\mathcal{J}$, then $\gamma(C)$ contains only one point.

Proof:

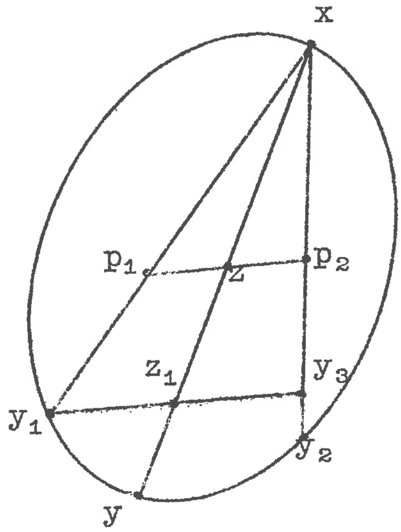
Assume $\gamma(C)$ contains two distinct points p_1 and p_2. Let

$$z = \tfrac{1}{2}(p_1 + p_2).$$

From Lemma 3:15:1, $z \in I(C)$ and, from (29), there exists $[x,y] \in L_z(C)$ such that

$$\alpha(C) = \alpha_z(C) = \alpha_z(x,y). \qquad (30)$$

Extend $[x,p_1]$ and $[x,p_2]$ to lines $[x,y_1]$ and $[x,y_2]$ through C. Construct lines through y_1 and y_2 parallel to $[p_1,p_2]$. One of these must cut $[x,y_2]$ or $[x,y_1]$. Let the line through y, parallel to $[p_1,p_2]$ cut $[x,y_2]$ in y_3. Let



$$z_1 = \tfrac{1}{2}(y_1 + y_2).$$

Then $z_1 \in [x,y]$.

Since C is strictly convex, there exists an $\varepsilon > 0$ such that $(1-t)x + tz_1 \in C$ for all $t \in [0,1+\varepsilon]$. Hence $z_1 \neq y$. Therefore, from (6),

$$\lambda_z(x,y) > \lambda_z(x,z_1).$$

Hence, from (8),

$$\alpha_z(x,y) < \alpha_z(x,z_1). \qquad (31)$$

But because $[y_1,y_3]$ is parallel to $[p_1,p_2]$, then

$$\alpha_z(x,z_1) = \alpha_{p_1}(x,y_1)$$

and, becuase $p_1 \in \gamma(C)$, from (10),

$$\alpha_{p_1}(x,y_1) \leqslant \alpha(C). \qquad (32)$$

(30), (31) and (32) gives rise to a contradiction. #

Note: In the application 3:14 $E(C)$ is closed. (This is because $E(S_a)$ is closed).

3:15:3 Theorem:

If C is uniformly convex, then $\gamma(C)$ contains only one point.

Proof:

Assume $\gamma(C)$ contains two distinct points p_1 and p_2. Let $z = \frac{1}{2}(p_1+p_2)$. Since $|p_1-p_2|$ is a positive constant, from the uniform convexity of C, there exists $\varepsilon > 0$ such that

$$(1-t)q + \frac{t}{2}(q_1+q_2) \in C,$$

for all q_1, q_2 and $q \in C$, such that $|q_1-q_2| \geqslant |p_1-p_2|$, and for all $t \in [0,1+\varepsilon]$.

Consider any $[x,y] \in L_z(C)$. Similarly to Theorem 3:15:2, construct $[x,y_1]$ and $[x,y_2] \in L(C)$ and let the line $[y_1,y_3]$ parallel to $[p_1,p_2]$ cut $[x,y_2]$ in y_3

Let

$$z_1 = \frac{1}{2}(y_1+y_3).$$

As before $z_1 \in [x,y]$.

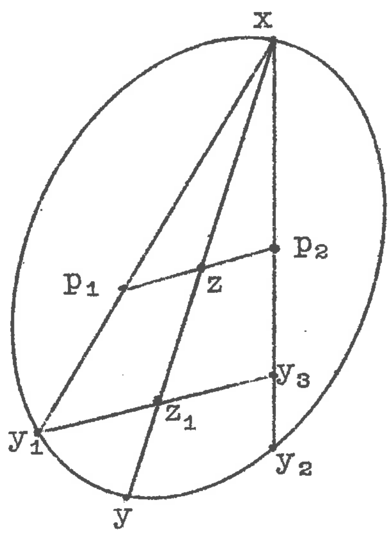
Since

$$|y_1-y_3| \geqslant |p_1-p_2|,$$

then

$$(1-t)x + tz_1 \in C,$$

for all $t \in [0,1+\varepsilon]$. Let



$$z_2 = -\varepsilon x + (1+\varepsilon) z_1 .$$

Then $z_2 \in [z_1, y]$ and $z_2 \neq z_1$. It is straightforward to show that

$$\lambda_z(x,z_2) = \frac{\lambda_z(x,z_1) + \varepsilon}{1 + \varepsilon} . \tag{33}$$

From (5), $\lambda_z(x,y) \geqslant \lambda_z(x,z_2)$. Hence $\alpha_z(x,y) \leqslant \alpha_z(x,z_2)$ and, from (12) and (33), it is straightforward to show that

$$\alpha_z(x,y) \leqslant \alpha_z(x,z_1)\left(1 - \frac{\varepsilon}{\lambda_z(x,z_1)+\varepsilon}\right). \tag{34}$$

Since $[y_1, y_3]$ is parallel to $[p_1, p_2]$ and $p_1 \in \gamma(C)$, then

$$\alpha_z(x,z_1) = \alpha_{p_1}(x,y_1) \leqslant \alpha(C).$$

Hence it can be shown, from (9), (10) and (34), that

$$\alpha_z(x,y) \leqslant \alpha(C) \left(1 - \varepsilon\left[\frac{\alpha(C)}{1+\alpha(C)} + \varepsilon\right]^{-1}\right).$$

Hence, from (12),

$$\alpha_z(C) \leqslant \alpha(C) \left(1 - \varepsilon\left[\frac{\alpha(C)}{1+\alpha(C)} + \varepsilon\right]^{-1}\right)$$

$$< \alpha(C),$$

a contradiction.

#

The above Theorems 3:15:2 and 3:15:3 with Theorem 3:11 give immediately the following two fixed point Theorems.

