GROUP THEORETICAL ASPECTS OF PARAFIELDS

by

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ABSTRACT

Three group theoretical problems associated with parafields are considered in this thesis.

The first two, discussed in Chapters 2 and 3, are similar since they deal with the restrictions imposed by physical requirements on theories whose commutation relations are more general than Bose or Fermi. In both cases the restrictions are a consequence of the properties of the algebras (or representations of these algebras) which are generated by the operators obeying the more general commutation relations.

The cluster property requires that the description of widely separated physical systems should be independent of each other. This property, formulated for parafields, is essentially a reduction of the groups generated by the parafield creation and annihilation operators to the appropriate sub-groups. The parafield representations of these groups are such that for p = 1 only, are the restrictions due to the cluster property equivalent to those placed upon the theory by the requirement of locality. This is not surprising since a comparison of locality and the cluster property shows that the two concepts are inequivalent.

In the non-relativistic case the cluster property restricts physical observables to those of the form $[\phi^*(x),\phi(y)]_\pm$. Restricting physical observables to elements of this form implies that the theory is just a convenient description of a system of

p fermions (or bosons). This also has the important consequence that, in the associated quantum mechanical space, physical observables are symmetric functions of their arguments. Thus, for parafields, the symmetry of observables is a result rather than an assumption, as it is usually stated. The relativistic case is treated as an extension of the non-relativistic one, and it is shown that it is necessary to decompose the wave function into positive and negative frequencies in order to construct physical observables. The S-matrix and Wightman formulations are also discussed in §2. For $p > 1$ only a very restricted set of vacuum expectation values of parafield operators factorize in accordance with the cluster property. The vacuum expectation values of pb2 operators are somewhat exceptional as a result of the commutation relations satisfied by these operators. Some attention is also given to a possible L.S.Z. formulation of parafield theory.

In §3 a quantization scheme recently proposed by Kademova and Kraev is shown to be inconsistent since it does not, in general, possess a vacuum state of lowest energy. This follows from the properties of the group generated by the annihilation and creation operators satisfying the proposed commutation relations. It is also shown in §3 that the requirement of unitary invariance of the algebra implies that for a theory with commutation relations of the form $a_k^{p+1} = 0$ and $[a_k, a_\ell]_- = 0$, there can only be p particles in the Universe.

The purely group theoretical problem of constructing the representations of a single parabose operator of any order is considered in §4. The representations corresponding to the Bargmann and harmonic oscillator representations of a boson operator are found. This is achieved by the introduction of an operator R satisfying

$$[R,z]_+ = [R,d_z]_+ = 0 .$$

The equivalence of the two representations is proved by the construction of a unitary integral transform connecting them.

STATEMENT

This thesis contains no material which has been accepted for the award of any degree, and to the best of my knowledge and belief, contains no material previously published or written by another person except where due reference is made in the text.

Douglas Andrew Gray

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§1. INTRODUCTION

An intriguing question in the description of elementary particles is why only Bose and Fermi statistics appear to be realized in Nature. While there does not seem to be any mathematical inconsistencies in theories describing particles more general than bosons or fermions, there is, as yet, no direct experimental evidence for the existence of any such particles. However physical assumptions may impose severe restrictions on a mathematically consistent theory. An approach to the problem of the existence of generalized particles is to determine which properties, if any, of the description of these particles are consistent with physical assumptions.

The first theoretical studies of generalized statistics were made by Gentile, Borsellino, Sommerfeld and others(1). These studies were mainly concerned with the statistical behaviour of ensembles of the generalized particles. One of the most important results that arose from these studies was the realization that generalized statistics were associated with higher dimensional representations of the symmetric group. However it was not until after Wigner(2) had shown that the Heisenberg equations of motion do not uniquely determine the commutation relations(3) that Green developed the first consistent description of generalized statistics(4). Using field theory Green derived a set of relationships for the creation and annihilation operators, more general than Bose or Fermi, which were consistent with Heisenberg's equation of motion. These statistics were termed "parastatistics", as distinct from generalized statistics, by Greenberg and Messiah(5) who investigated their properties. In particular the selection rules they derived implied that all elementary particles are either Bose or Fermi.

Galindo and Yndurain(6) showed that parafield states did not form a representation of the symmetric group. This objection was removed by Landshoff and Stapp(7), who after distinguishing between "particle" and "place" permutations, showed that parafields form a representation of the former only. The reduction of the Fock space to irreducible representations of the particle permutation operators was effected by Ohnuki and Kamefuchi(8). At that stage the relation of the quantum mechanical space to the Fock space of field theory was unknown and obscured discussions(9,10,11,12) on the significance of place and particle permutations. Ohnuki and Kamefuchi(10) derived the relationship between the quantum mechanical and quantum field descriptions and showed that the quantum mechanical space associated with a parafield permitted a sensible interpretation of the indistinguishability of identical particles.

Hartle and Taylor(13) were able to show that the quantum mechanical space of paraparticles was consistent with the cluster decomposition property. This property requires that widely separated systems of identical particles are non-interacting. This result was significant since Steinmann, Luders and others (14,15) had previously shown that not all generalized statistics which correspond to various collections of irreducible representations of S_n were consistent with the cluster property. The proof by Hartle and Taylor[13], formulated in the quantum mechanical space, assumes that observables are symmetric functions of their arguments. To avoid this assumption, and also the conceptual difficulties associated with labelling identical particles, it is easier to check the consistency of parastatistics with the cluster property within the framework of field theory rather than in the quantum mechanical space.

This is done in §2.1 to §2.6 of this thesis for a non-relativistic free parafield. The formulation of the cluster property in field theory becomes one of reducing certain groups, generated by the parafield operators, to their various subgroups. These groups; the orthogonal group for parafermi statistics and the symplectic group for parabose statistics, were first connected with parafields by Kamefuchi and Takahashi[16]. The application of the cluster property i.e., the reduction of these groups uses some important results that have recently been obtained for parafermi fields by Bracken and Green[17] and for parabose fields by Alabiso, Duimio and Redondo[18,19].

It is found that the cluster property severely restricts the form of permissible physical observables to functions constructed from elements of the unitary group.

Recently Drühl, Haag and Roberts[20] have classified the "local" observables of a parafermi field in terms of the associated non-Abelian gauge groups[21,22]. They have shown that a parafield may provide a convenient description of a system of p fermions provided the physical observables of the parafield are restricted to elements of $U(\nu)$. The results of this thesis show that the cluster property restricts the permissible physical observables of a parafield to the elements of $U(\nu)$. Combining these results it follows that the parafield description is nothing more than a convenient description of a system of p fermions.

In the associated quantum mechanical space the physical observables corresponding to elements of $U(\nu)$ are symmetric functions of their arguments. The proof of the cluster property by Hartle and Taylor corresponds to the proof given in this thesis that the elements of $U(\nu)$ are consistent with the cluster decomposition property.

The extension of these results to relativistic fields is also considered in §2. In this case, to construct physical observables which are consistent with the cluster decomposition property it is necessary to decompose the field operators into positive and negative frequencies. This is a more severe restriction than that obtained by Ohnuki and Kamefuchi[23] on the basis of locality requirements.

The S-matrix approach is briefly considered and it is shown that it is possible to construct S-matrices which are consistent with the cluster decomposition property.

The consequences of the cluster property within the Wightman axiomatic approach[24] have been widely investigated for Bose and Fermi statistics[25]. However little attention has been paid to the corresponding problem for parafields with $p > 1$. Using some general properties of the vacuum expectation values of parafield operators which were derived by Dell' Antonio, Greenberg and Sudarshan[26] and Govorkov,[27] it is shown that in general only a very restricted class of vacuum expectation values are consistent with the cluster decomposition property.

In view of the severe restrictions imposed on the vacuum expectation values of parafield operators a brief discussion on the possibility of formulating an L.S.Z.[28] theory of para-statistics concludes the first chapter.

Although parastatistics are sufficient to satisfy the Heisenberg commutation relations it does not necessarily follow that they are the only possibilities for generalized statistics. Other alternative statistics have been proposed by considering different sets of commutation relations[29,30,31]. Functions of the creation and annihilation operators satisfying these alternative relations are often associated with the well known classical groups. As with parafields, physical assumptions (such as the positive definiteness of states) are manifested by the choice of the classical group. This is particularly so for a set of commutation relations proposed by Kademova and Kraev[30]. It is shown in §3 that these statistics do not satisfy the accepted requirements of field theory. The restrictions placed on commutation relations of the form[31]: $a_k^{p+1} = 0$ and $[a_k, a_\ell]_- = 0$ by a less obvious physical assumption are also considered in §3.

The work presented in §4 does not follow on from the preceding chapters. It is concerned with the mathematical problem of actually constructing explicit representations of parabose operators. The construction of matrix representations of a single parafield operator was considered by O'Raifeartaigh and Ryan[32] and studies of the uniqueness of these were made by various other authors[33]. The construction of irreducible representations of ν parafermi operators was made by Ryan and Sudarshan[34] by means of the representations of the $O(2\nu+1)$ group associated with the parafermi operators. The matrix representation of a single parabose oscillator was constructed by Jordan, Mukunda and Pepper[35] and the generalization to ν operators has recently been effected by Alabiso and Duimio[19].

Two other representations of the boson algebra are the Bargmann[36] and harmonic oscillator[37] representations. For one degree of freedom Yang[38] has found a representation of the parabose algebra in terms of x, $\frac{d}{dx}$ and R, where R is an operator which anti-commutes with x and $\frac{d}{dx}$. By using Yang's expressions for the raising and lowering operators, harmonic oscillator representations of the parabose algebra are obtained in §4. The Bargmann space for a single parabose operator is also found and its equivalence to the harmonic oscillator representation is also shown. A discussion on the possibility of generalizing these results to ν degrees of freedom concludes §4.

§2. CLUSTER RESTRICTIONS ON PARAFIELD OPERATORS

§2.1 INTRODUCTION

A fundamental assumption about elementary processes is that the interaction between two bodies separated by a large distance is negligible. This assumption appears to have good experimental verification as there is no evidence that a system of identical particles localized on Earth is affected by the presence of another group of identical particles on Mars. It would, to say the least, be very difficult to describe electron-electron scattering on Earth if the effect of all other electron-electron scatterings in the Universe was to be accounted for. This decomposition of the Universe into separate non-interacting regions is termed "cluster decomposition" and any description of a system of identical elementary particles should exhibit this property. That is if the system is divided into two clusters C_1 and C_2, which are then separated by a large distance, then each subsystem may be described independently of the other. Not all theories will necessarily possess this property or alternatively the cluster decomposition property may place certain restrictions on a theory. The restrictions imposed on a parafield theory are investigated in this chapter.

Many authors have shown that attempts to generalize Bose and Fermi statistics within the quantum mechanical framework by considering higher dimensional representations of the symmetric group are not arbitrary but subject to restrictions imposed by the cluster property(13,14,15). Hartle and Taylor(13) and Doplicher, Haag and Roberts(15) have shown that the representations of the symmetric group afforded by the quantum mechanical space of para-particles are consistent with the cluster decomposition property. In particular Hartle and Taylor have shown that the cluster decomposition problem is essentially that of the reduction of S_{n+1} to S_n. Their proof , however, assumes that all operators in the quantum mechanical space are symmetric functions of their arguments.

On the other hand Ohnuki and Kamefuchi(10,23) have argued, from considerations based on field theory, that not all physical observables associated with a parafield theory are symmetric functions of their arguments. It is not obvious whether the proof given by Hartle and Taylor is capable of being modified to include observables of more general symmetry types. This problem is not tackled directly in this thesis, but an alternative approach is developed which does resolve the ambiguity.

The approach considered in this thesis is to formulate the cluster decomposition property within the field theoretic framework rather than, as is normally done, in the quantum mechanical space. This has the advantage of not only showing that parafields are consistent with the cluster decomposition property but also of demonstrating for which class of operators the cluster property holds. An additional advantage of the field theory formulation is that it allows a comparison with the results obtained by Ohnuki and Kamefuchi[23] which are based on apparently similar physical assumptions.

Similarly to the quantum mechanical case, the application of the cluster decomposition property to parafield theory is essentially the reduction of various groups to their appropriate subgroups. §2.2 contains a résumé of some recently discovered properties[17] of the representations of $O(2\nu+1)$, $O(2\nu)$ and $U(\nu)$ afforded by ν parafermi operators and those of the representations of $Sp(2\nu)$ and $U(\nu)$ afforded by ν parabose operators[19]. In order to emphasize the group theory involved the variables x_i of a non-relativistic parafield, $\phi(x_i)$, are restricted to a finite number of values labelled 1 to ν. The transition to continuous variables is discussed at the end of §2.2 and is also effected at the appropriate point in the discussion.

Ohnuki and Kamefuchi[23] have considered a "non-relativistic limit of locality" and have shown that it restricts terms in the interaction Hamiltonian to arbitrary functions of the following terms†

---

†Here, and throughout the thesis, upper signs refer to parabose fields and lower ones to parafermi fields.

$$[\phi^*(x),\phi(y)]_\pm \quad , \quad [\phi(x),\phi(y)]_\pm$$

and

$$[\phi^*(x),\phi^*(y)]_\pm \; .$$

They assume that this "non-relativistic limit" is equivalent to the cluster decomposition property and so imposes all the necessary restrictions on a theory. A comparison of the conditions of locality and cluster decomposition is given in §2.3 where it is shown that conceptually, at least, they are different. It is therefore not surprising that for parafields the restrictions on permissible physical observables due to cluster decomposition are different than those obtained by Ohnuki and Kamefuchi based on locality.

In §2.4 the cluster decomposition property is formulated within the framework of field theory rather than, as is usually the case, the quantum mechanical framework.

The restrictions on a non-relativistic parafield theory due to the cluster decomposition property are derived in §2.5. The results of §2.2 are of particular importance since in the parafermi case the problem is essentially that of reducing the appropriate representations of $O(2\nu+3)$, $O(2\nu+2)$ and $U(\nu+1)$ to those of the subgroups $O(2\nu+1)$, $O(2\nu)$ and $U(\nu)$ respectively. For the parabose case the appropriate reduction is from $Sp(2\nu+2)$ and $U(\nu+1)$ to $Sp(2\nu)$ and $U(\nu)$ respectively. In both the parafermi and the parabose case, permissible physical observables are restricted to the elements of $U(\nu)$ i.e., functions of operators of the form

$$[\phi^*(x_i),\phi(x_j)]_\pm .$$

This is obviously a more severe restriction than that obtained by Ohnuki and Kamefuchi. It also implies more severe selection rules than those obtained by previous authors[5,39] and in particular, for $p > 1$ it forbids the annihilation or creation of paraparticles.

The final section in §2.5 compares this restriction of physical observables to the unitary group with the classification of observables by means of the associated gauge groups that has recently been effected by Drühl, Haag and Roberts[20]. As a result of restricting physical observables to elements of $U(\nu)$ the results of the work of Drühl, Haag and Roberts may be used to show that a parafield is equivalent to a description of p fermions with certain restrictions imposed.

The discussion of a non-relativistic parafield is completed in §2.6 with a review of the associated quantum mechanical space with emphasis on the significance of the unitary rather than symmetric group. In the quantum mechanical space the physical observables corresponding to elements of $U(\nu)$ are symmetric functions of their arguments and hence commute with relabelling operators defined on that space. This proves, at least for parafields, the assumption of many authors that physical observables are symmetric functions of their arguments. The proofs of the consistency of the quantum mechanical space of parafields with cluster requirements may be taken to be quite general although it would obviously be possible to show the inconsistency of functions which are not symmetric functions of their arguments with cluster requirements. The significance of the unitary group is that it permits a resolution of the objections raised by Ohnuki and Kamefuchi[12] against the observability of particle permutations.

The generalization in §2.7 to a relativistic field is complicated by the introduction of an additional ν degrees of freedom for the description of anti-particles. The significance of the momentum representation is discussed and it is shown that permissible physical observables are those whose representations in momentum space are functions of the appropriate unitary algebra. To achieve this within the particle anti-particle formulation it is necessary to decompose the wave function into positive and negative frequencies. This is a direct generalization of the non-relativistic result. It is well known that the Fermi commutation relations are invariant under Bogoliubov transformations and Volkov[40] has observed that a similar result holds for parafermi statistics. Physically the Bogoliubov transformations relate the spaces of negative and positive energies. For $p > 1$ an interesting distinction between the types of physical observables that are admissible in each space results from an application of the cluster decomposition property. This illustrates the importance of considering the representation space associated with the algebra of operators as well as the algebra itself.

In the attempts to formulate an axiomatic description of elementary particles the cluster decomposition property has received considerably more attention than it has in the various field theoretic descriptions. In particular the use of "cluster amplitudes" to parametrize the S-matrix has proved extremely successful. The decomposition of the S-matrix was first considered by Wichmann and Crichton[41]. In §2.8 their approach is modified for parafields and it is expected that a similar parametrization of the S-matrix may be effected if its elements are constructed from functions of the appropriate unitary group.

Various proofs of the cluster decomposition properties of vacuum expectation values in the Wightman axiomatic formulation of field theory have been given[25]. However these proofs either directly or indirectly use the local commutativity condition that

$$[A(x), A(y)]_{\mp} = 0$$

when x and y have a space-like separation. Since this condition does not hold for parafields in general, not all vacuum expectation values necessarily satisfy the cluster decomposition property. Some examples are given in §2.9 which suggest that it would be difficult to find a sufficiently non-trivial set of vacuum expectation values for which the cluster property holds.

It is well known that for Bose fields the local commutativity condition implies that the S-matrix for two particle scattering is analytic[(42)]. For arbitrary order parabose fields the local commutativity condition is modified to

$$[[A(x),A(y)]_+, A(z)]_- = 0$$

when z is space-like with respect to both x and y. Whether or not this implies that the S-matrix is analytic is unresolved and in §2.10 some conjectures concerning this are discussed within the L.S.Z. formulation of S-matrix theory.

The final section contains a résumé and discussion of the various results obtained.

§2.2 SOME RELEVANT PROPERTIES OF A PARAFIELD

In order not to interrupt future discussion, some results about parafields which will be needed are given here.

A non-relativistic parafield $\phi(x_i,t)$ satisfies the following equal time commutation relations:

$$[\phi(x_i,t), [\phi^*(x_j,t), \phi(x_k,t)]_\pm]_- = 2\,\delta_{ij}\,\phi(x_k,t)$$

and

$$[\phi(x_i,t), [\phi(x_j,t), \phi(x_k,t)]_\pm]_- = 0$$

where x_i takes a finite number of values 1 to ν. In future the time variables will be omitted in the non-relativistic discussion; it being understood that all commutators (or anti-commutators) will be taken at equal times.

Bracken and Green[17] have recently obtained interesting results concerning the structure of the representations of $O(2\nu+1)$ generated by a_r^*, a_r, $\frac{1}{2}[a_r^*, a_s]_-$, $\frac{1}{2}[a_r^*, a_s^*]_-$ and $\frac{1}{2}[a_r, a_s]_-$ where the a_r^* and a_r are the ν creation and annihilation operators for parafermi fields. They have shown the following results:

(1) The vacuum, defined by

$$a_r \,|0\rangle = 0 \quad \text{for all } r$$

and

$$a_r\, a_s^* \,|0\rangle = p\, \delta_{rs} \,|0\rangle$$

where p is the order of the parafield, is the lowest weight vector of a finite dimensional, unitary irreducible representation of $O(2\nu+1)$ labelled $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$.

(2) The representation space is found by applying powers of a_r^*'s to the vacuum.

(3) This representation of $O(2\nu+1)$ reduces to $p + 1$ irreducible representations of the $O(2\nu)$ subgroup generated by $\frac{1}{2}[a_r^*, a_s]_-$, $\frac{1}{2}[a_r, a_s]_-$ and $\frac{1}{2}[a_r^*, a_s^*]_-$. These representations are labelled $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}, \frac{p}{2} - q')$ where $q' = 0, 1, \ldots, p$. Representations of this type will be termed "Fock" representations of $O(2\nu)$. More general representations not of this form will be termed "non-Fock" representations.

(4) The number conserving operators $N_{rs} = \frac{1}{2}[a_r^*, a_s]_-$ satisfy

$$[N_{ij}, N_{k\ell}]_- = \delta_{kj}\, N_{i\ell} - \delta_{i\ell}\, N_{kj}$$

which is the Lie algebra of the unitary group $U(\nu)$.

(5) Irreducible representations of $U(\nu)$ are denoted by $(\ell_1, \ell_2, \ldots, \ell_\nu)$ where ℓ_i may be regarded as the i^{th} row (or column when parabose fields are considered) of a corresponding Young tableau[43]. Bracken and Green have shown that for parafermi statistics each irreducible representation of $U(\nu)$ appears once only, that $p \geqslant \ell_1 \geqslant \ell_2 \geqslant \ldots \geqslant \ell_\nu \geqslant 0$ and

$$\sum_{j=1}^{\nu} (-1)^j \ell_j = -q$$

where $q = q'$ for ν even and $q = p - q'$ for ν odd.

The last result is particularly important since it implies that specifying the unitary labels of a state immediately determines the representation of $O(2\nu)$ to which the state belongs, i.e., the invariants of $O(2\nu)$ are determined by those of $U(\nu)$ by the above equation. This is a generalization of results obtained previously by Ohnuki and Kamefuchi[10].

For parabose fields the anti-commutators $\frac{1}{2}[a_r^*, a_s]_+$, $\frac{1}{2}[a_r^*, a_s^*]_+$ and $\frac{1}{2}[a_r, a_s]_+$ form a representation of the non-compact form $Sp(2\nu,R)$ of the symplectic group. As Alabiso and Duimio[19] have shown, the infinite dimensional space obtained by applying powers of a_r^* to the vacuum reduces into $p + 1$ irreducible representations of $Sp(2\nu)$. Similarly to the parafermi case each irreducible representation of $Sp(2\nu)$ contains all those representations of $U(\nu)$ (generated by the operators $\frac{1}{2}[a_r^*, a_s]_+$) with the same number of odd rows. Once again an important result holds; that specifying to which irreducible representation of $U(\nu)$ a state belongs immediately determines to which representation of $Sp(2\nu)$ it belongs. It is this property, common to both parafermi and parabose algebras, which is of importance in considerations of the cluster property.

Since the cluster property is formulated in configuration representation the above results, formulated in momentum space, should be restated in configuration representation. From the rather artificial definition of the $\phi(x_i)$'s the modification of the results is straightforward. For example the generators of the $O(2\nu)$ subgroup are

$$\tfrac{1}{2}[\phi^*(x_i), \phi^*(x_j)]_-, \tfrac{1}{2}[\phi(x_i), \phi(x_j)]_-$$

and

$$\tfrac{1}{2}[\phi^*(x_i), \phi(x_j)]_- .$$

The restriction of the domain of $\phi(x_i)$ to a finite set of values has been introduced in order to emphasize the group theoretical aspect of the cluster problem. The transition to the continuous case can be effected by replacing sums by integrals and the δ_{ij}'s by the appropriate delta functions. In particular, the set of states $|(x,y, \ldots z)\ell s_i>$ defined below still form a complete set of states and are labelled by operators of the form $\tfrac{1}{2}[\phi^*(x), \phi(y)]_{\pm}$. This is so because these properties are a direct consequence of the commutation relations and not of the restriction to a finite domain.

It is interesting to consider the number conserving operators N_{ij} as they form a generalization of the so-called particle permutations introduced by Landshoff and Stapp[7] and discussed in detail by many others (refs. 8, .., 13). In particular, Ohnuki and Kamefuchi[10] have shown that any parafield state can be expressed as a linear combination of states of the form

$$|\{\phi^*(x_{i_1}), \phi^*(x_{i_2}), \ldots \phi^*(x_{i_n})\}, \ell, s_i>$$

where

$$\ell = (\ell_1, \ell_2, \ldots, \ell_\nu)$$

denotes a particular representation of rank n of $U(\nu)$ and s_i labels the basis state of that representation. The arguments are to be symmetrized over the x_i's. In future the above expression will be abbreviated to $|(x)_n \ell s_i>$. Ohnuki and Kamefuchi interpret ℓ as denoting representations of particle permutations but their results may be suitably modified to interpret ℓ as denoting representations of $U(\nu)$. As will be considered in §2.4 labelling a state in this fashion introduces a redundancy into the description of a state since the projection operator P_n onto the n-particle space satisfies

$$P_n = \frac{1}{n!} \sum_{(x)_n} \sum_{\ell=1}^{h} \sum_{s_i=1}^{d_\ell} |(x)_n \ell s_i> <(x)_n \ell s_i|$$

$$= \sum_{(x)_n} \sum_{\ell=1}^{h} \frac{d_\ell}{n!} |(x)_n \ell s_j> <(x)_n \ell s_j| .$$

In this equation d_ℓ is the dimension of the ℓ^{th} irreducible representation of S_n and

$$\sum_{(x)_n}$$

denotes

$$\sum_{i_1=1}^{\nu} \sum_{i_2=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} .$$

The redundancy involved is not of any physical significance and could be removed by labelling a state by the chain $U(\nu) \supset U(\nu-1) \supset \ldots \supset U(1)$. In many cases the s_i's may be thought of as labels corresponding to the chain $U(\nu-1) \supset U(\nu-2) \supset \ldots \supset U(1)$. In future discussions "choosing s_i and s_j appropriately" will mean taking the particular basis states (or possibly combinations of them) which correspond to the symmetry labelled by $U(\nu-1) \supset U(\nu-2) \supset \ldots \supset U(1)$ which is being discussed.

§2.3 A COMPARISON OF THE CONDITIONS OF LOCALITY AND CLUSTER DECOMPOSITION

Two conditions common to a relativistic field theory are those of locality and cluster decomposition[25]. It is often assumed that these two conditions are equivalent but, as the results of this chapter will show, for $p > 1$ this is not the case. Even for Bose statistics, Sudarshan and Bardakci[44] have proposed an example of a field satisfying local commutavity but violating the cluster decomposition property. In this section the conceptual aspects of locality and the cluster decomposition property will be compared.

The condition of locality requires that events with space-like separation should not interfere since interaction effects are propagated at velocities less than or equal to the velocity of light. This condition is usually expressed mathematically[45] as

$$[H_I(x), H_I(y)]_- = 0 \tag{2.1a}$$

if $x \sim y$. The interaction Hamiltonian is denoted as $H_I(x)$ and $x \sim y$ denotes space-like separation. Oneda, Umezawa and Podolanski[46] have suggested that the above relation be supplemented by

$$[H_I(x), \phi(y)]_- = 0 \;. \tag{2.1b}$$

Ohnuki and Kamefuchi[23] have applied both these conditions to a parafield and have derived the restrictions on the field observables. However the above conditions may be criticized on two accounts. The first is that they have been derived using the interaction representation which is known to lead to inconsistencies. A criticism more relevant to parafields is that the derivation of (2.1a) assumes that the variations are commuting c-numbers (see p.421 of ref. 45). For $p > 1$ this is not the case and, as Kibble and Polkinghorne and Scharfstein[47] have shown, variations may be defined for only certain combinations of parafield operators. From this point of view the above conditions should not be applied to a parafield without further justification.

As discussed in the Introduction the cluster property requires that two widely separated clusters of identical particles should be non-interacting. Although, as Wichmann and Chrichton[41] have pointed out any large space-time separation is permissible; in general only large space-like separations will be considered in this thesis. The cluster property is expressed mathematically by the factorization of expectation values representing the results of measurements. This is a different set of mathematical restrictions imposed on the theory than those implied by locality and the equivalence of the two sets of restrictions may depend heavily on the mathematical structure of the theory.

