Plasma Stability Theory and Applications

by

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PLASMA STABILITY THEORY AND APPLICATIONS

SUMMARY

The hydromagnetic stability theory of a plasma is discussed both from a general viewpoint, and with application to a constricted discharge between electrodes having magnetic surfaces approximated by hyperboloids of one sheet.

The interchange of adjacent flux tubes in a low-pressure plasma is considered, with emphasis on systems terminated by conducting plates, in which only flux tubes containing equal magnetic flux can be interchanged. Provided the ends of the adjacent tubes are at the same magnetic scalar potential ψ, the stability criterion adopts a form involving the field-line curvature. However, assuming perfect electrical conductivity everywhere, complications arise because the above restriction cannot in general be met in open-ended systems.

The analysis is generalized, using the "double-adiabatic" theory of CHEW, GOLDBERGER and LOW. If stability exists in the system whose pressure always remains scalar, then the system in which isotropy cannot be maintained during perturbation is stable also. For the curvature-dependent perturbation, stability can be achieved in magnetic-well configurations, and if, in equilibrium, the sum $(p_\perp + p_{\parallel})$ of the pressure tensor components is constant along the field, the criterion reduces exactly to that obtained for scalar pressure.

Mechanical equilibrium in the discharge is examined and, in the low-pressure boundary region, the pressure gradient, to first order in the ratio of plasma pressure to magnetic pressure, is perpendicular to the hyperboloid surfaces. The same result is found for $(p_\perp + p_{\parallel})$ when the non-isotropic pressure tensor is used.

The stability criterion is applied to the constricted discharge, with careful attention to the problems which arise because the electrodes do not coincide with surfaces of constant ψ. The exact result obtained for the critical discharge current is interpreted geometrically, and is then reconciled with the earlier approximate expression of SEYMOUR.

Considering a general system with perfectly electrically conducting plasma separated from the vacuum by a surface sheet current, a discussion of boundary conditions is followed by an analysis concerning some aspects of the hydromagnetic energy principle of BERNSTEIN et al, for an arbitrary fluid displacement, $\underline{\xi}$. The approach adopted extends a derivation of VAN KAMPEN and FELDERHOF, to deal with a bounded plasma in contact with vacuum and with electrodes. The change δW_{BE}, in the external magnetic energy, calculated as work done against the pressure of the vacuum field at the plasma/vacuum interface, is used to complete the expression $\delta W(\underline{\xi},\underline{\xi})$, for the total system. To permit application of the result to the discharge between electrodes, the proof is generalized by taking account of necessary insulating supports, leading to some modification of the vacuum contribution.

The usual interpretation of the surface contribution to δW is shown to be incorrect, and for zero internal magnetic field it is established that neutral stability is obtained at best. The treatment by BERNSTEIN et al, extending the energy principle so that a constraint on $\underline{\xi}$ arising from the continuity of stress at the surface can be ignored, is considerably amplified by means of a rigorous mathematical approach.

Finally the energy principle is used to derive a stability criterion for a sheet-current version of the constricted discharge. For zero internal field, the necessary and sufficient condition for instability is that the current must exceed a critical value which, for identical external conditions, is found to be greater than that for the system with inter-diffused field and plasma. Treatment of the discharge with trapped axial field, but no internal electric current, leads to a sufficient condition for stability. A geometrical interpretation of the stability criterion is given, and an approximation for the critical current in terms of experimental parameters is obtained for the practical case of a thin, slightly constricted discharge.

STATEMENT

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and that to the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text.

(M. K. James)

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INTRODUCTION

Because of the presence of large numbers of free electrons and ions, the properties of a highly ionized gas, or plasma (TONKS and LANGMUIR, 1929), differ considerably from those of an unionized gas. On the microscopic scale, the interaction between particles changes character and obeys the long-range Coulomb law, so that the plasma dynamics on this scale must be treated as a many-body problem. Since electric currents can be sustained, the macroscopic behaviour of a plasma in its interaction with electromagnetic fields can often be treated in terms of magnetohydrodynamics (ALFVÉN and FÄLTHAMMAR, 1963, Chapter 3), especially when the motion of the gas is dominated by the magnetic body force, $\underline{j} \times \underline{B}$, where $\underline{j}$ is the electric current density and $\underline{B}$ is the magnetic field.

It is therefore clear that plasma dynamics provides a fruitful subject for study. This fact, coupled with estimates that this highly interesting state of matter accounts for all but a small fraction of the material in the universe (ALFVÉN and FÄLTHAMMAR, 1963, p.134), makes the study of plasmas, both experimental and theoretical, of major importance. However, further motivation arises from the possibilities of technological applications of plasmas, and the bulk of present day plasma research is directed towards such projects as the development of a controlled thermonuclear fusion reactor (ROSE and CLARK, 1961) for the generation of power; the direct conversion of kinetic energy to electrical energy in MHD generators (ROSA and KANTROWITZ, 1960; PETSCHEK, 1965); and the use of high-velocity ionized gas streams as propellants in rocket engines for long-term, reliable operation, at low thrust but high specific impulse (JAHN, 1968).

Experimentally, large, long-lived plasma systems such as the sun may be studied at a great distance, with obvious difficulties. However it is an ironical fact that while plasma occurs in enormous quantities in nature, in stellar and interstellar material, and in the earth's ionosphere, the laboratory experimenter is faced with great difficulty in gaining access to plasma, in the form of highly ionized gas, for useful periods of time. The main laboratory problem arises through cooling of the plasma by heat loss to the surroundings. The degree of ionization depends critically on the plasma temperature (SAHA and SAHA, 1934; SEYMOUR, 1961a, Fig. 1), and falls off rapidly as the gas is cooled through diffusion of particles to the walls of the chamber, and through other mechanisms such as bremsstrahlung radiation and charge exchange (SPITZER, 1962, p.147; THOMPSON, 1965a).

A serious problem has been encountered in attempts to prolong the life of laboratory plasma by keeping it out of contact with material walls through the use of suitably designed magnetic fields. This approach relies on the magnetic body force, $\underline{j} \times \underline{B}$, to balance the material forces tending to expand the plasma. It has proved a relatively simple matter to devise such confinement systems, mostly based on the pinch discharge (BENNETT, 1934; POST, 1956), the magnetic mirror machine (POST, 1958), and the Stellarator (SPITZER, 1958). However attempts at practical realization of these designs have so far been foiled by unstable motions which develop in the plasma, tending to destroy the configuration in times much shorter than the classical diffusion times (ROSENBLUTH, 1965).

The unstable motions may be broadly divided into two classes (LEHNERT, 1967). On the one hand are the localized microscopic, or kinetic instabilities. These small scale, high frequency oscillations in both position and velocity space are thought to cause turbulence and anomalously high diffusion rates (BOHM et al, 1949; TAYLOR, 1962; DRUMMOND and ROSENBLUTH, 1962; FOWLER, 1965; VEDENOV, 1968). They can result from differences in the motion of various particles occupying the same macroscopic volume, as in the two-stream instability (BOHM and GROSS, 1949). A class of micro-instabilities results from the anisotropy imposed by the (preferred) direction of the magnetic field itself (e.g. the "loss-cone" instability (ROSENBLUTH and POST, 1965; 1966)). Thus the micro-instabilities are due to properties of the particle velocity distribution function and are therefore difficult to control.

On the other hand are the macroscopic hydrodynamic or hydromagnetic instabilities, which may involve the unstable motion of macroscopic portions of the plasma. Such large scale motion of the plasma (e.g. the writhing of a plasma column or torus) can result in rapid destruction of the plasma body.

The problem of plasma instabilities has presented a particularly serious obstacle in the path of the development of a controlled thermonuclear fusion reactor, and it is with reference to this project that much of the research on instabilities is conducted. This is because a successful reactor must rely on the isolation of a body of very hot plasma (temperature $T \sim 10^8$ to 10^9°K) with a lifetime t and number density n sufficient to satisfy the criterion $nt > 10^{16}$ (LAWSON, 1957) for a D - D reaction, and $nt > 10^{14}$ for a D - T reaction. An excellent review of experimental and theoretical research on plasma instabilities, with emphasis on the thermonuclear fusion problem, is given by LEHNERT (1967). Further developments include the recent news of successful experiments with the toroidal Tokamak machines (ARTSIMOVICH, BOBROVSKY et al, 1969; PEACOCK et al, 1969; ARTSIMOVICH, ANASHIN et al, 1969), to which thermonuclear physicists are increasingly directing their attention (PEASE, 1970).

However it is not only in this sphere that the topic is important. Instabilities have proved a nuisance in the operation of some types of ion rocket engines (JAHN, 1968, p.133). Unstable microscopic oscillations are thought to be responsible for anomalous magnetron current flow above cut-off (HIRSCH, 1966), which represents a problem in the use of crossed-field devices (WELSH et al, 1960) in the direct conversion of heat to electrical energy. Unstable oscillations are important in the electron 'plasma' in solid-state devices (DRUMMOND, 1965). A number of unstable modes occur in situations of interest in astrophysics and in the earth's ionosphere. LEHNERT (1967) and PIDDINGTON (1969, p.16) provide lists of such phenomena, to which should be added the proposal (LIN and SHU, 1964; HOSKING, 1969) that hydromagnetic instability could play an important role in the formation of spiral structure in galaxies.

The full understanding of plasma instabilities is a challenge in itself. Also of great importance is the fact that experimental investigation of the properties of a very hot plasma is severely hindered by difficulties such as short life-time and diagnostic ambiguities introduced by instabilities. Clearly a stabilized high-temperature discharge would be of great value in obtaining in the laboratory experimental information on such phenomena as kinetic transport effects, which have received extensive theoretical investigation (THOMPSON, 1965b; c).

It has long been recognized that the hydromagnetic instabilities are strongly governed by the geometry of the confining magnetic field. A criterion for stability was proposed at an early stage by TELLER (see BISHOP, 1960), who related the curvature of the confining field lines to the tendency of the plasma to undergo unstable interchanges of adjacent flux tubes. Also, explanations of the fluting instability in terms of charged particle drifts associated with field gradients and curvature have been advanced

(ROSENBLUTH and LONGMIRE, 1957). The magnetic field geometry is a factor over which the experimenter has significant control. A large effort, both experimental and theoretical, has therefore been directed to the design of magnetic field configurations which are expected to provide stability against hydromagnetic disturbances. Examples in which some success has been achieved are the magnetic well configurations (TAYLOR, 1965), the stabilized mirror (IOFFE, 1965) and the cusp geometries (BERKOWITZ et al, 1958; KADOMTSEV and BRAGINSKY, 1958).

The use of a constricted discharge, stabilized by an externally applied magnetic field of favourable curvature, as a tool for the investigation of transport effects in a hot plasma and for the observation of a controlled transition from stability to instability, was proposed by SEYMOUR (1961). In a series of papers (SEYMOUR, 1961a; b; c; 1963) he investigated theoretically the temperature distribution, thermoelectric effects, stability and transport processes in a discharge between electrodes, the surface of which is shaped by the pressure of an external magnetic field so that it approximates a hyperboloid of one sheet. His stability analysis involved approximations introduced in the interests of mathematical tractability.

One of the main objects of the present work is to conduct a more detailed investigation of the hydromagnetic stability properties of this particular discharge geometry, which could also have relevance to the experiments being conducted on the dense plasma focus which forms at the end of the central electrode of a coaxial plasma gun (see, for example, BOSTICK et al, 1969; COMISAR, 1969).

The theoretical description of hydromagnetic instabilities relies on the assumption that the motion of particles in the same macroscopic volume can be represented by an average fluid velocity. That is, the plasma is assumed to behave like a conducting fluid. In a fluid the motion of the particles is restricted by collisions so that they tend to remain grouped in close association, and a given volume element retains its identity, being always composed of the same particles. In a plasma with a high enough collision rate, the pressure tensor $\underline{\underline{P}}$ is simply a scalar, and the equations of continuity and of motion, derived by taking moments of the Boltzmann equation, plus an equation of state linking the pressure p and the density ρ, form a complete set.

When collisions are not so frequent the situation is rather different. There is a lack of cohesion, and the local centre of mass motion has no meaning in terms of a fluid velocity. However in the presence of a magnetic field, the charged particles are forced to gyrate about the field lines, so that motion across the field is restricted. The situation with a strong magnetic field in a low-pressure plasma has been treated by CHEW, GOLDBERGER and LOW (1956), who use the collisionless Boltzmann equation to derive a system of one-fluid hydrodynamic equations for which closure is achieved by neglecting transport along the field. The random phase distribution of the gyrating particles provides the effect of an isotropic velocity distribution in the plane perpendicular to the field. The fluid velocity is just the zero-order electric drift, $\underline{v}_E = \frac{\underline{E} \times \underline{B}}{B^2}$, where $\underline{E}$ is the electric field, and in zero order in the spatial and time derivatives of $\underline{B}$, the pressure tensor is diagonal (SCHMIDT, 1966, p.76), with equal components for the isotropic motion perpendicular to $\underline{B}$:

$$\underline{\underline{P}} = p_\perp \underline{\underline{I}} + (p_\parallel - p_\perp)\underline{b}_o\underline{b}_o ,$$

where $\underline{\underline{I}}$ is the unit tensor and $\underline{b}_o = \frac{\underline{B}}{B}$. The system of equations is closed by two separate adiabatic equations of state, one for each of $p_\perp$ and $p_\parallel$.

Having adopted the fluid model, the plasma-magnetic field system in mechanical equilibrium is tested for stability with respect to the small perturbation $\underline{\xi}(\underline{r}_o,t)$ which represents the displacement of a fluid element from its equilibrium position $\underline{r}_o$. The equations describing the perturbed fluid are linearized with respect to the small perturbations in pressure, magnetic field and plasma density. In certain cases of great geometrical simplicity (e.g. plasma slab, linear pinch) these linearized equations can be solved by considering "normal modes":

$$\underline{\xi}_n(\underline{r}_o,t) = \underline{\xi}_n(\underline{r}_o)e^{i\omega_n t} ,$$

and obtaining a dispersion relation for the eigenvalues ω_n. Imaginary solutions for ω_n then indicate instability.

An alternative method which relies on the existence of an energy integral for the system avoids the usually difficult normal-modes solution procedure by exploiting the property of self-adjointness of the operator $\underline{F}$ in the linearized equation of motion:

$$\rho_o \ddot{\underline{\xi}} = \underline{F}(\underline{\xi}) \ ,$$

and the completeness of its eigenfunctions, $\underline{\xi}_n$, to show (BERNSTEIN et al, 1958; KULSRUD, 1964) that the second-order variation in potential energy of the system, $\delta W(\underline{\xi},\underline{\xi})$ can be negative if and only if there exists at least one negative value of ω_n^2. This then verifies what one expects on physical grounds – that if δW can be made negative by some perturbation field $\underline{\xi}$, the system is unstable. This variational approach is essentially no different from the normal-mode procedure, but has the advantage (BERNSTEIN et al, 1958) that if detailed knowledge of growth rates is not required, the question of the stability or otherwise of geometrically complicated systems can often be answered very directly. The normal-mode technique is the more general, however, since it can be applied in systems where dissipative effects are important, for which no potential function exists (THOMPSON, 1962). Since the stability analyses to be presented here hinge on the importance of magnetic field geometry, the problem will be approached via the energy principle of ideal hydromagnetics.

When the plasma pressure is small compared with the magnetic pressure, the potential energy function

$$W = \int d\tau \left(\frac{p}{\gamma - 1} + \frac{B^2}{2\mu_o}\right) ,$$

where γ is the usual ratio of specific heat and μ_o is the permeability of free space, is dominated by the magnetic energy term. Therefore δW will in general be dominated by the change in magnetic energy, which must be positive (LUNDQUIST, 1952; SEYMOUR, 1961c) if the magnetic field can be approximated as being closely curl-free. The only dangerous perturbations are therefore those which leave the magnetic field unchanged. The special perturbation which results in the interchange of adjacent flux tubes of equal flux meets this requirement, and was first discussed by ROSENBLUTH and LONGMIRE (1957). Their stability criterion, that the tube volume must decrease with decreasing pressure, is certainly a sufficient condition for stability against this interchange. However, for reasons discussed here in Chapter 1, it is doubtful if the condition $\delta\tau < 0$ (τ being the tube volume) is necessary and sufficient for stable interchanges of this type.

The Rosenbluth-Longmire analysis treats systems with planar field lines, where the ends of the tubes of matter interchanged lie at the same magnetic scalar potential, ψ. They write the criterion $\delta\tau < 0$ in terms of the radius of curvature of the field lines, giving a result in agreement with the curvature criterion of Teller. However, when considering systems in which the field lines enter terminating plates, there can arise situations where the ends of the flux tubes are not at the same ψ. Examples are systems terminated by end plates which are inclined to the field lines (COLGATE and YOSHIKAWA, 1964), and the discharge between electrodes, with both axial and azimuthal magnetic fields. Since the latter system is of particular interest here, it has been found necessary to consider important end effects which may arise through this geometrical effect. It has also been necessary to generalize the curvature form of the Rosenbluth-Longmire criterion to include the case where the field lines are twisted space curves, as in the constricted discharge. It is noted that some doubt may be felt with regard to the application of the interchange criterion to a system with shearing field lines. However, as explained by ROBERTS and TAYLOR (1965), shear should not stabilize the 'twisted, slicing' interchange mode in systems of *finite* length, but merely lead to a reduction of growth rate as kinetic energy is spent on rotating the flux tubes as well as displacing them vertically.

Assuming a volume distribution of current in the constricted discharge, there is expected to be a low-pressure region near the boundary (SEYMOUR, 1961c) where the pressure decreases slowly and smoothly to zero, and the Rosenbluth-Longmire criterion is applied. In this region, the hydrodynamics of CHEW, GOLDBERGER and LOW (1956) should apply, and for this reason the thermodynamic treatment of the interchange is here generalized for the case of the pressure tensor discussed above, using the 'double-adiabatic' equations of state to determine δW_p, the change in material energy produced by the interchange. As for the scalar pressure case, stability can be achieved in magnetic-well configurations. In fact if $\delta(p_\perp + p_{\parallel})$ is constant along the field lines, (where δ operates on equilibrium quantities, and gives the variation between the two interchanged flux tubes), the stability criterion becomes identical with that derived for the case of scalar pressure. It is therefore necessary to examine, in Chapter 2, the equilibrium structure of the discharge, particularly in the low-pressure boundary region. It is found that the assumption of azimuthal symmetry leads to the above requirement being met, to first order in the ratio of plasma pressure at a point to magnetic pressure at the same point. The symbol ν will be used for this ratio, since the familiar symbol β is more commonly used for the ratio of plasma pressure at a point in the plasma to magnetic pressure at another point outside the plasma.

The application of the stability criterion to the constricted discharge is covered in detail in Chapter 3. Careful attention is given to the problems which arise because the end plates do not coincide with surfaces of constant ψ. The analysis elaborates the earlier work of SEYMOUR (1961c) and the exact result obtained for the discharge current which is critical for the onset of unstable interchanges is reconciled with his approximate expression. The importance of the magnetic field geometry in relation to the critical current is emphasised by the interpretation of the result in terms of field-line curvature at the discharge boundary.

A contrasting version of the constricted discharge is that in which the current distribution is restricted to a very thin surface sheet. Thus the highly electrically conducting plasma is held in mechanical equilibrium by the interaction of this sheet current with the magnetic field on either side of the surface (KRUSKAL and SCHWARZSCHILD, 1954). In this discharge the plasma and magnetic field pressures are comparable. It is then more appropriate to consider stability with respect to the general perturbation $\underline{\xi}(\underline{r}_o, t)$. While it is recognised that such systems are ideal, and not likely to be achieved in practice, their analysis sheds light, at least in the sense of an approximation, on the stability properties of more realistic configurations in which, due to finite electrical conductivity of the plasma, inter-diffusion of field and plasma takes place.

Before analysing the constricted discharge in this form, detailed discussions of some basic aspects of the hydromagnetic energy principle (BERNSTEIN et al, 1958) are given in Chapters 4 and 5. This approach is based on a fluid theory in which the strength of collisions is assumed to be such that the pressure always remains scalar, but the electrical conductivity may be regarded as perfect. Other approaches, in which collisions are not considered to be so effective, are available (KULSRUD, 1964), but it is the fluid theory which is applied here since a relatively dense plasma is assumed. Also in this general analysis the electric current configuration combines a volume distribution within the plasma with a surface sheet current at the boundary.

Of particular interest is the application of the general theory to the discharge between electrodes. A compromise with reality must be made here, since the conduction of electricity is assumed to be perfect throughout the plasma whereas in practice the electrodes, which represent heat sinks, must cause significant cooling of the gas, leading (SPITZER, 1962, pp. 136-143) to a reduction of electrical conductivity. Of chief concern is the principle of 'freezing-in' of the magnetic field in the plasma (ALFVÉN and FÄLTHAMMAR, 1963, p.189), preventing slippage of matter across the field. The degree to which this approximation must hold depends on the time scale of the phenomena involved. At characteristic discharge temperatures the field-plasma diffusion time is in the region of milli-seconds (SEYMOUR, 1961c), so the approximation will be justified only for unstable motions occurring on a time-scale which is much shorter than this. In the past, configurations have been destroyed in times of a few micro-seconds (PEASE, 1970), so it appears reasonable to assume frozen-in fields.

The analysis of BERNSTEIN et al (1958) was not related to any specific geometry, but assumed only that the 'region of interest' was surrounded by a perfectly conducting shell. In application to the discharge between electrodes, it is clear that insulating supports must be present to avoid short-circuiting of the discharge. Thus it is important that the system geometry be generalized in this sense, to permit application of the result in the desired manner. As will be seen, this necessitates great care in applying Gauss' integral transform as is required at a number of places in the proof, and leads to some modification of the final expression for δW.

Bernstein et al obtain the expression for δW as a second-order functional of $\underline{\xi}$ effectively by integrating the second-order expression for $\frac{dW}{dt}$, given by

$$\frac{dW}{dt} = - \int_{\tau_p(o)} d\tau_o \; \underline{\dot{\xi}} \cdot \underline{F}(\underline{\xi}) \ ,$$

where $\tau_p(o)$ is the equilibrium volume of the plasma. The fact that $\underline{F}(\underline{\xi})$ is self-adjoint (which follows from the existence of an energy integral for the system, or which may be proved directly (KADOMTSEV, 1966)), enables the integration with respect to time to be carried out, yielding

$$\delta W = - \tfrac{1}{2} \int_{\tau_p(o)} d\tau_o \underline{\xi}(\underline{r}_o,t) \cdot \underline{F}(\underline{\xi}(\underline{r}_o,t)) \ .$$

An expression for δW may also be obtained by writing down the potential energy function and evaluating the second-order variation with respect to $\underline{\xi}$. However a search of the literature shows that as yet there is no complete derivation of δW by this method. VAN KAMPEN and FELDERHOF (1967) use this approach to derive the change in potential energy for a system in which the plasma is assumed to extend to infinity. In practice, of course, the plasma is a finite body. Therefore the work of Van Kampen and Felderhof is here extended to derive δW for the case of a finite system in which the plasma is in contact with a vacuum region and with electrodes. Essentially this necessitates a calculation of δW_{BE}, the change in the energy associated with the magnetic field external to the plasma, which is then added to the expression of Van Kampen and Felderhof, to obtain the final result. In this analysis the care which must be exercised in applying Gauss' theorem, because of the existence of necessary insulators, is most evident. Also, since δW_{BE} is evaluated as the work done by the perturbation against the pressure of the vacuum magnetic field at the plasma/vacuum interface, it is then possible to critically examine the interpretation commonly given (SCHMIDT, 1966, p.125) to the surface contribution δW_S (BERNSTEIN et al, 1958) as the work done against the surface current in deforming the boundary. This interpretation is found to be incorrect.

A final consideration concerns the 'extended energy principle' proposed by BERNSTEIN et al (1958) in order to allow one to ignore certain constraints on the perturbation when δW is minimized to find the most dangerous motion. Bernstein et al give a short proof of the validity of this apparently drastic step, while ROSE and CLARK (1961, pp.284-286) discuss the extension in physical terms, stressing its importance. On the other hand, however, SCHMIDT (1966, pp.144-149) in his analysis of the linear pinch, ignores the constraint arising from the continuity of stress across the plasma/vacuum interface, merely stating that '... this condition was used in deriving δW_S and is already incorporated in ...' the final result for δW. Because of this confusion in the literature as to the significance of the extended energy principle, and in view of the lack of a detailed rigorous mathematical treatment, such a proof is developed here, in Chapter 5.

Some of the simplest applications of the energy principle are to systems consisting of a field-free plasma in contact with a vacuum region. For these systems the energy principle reduces, in its interpretation, to the curvature criterion of Teller. Only the surface terms in δW are important, and the contribution at each point on the plasma surface is just the normal curvature of the field line at that point, weighted by the square of the magnetic field strength and the squared magnitude of the perturbation component ξ_n normal to the surface. It is evident that the choice of geometry of the plasma body is of particular importance.

In Chapter 6 the application to the field-free version of the constricted discharge is straight-forward, and yields a criterion for stability which, as before, can be expressed in terms of a critical discharge current. Also as before, the result lends itself very well to a geometrical interpretation in terms of field-line curvature on the surface. To complete the curvilinear generalization of the linear pinch (TAYLER, 1957; SCHMIDT, 1966, p.144), consideration is also given to the constricted discharge with trapped internal magnetic field but no internal electric current density (all current flows in the surface current sheet). A *sufficient* condition for stability is derived for this system, showing great similarity with the *necessary* and *sufficient* result for the field-free system, and indicating that the internal field in such a case has no effect on the form of δW_S.

In planning an experiment based on the constricted discharge, the technologist has at his disposal a number of parameters, including the size and shape of the system, the initial pressure and volume of the gas, and the magnetic field produced by coils external to the discharge. Given these parameters it would clearly be useful to have some idea of the maximum discharge current which may be passed without the system becoming unstable in the sense described above. For the geometry of interest in this thesis, it is possible to obtain an approximation for the critical current in terms of the initial conditions of the experiment, for the practical case of a thin discharge, with a constriction ratio of radius at the electrodes to radius at the median plane not very different from unity.

REFERENCES

ALFVÉN, H., and FÄLTHAMMAR, C. (1963) - "Cosmical Electrodynamics", 2nd ed. (Oxford University Press: Oxford).

ARTSIMOVICH, L. A., ANASHIN, A. M., GORBUNOV, E. P., IVANOV, D. P., PETROV, M. P., and STRELKOV, V. S. (1969) - JETP Letters, 10, 82.

ARTSIMOVICH, L. A., BOBROVSKY, G. A., GORBUNOV, E. P., IVANOV, D. P., KIRILLOV, V. D., KUZNETSOV, E. I., MIRNOV, S. V., PETROV, M. P., RAZUMOVA, K. A., STRELKOV, V. S., and SHCHEGLOV, D. A. (1969) - Nuclear Fusion Special Supplement, p.17. - Plasma Physics and Controlled Nuclear Fusion Research, Novosibirsk Conference Proceedings, 1968.

BENNETT, W. H. (1934) - Phys. Rev. 45, 890.

BERKOWITZ, J., FRIEDRICHS, K. O., GOERTZEL, H., GRAD, H., KILLEEN, J., and RUBIN, E. (1958) - Proc. 2nd U.N. Conf. on the Peaceful Uses of Atomic Energy 31, 171.

BERNSTEIN, I. B., FRIEMAN, E. A., KRUSKAL, M. D., and KULSRUD, R. M. (1958) - Proc. Roy. Soc. A224, 17.

BISHOP, A. S. (1960) - "Project Sherwood - The U.S. Program in Controlled Fusion", p.87. (Anchor Books, Doubleday & Co.: Garden City, New York).

BOHM, D., BURHOP, E. H. S., and MASSEY, H. S. W. (1949) - in A. GUTHRIE and R. K. WAKERLING, eds., "Characteristics of Electrical Discharges in Magnetic Fields", pp.62. (McGraw-Hill Book Co.: N.Y.).

BOHM, D., and GROSS, E. P. (1949) - Phys. Rev. 75, 1864.

BOSTICK, W. H., GRUNBERGER, L., and PRIOR, W. (1969) - "Third European Conference on Controlled Fusion and Plasma Physics: Utrecht", p.120. (Wolters-Noordhoff Publishing Co.: Groningen).

CHEW, G. F., GOLDBERGER, M. L., and LOW, F. E. (1956) - Proc. Roy. Soc. A236, 112.

COLGATE, S. A., and YOSHIKAWA, S. (1964) - Phys. Fl. 7, 1877.

COMISAR, G. G. (1969) - Phys. Fl. 12, 1000.

DRUMMOND, J. E. (1965) - "Plasma Physics", p.323. (IAEA: Vienna).

DRUMMOND, W. E., and ROSENBLUTH, M. N. (1962) - Phys. Fl. 5, 1507.

FOWLER, T. K. (1965) - Phys. Fl. 8, 459.

HIRSCH, E. H. (1966) - Int. J. Electronics 21, 521.

HOSKING, R. J. (1969) - Aust. J. Phys. 22, 505.

IOFFE, M. S. (1965) - "Plasma Physics", p.421 (IAEA: Vienna).

JAHN, R. G. (1968) - "Physics of Electric Propulsion". (McGraw-Hill: New York).

KADOMTSEV, B. B. (1966) - "Reviews of Plasma Physics", Vol. 2. (Ed.: M. A. LEONTOVICH), pp.158-61. (Consultants Bureau: New York).

KADOMTSEV, B. B., and BRAGINSKY, S. I. (1958) - Proc. 2nd U.N. Conf. on the Peaceful uses of Atomic Energy 32, 233.

KAMPEN, VAN N. G., and FELDERHOF, B. U. (1967) - "Theoretical Methods in Plasma Physics", Chapter VII. (North-Holland Pub. Co.: Amsterdam).

KRUSKAL, M., and SCHWARZSCHILD, M. (1954) - Proc. Roy. Soc. A223, 348.

KULSRUD, R. (1964) - International School of Physics - "Enrico Fermi". Proceedings, Course XXV, "Advanced Plasma Theory", (Ed.: M. N. ROSENBLUTH), p.54.

LAWSON, J. D. (1957) - Proc. Phys. Soc. (London) B70, 6.

LEHNERT, B. (1967) - Plasma Physics 9, 301.

LIN, C. C., and SHU, F. H. (1964) - Astrophys. J. 140, 646.

LUNDQUIST, S. (1952) - Ark. Fys. 5, 297.

PEACOCK, N. J., ROBINSON, D. C., FORREST, M. J., WILCOCK, P. D., and SANNIKOV, V. V. (1969) - Nature, London 224, 488.

PEASE, R. S. (1970) - Physics Bulletin 21, 538. (Institute of Physics and the Physical Society).

PETSCHEK, H. E. (1965) - "Plasma Physics", p.253. (IAEA: Vienna).

PIDDINGTON, J. H. (1969) - "Cosmic Electrodynamics". (John Wiley & Sons: New York).

POST, R. F. (1956) - Rev. Mod. Phys. 28, 338.

POST, R. F. (1958) - Proc. 2nd U.N. Conf. on the Peaceful Uses of Atomic Energy 32, 245.

ROBERTS, K. V., and TAYLOR, J. B. (1965) - Phys. Fl. 8, 315.

ROSA, R. J., and KANTROWITZ, A. (1960) - "Direct Conversion of Heat to Electricity", edited by J. KAYE and J. A. WELSH, Chapter 12. (John Wiley & Sons: New York).

ROSE, D. J., and CLARK, M., Jr. (1961) - "Plasmas and Controlled Fusion". (Wiley: New York).

ROSENBLUTH, M. N. (1965) - "Plasma Physics", p.485. (IAEA: Vienna).

ROSENBLUTH, M. N., and LONGMIRE, C. (1957) - Ann. Phys. 1, 120.

ROSENBLUTH, M. N., and POST, R. F. (1965) - Phys. Fl. 8, 547.

ROSENBLUTH, M. N., and POST, R. F. (1966) - Phys. Fl. 9, 730.

SAHA, M. N., and SAHA, N. K. (1934) - "A Treatise on Modern Physics", p.630. (Indian Press Ltd.: Allahabad and Calcutta).

SCHMIDT, G. (1966) - "Physics of High Temperature Plasmas". (Academic Press: New York).

SEYMOUR, P. W. (1961a) - Aust. J. Phys. 14, 129.

SEYMOUR, P. W. (1961b) - Aust. J. Phys. 14, 279.

SEYMOUR, P. W. (1961c) - Aust. J. Phys. 14, 321.

SEYMOUR, P. W. (1963) - Aust. J. Phys. 16, 216.

SPITZER, L., Jr. (1958) - Phys. Fl. 1, 253.

SPITZER, L., Jr. (1962) - "Physics of Fully Ionized Gases", 2nd ed. (Interscience: New York).

TAYLER, R. J. (1957) - Proc. Phys. Soc. (London) B70, 1049.

TAYLOR, J. B. (1962) - Nuclear Fusion Supplement, Part 2, 477.

TAYLOR, J. B. (1965) - "Plasma Physics", p.449. (IAEA: Vienna).

THOMPSON, W. B. (1962) - "An Introduction to Plasma Physics", p.98. (Pergamon Press: Oxford).

THOMPSON, W. B. (1965a) - "Plasma Physics", pp.199-200. (IAEA: Vienna).

THOMPSON, W. B. (1965b) - "Plasma Physics", p.7. (IAEA: Vienna).

THOMPSON, W. B. (1965c) - "Plasma Physics", p.207. (IAEA: Vienna).

TONKS, L., and LANGMUIR, I. (1929) - Phys. Rev. 33, 195.

VEDENOV, A. A. (1968) - "Theory of Turbulent Plasma", Chapter 9. (Iliffe Books Ltd.: London).

WELSH, J. A., HATSOPOULOS, G. N., and KAYE, J. (1960) - "Direct Conversion of Heat to Electricity", edited by J. KAYE and J. A. WELSH, Chapter 5. (John Wiley and Sons: New York).

CHAPTER I

THE STABILITY CRITERION FOR INTERCHANGES IN A LOW PRESSURE PLASMA

1.1 INTRODUCTION

The purpose of this chapter is to discuss from a general viewpoint the stability of an equilibrium plasma-magnetic field system with respect to the perturbation which interchanges adjacent magnetic flux tubes. The magnetic field is assumed to be embedded in a low pressure plasma of very high electrical conductivity. The discussion assumes an energy principle for stability, the net change in system potential energy produced by the interchange being evaluated. Since, in later work, particular interest will be taken in systems which are terminated by conducting end-plates (for example, electrodes), attention is here attracted to important effects which arise from the high conductivity of the end-plates and their orientation with respect to the magnetic field lines.

The usual 'thermodynamic' treatment (ROSENBLUTH and LONGMIRE, 1957), which assumes a scalar pressure, may be generalized under the assumption of a pressure tensor of the form

$$\underline{\underline{P}} = p_{\perp}\underline{\underline{I}} + (p_{\parallel} - p_{\perp})\underline{b}_o\underline{b}_o \ ,$$

where $\underline{B} = B\underline{b}_o$ and $\underline{\underline{I}}$ is the unit tensor, by using the 'double-adiabatic' equations of CHEW, GOLDBERGER AND LOW (1956).

1.2 THE VARIATION IN SYSTEM MAGNETIC ENERGY

The energy associated with the equilibrium magnetic field $\underline{B}_o$ in a flux tube is represented by the following integral over the volume of the tube:

$$W_B = \frac{1}{2\mu_o} \int_{\text{Tube}} B_o^2 d\tau \ . \tag{1.1}$$

Material in the volume element $d\tau_1$ at the point P_1 in tube number 1 is assumed to be interchanged with the material in $d\tau_2$ at P_2 in the adjacent tube number 2. Tube 2 is located in the direction of decreasing pressure from tube 1. It is a simple matter to show (see, for example, SCHMIDT, 1966, equation 5.107) that the change ΔW_B in system magnetic energy produced by this motion is given by

$$\Delta W_B = - \frac{\delta(\phi^2)}{2\mu_o} \delta \int_{\text{Tube}} \frac{B_o d\ell}{\phi} \ , \tag{1.2}$$

where ϕ is the flux through a tube (constant, of course, along the tube), and ℓ represents distance along the tube. The operator δ gives the variation in equilibrium quantities in travelling from tube 1 to tube 2. Equation (1.2) may be rewritten as

$$\Delta W_B = - \frac{\delta\phi\delta\Psi}{\mu_o} + \frac{(\delta\phi)^2\Psi}{\mu_o\phi} \ , \tag{1.3}$$

where

$$\Psi = \int_{\text{Tube}} B_o d\ell \ . \tag{1.4}$$

For low plasma pressure, the assumption of a closely curl-free magnetic field is made. Then for systems with closed field lines, such as toroids, Stokes' theorem gives $\delta\Psi = 0$, leaving

$$\Delta W_B = \frac{(\delta\phi)^2\Psi}{\mu_o\phi} , \tag{1.5}$$

which, in agreement with a theorem discussed by LUNDQUIST (1952) and SEYMOUR (1961), is positive and therefore, by the energy principle, stabilizing. It is then clear that, as discussed by numerous authors, the only dangerous interchange would be the one which leaves the magnetic field unchanged. This is achieved by exercising our freedom of choice and taking $\delta\phi = 0$.

For open-ended systems the situation is slightly different. Here, the field lines enter conducting end-plates which in general are not orthogonal to $\underline{B}_o$, so that $\delta\Psi \neq 0$. It may then be seen that here the choice of $\delta\phi$ is restricted, for if it were not, ΔW_B could be minimized with respect to $\delta\phi$ simply by setting $\frac{d(\Delta W_B)}{d(\delta\phi)}$ to zero and solving for $\delta\phi$, so that

$$\Delta W_B(\min) = - \frac{\phi(\delta\Psi)^2}{4\mu_o\Psi} \tag{1.6}$$

$$\leqslant 0 ,$$

violating the theorem mentioned above. In fact, on the short time-scale of unstable motions considered here, the magnetic field inside the conducting end-plates should be regarded as constant in time. Then the well-known condition of continuity of the normal component of $\underline{B}$ at an interface, and the fact that in a flux tube interchange the direction of $\underline{B}$ is unchanged, lead to the conclusion that the magnetic field in the plasma must be unchanged.

Thus one is *compelled* to choose $\delta\phi = 0$, so that ΔW_B vanishes. Note that, following ROBERTS and TAYLOR (1965), it is assumed in the above that there is a thin resistive layer at each end plate, so that lines of force are not tied and interchanges can occur.

1.3 THE VARIATION IN SYSTEM MATERIAL ENERGY

Assuming the adiabatic gas law,

$$p\tau^{\gamma} = \text{const.}, \tag{1.7}$$

γ being the usual ratio of specific heats, the energy associated with the matter in a flux tube is given by

$$W_p = \int_{\text{Tube}} \frac{p d\tau}{\gamma - 1} . \tag{1.8}$$

For the interchange described in Section 1.2, it is easy to show, by applying equation (1.7) to the elemental volumes $d\tau_1$ and $d\tau_2$, that the change in material energy of the system is

$$\Delta W_p = \int \left\{ \delta(d\tau)\delta p_o + \gamma p_o \frac{[\delta(d\tau)]^2}{d\tau} \right\} . \tag{1.9}$$

This is the most general form for ΔW_p, but it is noted that the expression commonly used is (ROSENBLUTH and LONGMIRE 1957)

$$\Delta W_p = \delta p_o \delta\tau + \frac{\gamma p_o (\delta\tau)^2}{\tau} , \tag{1.10}$$

obtained by applying equation (1.7) to the volumes τ_1 and τ_2 of the two flux tubes. It should be recognized then that the form (1.10) is arrived at by choosing a special perturbation which leaves the pressure constant along a field line *after* the interchange, as it was in mechanical equilibrium *before* the interchange, when the equation $\nabla p_o = \underset{\sim}{j}_o \times \underset{\sim}{B}_o$ was satisfied. This is, of course, a sensible choice to make since the most dangerous perturbation is that which would cause the greatest lowering of system potential energy. If, instead, the perturbation produced a pressure gradient along the field lines, matter could then flow to equalize the pressure, thereby lowering the potential energy of the system. The perturbed pressure in tube 1, for example, could be written as

$$p_1^* = p_2 \frac{d\tau_2{}^\gamma}{d\tau_1{}^\gamma}$$

$$= (p_1 + \delta p_o)\left(1 + \frac{\delta(d\tau)}{d\tau_1}\right)^\gamma \qquad (1.11)$$

Since p_1 and δp_o are constant along the field line, p_1^* will be constant if $\frac{\delta(d\tau)}{d\tau_1}$ is constant along the field line. However $\delta(d\tau)$ is subject to the constraint:

$$\int \delta(d\tau) = \delta\tau \ . \qquad (1.12)$$

Hence,if $\frac{\delta(d\tau)}{d\tau}$ is constant, it must follow that

$$\frac{\delta(d\tau)}{d\tau} = \frac{\delta\tau}{\tau} = \text{const.} \qquad (1.13)$$

Substituting equation (1.13) into (1.9) then leads to (1.10). The same result may be obtained by formally minimizing expression (1.9) with respect to $\delta(d\tau)$, subject to the constraint (1.12), by the standard procedure of the calculus of variations.

Having obtained expression (1.10), the usual argument (ROSENBLUTH and LONGMIRE 1957) proceeds by assuming that, if p_o is small enough, the following inequality is always satisfied:

$$\left|\frac{\delta p_o}{p_o}\right| > \left|\gamma \ \frac{\delta\tau}{\tau}\right| \ , \qquad (1.14)$$

so that the sign of

$$\left(\frac{\delta p_o}{p_o} + \gamma \ \frac{\delta\tau}{\tau}\right)$$

is just the sign of δp_o, which is negative. Then, since

$$\Delta W_p = p_o \delta\tau \left(\frac{\delta p_o}{p_o} + \gamma \ \frac{\delta\tau}{\tau}\right) \ , \qquad (1.15)$$

and the interchange is chosen so that $\Delta W_B = 0$, the necessary and sufficient condition for stability is that $\delta\tau$ be negative, a condition on the magnetic field geometry alone. However, the general validity of the inequality (1.14) is difficult to prove. It cannot be achieved by assuming singular behaviour of $\frac{\delta p_o}{p_o}$ since, with p_o decreasing smoothly towards the vacuum, $|\delta p_o|$ must always be less than p_o, so that $\left|\frac{\delta p_o}{p_o}\right|$ cannot diverge even as $p_o \to 0$.

An alternative view of this situation could however be obtained by considering the interchange of one tube containing plasma at the very low pressure p_o with another tube which is at practically zero pressure, so that $|\delta p_o| \approx p_o$. In deriving ΔW_p, p_2 would then be set to zero, resulting in

$$\Delta W_p = - p_o \delta\tau \ , \qquad (1.16)$$

which is still of "second-order" in the perturbation since $p_o \approx -\delta p_o$. Thus a necessary and sufficient condition for stability is again $\delta\tau < 0$.

However a difficulty arises here concerning the possibility of having $|\delta p_o| \approx p_o$, bearing in mind that the variation δ is taken over an infinitesimal distance. For, letting χ be a dimensionless coordinate measured normally to the nested magnetic surfaces, and ranging from zero on the magnetic axis, to unity at the outer limit of the system, the equation of mechanical equilibrium,

$$\nabla p_o = \underline{j}_o \times \underline{B}_o \qquad (1.17)$$

leads to

$$p_o = p_o(\chi) \; , \qquad (1.18)$$

so that

$$\delta p_o = p_o'(\chi)\delta\chi \; , \qquad (1.19)$$

where $\delta\chi \ll 1$. Therefore, in order to have $|\delta p_o| \approx p_o$, it would be necessary for

$$\left|\frac{p_o'(\chi)}{p_o}\right| \approx \left|\frac{1}{\delta\chi}\right| \gg 1. \qquad (1.20)$$

Also, from equation (1.17),

$$p_o'(\chi) = h_\chi |\underline{j}_o \times \underline{B}_o| \; , \qquad (1.21)$$

where h_χ is the scale factor for the coordinate χ. Further, the current density j_o is proportional to the number density of charge carriers, while the pressure p_o is proportional to the plasma particle number density. Since all charge carriers must be supplied by the low pressure plasma, it follows that if j_o were expanded in terms of the smallness parameter $\nu = \frac{2\mu_o p_o}{B_o^2}$, its leading term would have to be of at least first order in ν. This is why $\underline{B}_o$ may be approximated as being curl-free in the low pressure situation. Therefore, p_o' must be of at least first order in ν. Since, by definition, p_o is of order ν, it follows that the inequality (1.20) cannot hold.