3:15:4 Theorem:

If C is strictly convex, $E(C)$ is closed with respect to $\mathcal{T}$ and $\mathcal{H}$ is any set of 1:1 affine maps of C onto C; then the mappings of $\mathcal{H}$ have a common fixed point in C.

Remark: The norm is unnecessary for this fixed point Theorem.

3:15:5 Theorem:

If C is uniformly convex and $\mathcal{H}$ is any set of 1:1 affine maps of C onto C; then the mappings of $\mathcal{H}$ have a common fixed point in C.

Remark: These two theorems and Theorems 3:12 and 3:13 differ from most other fixed point Theorems in that the restrictions are on the set C and not on the set $\mathcal{H}$ of mappings.

This ends our discussion of sets with special convexity properties. We now end this chapter by noting a relationship between the diameter of $\gamma(C)$ and the diameter of C and by giving a counter example that shows that even in the finite dimensional Euclidean vector space there are nonempty convex compact sets for which the quotient defined centre contains more than one point.

Note: If X has a norm and we denote, for all subsets A of X, the diameter of A by diam(A), then it can be shown that

$$\operatorname{diam}(\gamma_k(C)) = \frac{\alpha(C) + \frac{1}{k} - 1}{\alpha(C) + \frac{1}{k} + 1} \operatorname{diam}(C)$$

for all k. Hence

$$\operatorname{diam}(\gamma(C)) = \frac{\alpha(C) - 1}{\alpha(C) + 1} \operatorname{diam}(C)$$

and, therefore, $\gamma(C)$ contains only one point when $\alpha(C)=1$.

3:16 Counter Example:

R^3 is the 3-dimensional Euclidean vector space with the usual norm topology. C is a wedgeshaped subset of R^3 with vertices (0,0,0), (1,0,0), (0,1,0), (0,0,1), (0,1,1) and (1,0,1). E(C) is closed. C contains internal points (e.g. $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$).

Let Δ be the triangle in R^2 whose vertices are $a = (0,0)$, $b = (1,0)$ and $c = (0,1)$. By straightchecking it is possible to show that, if $z = (\frac{1}{3},\frac{1}{3})$, then

$$\alpha_z(x,y) = \frac{|z-x|}{|z-y|} \leqslant 2$$

for all $[x,y] \in L_p(\Delta)$, that, if $d = (\frac{1}{2},\frac{1}{2})$,

$$\alpha_z(a,d) = 2$$

and that, for any $z_1 \in \Delta$, such that $z_1 \neq z$, there exists $[x,y] \in L_{z_1}(C)$ such that

$$\alpha_{z_1}(x,y) > 2.$$

Hence $\gamma(\Delta) = \{(\frac{1}{3},\frac{1}{3})\}$ and $\alpha(\Delta) = 2$.

If Δ' is any triangle in R^2 with vertices u,v and w, then we can construct a linear homeomorphism f from Δ onto Δ' with $f(a) = u$, $f(b) = v$ and $f(c) = w$. Using this, it is easy to show that $\gamma(\Delta') = \{f(\frac{1}{3},\frac{1}{3})\}$ and $\alpha(\Delta') = 2$.

The above results for a 2-dimensional triangle can be used to show that, if $p = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$ and $q = (\frac{1}{3},\frac{1}{3},\frac{2}{3})$, there exists $[x_1,y_1] \in L_p(C)$ and $[x_2,y_2] \in L_q(C)$ such that

$$\alpha_p(x_1,y_1) = \alpha_q(x_2,y_2) = 2,$$

that, for all $z \in C$ but not contained in $[p,q]$, there exists $[x_3,y_3] \in L_z(C)$ such that

$$\alpha_z(x_3,y_3) > 2$$

and that, for all $[x,y] \in L_p(C)$ and $[x',y'] \in L_q(C)$,

$$\alpha_p(x,y) \leqslant 2 \text{ and } \alpha_q(x',y') \leqslant 2.$$

Hence $\alpha(C) = 2$ and $\gamma(C)$ contains, at least, the two distinct points p and q. (Actually it can be shown that $\gamma(C) = [p,q]$.)

CHAPTER 4

COMPACT SEMIGROUPS OF BOUNDED CONTINUOUS MAPPINGS.

In [10] Sneperman proved the following fixed point Theorem:

Let K be a convex compact set in a locally convex topological space X and let $\mathcal{L}$ be a semigroup of linear transformations of K into K which are equicontinuous on K and such that $L_1\mathcal{L} \cap L_2\mathcal{L}$ is nonempty for all L_1 and $L_2 \in \mathcal{L}$ (i.e. $\mathcal{L}$ is left reversible). Then, in K, there exists a fixed point x_0 such that $\mathcal{L}x_0 = x_0$.

In a manner similar to that used by Sneperman in [24] to show the existence of a right invariant integral on a right reversible compact semigroup, we shall use this Theorem to show that there exists, on a locally compact metrizable space X, an integral which is right invariant under a left reversible semigroup G of bounded continuous mappings of X into X, which satisfies certain compactness conditions. Since the semigroup operations continuously map a topological semigroup into itself, this application includes the existence of a right invariant integral (under the semigroup operations) on a metrizable right reversible compact semigroup.

The existence of the right invariant integral is first shown when X is compact. A nonempty convex weakly compact set Γ of nontrivial positive bounded linear functionals and a left reversible semigroup $\mathcal{H}$ of linear transformations of Γ into Γ, corresponding to the members of G, are constructed to satisfy the conditions of Sneperman's fixed point Theorem. The resulting fixed point in Γ is an integral on X which is right invariant under G. When X is not compact, a right invariant integral on a compact subset Y, which is also mapped into itself by the members of G, of X, is extended to an integral on X and this extension is shown to be also right invariant under G.

X is a locally compact space with a metrizable topology. ρ is a metric generating this topology. G is a left reversible semigroup of bounded continuous mappings of X into X, compact with respect to the metric d defined by

$$d(\phi,\psi) = \sup_{x\in X} \rho(\phi(x), \psi(x)), \tag{1}$$

for all ϕ and $\psi \in G$, and such that, for all compact subsets C of X, $\phi^{-1}(C)$ is compact, for all $\phi \in G$.