Conceptually, locality is simply a kinematical requirement that a certain spatial separation is too great for interactions, propagated at less than the speed of light, to be transversed within a certain time interval. However it contains no information on how the strengths of these interactions may depend on the separation of the interacting bodies. It is exactly this information which is supplied by the cluster property which assumes that these interactions become negligible as the separation is increased.

The difference between locality and cluster decomposition can be illustrated by means of a "gendanken" or "thought" experiment. Two experiments are performed on two groups of similar particles, one group of which is on the Earth and the other which is on the Moon. If, in a chosen reference frame, both experiments are performed at the same time then both the cluster property and locality would imply that the two results are independent. However, if the experiment on the Moon was performed at a sufficiently later time such that the interactions propagated from Earth could reach the Moon then locality would no longer necessarily require the independence of the two experiments. On the other hand, the cluster property would still require the independence of the two experiments. This example shows the non-equivalence of the two restrictions.

In both cases the non-relativistic limits can easily be obtained. As discussed by Ohnuki and Kamefuchi the locality conditions may be modified by taking equal-time commutators and allowing the spatial separations to approach infinity. The cluster decomposition property is similarly modified by considering the results of measurements performed at the same time but with large spatial separation. The above discussions can be easily modified for these non-relativistic cases.

Having discussed the conceptual differences between the cluster property and locality the remainder of this chapter will be concerned with demonstrating their mathematical inequivalence for parafields, provided, as discussed at the beginning of this section, eqs. 2.1 are a reasonable expression of locality for parafields.

§2.4 FORMULATION OF CLUSTER DECOMPOSITION PROPERTY FOR PARAFIELDS

Since the cluster property is formulated in terms of systems that are spatially separated it is appropriate to use the configuration representation. Previous applications of the cluster property, while acknowledging this, have solely used the quantum mechanical framework. However, there is no reason why the cluster property cannot be applied to the field description. There are some differences between the quantum mechanical approach and that of the associated field theory which are relevant. One is that an operator in field theory, since it is expressed as a function of the ϕ's, takes the same form whether the system is described in terms of redundant particles or not. This has an advantage over the corresponding quantum mechanical description where, for example, an operator which is a symmetric function of its arguments in C_1 must also be symmetrized over the arguments of C_2 if the system C_1 is to be described in terms of the redundant cluster C_2. A point to be emphasized is that in the field theory, since the number of particles is not necessarily conserved, expectation values can be evaluated between superpositions of states with different numbers of particles.

Any state of a parafield may be represented by $|\alpha\rangle$ where

$$|\alpha\rangle = \sum_{n=0}^{\infty} \sum_{(x)_n} \sum_{\ell=1}^{h} f_{s_i}^{(\ell)} ((x)_n) \; |(x)_n \ell, s_i\rangle$$

and this may describe a state localized in C_1 provided the wave-functions $f_{s_i}^{(\ell)}((x_n))$ vanish outside C_1. Defining a physical observable† $F(C_1)$ by

$$F(C_1) = \sum_{(x_j)} f_{C_1}(x_1, x_2, \ldots, x_j)\, \psi(x_1)\, \psi(x_2) \ldots \psi(x_j)$$

where $\psi(x_i)$ stands for either $\phi(x_i)$ or $\phi^*(x_i)$ and f_{C_1} also has a vanishing support outside C_1. The result of a measurement performed on system C_1 is given by

$$<\alpha|\; F(C_1)\; |\alpha> .$$

The same measurement can be performed in the presence of a redundant cluster C_2. For the purpose of this thesis it will suffice to consider one additional particle, described by $\phi(x_R)$, which has a non-overlapping domain with $\phi(x_i)$, $i = 1,2, \ldots, \nu$. As already noted the form of the observable $F(C_1)$ is the same in combined system so that the only modification is in the description of the state $|\alpha>$. Only the modifications to the basis states $|(x)_n\ell, s_i>$ are considered; the general result follows directly.

As noted in §2.2 the state $|(x)_n\ell, s_i>$ is a basis state of an irreducible representation of $U(\nu)$ and so can be uniquely labelled by the invariants of $U(\nu)$, $U(\nu-1)$, ..., $U(1)$. Any state

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†In the limit of continuous x_i's the summation is replaced by an integral.

in n+1 particle system $C_1 \cup C_2$ which has the same labels for these invariants will then represent the same state of C_1. This is because the set of all states specified by the $U(\nu) \supset U(\nu-1) \supset \ldots \supset U(1)$ chain is complete. If C_1 is described by the state $|(x)_n \ell, s_i>$, then in terms of the combined system $C_1 \cup C_2$ a possible description of this state is the vector

$$|\alpha'> = \sum_{\ell'} c_{\ell'} \, |(x)_n (x_R) \ell' \;\; s_{i'}>$$

where

$$\sum_{\ell'} c^*_{\ell'} \, c_{\ell'} = 1 \; .$$

The summation is restricted to those ℓ' which on "removal of the last bose"(48) reduce to ℓ, and $s_{i'}$ is the basis state of the representation ℓ' which is appropriate to s_i. This ensures that the state $|\alpha'>$ has the same unitary labels as the state $|(x)_n \ell s_i>$. To describe a general superposition of states each basis state in the expansion is treated as above with one extra restriction. Rule: "Suppose $|\alpha>$ is a general superposition of basis states at least two of which, denoted as $|\ell s_i>$ and $|\ell s_j>$, belong to the same irreducible representation of $U(\nu)$. In the description of this state in the combined system $C_1 \cup C_2$, if $|\ell s_i>$ is described by

$$\sum_{\ell'} c_{\ell'} \, |\ell' \; s_{i'}>$$

and

$$|\ell s_j>$$

by

$$\sum_{\ell'} b_{\ell'} \, |\ell' s_j>$$

then

$$c_{\ell'} = b_{\ell'} \, .$$

The justification for this restriction is given in Appendix 1.

Denote by $|\alpha>$ any state of the n-particle system C_1 and by $|\alpha'>$ the same physical state described in terms of the combined n+1 particle system $C_1 \cup C_2$. The cluster property then requires that the results of measurements (i.e., the expectation values of physical observables) in C_1 should be independent of C_2. This requires

$$<\alpha| \; F(C_1) \; |\alpha> = <\alpha'| \; F(C_1) \; |\alpha'> \; .$$

§2.5 RESTRICTIONS ON THE PHYSICAL OBSERVABLES OF A NON-RELATIVISTIC PARAFIELD

For a non-relativistic parafield physical observables (e.g. the energy-momentum of the field or terms in the interaction Hamiltonian) may be represented as integrals of functions of the field operators $\phi(x_i)$ and $\phi^*(x_i)$. For parafermi fields these operator functions may be classified according to whether they are elements of the $O(2\nu+1)$, $O(2\nu)$ or $U(\nu)$ enveloping algebras. The appropriate algebras for parabose statistics are $Sp(2\nu)$ and $U(\nu)$. The states of a parafield form the basis for representations of these groups. The introduction of a redundant particle introduces an extra degree of freedom and so the cluster decomposition is essentially the reduction of these representations to those of the appropriate subgroup. The parafermi representations will be considered in detail and, since the argument for the parabose case is completely analogous, only the results in the latter case are indicated.

States localized in C_1 form representations of $O(2\nu+1)$, $O(2\nu)$ and $U(\nu)$ while for the combined system $C_1 \cup C_2$ the representations are $O(2\nu+3)$, $O(2\nu+2)$ and $U(\nu+1)$. The cluster property is a consideration of the reduction of $O(2\nu+3)$, $O(2\nu+2)$ and $U(\nu)$ to $O(2\nu+1)$, $O(2\nu)$ and $U(\nu)$ respectively. Each reduction is considered separately in sections (a), (b) and (c). The results for the parabose case are considered in (d). Finally, in (e), some obvious selection rules are discussed and in (f) an important consequence of the restrictions imposed by the cluster property is discussed.

(a) Unitary Group $U(\nu)$

As the elements of the unitary group leave ℓ unchanged the results of measurements are represented by

$$<(x')_n \ell s_i| \; F(U(\nu)) \; |(x)_n \ell s_j>$$

where $F(U(\nu))$ denotes an element of $U(\nu)$ as described in §2.4. In the presence of a redundant particle localized on C_2 the state $|(x)_n \ell s_j\rangle$ of C_1 may be described by

$$\sum_{\ell'} c_{\ell'} \, |(x)_n \, (x_R) \, \ell' \, s_{j'}\rangle$$

where the summation is as in §2.4 and

$$\sum_{\ell'} c^*_{\ell'} \; c_{\ell'} = 1$$

is the only restriction on the coefficients $c_{\ell'}$. By construction $x_R \neq x_1, x_2, \ldots, x_\nu$ so that the above state belongs to a representation of $U(\nu+1)$. By a standard reduction of $U(\nu+1)$ to $U(\nu)$ it follows that

$$\langle (x')_n (x_R) \ell' s_{i'} | \; F(U(\nu)) \; |(x)_n (x_R) \ell' s_{j'}\rangle$$

$$= \langle (x')_n \ell s_i | \; F(U(\nu)) \; |(x)_n \ell s_j\rangle$$

provided[†] $\ell'^* = \ell$ and $s_{i'}$ and $s_{j'}$ are chosen appropriately. This ensures that the cluster decomposition property is satisfied. For any combination of basis states, provided the restriction of §2.4 (discussed in Appendix 1) is observed the cluster property follows directly from above.

The generalization to a cluster C_2 of n' identical particles follows directly from the reduction of tensor representations of rank $n + n'$ of $U(\nu+\nu')$ to representations $U(\nu) \times U(\nu')$ of rank n and n' respectively. The crucial

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† ℓ^* denotes the Young tableau obtained from ℓ by "removing the last bose"[(48)].

point in these considerations is that there exist no para-field representations of $U(\nu+1)$ (or $U(\nu+\nu')$ in general) which, upon reduction, contain representations of $U(\nu)$ (or $U(\nu) \times U(\nu')$ in general) which are not themselves para-field representations. The generalization to two or more redundant clusters follows by induction.

(b) Orthogonal Group $O(2\nu)$

The cluster property in this case requires the reduction of $O(2\nu+2)$ to $O(2\nu)$.

Consider a state of the combined system $C_1 \cup C_2$ which belongs to the representation $\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}, \frac{p}{2} - 1\right)$ of $O(2\nu+2)$; this corresponds to a superposition of states of $U(\nu+1)$ with either zero or p odd columns. This reduces to the representation $\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2} - 1, \frac{p}{2} - 1\right)$ of $O(2\nu)$ by the following chain

$$\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}, \frac{p}{2}, \frac{p}{2} - 1\right) \supset \left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}, \frac{p}{2} - 1\right)$$

$$\supset \left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2} - 1, \frac{p}{2} - 1\right) .$$

This "non-Fock" representation of $O(2\nu)$ provides a representation (in general reducible) of $U(\nu)$. Consider a state C_1 belonging to a representation of $U(\nu)$ which is contained in both "Fock" and "non-Fock" representations of $O(2\nu)$. In terms of the combined system this state may be represented by a state of $U(\nu+1)$ which belongs to the representation

$\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}, \frac{p}{2} - 1\right)$ of $O(2\nu+2)$ and which reduces to the representation $\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2} - 1, \frac{p}{2} - 1\right)$ of $O(2\nu)$.

Now if the elements of $O(2\nu)$ are interpreted as observables it would be possible for an observer on C_1 to determine that, in the presence of C_2, the state belongs to the representation $\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2} - 1, \frac{p}{2} - 1\right)$ of $O(2\nu)$. For example, this could be achieved by evaluating the Casimir invariants of $O(2\nu)$. However, representations of this type do not occur if the system is described in terms of cluster C_1 alone. Thus an observer in C_1, determining by means of operators localized in C_1 that the state belongs to the representation $\left(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2} - 1, \frac{p}{2} - 1\right)$ of $O(2\nu)$, would be able to predict the existence of C_2. Also the relationship between the invariants of $O(2\nu)$ and $U(\nu)$ is lost for "non-Fock" representations, so that matrix elements of $O(2\nu)$ operators are not independent of the existence of C_2. Some explicit examples are given in Appendix 2.

If observables are restricted to functions of the unitary group then no contradiction between the two modes of description arises since it is never possible to determine to which representation of $O(2\nu)$ a state belongs.

That the above arguments hold only for $p > 1$ can be seen from the reduction of the $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$ representation of $O(2\nu+2)$ afforded by $p = 1$. The above difficulties do not arise since the only representation of $O(2\nu)$ obtained is labelled $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. No "non-Fock" representations occur and it is quite consistent to interpret the elements of $O(2\nu)$ as physical observables. This is a fundamental difference between Fermi and higher order parafermi statistics.

(c) Orthogonal Group $O(2\nu+1)$

Since $O(2\nu+1)$ contains of $O(2\nu)$ as a subgroup it is obvious from the previous section that elements of $O(2\nu+1)$ are not consistent with the cluster assumption. This is also apparent from the fact that $O(2(\nu+\nu')+1)$ does not include $O(2\nu+1) \times O(2\nu'+1)$ as a subgroup, so that even the non-relativistic limit of locality is not satisfied. This holds for all p and so reproduces the well-known result that it is impossible to create or annihilate an odd number of fermions.

(d) Non-Relativistic Parabose Fields

The arguments for the parabose fields are completely analogous to the parafermi case except that the relevant algebras are $Sp(2\nu)$ and $U(\nu)$. The only operators that are consistent with the cluster decomposition property are the elements of the unitary group generated by the N_{ij}'s where

$$N_{ij} = \tfrac{1}{2}[\phi^*(x_i), \phi(x_j)]_+ .$$

The examples of Appendix 2 can easily be modified by replacing commutators by anti-commutators and considering symmetric rather than anti-symmetric states. Once again these arguments hold for $p > 1$ only. By ordering all the particles in C_1 to the right and all in C_2 to the left, the cluster property can easily be shown to hold for any operator in Bose statistics.

(e) Selection Rules

Selection rules for parafields have been derived by a number of other authors[39] based on the locality requirements. The restriction of observables to functions of the unitary group obviously imposes more severe selection rules. Since elements of the unitary group are number conserving operators it follows that in any reaction the number of paraparticles on both sides of the reaction is conserved i.e., paraparticles are neither created or destroyed. This selection rule however, applies only in the case of a single parafield. To discuss the more general case of interacting fields the relative commutation relations between different fields must be considered. This is not discussed in this thesis.

(f) The Equivalence of a Parafermi Theory to p Fermion Fields

It is well known that a parafermi field of order p can be written as a sum of p commuting fermion fields, i.e..

$$\phi(x) = \sum_{\alpha=1}^{p} \phi^{(\alpha)}(x) \ .$$

These Green component fields can be related to a set of p anti-commuting fermion fields by means of Klein transformations. This indicates that a parafermi theory of order p may be equivalent to a system p fermions. Drühl, Haag and Roberts[20] (and Ohnuki and Kamefuchi[21]) have shown, since the Klein transformation is in general non-local, that the equivalence will only hold if certain restrictions are placed on the theory. The effect of these restrictions is to limit the choice of forms of operators representing physical observables.

The gauge transformations on the Green component fields that leave the parafermi commutation relations invariant form representations of U(p), O(p) and SO(p). Drühl, Haag and Roberts have shown that if the algebra of observables is invariant under U(p) transformations then the parafermi field is just an alternative description of a system of p fermions. However Bracken and Green[49] have shown that the group structure of the Green component fields may be characterized by the diagram

$$\begin{array}{ccccc} O(2\nu) & \supset & SO(2\nu) & \supset & U(\nu) \\ & & \otimes & & \\ U(p) & \supset & O(p) & \supset & SO(p) \end{array}$$

e.g., the $U(\nu)$ operators $\frac{1}{2}[a_r^*, a_s]_-$ are invariant under the transformations $a_r^{(i)} \to a_r^{(i)\prime} = \sum_{k=1}^{p} \alpha_{ik}\, a_r^{(k)}$, where α_{ik} is a unitary matrix. Thus the algebra corresponding to $U(p)$ is $U(\nu)$. Now, from the previous sections, the cluster property restricts the observables to elements of $U(\nu)$. But Drühl, Haag and Roberts have shown that this restriction is equivalent to describing a system of p fermions.

In other words, the cluster property ensures that a parafermi field is equivalent to a system of p fermions with certain conditions imposed. Exactly what these conditions are and what their physical interpretation would be is an interesting problem, which, in particular for $p = 3$, may have important consequences.

§2.6 THE QUANTUM MECHANICAL SPACE ASSOCIATED WITH A PARAFIELD

The restriction of physical observables in parafield theory to elements of the unitary group also imples a restriction on observables in the corresponding quantum mechanical space. In view of the recent concern of authors with the quantum mechanical space of paraparticles the results obtained in the previous sections are discussed within the quantum mechanical framework.

The quantum mechanical wave functions associated with a parafield are defined by[12]:

$$\phi^{(\ell)}((x)_n s_i, (k)_n s_j) \equiv \langle (x)_n \ell s_i \,|(k)_n \ell s_j\rangle \ .$$

This defines ϕ as a function of n variables $(x_1, x_2, \ldots, x_n)$ and as such provides a representation of permutations of these variables. Since permutations of the variables $(x_1, x_2, \ldots, x_n)$ only amount to a relabelling of the variables they do not change a physical state and hence cannot be interpreted as physical observables.

It is known that operators corresponding to functions of the N_{ij} are, in the associated quantum mechanical space[10], symmetric functions of their arguments. In standard discussions of permutation symmetry it is usually stated as an assumption that observables are symmetric functions of their arguments. This is unnecessary for parafields since the restriction of physical observables to functions of the N_{ij}, and hence to symmetric functions in the quantum mechanical space, follows directly from the cluster assumption. Since these observables commute with all relabelling operators they are unable to distinguish between states within an irreducible representation of the relabelling operators and hence the concept of a "generalized ray"[50] naturally arises. Apart from some modifications, such as the replacement of irreducible representations of S_n by reducible ones, the quantum mechanical space associated with a parafield is similar to that proposed by Greenberg and Messiah. The conjecture of Greenberg[51] that "each vector in $H_{F.T.}$ corresponds to an <u>entire</u> irreducible representation, belonging to the same Young tableau, in $H_{Q.M.}$" has been verified. His second conjecture that "the redundancy associated with the generalized rays that represent states of particles which are not Bose or Fermi in $H_{Q.M.}$ is removed in $H_{F.T.}$ and at the same time the unobservable permutation operators are eliminated" is also true since the $\Pi_x(\rho)$'s are not defined when acting on states in the field theory. A one-to-one correspondence between the quantum mechanical space and the parafield may be preserved by choosing a particular basis state in an irreducible representation of the relabelling operators and eliminating the rest. This has no physical effect since the concept of labelling identical particles is artificial and not of any physical significance. This so-called "elimination of the generalized ray" has been discussed in detail by Hartle and Taylor[13] and also by Stolt and Taylor[11].

Within the field representation the elements of the unitary group form a generalization of the particle permutation operators introduced by Landshoff and Stapp. In fact the restriction of observables to elements of the unitary group is a generalization of the suggestion by Landshoff and Stapp that physical observables are functions of particle permutations.

Ohnuki and Kamefuchi[12] have argued that particle permutations of the momentum labels, i.e. the $\Pi_k(\rho)$'s may only be interpreted as physical observables for a free (non-interacting) field since the $\Pi_k(\rho)$'s are not defined when acting on arbitrary n-particle states. However operators which are similar to particle permutations when acting on states with all labels different can be expressed in terms of the N_{ij} where

$$N_{ij} = \tfrac{1}{2}[a_i^*, a_j]_\pm \mp \frac{p}{2} .$$

The two such operators are

$$U_{ij} = N_{ij} N_{ji} - N_{ii}$$

and

$$\tilde{U}_{ij} = e^{i\frac{\pi}{2}(N_{ij}+N_{ji})}$$

Since both U_{ij} and $\tilde{U}_{ij}$ are defined as functions of generators of the unitary group they can be applied to any state within the Fock space and hence their interpretation as physical observables holds for both free and non-interacting fields. In particular the $\tilde{U}_{ij}$ form a representation of the permutation subgroup of $U(\nu)$ which corresponds to the Weyl reflections of the weight diagram of the appropriate representation of $U(\nu)$.

§2.7 RELATIVISTIC PARAFIELDS

The relativistic discussion of the cluster decomposition property is complicated by the introduction of an extra ν degrees of freedom for the description of anti-particles. For the parafermi fields the cluster property will be discussed in two representations; (a) the "negative energy picture" and (b) the "positive energy picture". The results obtained are different for the two representations. For parabose fields only the positive energy representation is applicable. In the following discussions the coordinate and, in general, the momentum variables will take a continuous range of values.

(a) Negative Energy States

The expansion of the parafermi wave function in the Heisenberg representation is

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3\underset{\sim}{p} \left(\frac{m}{E_{\underset{\sim}{p}}}\right)^{\frac{1}{2}} \left\{ \sum_{r=1}^{2} a_r(\underset{\sim}{p}) w^r(\underset{\sim}{p})\, e^{-ip\cdot x} + \sum_{r=3}^{4} a_r(-\underset{\sim}{p}) w^r(\underset{\sim}{p})\, e^{+ip\cdot x} \right\} \tag{2.2}$$

where $E_{\underset{\sim}{p}} = \sqrt{\underset{\sim}{p}^2 + m^2}$, $w^r(\underset{\sim}{p})$ are Dirac spinors, the $a_r(\underset{\sim}{p})$ satisfy parafermi commutation relations and $p\cdot x = p_o t - \underset{\sim}{p}\cdot\underset{\sim}{x}$. The $a_r^*(\underset{\sim}{p})$ are interpreted as creation operators for particles of positive (negative) energy for r = 1 or 2 (3 or 4). The "no-particle" state is defined by

$$a_r(\underset{\sim}{p})\, |\chi\rangle = 0 \qquad \forall_r, \underset{\sim}{p} \,.$$

Applying powers of $a_r^*(\underset{\sim}{p})$ to $|\chi\rangle$ defines a representation which will be termed the negative energy picture. It should not be confused with the hole theory of Dirac where a representation is defined by applying either creation or annihilation operators to a state that is completely filled with negative energy particles. Ignoring spin indices and restricting the momenta to a finite number; ν, of degrees of freedom the expansion (2.2) can be written

$$\psi(x) = \sum_{r=1}^{\nu} a_r \, u_r(x) + a_{r+\nu} \, u_{r+\nu}(x) = \sum_{\rho=1}^{2\nu} a_\rho \, u_\rho(x) \qquad (2.2')$$

By analogy with the non-relativistic case the cluster property restricts physical observables to functions of the form $[\bar{\psi}(x), \psi(y)]_-$. This could easily be checked by substituting the appropriate relativistics field operators in the examples of Appendix 2 and using the fact that $\Delta^+(x - y) \to 0$ as $x - y \to \infty$ in a space-like direction. However, since it is more convenient to use the momentum representation in this and future discussions, the significance of the momentum representation will be discussed here.

Suppose x and y have large space-like separation and consider an expectation value of the form

$$\langle\chi| \, \ldots \, \psi(x) \, \ldots \, \bar{\psi}(y) \, \ldots \, |\chi\rangle \; .$$

From expansion (2.2′) this becomes

$$\sum_{\rho=1}^{2\nu} \cdots \sum_{\sigma=1}^{2\nu} \langle\chi| \ldots a_\rho \ldots a_\sigma^* \ldots |\chi\rangle$$

$$\times \prod u_\rho(x)\bar{u}_\sigma(y) \ .$$

Using the commutation relations this can, in general, be reduced to expressions of the form

$$\sum_{\rho=1}^{2\nu} \cdots \sum_{\sigma=1}^{2\nu} \prod u_\rho(x)\bar{u}_\sigma(y)\ \delta_{\rho\sigma}\ \langle\chi| \ldots |\chi\rangle \ . \tag{2.3}$$

For example the relevant terms in

$$\langle\chi|\ \psi(x)\bar{\psi}(y)\ |\chi\rangle$$

reduce to

$$\sum_{\rho=1}^{2\nu} \sum_{\sigma=1}^{2\nu} u_\rho(x)\bar{u}_\sigma(y)\ \delta_{\rho\sigma} \ .$$

In the limit of a continuous range of momenta (2.3) can be written

$$\int dp\ e^{-ip\cdot(x-y)}\ f(w(p))\ \langle\chi| \ldots |\chi\rangle$$

where $f((w(p))$ is some function of Dirac spinors. This integral is of the form

$$\int e^{-ip\cdot(x-y)}\ F(p)dp$$

which by the Riemann-Lebesque lemma approaches zero as $|x - y|$ approaches infinity.

This means that in order to reduce the complexity of some of the subsequent algebras, terms containing $\delta_{\rho\sigma}$ or $\delta(p_\rho - p_\sigma)$ may be ignored on the understanding that the Riemann-Lebesque lemma is used where appropriate. It is only possible to use this lemma for those p_j and p_k which are conjugate variables to the x_j's and x_k's which have a large space-like separation.

In the momentum representation, by an argument analogous to the non-relativistic case, physical observables are restricted to elements of the $U(2\nu)$ generated by the $\frac{1}{2}[a^*_\rho, a_\sigma]_-$. The only operators in the configuration representation which reduces to the unitary group elements are functions of

$$[\bar{\psi}(x), \psi(x)]_- \ .$$

(b) Positive Energy States

The expansion of the wave-function in the positive energy representation is

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \left(\frac{m}{E_{\underset{\sim}{p}}}\right)^{\frac{1}{2}} \sum_{r=1}^{2} \{a_r(\underset{\sim}{p}) w^r(\underset{\sim}{p})\, e^{-ip\cdot x}$$

$$+ b^*_r(\underset{\sim}{p}) v^r(\underset{\sim}{p})\, e^{ip\cdot x}\} \tag{2.4}$$

$$= \psi^{(+)}(x) + \psi^{(-)}(x) \ ,$$

where $a^*_r(\underset{\sim}{p})$ and $b^*_r(\underset{\sim}{p})$ are interpreted as creation operators of positive energy particles and anti-particles respectively. A physical vacuum state is defined by:

$$a_r(\underset{\sim}{p})\ |0\rangle = b_r(\underset{\sim}{p})\ |0\rangle = 0 \qquad \forall\ r, \underset{\sim}{p}.$$

Restricting the momentum variables to a finite number; ν, of degrees of freedom and ignoring spin indices this can be written as

$$\psi(x) = \sum_{j=1}^{\nu} \{a_j\ u_j(x) + b^*_j\ u_{j+\nu}(x)\} \ .$$

Similarly to the negative energy representation the cluster property will be formulated in the momentum representation.

The algebra generated by the a_r's and b_r's is $O(4\nu+1)$ and a state of n particles in C_1 is characterized by the $U(2\nu)$ labels where the $U(2\nu)$ algebra is generated by

$$\tfrac{1}{2}[a_i^*, a_j]_-, \ \tfrac{1}{2}[a_i^*, b_j]_-, \ \tfrac{1}{2}[b_i^*, a_j]_-$$

and

$$\tfrac{1}{2}[b_i^*, b_j]_- . \tag{2.5}$$

The description of the system in terms of $C_1 \cup C_2$ introduces an extra degree of freedom because C_2 may contain a particle or anti-particle. The reduction of the representations of $U(2\nu+1)$ to $U(2\nu)$ ensures that the elements of the unitary group are permissible physical observables, while the appearance of "non-Fock" representations of the orthogonal group in the reduction of $O(4\nu+2)$ to $O(4\nu)$ precludes the elements of $O(4\nu)$ as physical observables. All this is quite straightforward and follows with only slight modifications to the argument used in the non-relativistic case.