In view of the above discussion, it is evident from a study of the literature that a rigorous proof showing the condition

$$\delta\tau < 0 \tag{1.22}$$

to be necessary and sufficient for stability has not yet been given for the realistic case of a smoothly decreasing pressure profile. If (1.22) is satisfied, the system is certainly stable, but if $\delta\tau > 0$, the system may or may not be stable, depending on the sign of

$$\left(\frac{\delta p_o}{p_o} + \gamma\,\frac{\delta\tau}{\tau}\right),$$

which depends not only on the geometry of the magnetic field (giving $\delta\tau$), but also on the structure of the discharge (defining δp_o). Summarizing,

(1) if $\delta\tau < 0$, $\Delta W_p > 0$, giving stability;

(2) if $0 < \delta\tau < \frac{\tau}{\gamma p_o}|\delta p_o|$, $\Delta W_p < 0$, giving instability; (1.23)

(3) if $\delta\tau > \frac{\tau|\delta p_o|}{\gamma p_o}$, $\Delta W_p > 0$, giving stability, (1.24)

and it is seen that a second region of stability given by (1.24) could possibly exist.

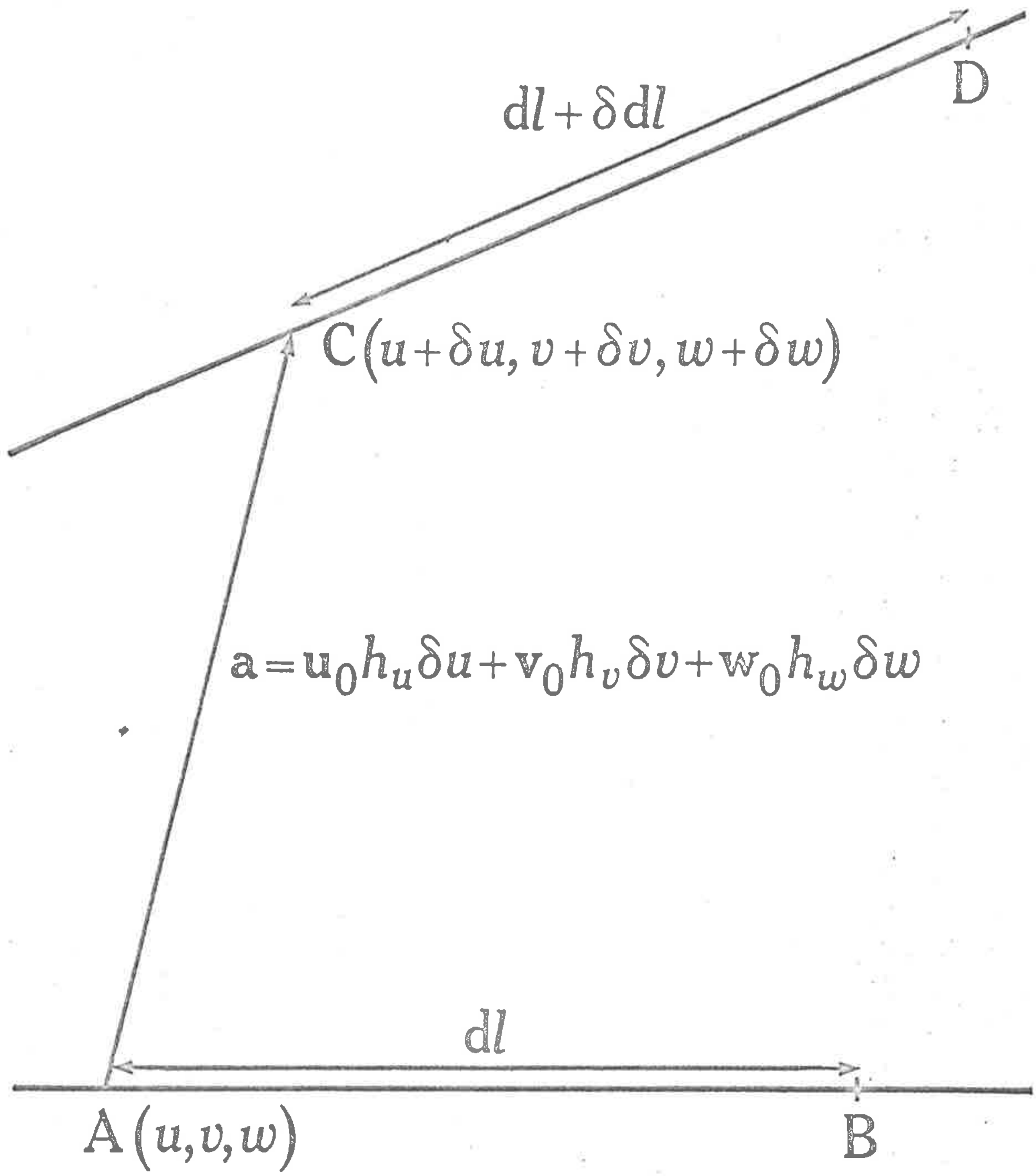

Fig. 1. Adjacent skew field lines AB and CD. The vector a is perpendicular to both lines, in this three-dimensional configuration.

1.4 THE IMPORTANCE OF FIELD LINE CURVATURE

Remembering that the flux ϕ is constant along a tube,

$$\tau = \int s\,d\ell = \int (\phi/B)\,d\ell = \phi \int d\ell/B \ ,$$

where s is the cross sectional area of the tube, and the subscript o on $\underline{B}$ has been dropped for convenience. Then, since $\delta\phi = 0$, the inequality (1.22) becomes

$$\phi\delta \int d\ell/B < 0 \ ,$$

and hence, with $\phi > 0$,

$$\delta \int d\ell/B < 0 \ . \tag{1.25}$$

Now consider two points A and C on adjacent field lines, but at the same magnetic scalar potential ψ ($\underline{B}$ is assumed curl-free), as in Fig. 1.

A and C are joined by the elemental vector $\underline{a}$, where, choosing an orthogonal curvilinear coordinate system (u,v,w), with $\underline{w}_o$ perpendicular to the magnetic surfaces,

$$\underline{a} = \underline{u}_o h_u \delta u + \underline{v}_o h_v \delta v + \underline{w}_o h_w \delta w \ . \tag{1.26}$$

Since A and C are at the same ψ,

$$\underline{a}\cdot\underline{B} = 0 \ . \tag{1.27}$$

The field magnitude on the line AB is B, while on the line CD it is B + δB. Points B and D are at potential $\psi + d\psi$.

Since $\nabla \times \underline{B} = 0$, it follows that, to first order

$$\delta(d\ell)/B = -\,d\ell\delta B/B^2 \ . \tag{1.28}$$

This implies that, referring to (1.4),

$$\delta\Psi = \delta \int B d\ell$$

$$= \int \{\delta B d\ell + B\delta(d\ell)\}$$

$$= 0 \ .$$

That is, the corresponding ends of the interchanged flux tubes must be at the same magnetic scalar potential.

From (1.28),

$$\delta(d\ell/B) = \delta(d\ell)/B - d\ell\delta B/B^2 = -2d\ell\delta B/B^2 \ . \qquad (1.29)$$

Equation (1.29) and the inequality (1.25) indicate the stability of so-called 'magnetic well' configurations with B increasing in the direction of decreasing pressure.

Now

$$\underline{B} \times (\nabla \times \underline{B}) = \nabla(\tfrac{1}{2}B^2) - \underline{B}\cdot\nabla\underline{B} \ ,$$

and so

$$\nabla(\tfrac{1}{2}B^2) = \underline{B}\cdot\nabla\underline{B} \ , \qquad (1.30)$$

in this case of curl-free $\underline{B}$.

Thus, introducing the Frenet-Serret set of unit vectors $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$, and setting $\underline{B} = \underline{e}_1 B$,

$$\nabla B = B\underline{e}_1\cdot\nabla\underline{e}_1 + \underline{e}_1\underline{e}_1\cdot\nabla B \ .$$

Since $\underline{e}_1\cdot\nabla\underline{e}_1 = \partial\underline{e}_1/\partial\ell$ is recognized as the curvature vector $\underline{K} = \underline{e}_2/R$, where R is the unsigned radius of curvature of the magnetic field line,

$$\nabla B - \underline{e}_1\underline{e}_1\cdot\nabla B = B\underline{K} \ ,$$

or

$$\nabla_{\perp} B = B\underline{K} \ , \tag{1.31}$$

where $\underline{\nabla}_{\perp}$ is the gradient operator perpendicular to the field line.

Thus

$$\delta B = \underline{a}\cdot\nabla_{\perp} B = B\underline{a}\cdot\underline{K} \ , \tag{1.32}$$

and so (1.29) becomes

$$\delta(d\ell/B) = - 2\underline{a}\cdot\underline{K}d\ell/B \ . \tag{1.33}$$

Thus stability is achieved if

$$\int d\ell \frac{\underline{a}\cdot\underline{K}}{B} > 0 \ . \tag{1.34}$$

The importance of the direction of the curvature vector $\underline{K}$ is now apparent. Expression (1.33) extends the expression for $\delta(d\ell/B)$ from the case of planar lines (ROSENBLUTH and LONGMIRE) to the more general case of non-planar, twisted lines. The inequality (1.34) permits a qualitative discussion of the stability against interchange, using Teller's familiar curvature criterion (BISHOP 1960, p.87).

1.5 SYSTEMS WITH $\delta\Psi \neq 0$

The form (1.33) only arises when $\underline{a}$ satisfies (1.27). That is, the ends of the tubes of matter supposed to be interchanged must lie at the same magnetic scalar potential ψ. Now in general, the ends of the corresponding flux tubes will not be at the same ψ.

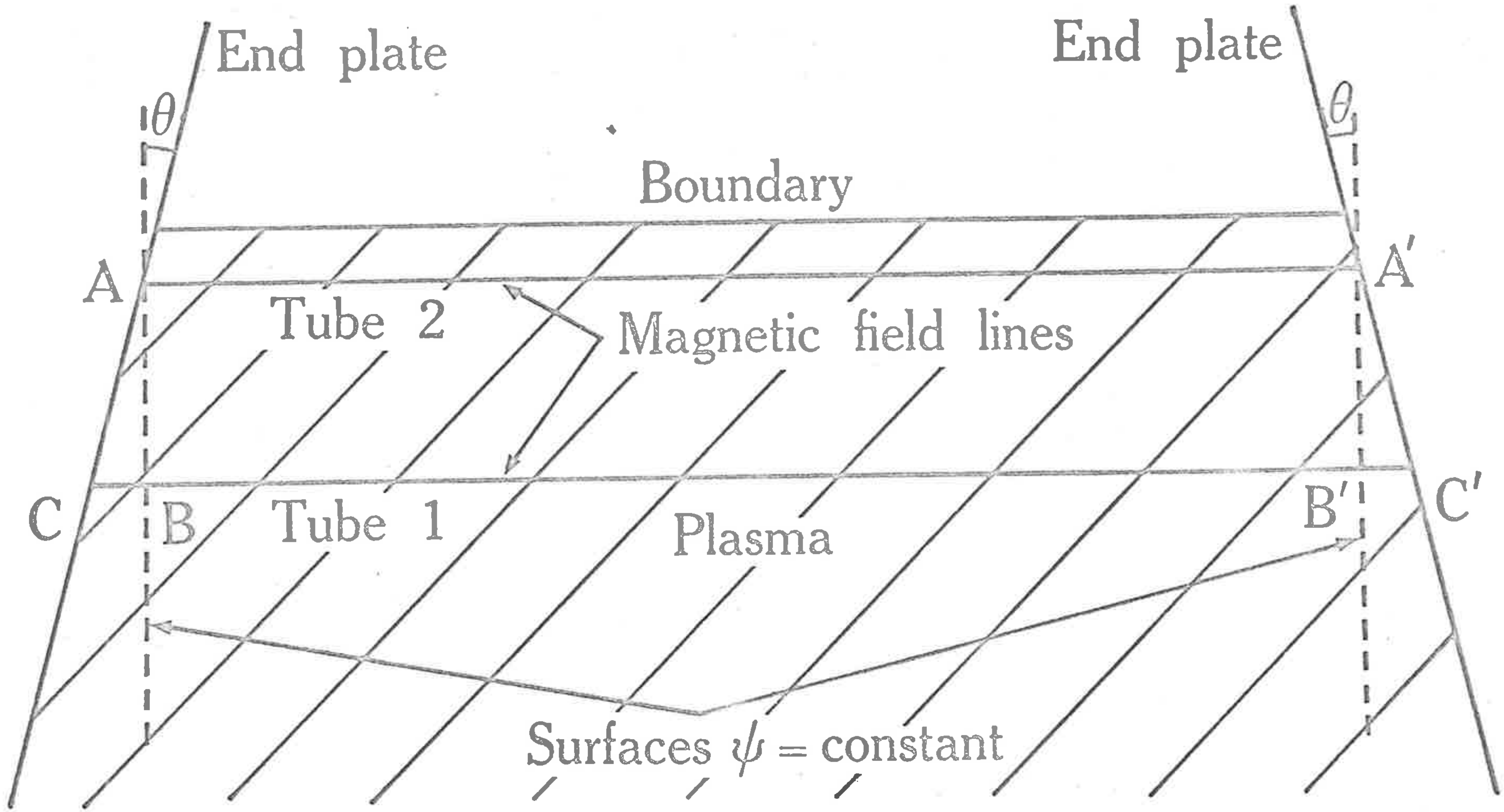

Fig. 2. Plasma slab bounded by slanted end plates (e.g. electrodes).

For example, consider a system with uniform, straight field lines, but terminated by slanted end plates (e.g. electrodes). Such a system (Fig. 2) has been considered by COLGATE and YOSHIKAWA (1964).

Points A and B are at the same ψ, but the ends of the corresponding flux tubes are at A and C, which are *not* at the same ψ. In this case, although Teller's criterion predicts neutral stability, the criterion $\delta\tau = \tau_2 - \tau_1 < 0$ predicts stability, since $\tau_1 < \tau_2$. If the end plates were slanted in the opposite directions, the opposite prediction would be made.

This sort of argument can be extended to a system with curved field lines, terminated by vertical plates. (Fig. 3). Again, points A and B are at the same ψ, while the end points A and C are not. The interchange which is immediately classified unstable by Teller's criterion is the one in which the matter in tube 2, between A and A′, is interchanged with the matter in tube 1, between B and B′ (*not* between C and C′).

Now if the condition of very high electrical conductivity applies throughout the plasma, the lines of force will be frozen into the matter, and it will be impossible for flux tube AA′ to interchange with flux tube CC′ without *all* the matter in each tube being interchanged. The effect which this constraint would have on the stability of the interchange is simply estimated by consideration of a special case, as follows.

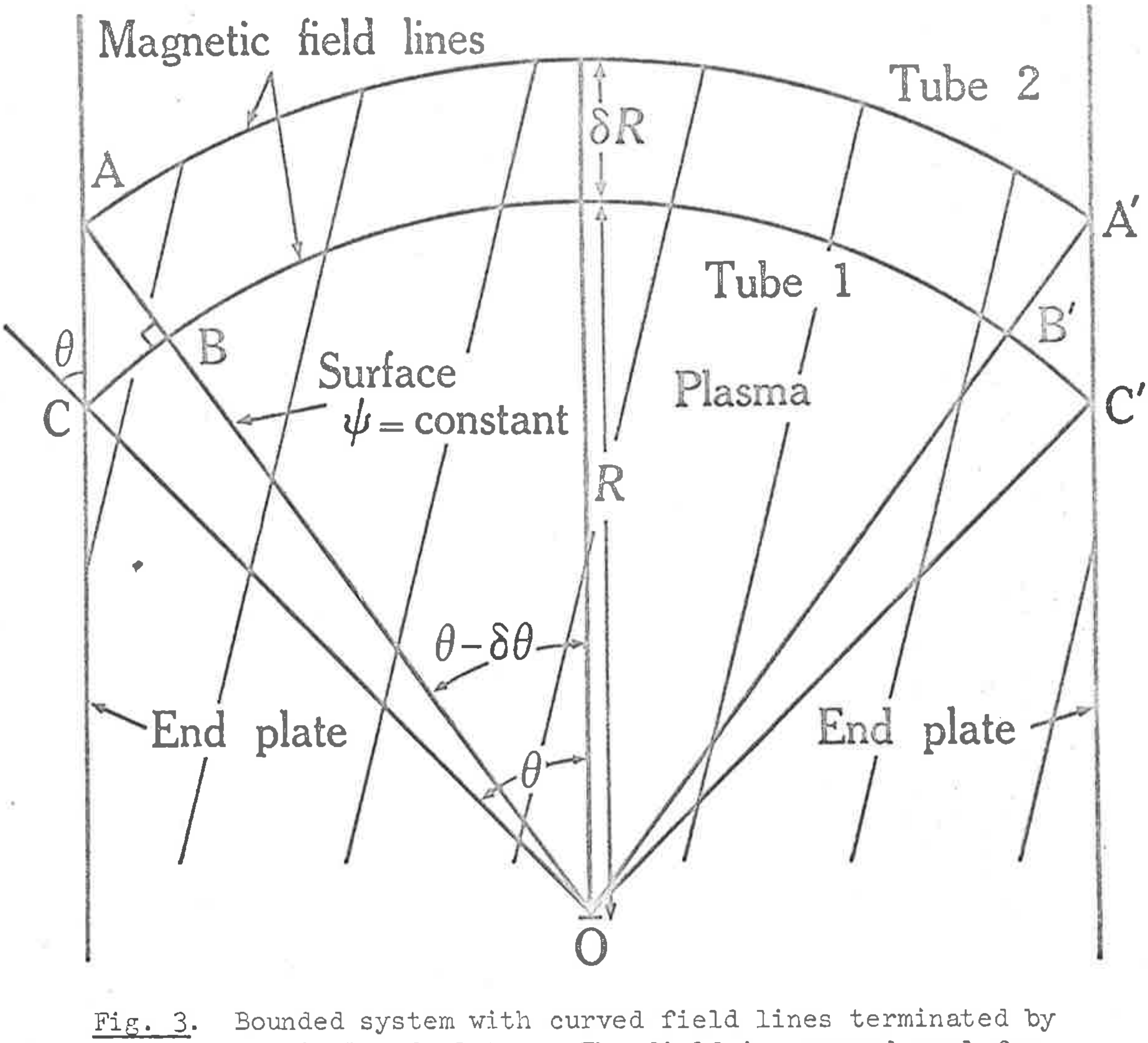

Fig. 3. Bounded system with curved field lines terminated by vertical end plates. The field is assumed curl-free, and $\delta\tau = \tau_2 - \tau_1$.

Suppose the field lines are arcs of concentric circles centred on O (Fig. 3). Then, since B is assumed curl-free, it follows that

$$B = C/R; \quad C = \text{constant} \ , \tag{1.35}$$

and the volume of flux tube 1 is, with flux ϕ = Bx (tube area),

$$\tau_1 = 2\phi R\theta/B = 2\phi R^2\theta/C \ , \tag{1.36}$$

while

$$\tau_2 = 2\phi(R + \delta R)^2(\theta - \delta\theta)/C \approx 2\phi(R^2\theta + 2R\theta\delta R - R^2\delta\theta)/C, \tag{1.37}$$

since both tubes contain the same flux ϕ.

Hence

$$\delta\tau = \tau_2 - \tau_1 = 2\phi(2R\theta\delta R - R^2\delta\theta)/C \ . \tag{1.38}$$

Thus stability is obtained if

$$2\delta R/R < \delta\theta/\theta \ . \tag{1.39}$$

Using Fig. 4, $\delta\theta$ can be expressed in terms of δR as follows:

$$\delta R/R\delta\theta = \tan\beta = \tan(90^\circ - \theta) = \cot\theta \ , \tag{1.40}$$

that is,

$$\delta R/R = \delta\theta/\tan\theta \ . \tag{1.41}$$

Substituting (1.41) into (1.39), it is found that stability is ensured by satisfying the inequality

$$2\theta < \tan\theta \ . \tag{1.42}$$

Thus it appears that the condition of very high conductivity would be strong enough for a stable situation to be achieved over a certain range of θ, even though the field curvature is unfavourable by Teller's criterion. Similarly, a system with favourable

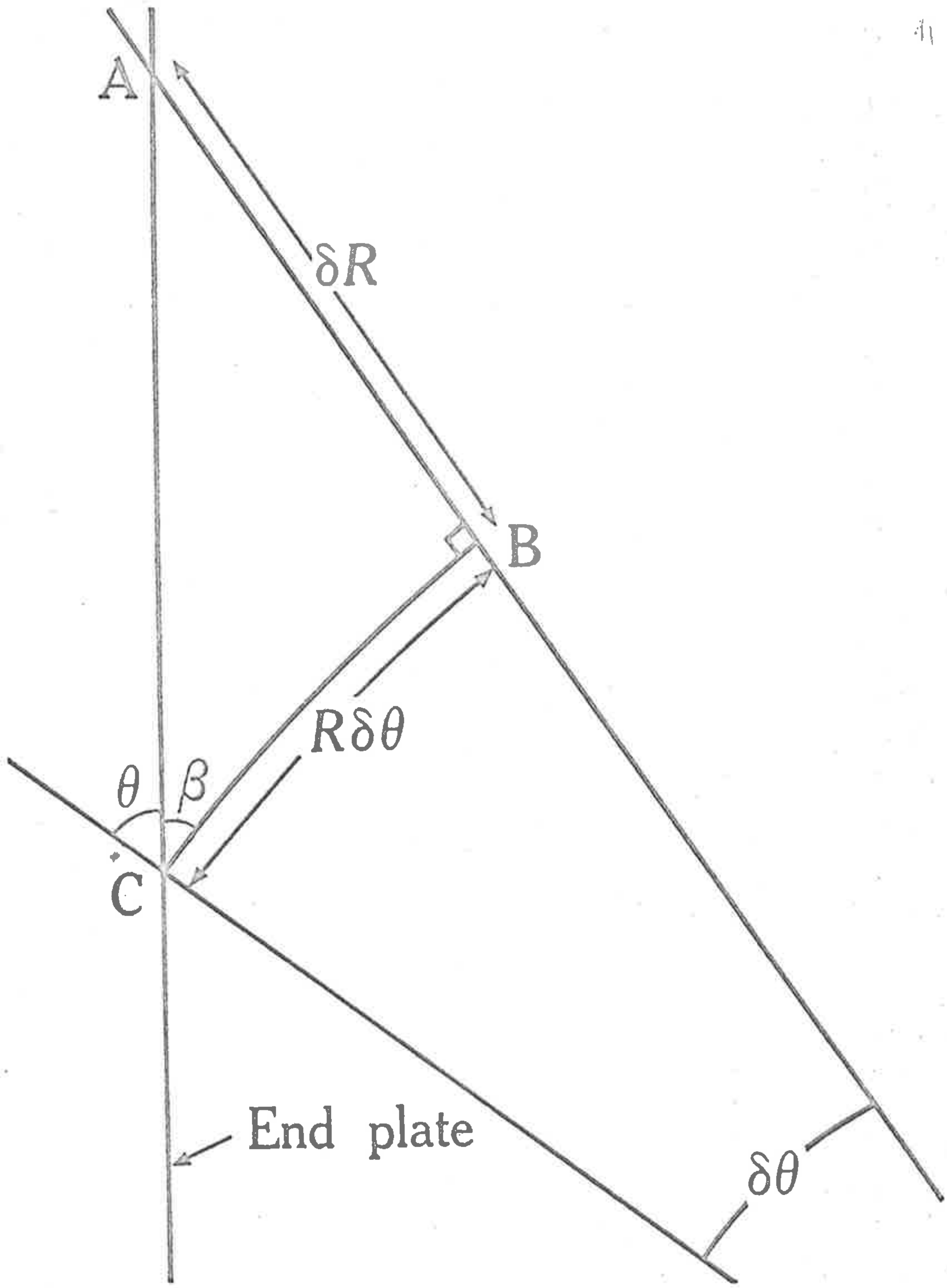

Fig. 4. Detail from Figure 3 for expressing δθ in terms of δR.

curvature may be unstable for a certain range of θ.

From equation (1.38) it is seen that if $\delta\theta$ is zero, then $\delta\tau > 0$, and Teller's criterion correctly predicts instability. $\delta\theta$ can be zero if the end-plates coincide with surfaces of constant ψ, or if the condition of very high conductivity is relaxed so that the field lines are no longer frozen into the plasma and the flux tubes may interchange without the matter between C and B, and between C′ and B′, taking part in the motion.

The foregoing analysis finds support in the theory of charged particle motions. Particle drifts resulting from the curvature of the field lines (ROSENBLUTH and LONGMIRE, 1957) produce charge separation and electric fields which, if the field curvature is unfavourable, will drive the fluting interchange instability. In a system with end-plates which are not orthogonal to the magnetic field lines, it is predicted (DIMOCK, 1966) that charge-dependent cross-field particle drifts could result from reflection at the end-plates. Depending on the end-plate orientation, these drifts could be destabilizing in exactly the same way as the curvature drifts discussed in the literature.

A simple picture of these drifts may be obtained by considering straight field lines passing into an end-plate which is inclined at 45° to $\underline{B}$, and off which the particles are specularly reflected. As described by DIMOCK (1966), a charged particle travelling along a field line, with no perpendicular velocity component, is reflected into a vertical plane, undergoes a half gyration before striking the plate a second time, when it is reflected back into the plasma, again travelling along the field. The net result of the two reflections is to displace the particle a distance of two gyro-radii perpendicular to the field, in a direction which depends on the sign of the particle's charge. This intuitive result is supported by the more general treatment (DIMOCK, 1966) which assumes arbitrary plate inclination and a realistic velocity distribution. However it must be noted that doubt has been cast on the occurrence of these drifts by the result of an experiment with a Q-device (DECKER, 1966) in which the inclination of the end-plate failed to have any observable effect on the stability of the plasma.

1.6 GENERALIZATION BY ASSUMING A DIAGONAL, ANISOTROPIC PRESSURE TENSOR

Under the assumption that heat flow along the magnetic field lines may be neglected (CHEW, GOLDBERGER and LOW, 1956), the collisionless Boltzmann equation may be used to derive one-fluid hydromagnetic equations for which the material stress tensor is diagonal, but not isotropic:

$$\underline{\underline{P}} = p_\perp \underline{\underline{I}} + (p_{\parallel} - p_\perp)\underline{b}_o\underline{b}_o \ , \qquad (1.43)$$

where $\underline{\underline{I}}$ is the unit tensor and $\underline{B} = B\underline{b}_o$. In this theory the adiabatic gas law for scalar pressure is replaced by two equations of state, one for $p_\perp$ and one for $p_{\parallel}$:

$$\frac{p_\perp \tau}{B} = \text{const.} \tag{1.44}$$

and

$$p_{\parallel} B^2 \tau^3 = \text{const.} \tag{1.45}$$

Within the framework of this theory we present a discussion of the flux tube interchange, analogous to that given previously for a plasma in which the pressure tensor always remains isotropic. Of course the magnetic energy term will in this case be no different from the expression (1.2): it is necessary only to evaluate the change in material energy of the system. Here the energy associated with the plasma in a flux tube is given by

$$W_p = \int_{\text{Tube}} (\tfrac{1}{2}p_{\parallel} + p_\perp)d\tau \ , \tag{1.46}$$

where the subscript 'o' for equilibrium quantities is omitted for convenience. Then the variation in system material energy resulting from the interchange in which plasma in $d\tau_1$ replaces plasma in $d\tau_2$, as in Section 1.2, is

$$\begin{aligned}\Delta W_p &= \int_{\text{Tube 1}} (\tfrac{1}{2}p^*_{\parallel}(1) + p^*_\perp(1))d\tau_1 + \int_{\text{Tube 2}} (\tfrac{1}{2}p^*_{\parallel}(2) + p^*_\perp(2))d\tau_2 \\ &\quad - \int_{\text{Tube 1}} (\tfrac{1}{2}p_{\parallel}(1) + p_\perp(1))d\tau_1 - \int_{\text{Tube 2}} (\tfrac{1}{2}p_{\parallel}(2) + p_\perp(2))d\tau_2 \\ &= \left[\int p^*_\perp(1)d\tau_1 - \int p_\perp(2)d\tau_2\right] + \left[\int p^*_\perp(2)d\tau_2 - \int p_\perp(1)d\tau_1\right] \\ &\quad + \tfrac{1}{2}\left[\int p^*_{\parallel}(1)d\tau_1 - \int p_{\parallel}(2)d\tau_2\right] + \tfrac{1}{2}\left[\int p^*_{\parallel}(2)d\tau_2 - \int p_{\parallel}(1)d\tau_1\right],\end{aligned} \tag{1.47}$$

where the asterisk is used to denote perturbed quantities.

Let

$$A = \int p_1^*(1)d\tau_1 - \int p_1(2)d\tau_2 . \qquad (1.48)$$

Relabelling for convenience the volume τ in (1.44) by $d\tau$ and then applying this equation of state to the material which is initially in the volume $d\tau_2$ and which occupies $d\tau_1$ after the interchange,

$$p_1^*(1)d\tau_1 = p_1(2) \frac{B_1^*}{B_2} d\tau_2$$

$$= p_1(2) \frac{B_1\phi_2}{B_2\phi_1} d\tau_2. \qquad (1.49)$$

Therefore,

$$A = \int p_1(2)d\tau_2 \left(\frac{B_1\phi_2}{B_2\phi_1} - 1\right)$$

$$= \int p_1(2)d\tau_2 \frac{\phi_2}{B_2} \left(\frac{B_1}{\phi_1} - \frac{B_2}{\phi_2}\right) . \qquad (1.50)$$

Next, let

$$C = \int p_1^*(2)d\tau_2 - \int p_1(1)d\tau_1 . \qquad (1.51)$$

By an argument similar to the above is obtained

$$C = \int p_1(1)d\tau_1 \frac{\phi_1}{B_1} \left(\frac{B_2}{\phi_2} - \frac{B_1}{\phi_1}\right) , \qquad (1.52)$$

so that, combining equations (1.50) and (1.52),

$$A + C = \int \left(p_1(2)d\tau_2 \frac{\phi_2}{B_2} - p_1(1)d\tau_1 \frac{\phi_1}{B_1}\right) \left(\frac{B_1}{\phi_1} - \frac{B_2}{\phi_2}\right)$$

$$= - \int \delta \left(\frac{p_1 d\tau\phi}{B}\right) \delta\left(\frac{B}{\phi}\right) , \qquad (1.53)$$

where δ has the same meaning as before.

Defining

$$2D = \int \overset{*}{p}_{11}(1)d\tau_1 - \int p_{11}(2)d\tau_2 , \qquad (1.54)$$

and making use of the equation of state (1.45), it can be shown that

$$2D = \int p_{11}(2) \frac{B_2{}^2 \ d\tau_2{}^3}{\phi_2{}^2} \left(\frac{\phi_1{}^2}{B_1{}^2 \ d\tau_1{}^2} - \frac{\phi_2{}^2}{B_2{}^2 \ d\tau_2{}^2}\right). \qquad (1.55)$$

Similarly the quantity

$$2F = \int \overset{*}{p}_{11}(2)d\tau_2 - \int p_{11}(1)d\tau_1 \qquad (1.56)$$

becomes

$$2F = \int p_{11}(1) \frac{B_1{}^2 \ d\tau_1{}^3}{\phi_1{}^2} \left(\frac{\phi_2{}^2}{B_2{}^2 \ d\tau_2{}^2} - \frac{\phi_1{}^2}{B_1{}^2 \ d\tau_1{}^2}\right) . \qquad (1.57)$$

Hence, combining equations (1.55) and (1.57),

$$D + F = \tfrac{1}{2} \int \left\{p_{11}(2) \frac{B_2{}^2 \ d\tau_2{}^3}{\phi_2{}^2} - p_{11}(1) \frac{B_1{}^2 \ d\tau_1{}^3}{\phi_1{}^2}\right) \left(\frac{\phi_1{}^2}{B_1{}^2 \ d\tau_1{}^2}\right.$$

$$\left. - \frac{\phi_2{}^2}{B_2{}^2 \ d\tau_2{}^2}\right) = - \tfrac{1}{2} \int \delta\left(p_{11} \frac{d\tau^3 B^2}{\phi^2}\right) \delta\left(\frac{\phi^2}{B^2 d\tau^2}\right) . \qquad (1.58)$$

Finally, combining equations (1.53) and (1.58), and using expression (1.47),

$$\Delta W_p = A + C + D + F$$

$$= - \int \left\{\delta \left(\frac{p_1 d\tau \phi}{B}\right) \delta \left(\frac{B}{\phi}\right) + \tfrac{1}{2}\delta \left(p_{11} \frac{d\tau^3 B^2}{\phi^2}\right) \delta \left(\frac{\phi^2}{B^2 d\tau^2}\right)\right\} . \qquad (1.59)$$

A situation which may be of interest in practice concerns a system for which the pressure tensor is a simple scalar in equilibrium, but adopts the anisotropic form when perturbed. That is, the collision rate is high enough to keep $\underline{\underline{P}}$ isotropic while the system is in equilibrium, but too low, on the short time scale of the instability, to maintain isotropy as the perturbation develops. The change in material energy in this situation would be given by expression (1.59) with the substitution $p_{11} = p_{\perp} = p$. Expansion of (1.59) then yields, after some algebra,

$$\Delta W_p = \int \left\{\delta p \delta(d\tau) + 3p \frac{[\delta(d\tau)]^2}{d\tau} + 3p d\tau \left(\frac{\delta X}{X}\right)^2 - 4p\delta(d\tau)\frac{\delta X}{X}\right\} ,$$

where $X = \frac{\phi}{B}$. Further rearrangement gives

$$\Delta W_p = \int \left\{\delta p \delta(d\tau) + \frac{5}{3} p \frac{[\delta(d\tau)]^2}{d\tau} + \frac{4}{3} p\, d\tau \left(\frac{\delta(d\tau)}{d\tau} - \frac{3}{2}\frac{\delta X}{X}\right)^2\right\}. \tag{1.60}$$

Comparing now expressions (1.9) and (1.60) it is noted that in the present case, ΔW_p differs by a positive definite term from the result obtained, with $\gamma = \frac{5}{3}$, for the system in which the stress tensor remains isotropic during the perturbation. Hence it is found, in agreement with BERNSTEIN et al. (1958, p.28), that if the system whose pressure **remains** a scalar is stable, then the system in which isotropy cannot be maintained during perturbation is also stable. Note also that the value of $\frac{5}{3}$ for γ corresponds to that of a gas with three degrees of freedom, where use is made of the kinetic theory result (SPITZER, 1962, p.17):

$$\gamma = \frac{2 + m}{m} , \tag{1.61}$$

m being the number of degrees of freedom.

Expression (1.60) can be reduced to the form (1.9) only if p is required to satisfy both equations (1.44) and (1.45), specialized for the case $p_{\parallel} = p_{\perp} = p$. Then elimination of p leads to

$$\tau^2 B^3 = \text{const.}$$

Thus, applying this equation to the volumes $d\tau_1$ and $d\tau_2 = d\tau_1 + \delta(d\tau)$, involved in the interchange, it is easy to show that

$$\delta\left(\frac{d\tau^2 B^3}{\phi^3}\right) = 0 \ .$$

Then, with $X = \frac{\phi}{B}$, it follows, by expansion, that

$$\frac{\delta X}{X} = \frac{2}{3}\,\frac{\delta(d\tau)}{d\tau} \ . \tag{1.62}$$

The result (1.62) clearly leads to the vanishing of the positive definite term $\frac{4}{3}\, p\, d\tau \left(\frac{\delta(d\tau)}{d\tau} - \frac{3}{2}\frac{\delta X}{X}\right)^2$ of (1.60), leaving expression (1.9).

While the foregoing procedure achieves the reduction of (1.60) to (1.9), it has doubtful physical significance. This is because, in the necessarily *collisional* system associated with the maintenance of isotropy of the stress tensor, equations (1.44) and (1.45) would not be valid, being derived (CHEW, GOLDBERGER and LOW, 1956) from the *collisionless* Boltzmann equation.

In general the complicated expression (1.59) is of little practical use. However, for the interchange considered in Section 1.4 a useful criterion may be derived. Following the earlier treatment for scalar pressure, a choice of the mapping vector <u>a</u>

such that $\underline{a} \cdot \underline{B} = 0$ leads, via equation (1.28) and the assumption $\delta\phi = 0$, to

$$\frac{\delta(d\tau)}{d\tau} = \delta\left(\frac{\phi d\ell}{B}\right) \frac{B}{\phi d\ell}$$

$$= \frac{\delta(d\ell)}{d\ell} - \frac{\delta B}{B}$$

$$= - \frac{2\delta B}{B} . \qquad (1.63)$$

With $\delta\phi = 0$, expression (1.59) may be expanded to give

$$\Delta W_p = - \int \{\delta\, p_\perp d\tau \frac{\delta B}{B} + p_\perp \delta(d\tau) \frac{\delta B}{B} - p_\perp d\tau \left(\frac{\delta B}{B}\right)^2$$

$$- \left[\frac{\delta p_{11}}{B} + \frac{3p_{11}}{B} \frac{\delta(d\tau)}{d\tau} + 2p_{11} \frac{\delta B}{B^2}\right] \left(\delta B d\tau + B\delta(d\tau)\right)\} . \qquad (1.64)$$

Use of equation (1.63) then leads to

$$\Delta W_p = - \int \{\delta p_\perp d\tau \frac{\delta B}{B} - 3p_\perp d\tau \left(\frac{\delta B}{B}\right)^2 + \left(\frac{\delta p_{11}}{B} - 4p_{11} \frac{\delta B}{B^2}\right) \delta B d\tau\}$$

$$= - \int d\tau \delta(p_{11} + p_\perp) \frac{\delta B}{B} + \int d\tau (3p_\perp + 4p_{11}) \left(\frac{\delta B}{B}\right)^2 . \qquad (1.65)$$

The second integral is positive and therefore stabilizing. A sufficient condition for stability against this interchange is therefore

$$\int d\tau \delta(p_\perp + p_{11}) \frac{\delta B}{B} < 0 . \qquad (1.66)$$

It is therefore found that, as in the scalar pressure case, stability would be ensured if confinement were in a magnetic well in the sense that $\delta(p_\perp + p_{11})\delta B < 0$. Also if the structure of the plasma is such that

$$\underline{B}\cdot\nabla(p_{11} + p_{\perp}) = 0 \ , \qquad (1.67)$$

then $\delta(p_{11} + p_{\perp})$ may be taken out of the integral in (1.66). Then, assuming $\delta(p_{11} + p_{\perp}) < 0$, the sufficient condition for stability reduces to the condition

$$\int d\tau \frac{\delta B}{B} > 0 \ , \qquad (1.68)$$

or, using (1.32) and the fact that $d\tau = \frac{\phi d\ell}{B}$, with $\phi > 0$,

$$\int d\ell \frac{\underline{a}\cdot\underline{K}}{B} > 0 \ , \qquad (1.69)$$

as for the system with scalar pressure.

1.7 REFERENCES

BERNSTEIN, I. B., FRIEMAN, E. A., KRUSKAL, M. D., and KULSRUD, R. M. (1958) - Proc. Roy. Soc. A224, 17.

BISHOP, A. S. (1960) - "Project Sherwood - The U.S. Program in Controlled Fusion". (Anchor Books, Doubleday and Co., Garden City, New York.)

CHEW, G. F., GOLDBERGER, M. L., and LOW, F. E. (1956) - Proc. Roy. Soc. A236, 112.

COLGATE, S. A., and YOSHIKAWA, S. (1964) - Physics of Fluids 7, 1877.

DECKER, J. A. (1966) - Physics of Fluids 9, 1421.

DIMOCK, D. (1966) - Physics of Fluids 9, 1110.

LUNDQUIST, S. (1952) - Ark. Fys. 5, 297.

ROBERTS, K. V., and TAYLOR, J. B. (1965) - Physics of Fluids 8, 315.

ROSENBLUTH, M. and LONGMIRE, C. (1957) - Ann. Phys. 1, 120.

SCHMIDT, G. (1966) - "Physics of High Temperature Plasmas". (Academic Press, New York).

SEYMOUR, P. W. (1961) - Aust. J. Phys. 14, 321.

SPITZER, L. Jr. (1962) - "Physics of Fully Ionized Gases". Second revised edition. (Interscience: New York.)

CHAPTER 2

MECHANICAL EQUILIBRIUM IN A CONSTRICTED DISCHARGE

2.1 INTRODUCTION

A constricted discharge is considered (SEYMOUR, 1961), the surface of which is shaped by interaction with an external magnetic field so that the discharge boundary approximates a hyperboloid of one sheet (Fig. 5). This surface coincides with the coordinate surface $w = w_b$ of the oblate spheroidal system (u,v,w) defined in terms of cylindrical polar coordinates by

$$\left.\begin{aligned} r &= k \cosh u \cos w \\ z &= k \sinh u \sin w \\ \theta &= v \end{aligned}\right\} . \qquad (2.1)$$

The domains of the variables are

$$0 \leqslant w \leqslant \frac{\pi}{2}$$
$$-\infty < u < \infty$$
$$0 \leqslant v \leqslant 2\pi \ ,$$

and the scale factors are

$$h_u = h_w = k(\sin^2 w + \sinh^2 u)^{\frac{1}{2}} \ , \qquad (2.2)$$

$$h_v = k \cosh u \cos w \ . \qquad (2.3)$$

The constant k is the distance off-axis of the common foci of the u and w coordinate surfaces. The system is symmetric about the median plane $u = 0$, and the electrodes occupy part of the coordinate surfaces $u = u_e$ and $u = -u_e$.