$C_0(X)$ is the normed linear space of all real valued continuous functions (the norm is the usual supremum norm). Λ is the vector space of all linear functionals on $C_0(X)$, endowed with the weak topology of the dual. (Note that the members of Λ need not be bounded.)

$\mathfrak{H}$ is the semigroup of transformations $T_\emptyset$, for each $\emptyset \in G$, of Λ into Λ defined, for all $\lambda \in \Lambda$ and $f \in C_0(X)$, by

$$T_\emptyset \lambda(f) = \lambda(f \circ \emptyset), \tag{2}$$

where $f \circ \emptyset \in C_0(X)$ $(\emptyset^{-1}(spt(f))$ is compact) and is defined by

$$(f \circ \emptyset)(x) = f(\emptyset(x)) \tag{3}$$

for all $x \in X$. It is straightforward to show that $\mathfrak{H}$ is left reversible and the members of $\mathfrak{H}$ are linear.

We assume for the time being that X is compact. Then $C_0(X)$ is a Banach space.

Define Γ to be the set of all positive bounded linear functionals λ in Λ such that

$$\lambda(1) = 1 \tag{4}$$

and

$$|\lambda(f)| \leqslant 1 \tag{5}$$

for all $f \in C_0(X)$, such that $|f| \leqslant 1$. Then, from Theorem 2:6:7, Γ is nonempty, convex and weakly compact. Obviously Γ does not contain the zero functional.

4:1 Theorem:

For each $\emptyset \in G$, $T_\emptyset$ maps Γ into Γ.

Proof:

Consider any $\emptyset \in G$ and $\lambda \in \Gamma$. Let f be any member of $C_0(X)$ with $|f| \leqslant 1$. Then, from (3),

$$|f\circ\emptyset| = \sup_{y\in X} |(f\circ\emptyset)(y)|$$

$$= \sup_{y\in X} |f(\emptyset(y))|$$

$$= \sup_{\emptyset(y)\in\emptyset(X)\subseteq X} |f(\emptyset(y))|$$

$$\leqslant |f| \leqslant 1.$$

Therefore, from (2) and (5),

$$|T_{\emptyset}\lambda(f)| = |\lambda(f\circ\emptyset)|$$

$$\leqslant 1.$$

Since $1\circ\emptyset = 1$, then, from (2) and (4),

$$T_{\emptyset}\lambda(1) = \lambda(1) = 1.$$

Hence $T_{\emptyset}\lambda \in \Gamma$.

#

4:2 Theorem:

For all $\emptyset \in G$, $T_{\emptyset}$ is weakly continuous on Λ.

Proof:

Let $\emptyset$ be any member of G. Consider any open neighbourhood

$$U(0,A,\varepsilon) = \{\mu \in \Lambda / |\mu(f)| < \varepsilon \text{ for all } f \in A\}$$

of 0, where ε is any positive real number and A is any finite subset of $C_o(X)$. Then

$$A\circ\emptyset = \{f\circ\emptyset / f \in A\}$$

is a finite subset of $C_o(X)$ and, if λ is a member of the open neighbourhood $V(0,A\circ\emptyset,\varepsilon)$ of 0, then, from (2),

$$|T_\emptyset \lambda(f)| = |\lambda(f \circ \emptyset)| < \varepsilon$$

for all $f \in A$, i.e.

$$T_\emptyset \lambda \in U(0, A, \varepsilon).$$

Hence $T_\emptyset$ is weakly continuous on Λ. #

4:3 Lemma:

If A is a finite subset of $C_0(X)$ and $\emptyset \in G$, then for every positive real number ε there exists a positive real number δ such that

$$|((T_\emptyset - T_\psi)\lambda)(f)| < \varepsilon,$$

for all $f \in A$, $\lambda \in \Gamma - \Gamma$ and $\psi \in G$, such that $d(\emptyset, \psi) < \delta$.

Proof:

Consider any finite subset A of $C_0(X)$ and $\emptyset \in G$. Let ε be an arbitrary positive real number. From the definition of Γ and the properties of norms,

$$|\lambda(g)| \leqslant 2|g| \tag{6}$$

for all $g \in C_0(X)$ and $\lambda \in \Gamma - \Gamma$. Hence, from (2), (6) and Theorem 4:1,

$$\begin{aligned} |((T_\emptyset - T_\psi)\lambda)(f)| &= |\lambda(f \circ \emptyset - f \circ \psi)| \\ &\leqslant 2|f \circ \emptyset - f \circ \psi|, \end{aligned}$$

for all $f \in A$, $\lambda \in \Gamma - \Gamma$ and $\psi \in G$. Therefore we need only show the existence of a positive real number δ such that

$$|f\circ\phi - f\circ\psi| < \tfrac{1}{2}\varepsilon, \qquad (7)$$

for all $f \in A$ and $\psi \in G$, such that $d(\phi,\psi) < \delta$. But, of course, if, for each $f \in A$, there exists positive real numbers δ_f such that

$$|f\circ\phi - f\circ\psi| < \tfrac{1}{2}\varepsilon,$$

for all $\psi \in G$, such that $d(\phi,\psi) < \delta_f$, then, since A is finite,

$$\delta = \min\{\delta_f / f \in A\}$$

will satisfy (7). Hence we need only show the existence of such δ_f.

Choose any $f \in A$. Assume that no δ_f exists, i.e. that there exists a sequence $\{\phi_r\}$ in G with limit $\phi \in G$ such that

$$|f\circ\phi - f\circ\phi_r| \geq \tfrac{1}{2}\varepsilon \qquad (8)$$

for all positive integers r. But then there exists a sequence $\{x_r\}$ in X such that, from (3),

$$|f\circ\phi - f\circ\phi_r| = |(f\circ\phi - f\circ\phi_r)(x_r)|$$

$$= |f(\phi(x_r)) - f(\phi_r(x_r))|$$

(we are assuming that X is compact). Let $\{x_{r_s}\}$ be the subsequence of $\{x_r\}$ which converges to limit x' in X. Then, from (8),

$$|f(\phi(x_{r_s})) - f(\phi_{r_s}(x_{r_s}))| \geq \tfrac{1}{2}\varepsilon \qquad (9)$$

for all positive integers s. But $\phi_{r_s}(x_{r_s}) \to \phi(x')$ and $\phi(x_{r_s}) \to \phi(x')$ as $s \to \infty$. This contradicts (9).