Within the coordinate representation this restriction implies that the only permissible observables are those which can be constructed from functions of the form (2.5). Decomposing the wave function into positive and negative frequencies the permissible combinations are

$$[\psi^{(+)}(x),\ \bar{\psi}^{(-)}(y)]_-,\ [\psi^{(-)}(x),\ \psi^{(+)}(y)]_-,$$

$$[\psi^{(-)}(x),\ \bar{\psi}^{(+)}(y)]_- \text{ and } [\bar{\psi}^{(+)}(x),\ \bar{\psi}^{(-)}(y)]_- .$$

However these are not the only possibilities since for example the operator

$$E = i \int (\bar{\psi}\gamma_4 \frac{\partial}{\partial t}\psi - \frac{\partial\psi}{\partial t}\gamma_4{}^t\ \bar{\psi})\ d^3\underset{\sim}{x} \tag{2.6}$$

has the momentum representation

$$E = \sum_{\underline{p},r} |\underline{p}_o|\ (N^{(+)}_{\underline{p},r} + N^{(-)}_{\underline{p},r} - 2) \tag{2.7}$$

where

$$N^{(+)}_{\underline{p},r} = \tfrac{1}{2}[\,a^*_r(\underline{p}),\ a_r(\underline{p})]_-$$

and

$$N^{(-)}_{\underline{p},r} = \tfrac{1}{2}[\,b^*_r(\underline{p}),\ b_r(\underline{p})]_- .$$

By the above arguments E is a permissible observable since the integration has removed the offending terms. However this integration implies a non-localizability which is not desirable in field theory since it implies that the whole of the Universe must be considered. Attempting to localize the description by integrating only over a finite volume would invalidate the reduction of (2.6) to (2.7) since the orthogonal terms, i.e., $[a^*_r(\underline{p}),\ a^*_r(\underline{k})]_-$ would not disappear. In this case the operator E would no longer be consistent with the cluster decomposition property. In order to construct operators which are consistent with the cluster decomposition property it is necessary to decompose the wave functions into positive and negative frequencies.

(c) A Comparison of the Positive and Negative Energy Representations

Parafermi commutation relations are invariant under Bogoliubov transformations of the form

$$a_{-k} \to b_k^* = a_{-k} \ .$$

Ignoring the possibility of myriotic representations these transformations are unitary. It would then appear, using the expansion (2.2) that since $[\bar{\psi}(x), \psi(x)]_-$ is compatible with the cluster property in the negative energy representation then it should also satisfy the cluster property in the positive energy picture if the expansion (2.4) is used. This is equivalent to requiring that if the operators $[a_\rho^*, a_\sigma]_-$ $(\rho, \sigma = 1 \ldots 2\nu)$ satisfy the cluster property in the negative energy representation then it follows the operators $[a_i, b_j]_-$, where $a_{r+\nu} = b_r^*$, satisfy the cluster property in the positive energy representation. In fact, as has been shown in the preceeding sections, this is not the case and in the positive energy representation, operators of the form $[a_i, b_j]_-$ do not satisfy the cluster property. The apparent contradiction is resolved by observing that transformations of the form $a_{r+\nu} \to b_r^*$ do not preserve the unitary symmetry of the basis states and hence a statement of the cluster property in one representation does not transform into the corresponding statement of the cluster property in the other representation. An example showing the necessity of considering the cluster property in each space is given in Appendix 3.

The parafield version of Dirac's hole theory, which has the expansion (2.2´) for the field operators and which can be considered intermediate between the positive and negative energy representations, can be easily shown to be equivalent to the positive energy representations as far as the cluster property is concerned. This is because the physical vacuum is defined in the hole theory to be

$$|\phi_o> = (a^*_{\nu+1})^p \; \dots \; (a^*_{2\nu})^p \; |\chi> \; .$$

Once again this indicates the importance of formulating the cluster property in that space since considering the expansion (2.2´) alone would incorrectly imply that $[\bar{\psi}(x), \psi(x)]_-$ is a permissible physical observable.

(d) An Example

In this section an example is given to show that within the particle-antiparticle representation operators of the form $[\bar{\psi}(x_1), \psi(x_2)]_-$ do not satisfy the cluster property.

Ignoring spin indices $\psi(x)$ has the expansion

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int_{k_o>0} \frac{1}{2k_o} \{a_{\underset{\sim}{k}} \, e^{-ik\cdot x} + b^*_{\underset{\sim}{k}} \, e^{ik\cdot x}\} \; .$$

The transition amplitude for an operator of the form $[\bar{\psi}(x_1), \psi(x_2)]_-$ between the vacuum and the anti-symmetric particle-antiparticle state

$$[\bar{\psi}^{(-)}(x_3),\ \psi^{(-)}(x_4)]_-\ |0\rangle$$

is $$p^2\Delta^+(x_1 - x_4)\Delta^+(x_2 - x_3)\ .$$

In the presence of a redundant particle the same process may be described by the following transition amplitude,

$$\langle 0|\ \psi^{(+)}(x_6 + \lambda a)\ [\bar{\psi}(x_1),\ \psi(x_2)]_-\ \{\bar{\psi}^{(-)}(x_3)\bar{\psi}^{(-)}(x_5 + \lambda a)$$

$$\times\ \ \psi^{(-)}(x_4) - \psi^{(-)}(x_4)\bar{\psi}^{(-)}(x_5 + \lambda a)\bar{\psi}^{(-)}(x_3)\}\ |0\rangle$$

where a is an arbitrary space-like vector and $\lambda \to \infty$. The asymptotic behaviour of the $\Delta^+(z)$ function for z space-like is given by

$$\Delta^+(z) \sim \frac{1}{(32\pi^3\ |z|^3)^{\frac{1}{2}}}\ e^{-z}$$

which can be ignored for large z. As $\lambda \to \infty$ the above vacuum expectation value approaches $p^2(p - 2)\Delta^+(x_2 - x_3)\Delta^+(x_1 - x_4)$ $\Delta^+(x_6 - x_5)$ which does not factorize according to

$$\langle 0|\ \psi^{(+)}(x_6)\bar{\psi}^{(-)}\ (x_5)\ |0\rangle\ \langle 0|\ [\bar{\psi}(x_1),\ \psi(x_2)]_-$$

$$\times\ [\bar{\psi}^{(-)}(x_3),\ \psi^{(-)}(x_4)]_-\ |0\rangle$$

as required by the cluster property.

It is easily seen that this is just a generalization of the underlying group theoretical concepts to the configuration representation.

(e) Relativistic Parabose Fields

Since Bose and, in general, parabose commutation relations are not invariant under transformations of the form $a_k \to a_k^*$ the relativistic parabose theory can only be formulated in terms of positive energy particles. Only the charged and neutral scalar fields are considered here; the generalization to vector and tensor fields can readily be effected.

The restrictions of observables to those whose Fourier transforms are functions of the $[a_{\underset{\sim}{k}}^*, a_{\underset{\sim}{\ell}}]_+$'s follows directly from the cluster property. Denote by $\phi(x)$ either

$$\frac{1}{\sqrt{(2(2\pi)^3}}\int_{k_o>0}\frac{d^3k}{k_o}\{a_{\underset{\sim}{k}}\, e^{ik\circ x} + a_{\underset{\sim}{k}}^*\, e^{ik\circ x}\}$$

or

$$\frac{1}{\sqrt{2(2\pi)^3}}\int_{k_o>0}\frac{d^3k}{k_o}\{b_{\underset{\sim}{k}}\, e^{-ik\cdot x} + a_{\underset{\sim}{k}}^*\, e^{ik\cdot x}\} .$$

For the charged and uncharged fields the cluster property implies that the following are permitted as physical observables:

$$[\phi^{(+)*}(x), \phi^{(-)}(y)]_+, [\phi^{(-)*}(x), \phi^{(-)}(y)]_+ ,$$

$$[\phi^{(+)}(x), \phi^{(-)}(y)]_+ \text{ and } [\phi^{(+)*}(x), \phi^{(-)*}(y)]_+$$

which is a straightforward generalization of the parafermi case. Also defining $\pi(x) = \partial_o\, \phi(x)$ then the following operator for the real scalar field is compatible with the cluster property

$$H = \tfrac{1}{2}\int d^3x\, \{m^2\, \phi^2(x) + \underset{\sim}{\nabla}\, \phi(x)\circ\underset{\sim}{\nabla}\, \phi(x) + \pi^2(x)\}$$

$$= \int d\Omega(k)\, \tfrac{1}{2}[a_{\underset{\sim}{k}}^*, a_{\underset{\sim}{k}}]_+\, w_{\underset{\sim}{k}} .$$

However, as for the parafermi case, this implies a non-localizability since the whole Universe must be considered. Once again, in order to construct operators whose domain is a finite volume of configuration space and which are consistent with the cluster property, it is necessary to decompose the wave-functions into positive and negative frequencies.

It is obvious that, as for the non-relativistic case, this implies more severe restrictions on the theory than would follow from locality. In particular it is difficult to construct interaction Hamiltonians $H_I(x)$ such that

$$[H_I(x), H_I(y)]_- = 0 \qquad \text{for } x \sim y \; .$$

However it may be possible that this is not satisfied but macrocausality is, since from §2.3 the vanishing of the above commutator may not be a reasonable expression of microcausality for a parafield. An interesting approach to this has been initiated by Napiórkowkis(52) who has shown that for C^* algebras the independence of observations in space-like regions (V_1 and V_2) does not necessarily imply microcausality, i.e., $[O(V_1), O(V_2)]_- = 0$.

§2.8 S-MATRIX THEORY

As an example of the cluster properties of theories other than field theory, the S-matrix description is considered. Even for Bose or Fermi statistics the cluster properties of S-matrix theory has received considerable attention. In particular

Wichmann and Crichton[41] have shown how to parametrize the S-matrix by means of "cluster amplitudes". Taylor[53] has modified their argument to consider phase factors, which may be relevant for different superselection sectors. For local Bose or Fermi fields the cluster amplitudes contributing to scattering processes are essentially the Feynman diagrams of the corresponding perturbation theory. This essentially verifies the cluster decomposition property for local fields. For parafields the S-matrix can be shown to be consistent with cluster properties provided its elements are restricted to functions of the unitary type operators $[a^*(\underset{\sim}{k}),\ a(\underset{\sim}{\ell})]_+$. The argument is a straightforward generalization of that given by Wichmann and Crichton. Since there are additional super-selection rules in a parafield theory the consideration of phase factors is comlicated and not discussed. This does not effect any conclusions since no attempt has been made to parametrize the S-matrix by cluster amplitudes or to correlate this parametrization for "para-local fields" with the appropriate Feynman diagrams.

Denote by W_n the set of all n-particle momentum space wave functions which are infinitely differentiable[24] and of "fast decrease". For every ϕ in W_n an operator $A^*(\phi)$ is defined by

$$A^*(\phi) = N_n \int_{(\infty)} d^3(\underset{\sim}{p}_1)\ldots d^3(\underset{\sim}{p}_n)\ \phi(\underset{\sim}{p}_1,\ldots,\underset{\sim}{p}_n)\ a^*(\underset{\sim}{p}_1)\ldots a^*(\underset{\sim}{p}_n)$$

where N_n is an appropriate normalization constant. The vacuum state is defined by

$$a(p_i)\ |0> = 0$$

and

$$a(p_i)\ a^*(p_j)\ |0> = p\delta(p_i - p_j)\ |0> .$$

An Hilbert space, H, can be constructed in the usual sense by applying $A^*(\phi)$'s to the vacuum. For parabose statistics it is more convenient to use symmetrized versions of the $A^*(\phi)$'s defined as

$$A^*(\phi)^{\ell}_{s_i} = N'_n \int_{(\infty)} d^3(p_1) \ \ldots \ d^3(p_n)$$

$$\times\ \phi^{\ell}_{s_i}(p_1, \ldots, p_n)\ a^*(p_1) \ \ldots \ a^*(p_n)$$

$$= N'_n \int_{(\infty)} d^3(p_1) \ \ldots \ d^3(p_n)$$

$$\times\ \phi(p_1, \ldots, p_n)\ \{a^*(p_1) \ \ldots \ a^*(p_n)\}^{\ell}_{s_i}$$

where the ℓ and s_i are S_n labels as discussed previously.

The S-matrix, S, is a unitary mapping of H onto itself and the plane-wave S-matrix elements are tempered distributions defined by

$$S^{\ell_m\ \ell_n}_{s_m\ s_n}\ (q_1, \ldots, q_m; p_1, \ldots, p_n)$$

$$= <0|\ \{a(q_1)\ \ldots\ a(q_m)\}^{\ell_m}_{s_m}\ S\{a^*(p_1)\ \ldots\ a^*(p_n)\}^{\ell_n}_{s_n}\ |0>$$

where ℓ_m, s_m and ℓ_n, s_n are symmetry labels referring to S_m and S_n respectively. The translation operator $U(I,z)$ is such that

$$U(I,z)\, A^*(\phi)\, U^{-1}(I,z) = A^*(\phi')$$

where for n = 1

$$\phi'(\underset{\sim}{p}_i) = \phi(\underset{\sim}{p}_i) \exp (iz\cdot p_i) .$$

It follows that

$$|\{(\phi';0)^{\ell_n}_{s_n} (\phi'';z)^{\ell_{n'}}_{s_{n'}}\}^{\ell_{n+n'}}_{s_{n+n'}}>$$

$$= \int_{(\infty)} d^3\underset{\sim}{p}_1 \ldots d^3\underset{\sim}{p}_{n+n'}$$

$$\times\, \phi^{\ell_{n+n'}}_{s_{n+n'}} ((\underset{\sim}{p}_1,\ldots,\underset{\sim}{p}_n)^{\ell_n}_{s_n} ; (\underset{\sim}{p}_{n+1},\ldots,\underset{\sim}{p}_{n+n'})^{\ell_{n'}}_{s_{n'}})$$

$$\exp (i \sum_{k=n}^{n+n'} z\cdot p_k)\, a^*(\underset{\sim}{p}_1) \ldots a^*(\underset{\sim}{p}_n)\, |0> .$$

The label $\ell_{n+n'}$ must be chosen such that it specifies an irreducible representation of $S_{n+n'}$ which reduces to the appropriate representations ℓ_n and $\ell_{n'}$ of S_n and $S_{n'}$ respectively. For example the function

$$\phi^{\ell_3}_{s_1}((\underset{\sim}{p}_1,\underset{\sim}{p}_2)^{\ell_2}_{s_1}, (\underset{\sim}{p}_3)^{\ell_1}_{s_1})$$

$$\equiv \phi(\underset{\sim}{p}_1,\underset{\sim}{p}_2,\underset{\sim}{p}_3) - \phi(\underset{\sim}{p}_2,\underset{\sim}{p}_3,\underset{\sim}{p}_1) + \phi(\underset{\sim}{p}_3,\underset{\sim}{p}_1,\underset{\sim}{p}_2)$$

$$- \phi(\underset{\sim}{p}_3,\underset{\sim}{p}_2,\underset{\sim}{p}_1)$$

has $\ell_3 = (2,1)$, $\ell_2 = (1,1)$ and $\ell_1 = (1)$.

With this notation the cluster requirements, as stated by Wichmann and Crichton are generalized to

$$\lim_{z\to\infty} \langle\{(\psi';0)^{\ell_n}_{s_n}\;(\psi'';z)^{\ell_{n'}}_{s_{n'}}\}^{\ell_{n+n'}}_{s_{n+n'}}\Big|S$$

$$\times\;\Big|\{(\phi';0)^{\ell_m}_{s_m}\;(\phi'';z)^{\ell_{m'}}_{s_{m'}}\}^{\ell_{m+m'}}_{s_{m+m'}}\rangle$$

$$= \langle(\psi';0)^{\ell_n}_{s_n}|S|(\phi';0)^{\ell_m}_{s_m}\rangle\;\langle(\psi'';0)^{\ell_{n'}}_{s_{n'}}|S|(\phi'';0)^{\ell_{m'}}_{s_{m'}}\rangle$$

and

$$\lim_{z\to\infty} \langle(\psi';0)^{\ell_n}_{s_n}|S|\{(\phi';0)^{\ell_m}_{s_m}(\phi'';z)^{\ell_{m'}}_{s_{m'}}\}^{\ell_{m+m'}}_{s_{m+m'}}\rangle = 0\;.$$

Only the first of these two restrictions will be considered. The application of the cluster decomposition property is quite simple and it follows the previous sections that only S-matrices whose elements in momentum space are functions of the unitary-type operators are permissible. In particular the second example, when modified for parabose operators, in Appendix 2 quite clearly shows that the elements of the functions associated with the orthogonal group violate the above factorization property.

It is straightforward to construct S-matrices from operators of the form $[a^*(\underset{\sim}{k}),\ a(\underset{\sim}{\ell})]_+$ which are consistent with the other axioms of S-matrix theory. The ease with which this can be done as compared with field theory is because S-matrix theory does not require the local behaviour of field theory. Due to the complicated nature of the commutation relations functional differentiation of parafields can only be defined for certain operators(47). The parametrization of the S-matrix by cluster amplitudes in a fashion similar to Bose statistics is not a trivial problem because of such difficulties and so is not considered in this thesis.

§2.9 WIGHTMAN FORMULATION

The Wightman formulation of parafield theory has been discussed by Dell' Antonio, Greenberg and Sudarshan(26) and also by Govorkov(27). These authors have shown that the usual conditions on vacuum expectation values (abbreviated to V.E.V.) such as weak local commutavity, T.C.P. and the spin-statistics theorem are satisfied by parafields. For parafermi fields Govorkov has obtained restrictions on the V.E.V.'s based on locality requirements. It would not be surprizing in view of the results obtained in previous sections if the cluster decomposition property did not impose more severe restrictions on the theory.

However Dell' Antonio, Greenberg and Sudarshan maintain that the cluster decomposition property holds for the V.E.V.'s of parafields. They maintain that this is so because proofs of the cluster property (i.e., Jost(25)) do not depend on local commutativity which is expressed as*

$$[A(x), A(y)]_- = 0 \qquad \text{for} \quad x \sim y \ .$$

Since these proofs involve a consideration of the matrix elements of the translation operator it would appear that the cluster

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*A(x) is used to denote a neutral scalar field.

decomposition property should hold for parafields. However for Bose fields it is possible to order the fields such that any V.E.V. of the form

$$<0|\ A(x_1)\ \dots\ A(x_i + \lambda a)\ \dots\ A(x_j)\ \dots\ A(x_n + \lambda a)\ |0>\ ,$$

as $\lambda \to \infty$ and for a an arbitrary space-like vector(*) can be ordered as

$$<0|\ A(x_1)\ \dots\ A(x_j)\ A(x_i + \lambda a)\ \dots\ A(x_n + \lambda a)\ |0> \qquad (2.8)$$

The cluster problem then reduces to evaluation of the matrix elements of the translation operator. For parafields the commutation relations are

$$[[A(x),\ A(y)]_+,\ A(z)]_- = 0$$
$$\text{for}\quad x \sim z \quad \text{and} \quad y \sim z\ ,$$

and in general these relationships are not sufficient to ensure that V.E.V.'s can always be ordered as in (2.8). Thus the standard arguments based solely on the matrix elements of the translation operator are not sufficient to prove the cluster decomposition properties of parafields.

In fact, as the examples in the previous sections and the Appendices show, the cluster decomposition property does not hold for all V.E.V.'s of parafield operators. This is because free field theories are an example of the Wightman axiomatic formulation.

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*Throughout this section a will denote an arbitrary space-like vector.

For interacting fields the Wightman formulation is conveniently investigated using the Green ansatz. This is defined by

$$A(x) = \sum_{\alpha=1}^{p} A^{\alpha}(x)$$

where

$$[A^{\alpha}(x), A^{\alpha}(y)]_{-} = 0 \qquad \text{for } x \sim y$$

and

$$[A^{\alpha}(x), A^{\beta}(y)]_{+} = 0 \qquad \text{if } \alpha \neq \beta \ .$$

Using the properties of the V.E.V.'s of these component fields it is shown in Appendix 4 that for $p = 2$

$$\lim_{\lambda\to\infty} \langle A(x_1)\, A(x_2 + \lambda a)\, A(x_3)\, A(x_4 + \lambda a)\rangle$$

$$\neq \langle A(x_1)\, A(x_3)\rangle \langle A(x_2)\, A(x_4)\rangle \ .$$

By the same method

$$\lim_{\lambda\to\infty} \langle A(x_1)\, A(x_2)\, A(x_3 + \lambda a)\, A(x_4 + \lambda a)\rangle$$

$$= \langle A(x_1)\, A(x_2)\rangle \langle A(x_3)\, A(x_4)\rangle \ .$$

Thus for $p = 2$, and in general any p, the cluster decomposition property is not necessarily satisfied. A non-trivial set of parafield operators whose V.E.V.'s do satisfy the cluster property is given by the set

$$A(x_i, x_{i+1}) = [A(x_i), A(x_{i+1})]_{+} \ .$$

For $x_i \sim x_j$, $x_i \sim x_{j+1}$, $x_{i+1} \sim x_j$ and $x_{i+1} \sim x_{j+1}$ it follows from the commutation relations for the $A(x)$'s that

$$[A(x_i, x_{i+1}), A(x_j, x_{j+1})]_{-} = 0 \ .$$

The proof of the cluster decomposition of the V.E.V.'s of the $A(x,y)$'s is analogous to Bose statistics. Alternatively, it could be proved using the Green ansatz.

In terms of symmetrized V.E.V.'s the cluster property may be formulated in a fashion similar to §2.8. The cluster property in this case would require that

$$\lim_{\lambda\to\infty} <\{(A(x_1) \ldots A(x_n))_{s_n}^{\ell_n} (A(y_1) + \lambda a) \ldots$$

$$A(y_{n'} + \lambda a))_{s_{n'}}^{\ell_{n'}}\}_{s_{n+n'}}^{\ell_{n+n'}}>$$

$$= <\{A(x_1) \ldots A(x_n)\}_{s_n}^{\ell_n}> <\{A(y_1) \ldots A(y_{n'})\}_{s_{n'}}^{\ell_{n'}}> .$$

Two examples are given to show that for $p > 2$ this restricted form of the cluster decomposition property does not hold. These examples are considered in Appendix 4.

It is an interesting peculiarity that the symmetrized V.E.V.'s of parabose operators of order two are consistent with the cluster requirements. This is due to a property of the representations of S_n (or in momentum space $U(n)$) afforded by the pb2 operators. These irreducible representations of S_n (or $U(n)$) are exactly those obtained by reducing the direct product of two totally symmetric representations S_N and S_N for $n = 2N$ and S_N and S_{N+1} for $n = 2N+1$. This is easy to see since in the reduction of the direct product of two totally symmetric representations S_N and $S_{N'}$ each representation of $S_{N+N'}$ with less than two rows occurs once only. However this is just the representation space of $S_{N+N'}$ afforded by pb2 operators and it is in this way that pb2 statistics are equivalent to the theory of two Bose fields. The reduction of some low order tensor products is given below.

$$(1) \times (1) \equiv (2,0) + (1,1)$$

$$(2,0) \times (1) \equiv (3,0) + (2,1)$$

$$(2,0) \times (2,0) \equiv (4,0) + (3,1) + (2,2)$$

etc., where (m,n) denotes the Young tableau with m boxes in the first row and n in the second. This property is unique to parabose statistics of order two and as the following example shows cannot be generalized to pb3 (and higher order statistics),

$$(1) \times (1) \times (1) \equiv (3,0,0) + 2(2,1) + (1,1,1) .$$

Since the "hook" diagram appears twice this cannot represent pb3 since the corresponding representation appears once only for any parafield. A type of statistics appropriate to the above reduction has been proposed by Carpenter[(54)]. It is this factorization property which ensures that the symmetrized V.E.V.'s of parabose operators of order two are consistent with the cluster decomposition property.

An important set of V.E.V.'s are the "truncated" V.E.V.'s introduced by Haag[(55)]. They are defined inductively by

$$W^{(n)}(x_1, \ldots, x_n) = W_T^{(n)} (x_1, \ldots, x_n)$$

$$+ \sum W_T^{(k)} (x_1, \ldots, x_k) \ldots W_T^{(n-k)}(\ldots x_n)$$

where the summation extends over all ways of dividing the $x_1, \ldots, x_n$ into more than one group such that the order within any group is the same as the left-hand side. The cluster property then requires that $W_T^{(n)} (x_1, \ldots, x_n) \to 0$ if any set of the arguments $x_1, \ldots, x_n$ have a large space-like separation. In particular, by explicitly constructing $W_T^{(3)}(x_1,x_2,x_3)$, it only holds that

$$\lim_{\lambda\to\infty} W_T^{(3)} (x_1,x_2 + \lambda a, x_3) = 0$$

iff

$$\lim_{\lambda\to\infty} W(x_1,x_2 + \lambda a, x_3) = W(x_1,x_3)\, W(x_2) \ .$$

As has been shown earlier for $p > 1$ this does not hold and so for the truncated V.E.V.'s the cluster decomposition property is not satisfied. The importance of this is indicated in the next section.

§2.10 CONJECTURES ON THE CAUSAL BEHAVIOUR OF PARAFIELDS

In a relativistic theory the commutation relations between field operators are no longer just an expression of permutation symmetry of identical particles. For example, the concept of causality is expressed for Bose fields as

$$[A(x), A(y)]_- = 0 \qquad \text{for } x \sim y \ .$$

It is well-known that this condition, within the L.S.Z. approach, implies the analyticity of the S-matrix[42]. From this analyticity follows dispersion relations and symmetries such as crossing (which is an obvious extension of the concept of interchanging identical particles).

For parabose fields the commutation relations are

$$[[A(x), A(y)]_+, A(z)]_- = 0 \tag{2.8$'$}$$

for $x \sim z$ and $y \sim z$. The previous chapters have been concerned with the consequences of this relation as far as the cluster property is concerned. Although this approach essentially only deals with the identity of particles it was found to place strong restrictions on the theory. To investigate fully the restrictions placed on a theory by (2.8$'$) the causality aspect of the commutation relations should be investigated. As in the Bose case the L.S.Z. approach would seem to be a natural framework to consider this. However the formulation of an L.S.Z. approach to parafields is not at all straightforward, so only some conjectures and relevant results will be discussed.

The L.S.Z. approach for parafields assumes the existence of an interpolating field $A(x)$ such that the asymptotic in and out fields are defined as

$$A_{\substack{out\\in}} = \lim_{x_o \to \pm\infty} A(x) \ .$$

Haag[55] and Ruelle[56] have shown that the asymptotic condition for $x_o \to \pm\infty$ can be interpreted as the vanishing of the interaction between two clusters as their spatial separation approaches infinity. In particular, a requirement for the existence of the asymptotic limits is that the truncated V.E.V. for equal time decrease more strongly than any power of R, where R is the radius of the smallest sphere enclosing all points in the group. However from the previous section the truncated V.E.V.'s do not exhibit this property for all separations of their arguments. It is not intended to go into a detailed discussion of whether the existence of the asymptotic fields can be proved from less stringent conditions for parabose or parafermi fields and for the remainder of this section it will be assumed that the limits do exist.