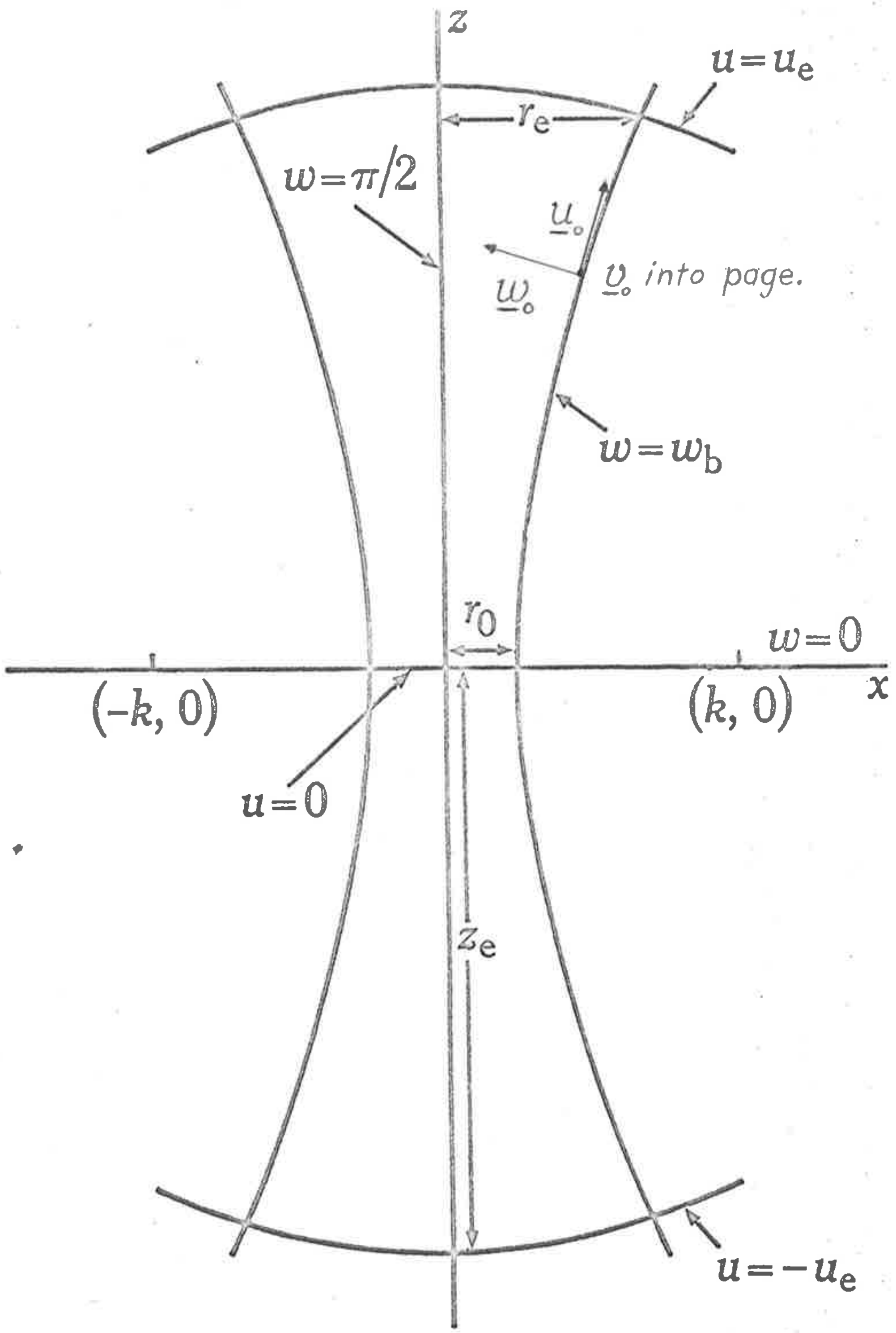

Fig. 5. In oblate spheroidal coordinates the discharge occupies the region

$$-u_e \leqslant u \leqslant u_e,$$

$$w_b \leqslant w \leqslant \pi/2 .$$

2.2 MECHANICAL EQUILIBRIUM WITH SCALAR PRESSURE

Interest centres on the low pressure region at the boundary, where it is proposed to apply the ROSENBLUTH-LONGMIRE (1957) stability criterion $\delta\tau < 0$, discussed in Chapter I. A basic assumption is that the main discharge current flows in a region closer to the axis than to the region of interest. Thus the total magnetic field in the low pressure region is sensibly curl-free and may be written

$$\underline{B} = \underline{B}^o + \underline{B}^1$$

where $\underline{B}^o$ is the curl-free part due to the main discharge current and the currents flowing in external coils, and $\underline{B}^1$ is a small perturbation produced by the small current $\underline{j}^1$ which flows in the boundary region. It is assumed that $\underline{B}^1$ is very small compared with $\underline{B}^o$. In terms of the orthogonal unit vectors $\underline{u}_o$, $\underline{v}_o$, $\underline{w}_o$ associated with the chosen coordinate system, it is noted that

$$\underline{B}^o = B^o_u \underline{u}_o + B^o_v \underline{v}_o \ , \qquad (2.4)$$

since it has been assumed that $w = w_b$ approximates the discharge surface. $\underline{B}^o$ satisfies the equations

$$\nabla \times \underline{B}^o = 0 \qquad (2.5)$$

and

$$\nabla \cdot \underline{B}^o = 0 \ . \qquad (2.6)$$

Using this pair of equations, analytical expressions for B^o_u and B^o_v may be derived. Equation (2.5) yields, under the assumption of azimuthal symmetry,

$$\frac{\partial}{\partial w}(h_v B_v^o) = 0 \;, \tag{2.7}$$

$$\frac{\partial}{\partial w}(h_u B_u^o) = 0 \tag{2.8}$$

and

$$\frac{\partial}{\partial u}(h_v B_v^o) = 0 \;. \tag{2.9}$$

Equations (2.7) and (2.9) give

$$B_v^o = \frac{C}{h_v} \;, \tag{2.10}$$

where the constant C is proportional to the total discharge current. Equation (2.8) gives

$$h_u B_u^o = F(u) \;, \tag{2.11}$$

while, with the assumption of azimuthal symmetry, equation (2.6) yields

$$\frac{\partial}{\partial u}(h_v h_w B_u^o) = 0 \;, \tag{2.12}$$

or

$$h_v h_w B_u^o = G(w). \tag{2.13}$$

Combining equations (2.11) and (2.13), and using equation (2.2) gives

$$k \cosh u \; F(u) = \frac{G(w)}{\cos w} = A \;, \tag{2.14}$$

where A is the separation constant.

Therefore

$$F(u) = \frac{A}{k \cosh u} \;,$$

and so using equation (2.2),

$$B_u^o = \frac{A}{k^2 \cosh u \,(\sin^2 w + \sinh^2 u)^{\frac{1}{2}}} \; . \tag{2.15}$$

The constant A can conveniently be determined from a measurement of B_u^o on the surface, at the median plane. Denoting this measured value by B_M, then

$$A = B_M k^2 \sin w_b \; . \tag{2.16}$$

At this stage the assumption of scalar pressure is made, and then the following equation of mechanical equilibrium must be satisfied in the steady-state plasma near the diffuse boundary:

$$\nabla p = \underline{j}^1 \times (\underline{B}^o + \underline{B}^1) \tag{2.17}$$

$$= \underline{u}_o(j_v^1 B_w^1 - j_w^1 B_v^o - j_w^1 B_v^1) + \underline{v}_o(j_w^1 B_u^o + j_w^1 B_u^1 - j_u^1 B_w^1)$$
$$+ \underline{w}_o(j_u^1 B_v^o + j_u^1 B_v^1 - j_v^1 B_u^o - j_v^1 B_u^1) \; . \tag{2.18}$$

Because of the assumption of azimuthal symmetry, $(\nabla p)_v$ must be zero. This leads, with $B_w^1 \neq 0$, to

$$j_w^1 = j_u^1 = 0 \; . \tag{2.19}$$

Then

$$\nabla p = \underline{u}_o j_v^1 B_w^1 - \underline{w}_o j_v^1 \,(B_u^o + B_u^1) \; . \tag{2.20}$$

Thus

$$\frac{\partial p}{\partial w} = h_w \underline{w}_o \cdot \nabla p = - h_w j_v^1 (B_u^o + B_u^1) \; , \tag{2.21}$$

so that, to first order in the small fields $\underline{j}^1$ and $\underline{B}^1$,

$$\frac{\partial p}{\partial w} = - h_w j_v^1 B_u^o \; .$$

Also,

$$\frac{\partial p}{\partial u} = h_u \underline{u}_o \cdot \nabla p = h_u j_v^1 B_w^1 \ , \tag{2.22}$$

which is of second order in $\underline{j}^1$ and $\underline{B}^1$, so that to first order, $\frac{\partial p}{\partial u}$ is zero. Thus, to first order in $\underline{j}^1$ and $\underline{B}^1$, the pressure is a function of w only and in this approximation may therefore be considered constant throughout an elemental flux tube of $\underline{B}^o$.

Equating mixed partial derivatives of p, using equations (2.21) and (2.22),

$$\frac{\partial}{\partial u} (h_w j_v^1 B_u^o + h_w j_v^1 B_u^1) = - \frac{\partial}{\partial w} (h_u j_v^1 B_w^1) \ . \tag{2.23}$$

As a first step in an iterative procedure, neglect second-order terms in (2.23) and write

$$\frac{\partial}{\partial u} (h_w j_v^1 B_u^o) = 0 \ , \tag{2.24}$$

so that

$$j_v^1 = \frac{J(w)}{h_w B_u^o} \ , \tag{2.25}$$

where J(w) is an unde termined function. It is possible to obtain an explicit form for J(w) by substituting (2.25) into (2.23) to give the second order equation

$$J(w) \frac{\partial}{\partial u} \left(\frac{B_u^1}{B_u^o}\right) = - \frac{\partial}{\partial w} \left(\frac{J(w) B_w^1}{B_u^o}\right) \ . \tag{2.26}$$

Using expressions (2.2) and (2.15), equation (2.26) becomes

$$J(w) \frac{\partial}{\partial u} (h_w \cosh u \ B_u^1) = - \cosh u \frac{\partial}{\partial w} (J(w) h_w B_w^1) \ . \tag{2.27}$$

Because of the solenoidal nature of the total magnetic field,

$$\nabla \cdot \underline{B}^1 = 0 \ . \tag{2.28}$$

With the assumption of azimuthal symmetry, this condition gives

$$\frac{\partial}{\partial u}(h_v h_w B_u^1) = -\frac{\partial}{\partial w}(h_u h_v B_w^1) \ , \tag{2.29}$$

which becomes, through the use of equations (2.2) and (2.3),

$$\frac{\partial}{\partial u}(h_w \cosh u \, B_u^1) = -\frac{\cosh u}{\cos w}\frac{\partial}{\partial w}(h_w \cos w \, B_w^1) \ , \tag{2.30}$$

so that equation (2.27) may be written

$$\frac{J(w)}{\cos w}\frac{\partial}{\partial w}(h_w \cos w \, B_w^1) = \frac{\partial}{\partial w}(J(w) h_w B_w^1). \tag{2.31}$$

Rearrangement of equation (2.31) leads to

$$J(w) h_w B_w^1 \tan w + J'(w) h_w B_w^1 = 0 \ . \tag{2.32}$$

From (2.2) it is clear that h_w is non-zero in the region of interest so that, with $B_w^1 \neq 0$, cancellation gives

$$J'(w) + J(w) \tan w = 0 \ . \tag{2.33}$$

Equation (2.33) is satisfied if

$$J(w) = K \cos w \ , \tag{2.34}$$

where K is a constant which must be regarded as 'small' in the same sense that j^1 is small since, using (2.25),

$$j_v^1 = \frac{K \cos w}{h_w B_u^o} \ . \tag{2.35}$$

Equation (2.21) then gives, in first order,

$$\frac{\partial p}{\partial w} = - K \cos w \tag{2.36}$$

or

$$p = - K \sin w. \tag{2.37}$$

Since $p > 0$ and, in the region of interest, $\sin w > 0$, it must follow that $K < 0$. Note that this gives $\frac{\partial p}{\partial w} > 0$, as required since in the present coordinates, w increases as one moves into regions of higher pressure.

Thus within the approximation $p \ll \frac{B^2}{2\mu_o}$, the assumption that the magnetic and pressure surfaces approximately coincide with the w coordinate surfaces is consistent with the equations of equilibrium and Maxwell's equations. However, in the plasma interior the pressure is expected to be so high that this approximation cannot be made. The complete equations without approximations must then be considered:

$$\nabla p = \underline{j} \times \underline{B} , \tag{2.38}$$

$$\mu_o \underline{j} = \nabla \times \underline{B} , \tag{2.39}$$

$$\nabla \cdot \underline{B} = 0 . \tag{2.40}$$

It is shown below that the assumption $p = p(w)$, $B_w = 0$, is not consistent with these equations, except for a trivial case.

The assumption $p = p(w)$, $B_w = 0$, requires $j_w = 0$. Thus, from the expanded form of equation (2.39),

$$\partial(h_v B_v)/\partial u = 0 ,$$

or

$$B_v = f(w)/h_v ,\qquad (2.41)$$

where f(w) is an arbitrary function.

From (2.40) is obtained

$$\partial(h_v h_w B_u)/\partial u = 0 ,\qquad (2.42)$$

or

$$B_u = g(w)/h_v h_w ,\qquad (2.43)$$

where g(w) is also an arbitrary function.

Only the w component of (2.38) remains, and this becomes, after use of (2.39) and some algebra,

$$\mu_o\, dp/dw = -\, f(df/dw)/h_v^2 - (g/h_u h_v h_w)\partial(h_u g/h_v h_w)/\partial w .\qquad (2.44)$$

Using (2.2) and (2.3) this reduces to

$$\mu_o\, dp/dw = -\, f(df/dw)/h_v^2 - F(w)/h_u^2 h_v^2 ,\qquad (2.45)$$

where

$$F(w) = gdg/dw + g^2 \sin w/\cos w .\qquad (2.46)$$

Substituting for the scale factors and rearranging gives

$$\mu_o k^2 \cos^2 w \cosh^4 u\, dp/dw + (fdf/dw - \mu_o k^2 \cos^4 w\, dp/dw)\cosh^2 u + F/k^2$$
$$- (fdf/dw)\cos^2 w = 0 .\qquad (2.47)$$

Equation (2.47) is satisfied for all u and w only if the coefficients of the different powers of cosh u vanish for all w, and this leads to the results:

$$\left.\begin{aligned} dp/dw &= 0 \ , \\ fdf/dw &= 0 \ , \\ F &= 0 \ . \end{aligned}\right\} \qquad (2.48)$$

These results are trivial because they imply that $j_u = j_v = 0$; i.e. they imply no physical discharge.

2.3 MECHANICAL EQUILIBRIUM WITH NON-SCALAR PRESSURE

The equation of mechanical equilibrium to be satisfied in this case is

$$\nabla \cdot \underline{\underline{P}} = \underline{j} \times \underline{B} \ , \qquad (2.49)$$

with the pressure tensor $\underline{\underline{P}}$ in the form (1.43). Under the same assumptions as before of low pressure, small electric current and azimuthal symmetry the argument following equation (2.18) applies, so that in first order,

$$(\nabla \cdot \underline{\underline{P}})_u = 0 \ , \qquad (2.50)$$

$$(\nabla \cdot \underline{\underline{P}})_v = 0 \ , \qquad (2.51)$$

and

$$(\nabla \cdot \underline{\underline{P}})_w = - \underline{w}_o j_v^1 B_u^o \ . \qquad (2.52)$$

The unit tensor $\underline{\underline{I}}$ is invariant under a transformation of axes (FERRARO and PLUMPTON, 1966, p.217), so that $\nabla \cdot \underline{\underline{I}} = 0$. Then, with reference to (1.43),

$$\nabla \cdot \underline{\underline{P}} = \nabla p_\perp + (p_{11} - p_\perp)\underline{b}_o \cdot \nabla \underline{b}_o + (p_{11} - p_\perp)\underline{b}_o \nabla \cdot \underline{b}_o$$
$$+ \underline{b}_o \underline{b}_o \cdot \nabla (p_{11} - p_\perp) \qquad (2.53)$$

where, in the present approximation,

$$\underline{b}_o = \frac{\underline{B}^o}{B^o} .$$

Thus, deleting the superscript for convenience,

$$\underline{b}_o \cdot \nabla \underline{b}_o = \left(\frac{B_u}{Bh_u}\frac{\partial}{\partial u} + \frac{B_v}{Bh_v}\frac{\partial}{\partial v}\right)\left(\frac{B_u}{B}\underline{u}_o + \frac{B_v}{B}\underline{v}_o\right)$$

$$= \underline{u}_o \frac{B_u}{Bh_u}\frac{\partial}{\partial u}\left(\frac{B_u}{B}\right) + \frac{B_u^2}{B^2 h_u}\frac{\partial}{\partial u}\underline{u}_o + \underline{v}_o \frac{B_u}{Bh_u}\frac{\partial}{\partial u}\left(\frac{B_v}{B}\right) + \frac{B_u B_v}{B^2 h_u}\frac{\partial \underline{v}_o}{\partial u}$$

$$+ \frac{B_u B_v}{B^2 h_v}\frac{\partial \underline{u}_o}{\partial v} + \frac{B_v^2}{B^2 h_v}\frac{\partial \underline{v}_o}{\partial v} , \qquad (2.54)$$

utilizing the assumption of azimuthal symmetry. Referring now to the expressions for the derivatives of the unit vectors $(\underline{a}_1, \underline{a}_2, \underline{a}_3)$ of the general orthogonal curvilinear coordinate system (ξ_1, ξ_2, ξ_3) (MORSE and FESHBACH, 1953, p.26):

$$\frac{\partial \underline{a}_1}{\partial \xi_1} = -\frac{\underline{a}_2}{h_2}\frac{\partial h_1}{\partial \xi_2} - \frac{\underline{a}_3}{h_3}\frac{\partial h_1}{\partial \xi_3} ,$$

$$\frac{\partial \underline{a}_1}{\partial \xi_2} = \frac{\underline{a}_2}{h_1}\frac{\partial h_2}{\partial \xi_1} ,$$

$$\frac{\partial \underline{a}_1}{\partial \xi_3} = \frac{\underline{a}_3}{h_1}\frac{\partial h_3}{\partial \xi_1}$$

(plus their cyclic counterparts), equation (2.54) may be rewritten as

$$\underline{b}_o \cdot \nabla \underline{b}_o = \underline{u}_o \left\{\frac{B_u}{Bh_u}\frac{\partial}{\partial u}\left(\frac{B_u}{B}\right) - \frac{B_v^2}{B^2 h_v h_u}\frac{\partial}{\partial u}h_v\right\} + \underline{v}_o \left\{\frac{B_u}{Bh_u}\frac{\partial}{\partial u}\left(\frac{B_v}{B}\right) + \frac{B_u B_v}{B^2 h_u h_v}\frac{\partial h_v}{\partial u}\right\}$$

$$- \underline{w}_o \left\{\frac{B_u^2}{B^2 h_w h_u}\frac{\partial h_u}{\partial w} + \frac{B_v^2}{B^2 h_v h_w}\frac{\partial h_v}{\partial w}\right\} . \qquad (2.55)$$

Further,

$$\nabla\cdot\underset{\sim}{b}_o = \nabla\cdot\left(\frac{\underline{B}}{B}\right)$$

$$= \underline{B}\cdot\nabla\left(\frac{1}{B}\right)$$

since $\nabla\cdot\underline{B} = 0$. Therefore, because of azimuthal symmetry,

$$\nabla\cdot\underset{\sim}{b}_o = \frac{B_u}{h_u}\frac{\partial}{\partial u}\left(\frac{1}{B}\right)$$

and

$$\underset{\sim}{b}_o\ \nabla\cdot\underset{\sim}{b}_o = \underset{\sim}{u}_o\frac{B_u^2}{Bh_u}\frac{\partial}{\partial u}\left(\frac{1}{B}\right) + \underset{\sim}{v}_o\frac{B_uB_v}{Bh_u}\frac{\partial}{\partial u}\left(\frac{1}{B}\right) . \qquad (2.56)$$

Also,

$$\underset{\sim}{b}_o\underset{\sim}{b}_o\cdot\nabla(p_{11} - p_\perp) = \frac{\underset{\sim}{u}_oB_u^2}{B^2h_u}\frac{\partial}{\partial u}(p_{11} - p_\perp) + \underset{\sim}{v}_o\frac{B_uB_v}{B^2h_u}\frac{\partial}{\partial u}(p_{11} - p_\perp) . \qquad (2.57)$$

Therefore, using equations (2.53), (2.55), (2.56) and (2.57) the components $(\nabla\cdot\underline{\underline{P}})_u$ and $(\nabla\cdot\underline{\underline{P}})_v$ may be finally obtained so that equations (2.50) and (2.51) become respectively

$$\frac{\partial p_\perp}{\partial u} + (p_{11} - p_\perp)\left\{\frac{B_u}{B}\frac{\partial}{\partial u}\left(\frac{B_u}{B}\right) - \frac{B_v^2}{B^2h_v}\frac{\partial h_v}{\partial u} + \frac{B_u^2}{B}\frac{\partial}{\partial u}\left(\frac{1}{B}\right)\right\}$$
$$+ \frac{B_u^2}{B^2}\frac{\partial}{\partial u}(p_{11} - p_\perp) = 0 , \qquad (2.58)$$

and

$$(p_{11} - p_\perp)\left\{\frac{\partial}{\partial u}\left(\frac{B_v}{B}\right) + \frac{B_v}{Bh_v}\frac{\partial h_v}{\partial u} + B_v\frac{\partial}{\partial u}\left(\frac{1}{B}\right)\right\} + \frac{B_v}{B}\frac{\partial}{\partial u}(p_{11} - p_\perp) = 0 . \qquad (2.59)$$

Since $\underline{B}$ is assumed to be approximately curl-free, the result (2.41) (obtained from the condition $j_w = 0$) must apply, yielding

$$\frac{\partial h_v}{\partial u} = f(w) \frac{\partial}{\partial u} \left(\frac{1}{B_v}\right)$$

$$= - \frac{h_v}{B_v} \frac{\partial B_v}{\partial u} . \qquad (2.60)$$

Equation (2.59) may therefore be written

$$(p_{11} - p_{\perp}) \left\{\frac{\partial}{\partial u} \left(\frac{B_v}{B}\right) - \frac{1}{B} \frac{\partial B_v}{\partial u} + B_v \frac{\partial}{\partial u} \left(\frac{1}{B}\right)\right\} + \frac{B_v}{B} \frac{\partial}{\partial u} (p_{11} - p_{\perp}) = 0,$$

which becomes, by carrying out the first differentiation inside the curly brackets,

$$2(p_{11} - p_{\perp}) \frac{\partial}{\partial u} \left(\frac{1}{B}\right) + \frac{1}{B} \frac{\partial}{\partial u} (p_{11} - p_{\perp}) = 0. \qquad (2.61)$$

This equation can be integrated to obtain

$$p_{11} - p_{\perp} = t(w)B^2 , \qquad (2.62)$$

where $t(w)$ is an arbitrary function whose magnitude is such that $\mu_o\ |t(w)|$ is much less than unity, so that $|p_{11} - p_{\perp}| \ll \frac{B^2}{2\mu_o}$.

Using equations (2.60) and (2.62), equation (2.58) may be written

$$\frac{\partial p_{\perp}}{\partial u} + t(w) \left\{BB_u \frac{\partial}{\partial u} \left(\frac{B_u}{B}\right) + B_v \frac{\partial B_v}{\partial u} + BB_u^2 \frac{\partial}{\partial u} \left(\frac{1}{B}\right) + \frac{B_u^2}{B^2} \frac{\partial B^2}{\partial u}\right\} = 0,$$

or

$$\frac{\partial p_{\perp}}{\partial u} + t(w) \left\{\frac{\partial}{\partial u} (\tfrac{1}{2}B_u^2) + \frac{\partial}{\partial u} (\tfrac{1}{2}B_v^2)\right\} = 0.$$

Therefore, since $B^2 = B_u^2 + B_v^2$,

$$\frac{\partial p_{\perp}}{\partial u} + t(w)B \frac{\partial B}{\partial u} = 0 ,$$

which integrates to give

$$p_{\perp} = -\tfrac{1}{2}t(w)B^2 + P(w) \ , \qquad (2.63)$$

where P(w) is another arbitrary function, whose magnitude is much less than $\frac{B^2}{2\mu_o}$.

Also, with the use of equation (2.62),

$$p_{11} = \tfrac{1}{2}t(w)B^2 + P(w) \qquad (2.64)$$

so that finally,

$$p_{11} + p_{\perp} = 2P(w) \ . \qquad (2.65)$$

Thus the present analysis shows that the combination $(p_{11} + p_{\perp})$ is constant along a flux tube of the magnetic field approximated by (2.4). Therefore, reverting to the stability criterion (1.66), it is seen that in the present system, the reduction of this criterion to the forms (1.68) and (1.69) is valid.

2.4 REFERENCES

FERRARO, V. C. A., and PLUMPTON, C. (1966) - "Magneto-Fluid Mechanics", 2nd Ed. (Clarendon Press: Oxford).

MORSE, P. M., and FESHBACH, H. (1953) - "Methods of Theoretical Physics", Part I, p.26 (McGraw-Hill).

ROSENBLUTH, M., and LONGMIRE, C. (1957) - Ann. Phys. 1, 120.

SEYMOUR, P. W. (1961) - Aust. J. Phys. 14, 321.

CHAPTER 3

STABILITY OF THE CONSTRICTED DISCHARGE

3.1 INTRODUCTION

The purpose of the curved external magnetic field in the discharge discussed in Chapter 2 is to provide extra compression of the plasma near the median plane and also to give a stabilizing contribution, because of its favourable curvature. It is expected that this stabilizing contribution will eventually be cancelled by the destabilizing effect of the azimuthal magnetic field produced by the discharge current, as that current is increased beyond some critical value (SEYMOUR 1961). A detailed stability analysis for the low pressure boundary region is presented in this chapter. The exact result obtained supports the above reasoning provided that in the interchange the ends of the tubes of matter are displaced along surfaces of constant magnetic scalar potential so that $\delta\Psi = 0$.

3.2 THE INTERCHANGE WITH $\delta\Psi \neq 0$

In the present geometry, the tubes of matter terminate on the electrodes $u = u_e$ and $u = - u_e$. However, the surfaces u = constant are not surfaces of constant ψ, since here the magnetic field has a component B_v as well as a component B_u. Thus the ends of the tubes are not at the same ψ. This means that if the approximation of infinite electrical conductivity is made, an interchange involving the condition $\underline{a} \cdot \underline{B} = 0$ is not possible. In fact, as discussed in Section 1.5, *all* the matter in each tube would have to take part in the interchange, and the relevant $\delta\tau$ would just be the difference in volume between flux tubes (of equal flux) on two surfaces $w = w_b$ and $w = w_b + \delta w$, each tube extending from one electrode to the other. Thus it would only be necessary to evaluate

$$\tau(w) = \phi \int \frac{d\ell}{B} , \tag{3.1}$$

and to then find

$$\delta\tau = \delta w \frac{d\tau}{dw} . \tag{3.2}$$

Equation (3.1) may be evaluated by using the expressions for the magnetic field components derived in Section 2.2. Since $d\underline{\ell}$ is an elemental vector tangent to a field line, it follows that

$$d\underline{\ell} \times \underline{B} = 0 , \tag{3.3}$$

which implies

$$h_u du/B_u = h_v dv/B_v = d\ell/B , \tag{3.4}$$

since

$$(d\ell)^2 = h_u^2 du^2 + h_v^2 dv^2$$

and

$$B^2 = B_u^2 + B_v^2 .$$

Therefore, using equations (2.2), (2.15) and (3.4), expression (3.1) becomes, after integration between the limits $u = -u_e$ and $u = +u_e$,

$$\tau(w) = 2k^3\phi\left(\sin^2 w \sinh u_e + (\sinh^3 u_e)/3\right)/A . \tag{3.5}$$

From equation (3.5) it follows immediately that, at $w = w_b$,

$$\delta\tau = 4k^3\phi \sin w_b \cos w_b \sinh u_e \, \delta w/A \, . \qquad (3.6)$$

Since the coordinate w falls from $\pi/2$ on the z axis to w_b at the discharge boundary, it follows from the definition of δ that $\delta w < 0$, and so equation (3.6) gives

$$\delta\tau < 0 \, . \qquad (3.7)$$

Hence, by (1.22), we always have stability! In spite of the unfavourable curvature of the azimuthal field, the 'freezing-in' effect of infinite conductivity gives stability against the interchange of flux tubes on adjacent surfaces w = constant, for all values of discharge current.

In the limiting case of a straight circular cylinder of length L, with field components B_z and B_θ, equations (2.5) and (2.6) give

$$B_\theta = C/r \qquad (3.8)$$

and

$$B_z = \text{constant}. \qquad (3.9)$$

Then equation (3.1) becomes

$$\tau = \tau(r) = \phi \int_0^L dz/B_z = \phi L/B_z = \text{constant} \, . \qquad (3.10)$$

From equation (3.10) it is seen that $\delta\tau = 0$, which implies neutral stability for the case of a linear pinch with infinite conductivity. But of course it is well known that such a pinch is *not* hydromagnetically stable; there are other perturbations which are unstable. Hence the important conclusion is reached that here we are by no means considering necessary and sufficient conditions for stability against all perturbations, but only stability against a rather special type of perturbation.

3.3 THE INTERCHANGE WITH $\underline{a} \cdot \underline{B} = 0$

Consideration is now given to an interchange whose stability has been described qualitatively with the aid of Teller's criterion (SEYMOUR 1961). This is the case of a variation with $\underline{a} \cdot \underline{B} = 0$. For this to be feasible, some resistivity must now be allowed, particularly in the region of the electrodes. This is in accordance with the practical situation, and the presence of a resistive sheath at the electrodes explains why "line-tying" can be overcome, so that interchanges can occur.

Guided by equation (3.5), in this case

$$\tau(u_+, u_-, w) = \tau_+ + \tau_- \qquad (3.11)$$

$$= \phi k^3(\sin^2 w \sinh u_+ + (\sinh^3 u_+)/3)/A$$

$$+ \phi k^3(\sin^2 w \sinh u_- + (\sinh^3 u_-)/3)/A \; . \qquad (3.12)$$

τ_+ is the volume of that part of the tube which lies in the positive u half of the system, while τ_- is the volume of that part which lies in the negative half. $u = u_+$ is the end point of the tube in the positive half, while $u = - u_-$ is the end point in the negative half region. $u_\pm$ may differ from u_e, as shall be seen.

Then

$$\delta\tau = \delta\tau_+ + \delta\tau_- \ .$$

Now τ is a function of the variables w, u_+ and u_-, which vary from one tube to the other. Thus $\delta\tau$ may be obtained by differentiation with respect to these variables:

$$\delta\tau = \delta w \frac{\partial\tau_+}{\partial w} + \delta u_+ \frac{\partial\tau_+}{\partial u_+} + \delta w \frac{\partial\tau_-}{\partial w} + \delta u_- \frac{\partial\tau_-}{\partial u_-} \ , \qquad (3.13)$$

$$= \phi k^3[2\sin w \cos w \sinh u_+ \delta w + \cosh u_+ (\sin^2 w + \sinh^2 u_+)\delta u_+]/A + \phi k^3[2\sin w \cos w \sinh u_- \delta w + \cosh u_- (\sin^2 w + \sinh^2 u_-)\delta u_-]/A \ , \qquad (3.14)$$

where $\delta u_\pm = \pm\, \delta u(\pm\, u_\pm, w)$, $\delta u(u,w)$ and δw being as defined in the elemental vector $\underline{a}$ given by equation (1.26), subject to the assumed condition $\underline{a}\cdot\underline{B} = 0$. $\delta\tau$ is to be evaluated at the boundary, $w = w_b$.

Further progress is made by expressing δu in terms of δw, as shown below.

From equations (1.26) and (1.27),

$$\delta v = -\, h_u B_u \delta u / h_v B_v \ . \qquad (3.15)$$

Another expression for δv is obtained by considering the equation of a field line. From (3.4) is first obtained

$$dv = h_u B_v du / h_v B_u \ . \qquad (3.16)$$

Substituting expressions (2.10) and (2.15) for B_v and B_u into (3.16) and integrating with w held constant, one obtains the equation of a field line,

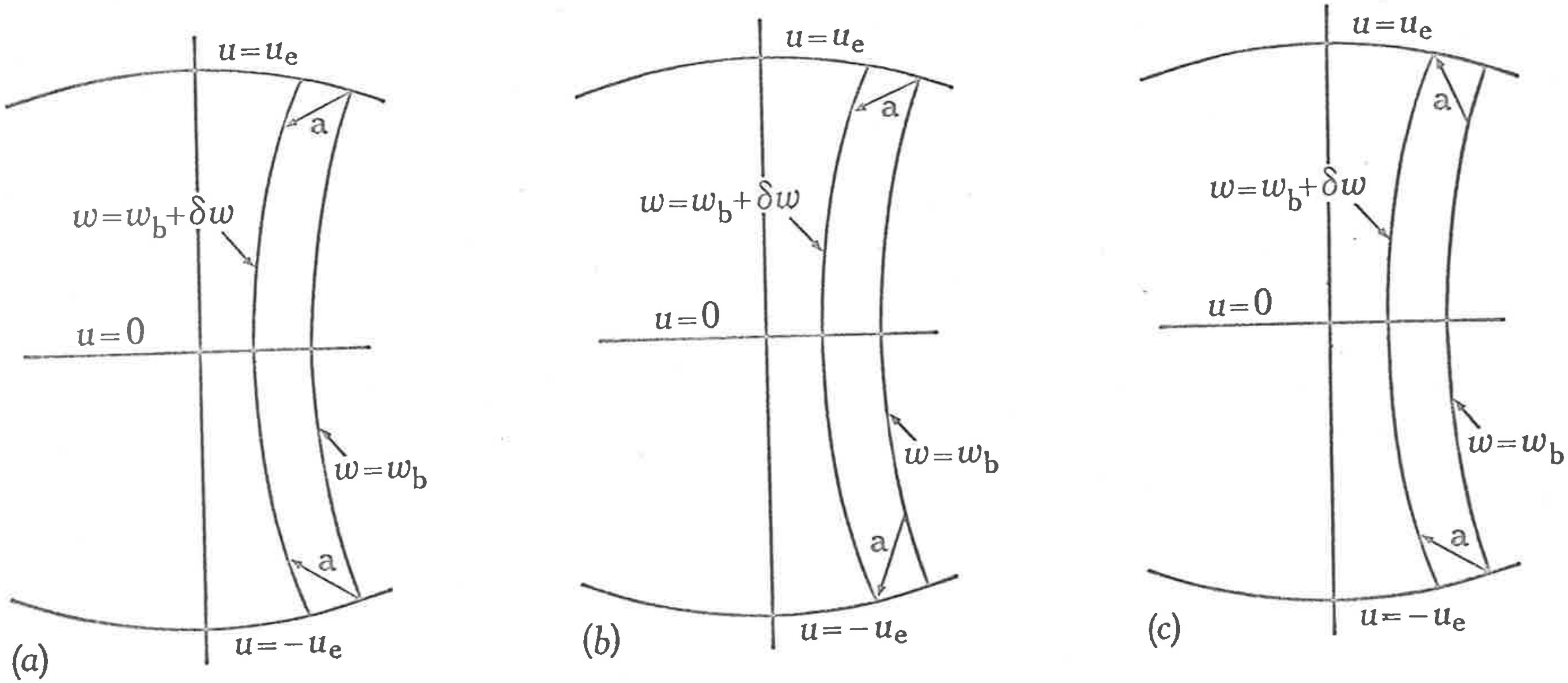

Fig. 6. Three possible cases corresponding to different ranges of δv_o, the azimuthal variation at $u = 0$: (a) $u_+ = u_e$, $u_- = u_e$; (b) $u_+ = u_e$, $u_- = u_e - |\delta u(-u_-, w_b)|$; (c) inversion of (b).

$$v = v_o + (kC/A\cos^2 w) \int_o^u (\cosh^2 u' - \cos^2 w)\,du'/\cosh u' \; . \qquad (3.17)$$

Therefore,

$$\delta v = \delta v_o + kC\cosh u \; \delta u/A\cos^2 w + 2kC \sinh u \sin w \; \delta w/A\cos^3 w$$

$$- kC \; \delta u/A\cosh u \; . \qquad (3.18)$$

Equating the two expressions for δv and solving for δu, gives

$$\delta u = - \delta v_o/\Gamma - \beta\delta w \; , \qquad (3.19)$$

where

$$\Gamma = \left[k^2C^2(\sin^2 w + \sinh^2 u) + A^2\cos^2 w\right]/AkC\cosh u \cos^2 w \qquad (3.20)$$

and

$$\beta = 2k^2C^2\sinh u \cosh u \sin w/\cos w\left[k^2C^2(\sin^2 w + \sinh^2 u)\right.$$

$$\left. + A^2\cos^2 w\right] \; . \qquad (3.21)$$

Γ is positive for all u and an even function of u, while β is an odd function of u and positive for positive u.

From this point, the presentation is a little clearer if the sign of δ is reversed, so that $\delta w > 0$. The criterion for stability is then

$$\delta\tau > 0 \; . \qquad (3.22)$$

It is now necessary to consider three cases which are illustrated in Fig. 6. Note that these plane diagrams do not give a true representation of the magnetic field lines, which are really twisted curves. Case (c) is just the inversion, about the median plane, of case (b), so that there are only two separate cases to

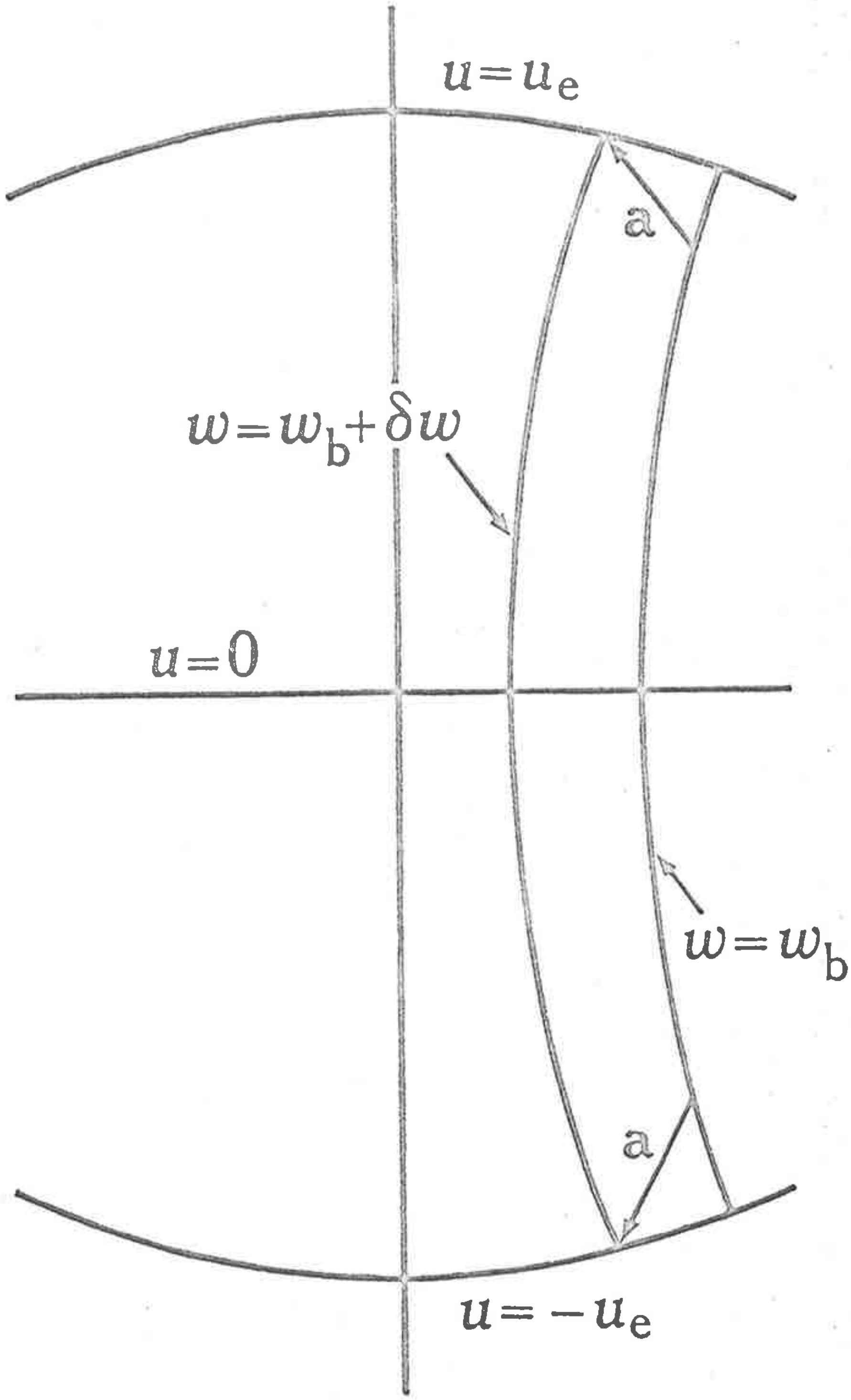

Fig. 7. A variation with $\delta u(u_+, w) > 0$, $\delta u(-u_-, w) < 0$, is not allowed by the field line geometry.

consider. Note also that there exists a fourth possible combination of the signs of $\delta u(u_+,w)$ and $\delta u(-u_-,w)$, namely

$$\delta u(u_+,w) > 0 \tag{3.23}$$

and

$$\delta u(-u_-,w) < 0 \ , \tag{3.24}$$

depicted in Fig. 7.

As shown below, this situation is not allowed by the geometry of the field lines. Thus, the inequality (3.23) gives

$$\Delta u(u_e,w_b + \delta w) < 0 \ , \tag{3.25}$$

where

$$\Delta \equiv -\delta \text{ and } \delta w > 0 \ .$$

That is,

$$\frac{-\Delta v_o}{\Gamma(u_e,w_b + \delta w)} - \beta(u_e,w_b + \delta w)\Delta w < 0 \ . \tag{3.26}$$

Let

$$\Gamma_{e,\delta} = \Gamma(u_e,w_b + \delta w) \ ,$$

$$\beta_{e,\delta} = \beta(u_e,w_b + \delta w) \ ,$$

and use the fact that $\Delta w < 0$.

Then (3.26) becomes

$$\Delta v_o > \Gamma_{e,\delta}\beta_{e,\delta} \ |\Delta w| \ . \tag{3.27}$$

The inequality (3.24) gives

$$\Delta u(-u_e,w_b + \delta w) > 0 \ ,$$

and this leads to

$$\Delta v_o < - \Gamma_{e,\delta}\beta_{e,\delta} \; |\Delta w| \; . \tag{3.28}$$

Clearly, since $\Gamma_{e,\delta} > 0$, $\beta_{e,\delta} > 0$, conditions (3.27) and (3.28) are not compatible. Hence the situation in Fig. 7 is not allowed.

Case (a)

Recalling the definitions of u_e, u_+, and u_-, Fig. 6(a) shows

$$u_+ = u_e \; , \tag{3.29}$$

$$u_- = u_e \; , \tag{3.30}$$

and

$$\delta u(u_e, w) < 0 \; , \tag{3.31}$$

$$\delta u(- u_e, w) > 0 \; . \tag{3.32}$$

Condition (3.31) gives

$$\delta v_o > - \Gamma(u_e, w)\beta(u_e, w)\delta w = - \Gamma_e\beta_e\delta w \; . \tag{3.33}$$

Condition (3.32) gives

$$- \delta v_o - \Gamma(- u_e, w)\beta(- u_e, w)\delta w > 0 \; ,$$

or

$$- \delta v_o + \Gamma(u_e, w)\beta(u_e, w)\delta w > 0 \; ,$$

using the parity properties of Γ and β.

Thus

$$\delta v_o < \Gamma_e\beta_e\delta w \; . \tag{3.34}$$

Since δw, Γ_e and β_e are all positive, conditions (3.33) and (3.34) are compatible, and show that case (a) covers the range

$$- \Gamma_e\beta_e\delta w \leqslant \delta v_o \leqslant \Gamma_e\beta_e\delta w \; . \tag{3.35}$$

Substituting into (3.14), using (3.29), (3.30), and the fact that

$$\delta u_{\pm} = \pm\ \delta u(\pm\ u_{\pm}, w) \ ,$$

evaluation at $w = w_b$ gives

$$\delta\tau = \phi k^3 \left[2\sin w_b \cos w_b \sinh u_e\ \delta w + \cosh u_e(\sin^2 w_b + \sinh^2 u_e)(-\ \delta v_o/\Gamma_e - \beta_e \delta w)\right]/A + \phi k^3\left[2\sin w_b \cos w_b \sinh u_e\ \delta w + \cosh u_e(\sin^2 w_b + \sinh^2 u_e)(\delta v_o/\Gamma_e - \beta_e \delta w)\right]/A \ .$$

The terms involving δv_o cancel, leaving

$$\delta\tau = 2\phi k^3(2\sin w_b \cos w_b \sinh u_e - \beta_e \cosh u_e(\sin^2 w_b + \sinh^2 u_e))\delta w/A \ .$$

Using (3.21), this becomes, after some algebra,

$$\delta\tau = \frac{4\phi k^3 \sin w_b \sinh u_e\left[A^2\cos^4 w_b - k^2C^2(\sin^2 w_b + \sinh^2 u_e)^2\right]\delta w}{A\cos w_b\left[k^2C^2(\sin^2 w_b + \sinh^2 u_e) + A^2\cos^2 w_b\right]} \ . \tag{3.36}$$

From (3.36) and (3.22) it is seen that, as expected from the discussion given by SEYMOUR (1961), a transition from stability to instability can occur as the discharge current (proportional to C) is increased.