#

4:4 Theorem:

$\mathcal{H}$ is weakly equicontinuous on Γ.

Proof:

Consider any open neighbourhood $U(0,A,\varepsilon)$ of 0 in Λ. We have to show that there exists an open neighbourhood V of 0 such that

$$T_\phi \lambda \in V,$$

for all $\phi \in G$ and $\lambda \in V \cap (\Gamma-\Gamma)$, i.e. such that

$$|T_\phi \lambda(f)| < \varepsilon,$$

for all $f \in A$, $\phi \in G$ and $\lambda \in V \cap (\Gamma-\Gamma)$.

Consider any $\phi \in G$. Then, from Theorem 4:2, there exists an open neighbourhood V_ϕ of 0 such that

$$|T_\phi \lambda(f)| < \tfrac{1}{2}\varepsilon, \tag{10}$$

for all $f \in A$ and $\lambda \in V_\phi$ (and so $\lambda \in V_\phi \cap (\Gamma-\Gamma)$). From Lemma 4:3, there exists a positive real number δ_ϕ such that

$$|((T_\phi - T_\psi)\lambda)(f)| < \tfrac{1}{2}\varepsilon, \tag{11}$$

for all $f \in A$, $\lambda \in \Gamma-\Gamma$ (and so $\lambda \in V_\phi \cap (\Gamma-\Gamma)$) and $\psi \in G$, such that $d(\phi,\psi) < \delta_\phi$. Let

$$N_\phi = \{\psi \in G / d(\phi,\psi) < \delta_\phi\}.$$

Then, from (10) and (11),

$$|T_{\psi}\lambda(f)| < \varepsilon, \tag{12}$$

for all $f \in A$, $\psi \in N_{\phi}$ and $\lambda \in V_{\phi} \cap (\Gamma-\Gamma)$. But

$$G = \bigcup_{\phi \in G} N_{\phi}$$

and G is compact, hence there exists $\phi_1, \dots, \phi_n \in G$ such that

$$G = \bigcup_{i=1}^{n} N_{\phi_i}. \tag{13}$$

Put

$$V = \bigcap_{i=1}^{n} V_{\phi_i}.$$

Then V is an open neighbourhood of 0 in Λ and, from (12) and (13),

$$|T_{\psi}\lambda(f)| < \varepsilon,$$

for all $f \in A$, $\psi \in G$ and $\lambda \in V \cap (\Gamma-\Gamma)$. #

We now have a nonempty convex weakly compact set Γ in a locally convex topological space Λ and a semi-group $\mathcal{H}$ of linear transformations of Γ into Γ which are weakly equicontinuous on K and such that $\mathcal{H}$ is left reversible. Hence Sneperman's fixed point Theorem is applicable and gives the following Theorem.

4:5 Theorem:

If X is compact, then there exists an integral on X which is right invariant under G.

Proof:

Sneperman's fixed point Theorem shows that there exists a positive bounded linear functional μ in Γ such that

$$T_{\emptyset}\mu = \mu$$

for all $\emptyset \in G$, i.e. such that

$$\mu(f \circ \emptyset) = \mu(f)$$

for all $f \in C_0(X)$ and $\emptyset \in G$. Thus μ is the integral on X which is right invariant under G. #

If X is not compact, we choose any $x \in X$ and consider the set

$$Y = \{a \in X / a = \emptyset(x), \emptyset \in G\}$$

in X. Since G is compact, it is easily shown that Y is compact and, furthermore, it is easily shown that

$$\emptyset(Y) \subseteq Y \tag{14}$$

for all $\emptyset \in G$. Hence Theorem 4:5 is applicable to Y and, therefore, there exists an integral μ on Y which is right invariant under G.

Extend μ to an integral η on X by, for all $f \in C_0(X)$,

$$\eta(f) = \mu(f'), \tag{15}$$

where f' is the restriction of f to Y.

4:6 Lemma:

For all $\phi \in G$ and $f \in C_o(X)$,

$$(f \circ \phi)' = f' \circ \phi.$$

Proof:

Let ϕ be any member of G and f be any member of $C_o(X)$. Consider any $y \in Y$. Then, from (14), $\phi(y) \in Y$ and, hence, from (3),

$$\begin{aligned}(f' \circ \phi)(y) &= f'(\phi(y)) \\ &= f(\phi(y)) \\ &= (f \circ \phi)(y) \\ &= (f \circ \phi)'(y).\end{aligned}$$

Hence

$$f' \circ \phi = (f \circ \phi)'.$$

#

4:7 Theorem:

η is right invariant under G.

Proof:

Let ϕ be any member of G. Consider any $f \in C_o(X)$. Then, from (15), Lemma 4:6 and the right invariance of μ,

$$\begin{aligned}T_\phi \eta(f) &= \eta(f \circ \phi) \\ &= \mu((f \circ \phi)') \\ &= \mu(f' \circ \phi) \\ &= \mu(f') \\ &= \eta(f).\end{aligned}$$

#

Therefore η is an integral on X which is right invariant under G. (Note that, since Γ does not contain the zero functional, μ and η are nontrivial.)

CHAPTER 5

LOCALLY COMPACT GROUPS

In two of the previous chapters, we have constructed centres for nonempty convex sets, which were compact in some sense, and used these to show the existence of fixed points. One of the motivations for this was to find a fixed point theorem that could be used to show right invariant integrals on a locally compact group. The existence of a right invariant integral on a locally compact group has, of course, been known since 1929 when it was shown by J. von Neumann in [26], but it was hoped to show a direct connection between a fixed point Theorem and the right invariant integral similarly to that shown for compact semigroups by Sneperman in [24] and also in this thesis in 2:6.

In this chapter we consider a locally compact topological group with a countable basis and a nonempty convex weakly compact subset Γ of positive linear functionals on $C_0(G)$, the space of real valued continuous functions on G with compact support. It is shown that the group operations take Γ onto Γ. We have been unable to obtain a fixed point Theorem which gives an element of Γ which remains fixed under the group operations, but a subset Γ' of Γ, consisting of some of the nonbounded linear functionals of Γ, can be shown to be nonempty, convex, weakly compact and invariant under the group operations, i.e. in nearly all respects identical to Γ, yet not to contain the linear functional, which is a common fixed point for the transformations corresponding to the group operations, i.e. not to contain the right invariant integral. This counter example demonstrates the difficulty of finding a fixed point Theorem applicable to the locally compact group and reveals some interesting directions further research for such a theorem could take.

