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QUANTISATION AND RENORMALISATION IN THE HOMOGENEOUS AXIAL GAUGE

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Abstract

In this work the quantisation and renormalisation of Yang-Mills (YM) gauge theories in axial gauges is examined. Faddeev-Popov ghosts are known to decouple in these gauges, and particular attention is given to the temporal gauge case. I first approach the gauge-field propagator ambiguity by parallel transport from the Coulomb gauge to the temporal gauge to obtain a propagator that breaks time-translation invariance. This result agrees with the propagator derived in the 'fully-fixed gauge' approach of other authors. Problems with implementing such propagators in perturbation theory are discussed with reference to the taking of limits and the Green's function properties of the propagator. The major impediment in this approach is that the breaking of time-translation invariance makes proving renormalisability in this gauge intractable. I thus turn to an alternative: Landshoff's alpha-prescription for the temporal gauge field propagator. I discuss two derivations of the prescription: one, due to this author, maintains BRST invariance in the temporal gauge-fixed Yang-Mills Lagrangian, but has non-local terms in the Lagrangian implying an infinite number of new vertices. This is unwieldy so I pursue a second, related, derivation of Przeszowski which sacrifices BRST invariance for locality and so generates no new vertices. Poincare invariance of the free Abelian theory is recovered in a subspace of the indefinite metric Hilbert space; the Gauss law is valid for matrix elements of states in this subspace. Tadpole graphs in the alpha-prescription, which do not vanish even in dimensional regularisation, are shown not to contribute to the Wilson loop. I compute, to one loop, the quark-self-energy and quark-gluon vertex correction in the alpha-prescription and make some progress on the renormalisation problem.

Declaration

Except as stated herein this thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, this thesis contains no copy or paraphrase of material previously published or written by another person, except where due reference is made in the text of this thesis.

I give my consent for this thesis to be made available for photocopying and loan.

A.C.Kalloniatis

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To my Adelaide Greeks:
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Chapter 1

Introduction

This thesis is concerned with work recently done by the author on the long-standing problem of quantising gauge-theories in non-covariant gauges. The purpose of this chapter is to outline the structure of the work and to place the key ideas preceding this work into some historical framework.

1.1 Covariant Formulations

Since the successes of Einstein's theory of gravitation, the Quantum Electrodynamics of Feynman, Schwinger and Tomonaga, and the Weinberg-Salam-Glashow Theory unifying electromagnetism and the weak nuclear force, the gauge principle has come to be accepted as the basis for a complete description of the strong-interaction and the unification of the fundamental forces.

The subject matter of this thesis, *non-covariant* gauges such as the temporal gauge, are as old as Quantum Mechanics itself; Weyl[1] considered such gauges in his early work on quantising electrodynamics. Despite this early foray into non-covariant manifestations, the earliest successes in electrodynamics used covariant gauges - in the work of Fermi[2] for example - where the desire to maintain the invariance properties of Special Relativity took strong precedence. Indeed, many of the issues that are raised in this thesis were encountered even then: such as the differences in implementing gauge constraints in the classical theory and in the quantum theory, the relevance of this to the structure of the quantum Hilbert space of states and indefinite metric structures. In the quantum theory of the free electromagnetic field this developed into the work of Gupta and Bleuler[3], Nakanishi[4], and takes its most rigorous form in the papers of Strocchi and Wightman[5] and in the C^*-algebra formalism by Grundling and Hurst[6].

When Yang and Mills[7] and Shaw[8] generalised the gauge-invariance of the electromagnetic potentials from the Abelian $U(1)$ group to non-Abelian gauge theories a vast array of researchers turned the area into the rich structure it is today. The significant discovery here was the identification of residual gauge symmetries in the gauge-fixed action in terms of Grassman transformations - the transformations of Becchi, Rouet, Stora and Tyutin [BRST] [9, 10]. The BRST formalism is now the most elegant way of treating covariant non-Abelian gauge theories in the covariant but indefinite metric formalism. Faddeev and Popov[14] were able to give an elegant explanation in both a canonical and path-integral approach for the observations made by Feynman[15] and DeWitt[16] - that in non-Abelian theories in covariant gauges extra (ghost) degrees of freedom are required in order to safeguard the unitarity of the S-matrix. These fictitious fields are mathematically necessary for a local formulation of quantised gauge theories where the massless gauge boson is treated in a covariant fashion with *redundant* degrees of freedom. The generalisation of the Gupta-Bleuler method for selecting the physical Hilbert space in the non-Abelian case was rather elegantly formulated by Kugo and Ojima[17]. Faddeev[12], dealing with so-called first class constraints, and Senjanovic[13], extending Faddeev's work to include second class constraints, were able to implement for non-Abelian gauge theories the generalised Hamiltonian dynamics of Dirac[11] in the path-integral formalism. Another approach to the implementation of constraints in the quantisation of relativistic field theories was developed in the combined works of a number of authors - Fradkin and Vilkovisky[18] and Batalin and Fradkin[19] and related works. Here the constrained system admits the use of dynamically 'active' Lagrange multipliers for the constraints, and ghost fields. Physical unitarity and gauge independence are gauranteed in this approach by the compensation of multipliers and ghosts due to their opposite statistics. I mention this work here for completeness and shall have no further recourse to this otherwise quite fruitful area. Henneaux[20] reviews all this work on the role of FP ghost fields and BRST invariance within these systematic approaches to constraints in gauge theories.

Thus at the formal level, non-Abelian gauge theories were examined in covariant gauges. 't Hooft and Veltman were able to prove the renormalisability of the theory, and the implementation of the theory in the theories of electro-weak and strong interactions (quantum chromodynamics) has been nothing short of thorough.

1.2 Noncovariant Gauges

1.2.1 The 1960s

A noncovariant gauge is one which introduces a breaking of manifest Lorentz invariance in the choice of gauge itself. As mentioned, noncovariant gauges such as the temporal gauge, or the Coulomb gauge are quite old. As well as the work of Weyl cited earlier, Heisenberg and Pauli utilised the temporal gauge in an early treatment of the quantisation of Maxwell's theory[21]. The temporal gauge in quantised electromagnetism was later studied in a preliminary work by Kummer [22]. However, given the advantages of a manifest Lorentz invariant formulation other authors did not take up axial gauges until the developments in non-Abelian gauge theories during the seventies.

I shall be particularly concerned with noncovariant gauges characterised by gauge-conditions involving $n \cdot A$ where n_μ is the gauge-vector defining the temporal gauge (n time-like), the light-cone gauge (n on the null-plane) or space-like axial gauge (n space-like). The *homogeneous* axial gauge choice is thus one where $n \cdot A = 0$.

The first paper applying noncovariant gauges to the Yang-Mills (YM) field is that of Arnowitt and Fickler[23] which shows that, in distinction to the Coulomb gauge [1], the constraints arising from the requirement that the Euler-Lagrange equations be consistent with the Heisenberg equations in Schwinger's action principle can be solved exactly.

1.2.2 The 1970s

Much of this work was on the formal machinery of quantising the theory. In the early seventies perturbative calculations in non-Abelian gauge theories were undertaken in earnest.

In non-covariant gauges this work began with the investigation of the form of the S-matrix generating functional in the space-like axial gauge by Fradkin and Tyutin[24] and applying the light-cone gauge to the anomalous dimensions of twist-four operators by Kainz, Kummer and Schweda[25] which I discuss in some detail now.

The first explicit statement of the decoupling of ghosts appears to be by in the above cited paper by Fradkin and Tyutin who, deriving the generating functional, state: "[t]he Feynman rules have no additional diagrams" arising from a non-trivial functional measure. Fradkin and Tyutin also write down the form of the gauge-field propagator but draw no comment on the appearance in this expression of an ambiguous pole in momentum space. This problem remains unresolved, certainly for the temporal gauge, even to this day. Examining some of the solutions proposed for this problem is the central goal of this thesis.

It was the paper by Kainz et al. that first appreciated the problem of the ambiguous pole in the gauge-field propagator. Curiously enough, their solution in that work echoes

[1] Arnowitt and Fickler in their paper actually refer to the *radiation gauge* which I shall take to be the gauge with both $\partial_i A_i(x) = 0$ and $A_0 = 0$. This can only be implemented without eliminating physical degrees of freedom in the case of free electromagnetism - see chapter 2.

a flavour of both the Leibbrandt-Mandelstam (LM) prescription - which arose in the eighties - and the principal value (PV) prescription popular in the seventies. Namely, they recognised even then that a useful approach would be to define the spurious pole such that the Wick rotation was still possible[2] But that left an ambiguity for zero momentum for which they used a PV regularisation [27].

The next paper of note was the application of the temporal gauge to a theory of scalar multiplets coupled to gauge-vector bosons by Delbourgo, Salam and Strathdee[26]. The propagator ambiguity was appreciated there also and the PV prescription was offered as a solution on the grounds of their being no absorptive part, thus no unphysical contributions to the unitarity of the S-matrix[28, 29]. There too it was realised that the non-Abelian Ward identities, or alternately the Slavnov-Taylor identities retained their simplest form in this gauge - a direct consequence of the decoupling of ghosts from the generating functional. This was realised independently by Kummer in [30], in which he also advocated the PV prescription. Cornwall[31], in an appendix to a paper on scalar meson theories, discussed the light-cone gauge and in particular noted that naive power counting breaks down with light-cone integrals; the divergence of integrals does not correspond with that obtained 'by adding tensor indices'. Chakrabarti and Darzens[32] report on similar results independently of Cornwall as does Crewther [33] who discusses the presence of logarithmic dependence on momentum in divergent parts. In short, these observations render the proof of renormalisability in the light-cone gauge intractable. In light of these comments, the view developed that the light-cone gauge might suffer from more devastating pathologies than the space-like and temporal axial gauges.

Crewther, in the above work[33], used the axial gauge in the operator product expansion. Unitarity was explored, again in the temporal gauge, by Konetschny and Kummer in [34]. Here, using the PV prescription, the imaginary part of the S-matrix was checked explicitly to see if the unphysical degrees of freedom cancel - which they do for the PV for precisely the reasons advocated by Delbourgo et al. Asymptotic freedom in the axial gauge was examined by Frenkel and Taylor[35] (and later, in the context of the Lehmann representation, by West[36]). A derivation for the PV from first principles was finally given by Frenkel[37].

The goal in this 'middle period', as one might call it, remains unaccomplished to this day: to set non-covariant gauges on an equal footing with covariant gauges. This is tantamount to demonstrating that the quantum theory in such gauges is fully equivalent and consistent with the covariant formulation, namely that it is unitary and renormalisable.

[2]This is discussed in the appendix of their paper: the prescription is given in their equation (A.2) in which, by the way, σ should be replaced by δ.

Moreover, if the seeming advantages of the gauge are to be verified for practical computations in field theory the task would seem to be showing that ghost-decoupling - indeed the minimisation of the number of redundant degrees of freedom - is consistent with the physical requirements of gauge independence of observables, unitarity and renormalisability.

1.2.3 The 1980s

As far as the non-light-cone gauges were concerned, the above work established the PV as the accepted prescription. In the early years axial gauges with PV prescription were applied to a number of areas including gravity[38, 39] and in deep inelastic scattering and parton theory by authors such as Dokshitzer et al[40], and Bassetto et al[41]. In this case, the decoupling of ghosts and the form of the gluon polarisation vectors rendered this gauge a more physically transparent gauge in the ladder approximation.

But the period of activity in which the field finds itself in today opened in the early to mid-eighties when the prevailing view on these gauges was challenged. As has been said, this view was that the non-light-cone gauges were considered to be in a more healthy state than the light-cone gauge. The significant step away from this position was in the work of Leibbrandt[42], and Mandelstam[43] who independently arrived at a new prescription for the gauge-ambiguity in the light-cone gauge - the Leibbrandt-Mandelstam (LM) prescription. Leibbrandt derived the prescription by requiring that one be able to Wick rotate the contour in energy space without crossing new poles generated by the prescription in the first and third quadrants. Leibbrandt tested this prescription in the computation in basic Feynman integrals in perturbation theory and verified that, unlike the PV case, naive power counting was indeed satisfied. In the same paper he discovered that, while naive power counting now succeeded, new problems were introduced in the guise of non-polynomial dependences on the external momentum in one-loop divergences such as the gluon self-energy. The divergences of logarithmic form detected in the previous decade[33] were not present in this case. Rather, one was now faced with divergences involving $1/(p \cdot n)$, where p is the external momentum. Non-polynomial or non-local divergences demand non-local counterterms and one generally has no control, by power counting arguments, over what types of terms may arise. The proof of renormalisability is thwarted. Vigorous activity by a number of groups enabled progress in understanding these divergence structures and developing a successful renormalisation program for the light-cone gauge. Work here included understanding constraints imposed on non-local counterterms by BRST invariance[44, 45], a derivation of the LM prescription in the context of Dirac generalised Hamiltonian dynamics by Bassetto et al.[48], and the demonstration of order by order renormalisability of the theory[49] based on the observation that the non-local divergences decouple from unamputated Green's functions. Work on the renormalisation of Yang-Mills theories in the light-cone gauge was also done in the Vienna school by [51, 52].

In the temporal gauge there were further surprises. Several papers had appeared up until this time arousing suspicion that all was not well with the PV prescription even off the light-cone[46, 47]. Bassetto et al. had also argued this in the case of space-like n_μ on the grounds of unitarity violation[50]. Caracciolo, Curci and Menotti [CCM] in [53] tested the PV prescription for the temporal gauge propagator in the gauge-invariant rectangular (static) Wilson-loop. The result, to order g^4 in perturbation theory, had been checked in various gauges by numerous authors[54, 55, 56, 57, 58]. The exponentiation property of this quantity in the large time-limit, which enables the determination of the inter-quark potential for static quarks, had been well-understood by these authors. Using the PV prescription the wrong result was obtained. The significance of this result should not be underestimated - the ghost free temporal gauge with PV prescription is *inconsistent* with gauge invariance. Indeed, CCM found that extra terms were needed to obtain the correct result - these terms breaking time-translation invariance thus rendering elegant momentum space techniques of conventional field theory inapplicable.

This provoked a number of strands of work: numerous authors further examined the role of non-translationally invariant propagators in the temporal gauge field theory [59, 60, 61, 62, 63, 64]. Bassetto et al[65], Nardelli[66] and Soldati[67] have studied, what has been termed, the Wilson loop *criterion* for non-covariant propagator prescriptions in more thorough mathematical detail. Cheng and Tsai have investigated the equivalence properties of gauge-invariant operators such as the Wilson loop and their relationship to ghost-gluon couplings [68, 69, 70]. Their approach has been applied in the computation of Wilson coefficients in the gluon condensate by Lavelle et al.[71]. Haller, using a formalism previously developed for spinor quantum electrodynamics in covariant gauges[72], has investigated non-translationally invariant structures in terms of ghost operators involving the non-physical components of the gauge field[73]. Slavnov and Frolov have given quite rigorous derivations of the CCM propagator and shown it to be consistent with the unitarity of the S-matrix[74].

Given the cumbersome properties of the non-translationally invariant propagators, Landshoff proposed a momentum space prescription, the alpha prescription, for the temporal gauge for which he was able to obtain the correct Wilson loop behaviour[75]. Steiner made an early effort at giving a derivation for Landshoff's propagator[76]. But no real progress has been made on the origins of the alpha-prescription until recently. This thesis will report on aspects of that work.

In light of the abandonment of the PV prescription, much work had to be redone in the non-light-cone gauges. In particular the generalisation of the LM prescription off the light-cone was studied by the Vienna schools[77, 78] and by Leibbrandt[79]. The same sorts of non-local divergence structures as for the light-cone gauge were encountered[80]. Unfortunately here no such decoupling can be shown to occur for Green's functions of the theory. Despite this Lazzizzera[81, 82] and others[83, 84] were able to derive the generalised prescription in non-light-cone gauges. Hüffel et al.[85] demonstrated that the generalised prescription satisfied the Wilson loop criterion, though recent work has called this result into question[86]. Pollack was able to show that for the S-matrix amplitude for quark-quark scattering the non-localities decouple[87]. Bagan and Martin were able to compute the three-gluon and four-gluon vertex corrections in the generalised LM prescription[88]. Most recently, the role of non-local divergences in the light-cone gauge using the LM prescription has been called into question by Burnel and Caprasse[89]. This issue is clearly contentious. Because in this thesis I will deal with alternatives to the LM prescription, as will become evident, I take no particular position on this development.

Leibbrandt published a review[90] on the subject in 1987 summarising the huge amount of material published up until the late eighties. Since then a workshop has been held on the subject[91], and another workshop on Gauge Theories on the Light-Cone which also included numerous papers in this field of activity. Finally, the Italian school under Bassetto have published a monograph essentially collating the contribution he and his collaborators have made to the field[92].

It is important to stress that the consequence of this research has been to establish the light-cone as a valid gauge. Off the light-cone, non-Abelian gauge theories suffer seeming irrepairable pathologies - words spoken by Bassetto himself[93]. It is to this I shall turn my attention in this work.

1.3 Aims

The numerous threads in the above historical overview can be drawn together by summarising what work in the last three decades on the non-light-like axial gauges has been generally directed towards.

1.Deriving from first principles an unambiguous theory in the temporal gauge - particularly where the propagators of the theory are well-defined.

2.Clarifying what unphysical degrees of freedom, including Faddeev-Popov ghosts, are or are not inherent in the formalism.

3.Demonstrating that the unitary S-matrix is retrievable.

4.Proving that observables calculated in this gauge are gauge-independent.

5.Proving that the theory can be consistently renormalised. This is particulaly important, despite the knowledge that non-Abelian gauge theories are renormalisable in covariant gauges, because there are so many subtleties in noncovariant gauges that one cannot be entirely sure that the theory is genuinely equivalent to the covariant theory until this program has been achieved. Classical equivalence of the theory in the two gauges is simply not a sufficient guarantee.

With the exception of (3), the work on this thesis is directed along precisely these lines. After a chapter introducing the relevant technical aspects that shall be employed in subsequent chapters, I turn in the second chapter to the approach of fully-fixed temporal gauges. I shall give my own derivation of the non-translationally invariant propagators in this gauge by parallel transport from the Coulomb Gauge where no ambiguities occur in the free electromagnetic theory. I shall clarify certain problems with this approach, particularly the Green's functions properties of the propagator which have not been explicitly noted in the literature before. When examining the non-Abelian theory I reiterate previously stated arguments why this approach to the temporal gauge is not, in the end, helpful in providing a ghost-free gauge in which one can prove renormalisability.

Given this state of affairs, in the remainder of the work I turn to the other prescription available in the temporal gauge: the alpha-prescription due to Landshoff. I give two approaches to deriving the prescription, one of which originates in my work. However I demonstrate that this - BRST invariant approach - again over-complicates the system by introducing non-local vertices. I thus follow the second approach, due to Przeszowski and demonstrate consistency between Poincare invariance in the candidate physical Hilbert space of states in the free electromagnetic theory. In the non-Abelian theory I complete Landshoff's proof of the Wilson loop exponentiation in the alpha prescription by showing that tadpole graphs, previously overlooked in Landshoff's original work, vanish safegaurding the exponentiation. In the final chapter I turn to the renormalisation in the alpha prescription. I develop techniques for performing loop integrations in the prescription which are applied to the basic one loop corrections in Yang-Mills theory coupled to fermions: the gauge-field self-energy, the fermion-gauge-field coupling correction and the fermion self-energy. The results are shown to be consistent with Ward Identities that are simple and suggest ghost decoupling is indeed a property of the alpha-prescription. The renormalisation is carried out to one loop order - though the form of the gauge boson self-energy necessitates the introduction of a BRST non-invariant counterterm. The problems this inspires are discussed. Nonetheless it shall become transparent that the alpha-prescription *is* an improvement on the generalised LM prescription in that no non-local ultraviolet (UV) divergences are found, and there is some promise that the theory will be renormalisable to all orders.

The original material is contained in (sub)sections 3.1.2, 3.2.1, 3.2.3, 3.3, 4.2, 5.1.2, 5.2.1, 5.2.3, 5.2.4, 6.2, 6.3, 6.4 and all of the material in appendix B. These results have been published or are in preprint form in [94, 95, 97, 96, 98].

There is a statement of conclusions reached on the non-translationally invariant approach at the end of chapter three, and a more extensive list of conclusions in the final chapter, suggesting that the alpha-prescription is a promising approach to the temporal gauge but with numerous problems remaining to be solved. The references are collected at the end and appendices include tables of integrals useful for the alpha-prescription.

Chapter 2

Background and Techniques

2.1 Outline

In this chapter I review some of the key tools used in aspects of this work. Techniques such as dimensional regularisation in ultraviolet (UV) divergent Feynman integrals, and the derivation of the perturbative expansion in gauge theories are by now quite standard in field theory. The reader is referred to any text on gauge theories for that material - for example[99, 100]. These texts are my source for the conventions and notation adopted through most of this work. I discuss the framework of non-Abelian gauge theories in order to establish aspects of this notation. I then briefly review the formal method of the Faddeev-Popov trick in the functional formalism with respect to non-covariant gauge choices. This will establish the naive theory in the axial gauge in which the propagator remains ambiguous. I examine some of the peculiarities of the temporal gauge choice. Finally, the Dirac method of constraints is reviewed, as the technique will be utilised in a later chapter.

2.2 Non-Abelian Gauge Theories

The most natural generalisation of the Maxwell theory, so effective in classical and quantum electrodynamics, is one where the gauge-transformations become elements of a non-Abelian group G, such as $SU(N)$. This means that the potentials are now *matrices* and are written as a linear combination

$$(A_\mu(x))_{ij} = A^a_\mu(x) T^a_{ij} \; . \tag{2.1}$$

The indices i, j label the components of an arbitrary matrix representation, R, of the matrices T^a. For brevity I shall often suppress these indices. These representation matrices are the generators of the semi-simple Lie algebra of the group G obeying the group algebra $[T^a, T^b] = if^{abc}T^c$ and f^{abc} are completely antisymmetric structure constants. The quadratic Casimir invariants relevant to the generic representation R are defined by

$$(T^a T^a)_{ij} = C_2(R)\delta_{ij} \tag{2.2}$$

while I shall also need the analogous expression for the adjoint representation,

$$f^{acd} f^{bcd} = C_2(G)\delta^{ab} \,. \tag{2.3}$$

The gauge-invariant Lagrangian density for an $SU(N)$ non-Abelian gauge theory with fermion interactions is given by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}^i (i\not{D}_{ij} - m\delta_{ij})\psi^j \tag{2.4}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2.5}$$

is the field-strength tensor in component form which transforms under the adjoint representation of the internal gauge symmetry group $G = SU(N)$, and the covariant derivative is defined by

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - igT^a_{ij} A^a_\mu \,. \tag{2.6}$$

I shall also have occasion to refer to the covariant derivative with respect to the adjoint representation

$$\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu . \tag{2.7}$$

2.3 Non-Covariant Gauge Fixing

In this section I largely follow the exposition given in the review by Leibbrandt[90]. I shall be concerned throughout this work with homogeneous non-covariant or axial gauges defined by the condition

$$n \cdot A^a = 0 \tag{2.8}$$

where n^μ is a fixed vector which falls into one of three regions depending on whether $n^2 = n^\mu n_\mu$ is positive (time like axial gauge), zero (light-cone gauge) or negative (space-like axial gauge). In particular I shall be concerned with the time-like case for which there is no loss of generality in choosing the vector $n = (1, 0, 0, 0)$ - and this explicit choice will be made at numerous points of this work. Thus my primary concern is with the *temporal gauge* choice

$$A^a_0 = 0 \,. \tag{2.9}$$

In the Faddeev-Popov approach[14, 101] this gauge-choice can be implemented by choosing the gauge-fixing functional

$$F^a[A] = n \cdot A^a \, . \tag{2.10}$$

Now an important quantity to be considered is the Faddeev-Popov determinant which is merely the Jacobian determinant for the formal path-integral with respect to infinitesmal gauge-transformations:

$$\det(M_F) = \det[\frac{\delta F^a(x)}{\delta \omega^b(y)}] \tag{2.11}$$

which, for the above choice for F, is evaluated to be

$$\det(n \cdot \partial \delta^{ab} + g f^{abc} n \cdot A^c) \, . \tag{2.12}$$

Recall the standard identity

$$\int \mathcal{D}g \det(M_F[{}^gA])\delta(F^a[{}^gA(x)]) = 1 \tag{2.13}$$

where $\mathcal{D}g$ is the invariant functional integral measure for an integration over the gauge group space

$$\mathcal{D}g = \prod_a dg^a \tag{2.14}$$

for group parameters g^a, with a running from 1 up to the dimension of the group. The formally gauge-invariant path-integral generating functional for Yang-Mills theory is given by

$$Z[J, \chi, \bar{\chi}] = \mathcal{N} \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi} \exp[i \int d^4x(\mathcal{L} + J \cdot A + \bar{\psi}\chi + \bar{\chi}\psi)] \tag{2.15}$$

where $\mathcal{L}$ is given by (2.4) and J^μ, χ and $\bar{\chi}$ are c-number sources for the gauge and fermion fields, permitting the generation of Green's functions of the theory. By introducing the trivial identity (2.13) into this expression one obtains

$$Z[J, \chi, \bar{\chi}] = \mathcal{N} \int \mathcal{D}A\mathcal{D}g\mathcal{D}\psi\mathcal{D}\bar{\psi} \det(M_F[{}^gA])\delta(F^a[{}^gA]) \exp[i \int d^4x(\mathcal{L} + J \cdot A)] \, . \tag{2.16}$$

Now $\mathcal{D}A$ is a measure invariant under local gauge transformations. The Faddeev-Popov determinant is also gauge invariant. Thus one can perform a gauge-transformation in the path-integral measure and integrand from ${}^gA \to A$ with the corresponding transformations on the fermion fields enabling the the infinite volume factor $\int \mathcal{D}g$ to decouple and be absorbed into the normalisation constant.