The in and out fields are related via the S-matrix by the following equation

$$A_{out} = S^* A_{in} S \ .$$

Substituting

$$A_{\beta\ \mathrm{out}} = \lim_{x_o \to \infty} A_\beta(x)$$

where

$$A_\beta(x_o) = i \int_{x_o} d^3x \; f_\beta^*(x) \overleftrightarrow{\frac{\partial}{\partial x_o}} \; A(x)$$

the four body S-matrix element $(A_{\mathrm{out}}^{\alpha\beta},\ A_{\mathrm{in}}^{\alpha'\beta'})$ may be reduced to a two body S-matrix element. It is no longer obvious that, as in the Bose case, the commutation relation for A(x) implies that these elements are analytic. It is not obvious whether a generalized reduction formula can be derived as in the Bose case. One reason for this can be seen from a consideration of the perturbation expansion of the S-matrix for higher order parabose fields. The S-matrix is usually expanded as

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \ldots dx_n \; \tau(x_1, \ldots, x_n)$$
$$\times :A_{\mathrm{in}}(x_1) \ldots A_{\mathrm{in}}(x_n):$$

where $\tau(x_1, \ldots, x_n)$ is a time-ordered V.E.V. and : : denotes normal ordering. For Bose fields the definition of normal ordering is unique and the S-matrix is complete (in an operator sense). However for parabose fields the definition of normal ordering is ambiguous; for example the normal product of the two operators a_k^* and a_ℓ could be defined as either

$$N(a_\ell a_k^*) = \tfrac{1}{2}[a_\ell,\ a_k^*]_+ - \frac{p}{2}\,\delta_{k\ell}$$

or

$$N(a_\ell a_k^*) = a_k^* a_\ell \ .$$

Both definitions have zero V.E.V. and for $p > 1$ are independent. It is not clear what extra properties, if any, a definition of normal ordering should possess which could differentiate between the two definitions. Although normal ordering is only an aid to evaluating the V.E.V.'s of S-matrix terms and so any consistent definitions should be satisfactory, it is quite possible that only certain definitions would permit the S-matrix elements to be analytic. However, irrespective of the choice of definition of normal ordering, the S-matrix is no longer complete. Similar difficulties in defining time-ordered and retarded-ordered products of field operators which are relativistically invariant also exist.

The causality condition for Bose fields is often expressed as

$$[j(x), j(y)]_- = 0 \qquad \text{for } x \sim y$$

where $j(x) = (\Box + m^2)\, A(x)$. This has been elegantly derived from a conceptual idea of causality of the S-matrix by Bogoliubov and Shirkov(57) using the concept of functional derivatives. Functional derivatives can only be defined for a certain class of functionals for parafields(47) and it is certainly not obvious that the relationship

$$[[j(x), j(y)]_+, j(z)]_- = 0 \qquad \text{for } x \sim z \text{ and } y \sim z$$

can be derived in a similar manner to the Bose case.

On the other hand a study of the S-matrix elements of quantum electrodynamics by Volkov, McCarthy and Amatuni[39] has shown that the matrix elements only differ from those of the Fermi case by certain numerical factors. This would seem to imply that the S-matrix elements are analytic.

Using the Green ansatz i.e.,

$$A(x) = \sum_{\alpha=1}^{p} A^{\alpha}(x)$$

and defining

$$S = S^{(1)} S^{(2)} \dots S^{(p)}$$

where $S^{(j)}$ is the S-matrix for the j^{th} component field then a standard S-matrix theory is defined provided

$$[S^{(i)}, S^{(j)}]_- = 0 .$$

It is straightforward to prove from the factorization of the V.E.V.'s of different component fields that the S-matrix elements are analytic. However it is difficult to find a suitable definition of normal product which enables S to be expanded solely in terms of parafield operators without resorting to the Green component fields.

Assuming analyticity to be an expression of causality it is not at all obvious from the above discussion whether a causal S-matrix theory of paraparticles can be developed. Evidence for either point of view has been given above. It may be conjectured that only those S-matrices which are consistent with the cluster decomposition property are causal.

§2.11 CONCLUSIONS

Although parafields satisfy the axioms of quantum mechanics in the sense that they form a sensible Hilbert space, severe restrictions are placed upon the theory by the cluster decomposition property. The cluster problem is essentially a reduction of various groups to their appropriate subgroups.

For a non-relativistic field the physical observables are restricted to those of the form

$$\int [\phi^*(x), O(x)\ \phi(x)]_\pm\ dx\ .$$

From this restriction of observables to elements of U(v) it follows that a parafermi (parabose) field of order p is equivalent to a set of p fermions (bosons). Another immediate consequence of this is that in the associated quantum mechanical space the corresponding observables are symmetric functions of their arguments.

For relativistic fields the only physical observables that are compatible with the cluster decomposition property are those whose momentum representations are functions of the operators $[a^*(\underset{\sim}{k}), a(\underset{\sim}{\ell})]_\pm$. One way to ensure this is to decompose the field operators into their positive and negative components and to construct interactions from these.

In a similar fashion an S-matrix theory which is compatible with the cluster decomposition property can be developed.

The restrictions derived are quite general and do not assume any particular model for the interactions.

Within the Wightman axiomatic formulation of parafield theory only certain V.E.V.'s are consistent with cluster decomposition. An attempt to form a generalized cluster decomposition property using symmetrized V.E.V.'s only works for $p = 2$ due to a peculiarity of the pb2 commutation relations.

An important aspect of field theory arises from the various discussions of the cluster property. The discussions made use of not just the algebra generated by a set of operators but also of the particular representations of the appropriate algebra. This emphasizes, at least for parafields, the importance of considering the representations of the algebra as well as the algebra itself. It is not surprising that violations of the cluster property are particularly obvious in the Wightman formulation since V.E.V.'s are essentially the matrix elements of operators.

For $p > 1$ the existence of an L.S.Z. formalism is questionable. It is also not obvious whether the two-body scattering amplitudes are necessarily analytic.

§3. PHYSICAL RESTRICTIONS ON SOME ALTERNATIVE METHODS OF QUANTIZATION

§3.1 Introduction

Although the most well-known, and most appealing, parafields do not exhaust the alternatives to Bose or Fermi quantization. Many other types of statistics have been proposed by postulating various commutation relations[29,30,31,54] between the creation and annihilation operators, a_r^* and a_r. This is relatively easy to do since in general the a_r and a_r^* may be associated with the lowering and raising operators of a Lie algebra.

In the preceding chapter it was shown that severe restrictions are placed on parafields as a consequence of the representations of the various groups afforded by a set of parafield operators. In general, physical restrictions, such as the cluster property, will impose restrictions on other attempts to generalize Bose and Fermi statistics. Often these restrictions, as for parafields, have a simple interpretation in terms of the algebra generated by the appropriate functions of the a_r and a_r^*.

Such is the case for a type of statistics recently proposed by Kademova and Kraev[30]. Their statistics are related to $O(2\nu,1)$, but as is shown in §3.2 this algebra is inappropriate for a description of quantum field theory. It is possible to modify their algebra to overcome the objections raised in §3.2 but it is still not clear whether these modified algebras are consistent with other properties of field theory.

In §3.3 commutation relations of the form[31]

$$a_r^{p+1} = 0$$

and

$$[a_r, a_s]_- = 0$$

are shown to be incompatible with invariance of the algebra under infinitesimal unitary transformations. However it is possible that this is too severe a restriction to be placed on a quantum field description. Finally as an example of the above commutation relations the statistics proposed by Parks[31] are considered and related to parafield operators.

§3.2 An Inconsistency in the Quantization Scheme of Kademova and Kraev

Kademova and Kraev have recently proposed a new quantization scheme for spin half fields which has commutation relations very similar to those of parafermi fields. Their scheme however, allows an unlimited number of identical spin-half particles in a given state. The commutation relations generate a representation of $O(2\nu,1)$ in distinction to the parafermi representations of $O(2\nu+1)$.

The proposed trilinear commutation relations for the creation operators a_i^* and their (assumed) hermitian conjugates, the annihilation operators a_i, $i = 1,2,\ldots,\nu$, are

$$[a_i, [a_j^*, a_k]_-]_- = -2\delta_{ij}a_k \; ,$$

$$[a_i, [a_j, a_k]_-]_- = 0 \qquad (3.1)$$

and the relations obtained from these through hermitian conjugation and application of Jacobi's identity. These differ from the parafermi commutation relations in the sign of the right-hand side of the first equation.

A "vacuum state" vector $|0\rangle$ is required to satisfy

$$a_i |0\rangle = 0 ,$$
$$a_i a_j^* |0\rangle = p\, \delta_{ij} |0\rangle \tag{3.2}$$

where p is some positive constant, the "order of the parastatistics". The operator $N_{pi} = \frac{1}{2}([a_i, a_i^*]_- - p)$ is then to be identified as "counting the number of particles in the i^{th} state", and is claimed to have a spectrum consisting of all non-negative integers.

The consistency of this scheme has been established only in the case where there is just one pair of creation and annihilation operators, a_1^* and a_1. It is easy to find an inconsistency when there is more than one such pair.

Consider the operators

$$S_1 = \tfrac{1}{4}[a_2 + a_2^*, a_1 - a_1^*]_- ,$$
$$S_2 = \tfrac{1}{4}i[a_2 + a_2^*, a_1 + a_1^*]_-$$
$$S_3 = \tfrac{1}{2}[a_1, a_1^*]_- ,$$

which in view of (3.1), satisfy the familiar angular momentum relations

$$[S_i, S_j]_- = i\, \varepsilon_{ijk} S_k ,$$

and which are hermitian operators in the representations under discussion. As proved in elementary quantum mechanics texts, any eigenvalue of S_3 in such a representation must be integral or half-odd-integral. More importantly, if λ is such an eigenvalue, so is $-\lambda$.

Now it is easily shown that the spectrum of S_3 is here unbounded above, except in the trivial representation $a_i = a_i^* = 0$, and it follows at once that it is also unbounded below. For suppose S_3 has a maximum eigenvalue $\lambda_{max} > 0$. Then, because $[S_3, a_1^*]_- = a_1^*$, there must exist a normalizable state vector $|\chi\rangle$ such that

$$S_3 \; |\chi\rangle = \lambda_{max} \; |\chi\rangle, \qquad a_1^* \; |\chi\rangle = 0.$$

But then

$$\begin{aligned}
\lambda_{max} \, \| \; |\chi\rangle \|^2 &= \langle\chi \; |S_3 \; |\chi\rangle \\
&= \tfrac{1}{2}\langle\chi| \; (a_1 a_1^* - a_1^* a_1) \; |\chi\rangle \\
&= - \tfrac{1}{2}\| a_1 |\chi\rangle \|^2 \\
&\leqslant 0,
\end{aligned}$$

which is contradictory.

In this way it may be shown that the spectrum of each operator $\tfrac{1}{2}[a_i, a_i^*]_-$ is unbounded above and below in the representations of interest. It is thus impossible by any addition process to ensure that S_3 has a positive spectrum. The same is therefore true of the spectrum of each N_{pi}, contrary to the claims of Kademova and Kraev, and as a result such operators are quite unsuitable for use as "number operators". Moreover, it follows that no representation of the algebra (3.1) in which a_i^* is the hermitian conjugate of a_i, contains a vector $|0\rangle$ satisfying eqs. (3.2).

As Kademova and Kraev have pointed out, the algebra (3.1) is isomorphic to the Lie algebra of $O(2\nu,1)$. In the case $\nu = 1$, the compact subalgebra contains only one element, $\frac{1}{2}[a_1, a_1^*]_-$, and it is possible to find an infinite-dimensional representation $D^+(-p)$[58] in which this operator is hermitian and has a spectrum bounded below by a positive constant p. It is a representation of this type which Kademova and Kraev wish to use for each operator $\frac{1}{2}[a_i, a_i^*]_-$. Unfortunately, as the preceding arguments show, it is impossible to find an infinite-dimensional representation of the $O(2\nu,1)$ algebra, $\nu > 1$, in which the operators $\frac{1}{2}[a_i, a_i^*]_-$ have these properties.

The inconsistency of (3.1) has also been recently demonstrated by Ohnuki, Yamada and Kamefuchi[59] who showed that certain two particle states have negative norms.

It is possible to modify the commutation relations to avoid the above difficulties. Since $O(2,1)$ is locally isomorphic to $SU(1,1)$ Bracken[60] has suggested the generalization to the algebra $SU(\nu,1)$. Denoting the generators of $SU(\nu,1)$ as N_{ij} the operators N_{io} and N_{oi} can be interpreted as creation and annihilation operators respectively. The relevant commutation relations are

$$[a_i,[a_j,a_k^*]_-]_- = \delta_{ik}a_j + \delta_{jk}a_i$$

and

$$[a_i,a_j]_- = 0.$$

Another generalization has been proposed by Lohe[61] who generalized the O(2,1) algebra to O(2,ν). The commutation relations are then those of the "modified boson operators" and are

$$[a_i,[a_j,a_k^*]_-]_- = -2\delta_{ij}a_k + 2\delta_{ik}a_j + 2\delta_{jk}a_i$$

and

$$[a_i,a_j]_- = 0.$$

Both cases would certainly lead to unusual, though possibly not incorrect, field theories.

§3.3 Statistics with a Maximum Occupancy

§3.3a Statistics with $a_k^{p+1} = 0$ and $[a_k,a_\ell]_- = 0$

The expansion of a field operator is

$$\phi(x) = \sum_k \phi^k(x)a_k ,$$

where the $\phi^k(x)$ are a complete set of one-particle wave-functions. Bialynicki-Birula[62] has observed that the commutation relations should be invariant under unitary transformations of the $\phi^k(x)$'s. For infinitesimal transformations this requires that the commutation relations for the a_k's should be invariant under the following transformations:

$$a_k \to a_k' = a_k + \sum_\ell \alpha_{k\ell}a_\ell$$

and

$$a_k^* \to a_k^{*\prime} = a_k^* - \sum_\ell \alpha_{\ell k}\, a_\ell^* \,. \qquad (3.3)$$

A simple physical interpretation of this condition would be that it requires if p is the maximum occupancy of a state in momentum space then p is also the maximum occupancy in configuration space. For example, in the case of Fermi statistics it requires that $a_k^2 = 0$ is equivalent to $(\phi(x))^2 = 0$. Bialynicki-Birula has observed that parastatistics are invariant under (3.3).

Statistics with a maximum occupancy of p are usually characterized by $a_k^{p+1} = 0$. Requiring the invariance of this relation under transformations of the type (3.3) it follows that

$$a_k a_r^p + a_r a_k a_r^{p-1} + \ldots + a_r^p a_k = 0 \,.$$

Requiring, in turn that this and subsequent relations are also invariant under transformations (3.3) it follows by induction that

$$\{a_{i_1}, a_{i_2}, \ldots, a_{i_{p+1}}\} = 0$$

where { } is the symmetrizer over the p+1 arguments contained within.

For statistics of the form $a_k^{p+1} = 0$ and $[a_k, a_\ell]_- = 0$ invariance under transformations (3.3) implies from above that $a_{i_1} a_{i_2} \ldots a_{i_{p+1}} = 0$ and hence that there are no more than p particles in the Universe.

It is possible that the requirement of unitary invariance is too strong, for Parks' statistics (to be discussed below) may have some significance in the B.C.S. theory of superconductivity[63].

It is interesting to note that Fermi statistics follow directly from $a_k^2 = 0$, irreducibility of the algebra and invariance under infinitesimal unitary transformations. Defining $\lambda = a^* a + a a^*$, then $a^2 = 0$ implies

$$[a,\lambda]_- = [a^*,\lambda]_- = 0 \; .$$

Irreducibility implies that $\lambda = cI$ where c may be chosen to be unity. Invariance of

$$a^2 = 0$$

and

$$aa^* + a^* a = 1$$

under (3.3) implies that

$$a_k a_\ell + a_\ell a_k = 0$$

and

$$a_k a_\ell^* + a_\ell^* a_k = \delta_{k\ell} \; .$$

However it is not possible to generalize this and derive a unique set of statistics from the conditions $a_k^{p+1} = 0$, irreducibility of the algebra and invariance under infinitesimal unitary transformations.

§3.3b Parks' Commutation Relations

Parks' commutation relations may be written as:

$$[a_k, a_\ell]_- = 0$$

$$[a_k^*, a_\ell]_- = 2\delta_{k\ell}(N_k - \tfrac{1}{2}n_o)$$

$$[a_k, N_\ell]_- = \delta_{k\ell} a_k \ .$$

For $n_o = 1$ realizations of this algebra can be constructed from parafermi operators. Denoting a parafermi operator by c_k then

$$a_k = \frac{1}{p!} c_k{}^p$$
$$= b_k{}^{(1)} b_k{}^{(2)} \ldots b_k{}^{(p)}$$

where $b_k{}^{(i)}$ is the i^{th} Green component field. With the convention of upper signs for p even and lower signs for p odd the a_k's satisfy

$$a_k^2 = 0 \ ,$$

$$[a_k, a_\ell]_{\mp} = 0 \ ,$$

$$[a_k^*, a_\ell]_{\mp} = 2\delta_{k\ell}(N_k - \tfrac{1}{2}) \qquad (3.4)$$

and

$$[a_k, N_\ell]_- = \delta_{k\ell} a_k \quad \text{for p even}$$

$$= 0 \quad \text{for p odd}$$

which are a generalization of Parks' commutation relations for $n_o = 1$. The proof is relegated to Appendix 5.

For $n_o > 1$ an ansatz similar to that for parafermi operators may be constructed. Define

$$a_k = \sum_{i=1}^{n_o} a_k^{(i)}$$

where

$$[a_k^{(i)}, a_\ell^{(j)}]_- = [a_k^{(i)}, a_\ell^{(j)*}]_-$$

$$= 0 \text{ for } i \neq j \ ,$$

and each $a_k^{(i)}$ satisfies (3.4) for p even. It follows directly that

$$a_k^{n_o+1} = 0 \ ,$$

$$[a_k, a_\ell]_- = 0 \ ,$$

$$[a_k^*, a_\ell]_- = 2\delta_{k\ell}(N_k - \frac{n_o}{2})$$

and

$$[a_k, N_\ell]_- = \delta_{k\ell} a_k$$

where

$$N_\ell = \sum_{i=1}^{n_o} N_\ell^{(i)} \ .$$

No attempt is made to generalize the statistics for p odd.

§4. BARGMANN AND HARMONIC OSCILLATOR REPRESENTATIONS OF A PARABOSE OPERATOR

§4.1 Introduction

Three well-known representations of the Bose commutation relations are; a) the matrix representation(64) of the operators, b) the Bargmann representation(36) in terms of complex variables, and c) the familiar quantum mechanical harmonic oscillator representation(37). It is of interest, especially in view of their association with representations of the symplectic group(16), to find the corresponding irreducible representations of parabose operators.

The matrix representations have recently been obtained by Alabiso and Duimio(19). They have calculated the matrix elements for a set of ν parabose operators. In this chapter the analogues of the Bargmann and harmonic oscillator representations are constructed and various properties of these representations are derived. The representations are restricted to those of a single operator and the generalization to ν degrees of freedom has not been effected.

The Bargmann representation of the Bose relations uses the following commutation relation:

$$[d_z, z]_- = 1 .$$

To construct the more general parabose representations it is necessary to introduce an extra operator; R, with the properties

$$[R,z]_+ = [R,d_z]_+ = 0 \ .$$

Expressions for a and a^* are

$$a^* = z \ ,$$

and

$$a = d_z + \frac{\tau}{z} R \ ,$$

where the simplest realization of R is the reflection operator. When $\tau = (-1)^{\alpha+1}\alpha$ the a and a^* form a representation of the parabose algebra of odd order since $p = 2\alpha + 1$. The choice $\tau = (-1)^{\alpha+1}(\alpha + \frac{1}{2})$ corresponds to an even order parabose algebra since, in this case, $p = 2\alpha + 2$. The introduction of R decomposes the representation space into subspaces of even and odd functions and the parabose operators take different functional forms in each subspace.

After the vacuum state is determined an important distinction between the representation for p odd and p even appears. For p odd the Bargmann space is of the form $z^\alpha f(z) (\alpha = 0,1,\ldots)$ where f(z) is an entire regular (analytic) function. However for p even the space is of the form $\sqrt{z\bar{z}}\ z^\alpha f(z)$; these functions are no longer differentiable at z = 0 and hence are not entire. Although it is possible to find analytic representations for p even, they are not considered because of the comparative simplicity of the non-analytic ones.

The metric is determined by requiring that a and a^* are adjoints. This results in the metric taking a 2 × 2 matrix form; its elements being modified Bessel functions of the third kind.

The discussion of the Bargmann space is completed by a calculation of a complete set of basis vectors and the determination of the representation of the unit element (reproducing kernel).

The harmonic oscillator representations for the parabose algebra may be determined by taking the following expressions for p and q:

$$p = - i(d_x + \frac{\tau}{x} R)$$

and

$$q = x ,$$

where a realization of R is the reflection operator. This space can also be decomposed into even and odd functions. This representation was proposed by Yang[38] who constructed, for p odd, a representation by applying powers of

$$\eta = \frac{-1}{\sqrt{2}} (q - ip)$$

to the state of lowest energy. However, Yang did not derive the commutation relations for the p and q, and so did not realize that they formed a representation of the parabose algebra.

A different method of obtaining the representation space is used in this thesis. If p and q are substituted in the Hamiltonian for the harmonic oscillator, the resulting Schrödinger equation may be used to determine the energy eigen-states. The energy is quantized in a similar manner to that for the Bose case by requiring that the wave-functions are bounded at infinity. For both p even and p odd the basis states are orthonormal Laguerre polynomials which are generalizations of the Hermite polynomials of Bose statistics.

The integral transform that expresses the equivalence of the Bargmann and harmonic oscillator representations is calculated by requiring that z and $d_z + \frac{\tau}{z} R$ correspond to the raising and lowering operators $\frac{-1}{\sqrt{2}}(q - ip)$ and $\frac{-1}{\sqrt{2}}(q + ip)$ respectively. This integral transform has a matrix structure corresponding to the decomposition into even and odd functions.

The generalized Bargmann and harmonic oscillator representations are shown to be equivalent by the proof that the integral transform is unitary.

The concluding remarks concern the possibility of generalising these representations for $\nu > 1$ degrees of freedom. A representation of the Green component fields is also briefly discussed.

§4.2 Bargmann Space of a Single Parabose Operator

§4.2.1 Representation of the Creation and Annihilation Operators

The parabose commutation relations for a single operator are

$$[a^2, a^*]_- = 2a$$

and

$$[a^{*2}, a]_- = -2a\ .$$

A realization of this algebra may be found in terms of the operators z, $\frac{d}{dz}$ and R where R satisfies

$$[R,z]_+ = [R,d_z]_+ = 0 \tag{4.1}$$

and the abbreviation $d_z = \frac{d}{dz}$ has been introduced. If the choice $a^* = z$ and $a = d_z + \frac{\tau}{z} R$ is made, where τ is at present an arbitrary constant, then

$$a^2 = d_z^2 - \frac{\tau^2}{z^2} R^2 - \frac{\tau}{z^2} R\ .$$

Thus

$$\begin{aligned}[a^2, a^*]_- &= [d_z^2, z]_- - \frac{\tau}{z^2}[R,z]_- \\ &= 2\left(d_z + \frac{\tau}{z} R\right) \\ &= 2a\ .\end{aligned}$$

Similarly,

$$\begin{aligned}[a^{*2}, a]_- &= [z^2, d_z]_- \\ &= -2z \\ &= -2a^*\ .\end{aligned}$$

In the above equations and throughout this chapter the Bose case may always be obtained by substituting $\tau = 0$.

A specific realization of R is given by the reflection operator. So in addition to satisfying (4.1), R also satisfies $R^2 = I$. The introduction of the reflection operator means that the space upon which the annihilation and creation operators act can be decomposed into even and odd functions. In the following it is convenient to consider any element of the space as a two component vector, i.e.,

$$f(z) = \begin{pmatrix} f_e(z) \\ f_o(z) \end{pmatrix}$$

where

$$f_e(z) = f(z) + f(-z)$$

and

$$f_o(z) = f(z) - f(-z) .$$

Any operator acting on the space will have a two-dimensional matrix structure, and for multiplication and addition of functions this structure will be

$$f(z) = \begin{pmatrix} f_e(z) & f_o(z) \\ f_o(z) & f_e(z) \end{pmatrix}.$$

It is straightforward to show that this structure is a polynomial ring. The representation of R is

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and hence the representations of a and a^* are

$$\begin{pmatrix} 0 & d_z - \frac{\tau}{z} \\ d_z + \frac{\tau}{z} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}$$

respectively. However, this is not the only representation of R possible. If the transformation $z = e^t$ is made then R may be represented by the following transformation:

$$R: e^t \rightarrow e^{t'} = e^{t-i\pi} \ .$$

Since $d_z = e^{-t} d_t$ it can easily be checked that the commutation relations hold since

$$\begin{aligned} [R,z]_+ &= Re^t + e^t R \\ &= e^{t-i\pi}R + e^t R \\ &= 0 \ , \end{aligned}$$

and similarly

$$[R,d_z]_+ = 0 \ .$$

The operator R no longer satisfies $R^2 = I$.

§4.2.2 Vacuum and Excited States

The vacuum state is defined by

$$a|0\rangle = 0$$

and

$$aa^*|0\rangle = p|0\rangle \ ,$$

where p is the order of the parabose algebra.

In the Bargmann representation these equations become

$$z\, d_z\, f^o(z) + \tau\, f^o(-z) = 0 \tag{4.2}$$

and

$$z\, d_z\, f^o(z) - \tau\, f^o(-z) = (p-1)\, f^o(z) \tag{4.3}$$

both of which are difference differential equations and where $f^o(z)$ is the vacuum state. A necessary condition that (4.2) and (4.3) are satisfied is

$$z\, d_z\, f^o(z) = \frac{p-1}{2}\, f^o(z)\ .$$

This implies that

$$f^o(z) = c(\bar{z})\, z^{\frac{p-1}{2}}$$

where $c(\bar{z})$ is a function of $\bar{z}$ $(x - iy)$, to be determined. However, although

$$[\, c(\bar{z}),\ z\, d_z]_- = 0$$

it does not necessarily follow that

$$[\, c(\bar{z}),\ d_z + \frac{\tau}{z} R]_- = 0$$

since

$$[\, \bar{z}, R]_- \neq 0\ .$$

Substituting $f^o(z)$ in (4.2) implies that

$$c(\bar{z})\, \frac{p-1}{2}\, z^{\frac{p-1}{2}} + \tau\, c(-\bar{z})(-z)^{\frac{p-1}{2}} = 0\ .$$

It is convenient to take different solutions of this equation depending on whether p is even or odd. For p odd (i.e., $p = 2\alpha + 1$)

take $c(\bar{z}) = 1$ and hence $\tau = (-1)^{\alpha+1}\alpha$ is the appropriate solution. For p even (i.e., $p = 2\alpha + 2$) take $c(\bar{z}) = \sqrt{\bar{z}}$ and hence $\tau = (-1)^{\alpha+1}(\alpha + \frac{1}{2})$, where the convention $\sqrt{-1} = i$ is adopted throughout this thesis. Thus the vacuum state may be written:

$$f^{o}(z) = z^{\alpha} \quad \text{when} \quad p = 2\alpha + 1 ,$$

and

$$= \sqrt{z\bar{z}}\; z^{\alpha} \text{ when } p = 2(\alpha + 1)$$

and

$$\alpha = 0,1, \ldots \quad .$$

The n^{th} excited state in each case can be written

$$|n\rangle = z^{\alpha+n} \qquad \text{for} \quad p \text{ odd}$$

and

$$|n\rangle = \sqrt{z\bar{z}}\; z^{\alpha+n} \qquad \text{for} \quad p \text{ even} .$$

There is however a major difference between these two spaces. For p odd, the basis of the Bargmann space consists of powers z^n, $n \geqslant \frac{p-1}{2}$ and hence span analytic functions. For p even, due to the appearance of the factor $\sqrt{z\bar{z}}$ the Bargmann space contains non-analytic functions. However, in this case, every element can be written as a product $\sqrt{z\bar{z}}\; f(z)$ where all $f(z)$'s are analytic functions.