The sign of $\delta\tau$ is governed by the sign of the factor in square brackets, which becomes negative if $C > C_{crit.}$, where

$$C_{crit.} = A\cos^2 w_b/k(\sin^2 w_b + \sinh^2 u_e) \ . \tag{3.37}$$

Thus, since $\delta w > 0$, condition (3.22) is violated if $C > C_{crit.}$, and so $C_{crit.}$ gives a measure of the current value which is critical for the onset of unstable interchanges. It is of interest to note that, in view of the discussion of Section 1.3, the system might be expected to enter a second stable régime as C is increased further so that the inequality (1.24) becomes satisfied.

The result (3.37) may be obtained by commuting δ with the integral sign in (1.25) and evaluating the resulting integral. This operation has been carried out, forming an important consistency check as presented in Appendix I.

Case (b)

From Fig. 6(b) it is seen that in this case,

$$u_+ = u_e \tag{3.38}$$

and

$$\begin{aligned} u_- &= u_e - |\delta u(-u_-, w_b)| \\ &= u_e - |\Delta u(u_e, w_b + \delta w)| , \end{aligned} \tag{3.39}$$

where $\Delta \equiv -\delta$, and $\delta w > 0$.

Also the conditions

$$\delta u(u_e, w_b) < 0 \tag{3.40}$$

and

$$\Delta u(-u_e, w_b + \delta w) > 0 \tag{3.41}$$

apply.

Condition (3.40) gives

$$\delta v_o > -\Gamma_e \beta_e \delta w , \tag{3.42}$$

while (3.41) is

$$\Delta u(-u_e, w_b) + \delta w \partial \Delta u / \partial w > 0 .$$

Neglecting the higher order terms,

$$\Delta u(-u_e, w_b) > 0 ,$$

or

$$\delta u(-u_e, w_b) < 0 .$$

This leads to

$$\delta v_o > \Gamma_e \beta_e \delta w . \qquad (3.43)$$

The conditions (3.42) and (3.43) are compatible and so the situations in Fig. 6(b) and Fig. 6(c) are allowed by the field line geometry. As in the case (a), using the fact that $u_+ = u_e$ and $\delta u_+ = \delta u(u_+, w_b)$,

$$\delta\tau_+ = \phi k^3 [2\sin w_b \cos w_b \sinh u_e \delta w - \cosh u_e (\sin^2 w_b + \sinh^2 u_e)(\delta v_o/\Gamma_e + \beta_e \delta w)]/A .$$

In this case, however, $u_- \neq u_e$ and

$$\delta\tau_- = \phi k^3 [2\sin w_b \cos w_b \sinh(u_e - \delta u_-)\delta w + \cosh(u_e - \delta u_-)(\sin^2 w_b + \sinh^2(u_e - \delta u_-))\delta u_-]/A$$

$$\approx \phi k^3 [2\sin w_b \cos w_b \sinh u_e \delta w + \cosh u_e (\sin^2 w_b + \sinh^2 u_e)\delta u_-]/A$$

$$= \phi k^3 [2\sin w_b \cos w_b \sinh u_e \delta w + \cosh u_e (\sin^2 w_b + \sinh^2 u_e) \times (-\delta u(-u_e + \delta u_-, w_b))]/A$$

$$\approx \phi k^3 [2\sin w_b \cos w_b \sinh u_e \delta w + \cosh u_e (\sin^2 w_b + \sinh^2 u_e) \times (-\delta u(-u_e, w_b))]/A$$

$$= \phi k^3\big[2\sin w_b \cos w_b \sinh u_e \delta w + \cosh u_e (\sin^2 w_b + \sinh^2 u_e)(\delta v_o/\Gamma_e - \beta_e \delta w)\big]/A \ .$$

Thus, in this approximation the form of $\delta\tau$ is the same as in case (a). Hence the critical current for instability is the same as before, and is given by equation (3.37).

3.4 RELATION TO FIELD LINE CURVATURE

An interesting feature of the result obtained here can be described in terms of the transition point Q (SEYMOUR 1961) which defines the value of u at which the normal curvature of the field lines in the surface $w = w_b$ vanishes. This is the value of u at which each field line becomes tangential to one of a family of straight lines which lie along the curved surface; each of these straight lines can be regarded as a generator of the ruled surface.

Perhaps the simplest method of locating Q is by direct evaluation of the normal curvature. Equation (1.31) gives

$$\begin{aligned} B^2\underline{K} &= \nabla(\tfrac{1}{2}B^2) - B\underline{e}_1\underline{e}_1\cdot\nabla B \\ &= \big[(\underline{u}_o/h_u)(\partial/\partial u) + (\underline{w}_o/h_w)(\partial/\partial w)\big]\tfrac{1}{2}B^2 \\ &\quad - (B_u\underline{u}_o + B_v\underline{v}_o)(B_u/Bh_u)(\partial B/\partial u) \ . \end{aligned} \tag{3.44}$$

The curvature vector $\underline{K}$ has two components, the normal curvature, $\underline{K}_n$, and the geodesic curvature, $\underline{K}_g$, which lies in the surface w = constant. $\underline{K}_n$ is just the w component of $\underline{K}$:

$$\underline{K}_n = (\underline{w}_o/h_w B^2)\partial(\tfrac{1}{2}B^2)/\partial w = K_n\underline{w}_o \ , \tag{3.45}$$

while

$$\underline{K}_g = (\underline{u}_o B_v - \underline{v}_o B_u)(B_v/h_u B^4)\partial(\tfrac{1}{2}B^2)/\partial u \ . \tag{3.46}$$

From (3.45) it is seen that $\underline{K}_n$ will be zero when

$$\partial(\tfrac{1}{2}B^2)/\partial w = 0 \ . \tag{3.47}$$

Using (2.10) and (2.15), equation (3.47) reduces to

$$\frac{2\sin w\left[k^2C^2(\sin^2 w + \sinh^2 u)^2 - A^2\cos^4 w\right]}{k^4\cosh^2 u \cos^3 w(\sin^2 w + \sinh^2 u)^2} = 0 \ .$$

Thus Q, on the surface $w = w_b$, is defined by the equation

$$k^2C^2(\sin^2 w_b + \sinh^2 u_Q)^2 - A^2\cos^4 w_b = 0 \ . \tag{3.48}$$

For $u < u_Q$, the normal curvature is negative and is directed out of the plasma, while for $u > u_Q$, the opposite holds.

Now (3.37) gives

$$k^2C^2_{crit.}(\sin^2 w_b + \sinh^2 u_e)^2 - A^2\cos^4 w_b = 0 \ . \tag{3.49}$$

Comparing (3.48) and (3.49), it will be noted that when $C = C_{crit.}$, then $u_Q = u_e$. The critical current is just that current which places the transition point Q on the electrode.

Basically there are two contributions to the integrand in $\int \delta(d\ell/B)$ as expressed in (1.33). They arise from the two components (normal and geodesic) of $\underline{K}$. The contribution to (1.33) made by the normal curvature is

$$-\ 2d\ell\underline{a}\cdot\underline{K}_n/B = -\ 2d\ell h_w \delta w K_n/B \ . \tag{3.50}$$

As noted after equation (3.48), K_n is negative in the region $u < u_Q$ while with the convention adopted immediately before equation (3.22), δw is positive everywhere. Hence, for $u < u_Q$, the contribution of $\underline{K}_n$ is positive, and therefore stabilizing in view of condition (3.22). That is, placing Q at the electrode allows only a stabilizing contribution from the normal curvature (SEYMOUR 1961).

The geodesic curvature makes the contribution

$$- 2d\ell \underline{a}\cdot\underline{K}_g/B \ .$$

From (3.46) is obtained

$$\underline{a}\cdot\underline{K}_g = (\delta u B_v - h_v \delta v B_u/h_u)(B_v/B^4)\partial(\tfrac{1}{2}B^2)/\partial u \ . \tag{3.51}$$

Using (3.15), equation (3.51) becomes

$$\underline{a}\cdot\underline{K}_g = (\delta u/B^2)\partial(\tfrac{1}{2}B^2)/\partial u = -(1/B^2)(\delta v_o/\Gamma + \beta\delta w)\partial(\tfrac{1}{2}B^2)/\partial u \ , \tag{3.52}$$

the last equation being obtained by using (3.19).

The complete geodesic term in the integrand of (1.33) becomes, by use of equations (3.4) and (3.52)

$$(2h_u du/B_u B^2)(\delta v_o/\Gamma + \beta\delta w)\partial(\tfrac{1}{2}B^2)/\partial u \ . \tag{3.53}$$

Now

$$\partial B^2/\partial u = \frac{\partial}{\partial u}\left[\frac{A^2}{k^4\cosh^2 u(\sin^2 w + \sinh^2 u)} + \frac{C^2}{k^2\cosh^2 u \cos^2 w}\right]$$

$$= \frac{-2\sinh u}{k^2\cosh^3 u}\left[\frac{C^2}{\cos^2 w} + \frac{A^2(\cosh^2 u + \sinh^2 u + \sin^2 w)}{k^2(\sin^2 w + \sinh^2 u)^2}\right] . \tag{3.54}$$

Thus:

(1) $\partial B^2/\partial u$ is an odd function of u ;

(2) for $u > 0$, $\partial B^2/\partial u < 0$.

Therefore, since Γ is an even function of u, the first term in (3.53) is an odd function of u, and will make no net contribution to the overall integral from $-u_e$ to u_e. This fact was seen earlier as the δv_o terms in $\delta\tau_+$ and $\delta\tau_-$ cancelled out when $\delta\tau$ was computed.

On the other hand, the second term in (3.53) is an even function of u and gives a net contribution to the integral which is negative (for $\delta w > 0$) and therefore destabilizing. It is this part of the geodesic contribution which just cancels the stabilizing effect of the normal curvature when $u_e = u_Q$; and it is this destabilizing effect of the geodesic curvature which makes the condition $u_Q \geqslant u_e$ not just a sufficient condition for stability against this interchange, as assumed by SEYMOUR (1961), but also a necessary condition.

The overall sign of the geodesic term depends on the sign and magnitude of δv_o compared with $\Gamma\beta\delta w$. As has been seen in Section 3.3, δv_o is largely arbitrary, but whatever the case, there is always a zero net contribution from the δv_o term and a net destabilizing effect from the other term, provided the system is symmetric in u, so that the integral (1.33) is evaluated between lower and upper limits of u symmetrically disposed about $u = 0$.

3.5 COMPARISON WITH THE EARLIER RESULT OF SEYMOUR

The expression for the critical current $I_{crit.}$, derivable from equation (3.37), may be compared with that obtained by SEYMOUR (1961). Seymour's expression was obtained by assuming that a sufficient condition for stability would be satisfied by placing Q at the electrode. The transition point was approximately located by equating $\tan\theta_{Be} = (B_v/B_u)_{electrodes}$ and $\tan\theta_g$, where θ_g is the inclination of the surface generator lines to the z axis. He obtained, for the critical current,

$$I_{crit.} = 5\phi_E \tan\theta_g/\pi r_e \ , \tag{3.55}$$

(equation (4.5) in his paper, 1961), where ϕ_E was the total external magnetic flux through the discharge, and was approximated as $\phi_E = B_M \pi r_o^2$, by assuming the magnetic field to be roughly constant over the median cross section.

By considering the geometry of the hyperboloid of one sheet it is easy to show that

$$\tan\theta_g = \cot w_b \ . \tag{3.56}$$

Hence (3.55) becomes

$$I_{crit.} = 5B_M r_o^2 \cot w_b/r_e \ . \tag{3.57}$$

The critical current is obtained from equation (3.37) as follows. In MKSC units, neglecting the displacement current,

$$\nabla \times \underline{B} = \mu_o \underline{j} \ , \tag{3.58}$$

and then it is easily shown that

$$C = \mu_o I/2\pi \ . \qquad (3.59)$$

Thus

$$I_{crit.} = 2\pi C_{crit.}/\mu_o = 2\pi A\cos^2 w_b/\mu_o k(\sin^2 w_b + \sinh^2 u_e) \ .$$

Using equations (2.1) and (2.16), some manipulation leads to

$$I_{crit.} = (2\pi B_M r_o^2 \cot w_b/\mu_o r_e)\left[\cosh u_e \sin^2 w_b/(\sin^2 w_b + \sinh^2 u_e)\right] \ . \qquad (3.60)$$

Thus, but for the units conversion factors, this expression differs from (3.57) by the factor in square brackets. The two expressions may be reconciled by considering the two approximations involved in Seymour's analysis:

(1) An approximate expression for B_u of the form

$$B_u \sim 1/r^2 \qquad (3.61)$$

was used, and

(2) The transition point was located in an approximate way by equating $\tan\theta_{Be}$ and $\tan\theta_g$.

These two approximations are dealt with below, where it is shown that by using more accurate expressions, the correction factor obtained above arises quite naturally by using Seymour's method.

Firstly, the approximate form for B_u yielded

$$\tan\theta_{Be} = \pi I_{crit.} r_e/5\phi_E \ , \qquad (3.62)$$

(equation (4.4) of Seymour's paper), or

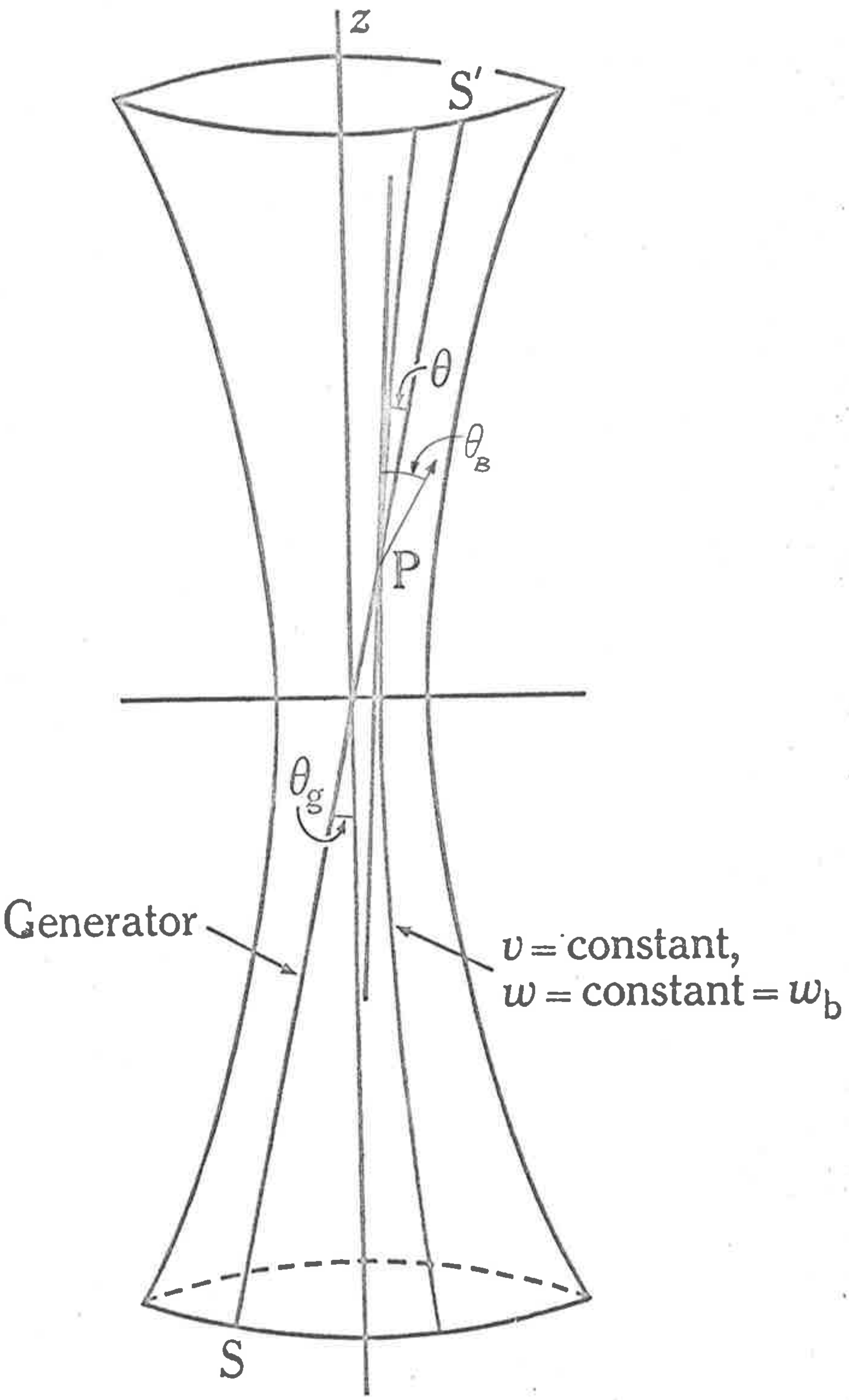

Fig. 8. The generator SS′ is at a distance r_o from the z axis, to which it is inclined at an angle θ_g.

$$\tan\theta_{Be} = I_{crit.}r_e/5r_o^2B_M \, . \tag{3.63}$$

If, on the other hand, expression (2.15) is used for B_u, then, with the help of equations (2.1) and (2.10)

$$\tan\theta_{Be} = (B_v/B_u)_{electrodes} = (\mu_o I_{crit.}r_e/2\pi B_M r_o^2)(\sin^2 w_b + \sinh^2 u_e)^{\frac{1}{2}}/\cosh u_e \sin w_b \, , \tag{3.64}$$

and it is noted that part of the correction factor in (3.60) is already emerging.

Secondly, to locate the transition point more accurately, one must equate $\tan\theta_{Be}$ with $\tan\theta_e$, where in general, θ is the angle between the generator line and the tangent to the curve v = constant, w = constant, at the point P (Fig. 8).

The expression for $\tan\theta$ is

$$\tan\theta = (h_v/h_u)dv(u;w_b)/du \, , \tag{3.65}$$

where $v = v(u;w_b)$ describes the generator passing through the point $(0,v_o,w_b)$, v_o being arbitrary (Fig. 9). For simplicity, take $v_o = 0$. Then, remembering that the generator is inclined at angle θ_g to the z axis and lies at a perpendicular distance r_o from it, the equations of the generator in the cartesian system (x,y,z) (Fig. 9) are:

$$x = r_o \tag{3.66}$$

and

$$y = z\tan\theta_g = z\cot w_b \, , \tag{3.67}$$

using equation (3.56).

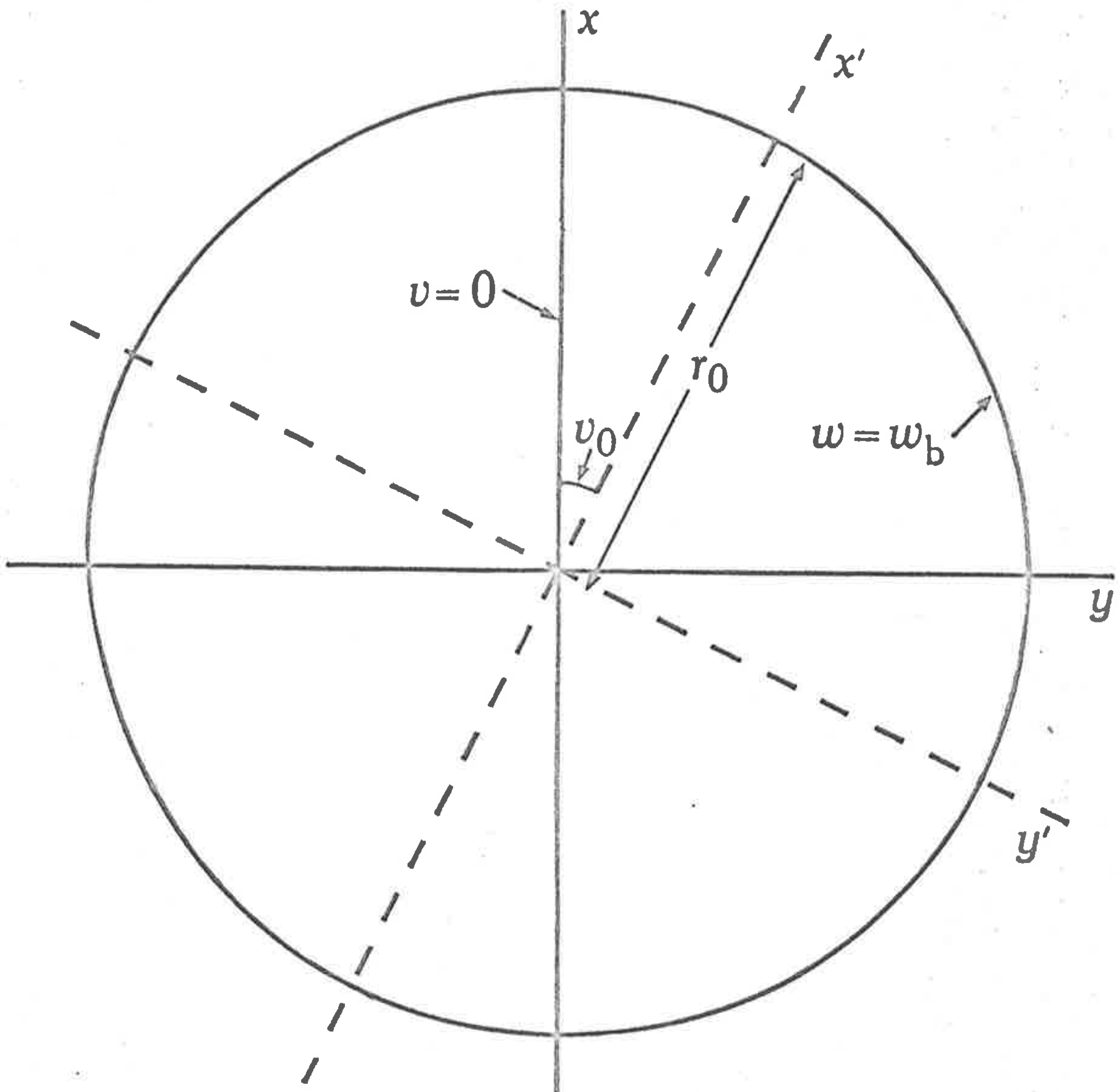

Fig. 9. The discharge cross section at the median plane.

Using equation (2.1) and the facts that

$$x = r \cos v$$

and

$$y = r \sin v ,$$

equations (3.66) and (3.67) yield

$$\tan v = \sinh u ,$$

from which equation (3.65) then becomes

$$\tan\theta = h_v/h_u \cosh u . \qquad (3.68)$$

Using equations (2.2) and (2.3) and evaluating at the electrode,

$$\tan\theta_e = \cot w_b \sin w_b/(\sin^2 w_b + \sinh^2 u_e)^{\frac{1}{2}} . \qquad (3.69)$$

The above treatment is easily generalized to the case of arbitrary v_o by use of the rotated cartesian system (x',y',z) (Fig. 9).

It can now be seen that in (3.69) the remaining part of the correction factor in (3.60) has emerged.

The final step is to equate $\tan\theta_{Be}$ with $\tan\theta_e$, and to solve for $I_{crit.}$, as below:

$$\frac{\mu_o I_{crit.} r_e (\sin^2 w_b + \sinh^2 u_e)^{\frac{1}{2}}}{2\pi B_M r_o^2 \cosh u_e \sin w_b} = \frac{\cot w_b \sin w_b}{(\sin^2 w_b + \sinh^2 u_e)^{\frac{1}{2}}} ,$$

which gives

$$I_{crit.} = \frac{2\pi B_M r_o^2 \cot w_b}{\mu_o r_e} \left(\frac{\cosh u_e \sin^2 w_b}{\sin^2 w_b + \sinh^2 u_e}\right) , \qquad (3.70)$$

in agreement with expression (3.60).

3.6 THE SLIGHTLY CONSTRICTED DISCHARGE

The derivation above of expression (3.70) shows how the correction factor, given by

$$F = \frac{\cosh u_e \, \sin^2 w_b}{\sin^2 w_b + \sinh^2 u_e} , \qquad (3.71)$$

arises when the two approximations in Seymour's analysis are taken into account. The conditions under which Seymour's expression is a good approximation to the result (3.70) may be seen as follows.

If

$$\sinh^2 u_e \ll \sin^2 w_b < 1 , \qquad (3.72)$$

then

$$F \approx \cosh u_e$$

$$\approx 1,$$

since u_e must be very small for (3.72) to hold. Therefore, under condition (3.72), the approximation is good. It is shown below that this condition is satisfied in the practical case of a discharge with semi-length z_e considerably greater than its radius r_o at the median plane, and a constriction ratio, of radius r_e at the electrodes to radius at the median plane, not very different from unity:

$$\frac{r_e}{r_o} \approx 1 . \qquad (3.73)$$

Equation (3.73) yields, through use of (2.1),

$$\cosh u_e \approx 1,$$

or

$$\sinh^2 u_e \ll 1, \qquad (3.74)$$

while the condition

$$r_o^2 \ll z_e^2$$

gives

$$\cos^2 w_b \ll \sin^2 w_b \sinh^2 u_e \ .$$

Therefore, since $\sin w_b < 1$, it follows that

$$\cos^2 w_b \ll \sinh^2 u_e \ll 1 \ , \qquad (3.75)$$

using (3.74). Condition (3.75) further indicates that $\sin w_b \approx 1$, so that, finally,

$$\sinh^2 u_e \ll \sin^2 w_b \ .$$

It is easy to show by partial differentiation that, for a given value of u_e, F increases monotonically with $\sin w_b$, in the region of interest. Thus for a given value of u_e, F cannot exceed the value $\frac{1}{\cosh u_e}$, obtained by substituting the maximum value of unity for $\sin w_b$ in equation (3.71). Since $\cosh u_e$ must always be greater than 1 ($u_e > 0$), it is clear that F cannot exceed the value unity, for any geometry.

However F may assume values much less than unity, in geometries for which Seymour's expression is not a good approximation to the exact result. For example, if u_e is large enough so that

$\cosh u_e \sim \sinh u_e \sim 10$, say, corresponding to the system with a high constriction ratio, then $F \lesssim \frac{1}{10}$.

3.7 CONCLUSION

The thermodynamic stability criterion of ROSENBLUTH and LONGMIRE (1957) has been applied to the interchange instability of the surface layer of a constricted discharge. The discharge boundary is shaped by an external magnetic field, to approximate an hyperboloid of one sheet. This gives rise to the possibility of stabilizing forces, under Teller's curvature criterion. The sign of the field line's normal curvature depends on the direction of the field line in relation to that of the surface generator. The balance of stabilizing forces, from the normal curvature, against destabilizing forces from the geodesic curvature gives rise to a critical current, above which interchange instability sets in. An expression for the critical current has been obtained in the form

$$\mu_o I_{crit.}/2\pi = C_{crit.} = A\cos^2 w_b/k(\sin^2 w_b + \sinh^2 u_e) ,$$

where $A = B_M k^2 \sin w_b$, B_M being the axial magnetic field component on the surface $w = w_b$ of the discharge, at the median plane. $I_{crit.}$ may be written in terms of the dimension-parameters of the discharge, r_o and z_e, where r_o is the discharge radius at the median plane, and z_e is the discharge semi-length. Using (2.1), is obtained

$$I_{crit.} = \frac{2\pi B_M r_o^2 (k^2 - r_o^2)^{\frac{1}{2}}}{\mu_o (z_e^2 + k^2 - r_o^2)} .$$

The destabilizing effect of the geodesic curvature of the field lines is such that the current must not exceed that value which places the transition point Q (at which the normal curvature changes sign) at the electrode. This means that the requirement that Q be at the electrode is not only a sufficient condition for stability, as assumed by SEYMOUR (1961), but also a necessary condition. The expression for $I_{crit.}$ obtained by SEYMOUR (1961), is a good approximation for the present expression, for a discharge with

$$r_e \approx r_o$$

and

$$r_o^2 \ll z_e^2 .$$

3.8 REFERENCES

JAMES, M. K. (1970) - Aust. J. Phys. 23, 275.

ROSENBLUTH, M., and LONGMIRE, C. (1957) - Ann. Phys. 1, 120.

SEYMOUR, P. W. (1961) - Aust. J. Phys. 14, 321.

CHAPTER 4

THE ENERGY PRINCIPLE FOR HYDROMAGNETIC STABILITY

4.1 INTRODUCTION

The analysis presented in the foregoing chapters dealt with the stability of low pressure plasma-magnetic field systems with respect to the specialized perturbation which results in the interchange of adjacent magnetic flux tubes. In particular an expression was derived for the discharge current which is critical for the onset of unstable interchanges in the low pressure boundary region of a diffuse constricted discharge. Attention is now given to the treatment of a more general class of fluid motions characterized by the small perturbation field, $\underline{\xi}$. $\underline{\xi}$ is usually taken to mean the displacement of a fluid element from its equilibrium position $\underline{r}_o$, and is written

$$\underline{\xi} = \underline{\xi}(\underline{r}_o,t) \quad . \qquad (4.1)$$

This approach is based essentially on a fluid theory in which the strength of collisions is assumed to be such that the pressure always remains scalar, but the electrical conductivity may be regarded as infinite.

An expression for $\delta W(\underline{\xi},\underline{\xi})$, the change in system potential energy produced by the perturbation, may be obtained by writing down the potential energy function and evaluating the second-order variation with respect to $\underline{\xi}$. Although an expression for δW has been obtained (BERNSTEIN et al, 1958) effectively by integrating the second-order expression for $\frac{dW}{dt}$, there does not appear to be in the literature a complete derivation of δW by the former method. VAN KAMPEN and FELDERHOF (1967) use this approach to derive $\delta \bar{\bar{W}}$ for a system in which the plasma extends to infinity. They are thus concerned only with a fluid domain, and certain terms, which by application of Gauss' theorem become integrals over the fluid surface, are assumed to give zero contributions as the surface is extended to infinity. Their work is here extended to derive δW for the case of a finite system in which the plasma is in contact with a vacuum region and with electrodes. The result is found to be in substantial agreement with that of BERNSTEIN et al (1958), although some modification arises because here the plasma region is not considered to be completely surrounded by a conducting shell, as in the above reference, but instead allowance is made for insulating supports for electrodes. This necessitates great care in applying Gauss' integral transform as is required at a number of places in the proof. Such a generalization of the system geometry then permits application of the result to the discharge between electrodes.

In the present chapter, and in the next, no particular configuration is assumed for the system, but in the final chapter the conditions for stability discussed here will be applied to the constricted discharge. Since in the later work a field-free discharge will be considered, some of the discussion here will be related to such systems.

4.2 STATIC AND DYNAMIC BOUNDARY CONDITIONS

Consider a highly conducting magnetically confined plasma, insulated from its surroundings by a vacuum region. The system will be described by orthogonal curvilinear coordinates (u_1, u_2, u_3) with unit vectors $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$. At the interface between field and plasma, $\underline{e}_3$ is taken to be the unit normal vector directed into the plasma.

Using a circumflex to indicate vacuum quantities, the vacuum magnetic field $\hat{\underline{B}}$ at all times satisfies Maxwell's equations without displacement or conduction current:

$$\nabla \cdot \hat{\underline{B}} = 0 \ , \qquad (4.2)$$

$$\nabla \times \hat{\underline{B}} = 0 \ ; \qquad (4.3)$$

and when the system is perturbed so that a time-dependent situation is produced,

$$\nabla \times \hat{\underline{E}} = - \frac{\partial \hat{\underline{B}}}{\partial t} \ . \qquad (4.4)$$

$$\nabla \cdot \hat{\underline{E}} = 0 \ . \qquad (4.5)$$

The boundary conditions to be applied at the interface are well known (KRUSKAL and SCHWARZSCHILD, 1954). At such an interface, one introduces a sheet current $\underline{j}^*$ and jump discontinuities in the magnetic field and particle pressure (and in the electric field, in the dynamical situation). Relations involving the discontinuities in the physical quantities are derived by integrating the appropriate equations across the thin layer which the surface sheet is assumed to approximate, and then allowing the layer to become vanishingly thin.

In the layer the following equations apply:-

The equation of motion, in the absence of charge accumulation and gravitational forces:

$$\rho \frac{d\underline{v}}{dt} = - \nabla p + \underline{j} \times \underline{B} \;; \tag{4.6}$$

the conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \;; \tag{4.7}$$

the infinite electrical conductivity approximation:

$$\underline{E} + \underline{v} \times \underline{B} = 0 \;; \tag{4.8}$$

the adiabatic equation of state:

$$\frac{d}{dt} (p\rho^{-\gamma}) = 0 \;; \tag{4.9}$$

Maxwell's equations without displacement current:

$$\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} \;, \tag{4.10}$$

$$\nabla \times \underline{B} = \mu_o \underline{j} \;, \tag{4.11}$$

and

$$\nabla \cdot \underline{B} = 0 \;. \tag{4.12}$$

These equations are listed, together with the conditions under which they are valid, by BERNSTEIN et al (1958). They are displayed here for convenience, all symbols having their usual significance.

Equation (4.6) becomes, on carrying out the above procedure,

$$\left\langle p + \frac{B^2}{2\mu_o} \right\rangle = 0 \quad , \qquad (4.13)$$

where $\langle X \rangle$ denotes the jump in the quantity X on crossing the interface in the direction of $\underline{n}$, where, in this coordinate system,

$$\underline{n} = - \underline{e}_3 \ . \qquad (4.14)$$

It is enlightening to examine the integration of the magnetic body force term $\underline{j} \times \underline{B}$ in equation (4.6). In the limit of zero layer thickness, $\underline{j}^*$ must lie in the interface since current cannot flow into or out of the vacuum. $\underline{B}$ must also lie in the interface in order to avoid infinite acceleration due to unbalanced tangential forces on the massless current sheet. Thus the surface layer may be considered as an assembly of surfaces in which lie the magnetic field and electric current vectors. This assembly reduces to the surface current sheet as the layer thickness, Δs, tends to zero.

The magnetic body force per unit area exerted on the layer is given by

$$\int ds \ \underline{j} \times \underline{B} \ , \qquad (4.15)$$

where s represents the path of integration normal to the magnetic surfaces in the layer. Quantitatively, $\underline{j}^*$ is defined as

$$\underline{j}^* = \int d\underline{j}^* = \int ds\ \underline{j}\ . \tag{4.16}$$

Hence the magnetic force per unit area is

$$\int d\underline{j}^* \times \underline{B} = \underline{j}^* \times \bar{\underline{B}}\ , \tag{4.17}$$

where $\bar{\underline{B}}$ as defined by (4.17) is an appropriate average of $\underline{B}$ for the layer. It is shown below that, as stated but not explicitly proved by KRUSKAL and SCHWARZSCHILD (1954), the appropriate average to take for $\bar{\underline{B}}$ after allowing Δs to become vanishingly small is just the arithmetic mean of the values on the two sides of the layer.

The result for $\bar{\underline{B}}$ is obtained by integrating expression (4.15), as follows. Making use of equation (4.11) and a standard vector identity, the integral assumes the form

$$\underline{j}^* \times \bar{\underline{B}} = \frac{1}{\mu_o} \int ds\ (\underline{B}\cdot\nabla\underline{B} - \nabla\tfrac{1}{2}B^2)\ . \tag{4.18}$$

On allowing Δs to tend towards zero, the first term of (4.18) vanishes, while the second term becomes

$$- \underline{n} < \frac{B^2}{2\mu_o} >\ . \tag{4.19}$$

Thus

$$\underline{j}^* \times \bar{\underline{B}} = - \underline{n} < \frac{B^2}{2\mu_o} >$$

$$= - \underline{n} < \underline{B} > \cdot \frac{1}{2\mu_o} (\underline{B}_i + \underline{B}_e)\ , \tag{4.20}$$

where $\underline{B}_i$ is the field on the inside of the interface, and $\underline{B}_e$ (strictly the vacuum quantity, $\hat{\underline{B}}$) is the field on the outside of the interface.

By means of standard procedure equation (4.11) can be integrated across a rectangular cross-sectional element of the layer to give, for $\Delta s \to 0$,

$$\mu_o \underline{j}^* = \underline{n} \times \langle \underline{B} \rangle , \qquad (4.21)$$

while, by means of the pill-box technique, the solenoidal property of $\underline{B}$ leads to

$$\underline{n} \cdot \langle \underline{B} \rangle = 0 . \qquad (4.22)$$

Therefore the vector product of $\underline{n}$ with equation (4.21) becomes

$$\mu_o \underline{n} \times \underline{j}^* = - \langle \underline{B} \rangle , \qquad (4.23)$$

upon expansion of the vector triple product and use of (4.22), and so equation (4.20) gives

$$\underline{j}^* \times \bar{\underline{B}} = \underline{n}\ (\underline{n} \times \underline{j}^*) \cdot \tfrac{1}{2}(\underline{B}_i + \underline{B}_e)$$

$$= \underline{n}\ \underline{n} \cdot \{\underline{j}^* \times \tfrac{1}{2}(\underline{B}_i + \underline{B}_e)\} . \qquad (4.24)$$

Since $\underline{n} \cdot \underline{j}^* = \underline{n}\cdot\underline{B}_i = \underline{n}\cdot\underline{B}_e = 0$, it follows therefore that

$$\underline{j}^* \times \bar{\underline{B}} = \underline{j}^* \times \tfrac{1}{2}(\underline{B}_i + \underline{B}_e) , \qquad (4.25)$$

thus identifying $\bar{\underline{B}}$ as the arithmetic mean of $\underline{B}_i$ and $\underline{B}_e$.

A further boundary condition of importance in the dynamical situation is

$$\underline{n} \times \langle \underline{E} \rangle = (\underline{n} \cdot \underline{v}) \langle \underline{B} \rangle , \qquad (4.26)$$

where $\underline{v}$ is the velocity of points on the interface. This equation, elegantly derived by JEFFREY (1966), expresses the continuity of the tangential component of $\underline{E}$ in the frame of reference moving with the interface.

Summarizing, the conditions of interest at the interface, in the special case of zero internal magnetic field, are essentially

$$\hat{B}^2 = 2\mu_o p \qquad (4.27)$$

from equation (4.13); and from equation (4.22),

$$\underline{e}_3 \cdot \underline{\hat{B}} = 0 , \qquad (4.28)$$

which apply at all times. In particular, using the subscript o to indicate quantities in the equilibrium state, the absence of an internal field leads to the condition that p_o is constant throughout the plasma. Equation (4.27) then implies that $\hat{B}_o^2$ is constant at all points on the interface.

Conditions (4.27) and (4.28) imply a geometric property of the magnetic lines of force on the interface in equilibrium; namely, that they are geodesics of $S_{pv}(o)$ where, for convenience, $S_{pv}(o)$ represents the equilibrium interface between the plasma and vacuum regions. The proof is as follows. First, application of a standard vector identity to the vacuum magnetic field $\underline{\hat{B}}$ gives

$$\underline{\hat{B}} \times (\nabla \times \underline{\hat{B}}) = \nabla(\tfrac{1}{2}\hat{B}^2) - \underline{\hat{B}} \cdot \nabla \underline{\hat{B}} , \qquad (4.29)$$

which, in view of equation (4.3), reduces to

$$\hat{\underline{B}}\circ\nabla\hat{\underline{B}} = \nabla(\tfrac{1}{2}\hat{B}^2) \ . \tag{4.30}$$

Introducing the curvature vector $\underline{K} = (\hat{\underline{b}}\circ\nabla)\hat{\underline{b}}$, where $\hat{\underline{b}}$ is the unit vector in the direction of $\hat{\underline{B}}$,

$$\hat{B}^2\underline{K} = \nabla(\tfrac{1}{2}\hat{B}^2) - \hat{\underline{b}}(\hat{\underline{b}}\circ\nabla)(\tfrac{1}{2}\hat{B}^2) \tag{4.31}$$

$$= (\underline{\underline{I}} - \hat{\underline{b}}\hat{\underline{b}})\circ\nabla(\tfrac{1}{2}\hat{B}^2) \ , \tag{4.32}$$

where the unit tensor $\underline{\underline{I}} = \underline{e_1}\underline{e_1} + \underline{e_2}\underline{e_2} + \underline{e_3}\underline{e_3}$. (4.33)

On the interface the condition (4.28) means that

$$\hat{\underline{b}} = \frac{\hat{B}_1\underline{e_1}}{\hat{B}} + \frac{\hat{B}_2\underline{e_2}}{\hat{B}} \ , \tag{4.34}$$

and so

$$\hat{B}^2\underline{K} = \left\{\underline{e_1}\underline{e_1} + \underline{e_2}\underline{e_2} + \underline{e_3}\underline{e_3} - \left(\frac{\hat{B}_1\underline{e_1}}{\hat{B}} + \frac{\hat{B}_2\underline{e_2}}{\hat{B}}\right)\left(\frac{\hat{B}_1\underline{e_1}}{\hat{B}} + \frac{\hat{B}_2\underline{e_2}}{\hat{B}}\right)\right\}\circ\nabla(\tfrac{1}{2}\hat{B}^2)$$

$$= \left\{\underline{e_1}\underline{e_1}\left(1 - \frac{\hat{B}_1{}^2}{\hat{B}^2}\right) + \underline{e_2}\underline{e_2}\left(1 - \frac{\hat{B}_2{}^2}{\hat{B}^2}\right) - \frac{\hat{B}_1\hat{B}_2}{\hat{B}^2}(\underline{e_1}\underline{e_2} + \underline{e_2}\underline{e_1})\right.$$

$$\left. + \underline{e_3}\underline{e_3}\right\}\circ\nabla(\tfrac{1}{2}\hat{B}^2) \ . \tag{4.35}$$

In view of the remarks following equation (4.28) it is seen that on $S_{pv}(o)$,

$$\underline{e_1}\circ\nabla(\tfrac{1}{2}\hat{B}_o^2) = \underline{e_2}\circ\nabla(\tfrac{1}{2}\hat{B}_o^2) = 0 \ ,$$

and so in equilibrium, on $S_{pv}(o)$,

$$\hat{B}_o^2\underline{K} = \underline{e_3}\underline{e_3}\circ\nabla(\tfrac{1}{2}\hat{B}_o^2) \ . \tag{4.36}$$

Thus on $S_{pv}(o)$ the vector curvature of the equilibrium field lines is everywhere normal to $S_{pv}(o)$. Using the language of differential geometry, the geodesic curvature of the field lines on $S_{pv}(o)$ is zero; thus, by definition, the field lines on $S_{pv}(o)$ are geodesics of $S_{pv}(o)$.

4.3 PERTURBATION OF THE VACUUM MAGNETIC FIELD

In practice the plasma is a finite body, either closed upon itself (torus), or terminated by electrodes. For the latter case it is assumed here that the electrodes are sufficiently hot for the plasma in their immediate vicinity to satisfy the infinite electrical conductivity approximation leading to equation (4.8). Although VAN KAMPEN and FELDERHOF concern themselves with an infinite plasma, for a finite plasma it is nevertheless possible to obtain from their work (1967, p.75) an expression for $\delta\bar{W}$, which represents the second-order variation in potential energy associated with the plasma and magnetic field within τ_p, the plasma volume. The complete expression for δW in the case of a finite plasma can be obtained by adding to $\delta\bar{W}$ the expression representing the second-order variation in W_{BE}, the energy of the external magnetic field. This derivation is carried through below.

For the discharge between electrodes, which will be the subject of the final chapter of this thesis, one cannot make the usual assumption that the region of interest - plasma and vacuum - is completely surrounded by a perfectly conducting wall. In fact if, as is usual for example in linear pinch experiments, a stabilizing conducting shell is used, coaxial with the discharge and continuous for the length of the discharge, then it is clear that insulating supports for the electrodes must also be present to avoid short-circuiting of the discharge. That is, the stabilizing shell should not be considered completely closed across the ends.