The task remains to cast this expression into a local form, amenable to perturbative diagrammatic computations by exponentiating terms into an effective action. The delta-function is easily dealt with by recognising the homogeneous gauge-condition (2.8) is a special case of the inhomogeneous non-covariant gauges

$$n \cdot A^a = C^a \tag{2.17}$$

where $C^a(x)$ is an arbitrary functional taking values in the gauge-group algebra. One may thus average the generating functional over $C^a(x)$ using a sharp Gaussian weight

$$\lim_{\lambda \to 0} \exp[-(i/2\lambda) \int d^4x (C^a(x))^2] \tag{2.18}$$

from which one recovers the homogeneous condition. This may alternately be achieved through a multiplier field $B^a(x)$ in a term in the Lagrangian of the form[4]

$$B^a n \cdot A^a \; . \tag{2.19}$$

Computation of the Euler-Lagrange equation for the multiplier field B^a shows the homogeneous condition (2.8) is returned.

Finally, the determinant is exponentiated by rewriting it as an integral over scalar Grassmann fields η^a and $\bar{\eta}^a$ which transform under the adjoint representation of $SU(N)$, giving the final expression for the effective Lagrangian

$$\mathcal{L}_{eff} = \mathcal{L}_{YM} - B^a n \cdot A^a + \bar{\eta}^a n \cdot D^{ab} \eta^b \; . \tag{2.20}$$

The Grassmann fields are the Faddeev-Popov ghost fields necessary in covariant non-Abelian gauge theories in order to retain unitarity of the S-matrix[15]. The crucial observation here is that the gauge condition $F^a[A] = n \cdot A^a$ does not involve a derivative ∂_μ. Implementing the gauge-condition at the level of the Lagrangian, one thus obtains for the ghost term

$$\mathcal{L}_{FPG} = \bar{\eta}^a n.\partial \delta^{ab} \eta^b \; . \tag{2.21}$$

Notice that the Feynman rules implicit in this expression do not involve a vertex between the ghosts and the gauge-boson. In other words, the ghosts have decoupled from the theory - much as they do in linear gauge quantum electrodynamics. This is the essential advantage of these gauges. In a BRST analysis, of course, the ghosts must be retained[90]. Thus the statement of decoupling applies to a perturbative Feynman diagram expansion.

There are other levels at which the ghost decoupling can be seen. The ghost term may be carried into the perturbative expansion where they appear in closed ghost-loop Feynman diagrams. For the inhomogeneous case ($\lambda \to 1$ in (2.18)), Frenkel demonstrates the decoupling by observing that the integrals are of a tadpole form which vanish in dimensional regularisation[102]. The decoupling can be seen to hold for the *planar* gauge (where $n \cdot A^a = V^a(x)$ for some arbitrary field V^a) by expanding about small gauge-transformations and finding that the formal Faddeev-Popov determinant is independent of the gauge-field and thus may be decoupled[40]. Finally, for the homogeneous case again, the decoupling is seen at the diagrammatic level in that the ghost-gauge field vertex is proportional to the gauge vector n^μ and so in unamputated graphs one obtains factors of the form

$$n^\mu D^{ab}_{\mu\nu}(p) \tag{2.22}$$

where $D^{ab}_{\mu\nu}(p)$ is the propagator or two-point Green's function for the gauge-field to which I turn in greater detail later - and for a true homogeneous axial gauge expression (2.22) must vanish reflecting the original gauge condition. Thus ghost diagrams drop out of S-matrix amplitudes.

Some of these arguments assume the propagator prescription for the ghost field is consistent with dimensional regularisation. I shall come back to this in the context of the alpha-prescription in chapter 4.

Turning to the remaining Feynman rules one finds that the gauge field and fermion vertices are exactly the same as in the covariant formulation (see appendix A for details). Only the gauge-field propagator has changed

$$D^{ab}_{\mu\nu}(p) = \frac{\delta^{ab}}{p^2}[g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n} + \frac{n^2 p_\mu p_\nu}{(p\cdot n)^2}] \tag{2.23}$$

where physical asymptotic conditions[103] dictate that the pole at $p^2 = 0$ should be treated in the sense of Feynman's prescription $p^2 \to p^2 + i\epsilon$ where ϵ is a positive infinitesmal quantity which suffices to define the deformed contour in p_0 space. Note that the result (2.23) is consistent with (2.22). In this case however there is an additional pole at $p \cdot n = 0$ - the source of the woe and effort discussed in the first chapter and to which I shall return in greater detail in subsequent chapters.

2.3.1 BRST Invariance

The effective Lagrangian, with non-covariant gauge-fixing term and ghost term *retained* have an additional BRST symmetry [9, 10] under transformations involving the Grassman anticommuting fields. Because I shall not make specific use of the transformations in this work I omit a detailed discussion. But an important aspect to note is that the BRST transformations are a subset of the gauge transformations, where the gauge part of the transformation is written in terms of the Grassman field. Thus the original gauge invariant piece of the Lagrangian is itself BRST invariant. The ghost term can be seen to be simply a consequence of BRST variations of the gauge-fixing term. Thus the sum of the gauge-fixing and ghost terms is also BRST invariant.

This symmetry provides a rich framework in which formal aspects of the theory such as renormalisation and the quantisation can be carried out in powerful and elegant ways. In the former case the Slavnov-Taylor identities [104, 105] or related BRST identities prove essential in dealing with the renormalisation of YM theory to all orders in perturbation theory. In the latter case, so called BRST quantisation permits an elegant framework in which to prove unitarity of the theory, and Poincare invariance. In particular the Kugo-Ojima criterion [17] where physical probability conserving states are distinguished from redundant states within the indefinite metric Hilbert space by being annihilated by the *BRST charge*, namely the Noether charge corresponding to invariance under the BRST transformations, is especially useful in the context of the functional approach to field theories.

2.4 The Temporal Gauge

Now I focus in more detail on peculiar aspects of the field theory in the temporal gauge theory defined, by $A_0^a = 0$.

The first observation is that this condition does not completely remove all gauge degrees of freedom - there is a residual invariance under time-independent gauge transformations

$$A_i^a(x) \to A_i^a(x) - D_i^{ab}\omega^b(\mathbf{x}) \,. \tag{2.24}$$

Residual gauge freedom is not of course peculiar to axial gauges. For example, covariant gauge QED retains a residual symmetry under gauge-transformations described by functions $\omega(x)$ satisfying

$$\Box\omega(x) = 0 \,. \tag{2.25}$$

As mentioned, the requirement of causal time-boundary conditions dictate how the remaining freedom is to be fixed and lead to the well-defined covariant gauge propagator with Feynman $i\epsilon$ prescription.

That the propagator ambiguity in the temporal gauge is related to the residual time-independent gauge freedom is most readily seen by considering that one fixes the gauge in perturbation theory in the first place in order to invert the quadratic differential operator in the Lagrangian or equations of motion. In coordinate space, for the temporal gauge one encounters the time-derivative operator, ∂_0, which must be inverted to derive the bare two-point Green's function of the free theory. This inversion implies an integral operator for which there is an arbitrary constant (with respect to time) of integration which remains unfixed. This is precisely the ambiguity in question. These 'constants' of integration are directly related to the asymptotic properties of the fields as $n \cdot x = x_0 \to \pm\infty$ which, in turn, enjoy the residual gauge freedom[36].

This means that one way of dealing with the propagator ambiguity is by removing this freedom and this approach will be discussed in the next chapter. But there are other approaches which retain the residual freedom, as implicit in the Leibbrandt-Mandelstam (LM) prescription[106] or in the method I shall adopt in later chapters with the alpha prescription.

Another issue in the temporal gauge is the loss of the Gauss law - an essential feature of the physical sector of a theory of massless gauge interactions.

To see how the Gauss' law arises in the unfixed Yang-Mills (YM) theory consider the canonical Hamiltonian in the classical theory

$$H_{YM} = \int d^3x[\frac{1}{2}F^a_{0i}F^a_{0i} + \frac{1}{4}F^a_{ij}F^a_{ij} + \bar{\psi}(i\gamma_i\partial_i + m)\psi + gA^a_i\bar{\psi}\gamma_iT^a\psi - A^a_0(g\bar{\psi}\gamma_0T^a\psi + D^{ab}_iF^b_{0i})] \tag{2.26}$$

where the last term was obtained after an integration by parts and the surface term discarded - permissible for YM theory in the context of perturbation theory.

One sees that the zero-component of the gauge-field, A^a_0, does not play a dynamical role in (2.26), rather it acts as a Lagrange multiplier for the Gauss law constraint

$$D^{ab}_i(x)\pi^b_i(x) + g\bar{\psi}\gamma_0T^a\psi = 0 \tag{2.27}$$

where $\pi^a_i = F^a_{0i}$.

In the covariant gauge quantum theory this constraint is implemented as an operator equation: the Gauss law operator is identically zero.

In the temporal gauge theory with A^a_0 absent from the classical formalism the Gauss law constraint cannot arise. So, as it must be present in the physical theory it must be imposed as a condition on the physical sector of the Hilbert space of states. Recall that in a general quantum field theory the total Hilbert space may be *too large*: there may be more states than are actually physically relevent for probability conservation (unitarity) and a positive definite Hamiltonian. The Gupta-Bleuler criterion[3] in covariant gauge QED serves to distinguish the physical sector from the redundant states in such an indefinite metric space. In non-Abelian covariant gauge theories the Kugo-Ojima[17] criterion fulfills the same purpose. In the temporal gauge there is an extra requirement that if the theory has an indefinite metric then the criterion that serves to distinguish states for which, for example, the Hamiltonian is positive definite must be consistent with the Gauss law being implemented on these states.

There are two ways in which this can be achieved which I refer to respectively as *strong* and *weak* conditions on physical states:

$$D^{ab}_i(x)\pi^b_i(x) + g\bar{\psi}\gamma_0T^a\psi|phys\rangle = 0 \tag{2.28}$$

or

$$\langle phys'|D^{ab}_i(x)\pi^b_i(x) + g\bar{\psi}\gamma_0T^a\psi|phys\rangle = 0\ . \tag{2.29}$$

Thus the first involves the Gauss law operator annihilating a physical state, while the second implementation enforces *matrix elements* of the operator between physical states to vanish. Various formulations that appear in the literature implement the Gauss law in one of these ways - some to more advantage than others[82].

I have clearly been somewhat cavalier in the above description with the role of constraints in the quantum theory. A more careful approach will be elucidated in the next section on the generalised Hamiltonian dynamics as formulated by Dirac.

I conclude this section by discussing the propagator structure specific to the temporal gauge. The temporal gauge condition, $A_0^a(x) = 0$, means that one is left with the spatial components of the gauge vector field, $A_i^a(x)$. For the remainder of this discussion I shall suppress the adjoint index, a. By the Helmholtz theorem the field can be decomposed into a sum of two terms - spatially transverse and longitudinal

$$A_i(x) = A_i^T(x) + A_i^L(x) \tag{2.30}$$

where $\partial_i A_i^T(x) = 0$ defines the transverse component. Idempotent operators can be constructed

$$P_{ij}^T = g_{ij} + \frac{\partial_i \partial_j}{\nabla^2} \tag{2.31}$$

$$P_{ij}^L = -\frac{\partial_i \partial_j}{\nabla^2} \tag{2.32}$$

which will project out the transverse and longitudinal parts

$$A_i^{T,L} = P_{ij}^{T,L} A^j \,. \tag{2.33}$$

The transverse and longitudinal free gauge fields satisfy the following equations of motion

$$\Box A_i^T = J_i^T \tag{2.34}$$

$$\Box A_i^L + \partial_i \partial_j A_j^L = J_i^L \tag{2.35}$$

which can be obtained by applying the projectors onto the Euler-Lagrange equation with the temporal gauge condition imposed. Observe that the transverse and longitudinal equations have *decoupled* implying that there are two Green's functions to be considered, with no mixed transverse-longitudinal Green's function to be dealt with. In the quantum theory this is reflected in the observation that there is a vanishing time-ordered vacuum expectation value between transverse and longitudinal quanta.

Going to momentum space, the propagators are easily derived, either directly as Green's functions of the decoupled equations or by applying the Fourier transforms of the projectors onto the full bare propagator:

$$D^T_{ij}(p) = \frac{1}{p^2+i\epsilon}(g_{ij}+\frac{p_ip_j}{|\mathbf{p}|^2}) \tag{2.36}$$

$$D^L_{ij}(p) = \frac{1}{p_0^2}\frac{p_ip_j}{|\mathbf{p}|^2} \tag{2.37}$$

and one notes that the transverse part is precisely the propagator used in the Coulomb gauge - unsurprising since the Coulomb gauge condition, $\partial_i A_i(x) = 0$, amounts to setting the longitudinal gauge field to zero.

More significantly also note in the above expressions that the propagator ambiguity is isolated in the longitudinal part which, for the unprescripted propagator, factorises into a spatial part and a term $1/p_0^2$ which I shall refer to as the time-dependent factor, or just the *time-factor*, of the longitudinal propagator. In momentum space I shall denote this generally as $D(p_0)$, and its Fourier transform as $D(x_0, y_0)$. I use the same symbol D here but it shall be clear from the context which is being dealt with. The situation will be encountered where a non-translationally invariant form for D is derived in coordinate space for which the corresponding Fourier transform in terms of a single momentum p_0 does not exist. The question of what prescription should be used in the propagator thus amounts to that of what form D should take when a prescription is introduced. I shall use subscripts to distinguish the different forms for D suggested by different authors.

As an example of one possible prescription that I mentioned in the first chapter, I give the Fourier transform of the PV prescription

$$\lim_{\epsilon\to 0}\frac{1}{2}\int\frac{dk_0}{2\pi}e^{-ik_0(x_0-y_0)}[\frac{1}{(k_0+i\epsilon)^2}+\frac{1}{(k_0-i\epsilon)^2}] = -\frac{1}{2}|x_0-y_0|\,. \tag{2.38}$$

This I denote by

$$D_{PV}(x_0,y_0) = -\frac{1}{2}|x_0-y_0|\,. \tag{2.39}$$

Note that using $D_{PV}(x_0, y_0)$ in the longitudinal propagator means the resultant propagator is a Green's function of the equation of motion (2.35). This is because D_{PV} satisfies the equation

$$\frac{\partial^2}{\partial {x_0}^2}D_{PV}(x_0,y_0) = -\delta(x_0-y_0)\,. \tag{2.40}$$

2.5 The Wilson Loop

2.5.1 General Loops

The Wilson loop is the non-local gauge-invariant operator defined by

$$W[\Gamma] = \langle Tr\ P\exp\ ig\oint_\Gamma dx\cdot A\rangle \tag{2.41}$$

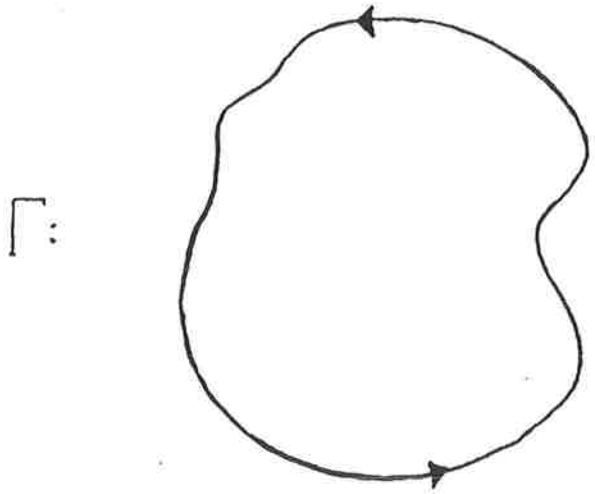

Figure 2.1: Wilson loop around an arbitrary closed contour Γ.

where Γ defines a closed contour in space-time (fig.2.1) around which the path-ordered exponential is defined. The gauge-invariance is a property of the contour being closed and traced. The closed-loop path-ordered exponential has its earliest applications in electrodynamics in the work of Mandelstam [107] but was adopted by Wilson in [54] as a useful order parameter to detect the onset of deconfinement in lattice QCD. The 'stringy' nature of the loop was the basis of the dual resonance model before QCD and thus formed the basis of the string theories that were the inspiration of much activity in the '80s[108, 109].

The Wilson loop can be brought into the domain of standard field theoretic techniques. The contour Γ may be represented by the curve $x_\mu(\eta)$ where $\eta_0 < \eta < \eta_1$ is some real quantity parametrising the contour such that, for a closed contour $x_\mu(\eta_0) = x_\mu(\eta_1)$. Then the path-ordered exponential is

$$W_{ij} = [P\exp\int_{\eta_0}^{\eta_1} d\eta\, ig\frac{dx_\mu(\eta)}{d\eta}A_\mu(x(\eta))]_{ij} \tag{2.42}$$

where i, j label the representation of the operators over which the trace is taken to give the final gauge-invariant quantity.

A perturbative expansion may be developed by obtaining Green's functions from the functional by considering the generating functional for the loop operator[108]

$$Z[\lambda, \bar{\lambda}] = \int \mathcal{D}z\mathcal{D}\bar{z}\exp\int d\eta[\bar{z}\partial_\eta z - ig\bar{z}A_\mu z\frac{dx_\mu(\eta)}{d\eta} + i\bar{\lambda}z + i\bar{z}\lambda] \tag{2.43}$$

where the Wilson loop becomes just

$$W = \frac{\delta^2}{\delta\lambda\,\delta\bar{\lambda}}\ln Z[\lambda, \bar{\lambda}]|_{\lambda=\bar{\lambda}=0}\,. \tag{2.44}$$

Here, z and $\bar{z}$ are one-dimensional Grassmann fields which exist on the contour; λ and $\bar{\lambda}$ are sources for these Fermi fields. The logarithm of the generating functional clearly

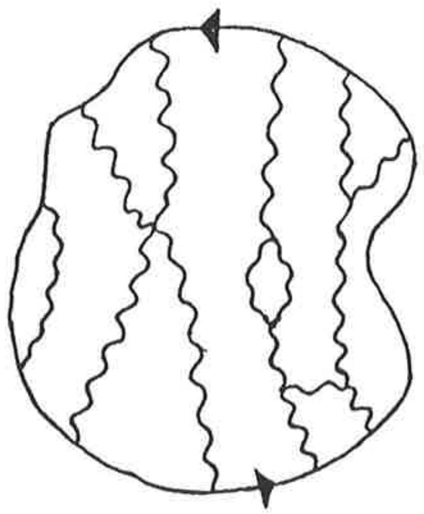

Figure 2.2: A diagram in the perturbative computation of an arbitrary Wilson loop.

enables the removal of vacuum graphs from the perturbative expansion. One sees that the Wilson loop computation has been reduced to a theory of YM gauge-fields coupled to one-dimensional fermions. The last step is to introduce into (2.43) the functional integral over A_μ with associated ghost fields by applying Faddeev-Popov quantisation to the system. Application of standard techniques to this yields a set of Feynman diagrams as illustrated in figure (2.2) which may be computed to give contributions to the loop operator to any order in perturbation theory.

Dotsenko and Vergeles[110], proving a conjecture by Polyakov[111], show that for smooth contours, the Wilson loop may be renormalised by absorbing infinities into the charge renormalisation. In particular they demonstrate the exponentiation property of the loop expansion: that each order of perturbation theory corresponds to a term in the expansion in the coupling constant, g, of an exponential.

2.5.2 Static Loop

The *static* Wilson loop, defined by a rectangular contour in, say, the $x_3 - t$ plane (fig.2.3) where the time-length T is taken to be very large, has proved useful in the study of heavy-quark bound systems such as charmonium[112].

The heavy nature of the charm and bottom quarks make a perturbative expansion of the loop a valid tool to study such systems - even for long range effects. From the above one can see that the one-dimensional fermions defined on the contour are indeed, in the static case, a pair of infinitely massive coloured fermions - static quark fields. The picture that seems consistent with this quantity is of a quark-antiquark pair created from the vacuum 'quasistatically' drawn apart to spatial separation L and held apart for some long

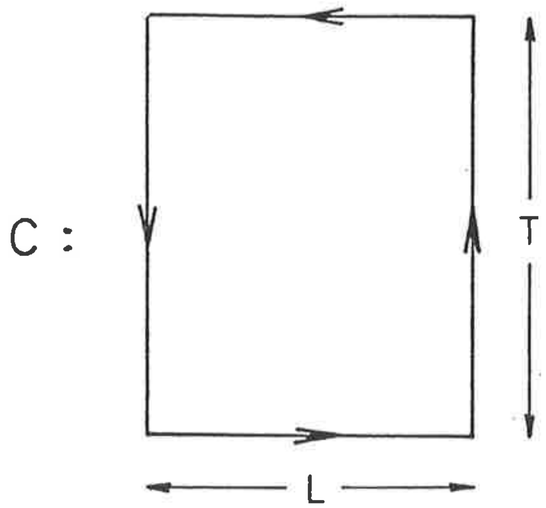

Figure 2.3: The static Wilson loop contour in the $x_3 - t$ plane.

time T whereupon they are drawn back together and are permitted to annihilate.

Using the functional methods mentioned in the previous section a set of simple Feynman rules may be developed to compute order by order contributions to the Wilson loop. These rules are summarised in appendix A. The types of diagrams are illustrated in figure (2.4). Because contributions from diagrams with closed *dynamical* fermion loops inserted in the Wilson loop are separately gauge-invariant, these can be ignored in the calculations I shall be performing later.

The choice of gauge is significant in this computation. For example, in a covariant gauge FP ghosts must be included in the computation[55, 56] with the appropriate ghost-loop insertions included in the diagrams. The computation has also been done in the Coulomb gauge by Appelquist et al.[57, 58]. The result obtained in all the above computations, to order g^4 in perturbation theory, is

$$W = e^{-iTV(L)} \tag{2.45}$$

where $V(L)$ is the potential between a static quark-antiquark pair:

$$V(L) = g^2 C_2(R) \int \frac{d^3p}{2(\pi)^3} \frac{e^{ip_3 L}}{|\mathbf{p}|^2} [1 + \frac{g^2}{16\pi^2} C_2(G) (\frac{11}{3} \log \frac{\mu^2}{|\mathbf{p}|^2} - \frac{11}{3}\gamma_E + \frac{31}{9})]. \tag{2.46}$$

Here μ is the renormalisation point and γ_E is the Euler constant.

2.5.3 The Temporal Gauge

The temporal gauge is an appropriate gauge in which to compute the static Wilson loop. The property of ghost-decoupling suggests a whole set of diagrams may be ignored. Moreover, the temporal gauge property of the propagator

$$D^{ab}_{00} = D^{ab}_{0i} = D^{ab}_{i0} = 0 \tag{2.47}$$

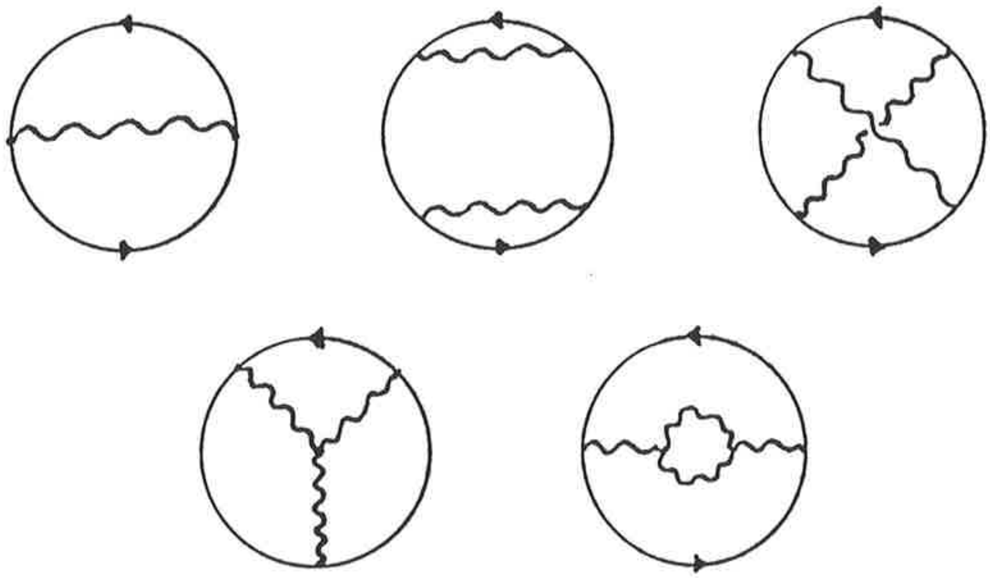

Figure 2.4: Diagrams contributing to the static Wilson loop to order g^4. As there are many diagrams, all different possible attachments of a propagator to the rectangle sides are summarised by the vertex of a gluon with a circular contour.

means that any diagrams with a gluon leg attaching to the vertical 'rungs' of the loop vanish (for some examples see fig.2.5).