It is possible however for the case p even to form a representation of the parabose operators in terms of analytic functions. Consider the alternative representation introduced in the previous section:

$$R: t \to t - i\pi \; , \; z = e^{t} \quad \text{and} \quad d_z = e^{-t}\, d_t \; .$$

If the vacuum state is denoted by $F(t)$ it satisfies the equation:

$$F'(t) = -\tau F(t - i\pi) .$$

A solution of the above is $F(t) = e^{\beta t}$ provided the following equation is satisfied

$$-\beta(-1)^{-\beta} = \tau .$$

The solutions to this transcendental equation when $\beta = 0,1,2,\ldots$ are just $\tau = 0, +1, -2, \ldots$ which have already been used for parabose algebras of odd order. It can be shown that for parabose algebras of even order the above equation also has solutions. In all cases the vacuum state, and hence all excited states, are analytic functions.

Since there exists an equivalence mapping from the non-analytic Bargmann space to the analytic space and because of the greater simplicity of the former, only the non-analytic space will be considered. This follows from the fact that the representations of the parabose algebra satisfying

$$a f^{o} = 0$$

and

$$aa^{*} f^{o} = pf^{o}$$

are unique to within a unitary equivalence[5]. It also follows from this, that any representation of the a and a^{*} in terms of more general functions of the z, d_z and R must be equivalent to the representation $a^{*} = z$ and $a = d_z + \frac{\tau}{z} R$.

§4.2.3 Metric

Since the Bargmann space has been decomposed into even and odd functions the metric may take a different form in each subspace. The metric which ensures a is the adjoint of a^* will be assumed to take the general form

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} .$$

Because the general form of the metrics for p even and p odd are different, each case will be treated separately.

(a) p odd

A scalar product is defined to be:

$$(f,g) = \iint dz\, d\bar{z}\, (\bar{f}_e,\bar{f}_o) \begin{pmatrix} \rho_{11}(z\bar{z}) & \rho_{12}(z\bar{z}) \\ \rho_{21}(z\bar{z}) & \rho_{22}(z\bar{z}) \end{pmatrix} \begin{pmatrix} g_e \\ g_o \end{pmatrix}$$

where the ρ_{ij} are to be determined by the condition that a and a^* are adjoint. Now

$$(f,ag) = \iint dz\, d\bar{z}\, (\bar{f}_e,\bar{f}_o) \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 0 & d_z - \frac{\tau}{z} \\ d_z + \frac{\tau}{z} & 0 \end{pmatrix} \begin{pmatrix} g_e \\ g_o \end{pmatrix}$$

$$= \iint dz\, d\bar{z}\, \{\bar{f}_e\, \rho_{12}(d_z + \tfrac{\tau}{z})g_e + \bar{f}_e\, \rho_{11}(d_z - \tfrac{\tau}{z})g_o$$

$$+ \bar{f}_o\, \rho_{22}(d_z + \tfrac{\tau}{z})g_e + \bar{f}_o\, \rho_{21}(d_z - \tfrac{\tau}{z})g_o\} . \qquad (4.4)$$

If it is assumed that the functions f and g do not grow too fast at infinity and are analytic so that $d_z \bar{f}_e = d_z \bar{f}_o = 0$, (4.4) reduces to

$$\iint dz\, d\bar{z}\, \{- \bar{f}_e \frac{d\rho_{12}}{dz} g_e + \bar{f}_e \frac{\tau}{z} \rho_{12}\, g_e - \bar{f}_e \frac{d\rho_{11}}{dz} g_o$$

$$- \bar{f}_e \frac{\tau}{z} \rho_{11}\, g_o - \bar{f}_o \frac{d\rho_{22}}{dz} g_e + \bar{f}_o \frac{\tau}{z} \rho_{22}\, g_e$$

$$- \bar{f}_o \frac{d\rho_{21}}{dz} g_o - \bar{f}_o \frac{\tau}{z} \rho_{21}\, g_o\} .$$

However

$$(a^* f,g) = \iint dz\, d\bar{z}\, \overline{\begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} \begin{pmatrix} f_e \\ f_o \end{pmatrix}} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} g_e \\ g_o \end{pmatrix}$$

$$= \iint dz\, d\bar{z}\, \{\bar{f}_e\, \bar{z}\, \rho_{21}\, g_e + \bar{f}_e\, \bar{z}\, \rho_{22}\, g_o +$$

$$+ \bar{f}_o\, \bar{z}\, \rho_{11}\, g_e + \bar{f}_o\, \bar{z}\, \rho_{12}\, g_o\} .$$

So $(a^* f,g) = (f,ag)$ for all f and g provided the following equations are satisfied by the ρ's:

$$\frac{d\rho_{12}}{dz} - \frac{\tau}{z} \rho_{12} = - \bar{z}\, \rho_{21} \tag{4.5a}$$

$$\frac{d\rho_{12}}{dz} + \frac{\tau}{z} \rho_{21} = - \bar{z}\, \rho_{12} \tag{4.5b}$$

and

$$\frac{d\rho_{11}}{dz} + \frac{\tau}{z} \rho_{11} = - \bar{z}\, \rho_{22} \tag{4.6a}$$

$$\frac{d\rho_{22}}{dz} - \frac{\tau}{z} \rho_{22} = - \bar{z}\, \rho_{11} . \tag{4.6b}$$

Elimination of ρ_{21} from 4.5 gives

$$\frac{d^2\rho_{12}}{dz^2} - (\bar{z}^2 + \frac{\tau(\tau - 1)}{z^2})\rho_{12} = 0 \tag{4.7}$$

which is a form of Bessel's equation with imaginary argument.

Since the derivation has required that terms of the form $f_e \, \rho_{12} \, g_e$ vanish at infinity the correct choice are the modified Bessel functions of the third kind which in Watson's notation[65] are denoted by $K_\nu(z)$. The asymptotic form:

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

ensures the assumed behaviour at infinity. The solution of (4.7) is:

$$\rho_{12}(z\bar{z}) = c\sqrt{z\bar{z}}\; K_{\tau-\frac{1}{2}}\,(z\bar{z}) \qquad (4.8a)$$

where c is an arbitrary constant. Equations (4.5) are recurrence relations for Bessel functions and it follows that

$$\rho_{21}(z\bar{z}) = c\sqrt{z\bar{z}}\; K_{\tau+\frac{1}{2}}\,(z\bar{z}) \;. \qquad (4.8b)$$

Similarly from equations (4.6) it follows that

$$\rho_{11}(z\bar{z}) = d\sqrt{z\bar{z}}\; K_{\tau+\frac{1}{2}}\,(z\bar{z}) \qquad (4.8c)$$

and

$$\rho_{22}(z\bar{z}) = d\sqrt{z\bar{z}}\; K_{\tau-\frac{1}{2}}\,(z\bar{z}) \;. \qquad (4.8d)$$

Since the order of the parafield depends only on τ no loss of generality would occur if c were to be set equal to zero. However if R is unitary i.e., $R^* R = I$ then

$$(Rf, Rg) = (f, g) \;.$$

It then follows that

$$[R, \rho(z,\bar{z})]_- = 0 \;.$$

Since $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the above commutator is only zero when the off-diagonal elements of $\rho(z,\bar{z})$ are zero i.e., $c = 0$. The value of the constant d is $\frac{1}{\pi}\sqrt{\frac{2}{\pi}}$.

The Bose case is obtained by substituting $\tau = 0$ and noting that

$$K_{\frac{1}{2}} = K_{-\frac{1}{2}} = \sqrt{\frac{\pi}{2z\bar{z}}}\, e^{-z\bar{z}} .$$

(b) <u>p even</u>

A scalar product is defined for p odd by:

$$(f,g) = \iint dz\, d\bar{z}\, (\bar{f}_e,\bar{f}_o)\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} g_e \\ g_o \end{pmatrix} .$$

Again the ρ_{ij} are determined by requiring that

$$(f,ag) = (a^* f,g) .$$

The derivation of the equations determining the metric is almost identical to the case of p odd. However the relation

$$\frac{d}{dz}\,\bar{f}(z) = 0 \tag{4.9}$$

no longer holds since $f(z)$ is not analytic. Since

$$f(z) = \sqrt{z\bar{z}}\; f_a(z)$$

where $f_a(z)$ is analytic (4.9) can be replaced by

$$\frac{d}{dz}\,\bar{f}(z) = \frac{1}{2z}\,\bar{f}(z) .$$

The equations determining ρ are now modified to

$$d_z \rho_{11} + (\tau + \tfrac{1}{2}) \frac{1}{z} \rho_{11} = - \bar{z} \rho_{22}$$

and

$$d_z \rho_{22} - (\tau - \tfrac{1}{2}) \frac{1}{z} \rho_{22} = - \bar{z} \rho_{11} .$$

Eliminating ρ_{22} the above equations can be reduced to

$$u^2 d_u^2 \rho_{11}(u) + u\, d_u \rho_{11}(u) - (u^2 + (\tau + \tfrac{1}{2})^2)\rho_{11}(u) = 0$$

where $u = z\bar{z}$. The above equation for ρ is Bessel's equation, and the choice of the Bessel function to give correct asymptotic behaviour implies:

$$\rho_{11}(z\bar{z}) = \frac{1}{\pi} \sqrt{\frac{2}{\pi}}\, K_{\tau+\frac{1}{2}} (z\bar{z}) \tag{4.10a}$$

and similarly

$$\rho_{22}(z\bar{z}) = \frac{1}{\pi} \sqrt{\frac{2}{\pi}}\, K_{\tau-\frac{1}{2}} (z\bar{z}) . \tag{4.10b}$$

As for p odd, the unitarity of R implies

$$\rho_{12} = \rho_{21} = 0 .$$

§4.2.4 <u>A Complete Orthonormal Basis</u>

(a) <u>Orthonormal Set of Basis States</u>

(a_1) <u>p odd</u>

The norm of any state $f(z)$ is defined by

$$(f,f) = \iint dz\, d\bar{z}\, (\overline{f_e}(z), \overline{f_o}(z)) \begin{pmatrix} \rho_{11}(z\bar{z}) & 0 \\ 0 & \rho_{22}(z\bar{z}) \end{pmatrix} \begin{pmatrix} f_e(z) \\ f_o(z) \end{pmatrix} .$$

For pairs of excited states of the form $z^{\alpha+n}$ and $z^{\alpha+m}$, the scalar product, since the metric is diagonal, is zero unless both states have the same parity i.e., both m and n are even or odd. In this case terms in the scalar product reduce to

$$I_{nm} = \iint dz\, d\bar{z}\, \bar{z}^{\alpha+n}\, \rho_{ii}(z\bar{z}) z^{\alpha+m} \ .$$

Substituting

$$z = re^{i\theta} \ , \quad dz\, d\bar{z} = rdrd\theta$$

and

$$\rho_{ii}(z\bar{z}) = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{z\bar{z}}\, K_{\tau\pm\frac{1}{2}}(z\bar{z})$$

the above integral becomes

$$I_{nm} = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\int_0^{\infty} r^{2\alpha+n+m+2}\, K_{\tau\pm\frac{1}{2}}(r^2)dr \int_{-\pi}^{\pi} e^{i\theta(n-m)}d\theta$$

$$= 0 \quad \text{for } m \neq n \ .$$

Thus if ρ is a function of $z\bar{z}$ only the orthogonality of the states is independent of the metric.

Then

$$I_n = I_{nn} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} r^{2\alpha+2n+2} K_{\tau\pm\frac{1}{2}}(r^2)dr \ .$$

Since $\tau = (-1)^{\alpha+1}\alpha$ and $K_{-\nu} = K_{\nu}$ it follows that the integral splits into two cases:

$$I_{2n} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} r^{2\alpha+4n+2}\, K_{\alpha-\frac{1}{2}}(r^2)dr$$

and

$$I_{2n+1} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} r^{2\alpha+4n+4}\, K_{\alpha+\frac{1}{2}}(r^2)dr \ .$$

On substituting $r^2 = u$ and using the result[66]

$$\int_0^\infty K_\nu(t)t^{\mu-1}dt = 2^{\mu-2}\,\Gamma\left(\frac{\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)$$

the above equations for I_m reduce to

$$I_{2n} = \frac{1}{\sqrt{\pi}}\,2^{\alpha+2n}\,\Gamma(n+1)\,\Gamma(\alpha+n+\tfrac{1}{2}) > 0 \qquad (4.11)$$

and

$$I_{2n+1} = \frac{1}{\sqrt{\pi}}\,2^{\alpha+2n+1}\,\Gamma(n+1)\,\Gamma(\alpha+n+\tfrac{3}{2}) > 0.$$

An orthonormal set of basis states is then

$$u^{\alpha+m} = \frac{1}{\sqrt{I_m}}\,z^{\alpha+m}\,.$$

(a_2) <u>p even</u>

The argument to show that

$$(\sqrt{z\bar{z}}\,z^{\alpha+m}, \sqrt{z\bar{z}}\,z^{\alpha+n}) = 0$$

when $m \neq n$ is exactly the same as for p odd and follows directly from the decomposition $z = r\,e^{i\theta}$. Once again, the evaluation of the norm, since $\tau = (-1)^{\alpha+1}(\alpha + \tfrac{1}{2})$ and $K_\nu = K_{-\nu}$, splits into the following two cases:

$$I_{2n} = (\sqrt{z\bar{z}}\,z^{\alpha+2n}, \sqrt{z\bar{z}}\,z^{\alpha+2n})$$

and

$$I_{2n+1} = (\sqrt{z\bar{z}}\,z^{\alpha+2n+1}, \sqrt{z\bar{z}}\,z^{\alpha+2n+1})\,.$$

In a fashion similar to p odd it can be shown that

$$I_{2n} = \sqrt{\frac{2}{\pi}} \int_0^\infty r^{2\alpha+4n+3} K_\alpha(r^2)dr$$

$$= \frac{1}{\sqrt{2\pi}} 2^{\alpha+2n} \Gamma(n + \alpha + 1) \Gamma(n + 1) \qquad (4.12)$$

and

$$I_{2n+1} = \sqrt{\frac{2}{\pi}} \int_0^\infty r^{2\alpha+4n+5} K_{\alpha+1}(r^2)dr$$

$$= \frac{1}{\sqrt{2\pi}} 2^{\alpha+2n+1} \Gamma(n + \alpha + 2) \Gamma(n + 1).$$

An orthonormal set of states is

$$u^{\alpha+m} = \frac{1}{\sqrt{I_m}} \sqrt{z\bar{z}} \quad z^{\alpha+m} .$$

For any order of the parabose algebra the Hilbert space spanned by the u's will be denoted by F.

(b) Completeness

The set of functions relevant to the parabose algebra have the form $z^\alpha f(z)$ for p odd and $z^\alpha \sqrt{z\bar{z}}\, f(z)$ for p even, where f(z) is an entire analytic function. In both cases it suffices to consider functions of the form $z^\alpha f(z)$ and, since any f(z) can be uniquely decomposed into even and odd components, only the even components need be considered. Thus, since f(z) is entire,

$$f'(z) = z^\alpha f_e(z)$$

$$= \sum_{m=0}^{\infty} \alpha_{2m} z^{2m+\alpha} .$$

Defining $\|f\| = (f,f)^{\frac{1}{2}}$ it follows that

$$\|f\|^2 = \sum_{m=0,\ldots}^{\infty} |\alpha_{2m}|^2 I_{2m}$$

where I_{2m} is given by (4.11) for p odd and (4.12) for p even. Every set of coefficients, α_{2m}, for which the above sum converges defines an entire function $f \in F$. Similarly the inner product of two functions f, g of parity $(-1)^\alpha$ is given by

$$(f,g) = \sum_m \bar{\alpha}_{2m} \beta_{2m} I_{2m}$$

where

$$g(z) = \sum_m \beta_{2m} z^{2m+\alpha} .$$

For any $f' \in F$,

$$(u_{2m}, f'(z)) = \sqrt{I_{2m}}\, \alpha_{2m} ,$$

which expresses the completeness of the system u_{2m}. In a similar way if

$$\begin{aligned} f''(z) &= z^\alpha f_o(z) \\ &= \sum_m \alpha_{2m+1} z^{2m+\alpha+1} \end{aligned}$$

then

$$\|f'\|^2 = \sum_m |\alpha_{2m+1}|^2 I_{2m+1}$$

and the set of u_{2m+1} are complete for the f'''s which have a finite norm.

(c) Some Inequalities

From the definition $\|f\| = (f,f)^{\frac{1}{2}}$ and the decomposition of the space into even and odd functions it follows that

$$\|f\| = \|f_e\| + \|f_o\| \ .$$

Inequalities can be derived for the even and odd components separately.

(c_1) p odd

The expansion

$$f_+(z) = \sum_m \alpha_{2m} \, z^{\alpha+2m}$$

has a parity $(-1)^\alpha$. By Schwarz' inequality

$$|f_+(z)|^2 \leqslant (\sum_m |\alpha_{2m} z^{\alpha+2m}|)^2$$

$$\leqslant (\sum_m I_{2m}|\alpha_{2m}|^2) \ (\sum_m \frac{|z^{\alpha+2m}|^2}{I_{2m}}) \ .$$

The second factor, which is evaluated, in the next section will be denoted by $I_+(z\bar{z})$. Then

$$|f_+(z)| \leqslant \{I_+(z\bar{z})\}^{\frac{1}{2}} \, \|f_+\| \ .$$

Similarly

$$|f_-(z)| \leqslant \{I_-(z\bar{z})\}^{\frac{1}{2}} \, \|f_-\| \ ,$$

where

$$I_-(z\bar{z}) = \sum_m \frac{|z^{\alpha+2m+1}|^2}{I_{2m+1}} \ .$$

(c_2) p even

The functions in this case are of the form $\sqrt{z\bar{z}}\, f(z)$ where $f(z)$ is analytic. Denoting

$$f_+(z) = \sqrt{z\bar{z}} \sum_m \alpha_{2m}\, z^{\alpha+2m}$$

then

$$|f_+(z)|^2 \leqslant \sum_m |\ \alpha_{2m} \sqrt{z\bar{z}}\ z^{\alpha+2m}|^2$$

$$\leqslant \Big(\sum_m |\alpha_{2m}|^2\ I_{2m} \Big)$$

$$\times \Big(\sum_m \frac{|\sqrt{z\bar{z}}\ z^{\alpha+2m}|^2}{I_{2m}} \Big)$$

$$\leqslant \| f_+(z) \|^2\ I_+(z\bar{z}) \ ,$$

where $I_+(z\bar{z})$ is evaluated in the next section. Similarly

$$|f_-(z)|^2 \leqslant I_-(z\bar{z})\ \| f_-(z) \|^2 ,$$

where

$$I_-(z\bar{z}) = \sum_m \frac{|\sqrt{z\bar{z}}\ z^{\alpha+2m+1}|^2}{I_{2m+1}} \ .$$

In both cases any general function $f(z)$ in the space satisfies

$$|f(z)| \leqslant w(z)\ \|f\| \ .$$

The usefulness of a relation of this form, apart from showing the equivalence of strong and pointwise convergence as discussed by Bargmann, is that it enables a set of principal vectors to be defined. The interpretation of $w(z)$ as the "reproducing kernel" is discussed in the next section.

§4.2.5 Reproducing Kernel

The last section showed that for any analytic function $f(z) \in \mathcal{F}$ the following relations hold:

$$|f_+(z)| \leq I_+(z) \, \| f_+ \|$$

and

$$|f_-(z)| \leq I_-(z) \, \| f_- \| \ .$$

In an identical manner to that used by Bargmann for Bose statistics, a set of "principal vectors" of $\mathcal{F}$ may be found in each subspace. These principal vectors are denoted by e_a^+ and e_a^- and are such that $\| e_a^+ \| = I_+(a)$ and $\| e_a^- \| = I_-(a)$.

The "reproducing kernel", $I_\pm$ is defined by

$$f_\pm(w) = \int I_\pm(w,z) \; f_\pm(z) \; d\rho(z\bar{z})$$

and is equal to $\bar{e}_w^\pm(z)$. It is the representation of the unit operator and in terms of any complete orthonormal system $v_1^\pm$, $v_2^\pm$, ... for each subspace

$$e_a^\pm(z) = \sum_k \bar{v}_k^\pm(a) \; v_k^\pm(z) \ .$$

Using the appropriate set of orthonormal functions in each subspace the reproducing kernel in each subspace is evaluated below.

(a) p odd

$$\begin{aligned} I_+(wv) &= \sum_{n=0,1,\ldots} \frac{1}{I_{2n}} (\bar{w}v)^{\alpha+2n} \\ &= \sqrt{\pi} \left(\frac{\bar{w}v}{2}\right)^{\alpha} \sum_{n=0,1,\ldots} \frac{1}{n!} \frac{1}{\Gamma(n+\alpha+\frac{1}{2})} \left(\frac{\bar{w}v}{2}\right)^{2n} \\ &= \sqrt{\frac{\pi}{2}} \sqrt{\bar{w}v} \; I_{\alpha-\frac{1}{2}} (\bar{w}v) \end{aligned}$$

where I_ν denotes a modified Bessel function of the first kind.

Similarly

$$I_-(wv) = \sum_{n=0,1,\ldots} \frac{1}{I_{2n+1}} (\bar{w}v)^{\alpha+2n+1}$$

$$= \sqrt{\pi} \left(\frac{\bar{w}v}{2}\right)^{\alpha+1} \sum_{n=0,1,\ldots} \frac{1}{n!} \frac{1}{\Gamma\left(n + \alpha + \frac{3}{2}\right)} \left(\frac{\bar{w}v}{2}\right)^{2n}$$

$$= \sqrt{\frac{\pi}{2}} \sqrt{\bar{w}v}\, I_{\alpha+\frac{1}{2}} (\bar{w}v) .$$

(b) <u>p even</u>

$$I_+(wv) = \sum_{n=0,1,\ldots} \frac{1}{I_{2n}} \sqrt{\bar{w}v}\sqrt{\bar{w}\ \bar{v}}\ (\bar{w}v)^{\alpha+2n}$$

$$= \sqrt{2\pi}\ \sqrt{\bar{w}v}\ \sqrt{w\bar{v}}\ I_\alpha(\bar{w}v) ,$$

and similarly

$$I_-(wv) = \sum_{n=0,1,\ldots} \frac{1}{I_{2n+1}} \sqrt{w\bar{w}}\ \sqrt{v\bar{v}}\ (\bar{w}v)^{\alpha+2n+1}$$

$$= \sqrt{2\pi}\ \sqrt{w\bar{w}}\ \sqrt{v\bar{v}}\ I_{\alpha+1}(\bar{w}v) .$$

For p odd the substitution $\alpha = 0$ gives the unit element $e^{\bar{w}v}$ decomposed into even and odd parts.

§4.3 HARMONIC OSCILLATOR REPRESENTATIONS OF A PARABOSE OPERATOR

§4.3.1 Representation of the Parabose Algebra

A representation of the parabose algebra is given by

$$p = -\ i(d_x + \frac{\tau}{x} R)$$

and

$$q = x .$$

Similarly to the Bargmann space R is such that $[R,x]_+ = [R,d_x]_+ = 0$ and $R^* = R$. The uniqueness of these expressions, to within a unitary equivalence, for p and q has been demonstrated by Yang. Raising and lowering operators, η and ξ respectively, are defined by

$$\eta = \frac{-1}{\sqrt{2}}(q - ip) = \frac{-1}{\sqrt{2}}(x - d_x - \frac{\tau}{x}R)$$

and*

$$\xi = \frac{-1}{\sqrt{2}}(q + ip)$$

$$= \frac{-1}{\sqrt{2}}(x + d_x + \frac{\tau}{x}R) .$$

---

* This definition differs from that used by Bargmann by a minus sign. This is because the results are expressed in terms of Laguerre polynomials rather than Hermite polynomials which are expressed as

$$H_{2n} = (-1)^n\, 2^{2n}\, n!\, L_n^{-\frac{1}{2}} ,$$

with the $(-1)^n$ factor compensating for the minus sign in the definitions used in this thesis.

As in the Bargmann space the representation space is decomposed into even and odd functions. The scalar product is then given by

$$(f,g) = \int_{-\infty}^{\infty} (f_e^*, f_o^*) \begin{pmatrix} g_e \\ g_o \end{pmatrix} dx$$

$$= \int_{-\infty}^{\infty} (f_e^* g_e + f_o^* g_o) dx \ .$$

Then

$$(f,\eta g) = - \int (f_e^*, f_o^*) \begin{pmatrix} 0 & x - d_x + \frac{\tau}{x} \\ x - d_x - \frac{\tau}{x} & 0 \end{pmatrix} \begin{pmatrix} g_e \\ g_o \end{pmatrix} dx$$

$$= - \int \Big[\left\{ \left(x + d_x + \tfrac{\tau}{x}\right) f_e^* \right\} g_o +$$

$$\left\{ \left(x + d_x - \tfrac{\tau}{x}\right) f_o^* \right\} g_e \Big] \ dx$$

and

$$(\xi f,g) = - \int (f_e^*, f_o^*) \begin{pmatrix} 0 & x + \overleftarrow{d}_x + \frac{\tau}{x} \\ x + \overleftarrow{d}_x - \frac{\tau}{x} & 0 \end{pmatrix} \begin{pmatrix} g_e \\ g_o \end{pmatrix} dx$$

$$= - \int \Big\{ f_e^* \left(x + \overleftarrow{d}_x + \tfrac{\tau}{x}\right) g_o +$$

$$f_o^* \left(x + \overleftarrow{d}_x - \tfrac{\tau}{x}\right) g_e \Big\} \ dx \ .$$

Thus

$$(f,\eta g) = (\xi f,g)$$

and similarly

$$(f,\xi g) = (\eta f,g)$$

so that η and ξ are adjoint operators.

The state of lowest energy is defined by

$$\xi\psi_0 = 0 \; .$$

On substituting for ξ and putting $\psi_0 = \exp\left(-\frac{x^2}{2}\right) g(x)$, g satisfies

$$g'(x) + \frac{\tau}{x} g(-x) = 0 \; .$$

This is the real form of the equation for the vacuum state of the Bargmann space. The solutions split into two cases

(a) p odd:

$$\psi_0 = \frac{1}{\sqrt{\Gamma(\alpha + \frac{1}{2})}} x^\alpha \exp\left(-\frac{x^2}{2}\right)$$

where $p = 2\alpha + 1$, and

(b) p even:

$$\psi_0 = \frac{1}{\sqrt{\Gamma(\alpha + 1)}} \sqrt{|x|}\, x^\alpha \exp\left(-\frac{x^2}{2}\right)$$

where $p = 2\alpha + 2$.

The coefficients α and τ are again related by

$\tau = (-1)^{\alpha+1}\alpha$ for p odd and $\tau = (-1)^{\alpha+1}(\alpha + \frac{1}{2})$ for p even.

§4.3.2 Schrödinger Equation

The raising and lowering operators η and ξ are analogous to the creation and annihilation operators a^* and a respectively. In a similar manner to that used for the Fock representations Yang has constructed a representation of the para-bose algebra by applying powers of η to ψ_0; the "state of lowest energy". An alternative method for constructing the representation space is to observe that the operator

$$H = \tfrac{1}{2}[\eta, \xi]_+ - \frac{p}{2}$$

satisfies

$$[H, \eta]_- = \eta$$

and so H can be interpreted as counting the powers of η in an arbitrary state, since $H\psi_0 = 0$. Thus the eigenstates of H form a representation of the parabose algebra. Defining $H' = H + \frac{p}{2}$ and substituting for η and ξ in terms of p and q gives

$$H' = \tfrac{1}{2}(q^2 + p^2)$$

which is the Schrödinger equation for the one-dimensional parabose harmonic oscillator in units of $m = \omega = \hbar = 1$.