G is a locally compact topological (Hausdorff) group whose topology has a countable basis. G is therefore separable and since the group operations are continuous with respect to the topology, then the function ϕ from $G \times G$ onto G defined by

$$\phi(x,y) = x^{-1}y,$$

for all x and $y \in G$, is continuous. Hence, if U and V are compact subsets of G, then $U \times V$ is a compact subset of $G \times G$ and $U^{-1}V$ is a compact subset of G. Similarly VU^{-1} is a compact subset of G. Therefore, from Theorem 1:22 on page 34 of [20], G is metrizable and there exists a metric ρ for G which is right invariant, i.e. a metric ρ such that

$$\rho(xa,ya) = \rho(x,y)$$

for all x,y and $a \in G$, and which generates the topology of G.

$C_o(G)$ is the vector space of all real valued continuous functions f on G with compact support $spt(f)$. $C_o(G)$ is endowed with the usual supremum norm. Since the topology of G has a countable basis, we can construct a sequence $\{V_n\}$ of compact subsets of G such that G is the limit of the interiors of the V_n. By Exercise IV 4.3 on page 246 of [12], for each positive integer n, the Banach space $C(V_n)$ of all real valued continuous functions on V_n is separable. Hence it can be shown that $C_o(G)$ is separable.

Let e be the identity element in G. Since G is locally compact, there exists a positive real number δ such that the set $\{x \in G/\rho(x,e) \leqslant \delta\}$ is compact. Define $f_o \in C_o(G)$ by

$$f_o(x) = \frac{\frac{1}{4}\delta - \rho(x,e)}{\frac{1}{4}\delta} \; ; \quad \text{when } x \text{ is such that } \rho(x,e) < \tfrac{1}{4}\delta,$$

$$= 0; \quad \text{otherwise.} \qquad (2)$$

Let $C_o^+(G)$ be the set of all positive members of $C_o(G)$. Obviously $f_o \in C_o^+(G)$. For each $f \in C_o(G)$ and $a \in G$, define the right translation f_a by

$$f_a(x) = f(xa)$$

for all $x \in G$. Since VU^{-1} is a compact subset of G, for all compact subsets V and U of G, and

$$[spt(f_a)]a = spt(f), \qquad (3)$$

i.e. $[spt(f_a)] = [spt(f)]\{a\}^{-1}$,

for all $f \in C_0(G)$ and $a \in G$, $spt(f_a)$ is compact and hence $f_a \in C_0(G)$ for all $f \in C_0(G)$ and $a \in G$. Let $\mathcal{F}$ be the set of functions g in $C_0^+(G)$ such that

$$g = (f_0)_a$$

for some $a \in G$.

We note, for all $f \in C_0(G)$, $g \in \mathcal{F}$ and a and $b \in G$, that

(i) $(f_a)_b(x) = f_a(xb) = f(xba) = f_{ba}(x)$

for all $x \in G$, i.e.

$$(f_a)_b = f_{ba}; \tag{4}$$

(ii) $g_a \in \mathcal{F}$;

(iii) $f_e = f$; and

(iv) $|f_a| = |f|$.

Define the diameter of any subset C of G by

$$\text{diam}(C) = \sup_{v,w \in G} \rho(v,w).$$

Then, from (3), it can easily be shown that

$$\text{diam}(spt(f_a)) = \text{diam}(spt(f)). \tag{5}$$

Hence, for any $g \in \mathcal{F}$,

$$\text{diam}(spt(g)) = \tfrac{1}{2}\delta. \tag{6}$$

Furthermore it can be shown that, for any $g \in \mathcal{F}$,

$$spt(g) = \{x \in G/\rho(x,a^{-1}) \leqslant \tfrac{1}{4}\delta\}$$

and

$$g(x) = \frac{\frac{1}{4}\delta - \rho(x,a^{-1})}{\frac{1}{4}\delta} \; ; \quad x \in \operatorname{spt}(g),$$

$$= 0; \quad \text{otherwise},$$

where $a \in G$ and is such that $g = (f_0)_a$.

Define a functional ν on $C_0^+(G)$ by:

$\nu(f) = \inf \sum_{i=1}^{k} C_i$; where the infimum is taken over all positive real numbers $C_1,\ldots,C_k$ for which there exists $f_1,\ldots,f_k \in \mathcal{F}$ such that $\sum_{i=1}^{k} C_i f_i \geqslant f$, (7)

for all $f \in C_0^+(G)$. Then it can be shown that

(i) ν is a sublinear functional;

(ii) $\nu(0) = 0$;

(iii) $\nu(f) \geqslant \nu(g)$, for all f and $g \in C_0^+(G)$ such that $f \geqslant g$; and

(iv) ν is right translation invariant, i.e.

$$\nu(f_a) = \nu(f)$$

for all $f \in C_0(G)$ and $a \in G$.

Now define, for any positive real number d, a d-packing on G to be:

a subset S of G such that

(i) $\rho(x,y) \geqslant d$ for all x and $y \in S$, and

(ii) there is no point $z \in G \sim S$ such that $\rho(z,x) \geqslant d$ for all $x \in S$. (8)

Then, given S is a d-packing on G for any d, it can be shown that:

(i) Sa is a d-packing on G for any $a \in G$ (using the right invariance of ρ); and

(ii) the intersection of S and any compact subset C of G is a finite set (using the sequential compactness of C).

5:1 Theorem:

There exists at least one d-packing on G for each d.

Proof:

Let d be any positive real number. Let $\mathscr{C}$ be the collection of all subsets C of G such that $\rho(x,y) \geq d$ for any x and $y \in C$. Order $\mathscr{C}$ by inclusion. Then $(\mathscr{C},\underline{C})$ is a preordering and any chain $\mathscr{C}'$ in $\mathscr{C}$ has an upper bound

$$C' = \bigcup_{C \in \mathscr{C}'} C$$

Thus Zorn's Lemma is applicable to $\mathscr{C}$ and $\mathscr{C}$, therefore, has a maximal element, C_o say.