Of course the problem remains the choice of prescription. The computation was first done by Caracciolo et al.[53] (CCM) who tested the PV prescription (2.38). The diagram with a gluon self-energy (fig.2.6) proved to be particularly sensitive. In fact using the PV prescription, namely D_{PV} in the longitudinal propagator, the exponentiation property was not recovered: the prescription failed to produce the correct gauge-independent result (2.46).

The remarkable result uncovered by CCM was that the correct Wilson loop behaviour to order g^4 in perturbation theory was recovered with a time-factor of the longitudinal part of the propagator

$$D_{CCM}(x_0, y_0) = -\frac{1}{2}|x_0 - y_0| \pm \frac{1}{2}(x_0 + y_0) + \gamma \tag{2.48}$$

where γ could take any real value. One notes that the first term corresponds to the PV prescription (2.38). But there is an additional term which breaks time-translational invariance. Thus the resultant propagator cannot be Fourier transformed in terms of a *single* four-momentum.

As a consequence of the surprising result that the Wilson loop, being a gauge independent object, was able to pick out problems with the PV prescription, the Wilson loop 'criterion' is now a key diagnostic tool in testing consistency of propagator prescriptions

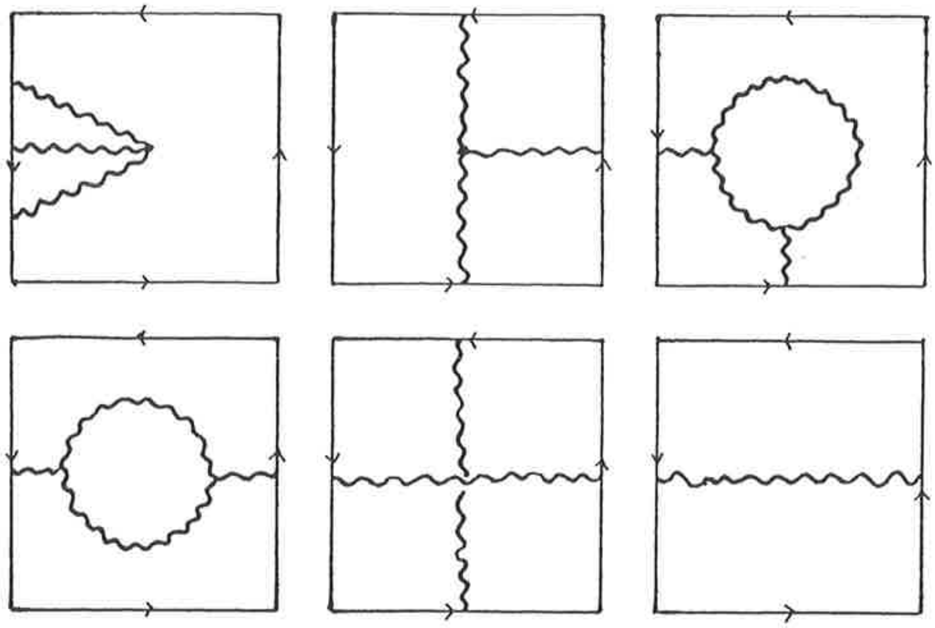

Figure 2.5: Some Wilson loop diagrams that vanish in the temporal gauge.

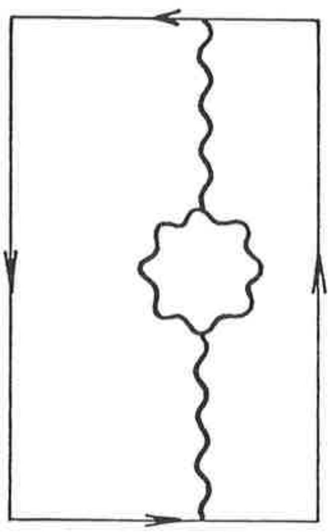

Figure 2.6: A self energy contribution to the Wilson loop in the temporal gauge.

in the temporal and other axial gauges[65, 66, 67, 113].

2.6 Dirac Method

There are numerous pedagogical expositions of the Dirac constraint method[114, 115] including lectures by Dirac himself[11]. I shall be content here with highlighting the main features. The work of Faddeev[12] and Senjanovic[13] has shown that the methods to be discussed here do indeed dovetail into the path-integral formalism used in the last section but I shall not go into these connections.

Consider a simple mechanical system of N degrees of freedom described by coordinates q_n, $n = 1, ..., N$. In general the Lagrangian will be *singular* - some of the conjugate momenta, which are functions of the velocities $\dot{q}_n$, may depend on each other. In other words there will exist $M < N$ constraints on some of the momenta, known as *primary* constraints as they arise directly from the Lagrangian,

$$\chi_m(q,p) = 0 \tag{2.49}$$

for $m = 1, ..., M$.

The simplest example of a system with an infinite number of degrees of freedom having the same behaviour is that of electromagnetism where the vanishing of the conjugate momentum $\pi_0(x) = 0$ by virtue of the antisymmetry of $F_{\mu\nu}$, is a trivial primary constraint: π_0 is a linear combination of the π_i momenta, where the coefficients are *zero*. Indeed this example illustrates the problems with constraints in quantising a theory in that the vanishing of π_0 contradicts the equal time commutation relations that one would like to impose based on the classical Poisson bracket relations[99].

Now the physics of the system with Hamiltonian function,

$$H = p_n \dot{q}_n - L \tag{2.50}$$

is the same as that for the same Hamiltonian with the constraints added via a set of undetermined multipliers, u_m

$$H_1 = H + u_m \chi_m \; . \tag{2.51}$$

For consistency, the primary constraints must be conserved in time. Consider the quantity

$$\psi_m = \{H, \chi_m\} \tag{2.52}$$

where the Poisson bracket is defined in the usual way (see appendix A). If it does not already vanish, 2.52 must be set equal to zero and this implies a number of secondary constraints, which can in turn be added to the Hamiltonian with associated mulipliers, v_α.

This procedure can be repeated until no more new secondary constraints are generated. To do this consistently the constraints must not be imposed until all Poisson brackets have been computed. This defines a *weak* constraint in the Dirac sense. Thus a strong constraint in the Dirac sense is one imposed before the computation of brackets. This is distinct from my use of the terms 'strong' and 'weak' in the last section to denote the two ways of implementing the Gauss law on a Hilbert subspace in the temporal gauge.

The indeterminacy of the system has now been re-expressed in the unknown multipliers, $u_m, v_\alpha, \ldots$ The set of constraints thus generated can be regrouped in another way - those for which the Poisson bracket between pairs of them gives another constraint are known as first class quantities (clearly there are an even number of these), and those that do not are second class. Following the above algorithm, the time-conservation requirement on the second class constraints will yield equations which can be solved for the undetermined multipliers. So the number of undetermined multipliers equals the number of first class constraints.

The second class constraints present quite distinct problems in generating a consistent quantum theory. In general, first class constraints in a classical theory are realised as conditions on states in the Hilbert space. Dirac gives a simple argument to demonstrate the problem posed by second class constraints. Consider a simple classical system with second class constraints on the position and momentum operators:

$$q = p = 0 \tag{2.53}$$

where these constraints are weak in the Dirac sense. As has been said, in the quantum theory these occur as conditions on states or wavefunctions

$$q|\psi\rangle = 0 \tag{2.54}$$

$$p|\psi\rangle = 0\,. \tag{2.55}$$

But then the quantity $(qp - pq)$ should annihilate a state (in the strong case, or have vanishing expectation value between states in the weak instance) - this contradicts the second class property of q and p, that their Poisson bracket (also second class) does not vanish. The classical Poisson bracket of second class quantities is not the appropriate quantity to be taken into the quantum commutator of the corresponding operators.

The solution given by Dirac is to introduce a modification of the classical Poisson brackets. If one forms a matrix from the set of second class constraints, χ_i, of some arbitrary system

$$C_{ij} = [\{\chi_i, \chi_j\}] \tag{2.56}$$

then the inverse of this matrix can be shown to exist. One then introduces the *Dirac bracket* between two quantities A and B:

$$\{A,B\}^* = \{A,B\} - \{A,\chi_i\}C_{ij}^{-1}\{\chi_j,B\} \,. \tag{2.57}$$

The repeated index, as usual, denotes a summation but can also mean a space-integration where A and B might be field variables. These brackets are consistent with the classical equations of motion. The Dirac bracket is now a first class quantity and can thus be appropriately taken into the equal time commutator to define a quantum theory. For a general set of constraints one can choose appropriate linear combinations of them that minimise the number of second class constraints enabling maximal use of the Poisson bracket. A simpler approach is to make *all* the constraints second class by expanding the set of constraints with some suitable subsidiary constraints and use the Dirac brackets *ab initio*. Thus each first class constraint requires the introduction of a subsidiary constraint or *gauge constraint* in order to get all constraints second class. Another way to see this is to recall that the dynamics is still only determined up to a number of Lagrange multipliers equalling the number of first class constraints. Introducing the subsidiary gauge constraints enables determination of these multipliers.

At this point, having constructed the Dirac brackets, all the constraints can be strongly implemented (in the Dirac sense) which enables one to deal with the *reduced* Hamiltonian - the Hamiltonian with the constraints implemented - and the Dirac brackets.

Thus in the case of free electromagnetism the problems identified earlier due to the vanishing of π_0 are satisfactorily resolved by noting that this generates a secondary constraint

$$\partial_i \pi_i = 0 \tag{2.58}$$

- and this is precisely the Gauss law constraint for the free theory. An important aside is that, implementing the constraint (2.58) weakly, the operator $\partial_i\pi_i$ becomes the generator of gauge transformations via the Dirac bracket.

One has two first class constraints. The constraint algorithm terminates at this point. To make these second class one needs two gauge constraints and one choice is the radiation gauge

$$A_0 = 0 \tag{2.59}$$

$$\partial_i A_i = 0 \tag{2.60}$$

and constructing the matrix of constraints with this system gives a classical Dirac bracket which avoids the contradictions which occur with the naive Poisson bracket. This approach gives precisely the canonical system for quantising the free Maxwell theory used in [116].

Chapter 3

Non-Translationally Invariant Propagators

3.1 Fully-Fixed Gauge

3.1.1 Subsidiary Conditions

I demonstrated in the last chapter that the origin of the propagator ambiguity in the temporal gauge is the residual gauge-freedom under time-independent transformations. This immediately suggests one approach to resolving the propagator ambiguity which I wish to briefly review here and then present a similar, though somewhat simpler approach in the next section. I shall follow, up to a certain point, the exposition of Girotti and Rothe (GR) [61], although other authors have followed related approaches such as Leroy et al. [63], (LMR) and Lavelle et al. (LSV) [64]. I have chosen not to follow these derivations as the first requires a preliminary discussion of the Feynman kernel formalism of Rossi and Testa [117, 118] and the latter uses the technique of stochastic quantisation, both of which are outside the scope of this thesis.

The essential step in all these approaches, nevertheless, is to simply fix all the gauge degrees of freedom. Now in free electromagnetism this is achieved by the two gauge conditions

$$A_0 = 0 \tag{3.1}$$

$$\partial_i A_i(x) = 0 \tag{3.2}$$

the first defining the temporal gauge, the second being the Coulomb gauge condition. The two conditions can only be imposed simultaneously in a free Abelian gauge theory.

In an interacting Abelian theory, with current $J^\mu(x)$, the above conditions cannot be imposed simultaneously. This is easily seen in the Euler-Lagrange equation for the A_0

field in the Coulomb gauge

$$A_0(x) = \int d^3y \frac{1}{4\pi|x-y|} J_0(y) \tag{3.3}$$

so one cannot also impose the temporal gauge condition without constricting the physical degrees of freedom contained in the charge density J_0.

However a weaker form is valid, where, along with the temporal gauge condition, one imposes the Coulomb gauge at a *specific* instant in time

$$\partial_i A_i(t_0, \mathbf{x}) = 0 \,. \tag{3.4}$$

This is equivalent to setting the longitudinal part of the photon to zero at the arbitrary time and indeed removes all remaining unphysical degrees of freedom. That this is the case can be seen by performing a gauge transformation on the longitudinal field at the time t_0:

$$\delta A_i^L(t_0, \mathbf{x}) = \partial_i \omega(\mathbf{x}) \,. \tag{3.5}$$

If this transformation is to be constrained so as not to take the field out of the gauge where (3.4) applies, then this must vanish. So $\partial_i \omega(\mathbf{x}) = 0$, and for fields with vanishing space-like boundary conditions, this means ω itself vanishes; there is no freedom left.

3.1.2 Non-Abelian Case

For non-Abelian gauge theories this argument is complicated by the presence of the gauge field in the covariant derivative which appears in the gauge variation:

$$\delta A_i^{a\,L}(t_0, \mathbf{x}) = \partial_i \omega^a(\mathbf{x}) + g f^{abc} \frac{\partial_i}{\nabla^2} [A_j^{b\,T}(t_0, \mathbf{x}) \partial^j \omega^c(\mathbf{x})] \,. \tag{3.6}$$

The transverse component has mixed into the longitudinal field under the transformation. So the vanishing of this variation implies, by vanishing space-like boundary conditions, that

$$\omega^a(\mathbf{x}) + g f^{abc} \frac{1}{\nabla^2} [A_j^{b\,T}(t_0, \mathbf{x}) \partial^j \omega^c(\mathbf{x})] = 0 \tag{3.7}$$

and the derivative in the second term may be applied on the product of the transverse field and ω. Moreover we may act on the whole expression with ∇^2 to obtain

$$\nabla^2 \omega^a(\mathbf{x}) + g f^{abc} \partial^j [A_j^{b\,T}(t_0, \mathbf{x}) \omega^c(\mathbf{x})] = 0 \,. \tag{3.8}$$

Finally we use $\nabla^2 = -\partial_j \partial^j$ and invoke boundary conditions to obtain

$$\partial_j \omega^a(\mathbf{x}) - g f^{abc} A_j^{b\,T}(t_0, \mathbf{x}) \omega^c(\mathbf{x}) = 0 \,. \tag{3.9}$$

Contracting $\omega^a(\mathbf{x})$ into this expression causes the second term to vanish by antisymmetry in f^{abc} and one is left with $\omega^a(\mathbf{x}) \partial_j \omega^a(\mathbf{x}) = 0$ or, by boundary conditions, $\omega^a(\mathbf{x}) \omega^a(\mathbf{x}) = 0$.

For real functions ω this has unique solution $\omega^a = 0$. Thus, again, there is no gauge freedom left.

Now the significance of this is that complete gauge fixing is achieved for *arbitrary* t_0. GR argue that the limit $t_0 \to \pm\infty$ is necessary for the two conditions to be imposed simultaneously in the non-Abelian theory, a statement which is not supported by this analysis.

However there is merit in GR's proposed limit in that it reduces the above analysis to the simpler one given for the Abelian theory by using the property that the transverse gauge field vanishes at time $\pm\infty$:

$$A_j^{a\,T}(\pm\infty, \mathbf{x}) = 0 \tag{3.10}$$

This property follows from the vanishing at this point of the transverse propagator which is a Green's function of the field A_i^T[119]. That the transverse propagator vanishes follows from the $i\epsilon$ prescription which causes damping in the coordinate x_0 at infinity (of course this only reflects the physical boundary conditions one adopts in perturbation theory anyway).

From this, and considerations I shall come to, it may seem advantageous to seek to take this otherwise unnecessary limit. However, there are problems associated with taking the limit, which shall be discussed later.

Nonetheless, in both the Abelian and the non-Abelian cases one has selected a specific time and one might already suspect that time-translational invariance is compromised in *gauge dependent quantities.* For any physical observable such as the S-matrix time-translational invariance, along with invariance under other Poincare transformations should be recovered.

3.1.3 Propagator

Now I proceed to the derivation of GR's form for the time factor in the longitudinal propagator, and thus of the prescription they propose.

Resuming with GR, I write down the generating functional for Green's functions using the Faddeev-Popov method for implementing the gauge conditions. The free part suffices for deriving the gauge field propagator:

$$Z_0[\mathbf{J}^T, \psi] = \int \mathcal{D}A^{T\,i}\mathcal{D}\phi \prod_{i,\mathbf{x}} \delta(\partial^i\phi(t_0,\mathbf{x})) \exp i(S + \int d^4x J_i^T A^{T\,i} + \int d^4x \psi(x)\phi(x)) \tag{3.11}$$

where ϕ is a scalar potential related to the longitudinal field by

$$A_i^L(x) = \partial_i\phi \tag{3.12}$$

and $\mathbf{J}^T$ and ψ are current sources for the transverse vector and scalar fields respectively. Clearly, the $A_0 = 0$ condition has been imposed from the outset - or alternately, has been imposed via a delta functional and the integral absorbed into a normalisation factor. This leaves the functional integration over A_i which has been split into integrations over the transverse and the longitudinal parts separately - the latter achieved by integration over the scalar field ϕ. I have slightly deviated from the presentation of GR, who maintain the integration over A_i - but specify that it amounts to a transverse integration by introducing the Coulomb condition $\partial_i A_i$. This I find to be slightly misleading and thus have described it in this way. The introduction of the functional $\delta(\partial_i\phi(t_0, \mathbf{x}))$ imposes the subsidiary condition (3.4).

The action, expressed in terms of the transverse and scalar fields, is given by

$$1/2 \int d^4x (A^{T\,i} g_{ij} \Box A^{T\,j} + \phi \partial_0^2 \nabla^2 \phi) \tag{3.13}$$

where terms have been dropped after integrations by parts - permissible in perturbation theory in a trivial sector of the gauge field space where surface terms can be ignored.

Now the transverse sector will simply yield the propagator in the Coulomb gauge after inversion of the kinetic operator in the action.

For the longitudinal sector I exponentiate the delta function in the integrand via

$$\prod_{i,\mathbf{x}} \delta(\partial^i \phi(t_0, \mathbf{x})) = \exp[\frac{i}{2\lambda} \int d^4x \int d^4y \phi(x) \delta(x_0 - t_0) \nabla_x^2 \delta(y_0 - t_0) \delta(\mathbf{x} - \mathbf{y}) \phi(y)] \tag{3.14}$$

where the limit as $\lambda \to 0$ is implicit.

One finds that the longitudinal propagator is given by

$$D_{ij}^L(x, y) = i \lim_{\lambda \to 0} \partial_i^x \partial_j^y G(x_0, y_0; \mathbf{x} - \mathbf{y}; t_0; \lambda) \tag{3.15}$$

where the spatial dependence factors neatly in

$$G(x_0, y_0; \mathbf{x} - \mathbf{y}; t_0; \lambda) = -D_\lambda(x_0, y_0; t_0) \nabla^{-2} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{3.16}$$

and D_λ satisfies the equation[61]

$$\frac{\partial^2}{\partial x_0^2} D_\lambda - \frac{1}{\lambda} \delta(x_0 - t_0) D_\lambda = -\delta(x_0 - y_0)\,. \tag{3.17}$$

The presence of the delta function $\delta(x_0 - t_0)$ suggests, already, that the subsidiary condition (3.4) has conspired to break time translation invariance. GR now proceed to the $\lambda = 0$ limit but go into no details on the taking of the limit, something I shall return to. But denoting this limit in D_λ by D_{GR} and requiring that, for fixed y_0, D_{GR} remain finite as $|x_0| \to \infty$ leads to the unambiguous solution

$$D_{GR}(x_0, y_0) = \frac{1}{2}\epsilon(x_0 - y_0)(x_0 - y_0) + \frac{1}{2}(x_0 + y_0) - t_0 - \theta(x_0 - t_0)(x_0 - t_0) - \theta(y_0 - t_0)(y_0 - t_0)\,. \tag{3.18}$$

This is the form that GR give in [61]. The result was reexpressed in a more compact form by LMR[63]:

$$D_{LMR}(x_0, y_0) = -\frac{1}{2}|x_0 - y_0| + \frac{1}{2}|x_0 - t_0| + \frac{1}{2}|y_0 - t_0| \,. \qquad (3.19)$$

As I have noted, it is unnecessary to take $t_0 \to \pm\infty$. But taking t_0 *large* but still finite

$$D_{LMR}(x_0, y_0) \sim -\frac{1}{2}|x_0 - y_0| \pm \frac{1}{2}(x_0 + y_0) - |t_0| \qquad (3.20)$$

which connects with the result used by CCM in the Wilson loop computation. I shall discuss more fully the limiting properties of the GR/LMR form in a later section.

The remaining Feynman rules for gauge boson vertices in the non-Abelian theory follow from standard perturbative techniques. I shall discuss the relevance or otherwise of FP ghosts later as well.

GR have checked the more general form (3.19) for the time part in the Wilson loop and find that the t_0 dependence drops out - thus the limiting procedure has no bearing on the Wilson loop consistency - and the correct result is obtained.

3.2 Alternative Derivation of Propagator

3.2.1 Derivation

In this section I discuss a more 'intuitive' derivation of the propagator (3.19). This is based on my original work with Crewther in [94, 96].

The essence of this approach is to treat the temporal gauge theory (naturally) as the parallel-transported or gauge transformed version of the theory in a gauge in which there are no ambiguities. Provided that the process of performing the gauge transformation invokes no ambiguities, the resultant theory in the temporal gauge should be ambiguity free. The intention is therefore to perform transformations on field operators themselves, *as well as* their Green's functions - such as the propagator.

I shall work in the Abelian theory which suffices for the determination of the gauge field propagator. There are a number of choices I could make for the starting gauge - covariant (Feynman) gauge or another noncovariant gauge which seems to be understood in greater detail, such as the Coulomb gauge. In fact I shall start with the latter.

I thus denote the Coulomb gauge potentials by $C_\mu(x)$, satisfying $\partial_i C_i(x) = 0$, the potentials in the temporal gauge by A_μ. The parallel transport along a time-like curve (line) is, in general, given by

$$A_\mu(x) = U^{-1} C_\mu(x) U + i g U^{-1} \partial_\mu U \qquad (3.21)$$

where, to connect the temporal with the Coulomb gauge, U may be written

$$U(x_0, \mathbf{x}) = P\exp(-(1/ig)\int_{t_0}^{x_0} d\tau C_0(\tau, \mathbf{x})) . \tag{3.22}$$

Such a choice automatically ensures $A_\mu(x)$ satisfies the temporal gauge condition, $A_0 = 0$, as U satisfies the differential equation

$$\partial U/\partial x_0 = -(1/ig)C_0 U . \tag{3.23}$$

Thus, in this scheme, the temporal gauge fields are identified with the Coulomb gauge fields at time $x_0 = t_0$, where t_0 is, as in the previous section, some arbitrary but fixed instant in time. At later times the temporal gauge fields are 'transported away' from their Coulomb gauge equivalents.