After a further substitution for p and q in terms of x, d_x and R, the eigenvalue equation

$$H\psi_\lambda = \lambda\psi_\lambda$$

becomes

$$\frac{d^2\psi_\lambda}{dx^2} + \left\{\lambda' - x^2 - \frac{\tau(\tau + R)}{x^2}\right\}\psi_\lambda = 0 \tag{4.13}$$

where $\lambda' = 2\lambda + p$.

Since $[H, R]_- = 0$ it follows that H and R form a complete set of commuting operators. The eigenstates of H can thus be classified according to their parity and are denoted by $\psi_\pm$ according to the equation

$$R\psi_{\pm\lambda} = \pm\psi_{\pm\lambda} .$$

The difference equation (4.13) is then reduced to a 2^{nd} order differential equation, the solution of which will be considered separately for p odd and p even.

(a) p odd

(a.1) Solution for even functions

The differential equation (4.13) becomes

$$\frac{d^2\psi_{+\lambda}}{dx^2} + \left(\lambda' - x^2 - \frac{\tau(\tau + 1)}{x^2}\right)\psi_{+\lambda} = 0 \qquad (4.14)$$

which can be recognized as the differential equation for the radial component of the three dimensional harmonic oscillator[37].

On substituting

$$\psi_{+\lambda} = x^{\tau+1} e^{-\frac{x^2}{2}} y_\lambda(x^2) \ ,$$

$y_\lambda(x)$ satisfies

$$\frac{xd^2y_\lambda(x)}{dx^2} + \left(\tau + \frac{3}{2} - x\right) \frac{dy_\lambda(x)}{dx} + \frac{1}{4} (\lambda' - 2\tau - 3)y_\lambda(x) = 0$$

which is Kummer's form of the confluent hypergeometric differential equation. To ensure reasonable behaviour at infinity the series must be terminated i.e.,

$$\frac{1}{4} (\lambda' - 2\tau - 3) = n \ .$$

As for the Bose case, this restricts (quantizes) the eigenvalues of H to the form

$$\lambda^{(1)} = \tfrac{1}{2}\{4n + 2\tau + 3 - p\} \ .$$

The solution of (4.14) is then

$$\psi^{(1)}_{+n}(x) = x^{\tau+1}\, e^{-\frac{x^2}{2}}\, L_n^{\tau+\frac{1}{2}}(x^2) \ .$$

It is readily checked that a second solution of (4.14) is

$$\psi^{(2)}_{+n}(x) = x^{-\tau}\, e^{-\frac{x^2}{2}}\, L_n^{-\tau-\frac{1}{2}}(x^2)$$

provided the eigenvalues of H are quantized according to

$$\lambda^{(2)} = \tfrac{1}{2}\{4n - 2\tau + 1 - p\} \ .$$

From Sansone[67] the Laguerre functions $L_n^{\nu}(x)$ have sensible behaviour around zero provided $\nu>-1$. This imposes a restriction on the combinations of $\psi^{(1)}_{+n}$ an $\psi^{(2)}_{+n}$ which are admissible. This restriction is surprisingly, identical to the requirement that the particular combination is an even function of x. Since $\tau = (-1)^{\alpha+1}\alpha$ the even solutions are for α even:

$$\psi_{+n}(x) = x^{\alpha}\, e^{-\frac{x^2}{2}}\, L_n^{\alpha-\frac{1}{2}}(x^2)$$

and for α odd:

$$\psi_{+n}(x) = x^{\alpha+1}\, e^{-\frac{x^2}{2}}\, L_n^{\alpha+\frac{1}{2}}(x^2)$$

with eigenvalues 2n and 2n+1 respectively.

(a.2) <u>Odd solutions</u>

The odd solutions can be simply derived from the even ones by noting that the differential equation for the odd solutions,

$$\frac{d^2\psi_{-\lambda}}{dx^2} + \left(\lambda' - x^2 - \frac{\tau(\tau - 1)}{x^2}\right)\psi_{-\lambda} = 0 ,$$

can be derived from (4.14) by the substitution $\tau \to \tau - 1$. Under the substitution $\tau \to \tau - 1$, the requirement of positive definiteness of the energy eigenvalues and odd parity of the eigenfunctions the solutions are for α even:

$$\psi_{-n}(x) = x^{\alpha+1} e^{-\frac{x^2}{2}} L_n^{\alpha+\frac{1}{2}}(x^2)$$

and for α odd:

$$\psi_{-n}(x) = x^{\alpha} e^{-\frac{x^2}{2}} L_n^{\alpha-\frac{1}{2}}(x^2) ,$$

with the eigenvalues of H being 2n+1 and 2n respectively.

If the even and odd solutions for a particular value of α are ordered with respect to the magnitude of the eigenvalues of H then, denoting ψ_m^α as the eigenvector corresponding to the m^{th} eigenvalue, for any value of α

$$\psi_{2n}^\alpha = x^\alpha e^{-\frac{x^2}{2}} L_n^{\alpha-\frac{1}{2}}(x^2)$$

and

$$\psi_{2n+1}^\alpha = x^{\alpha+1} e^{-\frac{x^2}{2}} L_n^{\alpha+\frac{1}{2}}(x^2) .$$

For n = 0 the "state of lowest energy" as calculated in §4.3.1 is obtained since $L_0^\nu(x^2) = 1$. The Bose case, once again, may be obtained by substituting $\alpha = 0$ and noting that

$$H_{2n}(x) = (-1)^n \, 2^{2n} \, n! \, L_n^{-\frac{1}{2}}(x^2)$$

and

$$H_{2n+1}(x) = (-1)^n \, 2^{2n+1} \, n! \, L_n^{\frac{1}{2}}(x^2)$$

where H_ν denotes Hermite polynomials.

(b) Solutions for p even

The solution of (4.13) for p even is very similar to that for p odd and so only the important distinctions will be considered.

(b.1) Even Solutions

A solution of (4.14) is:

$$\psi'_{+n}(x) = x^{\tau+\frac{1}{2}} \sqrt{x} \, e^{-\frac{x^2}{2}} \, L_n^{\tau+\frac{1}{2}}(x^2)$$

where $\tau = (-1)^{\alpha+1}(\alpha + \frac{1}{2})$. Now since $[R,H]_- = 0$ it follows that

$$\psi''_{+n}(x) = x^{\tau+\frac{1}{2}} \sqrt{-x} \, e^{-\frac{x^2}{2}} \, L_n^{\tau+\frac{1}{2}}(x^2)$$

is also a solution. These two solutions may be combined so that

$$\psi^{(1)}_{+n}(x) = \begin{matrix} \psi'_{+n}(x) & \quad x > 0 \\ \psi''_{+n}(x) & \quad x < 0 \end{matrix}$$

which may be written as

$$\psi^{(1)}_{+n}(x) = x^{\tau+\frac{1}{2}} \sqrt{|x|} \, e^{-\frac{x^2}{2}} \, L_n^{\tau+\frac{1}{2}}(x^2) \ .$$

Considering a second solution:

$$\psi^{(2)}_{+n}(x) = x^{-\tau+\frac{1}{2}} \sqrt{|x|}\; e^{-\frac{x^2}{2}}\; L_n^{-\tau+\frac{1}{2}}(x^2)$$

and requiring either of the three conditions

(a) even parity under R,

(b) sensible behaviour at the origin

or

(c) positive definiteness of the spectrum of H

implies that the solutions are for α even:

$$\psi_{+n}(x) = x^{\alpha} \sqrt{|x|}\; e^{-\frac{x^2}{2}}\; L_n^{\alpha}(x^2)$$

and for α odd:

$$\psi_{+n}(x) = x^{\alpha+1} \sqrt{|x|}\; e^{-\frac{x^2}{2}}\; L_n^{\alpha+1}(x^2)\;.$$

(b.2) <u>Odd Solutions</u>

Substituting $\tau \to \tau - 1$ the odd solutions are for α even:

$$\psi_{-n}(x) = x^{\alpha+1} \sqrt{|x|}\; e^{-\frac{x^2}{2}}\; L_n^{\alpha+1}(x^2)$$

and for α odd:

$$\psi_{-n}(x) = x^{\alpha} \sqrt{|x|}\; e^{-\frac{x^2}{2}}\; L_n^{\alpha}(x^2)\;.$$

This may be written in the compact form

$$\psi^{\alpha}_{2n} = x^{\alpha} \sqrt{|x|}\; e^{-\frac{x^2}{2}}\; L_n^{\alpha}(x^2)$$

and

$$\psi^{\alpha}_{2n+1} = x^{\alpha+1} \sqrt{|x|}\; e^{-\frac{x^2}{2}}\; L_n^{\alpha}(x^2)$$

where the ψ's are eigenvectors of H corresponding to eigenvalues 2n and 2n+1 respectively.

§4.3.3 A Complete Orthonormal Basis

It follows from construction that the even and odd functions are orthogonal. The normalization is different for ψ^{α}_{2n} and ψ^{α}_{2n+1} and must be considered separately.

(a) p odd

(1) $(\psi^{\alpha}_{2n}(x), \psi^{\alpha}_{2m}(x))$

$$= \int_{-\infty}^{\infty} x^{2\alpha} e^{-x^2} L_n^{\alpha-\frac{1}{2}}(x^2) L_m^{\alpha-\frac{1}{2}}(x^2)dx$$

$$= \delta_{mn} \frac{\Gamma(n + \alpha + \frac{1}{2})}{n!} .$$

(2) $(\psi^{\alpha}_{2n+1}(x), \psi^{\alpha}_{2m+1}(x))$

$$= \int_{-\infty}^{\infty} x^{2\alpha+2} e^{-x^2} L_n^{\alpha+\frac{1}{2}}(x^2) L_m^{\alpha+\frac{1}{2}}(x^2)$$

$$= \delta_{mn} \frac{\Gamma(n + \alpha + \frac{3}{2})}{n!}$$

where use has been made of the orthogonality relations for Laguerre polynomials[66]. The orthonormal set of states are

$$\phi^{\alpha}_{2n}(x) = (-1)^n \left[\frac{n!}{\Gamma(n + \alpha + \frac{1}{2})}\right]^{\frac{1}{2}} x^{\alpha} e^{-\frac{x^2}{2}} L_n^{\alpha-\frac{1}{2}}(x^2)$$

and

$$\phi^{\alpha}_{2n+1}(x) = (-1)^n \left[\frac{n!}{\Gamma(n + \alpha + \frac{3}{2})}\right]^{\frac{1}{2}} x^{\alpha+1} e^{-\frac{x^2}{2}} L_n^{\alpha+\frac{1}{2}}(x^2) .$$

A factor $(-1)^n$ has been included to ensure that

$$\phi_m^{\alpha}(x) = \frac{1}{\sqrt{I_m}} (\eta)^m \, \phi_o^{\alpha}(x),$$

where ϕ_o^{α} is the state of lowest energy. The $(-1)^n$ obviously does not affect the norm of the state.

(b) p even

(1) $(\psi_{2n}^{\alpha}(x), \psi_{2m}^{\alpha}(x))$

$$= \int_{-\infty}^{\infty} x^{2\alpha} \, |x| \, e^{-x^2} \, L_n^{\alpha}(x^2) \, L_m^{\alpha}(x^2) dx$$

$$= \delta_{mn} \frac{\Gamma(n + \alpha + 1)}{n!} .$$

(2) $(\psi_{2n+1}^{\alpha}(x), \psi_{2m+1}^{\alpha}(x))$

$$= \delta_{mn} \frac{\Gamma(n + \alpha + 2)}{n!} .$$

The orthonormal basis set is

$$\phi_{2n}^{\alpha}(x) = (-1)^n \left[\frac{n!}{\Gamma(n + \alpha + 1)}\right]^{\frac{1}{2}} x^{\alpha} \sqrt{|x|} \, e^{-\frac{x^2}{2}} \, L_n^{\alpha}(x^2)$$

and

$$\phi_{2n+1}^{\alpha}(x) = (-1)^n \left[\frac{n!}{\Gamma(n + \alpha + 2)}\right]^{\frac{1}{2}} x^{\alpha+1} \sqrt{|x|} \, e^{-\frac{x^2}{2}} \, L_n^{\alpha+1}(x^2) .$$

The Laguerre polynomials are a type of orthogonal polnomials and the completeness of these functions is proved in any text(67) on orthogonal functions. Thus for both p even and p odd any square integrable function may be expanded in terms of the complete set of basis states ϕ_{2n}^{α}

and ϕ^{α}_{2n+1}. The space spanned by these functions is denoted H and convergence in this space is not absolute but only convergence in the mean. This however is no objection.

§4.4 EQUIVALENCE OF THE BARGMANN AND HARMONIC OSCILLATOR REPRESENTATIONS

§4.4.1 An Integral Mapping

Since both the Bargmann and harmonic oscillator space are representations of the parabose algebra it immediately follows from general theory that they are unitarily equivalent. The mapping from H onto F is in terms of an integral transform:

$$f(z) = \int A(z,x)\psi(x)dx \ ,$$

where $A(z,x)$ is the kernel of the integral transform to be determined. Following Bargmann's analysis $A(z,x)$ can be found by the requirement that $\eta\psi$ is mapped into a^*f and $\xi\psi$ into af.

The most general form of $A(z,x)$ as a 2 × 2 matrix operator is

$$\begin{pmatrix} A_{11}(z,x) & A_{12}(z,x) \\ A_{21}(z,x) & A_{22}(z,x) \end{pmatrix}.$$

The mapping then requires that

$$\begin{aligned} a^*f &= \int (zA)\psi \ dx \\ &= \int A(\vec{\eta}\psi) \ dx \\ &= \int (A\overleftarrow{\eta})\psi \ dx \end{aligned}$$

where, for convenience, the arguments have been dropped and the arrow over η indicates the direction in which the d_x acts. Using the matrix representations the above equations determining A can be written

$$\begin{pmatrix} (x + d_x - \frac{\tau}{x})A_{12} & (x + d_x + \frac{\tau}{x})A_{11} \\ (x + d_x - \frac{\tau}{x})A_{22} & (x + d_x + \frac{\tau}{x})A_{21} \end{pmatrix} = -\sqrt{2}\begin{pmatrix} zA_{21} & zA_{22} \\ zA_{11} & zA_{12} \end{pmatrix} \tag{4.15}$$

Similarly

$$\begin{aligned} af &= \int \left(\frac{dA}{dz} + \frac{\tau}{z} RA\right)\psi dx \\ &= \int A(\overrightarrow{\xi}\psi)dx \\ &= \int (A\overleftarrow{\xi})\psi dx , \end{aligned}$$

which in matrix notation becomes

$$\begin{pmatrix} (x - d_x + \frac{\tau}{x})A_{12} & (x - d_x - \frac{\tau}{x})A_{11} \\ (x - d_x + \frac{\tau}{x})A_{22} & (x - d_x - \frac{\tau}{x})A_{21} \end{pmatrix}$$

$$= -\sqrt{2}\begin{pmatrix} (d_z - \frac{\tau}{z})A_{21} & (d_z - \frac{\tau}{z})A_{22} \\ (d_z + \frac{\tau}{z})A_{11} & (d_z + \frac{\tau}{z})A_{12} \end{pmatrix} . \tag{4.16}$$

From Appendix 6 the solutions of the partial differential equations for the diagonal elements are:

$$A_{\alpha-\frac{1}{2}}(z,x) = C_{\alpha-\frac{1}{2}}(\sqrt{2}zx)^{\frac{1}{2}}\, e^{-\frac{1}{2}(x^2+z^2)} \quad J_{\alpha-\frac{1}{2}}(i\sqrt{2}\,zx) \qquad (4.17a)$$

and

$$A_{\alpha+\frac{1}{2}}(z,x) = c_{\alpha+\frac{1}{2}}(\sqrt{2}\,zx)^{\frac{1}{2}}\, e^{-\frac{1}{2}(x^2+z^2)} \quad J_{\alpha+\frac{1}{2}}(i\sqrt{2}\,zx)\ . \qquad (4.17b)$$

The notation has been simplified here by denoting A_{ii} by $A_\nu(z,x)$; the integral transform which maps states of parity $(-1)^{\nu+\frac{1}{2}}$ in H onto states of the same parity in F. It is also shown in Appendix 6 that the off-diagonal elements are zero. $J_\nu(v)$ is the standard Bessel function and

$$c_\nu = \frac{(-1)^{-\frac{\nu}{2}}\pi^{\frac{1}{4}}}{\sqrt{2}}\ .$$

(b) <u>p even</u>

From Appendix 6:

$$A_\alpha(z,x) = c'_\alpha(\sqrt{2}\,z\bar{z}\ |x|)^{\frac{1}{2}}\, e^{-\frac{1}{2}(x^2+z^2)} \quad J_\alpha(i\sqrt{2}\,zx) \qquad (4.18a)$$

is the mapping for states of parity $(-1)^\alpha$ and

$$A_{\alpha+1}(z,x) = c'_{\alpha+1}\ (\sqrt{2}\,z\bar{z}\ |x|)^{\frac{1}{2}}\, e^{-\frac{1}{2}(x^2+z^2)} \quad J_{\alpha+1}(i\sqrt{2}\,zx) \qquad (4.18b)$$

is the mapping for states of parity $(-1)^{\alpha+1}$. The constant

$$c'_\nu = (-1)^{-\frac{\nu}{2}}\pi^{\frac{1}{4}}\ .$$

Similarly to p odd, the integral transform is diagonal.

§4.4.2 Unitarity of the Integral Transform

The equivalence of the two representations requires $A(z,x)$ be unitary. This is equivalent to the following conditions.

4A: $$\int A_\nu(z,x)\,\overline{A_\nu}\,(w,x)dx = I^{\pm}(zw),$$

where $I^{\pm}(zw)$ is the representation of the unit element in even or odd subspaces of F.

4B: $$\int A_\nu(z,x)\,\overline{A_\nu}\,(z,y)d\rho_\nu(z\bar{z}) = \delta^{\pm}(x-y)$$

where $\delta^{\pm}(x-y)$ is the decomposition of the δ-function into its even or odd components. Since it is more convenient to work in terms of well-defined integrals 4B is replaced by

4B′: $$\lim_{\lambda\to 1}\int A_\nu(\lambda z,x)\bar{A}_\nu(\lambda z,y)d\rho_\nu(z\bar{z}) = \delta^{\pm}(x-y)\ .$$

(1) Evaluation of 4A

(a) p odd

Substituting either (4.17a) or (4.17b) for $A_\nu(z,x)$ then

$$\int A_\nu(z,x)\bar{A}_\nu(w,x)dx$$

becomes

$$\sqrt{2}\ c_\nu\ \bar{c}_\nu \int_{-\infty}^{\infty} (z\bar{w})^{\frac{1}{2}}\ e^{-\frac{1}{2}(z^2+\bar{w}^2)}x\ e^{-x^2}\ J_\nu(i\sqrt{2}\ zx)$$
$$\times\ J_\nu(-\ i\sqrt{2}\ \bar{w}x)dx\ .$$

This equals

$$\sqrt{\frac{\pi}{2}}\ (z\bar{w})^{\frac{1}{2}}\ I_\nu(z\bar{w})$$

which is $I^{+}(zw)$ for $\nu = \alpha + \frac{1}{2}$ and $I^{-}(zw)$ for $\nu = \alpha - \frac{1}{2}$.
The identity[66]

$$\int_0^\infty J_\nu(\alpha t)\, J_\nu(\beta t)\, e^{-\gamma^2 t^2}\, dt$$

$$= \tfrac{1}{2}\gamma^{-2} \exp\left[-\tfrac{1}{4}\gamma^{-2}(\alpha^2+\beta^2)\right] I_\nu(\tfrac{1}{2}\alpha\beta\gamma^{-2})$$

has been used. This identity is valid provided $\mathrm{Re}\,\nu > -1$ and $\mathrm{Re}\,\gamma^2 > 0$. The choice $\alpha = -i\sqrt{2}\,z$, $\beta = i\sqrt{2}\,\bar{w}$, $\nu = \alpha \pm \frac{1}{2}$ and $\gamma = 1$ ensures that these inequalities are satisfied.

Since $I^{\pm}(zw)$ are the unit elements calculated in §4.2.5 condition 4A has been satisfied.

(b) p even

Substituting either (4.18a) or (4.18b) in

$$\int A_\nu(z,x)\, \bar{A}_\nu(w,x)\, dx$$

the integral becomes

$$\sqrt{2}\, c_\nu'\, \bar{c}_\nu' \int_{-\infty}^{\infty} (z\bar{z}\, w\bar{w})^{\frac{1}{2}}\, e^{-\frac{1}{2}(z^2+\bar{w}^2)}\, |x|\, e^{-x^2}$$

$$\times\, J_\nu(i\sqrt{2}\, zx)\, J_\nu(-i\sqrt{2}\,\bar{w}x)\, dx.$$

Since $J_\nu(-z) = (-1)^\nu J_\nu(z)$, the factor $\sqrt{|x|}$ (and not $\sqrt{x}$) in the expression for $A(z,x)$ ensures that the above integral is non-zero. Evaluation of the integrals in a similar manner to p odd gives

$$\int A_\nu(z,x)\, \bar{A}_\nu(w,x)\, dx = \sqrt{2\pi}\, (z\bar{z}\, w\, \bar{w})^{\frac{1}{2}} \quad I_\nu(z\bar{w}) = I^{\pm}(zw) \;,$$

which are the appropriate reproducing kernels for p even.

(2) Evaluation of $4B'$

In Appendix 7 it is shown that $\sigma_\nu(\lambda,x,y)$, defined by

$$\sigma_\nu(\lambda,x,y) = \int A_\nu(\lambda z,x)\, A_\nu(\lambda\bar{z},y)\, d\rho_\nu(z\bar{z}) \ ,$$

has the form

$$\frac{\lambda}{1-\lambda^4}\sqrt{xy}\; e^{-\frac{1}{2}(x^2+y^2)}\; e^{-\frac{\lambda^4(x^2+y^2)}{1-\lambda^4}}\; I_\nu\left(\frac{2\lambda^2 xy}{1-\lambda^4}\right)$$

for $\nu = \alpha \pm \frac{1}{2}$. Since the expression for σ_ν for $\nu = \alpha$ or $\alpha + 1$ is the above multiplied by a factor λ, and since $\lambda \to 1$, only the above expression need be discussed.

By inspection the argument of I_ν approaches infinity as $\lambda \to 1$. An asymptotic expansion of $I_\nu(z)$ is[66]:

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\left\{e^z \sum_{m=0}^{M-1} (-1)^m (\nu,m)(2z)^{-m} + i\, e^{-z+i\nu\pi} \sum_{m=0}^{M-1} (\nu,m)(2z)^{-m}\right\} .$$

Considering only the first order terms in the expansion the asymptotic form of $I_\nu(z)$ will be $\frac{1}{\sqrt{2\pi z}}e^z$ as $z \to \infty$ and $\frac{i}{\sqrt{2\pi z}}e^{-z+i\nu\pi}$ as $z \to -\infty$. Since in a domain where one of these is very large the other is very small it is convenient to incorporate both forms in the one expression:

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\{e^z + ie^{-z+i\nu\pi}\} ,$$

with the understanding that only the dominant term is considered in each case. As

$$z = \frac{2\lambda^2 xy}{1 - \lambda^4}$$

it follows that

$$\lim_{\lambda\to 1} \sigma_\nu(\lambda,x,y)$$

$$= \lim_{\lambda\to 1} \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{1 - \lambda^4}} \exp \{- \tfrac{1}{2}(x^2 + y^2) - \frac{\lambda^4(x^2 + y^2)}{1 - \lambda^4}\}$$

$$\times \{\exp \frac{2\lambda^2 xy}{1 - \lambda^4} + i(- 1)^\nu \exp (- \frac{2\lambda^2 xy}{1 - \lambda^4})\}$$

$$= \tfrac{1}{2} \lim_{\lambda\to 1} (\frac{1}{\sqrt{\pi(1 - \lambda^4)}} \exp \{- \tfrac{1}{4}\left[\frac{1 - \lambda^2}{1 + \lambda^2} (x + y)^2\right.$$

$$\left. + \frac{1 + \lambda^2}{1 - \lambda^2} (x - y)^2\right]\} + i(- 1)^\nu \frac{1}{\sqrt{\pi(1 - \lambda^2)}} \exp \{- \tfrac{1}{4}\left[\right.$$

$$\left.\frac{1 - \lambda^2}{1 + \lambda^2} (x - y)^2 + \frac{1 + \lambda^2}{1 - \lambda^2} (x + y)^2\right]\})$$

$$= \tfrac{1}{2} \lim_{\varepsilon\to 0} (\{(1 + \varepsilon^2)\, e^{-\varepsilon^2 s^2}\} \{(2\varepsilon\sqrt{\pi})^{-1}\, e^{-\, t^2/\varepsilon^2}\}$$

$$+ i(- 1)^\nu \{(1 + \varepsilon^2)\, e^{-\varepsilon^2 t^2}\} \{(2\varepsilon\sqrt{\pi})^{-1}\, e^{-\, s^2/\varepsilon^2}\})$$

where

$$\varepsilon = (\frac{1 - \lambda^2}{1 + \lambda^2})^{\frac{1}{2}} \quad , \quad s = \tfrac{1}{2}(x + y) \quad \text{and} \quad t = \tfrac{1}{2}(x - y).$$

As discussed by Bargmann, the first term approaches the one-dimensional delta functions; $\delta(x - y)$. Similarly the second term approaches $\delta(x + y)$. Substituting $\nu = \alpha \pm \tfrac{1}{2}$ gives

$$\sigma_{\alpha-\frac{1}{2}}(x,y) = \tfrac{1}{2}\{\delta(x - y) + (-1)^{\alpha}\, \delta(x + y)\}$$

and

$$\sigma_{\alpha+\frac{1}{2}}(x,y) = \tfrac{1}{2}\{\delta(x - y) - (-1)^{\alpha}\, \delta(x + y)\} ,$$

where the definition $\sigma_\nu(x,y) = \lim\limits_{\lambda\to1} \sigma_\nu(\lambda,x,y)$ is used. Since $\sigma_{\alpha-\frac{1}{2}}(x,y) + \sigma_{\alpha+\frac{1}{2}}(x,y) = \delta(x - y)$ the above expressions are just the decomposition of $\delta(x - y)$ into its appropriate representation in the spaces of even and odd functions.

There is only a slight modification involved in extending the above argument to p even. As $\lambda \to 1$, $\sigma_\nu(\lambda,x,y)$ approaches

$$\frac{1}{2\sqrt{\pi}}\,\frac{1}{\sqrt{1 - \lambda^4}} \exp\left\{-\tfrac{1}{2}(x^2 + y^2) - \frac{\lambda^4(x^2 + y^2)}{1 - \lambda^4}\right\} \sqrt{\frac{|xy|}{xy}}$$

$$\times \left\{\exp\left(\frac{2\lambda^2 xy}{1 - \lambda^4}\right) + i(-1)^{\nu} \exp\left(\frac{-2\lambda^2 xy}{1 - \lambda^4}\right)\right\} .$$

For $xy > 0$ the first term in the second bracket dominates and so $\sigma_\nu(\lambda,x,y) \to \tfrac{1}{2}\delta(x - y)$ since $\sqrt{\frac{|xy|}{xy}} = 1$. On the other hand for $xy < 0$ the second term dominates and hence $\sigma_\nu(\lambda,x,y) \to \tfrac{1}{2}(-1)^{\nu}\, \delta(x + y)$ since $\sqrt{\frac{|xy|}{xy}} = \frac{1}{i}$, and this cancels the extra i factor. Substituting $\nu = \alpha$ and $\alpha + 1$, and combining the results it follows that

$$\sigma_{\alpha}(x,y) = \tfrac{1}{2}\{\delta(x - y) + (-1)^{\alpha}\, \delta(x + y)\}$$

and

$$\sigma_{\alpha+1}(x,y) = \tfrac{1}{2}\{\delta(x - y) - (-1)^{\alpha}\, \delta(x + y)\} .$$

Once again this is the required decomposition of the δ-function into the appropriate even and odd functions.