Since $C_o \in \mathscr{C}$, then $\rho(x,y) \geq d$ for all x and $y \in C_o$. Let z be any member of G not in C_o. If $\rho(z,x) \geq d$ for all $x \in C_o$, then $C_o \cup \{z\} \in \mathscr{C}$, contradicting the maximality of C_o. Hence there is no $z \in G \sim C_o$ such that $\rho(z,x) \geq d$ for all $x \in C_o$. Hence C_o is a d-packing on G. #

Define a functional μ on $C_0^+(G)$ by:

$$\mu(f) = \inf \sum_{x \in S} f(x); \text{ where the infimum}$$

$$\text{is taken over all } \tfrac{3}{4}\delta\text{-packings on } G, \qquad (9)$$

for all $f \in C_0^+(G)$. Then it can be shown that

(i) μ is a superlinear functional;

(ii) $\mu(0) = 0$;

(iii) $\mu(f) \geqslant \mu(g)$; for all f and $g \in C_0^+(G)$ such that $f \geqslant g$;

(iv) μ is right translation invariant; and

(v) there exists at least one function $f \in C_0^+(G)$ such that $\mu(f) \geqslant \frac{1}{4}$ (e.g. the function $\phi_0 \in C_0^+(G)$ defined by

$$\phi_0(x) = \frac{\delta - \rho(x,e)}{\delta}; \text{ when } x \text{ is such that } \rho(x,e) < \delta,$$

$$= 0; \text{ otherwise}).$$

5.2 Lemma:

If $f \in C_0^+(G)$, S is a $\frac{3}{4}$ δ-packing on G and $C_1,\ldots,C_k$ are positive real numbers and $f_1,\ldots,f_k \in \mathcal{F}$ such that

$$\sum_{i=1}^{k} C_i f_i \geqslant f,$$

then

$$\sum_{x \in S} f(x) \leqslant \sum_{i=1}^{k} C_i.$$

Proof:

Let f be any member of $C_0^+(G)$ and a be any member of G. Let $C_1,\ldots,C_k$ be positive real numbers and

$f_1, \dots, f_k \in \mathcal{F}$ be such that

$$\sum_{i=1}^{k} C_i f_i \geqslant f.$$

Let

$$E = \{x \in S \ / \ f(x) > 0\}.$$

If E is empty, then

$$\sum_{x \in S} f(x) = 0$$

and is obviously less than or equal to $\sum_{i=1}^{k} C_i$. Therefore assume E is non empty.

Put

$I(x) = \{i \in$ set of positive integers $/ \ 1 \leqslant i \leqslant k, \ x \in spt(f_i)\}$,

for all $x \in E$. Choose any $x \in E$. Then

$$x \in spt(f) \subseteq \bigcup_{i=1}^{k} spt(f_i)$$

and so there exists a positive integer K such that $1 \leqslant K \leqslant k$ and $x \in spt(f_K)$. Hence $I(x)$ is non empty, for all $x \in E$. If there exists elements y and $z \in E$ such that $y \neq z$, then, since

$$diam[spt(f_i)] = \tfrac{1}{2}\delta$$

for all i, (from (6)) and

$$\rho(y,z) \geqslant \tfrac{3}{4}\delta,$$

both y and z can not belong to the same $spt(f)$, for any i such that $1 \leqslant i \leqslant k$. Hence the intersection of $I(y)$ and $I(z)$ is empty for all distinct y and $z \in E$.

Now for all $x \in E$, if j is a positive integer such that $1 \leqslant j \leqslant k$ and $j \notin I(x)$, then $x \notin spt(f_j)$ and therefore $f_j(x) = 0$. Hence

$$\sum_{i \in I(x)} C_i f_i(x) = \sum_{i=1}^{k} C_i f_i(x)$$

$$\geqslant f(x)$$

for all $x \in E$. But $f_0 \leqslant 1$ by definition. Therefore $f_i \leqslant 1$ for all $1 \leqslant i \leqslant k$. Hence

$$f(x) \leqslant \sum_{i \in I(x)} C_i$$

for all $x \in G$. Therefore

$$\sum_{x \in S} f(x) = \sum_{x \in E} f(x)$$

$$\leqslant \sum_{x \in E} \sum_{i \in I(x)} C_i$$

$$\leqslant \sum_{i=1}^{k} C_i,$$

since the $I(x)$ are disjoint. #

From the above Lemma 4:2 and the definitions of ν and μ ((7) and (9)), the following Theorem follows easily.

5:3 Theorem:

$$\nu(f) \geqslant \mu(f) \quad \text{for all} \quad f \in C_0^+(G).$$

Furthermore it is straightforward to show from the definitions that

$$\mu(f) - \nu(f+h) \leqslant \nu(g) - \mu(g+h) \qquad (10)$$

for all f,g and $h \in C_0^+(G)$.

Define a functional p on $C_0(G)$ by:

$$p(f) = \inf[\nu(\emptyset) - \mu(\psi)];$$

where the infimum is taken over all $\emptyset$ and $\psi \in C_0^+(G)$ such that $f = \emptyset - \psi$ (11)

for all $f \in C_0(G)$. Then, from the properties of ν and μ, it can be shown that

(i) $$p(f) \geq \mu(f) \quad (12)$$

and

$$p(-f) \leq -\mu(f) \quad (13)$$

for all $f \in C_0^+(G)$;

(ii) $p(f) \geq -\nu(|f|)$, and, hence, $-\infty < p(f) < \infty$ for all $f \in C_0(G)$;

(iii) $p(0) = 0$;

(iv) p is a sublinear functional; and

(v) p is right translation invariant.

Λ is the vector space of all linear functionals on $C_0(G)$ endowed with the weak topology $\mathcal{J}$. (Note that the members of Λ need not be bounded. Define Γ to be the subset of all linear functionals λ in Λ for which

$$\lambda(f) \leq p(f) \quad (14)$$

for all $f \in C_0(G)$.