In the Abelian case, the non-zero components of (3.21) reduce to

$$A_i(x) = C_i(x) - \partial_i^x \int_{t_0}^{x_0} d\tau C_0(\tau, \mathbf{x}) . \tag{3.24}$$

I now construct the temporal gauge propagator in a manner consistent with the Dyson-Wick expansion, as a time-ordered product of the vector fields

$$D_{ij}^{temp}(x,y) \equiv \theta(x_0 - y_0)\langle A_i(x)A_j(y)\rangle + \theta(y_0 - x_0)\langle A_j(y)A_i(x)\rangle \tag{3.25}$$

where the unordered (Wightman) functions are easily evaluated

$$\begin{aligned}\langle A_i(x)A_j(y)\rangle &= \langle [C_i(x) - \partial_i^x \int_{t_0}^{x_0} d\tau C_0(\tau, \mathbf{x})][C_j(y) - \partial_j^y \int_{t_0}^{y_0} d\tau' C_0(\tau', \mathbf{y})]\rangle \\ &= \langle C_i(x)C_j(y)\rangle + \partial_i^x\partial_j^y \int_{t_0}^{x_0} d\tau \int_{t_0}^{y_0} d\tau' \langle C_0(\tau, \mathbf{x})C_0(\tau', \mathbf{y})\rangle \end{aligned} \tag{3.26}$$

the 'mixed' terms vanishing as the one point functions $\langle C_i(x)\rangle = 0$. In the standard approach, C_0 is not expanded in *dynamical* fields. Inserting (3.26) and its reversed function into the ordered function (3.25), the various terms conspire, along with the fact that the instantaneous propagator is itself *unordered*, to return the original Coulomb gauge propagator components. Thus

$$D_{ij}^{temp}(x,y) = D_{ij}^{Coul}(x-y) + \partial_i^x\partial_j^y \int_{t_0}^{x_0} d\tau \int_{t_0}^{y_0} d\tau' D_{00}^{Coul}(\tau - \tau', \mathbf{x} - \mathbf{y}) . \tag{3.27}$$

Each term in this expression involves a well-defined Green's function; I can be confident then that the procedure has successfully given an unambiguous propagator. Using well-known momentum space representations of the individual terms[116] one finds:

$$D_{ij}^{temp}(x,y) = i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{1}{k^2 + i\epsilon}\left(g_{ij} + \frac{k_i k_j}{|\mathbf{k}|^2}\right)$$

$$- \quad i \quad \partial_i^x \partial_j^y \int_{t_0}^{x_0} d\tau \int_{t_0}^{y_0} d\tau' \int \frac{d^4k}{(2\pi)^4} e^{-ik_0(\tau-\tau')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \left(\frac{1}{|\mathbf{k}|^2}\right) . \tag{3.28}$$

The first term here is clearly the propagator for the transverse radiation fields, the second involving only the non-dynamical terms. The second piece shall be the focus of my attention, it being the longitudinal propagator I am interested in. Performing the derivatives and integrations gives

$$D_{ij}^L(x,y) = -i \int \frac{d^4k}{(2\pi)^4} \frac{k_i k_j}{|\mathbf{k}|^2} \frac{1}{k_0^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} (e^{-ik_0x_0} - e^{-ik_0t_0})(e^{ik_0y_0} - e^{ik_0t_0}) . \tag{3.29}$$

This factorises, as expected, in a space and the important time part, the latter taking the form:

$$D_{KC}(x_0,y_0) = \int \frac{dk_0}{(2\pi)} \frac{(e^{-ik_0x_0} - e^{-ik_0t_0})(e^{ik_0y_0} - e^{ik_0t_0})}{k_0^2} \tag{3.30}$$

where I now introduce the subscript KC to indicate this is our candidate propagator. Note that the integrand in this expression is well-defined at $k_0 = 0$. Performing the energy-integration gives our result for the time part of the longitudinal propagator

$$D_{KC}(x_0,y_0) = -1/2(|x_0 - y_0| - |x_0 - t_0| - |y_0 - t_0|) \tag{3.31}$$

which is precisely the expression derived by GR, LMR and also by LSV.

3.2.2 Connections with the CCM Result.

The question I primarily wish to address here is: Is the CCM result for D justified by the various approaches leading to propagator (3.31)?

Firstly, I wish to stress that expression (3.31) and the CCM result are *different*. However there are values of (x_0, y_0) for which they can be made to match. GR have observed that only in the regions

$$\begin{aligned} (x_0,y_0) &> t_0 \\ (x_0,y_0) &< t_0 \end{aligned} \tag{3.32}$$

does their (and my) propagator agree with the CCM result (2.48) if the arbitrary quantity γ in the CCM expression is identified with the arbitrary time, $|t_0|$. On the other hand these two forms for D disagree outside these regions; for $x_0 > t_0 > y_0$ and $y_0 > t_0 > x_0$ my result for the longitudinal propagator *vanishes*.

Now it is generally argued that if t_0 were chosen in the regions (3.32) then use of the CCM propagator would be justified by this approach. Hence the desire to take $t_0 \to \pm\infty$, so that t_0 will be in the regions (3.32) for all (x_0, y_0). But if the limit were to be taken directly in the propagator then the factor D immediately diverges. The option generally adopted is to take t_0 to be large, in other words satisfying either of the conditions in (3.32), but finite until the very end of a computation. In the context of the way in which a propagator is used in perturbation theory, as a kernel operator, it would appear that this implementation of the limit is also inadequate. In a typical calculation of, say, a scattering amplitude one would be integrating over *all* times x_0 and y_0, appearing in the propagator. One cannot keep the fixed value of t_0 both finite and outside the *infinite* region of integration which must be the case if one is to validly use the CCM propagator. So one is left with the remaining action of taking, *a priori*, t_0 infinite in, say, the negative time direction. In my parallel transport approach to an unambiguous temporal gauge theory, this effectively means evaluating the integral

$$\int \frac{dk_0}{2\pi} \int_{-\infty}^{x_0} d\tau \int_{-\infty}^{y_0} d\tau' \exp[-ik_0(\tau - \tau')] \tag{3.33}$$

but this integral is now *ambiguous* and I have thus failed again to obtain a fully regularised theory [1]. *There is no way to implement this limit.*

Before moving on I point out that Slavnov and Frolov (SF) [74] have given an alternative derivation of the CCM result within the context of functional integration, but not, apparently, as the limit of the propagator incorporating the time factor (3.31). The starting point of SF is to enforce Gauss' law as a constraint on states (strongly) and imposing the vanishing of the asymptotic transverse fields, just as GR have demanded. The CCM propagator was found by solving the classical equations of motion in the exact $A_0 = 0$ gauge in order to perform a (Gaussian) functional integral. The interesting point here is that the quantity γ in the CCM expression occurs also in the SF formalism but, though having the dimensions of 'length', is not easily interpreted as some arbitrary 'time', for example t_0. So, there may be an entirely different scenario in which the CCM propagator is applicable in the perturbative temporal gauge theory.

3.2.3 Green's Function Properties

Now I have to discuss the following issue. CCM give an argument in their work that the propagator must satisfy the same differential equation as (2.40) for the PV prescription:

$$(\partial^2/\partial x_0^2) D_{CCM}(x_0, y_0) = -\delta(x_0 - y_0) \, . \tag{3.34}$$

[1] This integral can be made unambiguous by introducing modulating factors in the exponents with opposite sign for each exponent. The resultant propagator in fact corresponds to the result in Steiner's *soft* temporal gauge [76], where $A_0 = 0$ is recovered only in the limit of the regulator being taken to zero. Thus taking the approach of parallel transport from the Coulomb gauge to the soft temporal gauge will automatically generate the modulating terms to give the above integral meaning. Because this is not an exact temporal gauge it is outside the scope of my concern in this work.

This was the basis on which they obtained their propagator; they solved (3.34) with the result

$$D_{CCM}(x_0, y_0) = -\frac{1}{2}|x_0 - y_0| + \alpha(x_0 + y_0) + \gamma \tag{3.35}$$

where α and γ remain unfixed by these considerations. Demanding Wilson loop consistency fixes α to be ± 1 giving the final CCM result (2.48). However my propagator (3.31) satisfies

$$(\partial^2/\partial x_0^2) D_{KC}(x_0, y_0) = -\delta(x_0 - y_0) + \delta(x_0 - t_0) \, . \tag{3.36}$$

One might argue that in the large $|t_0|$ limit, the second delta function term gives no contribution, implying that the result (3.31) gives a propagator which is a Green's function only, apparently, in this limit. However, as mentioned, there is no well-defined way in which to take this limit.

The quandary, which the literature does not explicitly discuss, would appear to be that the Green's function requirement on the propagator (and not completeness of the gauge-fixing) forces one to seek to take this impossible limit $|t_0| \to \infty$.

The subtlety that has not been stated clearly elsewhere is that associated with taking the limit $\lambda \to 0$ of the gauge parameter λ in (3.14).

Let us proceed with $\lambda \neq 0$ for the moment. Then the time dependent part $D_\lambda(x_0, y_0; t_0)$ of the longitudinal propagator satisfies (3.17). From (3.17), it is a simple matter to deduce the λ dependence of D_λ, viz.

$$D_\lambda(x_0, y_0; t_0) = D_{\lambda=0}(x_0, y_0; t_0) + \lambda \tag{3.37}$$

where $D_{\lambda=0}$ is required to satisfy the equations

$$\delta(x_0 - t_0) D_{\lambda=0} = 0 \tag{3.38}$$

and

$$\frac{\partial^2}{\partial x_0^2} D_{\lambda=0}(x_0, y_0; t_0) - \delta(x_0 - t_0) = -\delta(x_0 - y_0) \tag{3.39}$$

corresponding to terms proportional to λ^{-1} and those independent of λ when (3.37) is inserted into (3.17). Equation (3.38) is equivalent to the initial condition

$$D_{\lambda=0}(t_0, y_0; t_0) = 0 \, . \tag{3.40}$$

Now the $\lambda \to 0$ limit is trivial: the time factor in the longitudinal propagator becomes the quantity

$$D_{\lambda=0}(x_0, y_0; t_0) \tag{3.41}$$

which is defined by the λ independent constraints (3.39) and (3.40).

The differential equation (3.39) is equivalent to (3.36). Furthermore my candidate D_{KC} vanishes for $x_0 = t_0$, in agreement with (3.40). We conclude therefore

$$D_{\lambda=0}(x_0, y_0; t_0) = D_{KC}(x_0, y_0; t_0) \, . \tag{3.42}$$

3.3 Non-Abelian Theory

I now consider complications that arise in the context of a non-Abelian theory - in particular those introduced by FP-ghosts. This exposition is primarily developed from ideas in the unpublished thesis of Otto [120] who attempts to compute the pure YM two loop renormalisation group beta function in the temporal gauge using the CCM result for the time part (2.48). Though successful at the one loop level in obtaining the correct result [100]

$$\beta(g_R) = -\frac{g_R^3}{(4\pi)^2}\frac{11}{3}C_2(G) \tag{3.43}$$

the computation fails at two loops, or order g_R^5. One possible explanation raised in Otto's work for this failure is the absence of FP ghosts in the computation. The question that arises here is why ghosts should affect the two loop result and not that for one loop. However, my concern here is not so much for the beta function problem observed by Otto, whose calculation remains unpublished and unverified, but rather on the general role FP ghosts play in this approach.

That FP ghosts can arise is evident if I follow the FP method more carefully for the fully-fixed gauge conditions. The FP determinant is given by

$$\Delta_F^{-1}[A] = \int \mathcal{D}g \prod_{x_0,\mathbf{x}} \delta({}^gA_0(x_0,\mathbf{x})) \prod_{\mathbf{x}} \delta(\partial_i {}^gA^i(t_0,\mathbf{x})) \tag{3.44}$$

and this leads to the expression, in component form, for the implied FP matrix

$$M^{ab} = \delta^{ab}\nabla^2 + gf^{abc}A_i^c(t_0,\mathbf{x})\partial_i \, . \tag{3.45}$$

Now given that at $x_0 = t_0$ the longitudinal fields vanish, this expression only generates coupling between FP ghosts and the transverse gauge fields. For finite t_0, the ghosts remain coupled!

The need to take $t_0 \to \pm\infty$ has been dismissed, but given that $A_i^{a\,T}(\pm\infty,\mathbf{x}) = 0$ it may seem that the ghosts will decouple in this limit. But I have already discussed the dubiousness of the limiting argument as a justification for using the propagator with (2.48). The question one is now faced with is: are FP ghosts necessary when using the CCM propagator?

Does the Wilson loop offer any clues? In the Wilson loop computations with these types of propagators, which have all been carried out with t_0 *finite*, ghosts have clearly been unnecessary for the success of the exponentiation criterion. Indeed LMR [63] show that incorporating ghosts consistent with (3.45) does not affect the exponentiation: the ghosts amount to a factor independent of the time-length of the loop contour which does not contribute to the static potential. Thus the static Wilson loop is not sensitive to the relevance or otherwise of these Coulomb-type ghosts to more general perturbative computations, in particular the beta function computation attempted by Otto. The Wilson loop offers no information on this question.

The most rigorous derivation of the CCM propagator is that of Slavnov and Frolov[74] who obtain the CCM result, but do not address the question of ghost decoupling. In the absence of further information, one can only conclude from the above concerns about the limiting process that it is desirable to work *a priori* with t_0 finite using the propagator with time part (3.31), *with* ghosts. To date, there has been no work beyond the static Wilson loop employing these criteria.

3.4 Conclusions

To summarise the conclusions reached within the approach of fully-fixed temporal gauge theories:

- The question of the ambiguity in the propagator in the temporal gauge may be resolved by starting in a well-defined gauge such as the Coulomb gauge, and transforming the propagator directly into the temporal gauge.

- This inevitably yields a result non-invariant under time translations - because the time-slice from which one parallel transports remains in the formalism.

One must be cautious about limits with the initial time, t_0:

- The non-Abelian theory does not force one to take the limit $t_0 \to \pm\infty$ in order to achieve a valid complete gauge fixing. But were one to seek to employ the limit, it cannot be taken immediately (when parallel transporting) nor in the propagator itself.

- The approach of fully-fixing the gauge does not justify the use of the CCM propagator. Thus, though there may be other derivations giving the CCM result[74], the argument for the CCM result invoking the limit $t_0 \to \pm\infty$ in (3.31) is not valid.

- In this context then, the general form for the propagator longitudinal time part (3.31), as derived by GR, LSV, LMR and myself, should be used with t_0 kept finite.

- The factor (3.31) *is* a Green's function of the equations of motion with the subsidiary condition (3.4) properly taken into account.

- FP ghosts formally decouple in the FP determinant when $t_0 = \pm\infty$ based on the argument that the asymptotic transverse gauge field vanishes. However, because there is no well-defined way to employ this limit in computations, this does not permit the exclusion of ghosts from a general computation. Thus in general one should compute in this gauge with the propagator incorporating the factor (3.31) *and* FP ghosts, with t_0 kept finite.

One issue I have not taken into account are the problems that seem to be inherent in the Coulomb gauge itself [121, 122] which might manifest themselves while the arbitrary time t_0 is finite. The other, which I have already mentioned, is that there may be an alternate approach in which the CCM propagator is entirely valid [74].

The final conclusion I wish to draw out in this context is that the non-translational invariance and the presence of ghosts - though constituting a consistent approach to the quantisation of YM theory in the temporal gauge - render the theory cumbersome, overwhelm any advantage the naive temporal gauge might have presented, and make the task of proving renormalisability intractable.

In the following chapters I therefore abandon this approach and pursue the alpha-prescription as a possible method of dealing with the temporal gauge.

Chapter 4

The Alpha-Prescription

4.1 Introduction

The 'alpha-prescription', suggested by Landshoff [75], has been given scant attention in the literature, a surprising fact given the stalled progress achieved by other prescriptions in the temporal gauge. In the alpha prescription the regulated propagator takes the form

$$D^{ab}_{\mu\nu}(p) = \frac{\delta^{ab}}{(p^2+i\epsilon)}[g_{\mu\nu} - \frac{p\cdot n(p_\mu n_\nu + p_\nu n_\mu)}{(p\cdot n)^2+\alpha^2(n^2)^2} + \frac{n^2 p_\mu p_\nu - \alpha^2 n^2 n_\mu n_\nu}{(p\cdot n)^2+\alpha^2(n^2)^2}] \,. \tag{4.1}$$

The point at which the limit $\alpha \to 0$ is to be taken is a vexed question to which I shall return.

The features of this propagator are several: first, it exactly fulfills the condition for a 'true' temporal gauge propagator, in that it is orthogonal to the gauge-field vector, $n^\mu D_{\mu\nu}(p) = 0$ as an *algebraic* identity. This is especially useful in Wilson loop computations as I have discussed. In the next chapter I shall outline Landshoff's demonstration that (4.1) satisfies the Wilson loop criterion without requiring the introduction of FP ghost-fields. An important observation Landshoff notes is that keeping α non-zero until the very end of the computation enabled dangerous $1/\alpha$ terms to cancel, making the limit $\alpha \to 0$ safe, and *leaving behind* precisely those terms that gave the correct result.

To simplify things I consider the i, j components of the propagator for the case $n = (1,0,0,0)$ again. In the alpha prescription the propagator becomes:

$$D_{ij}(p) = \frac{1}{p^2+i\epsilon}\left[g_{ij} + \frac{p_i p_j}{|\mathbf{p}|^2}\frac{1}{p_0^2+\alpha^2}\right] \,. \tag{4.2}$$

Computing the longitudinal part of the propagator reveals that, unlike the prescriptions considered in the last chapter, the factorisation between the space and time dependences no longer occurs. Also, the Fourier transform with respect to the momentum p_0 contains a term of the form

$$\frac{p_i p_j}{|\mathbf{p}|^2+\alpha^2}\frac{1}{2\alpha}e^{-\alpha|x_0-y_0|} \tag{4.3}$$

and one sees the negative power of alpha that is a consequence of the energy (p_0) integration. So there is the potential for pathologies to arise in taking the limit $\alpha \to 0$.

This result conveys yet more information: firstly, that the dimension of α is that of *mass*, and secondly that the coordinate space two-point Green's function is in itself ill-defined in the limit $\alpha \to 0$. There is no gaurantee that Green's functions in general will be well defined in this limit, and that, if the alpha prescription is a valid prescription, one should only expect safe limit properties as pertaining to physical or gauge independent quantities - such as the S-matrix, or the Wilson loop. To expand on this point a little more: the expectation should be that for nonzero α a quantity computed with the alpha prescription would not itself constitute the gauge independent result. Rather, that once the (hopefully safe) limit $\alpha \to 0$ is performed the correct physical result should be restored.

4.2 A Derivation of the Alpha-Prescription

In seeking a derivation for the alpha prescription the first option that must be dismissed is interpreting the alpha prescription in the sense of *distributions*[27]. The result (4.3) contradicts this possibility for which one would expect that the limit $\alpha \to 0$ should be viable in the Fourier transform of the momentum space propagator.

It is clear then that the parameter α must appear in the Lagrangian from the outset. Steiner's derivation [76] is in this vein: he introduces a modification of the temporal gauge, the *soft* temporal gauge, where A_0 no longer vanishes but is proportional to α. In this derivation, the final propagator, though bearing the appropriate denominator $(p.n)^2 + \alpha^2(n^2)^2$, does not retain the same tensor structures as the form given by Landshoff and so does not exactly satisfy $n_\mu D^{\mu\nu}(p) = 0$. Certainly in the limit $\alpha \to 0$ the temporal gauge is restored, but it is not clear that at the quantum level this approach will reproduce the success of the alpha prescription, for example, in the Wilson loop.

Indeed any legitimate, gauge-fixing condition imposed on Yang-Mills theory is *a priori* ruled out by the Cheng-Tsai theorem [68, 69, 70], which cannot accomodate the tensor structure $n_\mu n_\nu$ in the propagator.

A recent preprint by Milgram[123] suggests that the prescription can be obtained by including a 'damping' term into the Lagrangian, like $\alpha^2 A^2$. But this generates an α^2 in place of the normal Feynman prescription for the $p^2 = 0$ pole, and it is known that having the same parameter in both Feynman and spurious prescriptions does not give the correct result for the Wilson loop[124].

Thus the answer to the origin of this prescription is available only outside the familiar structure of Yang-Mills + gauge-fixing in the Lagrangian.

With these considerations in mind I introduce the following path-integral formulation of the generating functional of Green's functions for an 'extended' gauge-theory:

$$Z[J] = N \int \mathcal{D}A \exp i \int d^4x[\mathcal{L}_{YM} + \mathcal{L}_\alpha + J \cdot A] \tag{4.4}$$

where $\mathcal{L}_{YM}$ is the usual Yang-Mills Lagrangian density, and the additional Lagrangian density is

$$\mathcal{L}_\alpha = -(1/2)\mathcal{D}^{ab\nu} K_n^{bc} n^\mu F^c_{\mu\nu} \left(\frac{\alpha^2 n^2}{Tr(\mathcal{D}^2)/(N^2-1) - \alpha^2 n^2}\right) \mathcal{D}^{ab'\nu'} K_n^{b'c'} n^{\mu'} F^{c'}_{\mu'\nu'} \,. \tag{4.5}$$

Here $\mathcal{D}^{ab}_\mu$ denotes the covariant derivative on the gauge group defined in the second chapter. By K_n^{ab} I mean the 'inverse' of the covariant derivative contracted with the gauge-vector, namely $(n \cdot \mathcal{D}^{ab})^{-1}$, which I shall expand perturbatively in the coupling constant, g:

$$K_n^{ab} = \delta^{ab}(1/n \cdot \partial) + g f^{acb}(1/n \cdot \partial) n \cdot A^c (1/n \cdot \partial) + \dots \,. \tag{4.6}$$

I am being rather cavalier in my treatment of the $1/n \cdot \partial$ factors. A careful treatment of this does give rise to some subtleties but I shall come to these later.

The other unusual quantity appearing in the alpha Lagrangian is the colour trace (Tr) of the square of the covariant derivative, suitably normalised by $\delta^{ab}\delta^{ab} = N^2 - 1$ in the adjoint representation:

$$Tr(\mathcal{D}^2)/(N^2-1) = \Box + (C_2(G)/(N^2-1))A^2 \,. \tag{4.7}$$

I observe a number of features in the extra Lagrangian term. For the bare Lagrangian $\mathcal{L}_\alpha$ the limit $\alpha \to 0$ is safe and it vanishes in that limit, restoring the original YM theory. The extra term breaks Lorentz invariance (via the presence of the gauge-vector n_μ). But, again, in the limit $\alpha \to 0$ the Lorentz symmetry is restored. One may hope that the quantised theory will respect this behaviour. The gauge-invariance of the extended theory also means I can apply the Faddeev-Popov procedure to fix the gauge and extract Feynman rules, and I shall fix in a non-covariant gauge. However, at this point there might be a concern that the operator K_n^{ab} involves an *infinite* number of gluon vertices, and moreover, with a knowledge of the infamous history of the propagator problem in these gauges in mind, the more alarming feature of $(1/n \cdot \partial)$ factors (or in momentum space, $1/p \cdot n$) reappearing in perturbative Green's functions. Have I attempted to 'prescript' these poles in the propagator, only to bring them back in, unprescripted, in the vertices? I shall show that the virtues of the alpha-presciption will enable one to overcome this particular problem.

I fix the gauge by following the method in chapter 2: introduce into the functional measure of (4.4) the delta function

$$\delta(n \cdot A^a - C^a) \tag{4.8}$$

and integrate over C^a with a sharp Gaussian weight. The standard Faddeev-Popov procedure indicates that FP ghosts decouple. This conclusion is actually correct, as will be shown later. Thus the FP method gives an effective Lagrangian

$$\mathcal{L}_{eff} = \mathcal{L}_{YM} + \mathcal{L}_{\alpha} - (1/2\lambda)(n\cdot A)^2 \, . \tag{4.9}$$

I shall be interested in the limit $\lambda \to 0$.

4.2.1 Propagator

I extract the kinetic operator in (4.9), go to momentum space, and invert the resulting quantity and I arrive at the gauge-field propagator:

$$D^{ab}_{\mu\nu}(p) = \frac{\delta^{ab}}{p^2}[g_{\mu\nu} - \frac{p\cdot n(p_\mu n_\nu + p_\nu p_\mu)}{(p\cdot n)^2+\alpha^2(n^2)^2} + \frac{n^2 p_\mu p_\nu - \alpha^2 n^2 n_\mu n_\nu}{(p\cdot n)^2+\alpha^2(n^2)^2} + \lambda\frac{p^2 p_\mu p_\nu}{(p\cdot n)^2}] \, . \tag{4.10}$$

Now one may register concern at the continued appearance of a $1/(p\cdot n)^2$ term here unprescripted. Were one made of sterner stuff and wished to pursue calculations with λ non-zero, this problem could be easily surmounted by employing a non-local gauge-fixing Lagrangian,

$$\mathcal{L}_{gf} = (-1/2\lambda) n\cdot A^a[1-\alpha^2(n^2)^2/(n\cdot\partial)^2] n\cdot A^a . \tag{4.11}$$

Inverting the quadratic part with this yields the same alpha-propagator with the λ term prescripted with the denominator, $(p\cdot n)^2 + \alpha^2(n^2)^2$. The ghost structure is now different but I shall discuss the decoupling even in this case a little later. Nonetheless, my intention is to take $\lambda \to 0$ and upon doing so I obtain the precise form of the alpha-prescription from (4.10).

4.2.2 New Vertices

The YM Lagrangian yields the usual 3-gluon and 4-gluon vertices. The specfically non-Abelian parts of the alpha Lagrangian demand new vertices in the theory. Of course, as far as generating the alpha prescription in the propagator is concerned these vertices are unimportant, and nor is the enforced gauge invariance. However, it would seem remarkable that a gauge-invariant additional term to the quadratic part suffices to generate an unambiguous propagator - and this gauge invariance (or BRST invariance once the gauge fixing term is introduced) may be a useful property to maintain in the non-Abelian theory, particularly to enable derivation of identities that may guide a proof of renormalisability at higher orders in perturbation theory. A question to be asked is: what price has to be paid for retaining BRST invariance? Does the retention of the symmetry force additional complications that may overwhelm any positive advantages in maintaining the symmetry?