In the development given below, the assumption of a closed wall is not made. Instead the general case is considered, in which the volume external to the plasma is assumed to comprise vacuum regions, perfect conductors and perfect insulators. One may also assume for generality that the confined plasma may be partly in contact with all three media, but since a very hot infinitely conducting plasma cannot remain in contact with an insulator without the occurrence of rapid quenching, it is assumed here that the plasma is in contact only with vacuum and with very hot electrodes. At the electrode surfaces the condition

$$\underline{n}\cdot\underline{\xi} = 0 \tag{4.37}$$

must apply.

For such a system the surface terms not taken into consideration by VAN KAMPEN and FELDERHOF must be included. The important result obtained here is that the form of δW derived for this system by extending the approach of VAN KAMPEN and FELDERHOF is in agreement with that obtained by BERNSTEIN et al (1958), except for some modification of the vacuum contribution. This modification arises simply because the entire region of interest is no longer confined to the interior of a closed conducting wall, but extends to infinity. The expression obtained in second order from VAN KAMPEN and FELDERHOF is, assuming zero gravitational field (VAN KAMPEN and FELDERHOF, p.75, equation (21) et seq.)

$$\delta\bar{W} = \tfrac{1}{2} \int_{\tau_p(o)} d\tau_o \, \{\frac{1}{\mu_o} |\underline{Q}|^2 - \underline{j}_o \cdot (\underline{Q} \times \underline{\xi}) + \gamma p_o (\nabla\cdot\underline{\xi})^2 + (\nabla\cdot\underline{\xi})\underline{\xi}\cdot\nabla p_o\} + \frac{1}{2\mu_o} \int_{S_p(o)} \Big[\Big(\mu_o p_o + \frac{B_o^2}{2}\Big)(\underline{\xi}\cdot\nabla\underline{\xi} - \underline{\xi}\nabla\cdot\underline{\xi}) + \{\underline{\xi}\cdot\underline{\xi}\cdot\nabla\underline{B}_o\}\, \underline{B}_o - \{\underline{\xi}\cdot\underline{B}_o\cdot\nabla\underline{B}_o\}\, \underline{\xi}\Big] \cdot d\underline{S}_o \, , \qquad (4.38)$$

where

$$\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B}_o) \, , \qquad (4.39)$$

$\tau_p(o)$ is the equilibrium plasma volume, and $S_p(o)$ encloses $\tau_p(o)$, with $d\underline{S}_o$ directed out of the plasma.

Consider the term

$$\frac{1}{2\mu_o} \int_{S_p(o)} \{\underline{\xi}\cdot\underline{\xi}\cdot\nabla\underline{B}_o\}\underline{B}_o\cdot d\underline{S}_o$$

in the surface integral of equation (4.38). Its integrand is zero on the plasma-vacuum interface $S_{pv}(o)$, since $\underline{B}_o\cdot d\underline{S}_o$ is zero there. It is also zero on the plasma-conducting electrode interface $S_{pc}(o)$, as may be seen from the following. On $S_{pc}(o)$ it is necessary to consider situations where the magnetic field does not lie in the interface, but actually enters the electrode. (e.g. a linear pinch with internal axial field.) At the interface the condition

$$\underline{dS} \times \underline{E} = 0$$

must apply. This becomes, through equation (4.8),

$$\underline{dS} \times (\underline{v} \times \underline{B}) = 0 \ ,$$

or

$$(\underline{dS} \circ \underline{B})\underline{v} - (\underline{dS} \circ \underline{v})\underline{B} = 0 \ .$$

Since the interface is rigid and fixed, $\underline{dS} \circ \underline{v} = 0$, and so

$$(\underline{dS} \circ \underline{B})\underline{v} = 0 \ . \qquad (4.40)$$

Integrated to first order, this gives

$$(\underline{dS}_o \circ \underline{B}_o)\underline{\xi} = 0 \ . \qquad (4.41)$$

Hence when $\underline{dS}_o \circ \underline{B}_o \neq 0$, the freezing-in effect of infinite conductivity leads to $\underline{\xi} = 0$ at the plasma-electrode interface. Clearly, then $\underline{\xi} \circ \underline{\xi} \circ \nabla \underline{B}_o$ must be zero on $S_{pc}(o)$, if $\underline{B}_o \circ \underline{dS}_o \neq 0$.

Since expression (4.38) represents $\delta W_B + \delta W_p$, where W_B is the energy of the magnetic field in the plasma and W_p is the material energy of the plasma, to extend the derivation here one must include the variation δW_{BE} in the energy of the magnetic field which occupies the volume external to the plasma.

This variation may be calculated, to second order in the perturbation, as the work done against the pressure of the vacuum magnetic field in deforming the surface $S_{pv}(t)$. The validity of this approach is established in the following discussion.

The magnetic energy external to the plasma may be written

$$W_{BE} = \frac{1}{2\mu_o} \int_{\tau_c+\hat{\tau}(t)+\tau_i} B^2 d\tau \quad , \tag{4.42}$$

where the subscripts c and i on the τ's refer respectively to the conductor and insulator regions. In the rigid material occupying τ_c the electrical conductivity is extremely high, and so $\frac{\partial \underline{B}}{\partial t}$ is sensibly zero there, at least on the rapid time-scale of unstable motions considered here.

Then

$$\frac{d}{dt} W_{BE} = \frac{1}{2\mu_o} \frac{d}{dt} \int_{\hat{\tau}(t)+\tau_i} B^2(\underline{r},t) d\tau \ . \tag{4.43}$$

Appendix II provides a rigorous analytical proof that the time derivative $\frac{dg(t)}{dt} = \frac{d}{dt} \int_{\tau(t)} f(\underline{r},t) d\tau$ may be expanded to the intuitively obvious form

$$\frac{dg(t)}{dt} = \int_{\tau(t)} \frac{\partial}{\partial t} f(\underline{r},t) d\tau + \int_{S(t)} f(\underline{r},t) \underline{v} \cdot d\underline{S} \ , \tag{4.44}$$

where S(t) is the surface enclosing $\tau(t)$, $\underline{v}$ is the velocity of a point on S(t), and $d\underline{S}$ is directed out of $\tau(t)$. Applying this result to equation (4.43) yields, therefore,

$$\frac{d}{dt} W_{BE} = \frac{1}{\mu_o} \int_{\hat{\tau}(t)+\tau_i} \frac{\partial}{\partial t} (\tfrac{1}{2}B^2) d\tau + \frac{1}{2\mu_o} \int_{S(t)} B^2 \underline{v} \cdot d\underline{S}' , \tag{4.45}$$

where $S(t)$ bounds the combined volumes of vacuum and insulator. That is,

$$S(t) = S_{pv} + S_{cv} + S_{pi} + S_{ci} + S_{\infty} , \qquad (4.46)$$

where S_{ab} is the interface between media 'a' and 'b', and S_{∞} is the "surface at infinity". In (4.45) the prime is introduced so that on S_{pv} there will be no confusion between $d\underline{S}'$, directed out of the vacuum, and $d\underline{S}$, directed out of the plasma consistent with its equilibrium form $d\underline{S}_o$ in (4.38).

Introducing the magnetic vector potential $\underline{A}$ such that $\underline{B} = \nabla \times \underline{A}$, the volume integral in equation (4.45) becomes, but for the factor $\frac{1}{\mu_o}$

$$\int_{\hat{\tau}(t)+\tau_i} \frac{\partial}{\partial t} (\tfrac{1}{2}B^2)d\tau = \int_{\hat{\tau}(t)+\tau_i} \underline{B} \cdot \frac{\partial \underline{B}}{\partial t} d\tau$$

$$= \int_{\hat{\tau}(t)+\tau_i} (\nabla \times \underline{A})\cdot(\nabla \times \frac{\partial \underline{A}}{\partial t})d\tau . \qquad (4.47)$$

Since no currents flow in $\hat{\tau}(t)$ and τ_i, $\nabla \times (\nabla \times \underline{A}) = 0$, and so

$$(\nabla \times \underline{A})\cdot\left(\nabla \times \frac{\partial \underline{A}}{\partial t}\right) = \nabla\cdot\left(\frac{\partial \underline{A}}{\partial t} \times (\nabla \times \underline{A})\right) ,$$

using a standard vector identity. Applying Gauss' theorem to the integral (4.47) therefore gives

$$\int_{\hat{\tau}(t)+\tau_i} \frac{\partial}{\partial t}(\tfrac{1}{2}B^2)\,d\tau = \int_{S(t)} d\underline{S}' \cdot \left(\frac{\partial \underline{A}}{\partial t} \times (\nabla \times \underline{A})\right)$$

$$= \int_{S(t)} (d\underline{S}' \times \frac{\partial \underline{A}}{\partial t}) \cdot \nabla \times \underline{A}$$

$$= - \int_{S(t)} (d\underline{S}' \times \underline{E}) \cdot \nabla \times \underline{A}\ , \qquad (4.48)$$

since the electric field $\underline{E} = - \frac{\partial \underline{A}}{\partial t}$.

Now $d\underline{S}' \times \underline{E}$ must vanish on the rigid conducting surfaces S_{cv} and S_{ci}, and furthermore it is assumed that the field quantities fall off rapidly enough for the contribution from the integral (4.48) over S_∞ to be vanishingly small. Additionally, conditions (4.8) and (4.26), together with the fact that $\underline{B}$ lies in the surface $S_{pv}(t)$ in the plasma/magnetic field model chosen leads to

$$d\underline{S}' \times \hat{\underline{E}} = \hat{\underline{B}}(\underline{v} \cdot d\underline{S}') \qquad (4.49)$$

on $S_{pv}(t)$.

Thus, assuming S_{pi} to be zero, equation (4.48) reduces to

$$\int_{\hat{\tau}(t)+\tau_i} \frac{\partial}{\partial t}(\tfrac{1}{2}B^2)\,d\tau = - \int_{S_{pv}(t)} \hat{\underline{B}}(\underline{v} \cdot d\underline{S}') \cdot \nabla \times \hat{\underline{A}}$$

$$= - \int_{S_{pv}(t)} \hat{B}^2 \underline{v} \cdot d\underline{S}' \qquad (4.50)$$

because $\hat{\underline{B}} = \nabla \times \hat{\underline{A}}$.

Considering now the surface integral in equation (4.45), it is noted that since S_{cv} and S_{ci} are rigid, $\underline{v}\cdot d\underline{S}' = 0$ at those surfaces. On the reasonable assumption that the integral over S_∞ is vanishingly small, and remembering that S_{pi} is here considered zero, this integral therefore reduces to

$$\frac{1}{2\mu_o}\int_{S(t)} B^2\underline{v}\cdot d\underline{S}' = \frac{1}{2\mu_o}\int_{S_{pv}(t)} \hat{B}^2\underline{v}\cdot d\underline{S}' \ . \qquad (4.51)$$

Thus, substituting the results (4.50) and (4.51) into equation (4.45),

$$\frac{d}{dt}W_{BE} = -\frac{1}{2\mu_o}\int_{S_{pv}(t)} \hat{B}^2\underline{v}\cdot d\underline{S}' \ . \qquad (4.52)$$

Since $\frac{\hat{B}^2}{2\mu_o}\, d\underline{S}'$ is the force directed into the plasma by the pressure of the vacuum magnetic field at the plasma/vacuum interface, integration of equation (4.52) with respect to time verifies that the change in external magnetic energy is just the work done against the pressure of the vacuum magnetic field in deforming the surface.

Originally the result (4.52) was derived by a method which involved the assumption of a form of equation of state for the plasma, and which made explicit use of the law of conservation of total system energy. The more general method presented above makes no assumptions concerning the material inside S_p. The original derivation is given, for interest, in Appendix III.

4.4 EVALUATION OF δW_{BE}

To find δW_{BE} to second order in the perturbation, equation (4.52) must be written to second order and integrated. To carry this integration through it is noted that, from the Lagrangian viewpoint, physical properties in a given fluid element at $(\underline{r},t)$ are functions of the initial position $\underline{r}_o$ of the fluid element, and of the time t. Thus to first order in $\underline{\xi}$, the vacuum magnetic field at the boundary at time t is (see, for example, SCHMIDT, 1966, p.123),

$$\underline{\hat{B}}(\underline{r},t) = \underline{\hat{B}}(\underline{r}_o,o) + \underline{\xi}\cdot\nabla\underline{\hat{B}}(\underline{r}_o,o) + \nabla \times \delta\underline{\hat{A}} \ , \qquad (4.53)$$

while

$$\underline{v}(\underline{r},t) = \frac{\partial}{\partial t}\,\underline{\xi}\,(r_o,t) \ . \qquad (4.54)$$

Here, $\delta\underline{\hat{A}}$ is the first-order perturbation in $\underline{\hat{A}}$. Further, to obtain the required expression for $d\underline{S}(\underline{r},t)$, one integrates $\frac{d}{dt}\,(d\underline{S})$ with respect to time. The expression for $\frac{d}{dt}\,(d\underline{S})$ on a deforming surface is usually derived by tensorial methods (see, for example, ERINGEN, 1962), but an easily understood vectorial proof has been developed to obtain the result (see Appendix I , JAMES and SEYMOUR (1971)),

$$\frac{d}{dt}\,(d\underline{S}) = (\nabla\cdot\underline{v})d\underline{S} - (\nabla\underline{v})\cdot d\underline{S} \ , \qquad (4.55)$$

where $\underline{v}$ is the velocity of points on the surface. Integrated to first order in the displacement, equation (4.55) leads to

$$d\underline{S}(\underline{r},t) = d\underline{S}(\underline{r}_o,o) + (\nabla\cdot\underline{\xi})d\underline{S}(\underline{r}_o,o) - \nabla\underline{\xi}\cdot d\underline{S}(\underline{r}_o,o) \ , \qquad (4.56)$$

Therefore, introducing the dummy variable t', insertion of equations (4.53), (4.54) and (4.56) into (4.52) gives

$$\mu_o \delta W_{BE} = - \int_o^t dt' \int_{S_{pv}(o)} \tfrac{1}{2}\left[\hat{B}_o^2 + \underline{\xi}\cdot\nabla(\hat{B}_o^2) + 2\underline{\hat{B}}_o\cdot\nabla\times\delta\underline{\hat{A}}\right] \frac{\partial\underline{\xi}}{\partial t'}\cdot\left[d\underline{S}_o' + \nabla\cdot\underline{\xi}\,d\underline{S}_o' - \nabla\underline{\xi}\cdot d\underline{S}_o'\right] , \qquad (4.57)$$

where $\underline{\hat{B}}_o = \underline{\hat{B}}(\underline{r}_o,o)$ and $d\underline{S}_o' = d\underline{S}_{pv}'(\underline{r}_o,o)$ (directed into the plasma). Since all quantities in (4.57) are functions of $(\underline{r}_o,t')$, and $\underline{r}_o$ is independent of time, the integrations can be commuted, and so rearranging and retaining terms in the integrand to second order in $\underline{\xi}$,

$$\mu_o \delta W_{BE} = - \int_{S_{pv}(o)} d\underline{S}_o' \cdot \int_o^t dt' \left\{\tfrac{1}{2}\hat{B}_o^2 \left[\frac{\partial\underline{\xi}}{\partial t'} + (\nabla\cdot\underline{\xi})\frac{\partial\underline{\xi}}{\partial t'} - \frac{\partial\underline{\xi}}{\partial t'}\cdot\nabla\,\underline{\xi}\right] + \underline{\xi}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)\frac{\partial\underline{\xi}}{\partial t'} + \underline{\hat{B}}_o\cdot\nabla\times\delta\underline{\hat{A}}\,\frac{\partial\underline{\xi}}{\partial t'}\right\} . \qquad (4.58)$$

Integration of the first term is trivial. Integration of the remaining terms is non-trivial but can be achieved in the following way. First consider the second, third and fourth terms together:

Let

$$I = - \int_{S_{pv}(o)} d\underline{S}_o' \cdot \int_o^t dt' \left\{\tfrac{1}{2}\hat{B}_o^2\left[(\nabla\cdot\underline{\xi})\frac{\partial\underline{\xi}}{\partial t'} - \frac{\partial\underline{\xi}}{\partial t'}\cdot\nabla\underline{\xi}\right] + \frac{\partial\underline{\xi}}{\partial t'}\,\underline{\xi}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)\right\} \qquad (4.59)$$

$$= - \int_{S_{pv}(o)} d\underline{S}_o' \cdot \int_o^t dt' \left\{\tfrac{1}{2}\hat{B}_o^2\left[\nabla\times\left(\frac{\partial\underline{\xi}}{\partial t'}\times\underline{\xi}\right) + \underline{\xi}\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'} - \cdots\right.\right.$$

$$- \underline{\xi}\cdot\nabla \frac{\partial\underline{\xi}}{\partial t'}\Big] + \frac{\partial\underline{\xi}}{\partial t'}\,\underline{\xi}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)\} \,, \qquad (4.60)$$

by application of the expansion of $\nabla \times \left(\frac{\partial\underline{\xi}}{\partial t'} \times \underline{\xi}\right)$. Recalling that

$$\nabla(\tfrac{1}{2}\hat{B}_o^2) \times \left[\frac{\partial\underline{\xi}}{\partial t'} \times \underline{\xi}\right] = \frac{\partial\underline{\xi}}{\partial t'}\,\underline{\xi}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2) - \underline{\xi}\,\frac{\partial\underline{\xi}}{\partial t'}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2) \;,$$

and using the pressure balance equation (4.13), equation (4.60) may be rewritten as

$$I = -\int_{S_{pv}(o)} d\underline{S}_o' \cdot \int_o^t dt' \left\{\tfrac{1}{2}\hat{B}_o^2\left(\underline{\xi}\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'} - \underline{\xi}\cdot\nabla\frac{\partial\underline{\xi}}{\partial t'}\right\} + \underline{\xi}\,\frac{\partial\underline{\xi}}{\partial t'}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)\right\}$$

$$- \int_o^t dt' \int_{S_{pv}(o)} d\underline{S}_o' \cdot \left\{(\mu_o p_o + \tfrac{1}{2}B_o^2)\nabla \times \left(\frac{\partial\underline{\xi}}{\partial t'} \times \underline{\xi}\right) + \nabla(\tfrac{1}{2}\hat{B}_o^2) \times \left(\frac{\partial\underline{\xi}}{\partial t'} \times \underline{\xi}\right)\right\} . \qquad (4.61)$$

It is further noted that

$$d\underline{S}_o'\cdot\left[\nabla(\tfrac{1}{2}\hat{B}_o^2) \times \left(\frac{\partial\underline{\xi}}{\partial t'} \times \underline{\xi}\right)\right] = \left(d\underline{S}_o' \times \nabla(\tfrac{1}{2}\hat{B}_o^2)\right)\cdot\left(\frac{\partial\underline{\xi}}{\partial t'} \times \underline{\xi}\right) . \qquad (4.62)$$

To progress further, consider $\nabla(\mu_o p_o + \tfrac{1}{2}B_o^2)$. Then $\nabla \times \nabla(\mu_o p_o + \tfrac{1}{2}B_o^2) = 0$; hence application of the theorem of Stokes, and interpretation of the vanishing line integral so obtained in relation to a small rectangular circuit with its longer sides situated on each side of a portion of the plasma/vacuum interface leads, with the aid of equation (4.13), to the result (see, for example, ROSE and CLARK, 1961)

$$<dS_o' \times \nabla(\mu_o p_o + \tfrac{1}{2}B_o^2) > \doteq 0,$$

or

$$dS_o' \times \nabla(\mu_o p_o + \tfrac{1}{2}B_o^2) = dS_o' \times \nabla(\tfrac{1}{2}\hat{B}_o^2) \ . \tag{4.63}$$

Equation (4.62) therefore becomes

$$dS_o' \circ \left[\nabla(\tfrac{1}{2}\hat{B}_o^2) \times \left(\frac{\partial \underline{\xi}}{\partial t'} \times \underline{\xi}\right)\right] = \left(dS_o' \times \nabla(\mu_o p_o + \tfrac{1}{2}B_o^2)\right) \circ \left(\frac{\partial \underline{\xi}}{\partial t'} \times \underline{\xi}\right)$$

$$= dS_o' \circ \left[\nabla(\mu_o p_o + \tfrac{1}{2}B_o^2) \times \left(\frac{\partial \underline{\xi}}{\partial t'} \times \underline{\xi}\right)\right] \ . \tag{4.64}$$

By making use of the expansion of $\nabla \times \left[(\mu_o p_o + \tfrac{1}{2}B_o^2)\left(\frac{\partial \underline{\xi}}{\partial t'} \times \underline{\xi}\right)\right]$, the second surface integral in equation (4.61) may now be written as

$$J = \int_{S_{pv}(o)} dS_o' \circ \nabla \times \left[(\mu_o p_o + \tfrac{1}{2}B_o^2)\left(\frac{\partial \underline{\xi}}{\partial t'} \times \underline{\xi}\right)\right] \ . \tag{4.65}$$

If $S_{pv}(o)$ is a closed surface (i.e. if the plasma is in contact with vacuum only), Gauss' theorem may immediately be applied to the integral J of (4.65) to show that it vanishes, since $\nabla \circ (\nabla \times \underline{N}) \equiv 0$ for all vectors $\underline{N}$. If, however, the plasma is in contact with electrodes, $S_{pv}(o)$ will not be a closed surface. In this case, suppose C(t) is the curve representing the intersection of $S_{pv}(t)$ and $S_{pc}(t)$. Then (4.65) may be transformed by the use of Stokes' theorem to obtain

$$J = \int_{C(o)} d\,\underline{l} \circ \left(\frac{\partial \underline{\xi}}{\partial t'} \times \underline{\xi}\right)(\mu_o p_o + \tfrac{1}{2}B_o^2) \ , \tag{4.66}$$

where $d\,\underline{l}$ is an element of path around C(o), its direction being specified by means of the right-hand screw rule used in relation to the direction of $\underline{dS}_{pv}$.

Since it has been shown that if, on $S_{pc}(o)$, the condition $\underline{B}_o \cdot \underline{dS}_c \neq 0$ applies, the 'freezing-in' effect of infinite conductivity leads to $\frac{\partial \underline{\xi}}{\partial t} = \underline{\xi} = 0$ there, it follows that the integral (4.66) correspondingly vanishes. If on the other hand $\underline{B}_o \cdot \underline{dS}_c = 0$, the perturbation $\underline{\xi}$ is constrained in such a manner that the displacement of fluid elements on S_{pc} must always be parallel to S_{pc}. That is,

$$\underline{v}(\underline{r},t) \cdot \underline{dS}_c(\underline{r}) = 0 \ .$$

But

$$\underline{v}(\underline{r},t) = \frac{\partial}{\partial t}\,\underline{\xi}(\underline{r}_o,t)$$

and, to first order,

$$\underline{dS}_c(\underline{r}) = \underline{dS}_c(\underline{r}_o) + \underline{\xi}(\underline{r}_o,t) \cdot \nabla \underline{dS}_c(\underline{r}_o) \ .$$

Hence in first order,

$$\frac{\partial \underline{\xi}}{\partial t} \cdot \underline{dS}_c(\underline{r}_o) = 0 \ ,$$

and then, integrating to first order,

$$\underline{\xi} \cdot \underline{dS}_c = 0 \ , \quad \text{since } \underline{\xi} = 0 \text{ at } t = 0 \ .$$

Combination of these results thus yields

$$\underline{dS}_c \times \left(\frac{\partial \underline{\xi}}{\partial t} \times \underline{\xi}\right) = 0 \ ,$$

and since $d\,\underline{l} \cdot \underline{dS}_c = 0$ (as C lies in S_{pc}), it follows that

$$d\underline{l} \cdot \left(\frac{\partial \underline{\xi}}{\partial t} \times \underline{\xi}\right) = 0 \ ,$$

leading to the result that the integral J in (4.66) above vanishes when $S_{pv}(o)$ is not a closed surface. At this stage the integral I of equation (4.61) reduces to

$$I = -\int_{S_{pv}(o)} d\underline{S}_o' \circ \int_0^t dt' \left\{ \tfrac{1}{2}\hat{B}_o^2 \left(\underline{\xi} \nabla \circ \frac{\partial \underline{\xi}}{\partial t'} - \underline{\xi} \circ \nabla \frac{\partial \underline{\xi}}{\partial t'} \right) + \underline{\xi} \frac{\partial \underline{\xi}}{\partial t'} \circ \nabla (\tfrac{1}{2}\hat{B}_o^2) \right\} ,$$

which can be integrated by parts to give

$$I = -\int_{S_{pv}(o)} d\underline{S}_o' \circ \left\{ \tfrac{1}{2}\hat{B}_o^2 \left(\underline{\xi} \nabla \circ \underline{\xi} - \underline{\xi} \circ \nabla \underline{\xi} \right) + \underline{\xi}\, \underline{\xi} \circ \nabla (\tfrac{1}{2}\hat{B}_o^2) \right\}$$

$$+ \int_{S_{pv}(o)} d\underline{S}_o' \circ \int_0^t dt' \left\{ \tfrac{1}{2}\hat{B}_o^2 \left(\frac{\partial \underline{\xi}}{\partial t'} \nabla \circ \underline{\xi} - \frac{\partial \underline{\xi}}{\partial t'} \circ \nabla \underline{\xi} \right) + \frac{\partial \underline{\xi}}{\partial t'} \underline{\xi} \circ \nabla (\tfrac{1}{2}\hat{B}_o^2) \right\} ,$$

a result which yields, by use of the definition (4.59),

$$I = -\tfrac{1}{2} \int_{S_{pv}(o)} d\underline{S}_o' \circ \left\{ \tfrac{1}{2}\hat{B}_o^2 \left(\underline{\xi} \nabla \circ \underline{\xi} - \underline{\xi} \circ \nabla \underline{\xi} \right) + \underline{\xi}\, \underline{\xi} \circ \nabla (\tfrac{1}{2}\hat{B}_o^2) \right\} . \qquad (4.67)$$

The final term in equation (4.58) is

$$K = -\int_{S_{pv}(o)} \int_0^t dt' \left(d\underline{S}_o' \circ \frac{\partial \underline{\xi}}{\partial t'} \right) (\hat{\underline{B}}_o \circ \nabla \times \delta \hat{\underline{A}}) . \qquad (4.68)$$

To simplify K one notes that use of equations (4.53), (4.54) and (4.56) enables equation (4.49) to be expressed to first order as

$$d\underline{S}_o' \times \frac{\partial \delta \hat{\underline{A}}}{\partial t'} = -\left(d\underline{S}_o' \circ \frac{\partial \underline{\xi}}{\partial t'} \right) \hat{\underline{B}}_o , \qquad (4.69)$$

and hence (4.68) becomes

$$K = \int_{S_{pv}(o)} \int_o^t dt' \left(d\underline{S}_o' \times \frac{\partial \delta \hat{\underline{A}}}{\partial t'}\right) \cdot \nabla \times \delta \hat{\underline{A}} \; . \tag{4.70}$$

Since $d\underline{S}_o' \times \frac{\partial \underline{A}}{\partial t}$ is zero on the conducting surfaces S_{cv} and S_{ci}, and the field quantities are assumed vanishingly small at infinity, the integral K of (4.70) may be taken over $S(o) = S_{pv}(o) + S_{cv} + S_{ci} + S_\infty$, the surface bounding the combined volumes of the vacuum and insulator regions. Commuting the surface integration and the integration with respect to time, and interchanging the dot and cross in (4.70),

$$K = \int_o^t dt' \int_{S(o)} d\underline{S}_o' \cdot \left(\frac{\partial \delta \underline{A}}{\partial t'} \times \nabla \times \delta \underline{A}\right) ,$$

which, upon permissible application of Gauss' theorem, further transforms to

$$K = \int_o^t dt' \int_{\hat{\tau}(o)+\tau_i} d\tau \nabla \cdot \left(\frac{\partial \delta \underline{A}}{\partial t'} \times \nabla \times \delta \underline{A}\right) . \tag{4.71}$$

Expansion of the integrand, and application of the vanishing electric current condition $\nabla \times \nabla \times \delta \underline{A} = 0$ enables (4.71) to be written as

$$K = \tfrac{1}{2} \int_o^t dt' \int_{\hat{\tau}(o)+\tau_i} d\tau \frac{\partial}{\partial t} (\nabla \times \delta \underline{A})^2$$

$$= \tfrac{1}{2} \int_{\hat{\tau}(o)+\tau_i} d\tau (\nabla \times \delta \underline{A})^2 \; . \tag{4.72}$$

Thus, from equations (4.58), (4.67) and (4.72),

$$\mu_o \delta W_{BE} = -\int_{S_{pv}(o)} d\underline{S}_o' \cdot \underline{\xi}(\tfrac{1}{2}\hat{B}_o^2) - \tfrac{1}{2}\int_{S_{pv}(o)} d\underline{S}_o' \cdot \{\tfrac{1}{2}\hat{B}_o^2(\underline{\xi}\nabla\cdot\underline{\xi} - \underline{\xi}\cdot\nabla\underline{\xi})$$

$$+ \underline{\xi}\ \underline{\xi}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)\} + \tfrac{1}{2}\int_{\hat{\tau}(o)+\tau_i} d\tau(\nabla \times \delta\underline{A})^2 \qquad (4.73)$$

4.5 DETERMINATION OF δW

To complete the expression for the second-order variation in the potential energy of the system, the second-order terms in equation (4.73) must now be added to equation (4.38), the result obtained by VAN KAMPEN and FELDERHOF. Noting that in equation (4.38) $S_p(o) = S_{pv}(o) + S_{pc}(o)$, and recalling that

(i) $\underline{B}_o \cdot d\underline{S}_o = 0$ on $S_{pv}(o)$,

(ii) $\underline{\xi} = 0$ on $S_{pc}(o)$ if $\underline{B}_o \cdot d\underline{S}_c \neq 0$ there,

(iii) $\mu_o p_o + \tfrac{1}{2}B_o^2 = \tfrac{1}{2}\hat{B}_o^2$ from equation (4.13),

equation (4.38) can be written in the form

$$\delta\bar{W} = \tfrac{1}{2}\int_{\tau_p(o)} d\tau_o \left\{\frac{|\underline{Q}|^2}{\mu_o} - \underline{j}_o\cdot(\underline{Q} \times \underline{\xi}) + \gamma p_o(\nabla\cdot\underline{\xi})^2 + (\nabla\cdot\underline{\xi})\underline{\xi}\cdot\nabla p_o\right\}$$

$$- \frac{1}{2\mu_o}\int_{S_{pv}(o)} d\underline{S}_o \cdot \{\tfrac{1}{2}\hat{B}_o^2(\underline{\xi}\nabla\cdot\underline{\xi} - \underline{\xi}\cdot\nabla\underline{\xi}) + (\underline{\xi}\cdot\underline{B}_o\cdot\nabla\underline{B}_o)\underline{\xi}\} . \qquad (4.74)$$

Remembering that $d\underline{S}_o = - d\underline{S}_o'$, summation of the terms in $\delta\bar{W}$ and the second-order terms in δW_{BE} gives the final expression for the variation of the system potential energy,

$$\delta W = \tfrac{1}{2} \int_{\tau_p(o)} d\tau_o \left\{ \frac{|\underline{Q}|^2}{\mu_o} - \underline{j}_o \cdot (\underline{Q} \times \underline{\xi}) + \gamma p_o (\nabla \cdot \underline{\xi})^2 + (\nabla \cdot \underline{\xi}) \underline{\xi} \cdot \nabla p_o \right\}$$

$$+ \frac{1}{2\mu_o} \int_{S_{pv}(o)} d\underline{S}_o \cdot \underline{\xi} \left\{ \underline{\xi} \cdot \nabla (\tfrac{1}{2} \hat{B}_o^2) - \underline{\xi} \cdot \underline{B}_o \cdot \nabla \underline{B}_o \right\} + \frac{1}{2\mu_o} \int_{\hat{\tau}(o)+\tau_i} d\tau_o (\nabla \times \delta \underline{A})^2 \ . \qquad (4.75)$$

The equilibrium quantities satisfy the equation of mechanical equilibrium,

$$\nabla p_o = \underline{j}_o \times \underline{B}_o \ ,$$

which, by a standard transformation, may be written as

$$\frac{1}{\mu_o} \underline{B}_o \cdot \nabla \underline{B}_o = \nabla (p_o + \frac{B_o^2}{2\mu_o}) \ . \qquad (4.76)$$

With the help of this result equation (4.75) may now conveniently be written in the form

$$\delta W = \delta W_F + \delta W_S + \delta W_E \ , \qquad (4.77)$$

where

(a) δW_F is the fluid contribution given by

$$\delta W_F = \tfrac{1}{2} \int_{\tau_p(o)} d\tau_o \left\{ \frac{|\underline{Q}|^2}{\mu_o} - \underline{j}_o \cdot (\underline{Q} \times \underline{\xi}) + \gamma p_o (\nabla \cdot \underline{\xi})^2 + (\nabla \cdot \underline{\xi}) \underline{\xi} \cdot \nabla p_o \right\} \ , \qquad (4.78)$$

(b) δW_S is the surface contribution given by

$$\delta W_S = \tfrac{1}{2} \int_{S_{pv}(o)} d\underline{S}_o \cdot \underline{\xi} \left\{ \underline{\xi} \cdot \nabla \left(\frac{\hat{B}_o^2}{2\mu_o} \right) - \underline{\xi} \cdot \nabla \left(p_o + \frac{B_o^2}{2\mu_o} \right) \right\} \ . \qquad (4.79)$$

By expressing the perturbation as

$$\underline{\xi} = \underline{n}_o\, \underline{n}_o \circ \underline{\xi} + (\underline{n}_o \times \underline{\xi}) \times \underline{n}_o \ ,$$

and inserting this form inside the curly bracket in (4.79)

$$\delta W_S = \tfrac{1}{2} \int_{S_{pv}(o)} d\underline{S}_o \circ \underline{\xi} \ \left\{ (\underline{n}_o \circ \underline{\xi}) \underline{n}_o \circ \nabla \left(\frac{\hat{B}_o^2}{2\mu_o} \right) - (\underline{n}_o \circ \underline{\xi}) \underline{n}_o \circ \nabla \left(p_o + \frac{B_o^2}{2\mu_o} \right) \right\} \ ,$$

since $(\underline{n}_o \times \underline{\xi}) \circ \underline{n}_o = 0$. Further, by writing $d\underline{S}_o = \underline{n}_o \, d\sigma_o$ (directed out of the plasma),

$$\delta W_S = \tfrac{1}{2} \int_{S_{pv}(o)} d\sigma_o (\underline{n}_o \circ \underline{\xi})^2 < \underline{n}_o \circ \nabla \left(p_o + \frac{B_o^2}{2\mu_o} \right) > . \qquad (4.80)$$

(c) δW_E is the contribution from regions external to the plasma given by

$$\delta W_E = \frac{1}{2\mu_o} \int_{\hat{\tau}(o)+\tau_i} d\tau_o (\nabla \times \delta \underline{A})^2 . \qquad (4.81)$$

Using the argument which appears prior to the result (4.52), the fact that here

$$\nabla \circ (\delta \hat{\underline{A}} \times \nabla \times \delta \hat{\underline{A}}) = (\nabla \times \delta \hat{\underline{A}})^2 \ ,$$

Gauss' theorem, and the boundary condition on $S_{pv}(o)$

$$d\underline{S}_o' \times \delta \hat{\underline{A}} = - (d\underline{S}_o' \circ \underline{\xi}) \hat{\underline{B}}_o \ , \qquad (4.82)$$

obtained by integration of equation (4.69), the result (4.81) can also be expressed in the interesting form

$$\delta W_E = - \frac{1}{2\mu_o} \int_{S_{pv}(o)} (d\underline{S}_o' \circ \underline{\xi}) \hat{\underline{B}}_o \circ \nabla \times \delta \hat{\underline{A}} \ . \qquad (4.83)$$

Thus the approach used by VAN KAMPEN and FELDERHOF has been extended to a system comprising a finite plasma body, with external rigid conductors and insulators and external magnetic field, to obtain the second order variation in potential energy originally obtained by BERNSTEIN et al. by a different approach. The present expression differs from that of Bernstein et al. in that the volume of integration for the expression (4.81) is not limited to the interior of a conducting shell, but extends to infinity and includes both vacuum and insulator regions.

4.6 DISCUSSION OF THE SURFACE CONTRIBUTION, δW_S

It is tempting to identify the surface term δW_S in equation (4.77) with the second-order part of the work performed against the surface current in displacing the boundary by $\underline{\xi}$, as has been done, for example, by SCHMIDT (1966). However, on making a closer examination of this term, it becomes doubtful if it is correct to make this identification. For example, in the special case of zero internal field, the "work done against the surface current" (Schmidt's phrase) is just the work done against the pressure of the vacuum magnetic field, given above to second-order by equation (4.73). Comparing the second-order part of (4.73) with expression (4.80) for δW_S, one notes the following striking difference: expression (4.80) is zero under the condition

$$\underline{n}_o \cdot \underline{\xi} = 0 \ , \qquad (4.84)$$

whereas under this condition expression (4.73), with the help of the result (4.83) reduces to

$$\delta W_{BE} = \frac{1}{2\mu_o} \int_{S_{pv}(o)} d\underline{S}_o' \cdot \tfrac{1}{2}\hat{B}_o^2 \ \underline{\xi} \cdot \nabla \underline{\xi} \ , \qquad (4.85)$$

which is not necessarily zero. This difference reflects the fact that condition (4.84) does not imply that the surface is undeformed to second order in $\underline{\xi}$. In fact, there remains a second-order deformation which requires a second order amount of work, given in (4.85). The condition which does ensure no deformation of the plasma surface is

$$\underline{v} \cdot d\underline{S}' = 0 \ . \qquad (4.86)$$

To discuss the effect of the condition (4.86) on δW_{BE}, it is helpful to transform (4.73) as follows: the change in plasma volume $\delta\tau$ resulting from perturbation in this case can be expressed as

$$\delta\tau = - \int_o^t dt' \int_{S_{pv}(o)} d\underline{S}' \cdot \underline{v} \ , \qquad (4.87)$$

and hence, by use of equations (4.54) and (4.56),

$$\delta\tau = - \int_o^t dt' \int_{S_{pv}(o)} \left(d\underline{S}_o' + (\nabla \cdot \underline{\xi}) d\underline{S}_o' - (\nabla \underline{\xi}) \cdot d\underline{S}_o' \right) \cdot \frac{\partial \underline{\xi}}{\partial t'}$$

$$= - \int_{S_{pv}(o)} d\underline{S}_o' \cdot \underline{\xi} - \int_{S_{pv}(o)} d\underline{S}_o' \cdot \int_o^t dt' \left((\nabla \cdot \underline{\xi}) \frac{\partial \underline{\xi}}{\partial t'} - \frac{\partial \underline{\xi}}{\partial t'} \cdot \nabla \underline{\xi} \right) \ . \qquad (4.88)$$

Using a standard vector identity, formulate

$$\int_{S_{pv}(o)} d\underset{\sim}{S}_o' \cdot \int_o^t dt' \left\{ \left(\frac{\partial \underline{\xi}}{\partial t'} \nabla \cdot \underline{\xi} - \frac{\partial \underline{\xi}}{\partial t'} \cdot \nabla \underline{\xi} \right) - \left(\underline{\xi} \nabla \cdot \frac{\partial \underline{\xi}}{\partial t'} - \underline{\xi} \cdot \nabla \frac{\partial \underline{\xi}}{\partial t'} \right) \right\}$$

$$= \int_o^t dt' \int_{S_{pv}(o)} d\underset{\sim}{S}_o' \cdot \nabla \times \left(\frac{\partial \underline{\xi}}{\partial t'} \times \underline{\xi} \right) = 0 \ , \qquad (4.89)$$

by Gauss' theorem.

Then, since

$$\frac{\partial}{\partial t} \left(\underline{\xi} \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla \underline{\xi} \right) = \left(\frac{\partial \underline{\xi}}{\partial t} \nabla \cdot \underline{\xi} - \frac{\partial \underline{\xi}}{\partial t} \cdot \nabla \underline{\xi} \right) + \left(\underline{\xi} \nabla \cdot \frac{\partial \underline{\xi}}{\partial t} - \underline{\xi} \cdot \nabla \frac{\partial \underline{\xi}}{\partial t} \right) ,$$

integration and use of (4.89) gives

$$\tfrac{1}{2} \int_{S_{pv}(o)} d\underset{\sim}{S}_o' \cdot \left(\underline{\xi} \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla \underline{\xi} \right) = \int_{S_{pv}(o)} d\underset{\sim}{S}_o' \cdot \int_o^t dt' \left((\nabla \cdot \underline{\xi}) \frac{\partial \underline{\xi}}{\partial t'} - \frac{\partial \underline{\xi}}{\partial t'} \cdot \nabla \underline{\xi} \right) , \qquad (4.90)$$

and so equation (4.88) becomes

$$\delta\tau = - \int_{S_{pv}(o)} d\underset{\sim}{S}_o' \cdot \underline{\xi} - \tfrac{1}{2} \int_{S_{pv}(o)} d\underset{\sim}{S}_o' \cdot \left(\underline{\xi} \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla \underline{\xi} \right) . \qquad (4.91)$$

Returning to equation (4.73), for the special case of zero internal magnetic field it has been seen from the argument following equations (4.27) and (4.28) that $\hat{B}_o^2$ is constant at all points on the plasma/vacuum interface. Thus, using this fact and the result (4.91), δW_{BE} may be written

$$\delta W_{BE} = \frac{\hat{B}_o^2}{2\mu_o}\,\delta\tau - \frac{1}{2\mu_o}\int_{S_{pv}(o)} (d\underline{S}_o'\cdot\underline{\xi})\underline{\xi}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)$$

$$- \frac{1}{2\mu_o}\int_{S_{pv}(o)} (d\underline{S}_o'\cdot\underline{\xi})\underline{\hat{B}}_o\cdot\nabla\times\delta\underline{\hat{A}}\ , \qquad (4.92)$$

where the volume integral appearing in (4.73) has been replaced by means of equation (4.83). The result (4.92) of course still assumes the form (4.85) when $\underline{n}_o\cdot\underline{\xi} = 0$, as is readily seen with the help of equation (4.91). On the other hand the stronger condition (4.86) (which implies the first-order result $\underline{n}_o\cdot\underline{\xi} = 0$ too) ensures the vanishing of δW_{BE} given by equation (4.92).

Obviously, if the identification of δW_S referred to above is to be correct in general, it has to be correct for the special case of zero internal magnetic field. The above discussion therefore reveals that the interpretation of the term δW_S as the work done against the surface current by displacing the boundary by $\underline{\xi}$ is not correct. Indeed, the manner in which we extend Van Kampen and Felderhof's expression (4.38) here by determining and including the second-order variation of the magnetic energy external to the plasma shows clearly that δW_S given by equation (4.80) is a composite term, made up from the second-order surface term appearing in δW_{BE} of equation (4.73), and the second-order surface term present in equation (4.74), a suitably modified form of equation (4.38).

4.7 REFERENCES

BERNSTEIN, I. B., FRIEMAN, E. A., KRUSKAL, M. D., and KULSRUD, R. M. (1958) - Proc. Roy. Soc. A224, 17.

ERINGEN, A. C. (1962) - "Nonlinear Theory of Continuous Media". p.71. (McGraw-Hill: New York).

JAMES, M. K., and SEYMOUR, P. W. (1971) - To be published.

JEFFREY, A. (1966) - "Magnetohydrodynamics", pp. 67-71. (Oliver and Boyd: London).

KRUSKAL, M., and SCHWARZSCHILD, M. (1954) - Proc. Roy. Soc. A223, 348.

ROSE, D. J., and CLARK, M., Jr. (1961) - "Plasmas and Controlled Fusion". pp.282-83. (Wiley: New York).

SCHMIDT, G. (1966) - "Physics of High Temperature Plasmas", p.125. (Academic Press: New York).