It is easily checked that

$$\sigma_\nu(x,y) = \sum_m \phi_m^\nu(x)\ \phi_m^\nu(y)$$

where for $\nu = \alpha - \frac{1}{2}$ or α the summation is over m even and for $\nu = \alpha + \frac{1}{2}$ or $\alpha + 1$ the summation is over m odd. For m even

$$\phi_{2m}^\nu(-\ x) = (-\ 1)^\alpha\ \phi_{2m}^\nu(x)$$

and so

$$\sigma_\nu(-\ x,y) = (-\ 1)^\alpha\ \sigma_\nu(x,y)$$

for the appropriate values of ν. This is a verification of the decomposition of $\sigma_\nu(x,y)$ into even or odd components.

§4.4.3 The Connection between Basis States of F and H

From the unitarity of $A(z,x)$ an explicit connection between the basis states of F and H can be derived.

a) p odd

From the expansion

$$I_\nu(b) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}b)^{\nu+2m}}{m!\Gamma(\nu + m + 1)}$$

it follows that

$$\frac{d^m}{d(b^2)^m}\left(2^\nu b^{-\nu} I_\nu(zb)\right)\bigg|_{b=0} = \frac{z^{2m}}{2^{2m}\Gamma(\nu + m + 1)} \quad .$$

But

$$I_\nu(zb) = \sqrt{\frac{2}{\pi}}\,(zb)^{-\frac{1}{2}} \int A_\nu(z,x)\, A_\nu(b,x)dx$$

from 3A.

Thus

$$z^{\nu+2m} = \sqrt{\frac{2}{\pi}}\; 2^{2m+\nu}\; \Gamma(\nu + m + 1) \int z^{-\frac{1}{2}}$$

$$\times\, A_\nu(zx)\, \{\frac{d^m}{d(b^2)^m}\, b^{-\nu-\frac{1}{2}}\, A_\nu(b,x)\}\Big|_{b=0} dx$$

$$= \sqrt{\frac{2}{\pi}}\, 2^{2m+\nu}\; \Gamma(\nu + m + 1) \int z^{-\frac{1}{2}}\, A_\nu(z,x)\; \tilde{\phi}(x)\; dx$$

where

$$\tilde{\phi}(x) = \frac{d^m}{d(b^2)^m}\, \{b^{-\nu-\frac{1}{2}}\, A_\nu(b,x)\}\Big|_{b=0}$$

$$= c_\nu\; 2^{\frac{1}{4}}\, x^{\frac{1}{2}}\, e^{-\frac{1}{2}x^2}\; \frac{d^m}{d(b^2)^m}\, \{b^{-\nu}\; e^{-\frac{1}{2}b^2}$$

$$\times\, J_\nu(i\sqrt{2}\, bx)\}\Big|_{b=0}$$

$$= \frac{\pi^{\frac{1}{4}}\; n!}{\Gamma(n + \nu + 1)}\; \frac{1}{2^{n + \frac{\nu}{2} + \frac{1}{4}}}\; (-\, 1)^n\; x^{\nu+\frac{1}{2}}\; e^{-\frac{1}{2}x^2}$$

$$\times\, L_n^\nu(x^2)\; ,$$

where the last line follows from the m differentiations of the modified generating relation for Laguerre polynomials[66]:

$$\sum_{n=0,1,\ldots} \frac{1}{\Gamma(n + \nu + 1)}\; \frac{L_n^\nu(x^2)z^{2n}(-\, 1)^n}{2^n}$$

$$= e^{-\frac{z^2}{2}}\; (xz)^{-\nu}\; 2^{\frac{\nu}{2}}\; (-\, 1)^{-\frac{\nu}{2}}\; J_\nu(i\sqrt{2}\; xz) \qquad (4.19)$$

Rearranging coefficients and substituting $\nu = \alpha - \frac{1}{2}$ it follows that

$$u^{\alpha+2m} = \int A_{\alpha-\frac{1}{2}}(z,x)\ \phi_{2m}(x)dx$$

where $u^{\alpha+2m}$ and ϕ_{2m} are the orthonormal basis states of F and H respectively. Similarly for $\nu = \alpha + \frac{1}{2}$;

$$u^{\alpha+2m+1} = \int A_{\alpha+\frac{1}{2}}(z,x)\ \phi_{2m+1}(x)dx\ .$$

As a particular example take

$$f(z) = \int_{-\infty}^{\infty} A_{\alpha-\frac{1}{2}}(z,x)\ \phi_o(x)dx$$

$$= \frac{2c_{\alpha-\frac{1}{2}}(\sqrt{2}\ z)^{\frac{1}{2}}\ e^{-\frac{1}{2}z^2}}{\{\Gamma(\alpha+\frac{1}{2})\}^{\frac{1}{2}}} \int_0^{\infty} x^{\alpha+\frac{1}{2}}\ e^{-x^2}\ J_{\alpha-\frac{1}{2}}\ (i\sqrt{2}\ zx)dx$$

Now(66)

$$\int_0^{\infty} J_\mu(\bar{\alpha}t)\ e^{-\gamma^2t^2}\ t^{\mu+1}\ dt$$

$$= (\bar{\alpha})^\mu\ (2\gamma^2)^{-\mu-1}\ \exp\ (-\ \tfrac{1}{4}\ \bar{\alpha}^2\gamma^{-2})\ .$$

Substituting $\mu = \alpha - \frac{1}{2}$, $\bar{\alpha} = i\sqrt{2}\ z$, $\gamma = 1$ then it follows that

$$f(z) = \frac{\pi^{\frac{1}{4}}}{\{\Gamma(\alpha+\frac{1}{2})\}^{\frac{1}{2}}}\ z^\alpha$$

which is the vacuum state of the Fock space for p odd.

(b) <u>p even</u>

From 4A,

$$I_\nu(zb) = (2\pi)^{-\frac{1}{2}}\ (z\bar{z}\ b\bar{b})^{-\frac{1}{2}} \int_{-\infty}^{\infty} A_\nu(z,x)\ A_\nu(b,x)dx$$

and so

$$z^{\nu+2m} = (2\pi)^{-\frac{1}{2}} 2^{2m+\nu} \Gamma(\nu + m + 1) \int_{-\infty}^{\infty} (z\bar{z})^{-\frac{1}{2}}$$

$$\times A_\nu(z,x) \left\{\frac{d^m}{d(b^2)^m} (\sqrt{b\bar{b}}\, b^\nu)^{-1} A_\nu(b,x)\right\}\Big|_{b=0} dx .$$

Substituting (4.18) for $A_\nu(z,x)$, differentiating (4.19) and collecting coefficients implies for $\nu = \alpha$,

$$u^{\alpha+2m} = \int A_\alpha(z,x)\, \phi_{2m}^{\alpha}(x)dx ,$$

which is the required connection between basis states. Similarly it follows that

$$u^{\alpha+2m+1} = \int A_{\alpha+1}(z,x)\, \phi_{2m+1}^{\alpha}(x)dx .$$

If using Dirac's bra and ket notation(68), the orthonormal basis states of F are denoted by $|u^m\rangle$ and those of H are denoted by $|x^m\rangle$ then equation (4.19) implies that

$$A(z,x) = \sum_n |u^n\rangle \langle x^n| .$$

The unitarity of $A(z,x)$ is implied by the orthonomality of the respective states and relations of the form

$$u^{\alpha+m} = \int A(z,x)\, \phi_m^{\alpha}(x)dx$$

can be expressed as

$$|u^m\rangle = \sum_n |u^n\rangle \langle x^n|x^m\rangle .$$

The validity of the above equation also follows from the orthonormality of the basis vectors.

§4.5 CONCLUSIONS

Two representation spaces for a single parabose operator have been constructed and their equivalence has been proved.

An obvious generalization of this is to construct the corresponding representations of an arbitrary number, ν, of operators. To effect this it is necessary to consider, in the appropriate representation space, functions of ν variables. It is not sufficient, unfortunately, to generalize the creation operators $\overset{*}{a}_1, \overset{*}{a}_2, \ldots, \overset{*}{a}_\nu$ to $z_1, z_2, \ldots, z_\nu$ respectively because this would imply $[\overset{*}{a}_i, \overset{*}{a}_j]_- = 0$ which only holds for $p = 1$. In order to satisfy $[[\overset{*}{a}_i, \overset{*}{a}_j]_+, \overset{*}{a}_k]_- = 0$ and $[\overset{*}{a}_i, \overset{*}{a}_j]_- \neq 0$ a more complicated representation of the $\overset{*}{a}_i$'s is needed. This more general representation has not yet been found.

For an arbitrary parabose field the elements $\tfrac{1}{2}[\overset{*}{a}_i, a_j]_+$ generate a representation of the unitary group and, when $i = j$, reduce to $z_i \dfrac{d}{dz_i} + 1$ if the Bargmann representation is used. The appropriate generalization of this could be expected to be $e_{ij} = z_i \dfrac{d}{dz_j}$. However the e_{ij} satisfy the additional commutation relations:

$$e_{ij}\, e_{k\ell} - e_{kj}\, e_{i\ell} = \delta_{k\ell}\, e_{ij} - \delta_{i\ell}\, e_{kj} .$$

As Louck(69) has shown this restricts the representations of $U(\nu)$ to totally symmetric tensors. This is only consistent with parabose operators for $\nu = 1$. Thus any generalization would then reduce $\frac{1}{2}[a_i^*, a_j]_+$ to something other than $z_i \frac{d}{dz_j}$. Not knowing the form of the unitary operators further complicates the problem of finding the general representation of the parabose algebra.

Another generalization is to form a representation of the Green component fields in terms of z, d_z and the appropriate reflection operators. For pb3 the Green ansatz is:

$$a^* = a^{1*} + a^{2*} + a^{3*}$$

and

$$a = a^1 + a^2 + a^3$$

where

$$[a^{i*}, a^{j*}]_+ = [a^{i*}, a^j]_+ = 0, \qquad i \neq j$$

and

$$[a^{i*}, a^i]_- = 1, \qquad \text{for all } i.$$

A representation of these operators is given by

$$a^{1*} = z , \quad a^{2*} = R_1 z_2 , \quad a^{3*} = R_1 R_2 z_3$$

and

$$a^1 = d_{z_1} , \quad a^2 = R_1 d_{z_2} , \quad a^3 = R_1 R_2 \, d_{z_3}$$

where R_i are reflection operators satisfying

$$[R_i , z_j]_- = [R_i , d_{z_j}]_- = 0, \qquad i \neq j$$

and

$$[R_i , z_i]_+ = [R_i , d_{z_i}]_+ = 0, \qquad \text{for all } i.$$

The generalization of this to higher order parabose fields is straightforward and the proof follows by induction. It is not intended to study these representations here, or their reduction into irreducible parabose representations. It is worth noting that, in contrast to the irreducible parabose operators constructed earlier, it is relatively easy by appropriate labelling to construct the reducible representations for ν degrees of freedom.

APPENDIX 1

RESTRICTIONS ON THE DESCRIPTION OF STATES OF C_1 IN TERMS OF STATES OF $C_1 \cup C_2$

The rule of §2.4 follows directly from the fact that any n-particle state $|\alpha\rangle$ satisfies[10]

$$|\alpha\rangle = \sum_{(x)_n} \sum_{\ell=1}^{h} d_\ell \, f_{s_j}^{(\ell)}((x)_n) \, |(x)_n \ell \; s_j\rangle$$

$$= \sum_{(x)_n} \sum_{\ell=1}^{h} \sum_{s_{i=1}}^{d_\ell} f_{s_i}^{(\ell)} \, ((x)_n) \, |(x)_n \ell \; s_i\rangle$$

for all $j = 1, \ldots, d_\ell$. Consider any $|\beta\rangle$ which is a combination of states of the form $|(x)_n \ell \; s_1\rangle$, $|(x)_n \ell \; s_2\rangle$, ... $|(x)_n \ell \; s_\ell\rangle$. By choosing an appropriate combination of the $f_{s_j}^{(\ell)} \, ((x)_n)$'s, any $|\beta\rangle$ may be expanded in terms of any one of the states $|(x)_n \ell \; s_j\rangle$. To ensure that the description of $|\beta\rangle$ in the combined system $C_1 \cup C_2$ is independent of the choice of s_i the rule of §2.4 must obviously be observed.

The following example indicates the necessity of this rule. Consider a state of the form

$$|\alpha\rangle = c_1 \, |(2,1,0,\ldots,0)\rangle + c_2 \, |(2,1,0,\ldots,0)'\rangle$$

where the usual arguments have been dropped and the prime distinguishes between the states. Then for an operator, say $\bar{F}$, which is an element of $U(\nu)$ and which effects the transition from $|(2,1,0,\ldots 0)\rangle$ to $|(2,1,0,\ldots 0)'\rangle$, it follows that $\langle\alpha|\ \bar{F}\ |\alpha\rangle \neq 0$. However if the rule of §2.4 is not observed and if the state $|\alpha\rangle$ is described in terms of a redundant particle by

$$|\alpha'\rangle = c_1|(2,2,0,\ldots,0)(2,1,0,\ldots,0)\rangle$$
$$+ c_2|(2,1,1,0,\ldots,0)(2,1,0,\ldots,0)'\rangle$$

then it immediately follows from the reduction of $U(\nu + 1)$ to $U(\nu)$ that $\langle\alpha'|\ \bar{F}\ |\alpha'\rangle = 0$ and the cluster property does not appear to hold. This is because

$$\langle(2,1,1,0,\ldots,0)(2,1,0,\ldots,0)'|\ \bar{F}\ |(2,2,0,\ldots,0)(2,1,0,\ldots,0)\rangle$$
$$\neq \langle(2,1,0,\ldots,0)'|\ \bar{F}\ |(2,1,0,\ldots,0)\rangle\ ,$$

which is quite appropriate since the L.H.S. describes in addition to the transition $(2,1,0,\ldots,0) \to (2,1,0,\ldots 0)'$ of C_1, an extra transition of the complete system from $(2,2,0,\ldots,0)$ to a $(2,1,1,0,\ldots,0)$ type symmetry. The R.H.S. contains no information about the symmetry of the combined system.

If the rule of §2.4 is imposed a permissible description of the state $|\alpha\rangle$ is by the vector

$$c_1|(2,2,0,\ldots,0)(2,1,0,\ldots 0)\rangle$$
$$+ c_2|(2,2,0,\ldots,0)(2,1,0,\ldots,0)'\rangle$$

and it follows directly from the reduction of the unitary group that the cluster property holds.

APPENDIX 2

VIOLATION OF THE CLUSTER PROPERTY BY ELEMENTS OF THE ORTHOGONAL GROUP

(a) Consider the case of cluster C_1 being restricted to two degrees of freedom, i.e. $\nu_1 = 2$ and cluster C_2 with $\nu_2 = 1$. The states of C_1 form representations of the groups U(2), O(4) and O(5). In particular the state $|\alpha\rangle$ where

$$|\alpha\rangle = \{\phi^\dagger(x_1)\ \phi^\dagger(x_2) - \phi^\dagger(x_2)\ \phi^\dagger(x_1)\} \quad |0\rangle$$

corresponds to the representation (1,1) of U(2) and hence to the representation $\left(\frac{p}{2}, \frac{p}{2}\right)$ of O(4) where p is the order of the para-field $\phi(x_i)$. The state $|\alpha\rangle$ is an eigenstate of the operator

$$[\phi^\dagger(x_1), \phi^\dagger(x_2)]_-\ [\phi(x_1), \phi(x_2)]_-$$

with eigenvalue $-4p$. In terms of the combined system $C_1 \cup C_2$ the state $|\alpha\rangle$ can be represented by any state which has the same U(2) labels as $|\alpha\rangle$. Any state having these required labels can be expressed as a combination of the two states

$$\begin{aligned}|\alpha'\rangle = p\{\phi^\dagger(x_1)\ \phi^\dagger(x_R)\ \phi^\dagger(x_2) - \phi^\dagger(x_2)\ \phi^\dagger(x_R)\ \phi^\dagger(x_1)\} \quad |0\rangle \\ - (p-2)\ \phi^\dagger(x_R)\ [\phi^\dagger(x_1), \phi^\dagger(x_2)]_- \quad |0\rangle\end{aligned}$$

and

$$|\alpha''\rangle = \phi^\dagger(x_R)\ [\phi^\dagger(x_1), \phi^\dagger(x_2)]_- \quad |0\rangle .$$

However the above states correspond to eigenvalues zero and $-4p$ respectively, of the above element of $O(2\nu)$. In particular if the elements of $O(2\nu)$ are permitted as physical observables and we describe a state of C_1 in terms of the combined system by $|\alpha'\rangle$ then the matrix elements of elements of $O(2\nu)$ differ from those obtained by a consideration of C_1 alone. A consistent interpretation can be obtained by restricting physical observables to elements of $U(2)$. This is due to the fact that the basis states of the combined system $C_1 \cup C_2$ form representations of $U(3)$, $O(6)$ and $O(7)$. In particular the state $|\alpha'\rangle$ belongs to the representation $\left(\frac{p}{2}, \frac{p}{2}, \frac{p}{2} - 1\right)$ of $O(6)$ and, upon restriction to the appropriate function of x_1 and x_2, to the representation $\left(\frac{p}{2} - 1, \frac{p}{2} - 1\right)$ of $O(4)$. The violation of the cluster property by elements of the orthogonal group is due to the appearance of "non-Fock" representations of the type $\left(\frac{p}{2} - 1, \frac{p}{2} - 1\right)$ which are not present when the cluster C_1 is considered alone.

The above example is readily generalized to the case of continuous range of degrees of freedom. A general antisymmetric (symmetric) state $|\alpha\rangle$ is written as

$$|\alpha\rangle = [\,\phi^{\dagger}(x), \phi^{\dagger}(y)]_{\pm} \quad |0\rangle$$

and the above number conserving operator is written as

$$\int dx\, dy\, f_{C_1}\, (x,y)\, [\,\phi^{\dagger}(x),\, \phi^{\dagger}(y)]_{\pm}\, [\,\phi(x),\, \phi(y)]_{\pm}$$

where $f_{C_1}\,(x,y)$ has a vanishing support outside C_1. The eigenvalue of this operator on the state $|\alpha\rangle$ is $\pm\, 4p\{f_{C_1}(x,y) + f_{C_1}(y,x)\}$. In the presence of the redundant cluster C_1 the above antisymmetric state can be represented by the vector

$$|\alpha'\rangle = p\{\phi^\dagger(x)\ \phi^\dagger(R)\ \phi^\dagger(y) \pm \phi^\dagger(y)\ \phi^\dagger(R)\ \phi^\dagger(x)\}\ |0\rangle$$

$$\pm\ (p - 2)\ \phi^\dagger(R)\ [\phi^\dagger(x),\ \phi^\dagger(y)]_\pm\ |0\rangle\ .$$

Since

$$f_{C_1}\ (R,y) = f_{C_1}\ (x,R) = f_{C_1}\ (R,R) = 0$$

the eigenvalue in the state $|\alpha'\rangle$ of the above operator is zero. Thus for these operators the cluster property does not hold.

(b) Another example is illustrative as it shows that for the "non-Fock" representations of $O(2\nu)$ the connection between the invariants of $U(\nu)$ and $O(2\nu)$ does not hold, or, in the notation of Kamefuchi and Ohnuki(23) the elements of $O(2\nu)$ do not conserve A. A is the number of odd columns in the Young tableau specifying the particular irreducible representation of $U(\nu)$. Take the case of $\nu_1 = 3$ and for simplicity consider $p = 2$. Then if $|\alpha\rangle$ represents the state

$$c_1\ |(1,1,0)\rangle + c_2\ |(2,1,1)\rangle$$

localized in cluster C_1 then it directly follows from the conservation of A that

$$\langle\alpha|\ F_{C_1}\ (O(2\nu))\ |\alpha\rangle = 0$$

if $F_{C_1}\ (O(2\nu))$ is of the form

$$[\phi(x_{i_1}),\ \phi(x_{i_2})]_-\ .$$

When the system is described in terms of the combined system $C_1 \cup C_2$ the associated group is O(8). It can readily be checked that the representation of O(6) labelled (1,0,0) contains both the representations (1,1,0) and (2,1,1) of U(3). Since the above element of O(6) behaves essentially like a lowering operator it has non-vanishing matrix elements between these states and thus to an observer localized on C_1 it would appear that the conservation law of A does not hold. As before this can be avoided by restricting observables to functions of the unitary group.

APPENDIX 3

BOGOLIUBOV TRANSFORMATIONS OF PARAFERMI FIELDS AND THE CLUSTER PROPERTY

Consider a set of three pf2 creation and annihilation operators a_r^*, a_r. Denote $|\chi\rangle$ as the solution of

$$a_1|\chi\rangle = a_2|\chi\rangle = a_3|\chi\rangle = 0 \ .$$

Consider 3 to label a redundant particle on C_2 and 1 and 2 to be labels referring to C_1. The anti-symmetric state of C_1 may be described in the presence of C_2 by either

$$|\phi_n^{(1)}\rangle = [\, a_1^*, \, a_2^*]_- \, a_3^*|\chi\rangle$$

or

$$|\phi_n^{(2)}\rangle = a_1^* \, a_3^* \, a_2^*|\chi\rangle \ .$$

Application of the cluster property would imply that the only permissible observables are elements of the U(2) algebra generated by $[\, a_1^*, \, a_1]_-$, $[\, a_2^*, \, a_2]_-$, $[\, a_1^*, \, a_2]_-$ and $[\, a_2^*, \, a_1]_-$.

Considering a Bogoliubov transformation

$$b_1 = a_1 , \quad b_2 = a_2^* , \quad b_3 = a_3^*$$

and defining $|0\rangle$ as the state such that

$$b_1|0\rangle = b_2|0\rangle = b_3|0\rangle = 0 \ ,$$

it follows that

$$|0\rangle = (a_2^*)^2 \, (a_3^*)^2|\chi\rangle .$$

A permissible physical observable i.e., $[a_1^*, a_2]_-$ is equal to the operator $[b_1^*, b_2^*]_-$. It would lead to a contradiction to consider this operator as a physical observable, since a similar application of the cluster decomposition property in the space of the b's would imply that observables are restricted to elements of the form $[b_i^*, b_j]_-$. Thus the $[b_1^*, b_2^*]_-$ cannot be interpreted as physical observables, because under the Bogoliubov transformations the physical interpretation of the theory alters. In particular for this example the anti-symmetric state $|\phi_n^{(1)}>$ becomes

$$|\phi_p^{(1)}> = [b_1^*, b_2]_- \, b_3(b_2^*)^2 \, (b_3^*)^2 \, |0>$$

$$= - 4\{b_1^*, b_2^*\} \, b_3^* \, |0> \, .$$

Thus, although

$$<\phi_p^{(1)}| \, [b_1^*, b_2^*]_- \, |\phi_p^{(1)}>$$

equals

$$<\phi_p^{(1)'}| \, [b_1^*, b_2^*]_- \, |\phi_p^{(1)'}>$$

where

$$|\phi_p^{(1)'}> = [b_1^*, b_2^*]_- \, |0> \, ,$$

it does not follow that $[b_1^*, b_2^*]_-$ satisfies the cluster property since $|\phi_p^{(1)}>$ no longer describes an anti-symmetric state on C_1.

In particular the physical content of the two representations is not invariant under the Bogoliubov transformations because the operators which determine the physical content i.e., the N_{ij} are not invariant under these transformations.

APPENDIX 4

EVALUATION OF VACUUM EXPECTATION VALUES

The Green ansatz for interacting parafields is particularly useful in the discussion of V.E.V's. This is because, as Dell' Antonio, Greenberg and Sudarshan and Govorkov have shown, the V.E.V. of a product of Green component fields factorizes to the product of the V.E.V.'s for each field. For example

$$\langle A^{(1)}(x_1)\ A^{(1)}(x_2)\ A^{(2)}(x_3)\rangle$$
$$= \langle A^{(1)}(x_1)\ A^{(1)}(x_2)\rangle\ \langle A^{(2)}(x_3)\rangle\ .$$

Since component fields with different superscripts anti-commute, V.E.V.'s of a product of operators in which there are an odd number of each component field, and there are at least two different species present, vanish. As an example:

$$\langle A^{(1)}(x_1)\rangle\ \langle A^{(2)}(x_2)\rangle$$
$$= \langle A^{(1)}(x_1)\ A^{(2)}(x_2)\rangle$$
$$= -\ \langle A^{(2)}(x_2)\ A^{(1)}(x_1)\rangle$$
$$= -\ \langle A^{(2)}(x_2)\rangle\ \langle A^{(1)}(x_1)\rangle$$
$$= 0\ .$$

Since the component fields are assumed to be equivalent the V.E.V. of a product of operators all labelled by the same superscript can be taken to be independent of that superscript. This implies that

$$<A^{(i)}(x_1)\ A^{(i)}(x_2)> = <A^{(j)}(x_1)\ A^{(j)}(x_2)> .$$

Since the V.E.V.'s of a product of Green component fields all with the same superscript have identical properties to a Bose field it will be assumed that these V.E.V.'s satisfy the cluster decomposition property.

Both properties discussed will be used in the following.

Example 1: Evaluation of $<A(x_1 + \lambda a)\ A(x_2)\ A(x_3 + \lambda a)\ A(x_4)>$ for $p = 2$ as $\lambda \to \infty$.

Substituting $A(x_i) = A^{(1)}(x_i) + A^{(2)}(x_i)$, the following results for the particular component fields are obtained:

$$\lim_{\lambda\to\infty} <A^{(1)}(x_1 + \lambda a)\ A^{(1)}(x_2)\ A^{(1)}(x_3 + \lambda a)\ A^{(1)}(x_4)>$$

$$= <A^{(1)}(x_1)\ A^{(1)}(x_3)>\ <A^{(1)}(x_2)\ A^{(1)}(x_4)> .$$

This assumes that the cluster decomposition property is valid for each component field.

Similarly

$$\lim_{\lambda\to\infty} <A^{(2)}(x_1 + \lambda a)\ A^{(2)}(x_2)\ A^{(2)}(x_3 + \lambda a)\ A^{(2)}(x_4)>$$

$$= <A^{(2)}(x_1)\ A^{(2)}(x_3)>\ <A^{(2)}(x_2)\ A^{(2)}(x_4)> .$$

V.E.V.'s of the form

$$<A^{(1)}(x_1 + \lambda a)\ A^{(2)}(x_2)\ A^{(2)}(x_3 + \lambda a)\ A^{(2)}(x_4)>$$

vanish.

The other non-vanishing V.E.V.'s are of the form

$$<A^{(1)}(x_1 + \lambda a)\ A^{(2)}(x_2)\ A^{(1)}(x_3 + \lambda a)\ A^{(2)}(x_4)>$$

$$= - <A^{(1)}(x_1 + \lambda a)\ A^{(1)}(x_3 + \lambda a)> <A^{(2)}(x_2)\ A^{(2)}(x_4)>$$

$$= - <A^{(1)}(x_1)\ A^{(1)}(x_3)> <A^{(2)}(x_2)\ A^{(2)}(x_4)> .$$

V.E.V.'s of the form

$$\lim_{\lambda\to\infty} <A^{(1)}(x_1 + \lambda a)\ A^{(1)}(x_2)\ A^{(2)}(x_3 + \lambda a)\ A^{(2)}(x_4)>$$

vanish as a result of cluster decomposition and factorization of the V.E.V.'s.