The new vertices implicit in the alpha Lagrangian (4.5) fall into two groups. In the first instance I concentrate on the structure in the alpha-Lagrangian:

$$\mathcal{D}^{ab\,\nu} K_n^{bc} n^\mu F^c_{\mu\nu} \tag{4.12}$$

which itself yields two types of terms, aside from that which directly contributes to the propagator. There is the term arising from the structure factor in the covariant derivative, and the first Abelian term from the field-strength tensor:

$$f^{abd} A^{d\nu} (\delta^{bc}/n \cdot \partial) n^\mu \partial_\mu A^c_\nu \tag{4.13}$$

which clearly reduces, assuming that $n{\cdot}\partial/n{\cdot}\partial = 1$, to an antisymmetric quantity multiplied by a symmetric piece, and a summation over colour indices; the quantity vanishes. Now the subtleties I have alluded to are connected with the naivety of this argument - but I defer this for the present.

What this leaves for non-propagator contributions from here are terms with an n_μ sitting in the resultant Feynman rule for a vertex. In any diagram this vertex attaching to a propagator of an internal gluon line will kill off most terms via the property of the alpha-prescription that $n^\mu D_{\mu\nu}(p) = 0$. If I make the restriction to S-matrix elements, characterised by the property that Feynman graphs contain gluon propagators in *external* legs, then all such diagrams built with these vertices are seen to decouple.

It should be said that the ordering of operators in (4.5) is non-unique as far as deriving the alpha-prescription is concerned. However, it is the unique property of the form given here that these (the most unpleasant) extra vertices decouple from unamputated S-matrix amplitudes.

The second type of vertex arises from the second non-local operator in (4.5). I am forced to actually extract the Feynman rule corresponding to this interaction part. In fact, there are also an infinite number of these, corresponding to an expansion of the operator in the coupling constant, g. Indeed, these give rise to gluon vertices involving 4,6,8..etc. gluons. I illustrate the structure of these vertices by examining the lowest order contribution - a four gluon vertex indicated in figure (4.1).

After extracting the term quartic in the gauge-field, and symmetrising, I can write in momentum space:

$$V^{abcd}_{\mu\nu\rho\lambda}(p,q,k,r) = -i\alpha^2 n^2 g^2 \frac{C_2(G)}{(N^2-1)} [\delta^{ab}\delta^{cd}(g_{\mu\nu}[k_\lambda, r_\rho] + g_{\lambda\rho}[p_\mu, q_\nu]) + \textit{cyclic perms.}] \tag{4.14}$$

where, in the interest of compactness, I introduce the notation:

$$[A, B] = \tilde{A}B + A\tilde{B} \tag{4.15}$$

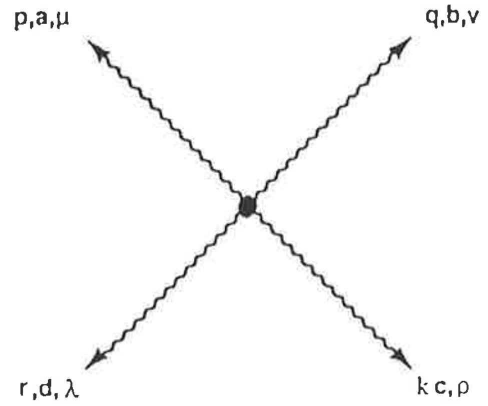

Figure 4.1: The lowest order new vertex for α non-zero: a four vertex.

and

$$\tilde{p}_\mu = p_\mu/(p^2 + \alpha^2 n^2)^2 \,. \tag{4.16}$$

The lowest order diagram that can be constructed from this graph is a non-vanishing tadpole graph. Scaling arguments reveal the truncated graph to be proportional to α^2. It remains to be seen whether this is sustained when inserted into the static Wilson loop or whether, after the limit $\alpha \to 0$ - if permissible - is taken, contributions survive which may be inconsistent with gauge-invariance.

4.2.3 Comments on BRST Quantisation

By virtue of the gauge invariance in the unfixed alpha Lagrangian a BRST approach to the quantisation of the theory may be taken. In particular it is possible to compute the conserved BRST charge consistent with the Lagrangian (4.5). Schematically this would give the form $Q^B_Y M + Q^B_{(\alpha)}$ where $Q^B_{(\alpha)}$ is the contribution arising from the specific alpha dependent term in the Lagrangian. Using this quantity the physical states of the Hilbert space may be defined as those annihilated by this charge [17] and I postulate that such states would transform into the physical states of the pure YM theory in the limit $\alpha \to 0$, where the charge $Q^B_{(\alpha)}$ itself vanishes. Further properties such as Lorentz invariance in the physical space may be demonstrated.

I have been content to outline this program schematically as the complications of the new vertices, and the subtleties described below cause one to question whether this approach to the alpha prescription is appropriate within the parameters outlined at the very start - of determining an unambiguous YM theory in the temporal gauge which *minimises the presence of additional fields and complications*, and this would include the new vertices. Thus the BRST quantisation of the theory with (4.5) has not been pursued in any detail.

4.2.4 Subtleties

To turn now to the complications implicit in the non-local operator $1/n \cdot \partial$. This quantity implies an integral operator involving some kernel, κ:

$$\frac{1}{n \cdot \partial} f(x_0, \mathbf{x}) = \int_{-\infty}^{+\infty} dt \kappa(t - x_0) f(t, \mathbf{x}) \tag{4.17}$$

where I now use the specific representation of the gauge-vector $n_\mu = (1, 0, 0, 0)$, and the kernel must satisfy the Green's function condition that

$$\frac{\partial}{\partial t} \kappa(t - x_0) = \delta(t - x_0) \, . \tag{4.18}$$

This plays its role in the factor in the alpha-Lagrangian (4.5):

$$\partial^\nu (1/n \cdot \partial) n^\mu F_{\mu\nu} \tag{4.19}$$

(I ignore for the moment gauge group indices). When the field strength tensor is expanded one can see that there are terms where the non-local operator acts on $n \cdot A$, which will be eliminated in the gauge-condition, so these do not concern us. The significant term in (4.19) is

$$\partial_i \frac{1}{\partial_0} \partial_0 A_i(x) \, . \tag{4.20}$$

Treating this naively earlier I set $\frac{1}{\partial_0}\partial_0$ to one, irrespective of the choice of kernel, κ. But in fact writing this with (4.17) and using integration by parts to truly act with the derivative on the kernel, one sees that one is left with a surface term depending on the value of the field $A_i(x)$ at *time* plus or minus infinity. Indeed choosing

$$\kappa(x) = \theta(x) \tag{4.21}$$

where θ is the usual Heaviside step function being 0 for negative values and 1 for positive, one sees that the value of the gauge-field at plus infinity is involved. I cannot ignore it. I find an extra term appearing in the kinetic term that I had previously ignored involving

$$\partial_i A_i(\infty, \mathbf{x}) \, . \tag{4.22}$$

Thus insofar as ignoring this term amounts to generating the alpha-prescription, a *fully fixed* gauge has been implicit all along - defined by the two gauge conditions that were the concern of the last chapter:

$$A_0 = 0 \tag{4.23}$$

$$\partial_i A_i(\infty, \mathbf{x}) = 0 \, . \tag{4.24}$$

The difference here is that the second equation in (4.24) is not implemented as the limit of some time $t_0 \to \pm\infty$. Rather the subsidiary condition is taken *a priori* at time infinity. Here it is permissible to take time infinity immediately as the subsequent propagator does not diverge; the arbitrary time t_0 has not been invoked at any stage of the analysis and thus the propagator does not depend on such a parameter. The reason why no time translational invariance is introduced rests on just this point: the boundary condition at $t_0 = \infty$ has been applied from the start and the non-zero alpha renders quantities well defined in this limit - unlike the ambiguous quantity (3.33). In other words, integrals have been regulated *within* the temporal gauge rather than by breaking the temporal gauge condition as in Steiner's approach [76].

The recognition of the subtlety in $1/\partial_0$ has potentially severe consequences - the introduction of yet more vertices. This problem can be minimised by staying with the choice of the kernel as a theta-function; this choice now only affects the form of the theory in the 'alpha-sector'; the YM vertices are unaffected by this choice. Since the hope is that in the limit $\alpha \to 0$ the non-YM sector drops out, the specific choice of κ is irrelevant, except that it is *this choice* that gives the alpha-prescription.

Reconsidering the full alpha-Lagrangian I find that the lowest order Feynman rule (again, there are an infinite number) arises from the structure-constant terms in the covariant derivative factor, acting on the non-local $\Box - \alpha^2 n^2$ factor in the middle, giving a term in the Lagrangian:

$$-(1/2)g^2 f^{abc} f^{ab'c'} A_i^c A_i^{out,b} \left(\frac{\alpha^2 n^2}{\Box - \alpha^2 n^2}\right) A_j^{c'} A_j^{out,b'} \tag{4.25}$$

where

$$A_i^{out,a} = A_i^a(\infty, \mathbf{x}) . \tag{4.26}$$

But now recall the result used in the last chapter: that the transverse fields $A_i^{T\,a}$ vanish at time $\pm\infty$. This, together with vanishing of the longitudinal part implicit in the complete gauge-fixing condition, mean that indeed $A_i^{out,a}$ vanishes, and this last set of vertices do not contribute.

One is still left with the infinite set of new vertices which includes (4.14). As mentioned, this has an overall factor of α^2 and one can only hope that this and the higher order corrections will only give vanishing contributions to *at least* the static Wilson loop.

4.2.5 FP Ghosts

I conclude this section with a further analysis of the ghost structure. I have mentioned that implementing the Faddeev-Popov method with the gauge invariant extended system (4.4) the standard ghosts can be seen to decouple in the Lagrangian itself using the arguments presented in chapter 2 for the homogeneous case, which is strictly all I am concerned with in this work. However, even in the inhomogeneous case the arguments presented in chapter 2 apply. Firstly, if the theory with gauge fixing Lagrangian (4.11) were pursued the argument used by [40] for the planar gauge would apply here with appropriate modification. Frenkel's approach[102] to the theory with gauge fixing Lagrangian $-(1/2\lambda)(n\cdot A)^2$ but $\lambda \neq 0$ might seem problematic. Recall that here one carries the ghosts into the perturbative formalism and argues the decoupling on the basis of integrals that vanish in dimensional regularisation. But the method I have used to generate the alpha prescription in the gauge field propagator has not correspondingly given meaning to the $p \cdot n = 0$ pole in the ghost propagator. This is as it should be. Recall that the ghosts arise from a BRST variation in the *gauge fixing term* and not in the gauge invariant (thus BRST invariant) part. In this theory I have generated the alpha prescription not by introducing alpha into the gauge fixing but by modifying the original gauge invariant theory in a gauge invariant way. Thus the fact that a prescription has been generated in the gauge field propagator should not (and does not) correspondingly generate a prescription for the ghosts - because the gauge fixing Lagrangian remains unchanged in the alpha prescription. Frenkel's argument for ghost decoupling may be seen to still apply by noting that ghosts really matter only if they contribute to loops. Hence any prescription which gives zero in dimensional regularisation for all loops will correspond to a decoupling of the ghosts.

But the subtleties noted in inverting ∂_0 also have a potentially more serious impact on the homogeneous case. The ghosts that seem to be carried in this formalism for the alpha prescription are the Coulomb ghosts at infinity. Here too in fact there is no problem: because the boundary condition (4.24) at infinity is not being implemented as a limit of $t_0 \to \pm\infty$ the argument for decoupling of these Coulomb ghosts discussed in the last chapter would apply in this case.

Thus the alpha prescription is indeed a ghost free approach to the temporal gauge, although, in this derivation, the insistence of BRST invariance for nonzero alpha introduces other complications that may outweigh this positive feature.

4.3 A Local Approach

The excessive complications encountered in the last section may be avoided in the approach to the derivation of the alpha prescription taken by Przeszowski in [125]. In this last section of this chapter I shall review his approach and follow it in the next chapter with my work checking the recovery of Poincare invariance of the Abelian theory.

The sources of unnecessary complexity in the last section were: the retention of gauge invariance, and the use of the nonlocal operator $(n \cdot \partial)^{-1}$ in the Lagrangian itself. Przeszowski surmounts the latter by introducing an additional scalar field, denoted by $Q(x)$, which couples to the gauge field. Though Przeszowski deals solely with the Abelian theory, I shall begin with a non-Abelian version of his theory, before outlining Przeszowski's work with the Abelian theory in a Dirac Hamiltonian approach.

4.3.1 NonAbelian Formulation

In this case, the Lagrangian may be written in the local form:

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu} - \frac{1}{2}\partial_\mu Q^a \partial^\mu Q^a + \frac{1}{2}\gamma^2 Q^a Q^a - \gamma \partial^\mu Q^a A^a_\mu + B^a A^a_0 \,. \tag{4.27}$$

This theory corresponds to the gauge field sector coupling, via a derivative coupling of strength γ, to a Klein-Gordon field Q - which carries a Lie group index - of mass γ. This quantity becomes the parameter regulating the gauge field propagator, which, when analytically continued to $-i\alpha$, generates the alpha prescription.

This theory is not gauge invariant. So there is no reason for the inclusion of the temporal gauge fixing term via the multiplier field $B^a(x)$ except that I precisely intend to investigate the hypothesis that the limit $\gamma \to 0$ generates, unambiguously, results corresponding to YM theory in the temporal gauge. Of course, if this Lagrangian does give a consistent S-matrix theory the limit $\gamma \to 0$ may only be well-defined in gauge independent quantities.

The Feynman rules could be formulated to include the usual rules for the YM sector, and additional rules corresponding to the Q field propagator - just the propagator for a Klein-Gordon field of mass γ - and a $Q - A$ propagator derived from the mixing term in (4.27). Alternately, in the path integral formalism, where the measure of the generating functional now includes a functional integration $\mathcal{D}Q$, I may perform this Gaussian integral and derive Feynman rules for the theory purely in terms of the YM sector. This produces a term in the effective Lagrangian of the form

$$\frac{1}{2}(\gamma \partial^\mu A^a_\mu)(\Box + \gamma^2)^{-1}(\gamma \partial^\nu A^a_\nu) \,. \tag{4.28}$$

Ignoring the extra terms in (4.5) put in to maintain gauge invariance, and dropping terms proportional to $n \cdot A$, the result is precisely the expression (4.28) in the temporal gauge with $\gamma^2 = -\alpha^2$. Thus I indeed extract the alpha prescription in this formulation, after continuing $\gamma \to -i\alpha$, but with no new self-coupled vertices for the gauge field, nor subtleties arising from inverting the operator $n \cdot \partial$. On the other hand the theory no longer obeys any manifest symmetry in the Lagrangian (not even a modified BRST symmetry).

So the task of dealing with the theory at higher orders of perturbation theory becomes difficult if not impossible.

In addition, because of the breaking of BRST invariance in this approach, I no longer have available the elegant approach of [17] to deal with the question of redundant states in the Hilbert space of the theory and the implementation of the Gauss law. This forces us to deal with an explicitly Hamiltonian approach to the theory, in particular using the Dirac constraint algorithm outlined in the second chapter. Unfortunately here the Hamilton equations of motion for the non-Abelian theory remain an obstacle to tackling these questions. For that reason I confine the remaining discussion to the free Abelian theory contained in Lagrangian (4.27) as considered by Przeszowski.

4.3.2 Free Theory and Method of Constraints

So I restrict the analysis to the local Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\partial_\mu Q\partial^\mu Q + \frac{1}{2}\gamma^2 Q^2 - \gamma\partial^\mu Q A_\mu + BA_0 \tag{4.29}$$

with $F_{\mu\nu}$ now just the Maxwell field strength tensor.

Przeszowski has given the Dirac constraint analysis for this system [125]. I reproduce the essential features of it here. With conjugate momenta to all the field variables defined in the usual way, the full system of primary and secondary constraints (all second class) is given by

$$\psi_1 = \pi_0 \tag{4.30}$$

$$\psi_2 = \pi_B \tag{4.31}$$

$$\psi_3 = \partial_i\pi_i + B + \gamma(\pi_Q + \gamma A_0) \tag{4.32}$$

$$\psi_4 = A_0 \tag{4.33}$$

and the inverse of the Poisson Bracket matrix of these constraints is given by

$$C^{-1}(x,y) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\gamma^2 \\ 0 & -1 & 0 & 0 \\ -1 & \gamma^2 & 0 & 0 \end{pmatrix} \delta^{(3)}(\mathbf{x}-\mathbf{y})\,. \tag{4.34}$$

It is useful to note that the naive Poisson Bracket approach to the theory suffices when calculating Poisson Brackets between functionals of the independent coordinates (A_i, π_i) and (Q, π_Q). The above matrix of constraints has the property that any corrections to the Dirac Brackets for such quantities will vanish. Thus for the most part one is permitted to use just the ordinary Poisson Brackets based on the definition in appendix A

$$\{u,v\} = \int d^3z[\frac{\delta u}{\delta A_i(z)}\frac{\delta v}{\delta \pi^i(z)} - \frac{\delta u}{\delta \pi_i(z)}\frac{\delta v}{\delta A^i(z)} + \frac{\delta u}{\delta Q(z)}\frac{\delta v}{\delta \pi_Q(z)} - \frac{\delta u}{\delta \pi_Q}\frac{\delta v}{\delta Q(z)}] . \tag{4.35}$$

The Dirac constraint analysis eventually leads to a reduced Hamiltonian density

$$\mathcal{H} = 1/2(\pi_i)^2 - 1/2(\pi_Q)^2 + 1/2(\partial_i A_j)^2 - 1/2(\gamma Q - \partial_i A_i)^2 - 1/2(\partial_i Q)^2 . \tag{4.36}$$

The canonical commutator structure for the A_i and Q fields and their conjugate momenta follow - after transposing Dirac Brackets into equal time commutators. More importantly working now with the reduced Hamiltonian and Dirac brackets one derives the following Hamilton equations of motion:

$$\begin{aligned} \dot{\pi}_i &= \triangle A_i + \partial_i(\gamma Q - \partial_j A_j) && (4.37)\\ \dot{\pi}_Q &= -\triangle Q + \gamma(\gamma Q - \partial_j A_j) && (4.38)\\ \dot{A}_i &= \pi_i && (4.39)\\ \dot{Q} &= \pi_Q . && (4.40) \end{aligned}$$

The Dirac brackets for the field operators $Q(x)$ and $A_i(x)$ can be taken to equal-time commutators to give the standard results

$$\begin{aligned} [A_i(t,\mathbf{x}), \pi_j(t,\mathbf{y})] &= -ig_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) && (4.41)\\ [Q(t,\mathbf{x}), \pi_Q(t,\mathbf{y})] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}) . && (4.42) \end{aligned}$$

Przeszowski computes the arbitrary time commutators by first rewriting these expressions in terms of the field variables only

$$\begin{aligned} \left[A_i(t,\mathbf{x}), \dot{A}_j(t,\mathbf{y})\right] &= -ig_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) && (4.43)\\ \left[Q(t,\mathbf{x}), \dot{Q}(t,\mathbf{y})\right] &= -i\delta^{(3)}(\mathbf{x}-\mathbf{y}) && (4.44)\\ \Box A_i &= \partial_i(\gamma Q - \partial_j A_j) && (4.45)\\ \Box Q &= -\gamma(\gamma Q - \partial_j A_j) . && (4.46) \end{aligned}$$

It is clear that the task is complicated by the coupled modes. So the independent degrees of freedom should be found and this can be facilitated by the orthogonal decomposition

$$A_i = \left(g_{ij} + \frac{\partial_i\partial_j}{\triangle}\right) A^j - \frac{\partial_i\partial_j}{\triangle} A^j \stackrel{\text{def}}{=} A_i^T + \frac{\partial_i}{\triangle}\phi \tag{4.47}$$

(although the scalar field ϕ here differs from that used in the last chapter by the operator $\triangle^{-1}$) and introducing new scalar fields X and Y

$$\begin{aligned} Y &= (-\triangle)^{-1/2}\left(-\gamma^2 - \triangle\right)^{-1/2}(\gamma\phi + \triangle Q) && (4.48)\\ X &= \left(-\gamma^2 - \triangle\right)^{-1/2}(\phi - \gamma Q) . && (4.49) \end{aligned}$$

The old fields are related to these by

$$\phi = \left(-\gamma^2 - \Delta\right)^{-1/2} (-\Delta)^{1/2} \left((-\Delta)^{1/2} X - \gamma Y\right) \quad (4.50)$$

$$Q = \left(-\gamma^2 - \Delta\right)^{-1/2} \left((\gamma X - (-\Delta)^{1/2} Y\right) . \quad (4.51)$$

I now ignore the transverse sector, A_i^T, for which the theory is quite standard and concentrate herein on the scalar fields X and Y. These now satisfy decoupled wave equations

$$\Box Y = 0 \quad (4.52)$$

$$\left(\partial_0{}^2 + \gamma^2\right) X = 0 \quad (4.53)$$

and have nonzero equal-time commutators:

$$\left[X(x_0, \mathbf{x}), \dot{X}(x_0, \mathbf{y})\right] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (4.54)$$

$$\left[Y(x_0, \mathbf{x}), \dot{Y}(x_0, \mathbf{y})\right] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (4.55)$$

In terms of the new field variables the reduced Hamiltonian density may be rewritten as

$$\mathcal{H} = \frac{1}{2}(\dot{A}_i^T)^2 + \frac{1}{2}(\partial_i A_j^T)^2 - \frac{1}{2}(\dot{Y})^2 - \frac{1}{2}(\partial_i Y)^2 + \frac{1}{2}(\dot{X})^2 + \frac{\gamma^2}{2} X^2 \quad (4.56)$$

where the four independent degrees of freedom enter explicitly. The negative sign associated with the field Y in the commutation relations and the Hamiltonian suggest it has a ghost-like nature.

Przeszowski then goes on to solve the equations of motion for each independent mode, using the approach to QED outlined by Nakanishi in [4]. From this he succeeds in finally computing the arbitrary time commutators for the scalar fields:

$$[Y(x), Y(y)] = -i\, d(x - y) \quad (4.57)$$

$$[Y(x), X(y)] = 0 \quad (4.58)$$

$$[X(x), X(y)] = -i \frac{\sin \gamma(x_0 - y_0)}{\gamma} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (4.59)$$

with commutators between X or Y and the transverse gauge field vanishing. The function $d(x - y)$ is defined by the three dimensional Fourier integral

$$d(x) = i \int d\Gamma(\mathbf{k})(e^{ik\cdot x} - e^{-ik\cdot x})e^{i\mathbf{k}\cdot\mathbf{x}} \quad (4.60)$$

with

$$d\Gamma(\mathbf{k}) = \frac{d^3k}{2|\mathbf{k}|(2\pi)^3} \quad (4.61)$$

and $k_0 = |\mathbf{k}|$. The unusual form of the commutator for the field X is a consequence of peculiar equation of motion, (4.53), it satisfies. The solutions for X and Y permit Fock space decompositions for the fields which are written:

$$A_i^T(x) = \int d\Gamma(\mathbf{k}) \left(a_i^T(\mathbf{k})e^{-ik\cdot x} + a^{T\dagger}(\mathbf{k})e^{ik\cdot x}\right) \quad (4.62)$$

$$Y(x) = \int d\Gamma(\mathbf{k}) \left(b(\mathbf{k})e^{-ik\cdot x} + b^\dagger(\mathbf{k})e^{ik\cdot x}\right) \quad (4.63)$$

$$\begin{aligned} X(x) &= \int d\Gamma(\mathbf{k})\sqrt{\frac{|\mathbf{k}|}{\gamma}} \left(c(\mathbf{k})e^{-ik_0t_0}e^{-i\gamma(x_0-t_0)}e^{i\mathbf{k}\cdot\mathbf{x}}\right. \\ &\quad \left. + c^\dagger(\mathbf{k})e^{ik_0t_0}e^{i\gamma(x_0-t_0)}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) . \end{aligned} \quad (4.64)$$

Then the operators $a_i^T(\mathbf{k})$, $a_i^{T\dagger}(\mathbf{k})$, $b(\mathbf{k})$, $b^\dagger(\mathbf{k})$, $c(\mathbf{k})$ and $c_i^\dagger(\mathbf{k})$ satisfy the commutation relations

$$\left[a_i^T(\mathbf{k}), a_j^{T\dagger}(\mathbf{p})\right] = \delta_{ij}^T\delta^\Gamma(\mathbf{k}-\mathbf{p}) \quad (4.65)$$

$$\left[b(\mathbf{k}), b^\dagger(\mathbf{p})\right] = -\delta^\Gamma(\mathbf{k}-\mathbf{p}) \quad (4.66)$$

$$\left[c(\mathbf{k}), c^\dagger(\mathbf{p})\right] = \delta^\Gamma(\mathbf{k}-\mathbf{p}) \quad (4.67)$$

with all other commutators vanishing, $\delta^\Gamma(\mathbf{k}-\mathbf{p}) = (2\pi)^3 2|\mathbf{k}|\delta^3(\mathbf{k}-\mathbf{p})$ and $\delta_{ij}^T = g_{ij} + k_ik_j/|\mathbf{k}|^2$. So a vacuum state can be consistently selected with the operators $a_i^T(\mathbf{k})$, $b(\mathbf{k})$ and $c(\mathbf{k})$ interpreted as annihilation operators:

$$a^T(\mathbf{k})|0\rangle = 0 \quad (4.68)$$

$$b(\mathbf{k})|0\rangle = 0 \quad (4.69)$$

$$c(\mathbf{k})|0\rangle = 0 \quad (4.70)$$

and $|0\rangle$ is the Fock vacuum, while conjugate quantities correspond to creation operators. The 'ghost' interpretation of the field Y, is reinforced by its abnormal commutation relations for its creation and annihilation operators.