KAMPEN, VAN, N. G., and FELDERHOF, B. U. (1967) - "Theoretical Methods in Plasma Physics". pp. 66-76. (North-Holland Publishing Co: Amsterdam).

CHAPTER 5

THE EXTENDED ENERGY PRINCIPLE

5.1 INTRODUCTION

In deriving the complete expression (4.75) for δW, the condition (4.13) of continuity of stress over the plasma boundary has been assumed to apply. As shown by BERNSTEIN et al (1958, equation (2.32)), continuity of stress over the boundary leads to a constraint relationship on $\underline{\xi}$ as follows:

$$- \gamma p_o \nabla \cdot \underline{\xi} + \frac{\underline{B}_o}{\mu_o} \cdot (\underline{Q} + \underline{\xi} \cdot \nabla \underline{B}_o) = \frac{\hat{\underline{B}}_o}{\mu_o} \cdot (\nabla \times \delta \hat{\underline{A}} + \underline{\xi} \cdot \nabla \hat{\underline{B}}_o) \ . \qquad (5.1)$$

This constraint relationship restricts the freedom of choice of $\underline{\xi}$. Hence, recalling that the sign of $\delta W_{min.}$ determines the stability of the plasma system, mathematical difficulties arise when minimization of δW is attempted with respect to all possible perturbations. However the energy principle may, in a sense, be extended so that this important constraint can be ignored, provided $\delta W(\underline{\xi},\underline{\xi})$ is written in the appropriate form.

5.2 A CONSEQUENCE OF THE CONTINUITY OF STRESS ACROSS S_{pv}

To obtain an idea of the consequences of (5.1) on plasma stability, the special case of a system having zero internal field is, for tractability, now considered. For this case,

$$\nabla p_o = 0 \ ,$$

so that p_o = constant,

$$\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B}_o)$$

$$= 0 ,$$

and then equations (4.78), (4.79) and (4.80) become respectively

$$\delta W_F = \frac{\gamma p_o}{2} \int_{\tau_p(o)} d\tau_o \; (\nabla\cdot\underline{\xi})^2 , \qquad (5.2)$$

$$\delta W_S = \frac{1}{2\mu_o} \int_{S_{pv}(o)} (d\underline{S}_o\cdot\underline{\xi}) \; \{\underline{\xi}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)\} , \qquad (5.3)$$

and

$$\delta W_S = \frac{1}{2\mu_o} \int_{S_{pv}(o)} d\sigma_o \; (\underline{n}_o\cdot\underline{\xi})^2 \; \underline{n}_o\cdot\nabla \; (\tfrac{1}{2}\hat{B}_o^2) , \qquad (5.4)$$

equations (4.81) and (4.83) for δW_E remain unchanged, while equation (5.1) reduces to

$$- \mu_o\gamma p_o \; \nabla\cdot\underline{\xi} = \hat{\underline{B}}_o\cdot\nabla \times \delta\hat{\underline{A}} + \underline{\xi}\cdot\nabla \; (\tfrac{1}{2}\hat{B}_o^2) , \qquad (5.5)$$

since $\hat{\underline{B}}_o\cdot\underline{\xi}\cdot\nabla\hat{\underline{B}}_o \equiv \underline{\xi}\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)$.

Combining equations (4.83) and (5.3), and using equation (5.5),

$$\delta W_S + \delta W_E = - \frac{\gamma p_o}{2} \int_{S_{pv}(o)} (d\underline{S}_o\cdot\underline{\xi})\nabla\cdot\underline{\xi}$$

$$= - \frac{\gamma p_o}{2} \int_{\tau_p(o)} d\tau_o \; \{\underline{\xi}\cdot\nabla(\nabla\cdot\underline{\xi}) + (\nabla\cdot\underline{\xi})^2\} , \qquad (5.6)$$

by use of the Gauss integral transformation.

Therefore, in this case, combination of equations (5.2) and (5.6) leads to

$$\delta W = \delta W_F + \delta W_S + \delta W_E = -\frac{\gamma p_o}{2} \int_{\tau_p(o)} d\tau_o \ \underline{\xi}\cdot\nabla(\nabla\cdot\underline{\xi}) \ . \qquad (5.7)$$

From the result (5.7) it is seen that a class of perturbations $\underline{\xi}$ exist, satisfying the condition

$$\underline{\xi}\cdot\nabla(\nabla\cdot\underline{\xi}) \equiv 0 \ , \qquad (5.8)$$

which leads to the vanishing of δW. This class of $\underline{\xi}$ clearly includes the incompressible perturbations, $\nabla\cdot\underline{\xi} = 0$. Hence, for this special plasma configuration it is evidently not possible for $\delta W > 0$ for all possible $\underline{\xi}$. In other words, such a system cannot be completely stable, but at best only neutrally stable. To examine this particular situation more specifically it is convenient to obtain the following alternative form of δW from equations (4.81), (5.2) and (5.4),

$$\delta W = \frac{\gamma p_o}{2} \int_{\tau_p(o)} d\tau_o \ (\nabla\cdot\underline{\xi})^2 + \frac{1}{2\mu_o} \int_{S_{pv}(o)} d\sigma_o \ (\underline{n}_o\cdot\underline{\xi})^2 \ \underline{n}_o\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)$$

$$+ \frac{1}{2\mu_o} \int_{\hat{\tau}(o)+\tau_i} d\tau_o \ (\nabla \times \delta\underline{A})^2 \ . \qquad (5.9)$$

The first and third terms (δW_F and δW_E) are always positive. If $\underline{n}_o \cdot \nabla(\frac{1}{2}\hat{B}_o^2)$ is positive at all points on $S_{pv}(o)$, the second term (δW_S) is always positive. Then δW will be zero only if each term is separately zero. This would require

$$\nabla \cdot \underline{\xi} \equiv \nabla \times \delta \underline{A} \equiv \underline{n}_o \cdot \underline{\xi} \equiv 0 \ . \qquad (5.10)$$

That is, the system would be stable for all perturbations except the one for which $\underline{n}_o \cdot \underline{\xi} \equiv 0$, for which it would be neutrally stable, in agreement with the argument stemming from the condition (5.8). Thus, while the plasma in this case is not completely stable, but only neutrally stable, the least favourable perturbation satisfying the condition $\underline{n}_o \cdot \underline{\xi} \equiv 0$ does not, to first order, physically disturb the plasma surface, and so does not have dire consequences from a practical viewpoint.

On the other hand, if $\underline{n}_o \cdot \nabla(\frac{1}{2}\hat{B}_o^2)$ is negative in some region R of $S_{pv}(o)$, it will be possible to find a perturbation for which the plasma is unstable. For example, consider a perturbation for which $\nabla \cdot \underline{\xi}$ is zero except in a very thin layer at the surface, and for which $\underline{n}_o \cdot \underline{\xi}$ is zero except in the region R, where it produces a fluting of the surface along the magnetic field lines. For such a perturbation, the destabilizing δW_S term could be made very large compared with the stabilizing terms δW_F and δW_E, and so the system would be unstable. Under these conditions the fact that there might exist a non-trivial perturbation which makes δW zero is irrelevant.

5.3 MINIMIZATION OF δW; EXTENSION OF THE ENERGY PRINCIPLE

The programme now is to minimize δW with respect to $\underline{\xi}$, subject to the various boundary conditions. The mathematical difficulties involved in taking account of equation (4.13), (which in first order yields equation (5.1)), and the equation

$$\nabla \times \nabla \times \delta\underline{\hat{A}} = 0 \ , \tag{5.11}$$

may be avoided by simply ignoring these conditions. The justification for taking this apparently drastic step is discussed from a somewhat physical viewpoint by ROSE and CLARK (1961, page 284). Using $d\underset{\sim}{S}_o' = - d\underset{\sim}{S}_o$ and equation (4.82), the remaining boundary conditions to be satisfied by the perturbation are

$$d\underset{\sim}{S}_o \times \delta\underline{\hat{A}} = - (d\underset{\sim}{S}_o \cdot \underline{\xi})\underset{\sim}{\hat{B}}_o \tag{5.12}$$

on $S_{pv}(o)$ and

$$d\underset{\sim}{S}_o \times \delta\underline{A} = 0 \tag{5.13}$$

on S_c. The set of vectors $\underline{\tilde{\xi}}$ which satisfy equations (5.1), (5.11), (5.12) and (5.13) is clearly a sub-set of the set of vectors $\underline{\xi}$ which satisfy (5.12) and (5.13), but not necessarily (5.1) and (5.11). Therefore the set of $\delta W(\underline{\tilde{\xi}},\underline{\tilde{\xi}})$ is contained in the set of $\delta W(\underline{\xi},\underline{\xi})$. Hence if $\delta W_{min}(\underline{\xi},\underline{\xi})$ and $\delta W_{min}(\underline{\tilde{\xi}},\underline{\tilde{\xi}})$ are the potential energy variations obtained by minimizing δW with respect to $\underline{\xi}$ and $\underline{\tilde{\xi}}$ respectively, then it may be concluded that

$$\delta W_{min}(\underline{\xi},\underline{\xi}) \leqslant \delta W_{min}(\underline{\tilde{\xi}},\underline{\tilde{\xi}}) \ . \tag{5.14}$$

Hence a sufficient condition for stability with respect to the real perturbation $\tilde{\underline{\xi}}$ is that $\delta W_{min}(\underline{\xi},\underline{\xi})$ be positive. While the argument leading to this conclusion is straightforward, it is not, however, so obvious that examination of the sign of $\delta W_{min}(\underline{\xi},\underline{\xi})$ (with $\underline{\xi}$ not constrained by (5.1)) actually yields a necessary and sufficient condition for stability. This 'extended energy principle' was also proposed by BERNSTEIN et al. (1958). A detailed mathematical proof of the extended energy principle, which does not appear to have been presented elsewhere, is developed as follows.

Consider the perturbation velocity

$$\underline{v}(\underline{r},t) = \frac{\partial}{\partial t}\,\xi(\underline{r}_o,t) + \varepsilon\,\frac{\partial}{\partial t}\,\underline{\eta}(\underline{r}_o,t) \;, \tag{5.15}$$

where $\underline{r} = \underline{r}_o + \underline{\xi}(\underline{r}_o,t) + \varepsilon\,\underline{\eta}(\underline{r}_o,t)$, ε is a parameter of smallness, and $\underline{\eta}$ is a vector of zero order in ε on the surface of the plasma, falling rapidly to zero in the distance ε from the surface. Also,

$$\frac{\partial}{\partial t}\,\underline{\eta}(\underline{r}_o,t) \times d\underline{S}(\underline{r},t) = 0 \;.$$

Thus $\frac{\partial}{\partial t}\,\underline{\eta}$ is non-zero only in a volume of order ε, and represents a motion of matter perpendicular to the perturbed fluid surface. $\underline{\eta}$ varies only slowly in any direction parallel to the surface, in such a way that the perturbed pressure and magnetic field satisfy equation (4.13). $\underline{\xi}(\underline{r}_o,t)$ is of zero order in ε, and varies only slowly in all directions.

The first-order form of equation (4.13) will now change from equation (5.1), additional terms appearing due to $\varepsilon\underline{\eta}$. It is shown below that such terms must arise because $\varepsilon\underline{\eta}$ in fact produces changes in the pressure p and magnetic field $\underline{B}$ of zero order in ε.

Consider first the standard fluid mechanics result

$$-\frac{1}{\gamma p}\frac{dp}{dt} = \nabla\cdot\underline{v} = \nabla\cdot\frac{\partial\underline{\xi}}{\partial t} + \varepsilon\nabla\cdot\frac{\partial\underline{\eta}}{\partial t}\ , \tag{5.16}$$

using equation (5.15). In view of the assumed properties of $\partial\underline{\xi}/\partial t$ the term $\nabla\cdot\frac{\partial\underline{\xi}}{\partial t}$ in (5.16) is of zero order in ε. Further, considering the second term in (5.16), to lowest order in ε

$$\varepsilon\nabla\cdot\frac{\partial\underline{\eta}}{\partial t} \sim \left|\frac{\partial\underline{\eta}}{\partial t}\right|\ , \tag{5.17}$$

a zero-order result in ε which is readily obtained by expressing nabla in the form

$$\nabla \equiv \underline{n}(\underline{n}\cdot\nabla) - \underline{n}\times(\underline{n}\times\nabla)\ , \tag{5.18}$$

substituting (5.18) into (5.17), and, bearing in mind the properties of $\partial\underline{\eta}/\partial t$ assumed above, permissibly neglecting in the resulting expression the term perpendicular to $\underline{n}$.

The result (5.17) shows that $\varepsilon\partial\underline{\eta}/\partial t$ of equation (5.15) gives a contribution to $\nabla\cdot\underline{v}$, and thus, from (5.16), to dp/dt, which is of zero order in ε: therefore changes in p due to $\varepsilon\underline{\eta}$ are of zero order.

For the magnetic field, equations (4.8) and (4.10) give the familiar infinite electrical conductivity result

$$\frac{\partial\underline{B}}{\partial t} = \nabla\times(\underline{v}\times\underline{B})$$

which, with $\nabla \cdot \underline{B} = 0$, enables the convective derivative of $\underline{B}$ to be expressed as

$$\frac{d\underline{B}}{dt} = \frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \nabla \underline{B} = \underline{B} \cdot \nabla \underline{v} - \underline{B} \nabla \cdot \underline{v} \ . \tag{5.19}$$

From the foregoing it is immediately seen that the second term on the right-hand side of (5.19) contains a contribution from $\varepsilon \partial \underline{\eta} / \partial t$ which is of zero order in ε. On the other hand, using equation (5.15),

$$\underline{B} \cdot \nabla \underline{v} = \underline{B} \cdot \nabla \frac{\partial \underline{\xi}}{\partial t} + \varepsilon \ \underline{B} \cdot \nabla \frac{\partial \underline{\eta}}{\partial t}$$

$$= \underline{B} \cdot \nabla \frac{\partial \underline{\xi}}{\partial t} + O(\varepsilon) \ , \tag{5.20}$$

since, because of the assumed properties of $\partial \underline{\eta} / \partial t$, $\underline{B} \cdot \nabla (\partial \underline{\eta} / \partial t)$ is of zero order in ε. Hence $\varepsilon \partial \underline{\eta} / \partial t$ gives rise only to terms of order ε in (5.20).

The nett result of these contributions is that in equation (5.19) terms of zero order in ε arise from $\varepsilon \partial \underline{\eta} / \partial t$: therefore changes in $\underline{B}$ due to $\varepsilon \underline{\eta}$ are of zero order. From these considerations it is clear that $\underline{\xi}$ here does not necessarily satisfy the constraint equation (5.1).

5.4 EVALUATION OF δW

The potential energy variation resulting from the perturbation $\underline{v}$ is

$$\delta W = -\int_0^t dt' \int_{\tau_p(t)} \underline{v}\cdot(\underline{j} \times \underline{B} - \nabla p)d\tau$$

$$= -\int_0^t dt' \int_{\tau_p(t)} \underline{v}\cdot\left[\frac{1}{\mu_o}\underline{B}\cdot\nabla\underline{B} - \nabla\left(p + \frac{B^2}{2\mu_o}\right)\right]d\tau \ , \qquad (5.21)$$

as can be seen from equation (4.18), (4.29) or (4.76).

Then application of Gauss' theorem to equation (5.21) gives

$$\delta W = -\int_0^t dt' \left\{ \int_{\tau_p(t)} \left[\frac{1}{\mu_o}\underline{v}\cdot\underline{B}\cdot\nabla\underline{B} + \left(p + \frac{B^2}{2\mu_o}\right)\nabla\cdot\underline{v}\right]d\tau \right.$$

$$\left. - \int_{S_{pv}(t)} \left(p + \frac{B^2}{2\mu_o}\right)\underline{v}\cdot d\underline{S}\right\} \qquad (5.22)$$

with the help of the expansion of $\nabla\cdot\left[\left(p + \frac{B^2}{2\mu_o}\right)\underline{v}\right]$.

(a) Evaluation of Volume Integral

Considering the volume integral inside the curly brackets of (5.22), the integration with respect to time is facilitated by the following transformation:

$$\int_{\tau_p(t)} \underline{v}\cdot\underline{B}\cdot\nabla\underline{B}\ d\tau = \int_{\tau_p(t)} v_i B_j (\partial_j B_i) d\tau$$

$$= \int_{\tau_p(t)} \left[\partial_j(v_i B_i B_j) - B_i \partial_j (v_i B_j)\right] d\tau \ \ldots\ldots$$

$$= \int_{\tau_p(t)} \left[\nabla\cdot(\underline{v}\cdot\underline{B}\ \underline{B}) - \underline{B}\cdot\underline{B}\cdot\nabla\underline{v} \right] d\tau \ , \tag{5.23}$$

since $\partial_j B_j = \nabla\cdot\underline{B} = 0$.

Applying Gauss' theorem, equation (5.23) becomes

$$\int_{\tau_p(t)} \underline{v}\cdot\underline{B}\cdot\nabla\underline{B}\ d\tau = - \int_{\tau_p(t)} \underline{B}\cdot\underline{B}\cdot\nabla\underline{v}\ d\tau + \int_{S_{pv}(t)} (\underline{v}\cdot\underline{B})\underline{B}\cdot d\underline{S}$$

$$= - \int_{\tau_p(t)} \underline{B}\cdot\underline{B}\cdot\nabla\underline{v}\ d\tau \ , \tag{5.24}$$

since in this sheet-current model $\underline{B}\cdot d\underline{S}$ vanishes at all points on S_{pv}.

In terms of this result the volume integral in equation (5.22) becomes

$$\Psi = \int_{\tau_p(t)} \left[\left(p + \frac{B^2}{2\mu_o}\right)\nabla\cdot\underline{v} - \frac{1}{\mu_o}\ \underline{B}\cdot\underline{B}\cdot\nabla\underline{v} \right] d\tau \ . \tag{5.25}$$

From the discussion relating to equation (5.16), $\nabla\cdot\underline{v}$ is of zero order in ε. Further, from equation (5.20), $\underline{B}\cdot\nabla\underline{v}$ is also of zero order. p and $\underline{B}$ being of zero order, it follows that the integrand of Ψ is of zero order, and thus if integrated over a volume of order ε, will yield a result which is of order ε.

Therefore if the domain of the volume integral Ψ is changed to $\tau(t) = \tau_p(t) - \tau_\varepsilon(t)$, where $\tau_\varepsilon(t)$ is the volume in which $\frac{\partial\underline{\eta}}{\partial t}$ is non-zero, an error of order ε is involved:

$$\Psi = \int_{\tau_p(t)} \left[\left(p + \frac{B^2}{2\mu_o}\right)\nabla\cdot\underline{v} - \frac{\underline{B}\cdot\underline{B}\cdot\nabla\underline{v}}{\mu_o}\right]d\tau$$

$$= \int_{\tau(t)} \left[\left(p + \frac{B^2}{2\mu_o}\right)\nabla\cdot\underline{v} - \frac{\underline{B}\cdot\underline{B}\cdot\nabla\underline{v}}{\mu_o}\right]d\tau + O(\varepsilon) \ . \qquad (5.26)$$

In $\tau(t)$, $\underline{v}(\underline{r},t) = \frac{\partial}{\partial t}\,\underline{\xi}(\underline{r}_o,t)$, where $\underline{\xi}(\underline{r}_o,t)$ has the same properties as in the usual treatments (e.g. BERNSTEIN et al.) where the equation of motion is linearized and perturbed quantities are expressed to first order in the perturbation, viz.,

$$p(\underline{r},t) = p(\underline{r}_o,0) - \gamma p(\underline{r}_o,0)\nabla\cdot\underline{\xi}$$

$$= p_o - \gamma p_o\nabla\cdot\underline{\xi} \ , \qquad (5.27)$$

$$\underline{B}(\underline{r},t) = \underline{B}(\underline{r}_o,0) + \underline{Q} + \underline{\xi}\cdot\nabla\underline{B}(\underline{r}_o,0)$$

$$= \underline{B}_o + \underline{Q} + \underline{\xi}\cdot\nabla\underline{B}_o \qquad (5.28)$$

$$d\tau = (1 + \nabla\cdot\underline{\xi})d\tau_o \ , \qquad (5.29)$$

$$\nabla_r = \nabla_o - \nabla_o\underline{\xi}\cdot\nabla_o \ , \qquad (5.30)$$

where as usual

$$\underline{Q} = \nabla\times(\underline{\xi}\times\underline{B}_o) \ , \qquad (5.31)$$

while in equations (5.27), (5.28) and (5.29), $\nabla \equiv \nabla_o$.

Using equations (5.27) to (5.30) the volume integral (5.26) may, with $\nabla \equiv \nabla_o$, be written to second order as

$$\Psi = \int_{\tau(o)} d\tau_o \Big\{ \Big(p_o + \frac{B_o^2}{2\mu_o}\Big)\Big[(\nabla\cdot\underline{\xi})\Big(\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'}\Big) - (\nabla\underline{\xi})\cdot\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'}\Big]$$

$$- \gamma p_o(\nabla\cdot\underline{\xi})\Big(\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'}\Big) + \frac{1}{\mu_o}\Big[\underline{B}_o\cdot\underline{Q}\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'}$$

$$+ \underline{B}_o\cdot\underline{\xi}\cdot(\nabla\underline{B}_o)\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'} - \underline{B}_o\cdot\underline{B}_o\cdot\Big(\nabla\frac{\partial\underline{\xi}}{\partial t'}\Big)\nabla\cdot\underline{\xi} +$$

$$\underline{B}_o\cdot\underline{B}_o\cdot(\nabla\underline{\xi})\cdot\nabla\frac{\partial\underline{\xi}}{\partial t'} - \underline{B}_o\cdot\underline{Q}\cdot\nabla\frac{\partial\underline{\xi}}{\partial t'} - \underline{B}_o\cdot\underline{\xi}\cdot(\nabla\underline{B}_o)\cdot\nabla\frac{\partial\underline{\xi}}{\partial t'}$$

$$- \underline{\xi}\cdot(\nabla\underline{B}_o)\cdot\underline{B}_o\cdot\nabla\frac{\partial\underline{\xi}}{\partial t'} - \underline{Q}\cdot\underline{B}_o\cdot\nabla\frac{\partial\underline{\xi}}{\partial t'}\Big]\Big\} + O(\varepsilon)\ , \qquad (5.32)$$

where the second-order terms have been retained, remembering that all first order terms in δW must sum to zero because the initial state is an equilibrium state, for which the potential energy function is stationary.

It is important now to show that the integration with respect to time of this expression (and of the corresponding second order expression for the surface integral in equation (5.22) can be carried out without requiring $\underline{\xi}$ to satisfy the first-order constraint equation (5.1). Since $\underline{r}_o$ is independent of time, the integrations with respect to time and volume may be commuted. The first time-integral to be evaluated is therefore

$$\int_o^t dt' \Big(\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'}\Big)(\nabla\cdot\underline{\xi})$$

$$= \tfrac{1}{2}\int_o^t dt' \frac{\partial}{\partial t'}(\nabla\cdot\underline{\xi})^2$$

$$= \tfrac{1}{2}(\nabla\cdot\underline{\xi})^2\ . \qquad (5.33)$$

The next integration is

$$- \int_0^t dt' \; (\nabla\underline{\xi})\cdot\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'}$$

$$= - \int_0^t dt' \; (\partial_i \xi_j)(\partial_j \dot{\xi}_i) \; .$$

Because of its symmetry in i and j this term is easily integrated by parts to give

$$- \tfrac{1}{2}(\partial_i \xi_j)(\partial_j \xi_i) = - \tfrac{1}{2}(\nabla\underline{\xi})\cdot\nabla\cdot\underline{\xi} \; . \qquad (5.34)$$

The terms involving $\underline{B}_o$ and $\underline{Q}$ in expression (5.32) can be integrated if $\underline{Q}$ is first expanded by the usual vector identity. Rearrangement then reduces these terms to

$$- \frac{1}{\mu_o} \int_0^t dt' \; (\underline{B}_o\cdot\nabla\underline{\xi} - \underline{B}_o\nabla\cdot\underline{\xi})\cdot\left(\underline{B}_o\cdot\nabla\frac{\partial\underline{\xi}}{\partial t'} - \underline{B}_o\nabla\cdot\frac{\partial\underline{\xi}}{\partial t'}\right)$$

$$= - \frac{1}{2\mu_o} \int_0^t dt' \; \frac{\partial}{\partial t'} \left|\underline{B}_o\cdot\nabla\underline{\xi} - \underline{B}_o\nabla\cdot\underline{\xi}\right|^2$$

$$= - \frac{1}{2\mu_o} \left|\underline{B}_o\cdot\nabla\underline{\xi} - \underline{B}_o\nabla\cdot\underline{\xi}\right|^2 \; . \qquad (5.35)$$

Combination of expressions (5.33), (5.34) and (5.35) then shows that

$$- \int_0^t \Psi dt' = - \int_0^t dt' \int_{\tau_p(t)} \left[\frac{1}{\mu_o} \underline{v}\cdot\underline{B}\cdot\nabla\underline{B} + \left(p + \frac{B^2}{2\mu_o}\right)\nabla\cdot\underline{v}\right] d\tau$$

$$= \tfrac{1}{2} \int_{\tau(o)} \left\{\frac{1}{\mu_o} \left|\underline{B}_o\cdot\nabla\underline{\xi} - \underline{B}_o\nabla\cdot\underline{\xi}\right|^2 - \left(p_o + \frac{B_o^{\,2}}{2\mu_o}\right)\left[(\nabla\cdot\underline{\xi})^2 - (\nabla\underline{\xi})\cdot\nabla\cdot\underline{\xi}\right]\right\} d\tau_o \; \ldots$$

$$+ \tfrac{1}{2} \int \gamma p_o (\nabla \cdot \underline{\xi})^2 \, d\tau_o + O(\varepsilon) + O(\xi^3) \ , \qquad (5.36)$$

the terms $O(\varepsilon)$ arising from equation (5.26), and $O(\xi^3)$ from the expansion of (5.26) leading to equation (5.32).

A residual error of order ε is involved if the domain of integration is here changed from $\tau(o)$ to $\tau_p(o)$, the equilibrium volume of the plasma. Except for an error of this order, the part of (5.36) of second order in $\underline{\xi}$ is precisely the $\delta\bar{W}$ expression derived by VAN KAMPEN and FELDERHOF (p.75, equation (20)) for a system comprising fluid only. It can be transformed to give expression (4.38) without requiring $\underline{\xi}$ to obey the constraint equation (5.1).

(b) Evaluation of Surface Integral

Attention is now directed to the surface integral in equation (5.22). Since p and B are such that equation (4.13) is satisfied, this term may be written

$$\sum = \int_o^t dt' \int_{S_{pv}(t)} \frac{\hat{B}^2}{2\mu_o} \underline{v} \cdot d\underline{S} \ , \qquad (5.37)$$

with $d\underline{S}$ directed out of the plasma.

Using the chain rule operator

$$\nabla_r \equiv \nabla_o - \nabla_r \underline{\tilde{\xi}} \cdot \nabla_o \ , \qquad (5.38)$$

the change occurring in $d\underline{S}$ as the perturbation develops is obtained from equation (4.55) as

$$\frac{d}{dt}\,d\underline{S} = (\nabla_o \cdot \underline{v})d\underline{S} - \nabla_o \underline{v} \cdot d\underline{S} - \nabla_r \tilde{\underline{\xi}} \cdot \nabla_o \cdot \underline{v}\ d\underline{S} + \nabla_r \tilde{\underline{\xi}} \cdot \nabla_o \underline{v} \cdot d\underline{S} \ , \qquad (5.39)$$

where $\tilde{\underline{\xi}} = \underline{\xi} + \varepsilon\, \underline{\eta}$. It will be clear that the third and fourth terms are of second order in the perturbation. Therefore to first order $d(d\underline{S})/dt$ becomes

$$\frac{d}{dt}\,d\underline{S} = (\nabla_o \cdot \underline{v})d\underline{S}_o - \nabla_o \underline{v} \cdot d\underline{S}_o \ ,$$

and so using equation (5.15) and the usual dot notation,

$$\frac{d}{dt}\,d\underline{S} = (\nabla_o \cdot \dot{\underline{\xi}})d\underline{S}_o - (\nabla_o \dot{\underline{\xi}}) \cdot d\underline{S}_o + \nabla_o \cdot (\varepsilon \dot{\underline{\eta}})d\underline{S}_o - \nabla_o (\varepsilon \dot{\underline{\eta}}) \cdot d\underline{S}_o \ . \qquad (5.40)$$

In a local Cartesian coordinate system with $\underline{e}_z \times d\underline{S}_o = 0$ and $\underline{e}_x$ and $\underline{e}_y$ lying in the surface, equation (5.40) can be expressed as

$$\begin{aligned}\frac{d}{dt}\,d\underline{S} &= (\nabla_o \cdot \dot{\underline{\xi}})d\underline{S}_o - (\nabla_o \dot{\underline{\xi}}) \cdot d\underline{S}_o + \varepsilon(\partial_i \dot{\eta}_i)d\underline{S}_o - \underline{e}_i \varepsilon(\partial_i \dot{\eta}_j)dS_{oj} \\ &= (\nabla_o \cdot \dot{\underline{\xi}})d\underline{S}_o - (\nabla_o \dot{\underline{\xi}}) \cdot d\underline{S}_o + \varepsilon(\partial_z \dot{\eta}_z)d\underline{S}_o - \underline{e}_z \varepsilon(\partial_z \dot{\eta}_j)dS_{oj} + 0(\varepsilon) \\ &= (\nabla_o \cdot \dot{\underline{\xi}})d\underline{S}_o - (\nabla_o \dot{\underline{\xi}}) \cdot d\underline{S}_o + \varepsilon(\partial_z \dot{\eta}_z)d\underline{S}_o - \varepsilon(\partial_z \dot{\eta}_z)\underline{e}_z dS_o + 0(\varepsilon) \ ,\end{aligned} \qquad (5.41)$$

since $d\underline{S}_o = \underline{e}_z dS_o$. Cancellation gives to first order in the perturbation

$$\frac{d}{dt}\,d\underline{S} = (\nabla_o \cdot \dot{\underline{\xi}})d\underline{S}_o - (\nabla_o \dot{\underline{\xi}}) \cdot d\underline{S}_o + 0(\varepsilon) \ . \qquad (5.42)$$

Integrating with respect to time then yields, in first order,

$$d\underline{S} = d\underline{S}_o + (\nabla_o \cdot \underline{\xi})d\underline{S}_o - (\nabla_o \underline{\xi}) \cdot d\underline{S}_o + 0(\varepsilon) \ . \qquad (5.43)$$

An expression for $\hat{\underline{B}}$ on the perturbed surface, to first order in $\underline{\xi}$, is still required. Two cases must be considered:-

(a) The perturbation at the surface is directed out of the plasma:-

In this case it is first necessary to consider the effect of the change in time occurring at the point $\underline{r}$ to which the fluid element is displaced:

$$\hat{\underline{B}}(\underline{r},t) = \hat{\underline{B}}(\underline{r},o) + \delta\hat{\underline{B}}(\underline{r},t) .$$

It is now permissible to apply a first-order Taylor expansion to $\hat{\underline{B}}(\underline{r},o)$, giving the change which occurs because of the spatial displacement. To first order in the perturbation

$$\begin{aligned}\hat{\underline{B}}(\underline{r},t) &= \hat{\underline{B}}(\underline{r}_o,o) + \tilde{\underline{\xi}}\cdot\nabla_o\hat{\underline{B}}(\underline{r}_o,o) + \delta\hat{\underline{B}}(\underline{r},t) \\ &= \hat{\underline{B}}(\underline{r},o) + \underline{\xi}\ \nabla_o\hat{\underline{B}}(\underline{r}_o,o) + \varepsilon\underline{\eta}\cdot\nabla_o\hat{\underline{B}}(\underline{r}_o,o) + \delta\hat{\underline{B}}(\underline{r},t) \\ &= \hat{\underline{B}}_o + \underline{\xi}\cdot\nabla_o\hat{\underline{B}}_o + \delta\hat{\underline{B}} + O(\varepsilon) . \qquad (5.44)\end{aligned}$$

(b) The perturbation at the surface is directed into the plasma:

In this case the spatial effect must be considered first, and so application of Taylor's expansion gives, to first order

$$\hat{\underline{B}}(\underline{r},t) = \hat{\underline{B}}(\underline{r}_o,t) + (\underline{\xi} + \varepsilon\underline{\eta})\cdot\nabla_o\hat{\underline{B}}(\underline{r}_o,t) . \qquad (5.45)$$

Expanding equations (4.2) and (4.3) in a local Cartesian coordinate system, with $\underline{e}_z \times d\underline{S}_o = 0$, and recalling that perturbed quantities have been assumed to vary slowly in directions parallel to the surface, it is found that in fact $\hat{\underline{B}}$ must vary slowly in *all* directions. Therefore equation (5.45) becomes,

$$\hat{\underline{B}}(\underline{r},t) = \hat{\underline{B}}(\underline{r}_o,t) + \underline{\xi}\cdot\nabla_o\hat{\underline{B}}(\underline{r}_o,t) + O(\varepsilon)$$

and consideration of the change of $\hat{\underline{B}}(\underline{r}_o,t)$ with time leads to

$$\hat{\underline{B}}(\underline{r},t) = \hat{\underline{B}}(\underline{r}_o,o) + \delta\hat{\underline{B}}(\underline{r}_o,t) + \underline{\xi}\cdot\nabla_o\hat{\underline{B}}(\underline{r}_o,o) + O(\varepsilon)$$

$$= \hat{\underline{B}}_o + \delta\hat{\underline{B}} + \underline{\xi}\cdot\nabla_o\hat{\underline{B}}_o + O(\varepsilon) \;, \qquad (5.46)$$

consistent with equation (5.44).

Using equations (5.43) and (5.46), $\sum$ of (5.37) becomes, with retention of second-order terms only,

$$\sum{}' = \frac{1}{\mu_o}\int_o^t dt' \int_{S_{pv}(o)} \{\tfrac{1}{2}\hat{B}_o^2 \frac{\partial\underline{\xi}}{\partial t'}\cdot(\nabla\cdot\underline{\xi}d\underline{S}_o - \nabla\underline{\xi}\cdot d\underline{S}_o) + \hat{\underline{B}}_o\cdot\delta\hat{\underline{B}}\frac{\partial\underline{\xi}}{\partial t'}\cdot d\underline{S}_o$$

$$+ \hat{\underline{B}}_o\cdot\underline{\xi}\cdot\nabla\hat{\underline{B}}_o \frac{\partial\underline{\xi}}{\partial t'}\cdot d\underline{S}_o\} + O(\varepsilon) \;. \qquad (5.47)$$

Since $\hat{\underline{E}} = -\dfrac{\partial\hat{\underline{A}}}{\partial t}$, the boundary condition (4.49) satisfied on $S_{pv}(t)$ is

$$- d\underline{S} \times \frac{\partial\hat{\underline{A}}}{\partial t} = (d\underline{S}\cdot\underline{v})\hat{\underline{B}} \;. \qquad (5.48)$$

To first order in the perturbation, equation (5.48) is, in the present case

$$d\underline{S}_o \times \frac{\partial\delta\hat{\underline{A}}}{\partial t} = -\left(d\underline{S}_o\cdot\frac{\partial\underline{\xi}}{\partial t}\right)\hat{\underline{B}}_o + O(\varepsilon) \;, \qquad (5.49)$$

where $\delta\hat{\underline{A}}$ is the first-order perturbation in $\hat{\underline{A}}$. Comparing equations (5.47) and (5.49) with equations (4.58) and (4.69) respectively, it is seen that the evaluation of $\sum{}'$ to zero order in ε reduces to the procedure followed earlier in this section for obtaining the second-order part of δW_{BE}, which did not invoke the use of equation (5.1), but where, of course, $\underline{\xi}$ was constrained by that equation. That is, $\sum'$ is given by the second-order part of equation (4.73), plus terms of order ε.

When the final result for $\sum'$ is combined with (5.36), transformed via equation (4.38) to the form (4.74), the second-order variation in potential energy becomes, with reference to equation (4.75),

$$\delta W(\tilde{\underline{\xi}}) = \delta W(\underline{\xi},\underline{\xi}) + O(\varepsilon)$$

$$= \delta W_F(\underline{\xi},\underline{\xi}) + \delta W_S(\underline{\xi},\underline{\xi}) + \delta W_E(\delta\hat{\underline{A}},\delta\hat{\underline{A}}) + O(\varepsilon) \ , \qquad (5.50)$$

where δW_F, δW_S and δW_E are the same functionals as appear in equation (4.77), but here $\underline{\xi}$ need not satisfy the constraint (5.1).

5.5 DISCUSSION

From the foregoing it is concluded that for a given functional $\delta W(\underline{\xi},\underline{\xi})$ of the small, slowly varying function $\underline{\xi}$, which does not necessarily satisfy equation (5.1), but which is such that equations (5.11), (5.12) and (5.13) are satisfied, there is a physically realizable perturbation $\tilde{\underline{\xi}}$ such that equation (4.13) is satisfied and which makes the second-order variation in potential energy arbitrarily close to $\delta W(\underline{\xi},\underline{\xi})$. Thus

$$\delta W_{min}(\tilde{\underline{\xi}}) = \delta W_{min}(\underline{\xi},\underline{\xi}) + O(\varepsilon) \ , \qquad (5.51)$$

and so a necessary and sufficient condition for stability can be obtained by examining the sign of $\delta W_{min}(\underline{\xi},\underline{\xi})$, while in minimizing $\delta W(\underline{\xi},\underline{\xi})$, the boundary condition (5.1) may be ignored.

It is important to note that the above analysis leads naturally to a functional of the form $\delta W_F + \delta W_S + \delta W_E$ given by equation (4.77). One concludes that it is not permissible to use the functional

$$\delta W = -\tfrac{1}{2} \int_{\tau_p(o)} \underline{\xi}\cdot\underline{F}(\underline{\xi})d\tau_o \ ,$$

where $\underline{F}(\underline{\xi})$ is the first-order unbalanced force in the fluid, since this form cannot be obtained from $(\delta W_F + \delta W_S + \delta W_E)$ unless $\underline{\xi}$ does in fact satisfy equation (5.1) (see BERNSTEIN et al. 1958, page 23).

On the other hand, because of the self-adjoint property of $\underline{F}$ with respect to $\underline{\tilde{\xi}}$ it can be shown (c.f. BERNSTEIN et al., 1958, page 22) that

$$\delta W(\underline{\tilde{\xi}},\underline{\tilde{\xi}}) = \tfrac{1}{2} \int_{\tau_p(o)} d\tau_o \cdot \underline{\tilde{\xi}}\cdot\underline{F}(\underline{\tilde{\xi}}) \ .$$

As implied by BERNSTEIN et al., this form of δW can be developed to give the result (5.50). This alternative proof, which, for rigour, contains much of the detail included in the above derivation, has also been completed. The present proof is, of course, more appropriate in the context of this thesis, and the alternative has been omitted to conserve space.

It should be noted also that to obtain a sufficient condition for instability one may use the same functional, but without requiring $\delta\hat{\underline{A}}$ to satisfy equation (5.11). This is possible because (5.11) is in fact the Euler equation for minimizing δW_E, subject to (5.12) and (5.13). Hence, if a function $\delta\hat{\underline{A}}$ appearing in (4.81) does not satisfy (5.11), another function $\delta\hat{\underline{A}}^*$, which does satisfy (5.11), would certainly decrease δW_E without changing δW_F or δW_S.

Therefore, if functions $\underline{\xi}$ and $\delta\hat{\underline{A}}$ are found which satisfy (5.12) and (5.13) but not necessarily (5.1) and (5.11), and which make the functional $\delta W(\underline{\xi},\underline{\xi})$ negative, then there is a physically realizable perturbation for which δW is certainly negative, and the system is unstable.

For a system with zero internal magnetic field, the potential energy variation corresponding to the form (5.9) becomes, from the extended energy principle result (5.50) and use of equations (5.2), (5.4) and (4.81)

$$\delta W(\tilde{\underline{\xi}}) = \frac{\gamma p_o}{2}\int\limits_{\tau_p(o)} (\nabla\cdot\underline{\xi})^2 d\tau_o + \frac{1}{2\mu_o}\int\limits_{S_{pv}(o)} d\sigma_o(\underline{n}_o\cdot\underline{\xi})^2\underline{n}_o\cdot\nabla(\tfrac{1}{2}\hat{B}_o^2)$$

$$+ \frac{1}{2\mu_o}\int\limits_{\hat{\tau}(o)+\tau_i} |\nabla\times\delta\underline{A}|^2 d\tau + O(\varepsilon)\ . \tag{5.52}$$

However, unlike $\underline{\xi}$ in equation (5.9), in this result $\underline{\xi}$ is not constrained by equation (5.1). Hence, reverting to the discussion following (5.9), there is freedom here to choose an incompressible perturbation $\underline{\xi}$, where $\underline{n}_o \cdot \underline{\xi}$ is non-zero only within the surface-fluting region R.

Using equations (4.36), (5.52) and the condition $\nabla \cdot \underline{\xi} = 0$,

$$\delta W(\tilde{\underline{\xi}}) = \frac{1}{2\mu_o} \int_{S_{pv}(o)} d\sigma_o \xi_n{}^2 \hat{B}_o^2 K + \frac{1}{2\mu_o} \int_{\hat{\tau}(o)+\tau_i} |\nabla \times \delta \underline{A}|^2 d\tau + O(\varepsilon) \ , \qquad (5.53)$$

where $\underline{K} = \underline{n}_o K$ is the vector curvature of the lines of magnetic force.

Note that here it is convenient to define the line curvature in terms of the *signed* quantity K, whereas, in the earlier work of Section 1.4, it was easier to use the *unsigned* radius of curvature, R, and define $\underline{K}$ in terms of the direction of $\underline{e}_2$, the unit principal normal to the field line. Thus the stability of the surface may be interpreted in terms of Teller's familiar curvature criterion (c.f. BERNSTEIN et al., 1958, pp.31-2).

As BERNSTEIN et al. show, it is possible to make the magnitude of the volume integral in (5.53) above arbitrarily small compared with that of the surface integral, and therefore a necessary and sufficient condition for instability can be obtained by examining the sign of the surface integral alone.

5.6 REFERENCES

BERNSTEIN, I. B., FRIEMAN, E. A., KRUSKAL, M. D., and KULSRUD, R. M. (1958) - Proc. Roy. Soc. A224, 17.

ROSE, D. J., and CLARK, M., Jr. (1961) - "Plasmas and Controlled Fusion". (Wiley: New York).

KAMPEN, VAN N. G., and FELDERHOF, B. U. (1967) - "Theoretical Methods in Plasma Physics". (North-Holland Publishing Co.: Amsterdam).

CHAPTER 6

APPLICATION TO THE CONSTRICTED DISCHARGE

6.1 INTRODUCTION

Consideration is now given to a constricted discharge between electrodes, in which the total discharge current is assumed to flow in a very thin layer on the plasma surface. It is interesting first to consider an idealized experiment in which the gas is initially at pressure p^o and occupies the whole discharge chamber of volume τ^o. The gas is ionized and made highly conducting by the application of an intense electric field. Breakdown occurs and at the same time external coils are energized, producing a magnetic field which is strongest near the median plane of the discharge column. Because of the very high electrical conductivity of the plasma, the discharge current is confined to a very thin layer which sweeps inwards because of the pinch effect and the pressure of the external field. The plasma is thus rapidly compressed, and the discharge column is centrally constricted.

Because the compression is so rapid it may be assumed, as in previous chapters, that the gas obeys the adiabatic law, so that the pressure and volume at any time are related to their initial values by the equation

$$p\tau^{\gamma} = p^{o}\tau^{o\gamma} , \tag{6.1}$$

where γ is the usual ratio of specific heats of the gas.