We note that

$$\lambda(-f) \leq p(-f),$$

and hence that

$$-p(-f) \leq \lambda(f),$$

giving, from (14),

$$-p(-f) \leq \lambda(f) \leq p(f), \tag{15}$$

for all $\lambda \in \Gamma$ and $f \in C_0(G)$.

If $f \in C_0^+(G)$, then from (13),

$$-p(-f) \geq \mu(f),$$

hence, from property (iii) of μ,

$$-p(-f) \geq 0,$$

and therefore

$$\lambda(f) \geq 0$$

for all $\lambda \in \Gamma$. Therefore all members of Γ are <u>positive</u> linear functionals.

It is straightforward to show that Γ is convex, weakly closed and does not contain the zero functional (use the function ϕ_0, defined in property (v) of μ, and (13)).

5:4 Theorem:

Γ is nonempty.

Proof:

Consider any $\psi \in C_0(G)$, which is not the zero function. Let M' be the linear manifold spanned by ψ. Define a linear functional η on M' by

$$\eta(\psi) = p(\psi).$$

We shall now show that

$$\eta(f) \leqslant p(f),$$

for all $f \in M'$.

Let f be any member of M'. Then there exists a real number α such that $f = \alpha\psi$. If α is non negative, then it can be shown, from the sublinearity of p, that

$$\eta(f) = p(f). \tag{16}$$

If α is negative, let ε be an arbitrary positive real number. Then, from (11), there exists ϕ_1 and $\phi_2 \in C_0^+(G)$ such that

$$\nu(\phi_1+\psi) - \mu(\phi_1) < p(\psi) + \tfrac{1}{2}\varepsilon$$

and

$$\nu(\phi_2) - \mu(\phi_2+\psi) < p(-\psi) + \tfrac{1}{2}\varepsilon.$$

Then, from (10) and (16),

$$\begin{aligned}\eta(-\psi) = -\eta(\psi) &= -p(\psi)\\ &< -\nu(\phi_1+\psi) + \mu(\phi_1) + \tfrac{1}{2}\varepsilon\\ &\leqslant \nu(\phi_2) - \mu(\phi_2+\psi) + \tfrac{1}{2}\varepsilon\\ &< p(-\psi) + \varepsilon.\end{aligned}$$

Since ε is arbitrary,

$$\eta(-\psi) \leqslant p(-\psi).$$

Hence, for negative α, it can be now easily shown (from the sub-linearity of p), that

$$\eta(f) \leqslant p(f).$$

Therefore, since p is a sublinear functional, by the Hahn-Banach Extension Theorem, there exists a linear functional η' on $C_0(G)$ such that

$$\eta'(f) = \eta(f)$$

for all $f \in M'$ and

$$\eta'(f) \leqslant p(f)$$

for all $f \in C_0(G)$. Hence $\eta' \in \Gamma$ and Γ is, therefore, non empty. #

5:5 Theorem:

Γ is weakly compact.

Proof: (This is similar to Theorem 4.61-A on page 228 of [25]).

For each $f \in C_0(G)$, $[-p(-f), p(f)]$ is a compact subset of the real line. Let

$$A = \prod_{f \in C_0(G)} [-p(-f), p(f)].$$

Then A is compact and, from (15), Γ is a subset of A. The topology induced on Γ by $\mathcal{J}$ is the same as that induced by the Cartesian Product topology of A. Therefore to show that Γ is compact, we have only to show that Γ is closed in A.

But Γ is weakly closed in Λ. Therefore to show that Γ is closed in A, we only have to show that, if ξ is an element of the closure of Γ in A, then ξ is linear, i.e. $\xi \in \Lambda$.

Suppose ξ is an element in the closure of Γ in A. Consider any positive real number ε and any f and $g \in C_0(G)$. Then

$U = \{\lambda \in A / |(\lambda-\xi)(f)| < \varepsilon, |(\lambda-\xi)(g)| < \varepsilon, |(\lambda-\xi)(f+g)| < \varepsilon\}$

is an open neighbourhood of ξ in A. This neighbourhood contains some element $\eta \in \Gamma$ and, therefore, since η is linear,

$$|\xi(f+g)-\xi(f)-\xi(g)| \leqslant |\eta(f+g)-\xi(f+g)| + |\eta(f)-\xi(f)| + |\eta(g)-\xi(g)| < 3\varepsilon.$$

Hence, since ε is arbitrary, $\xi(f+g) = \xi(f)+\xi(g)$. In a similar manner, it can be shown that $\xi(\beta f) = \beta\xi(f)$, for any real scalar β.

#

For each $a \in G$, define the linear transformation T_a of Λ into Λ by

$$T_a\lambda(f) = \lambda(f_a) \qquad (17)$$

for all $f \in C_0(G)$. $\mathcal{H}$ is the set of T_a, for all $a \in G$.

We note, for all $\lambda \in \Lambda$ and a and $b \in G$, that:

(i) $T_aT_b\lambda(f) = T_b\lambda(f_a)$

$= \lambda((f_a)_b)$

$= \lambda(f_{ba})$

$= T_{ba}\lambda(f).$

for all $f \in C_0(G)$ (from (8)), i.e.

$$T_aT_b = T_{ba}; \qquad (18)$$

(ii) T_e is the identity transformation;

(iii) $T_{a^{-1}}$ is the inverse of T_a (and hence T_a is 1:1);

(iv) $\mathcal{H}$ is a group isomorphic to G;

(v) T_a is continuous with respect to $\mathcal{J}$ (see Lemma 5 of chapter II of [24]); and

(vi) T_a is norm preserving.

5:6 Theorem:

$$\mathcal{H} \text{ maps } \Gamma \text{ onto } \Gamma$$

Proof:

Consider any $\lambda \in \Gamma$ and $a \in G$. Let f be any member of $C_o(G)$. Then, from (14) and the right invariance of p,

$$\begin{aligned} T_a\lambda(f) &= \lambda(f_a) \\ &\leqslant p(f_a) \\ &= p(f), \end{aligned}$$

and so $T_a\lambda \in \Gamma$. Furthermore, if we put

$$\mu = T_{a^{-1}}\lambda,$$

then, from (18),

$$T_a\mu = T_e\lambda = \lambda.$$

Hence T_a maps Γ onto Γ. #

5:7 Bounded linear functionals:

Define Γ' to be the subset of Γ consisting of all linear functionals in Γ which are bounded by a positive constant $K = \frac{C}{|\psi|}$, where ψ is the function from the proof of Theorem 5:4 and $C = \max\,[|p(\psi)|, |p(-\psi)|]$, i.e. for all $\lambda \in \Gamma'$ and for all $f \in C_o(G)$,

$$|\lambda(f)| \leqslant K|f|. \tag{19}$$

5:7:1 Theorem:

Γ' is nonempty, convex and weakly compact.