Substituting these results into equation (4.56) Przeszowski finds the normal ordered Hamiltonian

$$\begin{aligned} H &= \int d^3x : \mathcal{H}_D(\mathbf{x}) := \int d\Gamma(\mathbf{k}) \left[|\mathbf{k}| \left(a_i^{T\dagger}(\mathbf{k})a_i^T(\mathbf{k})\right.\right. \\ &\quad \left.\left. - b^\dagger(\mathbf{k})b(\mathbf{k})\right) + \gamma c^\dagger(\mathbf{k})c(\mathbf{k})\right] . \end{aligned} \quad (4.71)$$

Przeszowski observes that the complete Hilbert space of states generated by the Fock operators has an indefinite metric. Thus only a subspace of the states correspond to physical, probability conserving states. The question is can this subspace be uniquely defined such that the Gauss law will also be implemented for the 'candidate' physical photons?

It transpires that the choice of the subspace is not unique here. Przeszowski *begins* by demanding that the Gauss law be satisfied for the candidate physical states in a *weak* sense (as defined in chapter two). Thus he imposes

$$\langle phys'|\partial_i F_{0i}(x)|phys\rangle = 0 \tag{4.72}$$

for any time. This can be reexpressed in the form

$$\partial_i \dot{A}_i^{(+)}(x)|phys\rangle = \dot{\phi}^{(+)}(x)|phys\rangle = 0 \tag{4.73}$$

where $(+)$ denotes the positive frequency part of fields. As a consequence of the specific time dependences on X and Y, this condition may apply at arbitrary times only if

$$Y^{(+)}(x_0, \mathbf{x})|phys\rangle = X^{(+)}(x_0, \mathbf{x})|phys\rangle = 0 \tag{4.74}$$

or, in terms of the Fock operators,

$$c(\mathbf{k})|phys\rangle = b(\mathbf{k})|phys\rangle = 0 \,. \tag{4.75}$$

I shall denote the Hilbert space defined by these conditions as $\mathbf{H}_p$ whose state vectors are $|phys\rangle$. Thus, in $\mathbf{H}_p$ the field Q satisfies

$$Q^{(+)}(x_0, \mathbf{x})|phys\rangle = 0 \tag{4.76}$$

giving it a zero expectation value between physical state vectors.

Przeszowski verifies that, with this choice of the physical Hilbert space, the Hamiltonian is indeed positive definite and has only contributions from two transverse modes

$$\langle phys'|H|phys\rangle = \int d\Gamma(\mathbf{k})|\mathbf{k}|\langle phys'|\left(a_i^{T\dagger}(\mathbf{k})a_i^T(\mathbf{k})\right)|phys\rangle \,. \tag{4.77}$$

Moreover, the occupation number operator N which can be written in momentum space as

$$N = \int d\Gamma(\mathbf{k})\left(a_i^{T\dagger}(\mathbf{k})a_i^T(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}) - c^\dagger(\mathbf{k})c(\mathbf{k})\right) \tag{4.78}$$

also obtains positive definite matrix elements in $\mathbf{H}_p$,

$$\langle phys'|N|phys\rangle = \int d\Gamma(\mathbf{k})\ \langle phys'|a_i^{T\dagger}(\mathbf{k})a_i^T(\mathbf{k})|phys\rangle \tag{4.79}$$

This is significant because, unlike covariant gauge QED, there are choices of the candidate physical space for which positive definiteness applies for the Hamiltonian but *not* for the number operator.

Thus Przeszowski obtains a free dynamical system which contains only two photons as the physical excitations which satisfy the Gauss law - even for a non-zero γ parameter. That this occurs without requiring taking the limit $\gamma = -i\alpha \to 0$ is quite remarkable and probably should not be expected for the interacting theory. This is not to say that the limit $\gamma \to 0$ is unnecessary in the free theory; I shall show in the next chapter that the limit is required in order to satisfy Poincare invariance in the physical space as defined above.

I complete this chapter by verifying that the alpha prescription does indeed arise from this approach. This is most simply done by using the Fock expansions (4.62), (4.64) and (4.63), and the commutation relations (4.66) and (4.67) to compute time ordered vacuum expectation values for the field products $\phi\phi$, QQ and ϕQ and thus deriving propagators for the longitudinal gauge field A_i^L and Q. The results Przeszowski gives are

$$\langle 0|TA_i^L(x)A_j^L(y)|0\rangle = i\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\frac{1}{k_0^2-\mathbf{k}^2+i\epsilon}\frac{k_ik_j}{\mathbf{k}^2}\left(1-\frac{\mathbf{k}^2}{k_0^2+\alpha^2}\right) \quad (4.80)$$

$$\langle 0|TQ(x)Q(y)|0\rangle = -i\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\frac{1}{k_0^2-\mathbf{k}^2+i\epsilon}\left(1-\frac{\alpha^2}{k_0^2+\alpha^2}\right) \quad (4.81)$$

$$\langle 0|TA_i^L(x)Q(y)|0\rangle = -i\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\frac{k_i\alpha}{k_0^2-\mathbf{k}^2+i\epsilon}\frac{1}{k_0^2+\alpha^2}\,. \quad (4.82)$$

The result for the longitudinal propagator can be combined with the standard transverse result to give for the momentum space result of the full propagator

$$\frac{1}{k^2+i\epsilon}\left(g_{ij}+\frac{k_ik_j}{|\mathbf{k}|^2}\frac{1}{k_0^2+\alpha^2}\right)\,. \quad (4.83)$$

This indeed is the alpha prescription (4.2). In deriving these expressions Przeszowski has continued $\gamma \to -i\alpha$ in denominators of the form $k_0^2-\gamma^2+i\epsilon$. With this, the $i\epsilon$ is dropped as it ceases to be necessary: the poles are now off the real k_0 axis and the deformation in the contour induced by the $i\epsilon$ becomes superfluous. Observe also that in this approach the Feynman $i\epsilon$ arises in an entirely natural and correct way - no identification between ϵ and α or γ needs to be imposed as occurs in the derivation in [123].

In the next chapter I turn to further tests of the consistency of the alpha prescription in the Wilson loop and, in the context of this derivation, the recovery of Poincare invariance in the physical Hilbert space, $\mathbf{H}_p$.

Chapter 5

The Wilson Loop and Poincare Invariance

5.1 The Wilson Loop

As has been emphasised in the first two chapters, the most effective check for prescriptions in the temporal gauge is the rectangular Wilson loop as discussed in section 2.5. I shall first outline the key steps taken in Landshoff's calculation of the Wilson loop in the alpha prescription [75], before completing the proof of exponentiation in showing tadpole diagrams do not contribute. In the first part of the discussion I shall not discuss algebraic steps in any details - some of the tricks used by Landshoff will also be used in the second part.

5.1.1 Landshoff's Computation

The integral which plays a large role in computations in the alpha prescription is the time Fourier transform of the factor $1/(k_0^2 + \alpha^2)$ which is $(1/2\alpha)\exp[-\alpha|x_0|]$ and in the limit of small α gives

$$\frac{1}{2}(1/\alpha - |x_0|)\,. \tag{5.1}$$

Landshoff denotes this quantity by $I(x_0)$.

To order g^2 the exponentiation follows rather simply. The result from the lowest order diagrams is proportional to

$$I(T) - I(0) = T/2 \tag{5.2}$$

and this guarantees the exponentiation to this order.

The key step in proving that the alpha prescription does not violate the exponentiation property of the Wilson loop at the next order is to show that no terms proportional to

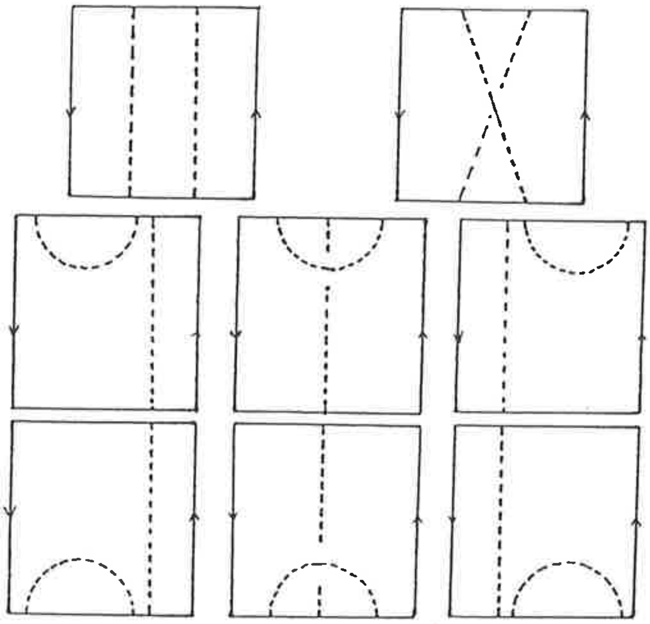

Figure 5.1: Diagrams contributing to the correct exponentiating behaviour of the loop - dashed lines indicate $k_i k_j$ terms from the propagator.

$C_2(G)C_2(R)g^4T^2$ occur, where T is the time length of the rectangular contour. It turns out (see the tadpole discussion) that the terms from the propagator that contribute to the correct large T behaviour of the loop are those with $k_i k_j$. These diagrams are given in figure (5.1) and are responsible for the desired $C_2(R)^2g^4T^2$ contributions to (2.46).

The violating terms, with the product of the two Casimir's, arise from diagrams with a three-gluon vertex inside the loop as indicated by figure (5.2).

By explicit calculation, Landshoff demonstrates that T^2 contributions drop out by direct cancellation of terms in different diagrams: the cancellation generally occured between a diagram with the two ends of the gauge propagator attached to opposite sides of the Wilson loop and and the same diagram but with the propagator ends attached to the same side. In particular, terms with pathological factors $1/\alpha$ and $1/\alpha^2$ cancelled permitting the limit $\alpha \to 0$ to be taken safely.

The final step taken by Landshoff was to show that the alpha prescription correctly reproduces the finite terms in (2.46) linear in T but with factors $C_2(G)C_2(R)$. Here Landshoff demonstrated that the alpha prescription gives the same result as the PV prescription which, as shown in [53], generates the correct result for such terms. Once again, the k_0 integrations generated a $1/\alpha$ term which cancelled between diagrams "where only one side of the Wilson rectangle is involved" [75].

Thus with these steps Landshoff was able to correctly reproduce the static potential (2.46) and at the same time demonstrate how a computation in the alpha prescription can work in a gauge independent quantity. The important step to this end was expanding for

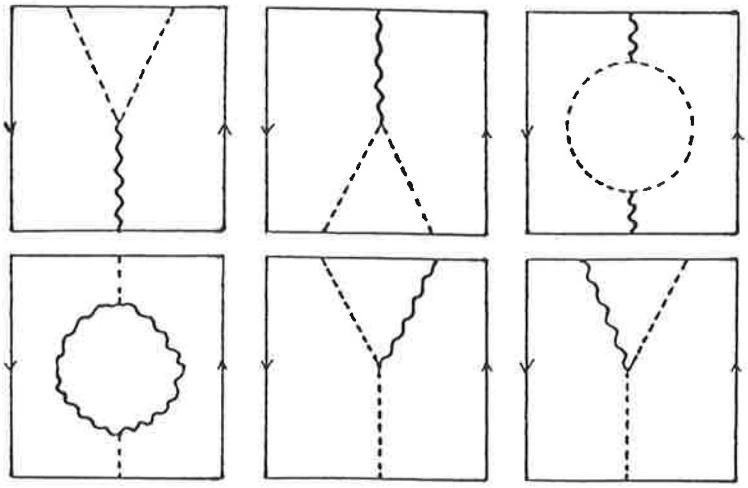

Figure 5.2: Diagrams with $C_2(G)C_2(R)T^2$ factors that could violate the Wilson loop exponentiation. The wavy line denotes the full propagator.

small α in expressions and finding that divergences at $\alpha = 0$ cancelled between diagrams. The remaining results were either independent of α - and these terms contained the correct gauge independent result - or of order α or higher, which vanished in the limit.

5.1.2 Tadpole Contributions

It is generally true that massless tadpole diagrams vanish in dimensional regularisation. However because the α in the Landshoff propagator has the dimensions of *mass*[1] it is not automatically true that such contributions vanish for the alpha-prescription, for finite α. Thus the above demonstration of Wilson loop exponentiation is incomplete unless one can show that the tadpoles do indeed give no extra contributions. Unlike the computation of Landshoff, in this calculation I shall use a manifestly covariant formalism throughout rather than deal with the ij components.

For the computation of these contributions both the UV-finite and UV-divergent parts of the truncated tadpole graph - figure (5.3) - are needed. Now I have computed the result for the tadpole using methods I discuss in the next chapter. Nyeo [126] has also computed this result but I slightly disagree with his result. For this computation though it suffices to give a schematic form for the result, and I postpone discussion of the above issue for the next chapter.

I thus may write for the truncated tadpole graph in the dimensional regularisation

[1]This is if $n^2 = 1$. Of course keeping n^2 general but timelike the statement is that $\alpha\sqrt{n^2}$ has the dimensions of mass. This will be discussed again in chapter 6.

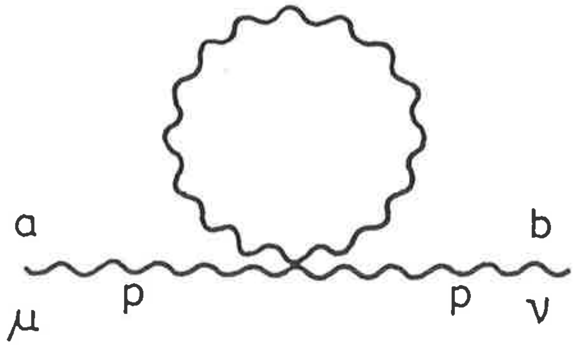

Figure 5.3: The tadpole graph.

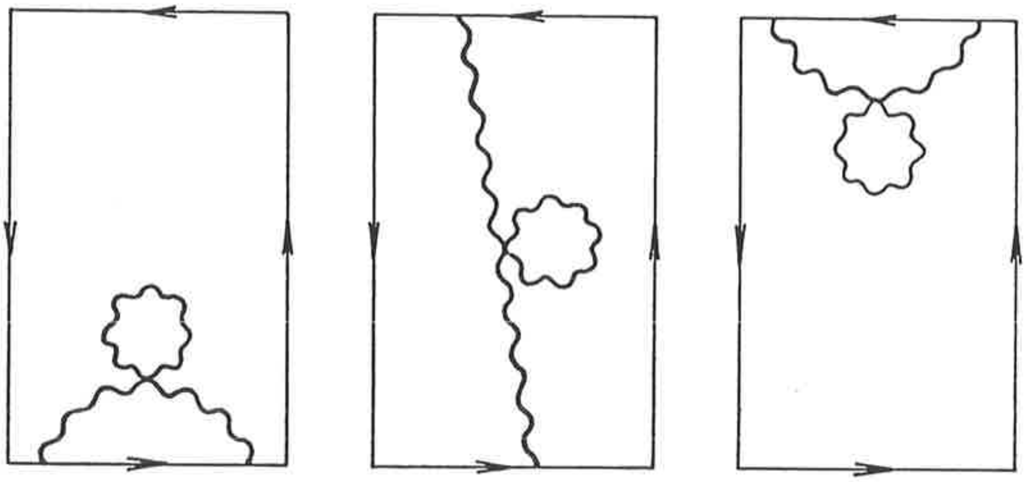

Figure 5.4: Tadpole diagram contributions to the Wilson loop.

scheme

$$T^{ab}_{\mu\nu}(p) = (\alpha^2)^{2\omega-3}\delta^{ab}C_2(G)\left[A(\omega)g_{\mu\nu} + B(\omega)n_\mu n_\nu\right]\chi(\omega) \tag{5.3}$$

and this general form is consistent with both my result, to be discussed in the next chapter, and Nyeo's result [126]. Here, A and B are finite, non-zero functions of the number of space-time dimensions. The function $\chi(\omega)$ contains sums and products of Euler gamma functions wherein the UV divergence is concentrated as a simple pole in $\varepsilon = 2 - \omega$. When this expression is inserted into the Wilson loop diagrams, the second term does not contribute since the gauge-vector, n_μ, exactly annihilates the external gluon propagator legs which themselves attach to the 'rungs' of the Wilson loop.

There are three diagrams to consider, given in figure (5.4). These can be summed together to give the result, after performing the Wilson loop contour integrations,

$$W_t \;=\; (\alpha^2)^{(2\omega-3)}g^4C_2(R)C_2(G)A(\omega)\chi(\omega)\times$$

$$\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{sin^2(p_3L)}{p_3^2}[1-\exp(-ip_0T)]D_3^\mu(p)D_{\mu 3}(p)\,. \tag{5.4}$$

It suffices to concentrate on the p_0 integration which reduces to a combination of three terms

$$(\alpha^2)^{(2\omega-3)}\int \frac{dp_0}{2\pi}\frac{(1-\exp(-ip_0T))}{(p^2+i\epsilon)^2}\times \left[-1+2\frac{p_3^2}{(p_0+i\alpha)(p_0-i\alpha)}-\frac{p_3^2\,|\mathbf{p}|^2}{(p_0+i\alpha)^2(p_0-i\alpha)^2}\right]\,. \tag{5.5}$$

Here, n^2 has been set to 1 and the overall factor of $(\alpha^2)^{(2\omega-3)}$ has been retained as it is this dependence which is important.

The Cauchy Residue theorem may be used to evaluate the energy integral if the contour is completed in the upper half complex p_0 plane for $T<0$, and the lower half plane for $T>0$. For large $|T|$ the residues from the Feynman pole do not contribute (this is the also the reason why, in the first part, I am able to isolate contributions from the k_ik_j terms). So the first term gives no contribution. The second term gives

$$(\alpha^2)^{(2\omega-\frac{7}{2})}\frac{p_3^2}{(|\mathbf{p}|^2+\alpha^2)^2}[1-\exp(-\alpha|T|)]\,. \tag{5.6}$$

Now I can expand in small $\varepsilon = 2-\omega$ using the well known result $a^\varepsilon = 1+\varepsilon\ln a+\ldots$. The first term (5.6) is proportional to α. The second is proportional to $\alpha\ln\alpha$ which vanishes as $\alpha\to 0$. Thus it is safe to take α to zero before the three-momentum integration, giving zero (presuming that is also safe to take the limit in the third term). A similarly vanishing contribution is included in the result of the third integration in (5.5). What remains finally is

$$-2i(\alpha^2)^{(2\omega-3)}\frac{p_3^2\,|\mathbf{p}|^2}{(2i\alpha)^3(|\mathbf{p}|^2+\alpha^2)^2}[\alpha|T|exp(-\alpha|T|)-(1-exp(-\alpha|T|)]\,. \tag{5.7}$$

Now one can expand the exponentials for small α and observe that most terms are of order α or higher or involve quantities such as $\alpha\ln\alpha$ once the expansion in ε is performed. It is safe to take the limit $\alpha\to 0$ for these. There are then just two terms linear in $|T|$ but involving factors $(\alpha^2)^{-\varepsilon}$. But these are equal in magnitude and opposite in sign. They cancel exactly. Indeed this cancellation is quite significant for simple scaling arguments would not have detected its occurence. Thus in the limit $\alpha\to 0$ the tadpole contributions vanish, and the exponentiation holds for the alpha prescription.

5.2 Poincare Invariance

In the last chapter I outlined the approach taken by Przeszowski in deriving the alpha prescription [125]. The key advantage of the generalised Hamiltonian approach taken there is that it permits discrimination between the redundant and physical parts of the Hilbert space. It was shown that the definition of the physical space was consistent with the implementation of the Gauss Law as a weak condition between physical states. I now wish to check that this choice of the physical space is also consistent with the restoration of Poincare invariance of the quantum states.

Given the problems in completing the derivation of the prescription for the non-Abelian interacting theory, mostly generated by the breaking of BRST invariance in the non-Abelian generalisation of Przeszowski's Lagrangian, (4.27), the following analysis shall be for the Abelian theory only - though I shall show that even here things become complex.

5.2.1 A First Attempt

In what follows it shall be useful to define a new variable

$$Z = \gamma Q - \phi \tag{5.8}$$

where Q and ϕ are the scalar fields used in Przeszowski's derivation of the alpha prescription discussed in the last chapter. Proceeding with the theory given by Lagrangian (4.29) I construct the energy-momentum tensor in the usual way,

$$T^{\mu\nu} = \partial^\nu A_\tau (\partial\mathcal{L}/\partial(\partial_\mu A_\tau)) + \partial^\nu Q(\partial\mathcal{L}/\partial(\partial_\mu Q)) - g^{\mu\nu}\mathcal{L}\,. \tag{5.9}$$

From this I obtain the generator of time translations

$$P_0 = \int d^3x T_{00}(x) \tag{5.10}$$

as just the reduced Hamiltonian, namely the canonical Hamiltonian with the constraints imposed

$$H_R = \int d^3x[\frac{1}{2}(\pi_i)^2 - \frac{1}{2}(\pi_Q)^2 + \frac{1}{2}(\partial_i A_j)^2 - \frac{1}{2}(\partial_i Q)^2 - \frac{1}{2}Z^2]\,. \tag{5.11}$$

The generator of space translations is found to be

$$P_i = \int d^3x(-\pi_k \partial_i A_k + \pi_Q \partial_i Q)\,. \tag{5.12}$$

With these results it is a simple matter to check that, using the Euler-Lagrange equations of motion

$$\begin{aligned} \partial_\mu F^{\mu\nu} - \gamma\partial^\nu Q + Bn^\nu &= 0 \qquad (5.13)\\ \Box Q + \gamma^2 Q + \gamma\partial^\mu A_\mu &= 0 \qquad (5.14)\end{aligned}$$

that the total divergence of the energy-momentum tensor vanishes:

$$\partial_\mu T^{\mu\nu} = 0 \tag{5.15}$$

For the angular-momentum tensor

$$\mathcal{M}^{\mu\nu\rho} = T^{\mu\rho}x^{\nu} - T^{\mu\mu}x^{\rho} + (\partial\mathcal{L}/\partial(\partial_{\mu}A_{\nu}))A^{\rho} - (\partial\mathcal{L}/\partial(\partial_{\mu}A_{\rho}))A^{\nu} \tag{5.16}$$

(the non-appearance of Q terms except via the energy-momentum tensor is because Q is a spin-zero field) one obtains

$$\mathcal{M}^{\mu\nu\rho} = T^{\mu\rho}x^{\nu} - T^{\mu\nu}x^{\rho} - F^{\mu\nu}A^{\rho} + F^{\mu\rho}A^{\nu} \ . \tag{5.17}$$

Calculating the total four-divergence of this expression, again using the Euler-Lagrange equations, gives

$$\partial_{\mu}\mathcal{M}^{\mu\nu\rho} = B(n^{\rho}A^{\nu} - n^{\nu}A^{\rho}) \ . \tag{5.18}$$

Using the third constraint relating B, G and π_Q, (4.33), this can be rewritten as

$$\partial_{\mu}\mathcal{M}^{\mu\nu\rho} = \dot{Z}(n^{\rho}A^{\nu} - n^{\nu}A^{\rho}) \ . \tag{5.19}$$

Now observe that the variable Z I have defined is directly related to the quantity X and so has its own Fock decomposition in terms of the fields c and $c^{\dagger}$. So between states in the physical Hilbert space $\mathbf{H}_p$, Z vanishes ensuring, as desired, that in this space the right-hand side of the above expression is zero. So far so good.