It is assumed that compression ends when a balance is reached between the plasma pressure and the magnetic field pressure at the surface. Thus an equilibrium situation is first considered, in which there is zero magnetic field in the plasma, and the plasma pressure is constant throughout the discharge, with a sharp drop to zero across the current sheet. This situation differs from that associated with a configuration of the same geometry investigated in earlier papers by SEYMOUR (1961) and in Chapters 2 and 3, in which a diffuse discharge was considered, with a large part of the discharge current flowing close to the axis.

Later the analysis will be extended by considering an axial magnetic field to be trapped within the plasma.

The discharge surface in equilibrium is shaped by the pressure of the external magnetic field and, as before, is approximated by a hyperboloid of one sheet (Fig. 5). Again, the system is described by the oblate spheroidal coordinates (u,v,w), where equations (2.1), (2.2) and (2.3) apply. The plasma occupies the region

$$\left.\begin{array}{l} w_b \leqslant w \leqslant \pi/2 \\ -u_e \leqslant u \leqslant u_e \\ 0 \leqslant v \leqslant 2\pi \end{array}\right\}, \qquad (6.2)$$

and all scalars are assumed independent of v in equilibrium.

In the vacuum region exterior to and within the conducting wall which surrounds the plasma column,

$$\underline{\hat{B}}_o = (\hat{B}_{ou}, \hat{B}_{ov}, \hat{B}_{ow})$$

or

$$\underline{B} = (B_u, B_v, B_w) \ , \tag{6.3}$$

it being convenient to omit the circumflex and the subscript 'o' here without confusion because in the analysis of the next section there is no magnetic field within the plasma, and equilibrium quantities only will appear. Further, on $S_{pv}(o)$, where $w = w_b$,

$$\underline{B}_b \cdot \underline{w}_o = 0 \tag{6.4}$$

or

$$\underline{B}_b = (B_{bu}, B_{bv}, 0) \ , \tag{6.5}$$

the subscript 'b' indicating quantities evaluated on the boundary.

6.2 APPLICATION OF THE ENERGY PRINCIPLE

In the present case

$$\underline{n}_o' = \underline{w}_o \ , \tag{6.6}$$

and so, with reference to δW_S given by the result (5.4), and using (6.5), on the surface $w = w_b$,

$$\underline{n}_o' \cdot \nabla(\tfrac{1}{2}B^2) = \frac{1}{2h_{wb}} \left(\frac{\partial B^2}{\partial w}\right)_b$$

$$= \frac{1}{h_{wb}} \left\{ B_{bu} \left(\frac{\partial B_u}{\partial w}\right)_b + B_{bv} \left(\frac{\partial B_v}{\partial w}\right)_b \right\}. \tag{6.7}$$

Equation (4.3) yields, with azimuthal symmetry,

$$\frac{\partial}{\partial u} (h_v B_v) = 0 \ , \tag{6.8}$$

$$\frac{\partial}{\partial w}(h_v B_v) = 0 \ , \qquad (6.9)$$

$$\frac{\partial}{\partial w}(h_u B_u) - \frac{\partial}{\partial u}(h_w B_w) = 0 \ . \qquad (6.10)$$

Equations (6.8) and (6.9) give

$$B_v = \frac{C}{h_v} \ , \qquad (6.11)$$

where C is a constant, proportional to the total discharge current I.

In fact C is the same constant as was used in Chapter 3 and, in M.K.S.C. units, is given by

$$C = \frac{\mu_o I}{2\pi} \ . \qquad (6.12)$$

From (6.11),

$$\left(\frac{\partial B_v}{\partial w}\right)_b = -\frac{C}{h_{vb}^2}\left(\frac{\partial h_v}{\partial w}\right)_b \ , \qquad (6.13)$$

and so equations (6.11) and (6.13) give

$$B_{bv}\left(\frac{\partial B_v}{\partial w}\right)_b = -\frac{C^2}{h_{vb}^3}\left(\frac{\partial h_v}{\partial w}\right)_b \ . \qquad (6.14)$$

From (6.5), $B_{bw} = 0$ on $S_{pv}(o)$, and so the second term in (6.10) must vanish when evaluated at $w = w_b$, leaving

$$\left\{\frac{\partial}{\partial w}(h_u B_u)\right\}_b = 0 \ ,$$

or

$$\left(\frac{\partial B_u}{\partial w}\right)_b = -\frac{B_{bu}}{h_{ub}}\left(\frac{\partial h_u}{\partial w}\right)_b \ . \qquad (6.15)$$

In the present case the condition (4.27) reduces to

$$B_{bu} = (2\mu_o p - B_{bv}^2)^{\frac{1}{2}} , \qquad (6.16)$$

and so, from (6.15) and (6.16),

$$B_{bu} \left(\frac{\partial B_u}{\partial w}\right)_b = - \frac{(2\mu_o p - B_{bv}^2)}{h_{ub}} \left(\frac{\partial h_u}{\partial w}\right)_b . \qquad (6.17)$$

Using equations (6.14) and (6.17), (6.7) becomes

$$\underline{n}_o' \cdot \nabla(\tfrac{1}{2}B^2) = - \frac{1}{h_{wb}} \left\{ \frac{(2\mu_o p - B_{bv}^2)}{h_{ub}} \left(\frac{\partial h_u}{\partial w}\right)_b + \frac{C^2}{h_{vb}^3} \left(\frac{\partial h_v}{\partial w}\right)_b \right\}. \qquad (6.18)$$

The expression (5.4) for δW_S now becomes, with $\underline{n}_o = - \underline{n}_o'$,

$$\delta W_S = \frac{1}{2\mu_o} \int_{S_{pv}(o)} d\sigma_o \frac{\xi_w^2}{h_{wb}} \left\{ \frac{(2\mu_o p - B_{bv})^2}{h_{ub}} \left(\frac{\partial h_u}{\partial w}\right)_b + \frac{C^2}{h_{vb}^3} \left(\frac{\partial h_v}{\partial w}\right)_b \right\} . \qquad (6.19)$$

Expression (6.19) is specialized to the oblate spheroidal geometry by the use of equations (2.2) and (2.3), yielding

$$\delta W_S = \frac{1}{2\mu_o} \int_{S_{pv}(o)} d\sigma_o \frac{\xi_w^2 \sin w_b}{h_{wb}} \left\{ \frac{(2\mu_o p - B_{bv}^2)\cos w_b}{(\sin^2 w_b + \sinh^2 u)} - \frac{C^2}{k^2 \cosh^2 u \cos^3 w_b} \right\} .$$

With the assistance of equations (2.2) , (2.3) and (6.11), manipulation yields

$$\delta W_S = \frac{1}{2\mu_o} \int_{S_{pv}(o)} d\sigma_o \xi_w^2 \frac{\sin w_b (2\mu_o p k^2 \cos^4 w_b - C^2)}{k^3 \cos^3 w_b (\sin^2 w_b + \sinh^2 u)^{3/2}} , \qquad (6.20)$$

where the integration is to be carried out over the surface $w = w_b$, from $u = - u_e$ to $u = u_e$. It is immediately apparent that the sign of the integrand in (6.20) is determined by the sign of the bracketed term in the numerator, and that the sign of this term is independent of position on the surface $S_{pv}(o) : w = w_b$. Therefore the sign of this bracketed term determines the sign of the complete integral. The necessary and sufficient condition for instability is that the integral (6.20) be negative, which leads to the condition

$$2\mu_o pk^2\cos^4 w_b - C^2 < 0 \ . \tag{6.21}$$

It is therefore clear that, as in the exact solution obtained previously (Chapter 3), a transition to instability can occur as the discharge current is increased. There exists a critical value for the discharge current, defined by

$$C_{crit.} = (2\mu_o p)^{\frac{1}{2}} k\cos^2 w_b \ , \tag{6.22}$$

above which, according to the theory of ideal hydromagnetics, instability of the surface develops. In M.K.S.C. units the critical current is, from equation (6.12),

$$I_{crit.} = \frac{2\pi}{\mu_o}(2\mu_o p)^{\frac{1}{2}} k\cos^2 w_b \ , \tag{6.23}$$

where, of course, p and w_b have values appropriate to the equilibrium configuration reached.

6.3 THE DISCHARGE WITH IMBEDDED AXIAL MAGNETIC FIELD

The discharge discussed in Section 6.2 is the curvilinear generalization of the cylindrical pinch discharge with an axial magnetic field in the vacuum region only. Stability analyses of such a cylindrical discharge by the normal mode approach (TAYLER, 1957) and by the energy principle approach (SCHMIDT, 1966, pp.144-149) show that it is always unstable. The instability arises, as expected, from the unfavourable curvature of the azimuthal magnetic field produced by the discharge current; stability within the framework of this theory can only be achieved if the discharge column is surrounded by a conducting metal shell, and an axial magnetic field is trapped within the plasma. This latter could be achieved by energizing the external coils just before breakdown occurs so that some external magnetic flux can be trapped as the current sheet implodes. Again, all the discharge current is assumed to flow in the surface current sheet. In this case, the stabilizing effect comes from the term δW_F in the expression (4.77) for δW. δW_F becomes, for zero internal current density, simply (see equation (4.78))

$$\delta W_F = \frac{1}{2} \int_{\tau_p(o)} d\tau_o \left(\frac{|\underline{Q}|^2}{\mu_o} + \gamma p(\nabla\cdot\underline{\xi})^2\right) , \qquad (6.24)$$

which is, of course, always positive. In arriving at expression (6.24) use has been made of the fact that in this case

$$\nabla p_o = \underline{j}_o \times \underline{B}_o = 0 .$$

In the curvilinear generalization of this linear discharge, the surface sheet current flows in the surface $w = w_b$, and there is a trapped internal magnetic field given by

$$\underline{B} = (B_u, 0, 0) \ , \tag{6.25}$$

with

$$\nabla \times \underline{B} = 0 \tag{6.26}$$

and

$$\nabla \cdot \underline{B} = 0 \ . \tag{6.27}$$

The analysis of Section 2.2 then applies, and gives, for the special case of oblate spheroidal geometry,

$$B_u = \frac{A}{k^2 \cosh u (\sin^2 w + \sinh^2 u)^{\frac{1}{2}}} \ . \tag{6.28}$$

However, as will be seen, there is no need to specialize the geometry in the following analysis which, in this respect, is quite general.

The energy-principle approach necessitates the minimization of δW_F with respect to $\underline{\xi}$, for a given boundary prescription of $\underline{\xi}$. As shown by BERNSTEIN et al (1958, Section 4), the Euler equation for this minimization is just

$$\underline{F}(\underline{\xi}) = 0 \tag{6.29}$$

where $\underline{F}(\underline{\xi})$ is the first-order unbalanced force generated in the plasma by the perturbation. For the case of zero internal electric current, equation (6.29) reduces to (SCHMIDT 1966, p.145)

$$\frac{1}{\mu_o} (\nabla \times \underline{Q}) \times \underline{B} + \gamma p \nabla (\nabla \cdot \underline{\xi}) = 0 \ . \tag{6.30}$$

The assumption of cylindrical symmetry in the special case of the linear discharge leads (SCHMIDT 1966, p.146) to a considerable simplification of equation (6.30), resulting in

$$\nabla^2(\underline{B}\cdot\underline{Q}) = 0 \ , \qquad (6.31)$$

which is easily solved. However in general, equation (6.30) may be written

$$\frac{1}{\mu_o}(-\nabla(\underline{B}\cdot\underline{Q}) + \underline{B}\cdot\nabla\underline{Q} + \underline{Q}\cdot\nabla\underline{B}) + \gamma p\nabla(\nabla\cdot\underline{\xi}) = 0 \ , \qquad (6.32)$$

by using the vector expansion of $\nabla(\underline{B}\cdot\underline{Q})$ and the condition (6.26). Taking the divergence of (6.32),

$$\nabla^2\left[\gamma p\nabla\cdot\underline{\xi} - \frac{\underline{B}\cdot\underline{Q}}{\mu_o}\right] + \frac{1}{\mu_o}\nabla\cdot\left[\underline{B}\cdot\nabla\underline{Q} + \underline{Q}\cdot\nabla\underline{B}\right] = 0 \ . \qquad (6.33)$$

Since, with $\nabla\cdot\underline{B} = \nabla\cdot\underline{Q} = 0$,

$$\nabla \times (Q \times \underline{B}) = \underline{B}\cdot\nabla\underline{Q} - \underline{Q}\cdot\nabla\underline{B} \ ,$$

it follows that

$$\nabla\cdot(\underline{B}\cdot\nabla\underline{Q}) = \nabla\cdot(\underline{Q}\cdot\nabla\underline{B}) \ .$$

Therefore equation (6.33) becomes

$$\nabla^2\left[\gamma p\nabla\cdot\underline{\xi} - \frac{\underline{B}\cdot\underline{Q}}{\mu_o}\right] + \frac{2}{\mu_o}\nabla\cdot\left[\underline{B}\cdot\nabla\underline{Q}\right] = 0 \ . \qquad (6.34)$$

This is the generalized counterpart of (6.31). Unlike (6.31) it is difficult to solve analytically, particularly in the oblate spheroidal geometry. Since the function $\underline{\xi}$ for which

$$\underline{Q} \equiv 0 \qquad (6.35)$$

and

$$\nabla \cdot \underline{\xi} \equiv 0 \tag{6.36}$$

minimizes the positive quantity δW_F to zero, it is of interest to see if this represents other than a trivial solution of (6.34). It is shown below that, because $\underline{\xi}$ has to satisfy certain boundary conditions, the only function for which (6.35) and (6.36) can be satisfied in the discharge is in fact the trivial solution, $\underline{\xi} = 0$.

Thus,

$$\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B})$$

$$= \underline{B} \cdot \nabla \underline{\xi} - \underline{\xi} \cdot \nabla \underline{B} - \underline{B} \nabla \cdot \underline{\xi}$$

since $\nabla \cdot \underline{B} = 0$. Then, with (6.36),

$$\underline{Q} = \underline{B} \cdot \nabla \underline{\xi} - \underline{\xi} \cdot \nabla \underline{B} \; . \tag{6.37}$$

Using (6.25), and writing $\underline{\xi} = (\xi_u, \xi_v, \xi_w)$,

$$\underline{B} \cdot \nabla \underline{\xi} = \frac{B_u}{h_u} \frac{\partial}{\partial u} (\underline{u}_o \xi_u + \underline{v}_o \xi_v + \underline{w}_o \xi_w)$$

$$= \frac{B_u \xi_u}{h_u} \frac{\partial \underline{u}_o}{\partial u} + \underline{u}_o \frac{B_u}{h_u} \frac{\partial \xi_u}{\partial u} + \frac{B_u \xi_v}{h_u} \frac{\partial \underline{v}_o}{\partial u} + \underline{v}_o \frac{B_u}{h_u} \frac{\partial \xi_v}{\partial u} + \frac{B_u \xi_w}{h_u} \frac{\partial \underline{w}_o}{\partial u}$$

$$+ \underline{w}_o \frac{B_u}{h_u} \frac{\partial \xi_w}{\partial u} \; . \tag{6.38}$$

Through use of the expressions following equation (2.54), for the derivatives of the unit vectors, and the assumption of azimuthal symmetry, equation (6.38) becomes

$$\underline{B} \cdot \nabla \underline{\xi} = \underline{u}_o \left(\frac{B_u}{h_u} \frac{\partial \xi_u}{\partial u} + \frac{B_u \xi_w}{h_u h_w} \frac{\partial h_u}{\partial w} \right) + \underline{v}_o \frac{B_u}{h_u} \frac{\partial \xi_v}{\partial u}$$

$$+ \underline{w}_o \left(\frac{B_u}{h_u} \frac{\partial \xi_w}{\partial u} - \frac{B_u \xi_u}{h_u h_w} \frac{\partial h_u}{\partial w} \right) . \tag{6.39}$$

Further,

$$\underline{\xi}\cdot\nabla\underline{B} = \left(\frac{\xi_u}{h_u}\frac{\partial}{\partial u} + \frac{\xi_v}{h_v}\frac{\partial}{\partial v} + \frac{\xi_w}{h_w}\frac{\partial}{\partial w}\right) \underline{u}_o B_u$$

$$= \underline{u}_o \frac{\xi_u}{h_u}\frac{\partial B_u}{\partial u} + \frac{B_u \xi_u}{h_u}\frac{\partial \underline{u}_o}{\partial u} + \frac{B_u \xi_v}{h_v}\frac{\partial \underline{u}_o}{\partial v} + \underline{u}_o \frac{\xi_w}{h_w}\frac{\partial B_u}{\partial w} + \frac{B_u \xi_w}{h_w}\frac{\partial \underline{u}_o}{\partial w} .$$

By a treatment similar to the above, this reduces to

$$\underline{\xi}\cdot\nabla\underline{B} = \underline{u}_o \left(\frac{\xi_u}{h_u}\frac{\partial B_u}{\partial u} + \frac{\xi_w}{h_w}\frac{\partial B_u}{\partial w}\right) + \underline{v}_o \frac{B_u \xi_v}{h_u h_v}\frac{\partial h_v}{\partial u} + \underline{w}_o \left(\frac{B_u \xi_w}{h_u h_w}\frac{\partial h_w}{\partial u} - \frac{B_u \xi_u}{h_u h_w}\frac{\partial h_u}{\partial w}\right) . \tag{6.40}$$

Substituting expressions (6.39) and (6.40) into (6.37),

$$\begin{aligned} \underline{Q} = \underline{u}_o & \left[\frac{B_u}{h_u}\frac{\partial \xi_u}{\partial u} + \frac{B_u \xi_w}{h_u h_w}\frac{\partial h_u}{\partial w} - \frac{\xi_u}{h_u}\frac{\partial B_u}{\partial u} - \frac{\xi_w}{h_w}\frac{\partial B_u}{\partial w}\right] \\ & + \underline{v}_o \left[\frac{B_u}{h_u}\frac{\partial \xi_v}{\partial u} - \frac{B_u \xi_v}{h_u h_v}\frac{\partial h_v}{\partial u}\right] \\ & + \underline{w}_o \left[\frac{B_u}{h_u}\frac{\partial \xi_w}{\partial u} - \frac{B_u \xi_w}{h_u h_w}\frac{\partial h_w}{\partial u}\right] . \end{aligned} \tag{6.41}$$

Then the w-component of equation (6.35) becomes, with $B_u \neq 0$,

$$\frac{\partial \xi_w}{\partial u} - \frac{\xi_w}{h_w}\frac{\partial h_w}{\partial u} = 0 ,$$

or

$$\frac{\partial}{\partial u}\left(\frac{\xi_w}{h_w}\right) = 0 . \tag{6.42}$$

Similarly, the v-component of (6.35) may be dealt with to give

$$\frac{\partial}{\partial u}\left(\frac{\xi_v}{h_v}\right) = 0 . \tag{6.43}$$

The u-component of (6.35) may be rearranged as

$$\frac{B_u{}^2}{h_u}\left(\frac{1}{B_u}\frac{\partial \xi_u}{\partial u} - \frac{\xi_u}{B_u{}^2}\frac{\partial B_u}{\partial u}\right) + \frac{B_u{}^2}{h_u}\frac{\xi_w}{h_w}\left(\frac{1}{B_u}\frac{\partial h_u}{\partial w} - \frac{h_u{}^2}{B_u{}^2}\frac{\partial B_u}{\partial w}\right) = 0 \ ,$$

or

$$\frac{\partial}{\partial u}\left(\frac{\xi_u}{B_u}\right) + \frac{\xi_w}{h_w}\frac{\partial}{\partial w}\left(\frac{h_u}{B_u}\right) = 0 \ . \qquad (6.44)$$

Now, as discussed in Section 4.3, the freezing-in effect of the high electrical conductivity of the plasma leads to the condition $\underline{\xi} = 0$ at $u = \pm u_e$, where the lines of force of $\underline{B}$ enter the electrodes. Conditions (6.42) and (6.43) therefore give

$$\xi_v \equiv \xi_w \equiv 0 \qquad (6.45)$$

throughout the plasma. This condition in turn reduces equation (6.44) to

$$\frac{\partial}{\partial u}\left(\frac{\xi_u}{B_u}\right) = 0 \qquad (6.46)$$

so that, since $\xi_u\ (\pm\ u_e) = 0$, ξ_u must also be zero throughout the plasma. That is, $\underline{\xi}$ has to be identically zero for conditions (6.35) and (6.36) to hold, so that the solution of (6.34) represented by (6.35) and (6.36) is trivial.

Although equation (6.34) poses analytical difficulties, it is fortunately not necessary to effect its solution because a useful result in the form of a sufficient criterion for stability can be obtained by an alternative method, as follows. Since, for this system, δW_F (equation (6.24)) and δW_E (equation (4.81)) are both positive, this sufficient condition for stability is derived by requiring δW_S (equation (4.80)) to be positive also. Here, the constricted discharge, with surface approximating a hyperboloid of one sheet, contrasts with the cylindrical discharge, because the favourable curvature imposed by the external magnetic field makes it possible, as will be seen, for δW_S to be positive provided a certain transitional discharge current is not exceeded. For the cylindrical discharge (TAYLER, 1957; SCHMIDT, 1966) δW_S must always be negative for $\xi_n = \underline{n}_o \cdot \underline{\xi} \neq 0$.

The derivation of this condition is entirely analogous to that given for the field-free system in Section 6.2. Equations (6.25) and (6.26) yield, as in Section 2.2,

$$B = \frac{F(u)}{h_u} , \qquad (6.47)$$

it being convenient here to omit the subscript 'u' on B since the v- and w- components of the internal field $\underline{B}$ are zero. Then

$$\underline{n}_o \cdot \nabla(\tfrac{1}{2}B^2) = - \frac{B_b}{h_{wb}} \left(\frac{\partial B}{\partial w}\right)_b$$

$$= \frac{B_b{}^2}{h_{ub}h_{wb}} \left(\frac{\partial h_u}{\partial w}\right)_b . \qquad (6.48)$$

The term $\underline{n}_o \cdot \nabla(\tfrac{1}{2}\hat{B}^2)$ is derived in exactly the same way as before, yielding, through use of the pressure balance equation

$$2\mu_o p + B_b^2 = \hat{B}_{bu}^2 + \hat{B}_{bv}^2 ,$$

the form

$$\underline{n}_o\cdot\nabla(\tfrac{1}{2}\hat{B}^2) = \frac{1}{h_{wb}}\left\{\frac{(2\mu_o p + B_b^2 - \hat{B}_{bv}^2)}{h_{ub}}\left(\frac{\partial h_u}{\partial w}\right)_b + \frac{C^2}{h_{vb}^3}\left(\frac{\partial h_v}{\partial w}\right)_b\right\}, \quad (6.49)$$

corresponding to equation (6.18). Combining equations (6.48) and (6.49) to determine $< \underline{n}_o\cdot\nabla(\tfrac{1}{2}B^2) >$, it is noted that the terms involving B_b cancel, leaving

$$< \underline{n}_o\cdot\nabla(\tfrac{1}{2}B^2)> = \frac{1}{h_{wb}}\left\{\frac{(2\mu_o p - \hat{B}_{bv}^2)}{h_{ub}}\left(\frac{\partial h_u}{\partial w}\right)_b + \frac{C^2}{h_{vb}^3}\left(\frac{\partial h_v}{\partial w}\right)_b\right\}. \quad (6.50)$$

Finally, making use of the fact that in equilibrium $\nabla p = \underline{j}\times\underline{B} = 0$ in this system with zero internal current density, expression (4.80) becomes

$$\delta W_S = \frac{1}{2\mu_o}\int_{S_{pv}(o)} d\sigma_o\,(\underline{n}_o\cdot\underline{\xi})^2 < \underline{n}_o\cdot\nabla(\tfrac{1}{2}B^2) >$$

$$= \frac{1}{2\mu_o}\int_{S_{pv}(o)} d\sigma_o\,\frac{\xi_w^2}{h_{wb}}\left\{\frac{(2\mu_o p - \hat{B}_{bv}^2)}{h_{ub}}\left(\frac{\partial h_u}{\partial w}\right)_b + \frac{C^2}{h_{vb}^3}\left(\frac{\partial h_v}{\partial w}\right)_b\right\}. \quad (6.51)$$

It is noted that expression (6.51) is of exactly the same form as that obtained for the system with zero internal field (equation (6.19)). Thus, specializing to the oblate spheroidal geometry and noting that $\hat{B}_{bv}$ is again given by (6.11), it is clear that the expression for the transition current $C_{trans.}$ above which δW_S is negative will be the same as that given before for $C_{crit.}$. The difference between the two systems is that $C_{trans.}$ here is not critical for the onset of instability, but merely marks the value of discharge current below which the system is certainly stable, within the framework of this theory. Of course, in the limit of zero internal field, the extra stabilization represented by the positive quantity δW_F is lost and $C_{trans.}$ reduces to the critical current, $C_{crit.}$.

Reverting to expressions (6.19) and (6.51) for δW_S, and bearing in mind that they refer to structurally different systems, it is of interest to consider their equivalence of form. The reason for the similarity lies in the fact that δW_S is, essentially, a field-line curvature term (equation (5.53)). In the integrand of δW_S, the internal field curvature evaluated on the plasma side of the infinitesimally thin surface current sheet and weighted by the square of the internal field magnitude at the same point, is subtracted from the vacuum field curvature, weighted by $\hat{B}^2$, evaluated at the corresponding point on the vacuum side of the surface current sheet. Although, compared with the field-free system, the discharge with internal axial field has extra u-component fields both inside and, to maintain pressure balance, outside the surface, the curvatures of these extra field components just on either side of the vanishingly thin surface are clearly the same. Then, when the integrand of δW_S is obtained as outlined above, the contribution of the internal field at the surface is cancelled by that of the extra vacuum field needed just outside the surface to satisfy pressure balance.

6.4 CRITICAL CURRENT IN TERMS OF THE INITIAL CONDITIONS OF THE EXPERIMENT

An approximation for the critical current in terms of the initial conditions of the experiment (described at the beginning of Section 6.1) is useful, and is obtained for the oblate spheroidal geometry in the following analysis. The discussion is, for simplicity, restricted to the field-free system. This serves to illustrate the method used and the approximations involved, which could then be applied to the more complicated system with internal field, if required.

The discharge volume τ can be computed in terms of the plasma geometry parameters k, u_e and w_b. The adiabatic equation of state (6.1) may then be expressed

$$p = p(p^o, \tau^o, k, u_e, w_b) \; . \tag{6.52}$$

At equilibrium, the pressure balance equation is

$$\begin{aligned} 2\mu_o p &= B_b^2 \\ &= B_{bu}^2 + B_{bv}^2 \end{aligned}$$

and so if $B_{bu} = B_M$, the external magnetic field component at the median plane u = 0,

$$2\mu_o p = B_M^2 + \frac{C^2}{k^2 \cos^2 w_b} \; , \tag{6.53}$$

with the aid of equation (6.11).

Referring now to the result (6.22), let it be assumed here that equilibrium of the plasma/magnetic field configuration is reached at the critical current. Then equations (6.52), (6.22) and (6.53) may be solved for $C_{crit.}$ and w_b in terms of the quantities p^o, τ^o, k and B_M. To this end it is first necessary to obtain an expression for the volume τ of the discharge,

$$\tau = \int_{w_b}^{\pi/2} \int_{o}^{2\pi} \int_{-u_e}^{u_e} h_u h_v h_w \, du \, dv \, dw \ , \tag{6.54}$$

the integration limits being clear from the inequalities (6.2).

The integration procedure is straightforward, and yields

$$\tau = \frac{4}{3} \pi k^3 \sinh u_e \{1 - (1 - X^2)^{3/2} + \sinh^2 u_e \ [1 - (1 - X^2)^{\frac{1}{2}}]\}, \tag{6.55}$$

where $X = \cos w_b$. Recalling the discussion in Section 3.6, the degree of radial constriction is not expected to be large, so that it is reasonable to take

$$w_b \approx \pi/2$$

or

$$X^2 \ll 1 \ . \tag{6.56}$$

The inequality (6.56) enables (6.55) to be approximated to second order in X by binomial expansion, giving

$$\tau = \frac{2}{3} \pi k^3 \sinh u_e (3 + \sinh^2 u_e) X^2 \ . \tag{6.57}$$

Defining

$$F_e = \frac{2}{3} \pi k^3 \sinh u_e (3 + \sinh^2 u_e) \ , \tag{6.58}$$

$$\tau = F_e X^2 \ . \tag{6.59}$$

Combination of equations (6.1) and (6.59) results in the explicit form of (6.52) required,

$$p \approx \frac{p^o \tau^{o\gamma}}{F_e{}^{\gamma} X^{2\gamma}} = qX^{-2\gamma} \ , \tag{6.60}$$

where

$$q = p^o \tau^{o\gamma} F_e{}^{-\gamma} \ . \tag{6.61}$$

Substitution of the result (6.60) into equations (6.22) and (6.53) gives, in terms of $X = \cos w_b$, the respective results

$$C^2_{crit.} = 2\mu_o q k^2 X^{2(2-\gamma)} \ , \tag{6.62}$$

and

$$C^2_{crit.} = 2\mu_o q k^2 X^{2(1-\gamma)} - B_M^2 k^2 X^2 \ . \tag{6.63}$$

Elimination of $C^2_{crit.}$ from the last two equations leads to

$$\frac{B_M{}^2}{2\mu_o q} X^{2\gamma} + X^2 - 1 = 0 \ . \tag{6.64}$$

(a) Solution for $\gamma = 2$.

In terms of m, the number of degrees of freedom over which the energy of compression becomes distributed, the adiabatic index in equation (6.1) is given by equation (1.61).

If the compression is assumed to be two-dimensional, as would seem appropriate for the case of implosion of a cylindrical current sheet onto a collisionless plasma, $\gamma = 2$, and so, from equation (6.62)

$$C^2_{crit.} = 2\mu_o q k^2 \ . \tag{6.65}$$

Thus, for a two-dimensional compression the critical current obtained from equations (6.12) and (6.65), is

$$I_{crit.} = \frac{2\pi}{\mu_o} k(2\mu_o q)^{\frac{1}{2}} . \qquad (6.66)$$

In this case $I_{crit.}$ is therefore simply determined by the initial conditions of the experiment, and is independent of the external quantity B_M. In fact, for $\gamma = 2$, equation (6.64) reduces to

$$\frac{B_M^{\ 2}}{2\mu_o q} X^4 + X^2 - 1 = 0 , \qquad (6.67)$$

with solution for X^2 of form

$$X^2 = \frac{\mu_o q}{B_M^{\ 2}} \{\pm (1 + 2 \frac{B_M^{\ 2}}{\mu_o q})^{\frac{1}{2}} - 1\} . \qquad (6.68)$$

From equations (6.58) and (6.61),

$$q > 0 , \qquad (6.69)$$

and so to avoid X becoming imaginary the positive sign must be taken in equation (6.68). Hence for $\gamma = 2$, X^2 is always real and positive, irrespective of the value of B_M, and the existence of a real solution of equation (6.67) does not impose a condition on B_M.

(b) Solution for $\gamma = 3$.

It is doubtful if the assumption of a two-dimensional compression would be valid in the present geometry. Reflections of plasma particles off the curved imploding current sheet and off the curved end plates would tend to produce distribution of the energy of compression over all three translational degrees of freedom, even in a collisionless plasma. Thus, if a value of 3 is assumed for m, $\gamma = 5/3$ and equation (6.64) becomes

$$\frac{B_M{}^2}{2\mu_o q} X^{10/3} + X^2 - 1 = 0 , \qquad (6.70)$$

which is difficult to solve exactly.

However, since the inequality (6.56) applies it is possible to obtain an approximate solution to the more general form (6.64), as follows. The first term in (6.64) becomes

$$\frac{B_M{}^2}{2\mu_o q} X^{2\gamma} = \frac{B_M{}^2}{2\mu_o p} , \qquad (6.71)$$

by means of equation (6.60), which applies when $X^2 \ll 1$.

Hence, regarding p and w_b as "critical" quantities, equation (6.64) yields

$$\frac{B_M{}^2}{2\mu_o p} = 1 - X^2 = \sin^2 w_b \approx 1 , \qquad (6.72)$$

in view of (6.56). The implication of this result is that B_{bv}, given for the critical current by equation (6.11) as

$$B_{bv}(c,0) = \frac{C_{crit.}}{k \cos w_b} = \frac{C_{crit.}}{k\ X} \qquad (6.73)$$

at the median plane, satisfies, with reference to (6.53), the condition

$$B_{bv}^2(c,0) \ll B_M^2 . \qquad (6.74)$$

That this is a reasonable result under the conditions cited follows from the fact that for such a plasma/magnetic field geometry the stabilizing effect of the external field is quite small because the shape of the discharge surface is not greatly altered from that of a cylindrical discharge, and hence, relatively, the destabilizing effect of the azimuthal self-field must be kept small to avoid instability. This fact is amplified in Section 6.5.

Noting that q does not depend upon w_b, equations (6.71) and (6.72) give the first approximation for w_b as

$$X^{2\gamma} \approx \frac{2\mu_o q}{B_M{}^2} . \tag{6.75}$$

If now the term X^2 is not neglected in equation (6.64), but is expressed in terms of the first approximation (6.75), there results a second approximation to $X^{2\gamma}$, given by

$$X^{2\gamma} \approx \frac{2\mu_o q}{B_M{}^2} \{1 - \left(\frac{2\mu_o q}{B_M{}^2}\right)^{1/\gamma} \} , \tag{6.76}$$

where, from (6.56),

$$\left(\frac{2\mu_o q}{B_M{}^2}\right)^{1/\gamma} \ll 1 . \tag{6.77}$$

Employing the first approximation (6.75), equations (6.12) and (6.62) lead to

$$I_{crit.} \approx \frac{2^{\left(\frac{1 + \gamma}{\gamma}\right)} \pi k (\mu_o q)^{\frac{1}{\gamma}}}{\mu_o \; B_M{}^{\left(\frac{2 - \gamma}{\gamma}\right)}} . \tag{6.78}$$

This expression gives an approximation for the critical current in terms of the initial condition quantities p^o, τ^o, u_e, and the measurable quantity B_M.

When $\gamma = 2$, (6.78) reduces to equation (6.66) as required; and when $\gamma = 5/3$, equation (6.78) becomes

$$I_{crit.} \approx \frac{2^{\frac{8}{5}}\pi k(\mu_o q)^{\frac{3}{5}}}{\mu_o B_M^{\frac{1}{5}}} . \qquad (6.79)$$

6.5 GEOMETRICAL INTERPRETATION OF THE CRITICAL CURRENT

There is a simple interpretation of the result (6.22). From the geometry of the discharge surface (SEYMOUR, 1961) it is known that if the field line direction lies to the left of a surface generator (Fig. 8) at a given point, then the field line curves away from the plasma at that point and should, by Teller's curvature criterion, provide a stabilizing effect. The opposite holds if the field line direction lies to the right of the generator. To determine the field line direction relative to that of a chosen generator, comparison is made between two angles θ and θ_B, defined as follows.

As in Section 3.5, θ is the angle between the straight line generator passing through the point $P(u,v,w_b)$ and the tangent to the curve v = const., w = const. through P; θ_B is defined by the equation

$$\tan\theta_B = \frac{B_{bv}(P)}{B_{bu}(P)} , \qquad (6.80)$$

and is the angle between the direction of the field line at P and the same tangent.

From equations (3.68), (2.2) and (2.3), on the surface $w = w_b$,

$$\tan\theta = \frac{\cos w_b}{(\sin^2 w_b + \sinh^2 u)^{\frac{1}{2}}} \quad . \tag{6.81}$$

For $w = w_b$, equations (6.11) and (6.16) give

$$B_{bv} = \frac{C}{k \cosh u \cos w_b} \ , \tag{6.82}$$

and

$$B_{bu} = \frac{(2\mu_o p k^2 \cosh^2 u \cos^2 w_b - C^2)^{\frac{1}{2}}}{k \cosh u \cos w_b} \ , \tag{6.83}$$

each as a function of the coordinate u.

Thus substitution of the results (6.82) and (6.83) into equation (6.80) gives

$$\tan\theta_B = \frac{B_{bv}}{B_{bu}} = \frac{C}{(2\mu_o p k^2 \cosh^2 u \cos^2 w_b - C^2)^{\frac{1}{2}}} \quad . \tag{6.84}$$

To compare θ and θ_B it is convenient to introduce a quantity Δ, defined as

$$\Delta = \tan^2\theta_B - \tan^2\theta$$

$$= (\tan\theta_B - \tan\theta)(\tan\theta_B + \tan\theta) \ , \tag{6.85}$$

and examine its sign.

Using equations (6.81) and (6.84), equation (6.85) can be manipulated to the form

$$\Delta = \frac{\cosh^2 u (C^2 - 2\mu_o p k^2 \cos^4 w_b)}{(\sinh^2 u + \sin^2 w_b)(2\mu_o p k^2 \cosh^2 u \cos^2 w_b - C^2)} \quad . \tag{6.86}$$

To determine the sign of Δ it is found advantageous to eliminate $(2\mu_o pk^2 \cosh^2 u \cos^2 w_b - C^2)$ from the denominator of (6.86), using (6.83) again. Thus

$$\Delta = \frac{C^2 - 2\mu_o pk^2 \cos^4 w_b}{k^2 B_{bu}^2 \cos^2 w_b (\sinh^2 u + \sin^2 w_b)} \, . \tag{6.87}$$

From this form it is seen that the denominator of Δ is always positive; hence the sign of Δ is completely determined by the sign of the numerator in (6.87). In turn, from equation (6.85) the sign of Δ corresponds to the sign of $\tan\theta_B - \tan\theta$, since, for the geometry under consideration,

$$\theta < \frac{\pi}{2} \, , \quad \theta_B < \frac{\pi}{2} \, ,$$

and so in (6.85)

$$\tan\theta_B + \tan\theta > 0 \, . \tag{6.88}$$

It now follows that

$$\tan\theta_B > \tan\theta \quad \text{if} \quad C^2 > 2\mu_o pk^2 \cos^4 w_b \, . \tag{6.89}$$

With reference to Fig. 8, if (6.89) holds, so that $\theta_B > \theta$, the field line direction lies to the right of the generator, and the field line curves towards the plasma, giving a destabilizing effect. Note that condition (6.89) is independent of position on the surface, so that if it is satisfied the field lines everywhere curve towards the plasma and the entire system is unstable. Further, when the magnetic field resultant lies precisely along a generator, equation (6.89) confirms the existence of a critical current given by expression (6.22), derived by means of the energy principle in Section 6.2. Physically, the transition to instability occurs as a result of the increase in the destabilizing azimuthal field, which swings the direction of the resultant magnetic field across the surface generator into the unstable region. Obviously, if the surface is only slightly constricted, the surface generator angle θ will be small, and instability will set in at a relatively low value of discharge current. Hence the remarks following equation (6.74).

6.6 DISCUSSION OF THE APPROXIMATION

Considering now the critical current for $\gamma = 5/3$ given by the approximation (6.79), it is seen that $I_{crit.}$ decreases with increasing B_M, for given dimensions and initial conditions. At first this result seems paradoxical, since B_M gives a measure of the strength of the stabilizing field. However, for a given initial pressure and volume, increasing B_M must give increased compression of the plasma, leading to a higher value of w_b, and a lower value of $X = \cos w_b$. In fact, in the vicinity of the critical discharge current, equation (6.75) holds approximately, for $X^2 \ll 1$. As X is reduced, the stabilizing effect of the favourable curvature imposed by the external field is reduced. This is seen from expression (6.81) for the tangent of the generator angle θ, evaluated at the median plane,

$$\tan\theta_o = \frac{X}{(1 - X^2)^{\frac{1}{2}}} \approx X \ , \tag{6.90}$$

for $X^2 \ll 1$. As θ_o becomes smaller, it is easier for the azimuthal self-field component to swing the resultant field vector across the direction of the generator. At the same time, the surface value of this destabilizing azimuthal field increases as C increases and X decreases, as is evident from equation (6.82), tending to increase the angle θ_B of the resultant magnetic field, as shown by equation (6.84) when $u = 0$,

$$\tan\theta_{Bo} = \frac{C}{(2\mu_o pk^2X^2 - C^2)^{\frac{1}{2}}}$$

$$= \frac{C}{kB_M X} \ , \tag{6.91}$$

since at $u = 0$ equation (6.83) gives

$$B_{bu} = B_M = \frac{(2\mu_o pk^2X^2 - C^2)^{\frac{1}{2}}}{kX} \ . \tag{6.92}$$

At the critical current, $\tan\theta_o$ of (6.90) becomes equal to $\tan\theta_{Bo}$ of (6.91), so that

$$C_{crit.} = kB_M X^2 \ , \tag{6.93}$$

a result which can also be obtained by eliminating $2\mu_o q$ from equation (6.62) by means of equation (6.75). Equation (6.93) cannot be obtained from (6.63) using the first approximation (6.75) only, since equation (6.63), being of different order in X from (6.62), yields $C_{crit.} = 0$. Of course, using the second approximation (6.76), equations (6.62) and (6.63) consistently yield, to fourth order in X, $C^2_{crit.} = k^2B_M^2X^4$, in agreement with (6.93). To summarise, in planning an experiment here, one may fix the initial quantities, p^o, τ^o, k and u_e, thus determining F_e of (6.58) and q of (6.61), and then, depending on the compressibility of the plasma (a measure of which is γ of (1.61)), it is of interest to consider how the final value of X attained depends on the initial choice of B_M. It is clear from the relationship (6.93) that, for given k, $C_{crit.}$ varies linearly with the product B_MX^2. In turn, by writing equation (6.75) in the form

$$B_MX^2 = \frac{(2\mu_o q)^{\frac{1}{\gamma}}}{B_M^{\frac{2-\gamma}{\gamma}}}, \qquad (6.94)$$

it is seen that the γ-dependence of B_MX^2 is as follows:

(i) For m = 1 in (1.61), $\gamma = 3$ and B_MX^2 increases with B_M.

(ii) For m = 2, $\gamma = 2$ and B_MX^2 is independent of B_M.

(iii) For m = 3, $\gamma = 5/3$ and B_MX^2 decreases as B_M increases.

From (i), (ii), (iii) and (6.93) it follows that for an assumed one degree of freedom $C_{crit.}$ increases with B_M; for two degrees of freedom $C_{crit.}$ is not affected by B_M; and for three degrees of freedom $C_{crit.}$ decreases as B_M is increased.

The case (ii) is somewhat curious, and it might be concluded that since $C_{crit.}$ is independent of B_M, the choice of B_M is in no way restricted. But it must be recalled that the degree of radial constriction of the discharge has been assumed small, so that (6.56) applies. Correspondingly, the inequality

$$(2\mu_o q)^{\frac{1}{2}} \ll B_M , \qquad (6.95)$$

obtained from (6.77) for $\gamma = 2$, must be satisfied in case (ii).

Equation (6.78) may be used to determine how $I_{crit.}$ depends on k, the distance away from the system symmetry axis of the common foci of the u and w coordinate surfaces. k may be usefully regarded as a system scaling factor. The initial volume of the discharge would scale as k^3, and hence q of equation (6.61) would, with the assistance of the expression (6.58) for F_e, scale as k^o, i.e. q remains constant if p^o does not scale with k, but is regarded as an independent parameter. Therefore equation (6.78) shows that $I_{crit.}$ would scale as k for a given value of B_M. This conclusion reflects the fact that the destabilizing azimuthal magnetic field $B_{bv}(c,0)$ of equation (6.73) produced by $I_{crit.}$ scales as k^{-1}. If k is varied, equation (6.90) shows that the tangent of the generator angle at the median plane is not altered, and hence, for the critical current condition, $\tan\theta_{Bo}$ must not change with k. But $\tan\theta_{Bo}$ is, from equation (6.84), simply $B_{bv}(c,0)/B_M$. Thus, for an allocated value of B_M, $B_{bv}(c,0)$, given by equation (6.73), must not change. This condition is met if $C_{crit.}$ varies as k, as was deduced from equation (6.78).