Proof:

In the proof of Theorem 5:4, the linear functional η on M', the linear manifold spanned by $\psi \in C_0^+(G)$, is bounded, in fact,

$$|\eta(f)| \leq \frac{C}{|\psi|} |f|,$$

where $C = \max[|p(\psi)|, |p(-\psi)|]$, for all $f \in M'$. Define functional q on $C_0(G)$ by

$$q(f) = \min[\frac{C}{|\psi|} |f|, p(f)] \tag{20}$$

for each $f \in C_0(G)$. It is straight forward to show that q is a sublinear functional. Therefore, in a similar manner as in the proof of Theorem 5:4, we can use the Hahn-Banach Extension Theorem to show that there exists a linear functional ξ on $C_0(G)$ such that

$$\xi(f) = \eta(f)$$

for all $f \in M'$ and

$$\xi(f) \leq q(f) \tag{21}$$

for all $f \in C_0(G)$. From (20) and (21), $\xi(f) \leq p(f)$ for all $f \in C_0(G)$, hence $\xi \in \Gamma$, and

$$\xi(f) \leq \frac{C}{|\psi|} |f| \tag{22}$$

for all $f \in C_0(G)$. But, from (21), for all $f \in C_0(G)$,

$$-\xi(f) = \xi(-f)$$
$$\leqslant q(-f),$$

and from (20), this means that

$$-\xi(f) \leqslant \frac{C}{|\psi|}\,|-f|$$
$$= \frac{C}{|\psi|}\,|f|. \qquad (23)$$

Hence, from (22) and (23), ξ is bounded by $\frac{C}{|\psi|}$, and, therefore, is a member of Γ'. Hence Γ' is nonempty.

If λ and $\eta \in \Gamma$ and are bounded, it is straightforward to show that $t\lambda + (1-t)\eta$ (which is a member of Γ, since Γ is convex) is bounded, for all real numbers t between 0 and 1. Hence Γ' is convex.

Let F be the subset of Λ consisting of all members of Λ which are bounded by K. Then, from Theorem 4.61-A on page 228 of [25], F is weakly compact. Γ' is the intersection of Γ and F and, hence, is also weakly compact.

#

5:7:2 Theorem:

$\mathcal{H}$ maps Γ' onto Γ'.

Proof:

Consider any $\lambda \in \Gamma'$ and $a \in G$. From Theorem 5:6, $T_a\lambda \in \Gamma$ and there exists $\mu = T_{a^{-1}}\lambda \in \Gamma$ such that $T_a\mu = \lambda$. We, therefore, need only show that $T_a\lambda$ and $T_{a^{-1}}\lambda \in \Gamma'$, i.e. are bounded. Let f be any member of $C_0(G)$. It is straightforward to show that $|f_a| = |f|$. Hence, with (19), this gives

$$\begin{aligned} |T_a\lambda(f)| &= |\lambda(f_a)| \\ &\leqslant K\,|f_a| \\ &= K\,|f|, \end{aligned}$$

and, similarly,

$$|T_{a^{-1}}\lambda(f)| \leqslant K\,|f|.$$

#

Therefore Γ and Γ' are nearly identical in this situation. But as will be shown, the common fixed point for the mappings of $\mathcal{H}$ (i.e. the right invariant integral) cannot be a member of Γ' when G is not compact. This exemplifies the difficulty of finding a fixed point Theorem that can be used to show the existence of a right invariant integral on a locally compact group. In fact, since $\mathcal{H}$ is unchanged in its action on Γ and Γ', this points to the need to consider conditions on the nonempty convex weakly compact set Γ (as in Chapter 3) when looking for such a fixed point Theorem. For example, the answer may be, as in Chapter 3, with conditions on the convex compact set or with a topology in which Γ is compact and Γ' is not.

5:7:3 Theorem:

If G is not compact and ξ is a common fixed point, in Γ, for the mappings of $\mathcal{H}$, then ξ is unbounded, i.e. $\xi \notin \Gamma'$.

Proof:

Assume ξ is a common fixed point, in Γ, for the mappings of $\mathcal{H}$. Consider the function $\phi_0 \in C_0^+(G)$ as defined in property (v) of sublinear functional μ, i.e.

$$\phi_0(x) = \frac{\delta-\rho(x,e)}{\delta}; \quad \text{when } x \text{ is such that } \rho(x,e) < \delta,$$

$$= 0; \quad \text{otherwise.}$$

Then $|\phi_0| = 1$ and

$$\text{spt}(\phi_0) = \{x \in G/\rho(x,e) \leqslant \delta\}. \tag{24}$$

From (13) and (15),

$$\xi(\phi_0) \geqslant \mu(\phi_0) \geqslant \frac{1}{4}. \tag{25}$$

Since G is not compact, there exists sequence $\{a_n\}$ in G such that $e = a_1$ and $\rho(a_n, a_m) \geqslant \delta$, for all distinct positive integers n and m. For each n, let

$$g_n = (\phi_0)_{a_n^{-1}}.$$

Then, for each n, $g_n \in C_0^+(G)$, $|g_n| = 1$, from (24),

$$\text{spt}(g_n) = \{x \in G/\rho(x,a_n) \leqslant \delta\}$$

and, since ξ is a common fixed point for the mappings of $\mathcal{H}$ and (25) holds,

$$\xi(g_n) = \xi((\phi_o)_{a_n^{-1}})$$

$$= \xi(\phi_o) \geq \frac{1}{4} .$$

For each n, put

$$h_n = \sum_{i=1}^{n} g_i .$$

Then, for each n, $h_n \in C_o^+(G)$, $|h_n| \leq 1$ and

$$\xi(h_n) = \sum_{i=1}^{n} \xi(g_i) \geq \frac{n}{4} .$$

Hence ξ is unbounded. #

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