For the generators of Lorentz rotations and boosts

$$M_{\mu\nu} = \int d^3x \mathcal{M}_{0\mu\nu} \tag{5.20}$$

I obtain the following explicit expressions

$$M_{0i} = \int d^3x[(-\pi_k\partial_i A_k + \pi_Q\partial_i Q)x_0 - \mathcal{H}_R x_i] \tag{5.21}$$

$$M_{ij} = \int d^3x[(-\pi_k\partial_j A_k + \pi_Q\partial_j Q)x_i + (\pi_k\partial_i A_k - \pi_Q\partial_i Q)x_j + \pi_i A_j - \pi_j A_i] \tag{5.22}$$

The generators satisfy the following bracket algebra

$$\{P_{\mu}, P_{\nu}\} = 0 \tag{5.23}$$

$$\{P_0, M_{0i}\} = P_i - \int d^3x \pi_i Z \tag{5.24}$$

$$\{P_i, M_{0j}\} = -g_{ij}P_0 \tag{5.25}$$

$$\{P_0, M_{ij}\} = 0 \tag{5.26}$$

$$\{P_i, M_{jk}\} = g_{ij}P_k - g_{ik}P_j \tag{5.27}$$

$$\{M_{0i}, M_{0j}\} = -M_{ij} + \int d^3x[(\pi_i x_j - \pi_j x_i)Z + \pi_i A_j - \pi_j A_i] \tag{5.28}$$

$$\{M_{0i}, M_{jk}\} = g_{ij}M_{0k} - g_{ik}M_{0j} \tag{5.29}$$

$$\{M_{ij}, M_{kl}\} = g_{il}M_{jk} + g_{kj}M_{il} + g_{ki}M_{lj} + g_{jl}M_{ki} \tag{5.30}$$

Clearly in all but anomalous expression (5.28) the violation of the Poincare Lie algebra is connected with the Lorentz non-invariant expression that appears in the angular-momentum conservation expression, $\gamma Q - \phi$. The problem with (5.28) is the absence from the final result of the terms

$$\int d^3x(\pi_i A_j - \pi_j A_i)$$

hence they are added and taken away to enable one to write M_{ij} in the result. It is evident that in (5.28) the correct Poincare algebra is not satisfied in $\mathbf{H}_p$.

5.2.2 Reduced Lagrangian Approach

Przeszowski[127] has offered a way out of the dilemma just noted. Here his suggestion is to start with the *reduced* Lagrangian density, denoted $\mathcal{L}_R$, where the temporal gauge condition is implemented from the start. So far I have been happy to use the Lagrangian in which the gauge condition is generated by an extra term in the Lagrangian - either with a Lagrange multiplier field explicitly present or integrated out. But there is nothing sacrosanct about this approach - it merely enables an elegant covariant implementation of the gauge. In this case, the difference between the reduced Lagrangian and the original form used (4.29) leads to a total divergence when computing the Euler-Lagrange equations and so, at this level, there is no physical consequence in adopting this starting point; indeed for the latter it is clear that there are some definitely unphysical consequences.

Following Przeszowski, I adopt the following Lagrangian to repeat the above analysis:

$$\begin{aligned}\mathcal{L}_R &= \frac{1}{2}\partial^\mu A_i \partial_\mu A_i + \frac{1}{2}\partial_i A_j \partial_j A_i - \frac{1}{2}(\dot{Q})^2 \\ &\quad + \frac{1}{2}\partial_i Q \partial_i Q + \frac{1}{2}\gamma^2 Q^2 + \gamma \partial_i Q A_i \,.\end{aligned} \tag{5.31}$$

With this Lagrangian one obtains the 00 components of the energy-momentum tensor

$$T^{00} = \dot{A}_j \frac{\partial \mathcal{L}_R}{\partial \dot{A}_j} + \dot{Q}\frac{\partial \mathcal{L}_R}{\partial \dot{Q}} - g^{00}\mathcal{L}_R \tag{5.32}$$

which differs from the original expression (5.11):

$$T^{00} = \mathcal{H}_R + \partial_i \left[\frac{1}{2}A_i \partial_j A_j - \frac{1}{2}A_j \partial_j A_i - A_i \gamma Q\right] \tag{5.33}$$

Observe in this expression the total divergence alluded to above and thus one concludes that in computing the Hamiltonian, which involves integrating over $\mathbf{x}$, the extra term will not contribute.

This energy tensor still satisfies the correct conservation relation as evidenced by the identity

$$\partial_0 T^{00} = -\partial_i T^{i0} \tag{5.34}$$

where now the momentum density T^{i0} is given by

$$T^{i0} = -\pi_j(\partial^i A_j - \partial_j A^i) + \pi_Q(\partial^i Q + \gamma A^i) \tag{5.35}$$

However, now the canonical energy momentum tensor is not symmetric. Reversing the indices one obtains a different result:

$$T^{0i} = -\pi_j \partial^i A_j + \pi_Q \partial^i Q \tag{5.36}$$

which is the expression derived in the naive approach so the space translation generator also remains unchanged in the reduced Lagrangian approach. The assymmetry does not violate the conservation relation

$$\partial_0 T^{0i} = -\partial_k T^{ki} \tag{5.37}$$

where the result for the space components of the energy-momentum tensor is

$$T^{ki} = \partial^i A_j(\partial^k A_j - \partial_j A^k) + \partial^i(\partial^k Q + \gamma A^k) - g^{ik}\mathcal{L}_R\,. \tag{5.38}$$

Next Przeszowski gives the Poincare generators based on the new angular momentum tensor. With the reduced Lagrangian density the only significant change is in the Lorentz boost generator which becomes

$$M^{0i} = \int d^3x[T^{0i}x^0 - \mathcal{H}_R x^i + A^i Z] \tag{5.39}$$

for which he finds the surprising result that the generator is not *time-independent*:

$$\frac{d}{dt}M^{0i} = \int d^3x A^i \dot{Z}\,. \tag{5.40}$$

Przeszowski notes the significance of this: for this generator one may calculate Poisson (Dirac) brackets only for equal time. Of course, as one would have hoped, between physical states the quantity on the right hand side vanishes - so in the space $\mathbf{H}_p$ time independence is recovered. The other generators remain time independent at the operator level thus their Poisson brackets are valid in general. At equal time, then, he obtains the Poisson bracket relation to replace (5.28)

$$\{M_{0i}(t), M_{0j}(t)\} = -M_{ij} + \int d^3x(x_i A_j - x_j A_i)(t,\mathbf{x})\dot{Z}(t,\mathbf{x})\,. \tag{5.41}$$

The Poincare algebra is recovered when one takes this expression between physical states.

Another suggestion of Przeszowski[127] is to introduce an additional modification of the boost generator such that the new choice regains time independence. However for this choice the bracket relation between two boosts retains an anomalous term even between physical states. In the limit $\gamma \to 0$ the anomaly safely vanishes. On the other hand I shall show below that further properties of the physical free theory are also recovered in this limit using the approach just outlined. There thus appears to be no *a priori* reason to choose this second modification over the theory based on the reduced Lagrangian alone. For this reason I do not outline this additional aspect of Przeszowski's work, but continue with Przeszowski's original suggestion.

5.2.3 Transformations of the Fields

Under the Lorentz transformations, the 'canonical' fields Q and A_i transform as follows:

$$\{A_j(x), P_i\} = \partial_i A_j(x) \tag{5.42}$$

$$\{Q(x), P_i\} = \partial_i Q(x) \tag{5.43}$$

$$\{A_j(x), M_{\mu\nu}\} = (x_\mu \partial_\nu - x_\nu \partial_\mu) A_j(x) + g_{\mu j} A_\nu - g_{\nu j} A_\mu \tag{5.44}$$

$$\{Q(x), M_{\mu\nu}\} = (x_\mu \partial_\nu - x_\nu \partial_\mu) Q(x)\,. \tag{5.45}$$

This is consistent with expectations.

5.2.4 Mapping of Physical Space

The final task is to check that Poincare transformations map the physical Hilbert space $\mathbf{H}_p$ into itself. States in the space transform under the translations P_μ

$$|phys\rangle \to U(a)|phys\rangle \tag{5.46}$$

where

$$U(a) = \exp(-iP^\mu a_\mu)\,. \tag{5.47}$$

Under Lorentz transformations $M_{\mu\nu}$ the states transform according to

$$|phys\rangle \to \exp(-i\theta^{\mu\nu} M_{\mu\nu})|phys\rangle\,. \tag{5.48}$$

For infinitesmal transformations these criteria simplify into requiring that $P_\mu|phys\rangle$ and $M_{\mu\nu}|phys\rangle$ are 'physical', in other words that they themselves are annihilated by the Fock operators b and c.

Beginning with time-translations, the simplest way to proceed is to use the expression for the Hamiltonian in terms of the Fock space operators. The normal ordered expression is

$$P_0 = \int d\Gamma(\mathbf{k})[|\mathbf{k}|(a_i^{T\dagger}(\mathbf{k})a_i^T(\mathbf{k}) - b^\dagger(\mathbf{k})b(\mathbf{k})) + \gamma c^\dagger(\mathbf{k})c(\mathbf{k})] \tag{5.49}$$

where once again $d\Gamma(\mathbf{k}) = (d^3k/(2\pi)^3)(1/2|\mathbf{k}|)$. Consider now the commutator of the Fock operators $b(\mathbf{k})$ and $c(\mathbf{k})$ with the generator P_0:

$$[b(\mathbf{k}), P_0] = |\mathbf{k}|b(\mathbf{k}) \tag{5.50}$$

$$[c(\mathbf{k}), P_0] = \gamma c(\mathbf{k}) \,. \tag{5.51}$$

Acting with this commutator on the candidate physical states $|phys\rangle \in \mathbf{H}_p$ will annihilate them. In other words, the time-translated states are annihilated by $b(\mathbf{k})$ and $c(\mathbf{k})$ and thus by the above criteria are in $\mathbf{H}_p$ themselves.

For space translations I derive for the momentum space expansion of the normal ordered generator

$$P_i = \int d\Gamma(\mathbf{k}) k_i [a_i^{T\dagger}(\mathbf{k}) a_i^T(\mathbf{k}) - b^\dagger(\mathbf{k}) b(\mathbf{k}) + c^\dagger(\mathbf{k}) c(\mathbf{k})] \,. \tag{5.52}$$

The argument is identical here - the commutator between the generator and the fictitious field Fock operators are proportional to the operators themselves and thus annihilate the physical states.

It is clear that much tedious algebra can be avoided with the case of the boost and rotation operators by noticing that the only place where the above arguments would fail is when in M_{0i} and M_{ij} I find operators b and c multiplying the transverse photon operators, a_i^T. To illustrate the significance of this statement of the problem, consider the possibility that there is a term appearing in $M_{\mu\nu}$ of the form

$$\chi^\dagger(\mathbf{k}) a_i^T(\mathbf{k}) \tag{5.53}$$

where $\chi(\mathbf{k})$ generically denotes the Fock operators b and c. Then the commutator of χ with the generator yields a term with a transverse photon annihilation operator at momentum $\mathbf{k}$. Acting with this on a physical state of N transverse photons will yield a state which is a sum of $N-1$ photon states. Such a state clearly does not vanish.

The analysis for the boost generator is somewhat more tedious, so I consider it before the rotations. The first thing to observe is that for γ non-zero there are indeed mixing terms. However the coefficients in this case involve γ or $\gamma^{1/2}$ for which it is evidently safe to take the limit. For this reason I do not write down the full expression but disregard these otherwise cumbersome contributions. Note that in this computation I am dealing with a *physical* quantity - and thus it is entirely within the 'philosophy' established above, in the Wilson loop analysis, to take the limit γ (or α) $\to 0$. Taking the limit also, single-handedly, eliminates much of the surviving time-dependence.

I thus obtain, for the momentum space expansion of the normal ordered boost generator, the expression

$$\begin{aligned} M_{0i} &= i \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} \left[a_j^{T\dagger}(\mathbf{k}) \left(k_0 \frac{\partial}{\partial k^i} \right) a_j^T(\mathbf{k}) - b^\dagger(\mathbf{k}) \left(k_0 \frac{\partial}{\partial k^i} \right) b(\mathbf{k}) \right. \\ &\qquad \left. - c^\dagger(\mathbf{k}) \left(\gamma \frac{\partial}{\partial k^i} \right) c(\mathbf{k}) \right] + x_0 \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} k_i c^\dagger(\mathbf{k}) c(\mathbf{k}) \end{aligned} \tag{5.54}$$

In writing this expression I have kept one of the γ-dependent terms, despite having taken $\gamma \to 0$ for the other terms, in order to show the role the c-particles play in the boost generator. The last term simply reveals the γ-independent part of the boosts' time-dependence. This form is consistent with my earlier observation that the time-dependent part vanishes in the physical sector of Hilbert space.

I should also mention that in the full expression there are terms with dependence on the arbitrary time, t_0 which arise from the Fock space expansion of the field X, (4.64). These multiply the product of fields $c^\dagger(\mathbf{k})c(\mathbf{k})$; so they do not contribute between physical states. Indeed it is the presence of the time t_0 which is the original cause of the time dependence outside the physical space: the time dependence in (5.39) from the piece $x_0 T_{0i}$ would normally cancel with other terms generated by derivatives $\partial/\partial k^i$ acting on $k_0 = |\mathbf{k}|$ in an expontial, dropping a factor x_0 - but here t_0 not x_0 multiplies k_0 in the exponent of the Fock expansion (4.64) of X.

For completeness I give the explicit momentum expansion for the normal ordered rotation generator M_{ij}. I obtain

$$M_{ij} = M^a_{ij} + M^b_{ij} + M^c_{ij} \tag{5.55}$$

where the terms represent the contribution to the boosts from the transverse photons, the b particles and the c particles. The photon term is

$$\begin{aligned} M^a_{ij} = i \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} [a^{T\,\dagger}_k(\mathbf{k}) \left(k_i \frac{\partial}{\partial k^j} - k_j \frac{\partial}{\partial k^i}\right) a^T_k(\mathbf{k}) \\ - \left(a^{T\,\dagger}_i(\mathbf{k}) a^T_j(\mathbf{k}) - a^{T\,\dagger}_j(\mathbf{k}) a^T_i(\mathbf{k})\right)] \end{aligned} \tag{5.56}$$

where we see the standard orbital and spin angular momentum terms displayed. The fictitious fields give the contributions:

$$M^b_{ij} = -i \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} \left[b^\dagger(\mathbf{k}) \left(k_i \frac{\partial}{\partial k^j} - k_j \frac{\partial}{\partial k^i}\right) b(\mathbf{k})\right] \tag{5.57}$$

$$M^c_{ij} = i \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} \left[c^\dagger(\mathbf{k}) \left(k_i \frac{\partial}{\partial k^j} - k_j \frac{\partial}{\partial k^i}\right) c(\mathbf{k})\right] . \tag{5.58}$$

In this case no mixed χa^T terms occur - they cancel out - but mixed $b - c$ terms do occur and retain a time dependence arising from the exponentials in the field operator expansion. These terms have coefficients of $\gamma^{1/2}$ or $\gamma^{3/2}$. Thus in the limit $\gamma \to 0$ they vanish. Alternately in the physical space they and their commutator with themselves vanish in the physical subspace. Thus $\mathbf{H}_p$ is invariant under rotations.

I note that all the formulae given for the momentum expansions of the generators in the physical space in the limit $\gamma \to 0$ are consistent with the results given for quantised electromagnetism in various works such as [99, 116]. Thus the theory described by the

Przeszowski Q-Lagrangian does satisfy Poincare invariance in the limit $\gamma \to 0$, and the true degrees of freedom of quantised electromagnetism can be recovered consistently; the alpha prescription does generate a physical theory in the Abelian temporal gauge.

Chapter 6

UV-Divergences in the Alpha-Prescription

6.1 Introduction

It is the issue of non-local UV divergences in Green's functions which continues to frustrate the demonstration that the generalised LM prescription in non-light-cone gauges is consistent with a renormalisable quantum field theory of gauge-interactions[78, 80]. Given the problems I have outlined in the second and third chapters with the PV prescription and the non-translational propagator approach, it would seem that the only remaining way forward in the temporal gauge is the alpha prescription. Whether it too is plagued by the problems that beset the LM prescription is the concern of this chapter.

Of course the question of the derivation of the prescription does influence the renormalisation problem. The position reached at the end of chapter 4 indicates that either new vertices must be included in order to maintain BRST invariance with the alpha prescription, or BRST invariance must be sacrificed altogether while $\alpha \neq 0$. It shall transpire that the renormalisation is best done for $\alpha \neq 0$, and that this will require BRST non-invariant counterterms. Of course the invariance is ultimately useful to proceed to higher loops via Slavnov-Taylor identities. At present there seems no way around this difficulty but it shall be possible to make some conclusions about the renormalisability despite this sacrifice. Thus in the computations presented in this chapter the BRST non-invariance for $\alpha \neq 0$ will be accepted and no account shall be taken of the new vertices discussed in chapter 4.

6.2 Feynman Integrals in the Alpha Prescription

Feynman graphs will generally involve a product of two or more gauge boson propagators. In noncovariant gauges characterised by the condition $n \cdot A = 0$ this generally involves integrands with products of noncovariant 'denominators':

$$\frac{1}{(q \cdot n)[(q-p) \cdot n]} \tag{6.1}$$

where q is the integration variable and of course a prescription must be provided for the poles implicit in this expression. In order to simplify such Feynman integrations a useful tool invoked, for example with the PV and LM prescriptions, is a *decomposition* formula which enables the product to be split and the integral decomposed into more simple expressions involving a single noncovariant denominator.

Unlike the PV prescription and one form of the LM prescription (that given by Mandelstam [43]), the alpha prescription involves a basic denominator that is now *quadratic* in the momenta: $(q \cdot n)^2 + \alpha^2 (n^2)^2$. So double denominators become significantly more complex. It is also crucial to treat α in an algebraic manner. For these reasons, though one is forced to deal with significantly more terms in any given computation, I have chosen to use a partial fractions expansion as the basis for a decomposition formula in the alpha prescription. So for a single 'Landshoff denominator', I use the expansion

$$\frac{1}{(q \cdot n)^2 + \alpha^2} = \frac{1}{(2i\alpha)}\left[\frac{1}{(q \cdot n - i\alpha)} - \frac{1}{(q \cdot n + i\alpha)}\right] \tag{6.2}$$

and for a double denominator,

$$\begin{aligned} \frac{1}{[(q \cdot n)^2 + \alpha^2][((q-p) \cdot n)^2 + \alpha^2]} &= \frac{1}{4\alpha^2}\Big[\frac{1}{p \cdot n}\Big(\frac{1}{q \cdot n + i\alpha} - \frac{1}{(q-p) \cdot n + i\alpha}\Big) \\ &- \frac{1}{p \cdot n + 2i\alpha}\Big(\frac{1}{q \cdot n + i\alpha} - \frac{1}{(q-p) \cdot n - i\alpha}\Big) \\ &- \frac{1}{p \cdot n - 2i\alpha}\Big(\frac{1}{q \cdot n - i\alpha} - \frac{1}{(q-p) \cdot n + i\alpha}\Big) \\ &+ \frac{1}{p \cdot n}\Big(\frac{1}{q \cdot n - i\alpha} - \frac{1}{(q-p) \cdot n - i\alpha}\Big)\Big] \,. \end{aligned} \tag{6.3}$$

Here the $(n^2)^2$ factor has been absorbed into α^2. This then reduces the problem to computing Feynman integrals of the form

$$I_{\mu_1 \mu_2 \ldots \mu_n} = \int \frac{d^{2\omega} q}{(2\pi)^{2\omega}} \frac{q_{\mu_1} q_{\mu_2} \cdots q_{\mu_n}}{(q^2 + 2q \cdot p - L)^\sigma (q \cdot n + i\alpha)} \tag{6.4}$$

where, unlike the Feynman $i\epsilon$ implicit in this expression, α is to be treated as a finite non-zero quantity and maintained as such throughout all integrations.

Most authors dealing with Feynman integrals in noncovariant gauges usually proceed now by employing an exponential parametrisation [90]. Here I prefer to use the more conventional tool of Feynman parametrisation. To that end I adopt the techniques used in the paper by Konestchny [128] with the proviso of non-zero, finite α. To review this I consider the integral

$$I(a,b) = \int d^{2\omega}q(q^2 + 2p\cdot q - L + i\epsilon)^{-a}(q\cdot n + i\alpha)^{-b}\,. \tag{6.5}$$

The covariant denominator in this expression and in (6.4) is generally the result of combining various massive and massless covariant denominators, in this case using the standard Feynman method. I now combine the denominators in the integrand and obtain

$$I(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int d^{2\omega}q\int_0^1 dx x^{a-1}(1-x)^{b-1}\times \tag{6.6}$$
$$[(q^2+2p\cdot q - L)x + q\cdot n(1-x) + i\epsilon x + i\alpha(1-x)]^{-a-b}$$

and, for nonzero x the momentum integral may be written as

$$\int d^{2\omega}q\left[(q+p+n\frac{1-x}{2x})^2 - L - (p+n\frac{1-x}{2x})^2 + i\eta\right]^{-a-b} \tag{6.7}$$

where $\eta = \epsilon + \alpha(1-x)/x$. Now I can perform the momentum integration using standard dimensionally regularised integrals [129] - either by incorporating the $i\eta$ in a complex mass and using Minkowski space integrals, or by performing a Wick rotation as Konetschny does. The result has the form

$$I(a,b) = (-1)^{a+b} i\pi^{\omega}4^{a+b-\omega}\frac{\Gamma(a+b-\omega)}{\Gamma(a)\Gamma(b)}\int_0^1 dx x^{2a+b-1-2\omega}(1-x)^{b-1}[M(x)]^{\omega-a-b}\,. \tag{6.8}$$

The function M is given by

$$M(x) = 4(L+p^2)x^2 + 4(p\cdot n - i\alpha)x(1-x) + n^2(1-x)^2\,. \tag{6.9}$$

Having completed the momentum integral I have taken $\epsilon \to 0$ as one conventionally does in one loop covariant integrals.

Konetschny then exploits the factorisability of M -

$$M(x) = n^2(1-ux)(1-vx) \tag{6.10}$$

where u,v are given by

$$u,\ v = \left[n^2 - 2(p\cdot n - i\alpha \pm 2[(p\cdot n - i\alpha)^2 - n^2(p^2+L)]^{1/2}\right]/(p.n - i\alpha) \tag{6.11}$$

- and the properties of the Appel function[130] F_1 in order to eventually write the result in terms of the Gauss hypergeometric function[131], ${}_2F_1$. The result in this case for $I(a,b)$ is

$$I(a,b) = (-1)^{a+b} i\pi^{\omega+1/2}\frac{\Gamma(2A)}{\Gamma(a)\Gamma(C)}(n^2)^{a-\omega}(2)^{-2A+1}(p\cdot n - i\alpha)^{-2A}\times \tag{6.12}$$
$${}_2F_1\left[A, B; C; 1 - \frac{L+p^2}{p_L^2}\right]$$

where

$$
\begin{aligned}
A &= a + b/2 - \omega \\
B &= a + (b+1)/2 - \omega \\
C &= a + b + 1/2 - \omega
\end{aligned}
\tag{6.13}
$$

and $p_L^2 = (p \cdot n - i\alpha)^2 / n^2$.

Similar results can be obtained for (6.4) and for the purposes of the results to one loop presented in the next section I computed these integrals up to the case of four Lorentz indices, that is up to $I_{\mu\nu\lambda\rho}$. These results are among those presented in appendix B.

Turning to one loop order diagrams and the types of covariant denominators that will appear, for massless diagrams involving, say, a gauge boson we have

$$
\frac{1}{q^2(q-k)^2} \tag{6.14}
$$

so that, combining denominators à la Feynman with parameter y, we must make the following replacement in the above form (6.13)

$$
\begin{aligned}
L &= -yk^2 \\
p_\mu &= -yk_\mu \, .
\end{aligned}
\tag{6.15}
$$

For diagrams with a massive fermion the corresponding covariant part of the denominator will have the form

$$
\frac{1}{q^2[(q-k)^2 - m^2]} \tag{6.16}
$$

and the replacement in (6.13) is

$$
\begin{aligned}
L &= -y(k^2 - m^2) \\
p_\mu &= -yk_\mu \, .
\end{aligned}
\tag{6.17}
$$

The crucial observation to be made here is that in the argument for the hypergeometric function in (6.13) one has the factor

$$
(yk \cdot n + i\alpha)^a \tag{6.18}
$$

for some power a. Because one is treating α as a finite parameter, rather than as an infinitesmal, one is not at liberty to factor the parameter y out of this expression in order to perform the parameter integrals, otherwise algebraic consistency is lost.

The complexity of the computation is now manifest. In order to obtain both UV finite and infinite parts of integrals and diagrams the task devolves to computing Feynman parameter integrals over the hypergeometric functions. In some cases, as in the computation of the tadpole diagram, the hypergeometrics reduce to ordinary Euler gamma functions rendering it easy to extract both finite and divergent parts. To proceed further is a significantly arduous task which I have not pursued beyond qualitative verification that negative powers of the parameter α do arise from these integrals. Hence the limit $\alpha \to 0$ must not be taken even at the completion of individual momentum integrals. But this is no surprise as I showed in the last chapter that even for entire diagrams relevant for the Wilson loop the limit was unsafe. I should expect that in the computation of loop amplitudes the limit should not be taken until individual graphs have been summed into a gauge independent quantity.

Because the primary question I wish to investigate in this work is whether the alpha prescription avoids the nonlocal momentum problems besetting the LM prescription, it suffices in this work to consider just the UV divergent parts. Thus I have been content to extract the infinite parts of integrals as $\omega \to 2$.

Once these parts are obtained for the individual integrals with denominators $(q \cdot n \pm i\alpha)$ they can be recombined via quite tedious algebra into the decomposition formulae (6.2) and (6.3).