Collecting terms it holds that

$$\lim_{\lambda\to\infty} <A(x_1 + \lambda a)\ A(x_2)\ A(x_3 + \lambda a)\ A(x_4)>$$

$$= \sum_{i=1}^{2} <A^{(i)}(x_1)\ A^{(i)}(x_3)> <A^{(i)}(x_2)\ A^{(i)}(x_4)>$$

$$- <A^{(1)}(x_1)\ A^{(1)}(x_3)> <A^{(2)}(x_2)\ A^{(2)}(x_4)>$$

$$- <A^{(2)}(x_1)\ A^{(2)}(x_3)> <A^{(1)}(x_2)\ A^{(1)}(x_4)>$$

$$\neq <A(x_1)\ A(x_2)> <A(x_3)\ A(x_4)> .$$

In a similar manner for $p > 2$ the same example may be used to show that the cluster property is not satisfied.

If it is also assumed that

$$<A^{(k)}(x_i)\ A^{(k)}(x_j)>$$

$$= <A^{(\ell)}(x_i)\ A^{(\ell)}(x_j)>$$

then

$$\lim_{\lambda\to\infty} \langle A(x_1 + \lambda a)\ A(x_2)\ A(x_3 + \lambda a)\ A(x_4)\rangle = 0 \ .$$

This holds only for pb2 statistics.

Example 2

Let $W(x_1x_2 \ldots x_n)$ denote $\langle A(x_1)\ A(x_2)\ \ldots\ A(x_n)\rangle$ and define

$$W^{(2,1)}(x_1x_2x_3) = W(x_1x_3x_2)$$

$$- W(x_2x_3x_1) + W(x_3x_1x_2) - W(x_3x_2x_1)$$

and

$$W^{(1,1)}(x_1x_2) = W(x_1x_2) - W(x_2x_1) \ .$$

It follows that

$$\lim_{\lambda\to\infty} W^{(2,1)}(x_1x_2x_3 + \lambda a) = W^{(1,1)}(x_1x_2)\ W(x_3)$$

iff

$$\lim_{\lambda\to\infty} W(x_1x_3 + \lambda a\ x_2) - W(x_2x_3 + \lambda a\ x_1) = 0 \ .$$

Substituting

$$A(x_j) = \sum_{\alpha=1}^{p} A^{\alpha}(x_j)$$

and considering the various terms it can be shown by an argument similar to that used in the previous example that only for $p = 2$ does

$$\lim_{\lambda\to\infty} W(x_1x_3 + \lambda a\ x_2) - W(x_2x_3 + \lambda a\ x_1) = 0 \ .$$

Thus only for $p = 2$ does a restricted cluster property hold.

Example 3

Since the free scalar field $A(x)$ satisfies the Wightman axioms the V.E.V.'s of the $A(x_i)$'s should also satisfy any form of the cluster decomposition property. Substituting

$$A(x) = \frac{1}{(2(2\pi)^3)^{\frac{1}{2}}} \int_{k_o>0} \frac{1}{k_o} \left(a_{\underset{\sim}{k}}\, e^{-ik\cdot x} + a^*_{\underset{\sim}{k}}\, e^{ik\cdot x}\right) d^3k$$

in the V.E.V. $W(x_1x_2x_3x_4)$ gives the result

$$W(x_1x_2x_3x_4) = p^2\{\Delta^+(x_1 - x_2)\Delta^+(x_3 - x_4)$$
$$+ \Delta^+(x_2 - x_3)\Delta^+(x_1 - x_4)\} - p(p - 2)\Delta^+(x_2 - x_4)$$
$$\times \Delta^+(x_1 - x_3) .$$

Define

$$W^{(2,2)}(x_1x_2x_3x_4) =$$
$$W(x_1x_3x_2x_4) - W(x_2x_3x_1x_4) + W(x_2x_4x_1x_3) - W(x_1x_4x_2x_3)$$
$$+ W(x_3x_1x_2x_4) - W(x_3x_2x_1x_4) + W(x_4x_2x_1x_3) - W(x_4x_1x_2x_3)$$
$$+ W(x_1x_3x_4x_2) - W(x_2x_3x_4x_1) + W(x_2x_4x_3x_1) - W(x_1x_4x_3x_2)$$
$$+ W(x_3x_1x_4x_2) - W(x_3x_2x_4x_1) + W(x_4x_2x_3x_1) - W(x_4x_1x_3x_2) .$$

Then, ignoring $\Delta^+(\xi)$ as ξ becomes large spacelike,

$$\lim_{\lambda\to\infty} W^{(2,2)}(x_1x_2x_3 + \lambda a\ x_4 + \lambda a)$$
$$= 2p(\Delta^+(x_1 - x_2) - \Delta^+(x_2 - x_1))(\Delta^+(x_3 - x_4)$$
$$- \Delta^+(x_4 - x_3))$$
$$= 2p\Delta(x_1 - x_2)\Delta(x_3 - x_4) .$$

However

$$W^{(1,1)}(x_1x_2)\, W^{(1,1)}(x_3x_4)$$

$$= p^2\Delta(x_1 - x_2)\Delta(x_3 - x_4)$$

and so once again the factorization of V.E.V.'s only holds for $p = 2$ since for $p > 2$

$$\lim_{\lambda\to\infty} W^{(2,2)}(x_1x_2x_3 + \lambda a\; x_4 + \lambda a)$$

$$\neq W^{(1,1)}(x_1x_2)\, W^{(1,1)}(x_3x_4)\ .$$

APPENDIX 5

A REPRESENTATION OF PARKS' COMMUTATION RELATIONS BY PARAFERMI OPERATORS

From §3.3b a_k is expressed as

$$a_k = b_k^{(1)} b_k^{(2)} \ldots b_k^{(p)} \tag{A5.1}$$

where

$$[b_k^{(i)}, b_\ell^{(j)}]_- = [b_k^{(i)}, b_\ell^{(j)*}]_-$$

$$= 0 \qquad \text{for } i \neq j ,$$

$$[b_k^{(i)}, b_\ell^{(i)}]_+ = 0$$

and

$$[b_k^{(i)}, b_\ell^{(i)*}]_+ = \delta_{k\ell} .$$

Since $(b_k^{(i)})^2 = 0$ it directly follows that $a_k^2 = 0$. From the expression for a_k it follows that $[a_k, b_j^{(i)}]_+ = 0$ and hence that $[a_k, a_j]_\mp = 0$ where the upper sign corresponds to p even and the lower sign to p odd.

From (A5.1) it follows that

$$\begin{aligned}
a_k a_\ell^* &= b_k^{(1)} b_\ell^{(1)*} b_k^{(2)} b_\ell^{(2)*} \ldots b_k^{(p)} b_\ell^{(p)*} \\
&= (\delta_{k\ell} - b_\ell^{(1)*} b_k^{(1)}) (\delta_{k\ell} - b_\ell^{(2)*} b_k^{(2)}) \\
&\quad \ldots(\delta_{k\ell} - b_\ell^{(p)*} b_k^{(p)}) \\
&= (-1)^p a_\ell^* a_k + \delta_{k\ell} \{1 - n_k^{(1)} - n_k^{(2)} - \ldots \\
&\quad - n_k^{(p)} + n_k^{(1)} n_k^{(2)} + \ldots (-1)^{p-1} n_k^{(2)} n_k^{(3)} \\
&\quad \ldots n_k^{(p)}\}
\end{aligned}$$

$$= (-1)^p\, a_\ell^* \, a_k + \delta_{k\ell} \{1 - N_k\} \; , \qquad \text{(A5.2)}$$

where $n_k^{(i)}$ is the number operator for the i^{th} component field. It is only necessary to show that

$$[a_k, N_\ell]_- = 2\delta_{k\ell}\, a_k \; ,$$

since it follows trivially that $[N_i, N_j]_- = 0$ and $N_i^* = N_i$. Now

$$[a_k, n_\ell^{(i)}]_- = b_k^{(1)} \ldots [b_k^{(i)}, n_\ell^{(i)}]_- \ldots b_k^{(p)} = \delta_{k\ell}\, a_k \; .$$

In general the relation

$$[a_k, n_\ell^{(i)} \ldots n_\ell^{(q)}]_- = \delta_{k\ell}\, a_k$$

can be shown to hold by induction. Thus $[a_k, N_\ell]_-$ will be a linear combination of $\delta_{k\ell}\, a_k$'s whose coefficients are given by the expansion (A5.2). From (A5.2) it follows that the sum of the coefficients will be the same as the sum of the coefficients in the expansion of $-(1 - x)^p$ with the highest and lowest powers of x removed. Now the sum of <u>all</u> the coefficients in $(1 - x)^p$ is zero for all p. For p even, 1 and x^p have the same signs so the sum of the remaining term is -2. Thus

$$[a_k, N_\ell]_- = + 2\delta_{k\ell}\, a_k \; .$$

For p odd, 1 and x^p have opposite signs and so the sum of the remaining terms is zero. This implies the unusual result:

$$[a_k, N_\ell]_- = 0.$$

The resulting commutation relations are

$$a_k^2 = 0 \ ,$$

$$[\, a_k ,\ a_\ell]_{\mp} = 0 \ ,$$

$$[\, a_k ,\ a_\ell^{*}]_{\mp} = 2\delta_{k\ell} \ (\tfrac{1}{2} - N_k')$$

and

$$[\, a_k ,\ N_\ell']_{-} = \delta_{k\ell} \ a_k \qquad \text{for p even}$$

and

$$[\, a_k ,\ N_\ell']_{-} = 0 \qquad \text{for p odd}$$

where $N_k' = \frac{1}{2} N_k$ is the number operator.

APPENDIX 6

DETERMINATION OF THE INTEGRAL TRANSFORM A(z,x)

Equations (4.15) and (4.16) of §4.4.1 can be rearranged into two sets each containing two pairs of coupled partial differential equations. The equations are

(A).
$$(x + d_x + \frac{\tau}{x})A_{11} = - \sqrt{2}\, z\, A_{22} \qquad \text{(A6.1a)}$$

$$(x + d_x - \frac{\tau}{x})A_{22} = - \sqrt{2}\, z\, A_{11} \qquad \text{(A6.1b)}$$

and

$$(x - d_x - \frac{\tau}{x})A_{11} = - \sqrt{2}\, (d_z - \frac{\tau}{z})A_{22} \qquad \text{(A6.2a)}$$

$$(x - d_x + \frac{\tau}{x})A_{22} = - \sqrt{2}\, (d_z + \frac{\tau}{z})A_{11}\ . \qquad \text{(A6.2b)}$$

(B).
$$(x + d_x - \frac{\tau}{x})A_{12} = - \sqrt{2}\, z\, A_{21} \qquad \text{(A6.3a)}$$

$$(x + d_x + \frac{\tau}{x})A_{21} = - \sqrt{2}\, z\, A_{12} \qquad \text{(A6.3b)}$$

and

$$(x - d_x + \frac{\tau}{x})A_{12} = - \sqrt{2}\, (d_z - \frac{\tau}{z})A_{21} \qquad \text{(A6.4a)}$$

$$(x - d_x - \frac{\tau}{x})A_{21} = - \sqrt{2}\, (d_z + \frac{\tau}{z})A_{12}. \qquad \text{(A6.4b)}$$

Solution of set A

Differentiating (A6.1a) and substituting for A_{22} and $d_x\, A_{22}$ from (A6.1b) the following equation is obtained:

$$d_x^2\, A_{11} + 2x\, d_x\, A_{11} + (1 - 2z^2 + x^2$$
$$- \frac{\tau(\tau + 1)}{x^2})A_{11} = 0\ . \qquad \text{(A6.5)}$$

In a similar manner (A6.2) reduce to

$$d_z^2 A_{11} + 2z\, d_z A_{11} + (1 - 2x^2 + z^2 - \frac{\tau(\tau + 1)}{z^2}) A_{11} = 0 . \qquad (A6.6)$$

Substituting

$$A_{11} = e^{-\frac{1}{2}(x^2+z^2)} \sqrt{\xi}\, u(\xi) ,$$

where $\xi = (\sqrt{2}zx)$ both (A6.5) and (A6.6) become

$$\xi^2 u''(\xi) + \xi u'(\xi) - (\xi^2 + (\tau + \tfrac{1}{2})^2) u(\xi) = 0 ,$$

which is Bessel's equation with imaginary argument. The solution is

$$A_{11} = a_{\tau+\frac{1}{2}} e^{-\frac{1}{2}(x^2+z^2)} (\sqrt{2}\, zx)^{\frac{1}{2}} Q_{\tau+\frac{1}{2}} (\sqrt{2}\, zx)$$

where $Q_{\tau+\frac{1}{2}} (+ \sqrt{2}\, zx)$ is any combination of Bessel functions with imaginary argument and index $\tau + \frac{1}{2}$ or $-(\tau + \frac{1}{2})$ and a_ν is an arbitrary coefficient.

Similarly equations (A6.2) reduce to

$$d_x^2 A_{22} + 2x\, d_x A_{22} + (1 - 2z^2 + x^2 - \frac{\tau(\tau - 1)}{x^2})A_{22} = 0$$

and

$$d_z^2 A_{22} + 2z\, d_z A_{22} + (1 - 2x^2 + z^2 - \frac{\tau(\tau - 1)}{z^2})A_{22} = 0 .$$

Since these equations are symmetric with respect to interchange of x and z the solution is

$$A_{22} = a_{\tau-\frac{1}{2}} \, e^{-\frac{1}{2}(x^2+z^2)} \, (\sqrt{2}\, zx)^{\frac{1}{2}} \, Q_{\tau-\frac{1}{2}} \, (\sqrt{2}\, zx)$$

where Q is defined above.

Ignoring the A_{12} and A_{21} terms (which will be shown to be equal to zero) the appropriate choices for the Q's are:

(1) <u>α odd</u>

$$Q_{11} = J_{\substack{\alpha+\frac{1}{2} \\ \alpha+1}} \, (i\sqrt{2}\, zx) \quad \text{and}$$

$$Q_{22} = J_{\substack{\alpha-\frac{1}{2} \\ \alpha}} \, (i\sqrt{2}\, zx)$$

where the upper sign corresponds to p odd and the lower sign to p even.

(2) <u>α even</u>

$$Q_{11} = J_{\substack{\alpha-\frac{1}{2} \\ \alpha}} \, (i\sqrt{2}\, zx) \quad \text{and}$$

$$Q_{22} = J_{\substack{\alpha+\frac{1}{2} \\ \alpha+1}} \, (i\sqrt{2}\, zx) \; .$$

There is however an important modification to the solutions for p even. Denoting R_x as the reflection operator for the x variable it holds that

$$[R_x, \, d_x^2 + 2x\, d_x + (1 - 2z^2 + x^2 - \frac{\tau(\tau+1)}{x^2})]_- = 0 \; .$$

By a similar technique to that applied to the solutions of the Schrödinger equation in §4.3.2b the $\sqrt{x}$ factor in the solution may be modified to $\sqrt{|x|}$. Similarly since

$$[\bar{z}, \, d_z^2 + 2z\, d_z + (1 - 2x^2 + d_z^2 - \frac{\tau(\tau+1)}{z^2})]_- = 0$$

an extra $\sqrt{z}$ factor may be added to the solution. The modified solution now becomes

$$A_{11} = a_{\tau-\frac{1}{2}} \, e^{-\frac{1}{2}(x^2+z^2)} \, (\sqrt{2} \, z\bar{z} \, |x|)^{\frac{1}{2}} \, Q_{\tau-\frac{1}{2}} \, (\sqrt{2} \, zx)$$

and

$$A_{22} = a_{\tau+\frac{1}{2}} \, e^{-\frac{1}{2}(x^2+z^2)} \, (\sqrt{2} \, z\bar{z} \, |x|)^{\frac{1}{2}} \, Q_{\tau+\frac{1}{2}} \, (\sqrt{2} \, zx) \; .$$

Solution for Set B

Premultiplying (A6.4b) by $-\sqrt{2}z$ gives

$$2z \, (d_z + \frac{\tau}{z})A_{12} = (x - d_x - \frac{\tau}{x}) \, (-\sqrt{2} \, z)A_{21}$$

and substituting (A6.3a) this becomes

$$2z \, (d_z + \frac{\tau}{z})A_{12} = (x - d_x - \frac{\tau}{x}) \, (x + d_x - \frac{\tau}{x})A_{12} \; . \qquad \text{(A6.7)}$$

Similarly from (A6.3b) and (A6.4c)

$$2 \, (d_z - \frac{\tau}{z})z \, A_{12} = (x + d_x + \frac{\tau}{x}) \, (x - d_x + \frac{\tau}{x})A_{12} . \qquad \text{(A6.8)}$$

Then (A6.7) - (A6.8) implies

$$2 \, ([z, d_z]_- + 2\tau)A_{12} = 2 \, ([x, d_x]_- - 2\tau)A_{12}$$

and hence

$$\tau A_{12} = - \tau A_{12} .$$

For $\tau \neq 0$ this implies $A_{12} = 0$. Similarly $A_{21} = 0$ and hence $A(z,x)$ is diagonal.

In terms of the algebras a, a^* and ξ,η the above argument is equivalent to requiring that

$$\frac{1}{2\tau}\{1 - [a, a^*]_-\}$$

is mapped into

$$\frac{1}{2\tau}\{1 - [\xi, \eta]_-\} .$$

Substituting the appropriate expressions for a, a^*, ξ and η this requirement is that R is mapped into R i.e.,

$$f(-z) = \int A(z,x)\,\psi(-x)\,dx$$

which implies that

$$[R, A(z,x)]_- = 0 .$$

As for the metric, vanishing of the above commutator implies that A(z,x) is diagonal.

Defining

$$N_z = \tfrac{1}{2}[a^*, a]_+ - \tfrac{1}{2}$$

and

$$N_x = \tfrac{1}{2}[\eta, \xi]_+ - \tfrac{1}{2}$$

it follows from the definition of A(z,x) that

$$e^{i\pi N_z} A(z,x) = A(z,x)\, e^{i\pi N_x} . \qquad \text{(A6.9)}$$

Since $R = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is the simplest closed extension of both $e^{i\pi N_z}$ and $e^{i\pi N_x}$ then (A6.9) reduces to RA = AR and hence A is diagonal.

APPENDIX 7

EVALUATION OF AN INTEGRAL

To show that the integral transform $A_\nu(z,x)$ is unitary it is necessary to evaluate the following integral:

$$\sigma_\nu(\lambda,x,y) = \int A_\nu(\lambda z,x)\, A_\nu(\lambda\bar{z},y)d\rho_\nu(z\bar{z}) \,.$$

A detailed evaluation of σ for p odd is given below, the relevant modifications for p even being discussed in (b).

(a) p odd

Substituting (4.17) for A_ν and (4.8) for ρ_ν the integral becomes

$$\sigma_\nu(\lambda,x,y) = \frac{\lambda}{\pi}\sqrt{xy}\, e^{-\frac{1}{2}(x^2+y^2)} \int dz\, d\bar{z}\, z\bar{z}$$

$$\times\, e^{-\frac{\lambda^2}{2}(z^2+\bar{z}^2)}\, J_\nu(i\sqrt{2}\lambda zx)\, J_\nu(-\, i\sqrt{2}\lambda\bar{z}y)\, K_\nu(z\bar{z}) \,.$$

$$= \frac{\lambda^{2\nu+1}}{\pi}(xy)^{\nu+\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2)} \sum_n \sum_m (-\,1)^n\, (-\,1)^m$$

$$\times\, 2^{-\nu-m-n} \frac{L_n^\nu(x^2)\, L_m^\nu(y^2)}{\Gamma(n+\nu+1)\,\Gamma(m+\nu+1)}$$

$$\times \int dz\, d\bar{z}\, z\bar{z}\, \bar{z}^{2m+\nu}\, z^{2n+\nu}\, K_\nu(z\bar{z}) \,,$$

since[66]

$$J_\nu(2(xz)^{\frac{1}{2}}) = (xy)^{\frac{\nu}{2}} e^{-z} \sum_{n=0}^{\infty} \frac{L_n^\nu(x)z^n}{\Gamma(n+\nu+1)} \,.$$

Evaluating the integral (which is of the form $(\bar{z}^{2m+\nu+\frac{1}{2}},\, z^{2n+\nu+\frac{1}{2}})$) and substituting gives the following expression for σ:

$$\sigma_\nu(\lambda,x,y) = \lambda^{2\nu+1} (xy)^{\nu+\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2)}$$

$$\times \sum_n \frac{L_n^\nu(x^2)\, L_n^\nu(y^2)\, \lambda^{4n}\, n!}{\Gamma(n+\nu+1)}$$

$$= \frac{\lambda}{1-\lambda^4} \sqrt{xy}\, e^{-\frac{1}{2}(x^2+y^2)}\, e^{-\frac{\lambda^4(x^2+y^2)}{1-\lambda^4}}$$

$$\times I_\nu\left(\frac{2\lambda^2\, xy}{1-\lambda^4}\right) .$$

The last line follows from the fact that the Bessel function I_ν is a generating function for products of Laguerre polynomials[66] i.e.,

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\nu+1)} L_n^\nu(x)\, L_n^\nu(y)\, \lambda^n$$

$$= (1-\lambda)^{-1} \exp\left(-\frac{\lambda(x+y)}{1-\lambda}\right) (xy\lambda)^{-\frac{\nu}{2}}\, I_\nu\left[\frac{2(xy\lambda)^{\frac{1}{2}}}{1-\lambda}\right]$$

for $|\lambda| < 1$.

Substituting $\nu = \alpha \pm \frac{1}{2}$ then gives the required result.

(b) p even

Substituting (4.18) for A_ν and (4.10) for ρ_ν implies

$$\sigma_\nu(\lambda,x,y) = \frac{2}{\pi} \lambda^2\, |xy|^{\frac{1}{2}}\, e^{-\frac{1}{2}(x^2+y^2)} \int dz\, d\bar{z}$$

$$\times z\bar{z}\, J_\nu(i\sqrt{2}\,\lambda z x)\, J_\nu(-i\sqrt{2}\,\lambda \bar{z} y)$$

$$\times e^{-\frac{\lambda^2}{2}(z^2+\bar{z}^2)}\, K_\nu(z\bar{z}) .$$

which, upon repeating the same steps as for p odd, reduces to

$$\sigma_\nu(\lambda,x,y) = \frac{\lambda^2}{1-\lambda^4}\sqrt{|xy|}\; e^{-\frac{1}{2}(x^2+y^2)}\; e^{-\frac{\lambda^4(x^2+y^2)}{1-\lambda^2}}$$

$$\times\, I_\nu\left(\frac{2\lambda^2\, xy}{1-\lambda^4}\right)$$

for $|\lambda| < 1$.

Substituting $\nu = \alpha$ or $\alpha + 1$ gives the required result.

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Cluster Restrictions on Parafermi Operators

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INCONSISTENCY OF THE QUANTIZATION SCHEME OF KADEMOVA AND KRAEV

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The quantization scheme recently proposed by Kademova and Kraev is shown to be inconsistent. There is no apparent way in which it can be modified in order to provide a scheme liable to a reasonable physical interpretation.

Kademova and Kraev [1] have recently proposed a new quantization scheme for spin-half fields, which would allow an unlimited number of identical spin-half particles to occupy one and the same state.

The trilinear commutation relations they suggest for the creation operators a_i^* and their (assumed) hermitean conjugates, the annihilation operators a_i, $i = 1,2,\ldots$, are

$$[a_i,[a_j^*,a_k]] = -2\delta_{ij}a_k\,, \qquad [a_i,[a_j,a_k]] = 0\,, \quad (1)$$

and the relations obtained from these through hermitean conjugation and application of Jacobi's identity. These differ from Green's parafermion commutation relations [2] in the sign of the right-hand side of the first equation.

According to Kademova and Kraev, one should require in addition to (1) the existence of a "vacuum state" vector $|0\rangle$ satisfying

$$a_i|0\rangle = 0\,,$$
$$a_i a_j^*|0\rangle = p\delta_{ij}|0\rangle\,, \quad (2)$$

where p is some positive constant, the "order of the parastatistics". The operator $N_{pi} = \frac{1}{2}([a_i,a_i^*]-p)$ is then to be identified as "counting the number of particles in the ith state", and is claimed to have a spectrum consisting of all non-negative integers.

The consistency of this scheme has been established only in the case where there is just one pair of creation and annihilation operators, a_1^* and a_1. It is easy to find an inconsistency when there is more than one such pair.

Consider the operators

$$S_1 = \tfrac{1}{4}[a_2+a_2^*, a_1-a_1^*]\,,$$

$$S_2 = \tfrac{1}{4}\mathrm{i}[a_2+a_2^*, a_1+a_1^*]$$

$$S_3 = \tfrac{1}{2}[a_1,a_1^*]\,,$$

which, in view of (1), satisfy the familiar angular momentum relations

$$[S_i,S_j] = \mathrm{i}\,\epsilon_{ijk}S_k\,,$$

and which are hermitean operators in the representations under discussion. As proved in elementary quantum mechanics texts, any eigenvalue of S_3 in such a representation must be integral or half-odd-integral. More importantly, if λ is such an eigenvalue, so is $-\lambda$.

Now it is easily shown that the spectrum of S_3 is here unbounded above, except in the trivial representation $a_i = a_i^* = 0$, and it follows at once that it is also unbounded below. For suppose S_3 has a maximum eigenvalue $\lambda_{\max} > 0$. Then, because $[S_3,a_1^*] = a_1^*$, there must exist a normalizable state vector $|\chi\rangle$ such that

$$S_3|\chi\rangle = \lambda_{\max}|\chi\rangle\,, \qquad a_1^*|\chi\rangle = 0\,.$$

But then

$$\begin{aligned}\lambda_{\max}\,\||\chi\rangle\|^2 &= \langle\chi|S_3|\chi\rangle\\ &= \tfrac{1}{2}\langle\chi|(a_1a_1^*-a_1^*a_1)|\chi\rangle\\ &= -\tfrac{1}{2}\|a_1|\chi\rangle\|^2\\ &\leqslant 0\,,\end{aligned}$$

which is contradictory.

In this way it may be shown that the spectrum

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of each operator $\frac{1}{2}[a_i, a_i^*]$ is unbounded above and below in the representations of interest. The same is therefore true of the spectrum of each N_{pi}, contrary to the claims of Kademova and Kraev, and as a result such operators are quite unsuitable for use as "number operators". Moreover, it follows that no representation of the algebra (1) in which a_i^* is the hermitean conjugate of a_i, contains a vector $|0\rangle$ satisfying eqs. (2).

As Kademova and Kraev have pointed out, the algebra (1), with n pairs of creation and annihilation operators, is isomorphic to the Lie algebra of $O(2n, 1)$. In the case $n = 1$, the compact subalgebra contains only one element, $\frac{1}{2}[a_1, a_1^*]$, and it is possible to find [3] an infinite-dimensional representation $D^+(-p)$ in which this operator is hermitean and has a spectrum bounded below by a positive constant p. It is a representation of this type which Kademova and Kraev wish to use for each operator $\frac{1}{2}[a_i, a_i^*]$. Unfortunately, as the preceding arguments show, it is impossible to find an infinite-dimensional representation of the $O(2n, 1)$ algebra, $n > 1$, in which the operators $\frac{1}{2}[a_i, a_i^*]$ have these properties.

We may conclude by saying that, not only is the quantization scheme of Kademova and Kraev inconsistent, but also there is no apparent way in which one could amend it in order to obtain a scheme liable to a reasonable physical interpretation.

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