6.7 COMPARISON WITH THE RESULT FOR THE DIFFUSE DISCHARGE

Finally, for a given stabilizing field B_M, a comparison may be made between the critical current given by equation (6.22), and that obtained earlier in Chapter 3 for the diffuse discharge. From equation (3.37) and the defined parameter $A = B_M k^2 \sin w_b$, the critical current for the diffuse discharge can be written

$$\frac{\mu_o}{2\pi} I^*_{crit.} = C^*_{crit.} = \frac{kB_M \cos^2 w_b \sin w_b}{\sin^2 w_b + \sinh^2 u_e} , \qquad (6.96)$$

whereas here, for the sheet-current discharge, equation (6.22) becomes, by means of equation (6.72),

$$C_{crit.} = \frac{kB_M \cos^2 w_b}{\sin w_b} . \qquad (6.97)$$

Comparison of equations (6.96) and (6.97) shows that

$$C^*_{crit.} < C_{crit.} . \qquad (6.98)$$

It is therefore concluded that the diffuse discharge would be unstable for a lower discharge current than would the field-free discharge. Also, the presence of the internal field in the diffuse discharge would result in a loss of compression of the plasma, compared with the field-free case.

6.8 REFERENCES

BERNSTEIN, I. B., FRIEMAN, E. A., KRUSKAL, M. D., and KULSRUD, R. M. (1958) - Proc. Roy. Soc. A224, 17.

SCHMIDT, G. (1966) - "Physics of High Temperature Plasmas". (Academic Press: New York.).

SEYMOUR, P. W. (1961) - Aust. J. Phys. 14, 321.

TAYLER, R. J. (1957) - Proc. Phys. Soc. (London) B70, 1049.

CONCLUDING REMARKS

A detailed summary of the material presented in this thesis has already been given in the Introduction. To complete the discussion at this point, the following comments are made.

Some of the analysis (e.g. Sections 2.3 and 6.3) has been carried out in the context of a general orthogonal curvilinear coordinate system (u,v,w), with the assumption of symmetry with respect to one coordinate (v) only. The results obtained therefore have a much wider application than that employed here. Also, general conclusions can be drawn from the detailed discussion of Chapter 3, concerning the dangers in neglecting the effect of geodesic curvature when considering destabilizing mechanisms arising from field-line curvature in magnetostatic systems with twisted field lines.

Specialized results have been obtained by application of the general theory to a specific geometry of constricted discharge. For a numerical illustration, consider the field-free discharge (Chapter 6) characterized by values of w_b close to $\frac{\pi}{2}$. Assuming a stabilizing field B_M of 1.0 weber /metre^2, a scaling parameter k of order 1.0 metre, and a value of 0.1 for cos w_b, expressions (6.97) and (6.12) yield $I_{crit.} \sim 10^5$ amps. Bearing in mind the limitations of the model used here, and assuming that the remaining formidable technological difficulties can be overcome, theory therefore predicts that it should be possible to pass currents of a magnitude sufficient to produce a considerable pinching effect, without encountering the severe hydromagnetic instabilities common in cylindrical pinches.

In this thesis, the specific aims outlined in the Introduction have been achieved. However, experience gained in the course of the work indicates that there exist further difficult but worth-while problems for future solution in this interesting geometry, which could be highly relevant to recent experiments with non-cylindrical z-pinches and dense plasma foci. For example, a stability analysis of the curvilinear analogue of the cylindrical system treated theoretically by SUYDAM (1958) and NEWCOMB (1960) could be a project of considerable importance. Particular difficulties will arise here, for two reasons. Firstly, the functions appropriate to this coordinate system are the oblate spheroidal wave functions, which are not easy to handle in general. A second problem arises because magnetostatic equilibrium is necessary for application of the hydromagnetic energy principle (SIMON, 1959), whereas it has been shown here in Section 2.2 that the magnetostatic equations and Maxwell's equations are not satisfied by the configuration in which the magnetic and electric-current surfaces are hyperboloids of one sheet on which the plasma pressure (comparable in magnitude to the magnetic pressure) is constant. The necessity for such equilibrium has apparently been ignored by COMISAR (1969) in his analysis of the dense plasma focus, although there seems to be excellent agreement of his results with those of experiment. In the absence of a static equilibrium, it may be possible to study this geometry as a dynamical stability problem, using normal mode analysis and numerical methods.

Finally, it should be recalled that the ideal hydromagnetic stability theory has severe limitations in its application to plasmas, being based on a somewhat unrealistic model which neglects the effects of transport processes, and of factors such as the finite gyroradius of the plasma particles. The inclusion of such plasma properties greatly enlarges the number of possible instabilities (LEHNERT, 1967). Of particular importance is the fact that finite particle mass, the Hall effect, pressure gradients, electric fields parallel to the magnetic field, as well as finite resistivity, remove the constraint of frozen-in fields. This changes the character of possible motions and leads to a range of resistive instabilities, most of which have been detected in linear pinches (LEHNERT, 1967), and should be relevant in the constricted discharge. Considering the temperature distributions derived by SEYMOUR (1961), there may be instabilities associated with finite heat conductivity along the magnetic field (GALEEV et al, 1963). In addition, of course, numerous microinstabilities could be important, particularly the 'universal' instabilities driven by inhomogeneities in plasma temperature and density.

REFERENCES

COMISAR, G. G. (1969) - Phys. Fl. 12, 1000.

GALEEV, A. A., ORAEVSKII, V. N., and SAGDEEV, R. Z. (1963) - Soviet Phys. JETP 17, 615.

LEHNERT, B. (1967) - Plasma Physics 9, 301.

NEWCOMB, W. (1960) - Ann. Phys. 10, 232.

SEYMOUR, P. W. (1961) - Aust. J. Phys. 14, 129; ibid, 279; ibid, 321.

SIMON, A. (1959) - "An Introduction to Thermonuclear Research", p.148. (Pergamon Press: London).

SUYDAM, B. R. (1958) - Proc. 2nd U.N. Conf. on the Peaceful Uses of Atomic Energy 31, 157.

APPENDIX I

As mentioned in Section 3.3, it is possible to evaluate the variation $\delta \int d\ell/B$ by commuting δ and the $\int$ sign, and carrying out the resulting integration. Equation (1.29) is

$$\delta(d\ell/B) = -\, 2d\ell(\delta u\partial B/\partial u + \delta w\partial B/\partial w)/B^2\,, \qquad (1)$$

where $h_u\delta u$ and $h_w\delta w$ are components of the elemental vector $\underline{a}$ which defines the variation such that $\underline{a}\cdot\underline{B} = 0$. It is also known that

$$\delta u = -\,\delta v_o/\Gamma - \beta\delta w \qquad (2)$$

(equation (3.19) et. seq.).

From equation (3.54),

$$(1/B)\partial B/\partial u = \frac{-\,\sinh u}{\cosh u(A^2\cos^2 w + S^2k^2C^2)}\left[\frac{A^2\cos^2 w\; S^2 + A^2\cosh^2 u\,\cos^2 w}{S^2} + S^2k^2C^2\right], \qquad (3)$$

where $S^2 = \sin^2 w + \sinh^2 u$.

Combining (2) and (3) gives

$$(\delta u/B)\partial B/\partial u = \frac{2k^2C^2\sinh^2 u\,\sin w\;\delta w}{\cos w(A^2\cos^2 w + S^2k^2C^2)}\left[1 + \frac{A^2\cosh^2 u\,\cos^2 w}{S^2(A^2\cos^2 w + S^2k^2C^2)}\right] - \frac{\delta v_o}{B\Gamma}\frac{\partial B}{\partial u}\,. \qquad (4)$$

As discussed in Section 3.4, the term in δv_o gives no net contribution to the integral from $-\,u_e$ to u_e, and may therefore be omitted.

A treatment similar to the above gives

$$(\delta w/B)\,\partial B/\partial w = \frac{\sin w\ \delta w(S^4k^2C^2 - A^2\cos^4 w)}{S^2\cos^2 w(A^2\cos^2 w + S^2k^2C^2)} \ . \tag{5}$$

The complete integrand is, using equations (3.4), (1), (4) and (5),

$$-\frac{2k^3}{A}\tan w\ \cosh u\ \delta w du \left[\frac{S^4k^2C^2 - A^2\cos^4 w + 2S^2k^2C^2\sinh^2 u}{A^2\cos^2 w + S^2k^2C^2}\right.$$

$$\left. + \frac{2A^2k^2C^2\ \sinh^2 u\ \cosh^2 u\ \cos^2 w}{(A^2\cos^2 w + S^2k^2C^2)^2}\right] \ . \tag{6}$$

Defining M and N as:

$$M = C^2k^2S^4 - A^2\cos^4 w \ ,$$

$$(dM)_w = 4C^2k^2\sinh u\ \cosh u\ S^2 du \ ;$$

$$N = A^2\cos^2 w + C^2k^2S^2 \ ,$$

$$(dN)_w = 2C^2k^2\sinh u\ \cosh u\ du \ ,$$

lengthy manipulation will reduce the integrand, on the surface $w = w_b$, to

$$-\frac{2k^3}{A}\tan w_b\ \delta w d\left(\frac{M\sinh u}{N}\right) \ .$$

The integral is therefore

$$-2\frac{k^3}{A}\tan w_b\ \delta w\int_{-u_e}^{u_e} d\left(\frac{M\sinh u}{N}\right)$$

$$= -4\frac{k^3}{A}\tan w_b\ \delta w\frac{M(u_e,w_b)\sinh u_e}{N(u_e,w_b)} \ .$$

The critical current is obtained by equating this expression to zero and solving for $C_{crit.}$. This operation reduces to solving

$$M(u_e, w_b) = 0 \ ,$$

yielding

$$C_{crit.} = A \cos^2 w_b / k(\sin^2 w_b + \sinh^2 u_e) \qquad (7)$$

which is the same as equation (3.37).

APPENDIX II

While equation (4.44) of Section 4.3 may be regarded as an intuitively obvious result, in the interest of rigour an analytical derivation is provided here.

Generally, if g(t) is the time-dependent function given by

$$g(t) = \int_{\tau(t)} f(\underline{r},t)d\tau \quad , \qquad (1)$$

where $\tau(t)$ is a simple closed volume, bounded by the surface S(t), the time derivative dg/dt may be obtained by employing orthogonal curvilinear coordinates to express g(t) as a triple integral possessing time-dependent limits of integration. However, the proof is greatly simplified by choosing, in particular, spherical polar coordinates. This choice occasions no loss of generality, since all points on any closed surface S(t) may be described in terms of coordinates r, θ,ϕ having their origin within S(t), by the equation

$$r_1 = r_1(\theta,\phi,t) \ . \qquad (2)$$

For simplicity it is assumed that r_1 is a single-valued function of θ and ϕ. In terms of the elementary volume so defined, any arbitrary simply-connected volume may then be treated by summing a number of such elementary volumes.

In terms of r, θ and φ, g(t) becomes the triple integral

$$g(t) = \int_\tau f(r,\theta,\phi,t)d\tau$$

$$= \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{r_1(\theta,\phi,t)} f(r,\theta,\phi,t)r^2 \sin\theta \, dr$$

$$= \int_0^\pi d\theta \int_0^{2\pi} d\phi \, F(\theta,\phi,t) \ , \qquad (3)$$

where

$$F(\theta,\phi,t) = \int_0^{r_1(\theta,\phi,t)} f(r,\theta,\phi,t)r^2 \sin\theta \, dr \ . \qquad (4)$$

Observing that the limits of integration with respect to θ and φ are constants,

$$\frac{dg}{dt} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \, \frac{\partial}{\partial t} F(\theta,\phi,t) \ . \qquad (5)$$

From (4) it is seen that F(θ,φ,t) is a one-dimensional integral whose upper limit depends on the three coordinates θ, φ and t. From (5) it is seen that the partial derivative of F with respect to t is required here, θ and φ being held constant. The theory of differentiation of a definite integral depending on a parameter, say x, and having limits also dependent on x is therefore applicable, the required formula being (see, for example, HILDEBRAND, 1963),

$$\frac{d}{dx} \int_{A(x)}^{B(x)} p(x,s)ds = \int_{A(x)}^{B(x)} \frac{\partial}{\partial x} p(x,s)ds + p(x,B) \frac{dB}{dx} - p(x,A) \frac{dA}{dx} \ . \qquad (6)$$

Therefore, since θ and ϕ are held constant, application of (6) to (4) gives

$$\frac{\partial}{\partial t} F(\theta,\phi,t) = \frac{\partial}{\partial t} \int_0^{r_1(\theta,\phi,t)} f(r,\theta,\phi,t) r^2 \sin\theta \, dr$$

$$= \int_0^{r_1(\theta,\phi,t)} \frac{\partial}{\partial t} \left[f(r,\theta,\phi,t) r^2 \sin\theta \right] dr + f(r_1,\theta,\phi,t) r_1^2 \sin\theta \times \frac{\partial}{\partial t} r_1(\theta,\phi,t) \,, \qquad (7)$$

since A(x) of equation (6) is zero here.

Substitution of (7) into (5) gives

$$\frac{dg}{dt} = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{r_1(\theta,\phi,t)} \frac{\partial}{\partial t} \left[f(r,\theta,\phi,t) r^2 \sin\theta \right] dr$$

$$+ \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \; f(r_1,\theta,\phi,t) r_1^2 \sin\theta \, \frac{\partial}{\partial t} r_1(\theta,\phi,t)$$

$$= \int_{\tau(t)} \frac{\partial}{\partial t} f(\underline{r},t) d\tau + \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \; f(r_1,\theta,\phi,t) r_1^2 \sin\theta \times \frac{\partial}{\partial t} r_1(\theta,\phi,t) \,. \qquad (8)$$

Now r_1, θ, ϕ and t are the coordinates of a point P on the surface S(t) at time t. As time passes the surface deforms, and the velocity of P is given by

$$\underline{v}_p = \frac{dr_1}{dt} \underline{e}_r + r_1 \frac{d\theta}{dt} \underline{e}_\theta + r_1 \sin\theta \frac{d\phi}{dt} \underline{e}_\phi \,. \qquad (9)$$

In the particular curvilinear coordinate system chosen here the vector element of surface area $d\underline{S}$ on the surface described by (2) is readily determined as follows. The elements of length associated

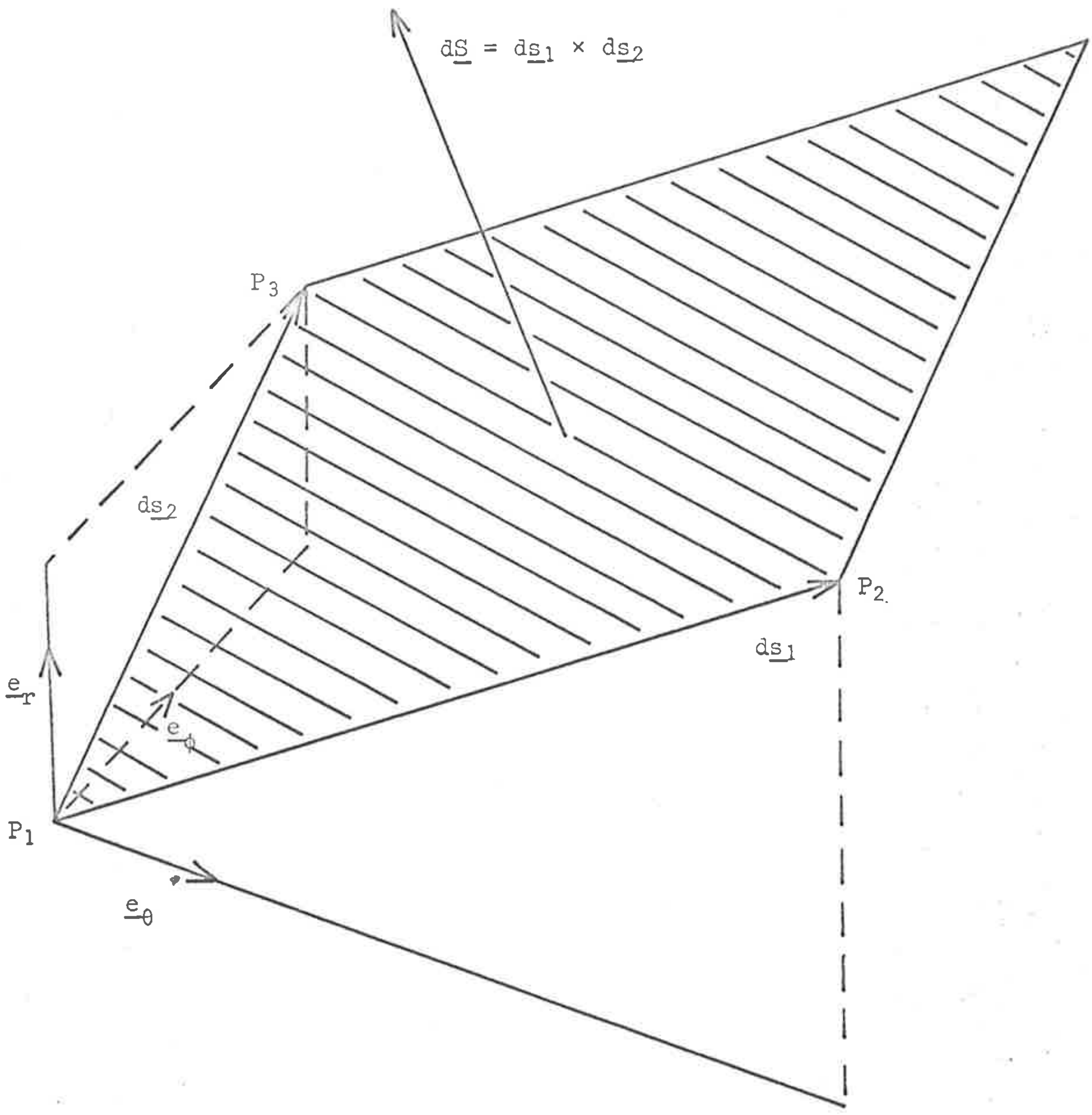

Fig. 10. An element of surface area on $S(t)$ is defined by the surface points P_1, P_2, P_3, chosen so that $d\underline{s}_1 = \overline{P_1P_2}$ is orthogonal to $\underline{e}_\phi$, and $d\underline{s}_2 = \underline{P_1P_3}$ is orthognal to $\underline{e}_\theta$.

with dr, dθ and dϕ are, in terms of the scale factors h_r, h_θ, h_ϕ, respectively

$$\left.\begin{aligned} h_r\, dr &= dr \;, \\ h_\theta\, d\theta &= r d\theta \;, \\ h_\phi\, d\phi &= r \sin\theta\, d\phi \;, \end{aligned}\right\} \qquad (10)$$

since $h_r = 1$, $h_\theta = r$ and $h_\phi = r \sin\theta$ in the spherical polar co-ordinate system chosen.

From Fig. 10, if P_1, P_2 and P_3 are three closely associated points on the surface $\underline{r_1} = \underline{r_1}\ (\theta,\phi,t)$, a vector surface element $d\underline{S}$ normal to this surface as shown is given by

$$d\underline{S} = d\underline{s_1} \times d\underline{s_2} \;, \qquad (11)$$

where $d\underline{s_1}$, with tail at P_1 and tip at P_2, has no ϕ-component, and $d\underline{s_2}$, with tail also at P_1 and tip at P_3, has no θ-component. In other words, $d\underline{s_1}$, lying in the tangent plane to the coordinate surface ϕ = constant at P_1, has components $r_1\, d\theta\, \underline{e}_\theta$ and, because r_1 depends on θ alone under these conditions,

$$\frac{\partial r_1}{\partial \theta}\, d\theta\, \underline{e}_r \;,$$

i.e.

$$d\underline{s_1} = r_1\, d\theta\, \underline{e}_\theta + \frac{\partial r_1}{\partial \theta}\, d\theta\, \underline{e}_r \;. \qquad (12)$$

Similarly, for the coordinate surface θ = const. and associated tangent plane at P_1, $d\underline{s_2}$ has the form

$$d\underline{s_2} = r_1 \sin\theta\, d\phi\, \underline{e}_\phi + \frac{\partial r_1}{\partial \phi}\, d\phi\, \underline{e}_r \;. \qquad (13)$$

Hence, from (11), (12) and (13), $d\underline{S}$ normal to the surface $r_1 = r_1(\theta,\phi,t)$ becomes

$$d\underline{S} = r_1^2 \sin\theta \, d\theta \, d\phi \, \underline{e}_r - r_1 \frac{\partial r_1}{\partial \theta} \sin\theta \, d\theta \, d\phi \, \underline{e}_\theta - r_1 \frac{\partial r_1}{\partial \phi} d\theta \, d\phi \, \underline{e}_\phi , \qquad (14)$$

since

$$\underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r , \; \underline{e}_r \times \underline{e}_\phi = -\underline{e}_\theta , \; \underline{e}_\theta \times \underline{e}_r = -\underline{e}_\phi .$$

From (9) and (14),

$$\underline{v}_p \cdot d\underline{S} = r_1^2 \sin\theta \, d\theta \, d\phi \left(\frac{dr_1}{dt} - \frac{\partial r_1}{\partial \theta}\frac{d\theta}{dt} - \frac{\partial r_1}{\partial \phi}\frac{d\phi}{dt}\right)$$

$$= r_1^2 \sin\theta \, d\theta \, d\phi \, \frac{\partial r_1}{\partial t} , \qquad (15)$$

by use of the chain rule for differentiation of dr_1/dt .

In the light of the result (15), the double integral in equation (8) may be written

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \; f(r_1,\theta,\phi,t) r_1^2 \sin\theta \, \frac{\partial r_1}{\partial t} = \int_{S(t)} f(\underline{r}_p,t)\underline{v}_p \cdot d\underline{S} , \qquad (16)$$

and so (8) becomes finally

$$\frac{dg}{dt} = \int_\tau \frac{\partial}{\partial t} f(\underline{r},t) d\tau + \int_{S(t)} f(\underline{r},t)\underline{v} \cdot d\underline{S} , \qquad (17)$$

a result which rigorously establishes equation (4.44) of Section 4.3.

REFERENCE:

HILDEBRAND, F. B. (1963) - "Advanced Calculus for Applications". pp.359-60. (Prentice-Hall, Inc.: New Jersey).

APPENDIX III

As mentioned in Section 4.3, it is possible to derive the result (4.52) by the following method, which involves the assumption of a form of equation of state for the plasma.

If W represents the total potential energy of the system and $\underline{F}$ is the force per unit volume acting in the fluid, then

$$\frac{dW}{dt} = - \int_{\tau_p(t)} \underline{v} \circ \underline{F} \, d\tau \ , \qquad (1)$$

where $\underline{v}$ is the perturbation fluid velocity. Neglecting gravitational forces,

$$\frac{dW}{dt} = - \int_{\tau_p(t)} d\tau \ \underline{v} \circ (\underline{j} \times \underline{B} - \nabla p)$$

$$= - \int_{\tau_p(t)} d\tau \ \underline{v} \circ \left[\frac{1}{\mu_o} \underline{B} \cdot \nabla \underline{B} - \nabla \left(p + \frac{B^2}{2\mu_o} \right) \right] \ , \qquad (2)$$

by use of equation (4.11) and a standard vector identity. Application of Gauss' theorem gives

$$\frac{dW}{dt} = - \int_{\tau_p(t)} d\tau \ \left\{ \frac{1}{\mu_o} \underline{v} \cdot \underline{B} \cdot \nabla \underline{B} + \left(p + \frac{B^2}{2\mu_o} \right) \nabla \circ \underline{v} \right\}$$

$$+ \int_{S_p(t)} d\underline{S} \cdot \underline{v} \ \left(p + \frac{B^2}{2\mu_o} \right) \ , \qquad (3)$$

where $d\underline{S}$ is directed out of the plasma.

Now

$$W = W_p + W_{Bp} + W_{BE}$$

$$= W_i + W_{BE} , \qquad (4)$$

where W_p is the potential energy ascribable to the plasma material and W_{Bp} is the energy associated with the magnetic field imbedded in the plasma. It is assumed, following VAN KAMPEN and FELDERHOF (1967) p.19, that W_p may be written

$$W_p = \int_{\tau_p(t)} \rho\psi d\tau , \qquad (5)$$

where ψ is the compression energy per unit mass, given by

$$\psi = - \int_{\rho_o}^{\rho} \frac{p}{\rho'^2} d\rho' . \qquad (6)$$

The equation of state of the plasma is assumed to be of the form

$$p = p(\rho, T)$$

$$= p(\rho(t), T(t)) , \qquad (7)$$

where T is the temperature. Hence ψ may be written as a function of time:

$$\psi(t) = - \int_{o}^{t} \frac{p}{\rho^2} \frac{d\rho}{dt'} dt' , \qquad (8)$$

so that

$$\frac{d\psi}{dt} = - \frac{p}{\rho^2} \frac{d\rho}{dt} . \qquad (9)$$

Then

$$W_i = \int_{\tau_p(t)} \left(\rho\psi + \frac{B^2}{2\mu_o}\right) d\tau$$

and

$$\frac{dW_i}{dt} = \int_{\tau_p(t)} \left\{\frac{d\psi}{dt}\,\rho d\tau + \psi\,\frac{d}{dt}\,(\rho d\tau) + \frac{d}{dt}\left(\frac{B^2}{2\mu_o}\right)\,d\tau + \frac{B^2}{2\mu_o}\,\frac{d}{dt}\,(d\tau)\right\} . \qquad (10)$$

Here, $\frac{d}{dt}$ is the total derivative $\left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla\right)$, measured by an observer moving with the fluid. As the flow develops, a given fluid element $d\tau$ will deform, but will retain its identity, being always composed of the same particles. Its mass, $\rho d\tau$, is then a constant of its motion:

$$\frac{d}{dt}\,(\rho d\tau) = 0 . \qquad (11)$$

Equation (10) then becomes

$$\frac{dW_i}{dt} = \int_{\tau_p(t)} \left\{ \frac{d\psi}{dt}\,d\tau + \frac{d}{dt}\left(\frac{B^2}{2\mu_o}\right)d\tau + \frac{B^2}{2\mu_o}\,\frac{d}{dt}\,(d\tau)\right\} . \qquad (12)$$

Expanding equation (11), and using the equation of conservation of matter, (4.7), the standard hydrodynamic result is obtained

$$\frac{d}{dt}\,(d\tau) = (\nabla\cdot\underline{v})d\tau . \qquad (13)$$

Also, by use of Maxwell's equations and the high electrical conductivity approximation (4.8),

$$\frac{d\underline{B}}{dt} = \frac{\partial \underline{B}}{\partial t} + \underline{v}\cdot\nabla\underline{B}$$

$$= \nabla \times (\underline{v} \times \underline{B}) + \underline{v}\cdot\nabla\underline{B}$$

$$= \underline{B}\cdot\nabla\underline{v} - \underline{B}\nabla\cdot\underline{v} \ . \qquad (14)$$

Therefore

$$\frac{d}{dt}\left(\frac{B^2}{2\mu_o}\right) = \frac{\underline{B}}{\mu_o}\cdot\frac{d\underline{B}}{dt}$$

$$= \frac{1}{\mu_o}\underline{B}\cdot\underline{B}\cdot\nabla\underline{v} - \frac{B^2}{\mu_o}\nabla\cdot\underline{v} \ . \qquad (15)$$

Using equations (9), (13) and (15), equation (12) may be written

$$\frac{dW_i}{dt} = \int_{\tau_p(t)} d\tau \ \left\{\frac{1}{\mu_o}\underline{B}\cdot\underline{B}\cdot\nabla\underline{v} - \frac{B^2}{2\mu_o}\nabla\cdot\underline{v} - p\nabla\cdot\underline{v}\right\} \ , \qquad (16)$$

where equation (4.7) has been used also.

As shown in Section 5.4(a),

$$\int_{\tau_p(t)} d\tau \ \underline{B}\cdot\underline{B}\cdot\nabla\underline{v} = -\int_{\tau_p(t)} d\tau \ \underline{v}\cdot\underline{B}\cdot\nabla\underline{B} \ , \qquad (17)$$

so that, finally,

$$\frac{dW_i}{dt} = -\int_{\tau_p(t)} d\tau \ \left\{\frac{1}{\mu_o}\underline{v}\cdot\underline{B}\cdot\nabla\underline{B} + \left(p + \frac{B^2}{2\mu_o}\right)\nabla\cdot\underline{v}\right\} \ . \qquad (18)$$

Comparing equations (3) and (18), it is seen that

$$\frac{dW}{dt} = \frac{dW_i}{dt} + \int_{S_p(t)} d\underline{S}\cdot\underline{v}\left(p + \frac{B^2}{2\mu_o}\right) \quad \ldots.$$

$$= \frac{dW_i}{dt} + \int_{S_{pv}(t)} d\underline{S}\cdot\underline{v} \left(p + \frac{B^2}{2\mu_o}\right) , \qquad (19)$$

since $d\underline{S}\cdot\underline{v}$ is zero on $S_{pc}(t)$. But equation (4) gives

$$\frac{dW}{dt} = \frac{dW_i}{dt} + \frac{dW_{BE}}{dt}$$

and so

$$\frac{dW_{BE}}{dt} = \int_{S_{pv}(t)} d\underline{S}\cdot\underline{v} \left(p + \frac{B^2}{2\mu_o}\right)$$

$$= -\frac{1}{2\mu_o} \int_{S_{pv}(t)} d\underline{S}'\cdot\underline{v}\ \hat{B}^2 ,$$

in agreement with (4.52), where $d\underline{S}' = -d\underline{S}$, and the pressure balance equation (4.13) has been used.

REFERENCE:

KAMPEN, VAN N. G., and FELDERHOF, B. U. (1967) - "Theoretical Methods in Plasma Physics". (North-Holland Publishing Co.: Amsterdam).

BIBLIOGRAPHY

Following is a list of articles and books which have been consulted during the course of the work, but to which specific reference has not been necessary in the text of the preceding chapters.

ABRAMOWITZ, M., and STEGUN, I. A. (1965) - "Handbook of Mathematical Functions". (National Bureau of Standards).

AKHEIZER, A. I., AKHEIZER, I. A., POLOVIN, R. V., SITENKO, A. G., and STEPANOV, K. N. (1967) - "Collective Oscillations in a Plasma". (Pergamon Press: Oxford).

ALLEN, C. D., SENIOR, G. A., and ZOLLERS, S. M. (1961) - R-F Techniques for Controlled Thermonuclear Fusion, Electronics 34, (October), 50.

ALLIS, W. P. (1960) - "Nuclear Fusion". (Van Nostrand: Princeton).

ANICIN, B. A. (1963) - On the motion of a charged particle under the action of rectangular pulses of a magnetic field. J. Nucl. Energy Part C 5, 341.

ARTSIMOVICH, L. A. (1964) - "Controlled Thermonuclear Reactions". (Oliver and Boyd: Edinburgh and London).

BERSHADER, D., ed. (1962) - "Plasma Hydromagnetics". (Stanford University Press: California).

BERTOTTI, B., and NOCENTINI, A. (1970) - Collisionless drift instabilities in a bounded plasma. Plasma Physics 12, 39.

BOLOTIN, V. V. (1964) - "The Dynamical Stability of Elastic Systems". (Holden-Day Inc.: San Francisco).

BOYD, T. J. M., and SANDERSON, J. J. (1969) - "Plasma Dynamics". (Nelson: London).

BRAND, L. (1947) - "Vector and Tensor Analysis". (J. Wiley & Sons: New York).

BRANDSTATTER, J. J. (1963) - "An Introduction to Waves, Rays and Radiation in Plasma Media". (McGraw-Hill: New York).

BUNEMAN, O., and DUNN, D. A. (1966) - Computer experiments in plasma physics. Science J. 2, 34.

CAMBEL, A. B. (1963) - "Plasma Physics and Magnetofluid Mechanics". (McGraw-Hill: New York).

CHANDRASEKHAR, S. (1953) - The stability of viscous flow between rotating cylinders in the presence of a magnetic field. Proc. Roy. Soc. A216, 293.

CHANDRASEKHAR, S. (1961) - "Hydrodynamic and Hydromagnetic Stability". (Clarendon: Oxford).

CHANDRASEKHAR, S., KAUFMAN, A. N., and WATSON, K. M. (1958) - The stability of the pinch. Proc. Roy. Soc. A245, 435.

CHANDRASEKHAR, S., and TREHAN, S. K. (1960) - "Plasma Physics". (University of Chicago Press).

CHAPMAN, S., and COWLING, T. G. (1970) - "The Mathematical Theory of Non-Uniform Gases", 3rd edition. (University Press: Cambridge).

CLAUSER, F. H., ed. (1960) - "Symposium of Plasma Dynamics". (Addison-Wesley: Massachusetts).

COBINE, J. D. (1958) - "Gaseous Conductors". (Dover Publications: New York).

COMBES, L. S., GALLAGHER, C. C., and LEVINE, M. A. (1962) - Plasma behaviour in cusp-mirror magnetic field. Phys. Fl. 5, 1070.

COPPI, B. (1966) - Geodesic curvature in closed plasma configurations. Nucl. Fus. 6, 293.

COURANT, R., and HILBERT, D. (1953) - "Methods of Mathematical Physics". (Interscience Publishers: New York).

COWLING, T. G. (1957) - "Magnetohydrodynamics". (Interscience Publishers: New York and London).

CURRY, D., and NEWMAN, B. R. (1960) - "The Challenge of Fusion". (Van Nostrand: Princeton).

DELCROIX, J. L. (1960) - "Introduction to the Theory of Ionized Gases". (Interscience Publishers: New York and London).

DELCROIX, J. L. (1965) - "Plasma Physics". (J. Wiley & Sons: London).

DWIGHT, H. B. (1957) - "Tables of Integrals and other Mathematical Data". (The MacMillan Co.: New York).

ERDÉLYI, A. ed. (1953) - "Higher Transcendental Functions" - Bateman Manuscript Project. (McGraw Hill: New York).

ESKINAZI, S. (1967) - "Vector Mechanics of Fluids and Magnetofluids". (Academic Press: New York and London).

FOWLER, T. K. (1963) - Lyapunov's stability criterion for plasmas. J. Math. Phys. 4, 559.

GILINSKY, V., and HUBBARD, H. (1968) - Economics of power from pulsed fusion reactors. Nucl. Fus. 8, 69.

GLASSTONE, S., and LOVBERG, R. H. (1960) - "Controlled Thermonuclear Reactions". (Van Nostrand: New York).

GOLDSTEIN, H. (1950) - "Classical Mechanics". (Addison-Wesley: Massachusetts).

GREEN, H. S. (1959) - Ionic theory of plasmas and magnetohydrodynamics. Phys. Fl. 2, 341.

GREEN, H. S. (1961) - Theories of transport in fluids. J. Math. Phys. 2, 344.

GREEN, H. S., and LEIPNIK, R. B. (1965) - Diffusion and conductivity of plasmas in strong external fields. Int. J. Engng. Sci. 3, 491.

GREEN, H. S., and LEIPNIK, R. B. (1970) - "Sources of Plasma Physics". Vol. 1. (Wolters-Noordhoff Pub. Co.: Groningen).

GREEN, T. S. (1963) - "Thermonuclear Power". (George Newnes Ltd.: London).

GUEST, G. E., and BEAZLEY, C. O. Jr., (1966) - Cold-plasma effect in finite length plasmas. Phys. Fl. 9, 1798.

HERCUS, E. O. (1950) - "Elements of Thermodynamics and Statistical Mechanics". (University Press: Melbourne).

HIRSCHFELDER, J. O., CURTISS, C. F., and BIRD, R. B. (1954) - "Molecular Theory of Gases and Liquids". (J. Wiley & Sons: New York).

HOLT, E. H., and HASKELL, R. E. (1965) - "Plasma Dynamics". (The MacMillan Co.: New York).

HOSKING, R. J. (1965) - New instabilities due to Hall effect. Phys. Rev. Lett. 15, 344.

HOSKING, R. J. (1967) - Stability of a self-gravitating plasma cylinder including Hall effect. Phys. Fl. 10, 588.

HOSKING, R. J. (1967) - Incompressible 'tearing' instability in the hard-core pinch. Phys. Fl. 10, 1654.

HOSKING, R. J. (1968) - Rayleigh-Taylor problem for a Hall plasma. J. Plasma Phys. 2, 613.

HOSKING, R. J., and BLACKWELL, J. H. (1966) - Axisymmetric resistive instabilities in the hard-core pinch. Canad. J. Phys. 44, 1037.

IAEA, Vienna (1966) - "Plasma Physics and Controlled Nuclear Fusion Research", Vols. I and II. (Proceedings of Culham Conference).

IKUTA, K. (1966) - A stabilized magnetic mirror field. J. Phys. Soc. Japan 21, 820.

JAHNKE-EMDE-LOSCHE (1960) - "Tables of Higher Functions". 6th ed. (McGraw-Hill: New York).

JANCEL, R., and KAHAN, Th. (1966) - "Electrodynamics of Plasmas". (J. Wiley & Sons: New York).

JEANS, J. H. (1954) - "The Dynamical Theory of Gases". (Dover: New York).

JEFFREY, A., and TANIUTI, T. (1966) - "Magnetohydrodynamic Stability and Thermonuclear Containment" - a collection of reprints. (Academic Press: New York and London).

JOHNSON, J. L. (1968) - Use of two-dimensional slab models in hydromagnetic stability theory. Phys. Fl. 10, 1519.

JOOS, G. (1958) - "Theoretical Physics". (Blackie & Son: London).

JUKES, J. D. (1959) - "Man-Made Sun. The Story of Zeta". (Abelard-Schuman: London and New York).

KADOMTSEV, B. B. (1965) - "Plasma Turbulence". (Academic Press: London and New York).

KASH, S. W., ed. (1960) - "Plasma Acceleration". (Stanford University Press: California).

KILLEEN, J., and WHITEMAN, K. J. (1966) - Computation of hydromagnetic equilibria with finite pressure in minimum B systems. Phys. Fl. 9, 1846.

KNOX, F. B. (1961) - On the validity of treating a dense plasma as a metal in the radio-frequency containment of plasmas. J. Electronics and Control, 10, 1.

KRUSKAL, M., and TUCK, J. L. (1958) - The instability of a pinched fluid with a longitudinal magnetic field. Proc. Roy. Soc. A245, 222.

KUNKEL, W. B., ed. (1966) - "Plasma Physics in Theory and Application". (McGraw-Hill: New York).

LAMB, H. (1932) - "Hydrodynamics". (University Press: Cambridge).

LANDSCHOFF, R. K. M., ed. (1957) - "Magnetohydrodynamics". (Stanford University Press: California).

LEHNERT, B. (1964) - "Dynamics of Charged Particles". (North-Holland Pub. Co.: Amsterdam).

LEONTOVICH, M. A., ed. - "Reviews of Plasma Physics", Vols. 1, 3, 4, 5. (Consultants Bureau: New York).

LINHART, J. G. (1960) - "Plasma Physics". (North-Holland Pub. Co.: Amsterdam).

LUNDQUIST, S. (1952) - On the stability of magneto-hydrostatic fields. Phys. Rev. 83, 307.

MARGENAU, H., and MURPHY, G. M. (1964) - "The Mathematics of Physics and Chemistry". (Van Nostrand: Princeton).

MOON, P., and SPENCER, D. E. (1960) - "Foundations of Electrodynamics". (Van Nostrand: Princeton).

MOON, P., and SPENCER, D. E. (1961) - "Field Theory for Engineers". (Van Nostrand: Princeton).

PAI, S. (1962) - "Magnetogasdynamics and Plasma Dynamics". (Springer-Verlag: Vienna).

PAI, S. (1966) - "Dynamics of Fluids and Plasmas". (Academic Press: New York and London).

PANOFSKY, W. K. H., and PHILLIPS, M. (1962) - "Classical Electricity and Magnetism". 2nd ed. (Addison-Wesley: Massachusetts).

PEASE, R. S. (1957) - Equilibrium characteristics of a pinched gas discharge cooled by bremsstrahlung radiation. Proc. Phys. Soc. B70, 13.

PLONSEY, R., and COLLIN, R. E. (1961) - "Principles and Applications of Electromagnetic Fields". (McGraw-Hill, New York, and Kogakusha, Tokyo).

ROBERTS, P. H. (1967) - "An Introduction to Magnetohydrodynamics". (Longmans: London).

ROUBINE, E., ed. (1970) - "Mathematics Applied to Physics". (Springer-Verlag: Berlin).

RYE, B. J., and TAYLOR, J. C., eds. (1970) - "Physics of Hot Plasmas". (Oliver and Boyd: Edinburgh).

SAGDEEV, R. Z., and GALEEV, A. A. (1969) - "Nonlinear Plasma Theory". Revised and edited by T. M. O'Neil and D. L. Book. (W. A. Benjamin Inc.: New York).

SEYMOUR, P. W. (1964) - Motions of charged particles in plasmas. Int. J. of Engng. Sc. 1, 423.

SEYMOUR, P. W. (1966) - On the adiabatic motion of a charged particle in the quasi-static magnetic field of a solenoid. Aust. J. Phys. 19, 735.

SEYMOUR, P. W. (1970) - Exact drift velocity solutions for charged particle motion in a static axially symmetric magnetic field. Aust. J. Phys. 23, 133.

SEYMOUR, P. W., LEIPNIK, R. B. and NICHOLSON, A. E. (1965) - Charged particle motion in a time-dependent axially symmetric magnetic field. Aust. J. Phys. 18, 553.

SHAFRANOV, V. D. (1957) - On the stability of a cylindrical gaseous conductor in a magnetic field. J. Nucl. Energy 5, 86.

SHAFRANOV, V. D., and YURCHENKO, E. I. (1968) - Condition for flute instability of a toroidal-geometry plasma. JETP 26, 682.

SHKAROFSKY, I. P., JOHNSTON, T. W., and BACHYNSKI, M. P. (1966) - "The Particle Kinetics of Plasmas". (Addison-Wesley: Massachusetts).

SIMON, A., and THOMPSON, W. B., eds. (1968) - "Advances in Plasma Physics", Vols. 1 and 2. (Interscience Publishers: New York).

SINEL'NIKOV, K. D., ed. (1966) - "Plasma Physics and Controlled Thermonuclear Fusion". (Israel Program for Scientific Translations: Jerusalem).

SPIEGEL, M. R. (1959) -"Vector Analysis and an Introduction to Tensor Analysis". (Schaum Pub. Co.: New York).

STRATTON, J. A. (1941) - "Electromagnetic Theory". (McGraw-Hill: New York and London).

STRATTON, J. A., MORSE, P. M., CHU, L. J., LITTLE, J. D. C., and CORBATO, F. J. (1956) - "Spheroidal Wave Functions". (Technology Press of M.I.T. and J. Wiley & Sons, New York).

TAYLER, R. J. (1957) - Hydromagnetic instabilities of an ideally conducting fluid. Proc. Phys. Soc. B70, 31.

TAYLOR, J. B. (1963) - Some stable plasma equilibria in combined mirror-cusp fields. Phys. Fl. 6, 1529.

TONKS, L. (1968) - The birth of 'Plasma'. Amer. J. Phys. 35, 857.

TRUBNIKOV, B. A. (1966) - Plasma stability in a mirror with stabilizing rods. J. Nucl. Energy Part C 8, 583.

VANDENPLAS, P. E. (1968) - "Electron Waves and Resonances in Bounded Plasmas". (Interscience Publishers: London).

WANGSNESS, R. K. (1963) - "Introductory Topics in Theoretical Physics". (J. Wiley & Son: New York).

WEATHERBURN, C. E. (1957) - "Advanced Vector Analysis". (G. Bell & Sons: London).

WESTON, V. H. (1957) - Solutions of the Helmholtz equation for a class of non-separable cylindrical and rotational coordinate systems. Quarterly of Applied Maths. 15, 420.

WHITTAKER, E. T., and WATSON, G. N. (1958) - "A Course of Modern Analysis". (University of Cambridge Press).

WRIGHT, J. K., MEDFORD, R. D., and CHAMBERS, B. (1961) - Equilibrium configuration of the plasma in theta discharges with reversed trapped magnetic fields. J. Nucl. Energy Part C 3, 242.