6.2.1 Other Tricks

As mentioned, I have only computed integrals up to the case of four Lorentz indices. However in the graphs I consider below integrals are required with five and even six Lorentz components - a nasty property of the above noted fact that the alpha prescription denominator is quadratic. Nonetheless, the integrals I have computed suffice to generate all other required divergent parts. Thus, though integrals with five or six powers of momentum in the numerator could be derived from first principles this necessitates a massive number of permutations of all free indices.

A simple trick I have used has been to exploit the specific form of integrals actually appearing in diagrams which involve factors in the numerator such as powers of $(q \cdot n)^2$. Such expressions may be *reduced* by using the substitution

$$(q \cdot n)^2 \to [(q \cdot n)^2 + \alpha^2(n^2)^2] - [\alpha^2(n^2)^2] \tag{6.19}$$

and then algebraically cancelling corresponding factors in the denominator. The resulting expression is then an integral with four or fewer factors of momentum in the numerator for which the results up to $I_{\mu\nu\lambda\rho}$ may be used. Note that this does not invoke delicate properties of distributions, and is thus an entirely safe procedure. Similarly, integrands with powers of (q^2) in the numerator may be simplified by cancelling in the denominator - which the Feynman $i\epsilon$ permits.

Figure 6.1: The gauge boson self-energy diagram.

A list of divergent parts of integrals used in the computations of the next section, is presented in appendix B.

6.3 UV Divergent Parts of Diagrams

In the alpha-prescription I find the following expression for the gluon self-energy (figure (6.1)) UV divergent part

$$\Pi^{ab}_{\mu\nu}(p) = \frac{ig^2\delta^{ab}C_2(G)}{16\pi^2(2-\omega)}[\frac{11}{3}(p^2 g_{\mu\nu} - p_\mu p_\nu) - \frac{2}{3}\alpha^2(5n^2 g_{\mu\nu} + 7n_\mu n_\nu)] \,. \tag{6.20}$$

Of course, as discussed in chapter 5, the tadpole contribution cannot be *a priori* neglected. I obtain for the UV divergent part

$$T^{ab}_{\mu\nu}(p) = \frac{ig^2\delta^{ab}C_2(G)}{16\pi^2(2-\omega)}\frac{2}{3}\alpha^2[2n^2 g_{\mu\nu} + n_\mu n_\nu] \,. \tag{6.21}$$

In the last chapter I alluded to a disagreement between my result and that of Nyeo [126]. Nyeo obtains, as $2\omega \to 4$, 4 for the coefficient of $n^2 g_{\mu\nu}$ and the $n_\mu n_\nu$ term has the opposite sign to that in my result. My analysis suggests that both these differences arise from a single sign error in Nyeo's tadpole computations. As far as the discussion that follows is concerned this issue is not particularly significant.

The sum of (6.20) and (6.21) represents the divergent part of the one-loop gluonic two-point function (figure (6.2))

$$G^{ab}_{\mu\nu}(p) = \frac{ig^2\delta^{ab}C_2(G)}{16\pi^2(2-\omega)}[\frac{11}{3}(p^2 g_{\mu\nu} - p_\mu p_\nu) - 2\alpha^2(n^2 g_{\mu\nu} + 2n_\mu n_\nu)] \,. \tag{6.22}$$

Note the lack of gauge or BRST invariance in (6.22); it does not satisfy the naive Ward identity

$$p^\mu G_{\mu\nu}(p) = 0 \,. \tag{6.23}$$

In light of the results of the chapter 4 this is entirely expected: without additional vertices in the theory, the alpha prescription does not arise from a BRST invariant theory [97, 125].

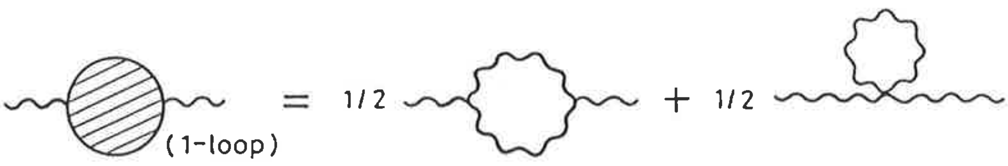

Figure 6.2: The gauge boson two-point function to one loop.

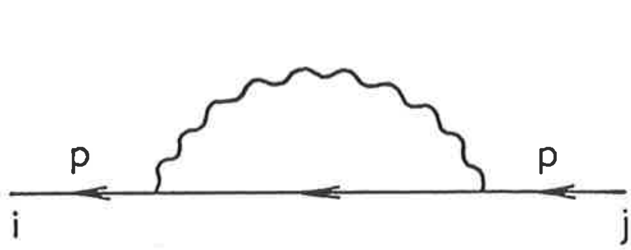

Figure 6.3: The fermion self energy to one loop.

On the other hand, the expression for the self-energy is far simpler than previous expressions derived for the LM, or generalised LM prescriptions. Moreover, it is purely local in the external momentum, a feature not shared by the result in the LM prescriptions.

Turning to the matter fields now: the fermion-loop contribution to the two point function does not involve an integration over the gauge-boson propagator. Indeed, it yields the usual transverse term thus modifying only the coefficient of the transverse piece in the above expression. I thus omit any discussion of this contribution.

The fermion self-energy divergent part (figure [6.3]) is evaluated

$$-\Sigma_{ij}(p) = \frac{g^2\delta_{ij}}{16\pi^2(2-\omega)}[3C_2(R)]\not{p}\,. \tag{6.24}$$

In fact, this is made up from two terms

$$(-\not{p}+4m)+4(\not{p}-m)=3\not{p} \tag{6.25}$$

the first being the familiar expression arising from the Feynman gauge piece in the gauge boson propagator, and the second being directly proportional to a contribution which vanishes, by the Dirac equation, for on-shell fermions when external lines are included. Thus, the surprising appearance of mass independence in the full expression is only spurious.

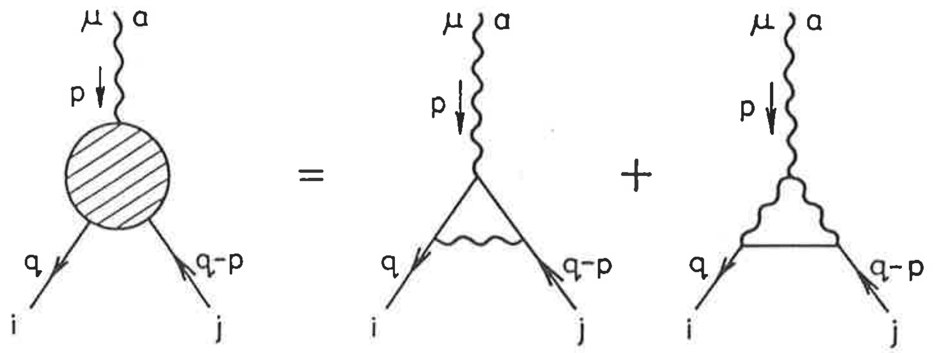

Figure 6.4: The fermion-boson vertex to one loop order.

The one-loop correction to the fermion-boson vertex has two parts: a QED-like piece and a strictly non-Abelian contribution involving the non-Abelian three-gluon vertex (figure [6.4]). For the first, one obtains

$$\Gamma^{(1)a}_{\mu\,ij} = (g\gamma_\mu T^a_{ij})\frac{g^2}{16\pi^2(2-\omega)}[\frac{3}{2}C_2(G) - 3C_2(R)] \tag{6.26}$$

and for the second

$$\Gamma^{(2)a}_{\mu\,ij} = (g\gamma_\mu T^a_{ij})\frac{g^2}{16\pi^2(2-\omega)}[-\frac{3}{2}C_2(G)]\,. \tag{6.27}$$

For clarity, the tree-level Feynman rule for the vertex has been written as a separate factor in these expressions for the amplitude. Summing these gives the divergent part of the full one-loop correction to the fermion-gauge-boson vertex

$$\Gamma^{a}_{\mu\,ij} = (g\gamma_\mu T^a_{ij})\frac{g^2}{16\pi^2(2-\omega)}[-3C_2(R)]\,. \tag{6.28}$$

These expressions are extremely attractive for a number of reasons: they are independent of gauge-related quantities such as α or n_μ and there are no non-local dependences on the external momentum. Indeed, to one loop the divergence in the fermion-gauge-boson vertex is momentum independent. Once again, these features are not shared by the corresponding results in the generalised LM prescription[42].

Unlike the gauge-boson two point correction, these expressions do satisfy the naive Ward identity relating fermion self-energy to the vertex parts

$$(k-p)^\mu \Lambda_\mu(k,p) = -\Sigma(k) + \Sigma(p) \tag{6.29}$$

where Λ and Σ (with no gauge group indices) are related to the original amplitudes by

$$\Gamma^a_{\mu\,ij} = gT^a_{ij}\Lambda_\mu \tag{6.30}$$

$$-\Sigma_{ij}(p) = \Sigma(p)\delta_{ij}\,. \tag{6.31}$$

6.4 Renormalisation

6.4.1 General Considerations

Before embarking on the renormalisation itself I first consider the mass dimensions of the various quantities that have been implicit in the alpha prescripted propagator (4.1). Maintaining $n^2 > 0$ my previous statement that α has mass dimensionality one must be altered. Adopting the notation $dim(Q)$ to denote the mass dimensionality of some quantity Q, one finds

$$dim(\alpha\sqrt{n^2}) = 1 \tag{6.32}$$

Now there is a certain amount of freedom in choosing the dimensions of α and n individually. For example if the gauge parameter λ is taken to be dimensionless then $dim(n) = 1$ and α is also dimensionless. In what follows however there is no need to make such a specific choice.

The other issue is that of power counting. The explicit computations outlined in the previous sections verify that naive power counting is valid to one loop for the two-point corrections considered; the UV divergences obtained are consistent with expectations based on the counting of momenta in the integrals. At this stage no proof of the *general* validity of naive power counting exists for Feynman integrals with the alpha prescription. The reason why one should not in general expect this is because in the Feynman integrals the denominator $(p \cdot n)^2 + \alpha^2(n^2)^2$ arises. In some directions in momentum space this scales like p^2 for high momenta, but not in the directions in which $p \cdot n$ is constant (for example if $n_\mu = (1,0,0,0)$ and $p_\mu = (v_0, \eta\ \mathrm{v})$, for fixed v_0 and v, where η is the scaling variable). Nevertheless, for the specific case of my one loop results naive power counting works. Moreover, the divergences are polynomial in the external momentum. This means that I can introduce, at the one loop level, counterterms that are polynomial in the fields and derivatives. If one *assumes* that this is sustained at all orders then one can write down the general form of counterterms appearing in the Lagrangian, to any loop order. The first types of counterterm that will appear are those of the same form as the original terms in the bare Lagrangian. These enable removal of divergences by multiplicative renormalisation. The second types, which do not correspond with terms in the original Lagrangian will remove divergences subtractively. Thus power counting and the dimensionlessness of the action give for these counterterms

$$\mathcal{L}_C \sim (\alpha\sqrt{n^2})^{4-D}\mathcal{O}_n^{(D)} \tag{6.33}$$

Here D is the operator dimension of $\mathcal{O}_n^{(D)}$ counting the number of fields and derivatives in the operator. By the assumption of power counting $D \leq 4$. I take $\mathcal{O}_n^{(D)}$ to be *homogeneous* in n_μ (namely, independent of scaling of n_μ).

Generally if these counterterms are treated as vertex insertions in Feynman graphs, they will generate new counterterms at the next order in perturbation theory. Will there be a finite or infinite number of these? In (6.33) the factor $(\alpha\sqrt{n^2})^{4-D}$ plays the role of a coupling constant - with *positive* mass dimension. Renormalisability to all orders in perturbation theory is indeed guaranteed[132].

Other issues such as the breaking of BRST invariance implicit in (6.33) and the case $D = 2$ shall be discussed below.

6.4.2 Counterterms

I now deduce the specific forms of counterterms necessary to cancel divergences in the fermion and gluon two-point functions and the fermion-gluon vertex to one loop order.

Beginning with the fermionic sector where matters are quite simple. The coupling-constant, fermion-field and mass renormalisation constants for constructing counterterms are simple to write down in the MS scheme:

$$Z_{1F} = 1 + \frac{g_R^2}{(4\pi)^2} 3C_2(R)/\varepsilon \tag{6.34}$$

$$Z_2 = 1 + \frac{g_R^2}{(4\pi)^2} 3C_2(R)/\varepsilon \tag{6.35}$$

$$Z_m = 1 - \frac{g_R^2}{(4\pi)^2} 3C_2(R)/\varepsilon \tag{6.36}$$

where, as usual, $\varepsilon = 2 - \omega$. These are consistent with the naive Slavnov-Taylor form for the Ward identity discussed in the previous section

$$Z_{1F} = Z_2 \,. \tag{6.37}$$

For the gauge-field renormalisation matters are complicated by the non-transverse term in the gluon self-energy result. Introducing the usual field renormalisation constant, Z_3, defined by

$$Z_3 = 1 - \frac{g_R^2}{(4\pi)^2}[\frac{11}{3}\frac{C_2(G)}{\varepsilon}] \tag{6.38}$$

will eliminate the transverse part of the divergence.

For the longitudinal part one is now forced to resort to a counterterm consistent with (6.33) as neither the Yang-Mills Lagrangian nor the alpha (or Q) Lagrangian of chapter 4 give a term of the appropriate form to cancel the divergence in (6.22). *To one loop* my result for the counterterm Lagrangian is

$$\mathcal{L}_C^{(\alpha)\ one-loop} = \alpha^2 \frac{g_R^2}{(4\pi)^2}\frac{C_2(G)}{\varepsilon}(n^2A^2 + 2(n\cdot A)^2) \,. \tag{6.39}$$

If divergences at higher orders remain polynomial in the external momenta then to higher orders the counterterm can be written as

$$\mathcal{L}_C^{(\alpha)} = \alpha^2(\mathcal{Z}_1 n^2 A^2 + \mathcal{Z}_2 (n \cdot A)^2) \tag{6.40}$$

for appropriate (infinite) renormalisation constants $\mathcal{Z}_1$ and $\mathcal{Z}_2$. It is clear that the two terms in this expression are consistent with the general form (6.33) for the case $D = 2$.

This counterterm may be implemented in one of two ways (as opposed to the cases where $D > 2$): either by including it in a new kinetic term and generating a new propagator for diagrams, or by inserting it as a two-point vertex of order g_R^2 at each loop order in 1PI graphs in perturbation theory. The latter course is related to the former in that the new propagator will be the consequence of an infinite sum of a geometric series of insertions; the first method truncates the series implicit in the second approach.

Following the course of defining a new propagator is undesirable. Consider first that this situation is similar to that found in massive QED/QCD, where a mass term in the Lagrangian $M^2 A^\mu A_\mu$ is is seen to generate a term proportional to $p_\mu p_\nu / p^2 M^2$ in the corrected propagator. The difficulty that arises here is that power counting is *lost* in the propagator and the theory is not manifestly renormalisable[132]. In the alpha prescription, $\alpha\sqrt{n^2}$ behaves precisely like a mass and the same danger is apparent. The danger of nonrenormalisability is however avoided by truncating the series as stated above. The second reason defining a new propagator in order to introduce the counterterm is undesirable is because the object of fundamental concern in this work is precisely the *Landshoff* form of the propagator (4.1); it would merely obscure matters to follow the course of defining a new propagator.

The other issue to be dealt with is that of the breaking of BRST invariance for non-zero α in the new counterterm (6.39) or more generally in the counterterms (6.33). Because the gauge-dependent divergences found are proportional to α^2 it might seem reasonable to take the limit $\alpha \to 0$ before the limit $2\omega \to 4$, provided it is done in gauge independent objects for which we should expect, if the theory is consistent, the limit $\alpha \to 0$ to be viable. In the Wilson loop computation it was possible to take the limit $\alpha \to 0$ for the *sum* of all diagrams before $2\omega \to 4$. What was occurring in that calculation, had one been explicit in dealing with the divergences in the loop, was that after the first limit (which had no singular behaviour) one would have been left with divergences in the Wilson loop amplitude which were alpha independent and which could be absorbed in the coupling constant renormalisation [111].

However, the legitimacy of the limit $\alpha \to 0$ in all gauge independent quantities is a formidable question in its own right and the ideal approach would be to disentangle it from the renormalisation problem. The introduction of the BRST non-invariant counterterm enables the one-loop renormalisation now for $\alpha \neq 0$ and so facilitates this disentangling of problems to one loop order at least.

The price to be paid for this unravelling of two problems is that, alluded to before, of characterising divergences at the higher loop level via Slavnov-Taylor or equivalent BRST identities. In admitting the counterterm structure (6.39) such a tool is surrendered. Moreover, for non-zero α unitarity would be violated for the non-Abelian theory. And the verification that the limit $\alpha \to 0$ restores unitarity is a long way off.

At one loop, nevertheless, the renormalisation of the fermion-gluon vertex and the fermion and gluon propagators is now complete.

Chapter 7

Conclusions - The Alpha Prescription

7.1 Results

I have discussed in depth two approaches to a well-defined YM theory in the temporal gauge in perturbation theory. The former, based on fully-fixing the remaining gauge degrees of freedom, invokes complications which undermine any usefulness that the temporal gauge might present.

I have therefore focussed in greater detail on the second approach - that of the alpha prescription.

I may conclude that it is possible to derive the exact alpha prescription based on a Lagrangian theory. The two forms this may take presented in this work involve either a BRST invariant Lagrangian in the temporal gauge which necessarily generates an infinite number of, well-defined, vertices that must be included in the Feynman rules, or a BRST non-invariant theory which nonetheless simplifies the number of additional rules required in perturbative calculations.

The generation of a new four gluon vertex rule in the first derivation has not been checked in the static Wilson loop, which Landshoff shows to be consistent with the prescription provided the regulating parameter is kept non-zero until all diagrams are summed. Landshoff's work omits any reference to tadpole graphs which are not necessarily zero for non-zero alpha. But I have shown in this work that these also do not contribute to the Wilson loop in the limit $\alpha \to 0$. Also in this approach to the derivation I have outlined how the BRST quantisation could proceed for the complete non-Abelian theory - but the presence of the new vertices again causes one to question whether this is a satisfactory approach to the temporal gauge; though FP ghosts are shown to decouple, the new vertices again overcomplicate the theory.

In the BRST non-invariant approach, due to Przeszowski, I outlined his Hamiltonian approach to the quantisation which becomes difficult to solve in the interacting theory. Thus limiting myself to the free Abelian theory I reviewed Przeszowski's work showing that in a subspace of the indefinite metric Hilbert space it is possible to recover the physical requirements of the positivity of the Hamiltonian and number operator and the weak implementation of the Gauss law. I discussed my joint work with him demonstrating the recovery of Poincare invariance in the limit $\alpha \to 0$ in the same subspace of physical states. In particular, this subspace does map into itself under space-time translations, rotations and Lorentz boosts.

Finally I initiated work on renormalising YM theory in the temporal gauge using the alpha prescription. I developed techniques for performing Feynman integrals for loop amplitudes in this prescription, and used these results in computing the ultraviolet divergent parts of the gluon self-energy, the fermion self-energy and the fermion-gluon vertex to one loop. These results were seen to be eminently more simple than corresponding results in the most seriously considered alternative prescription in the literature thus far - the generalised Leibbrandt-Mandelstam prescription.

The renormalisation constants for the fermion and gauge boson fields, the mass and coupling constant were computed. A counterterm, which is not BRST invariant, was constructed to remove the longitudinal divergence in the gauge field two-point function. Thus the renormalisation of the gluon and fermion propagators and the fermion-gluon vertex was demonstrated at the one loop level. This renormalisation was achieved, moreover, for non-zero α. Dimensional arguments and the assumption of power counting, valid at least for the graphs considered so far, dictated the general form for these BRST non-invariant counterterms. Introducing these as vertex insertions at the next loop order was shown to be consistent with a renormalisable theory.

7.2 Outlook

Most of the problems that arise directly from this work rest on two outstanding difficulties for their resolution:

- The completion of the Hamiltonian quantisation for the full interacting theory.

Knowing the full Hilbert space in light of this would show that the temporal gauge is a physically consistent and unambiguous theory, from which the unphysical degrees of freedom - such as Faddeev-Popov ghosts - have decoupled. Thus, turning to the goals established in the opening chapter, the fulfillment of conditions (1) and (2) rest on the completion of this work.

- The verification that the limit $\alpha \to 0$ is permissible for, *at least*, all gauge independent observables.

Condition (4) in chapter 1 is tied to the completion of this task, but also invokes the requirement that, upon taking the limit, the gauge-independent amplitude for an observable results. In the absence of any elegant method, one appears forced to check this for all conceivable gauge independent observables. The root of this problem is that conventional field theoretic work is based on Green's functions as the primary constructs of the theory. The limit $\alpha \to 0$ is ill-defined at the level of the coordinate space Green's functions. The check that the higher order vertices implicit in my derivation do not affect the physical theory involves the limit. This involves two subsidiary questions: whether the new vertices spoil gauge-independence in observables, and whether divergences arising from them ruin renormalisability. The demonstration of unitarity of the S-matrix as $\alpha \to 0$ (condition (3) of chapter 1) is, of course, intimately tied into these questions, and no work has been done in this respect.

Another question is what form will divergences arising from the three-gluon and four-gluon vertices at one loop take? Is naive power counting sustained in integrals with the Landshoff denominator? This would secure the renormalisability to all orders in perturbation theory on the basis of general arguments. The explicit demonstration of renormalisibility is hindered by the numerous places where BRST non-invariant terms have been added into the Lagrangian - either to derive the alpha prescription, or to renormalise.

Numerous outstanding difficulties remain. Nonetheless, the work presented in this thesis suggets the alpha prescription is a way forward for a physically consistent, and renormalisable theory of the Yang-Mills field in the temporal gauge. That the final form this theory will take is not much simpler than the theory in covariant gauges seems an unavoidable conclusion. *Immo id, quod aiunt, auribus teneo lupum nam neque quomodo a me amittam, invenio: neque, uti retineam scio*[1].

[1] Indeed it is as they say, I have got a wolf by the ears; How to loose him from me I don't see, how to hold him I can't tell. Terentius, *Phormia* 3, 2, 21.

Appendix A

Conventions and Rules

A.1 Metric Conventions

The following conventions - largely adopted from Itzykson and Zuber [99] - have been used for the metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{A.1}$$

where derivatives with respect to contravariant (x^μ) or covariant (x_μ) coordinates are written in the form

$$\begin{aligned} \partial_\mu &= \frac{\partial}{\partial x^\mu} \\ \partial^\mu &= \frac{\partial}{\partial x_\mu} . \end{aligned} \tag{A.2}$$

Repeated Lorentz (Greek) or spatial (Latin) indices indicate summation

$$A \cdot B = A_\mu B^\mu = A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B} = A^0 B^0 - A^i B^i \tag{A.3}$$

where a boldface letter represents a three-vector and the Latin index runs over $(1, 2, 3)$. The three dimensional gradient operator is denoted

$$\nabla = (\partial_1, \partial_2, \partial_3) = (\partial_i) = \left(-\partial^i\right) \tag{A.4}$$

The Laplacian operator is written throughout in either of a number of ways:

$$\nabla^2 = \partial_i \partial_i = \triangle = -\partial^i \partial_i \tag{A.5}$$

and the d'Alembertian operator is then

$$\Box = \partial^\mu \partial_\mu = \partial_0^2 - \nabla^2 = \partial_0^2 - \triangle . \tag{A.6}$$

With these metric conventions we adopt the following definition of the Poisson bracket for a theory with a vector field A_μ and a scalar field ϕ:

$$\{u,v\} = \int d^3z [\frac{\delta u}{\delta A_i(z)}\frac{\delta v}{\delta \pi^i(z)} - \frac{\delta u}{\delta \pi_i(z)}\frac{\delta v}{\delta A^i(z)} + \frac{\delta u}{\delta \phi(z)}\frac{\delta v}{\delta \pi_\phi(z)} - \frac{\delta u}{\delta \pi_\phi}\frac{\delta v}{\delta \phi(z)}] \,. \qquad (A.7)$$

A.2 Feynman Rules for Yang-Mills Theory

As mentioned at the beginning of this work, the conventions of Muta [100] have been adopted for the Feynman rules, with appropriate modifications to the propagator for working in an axial gauge. Because ghost-decoupling is a sought for aspect of the work FP ghost Feynman rules are not given.

- Gauge Field Propagator:

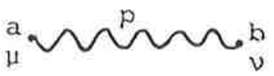

Figure A.1: Gauge Field Propagator

$$D^{ab}_{\mu\nu}(p) = \frac{\delta^{ab}}{(p^2 + i\epsilon)} d_{\mu\nu}(p) \qquad (A.8)$$

where $d_{\mu\nu}(p)$ is the appropriate factor for either the bare axial gauge propagator (no prescription) or with the alpha prescription - see the main body of the text for details.

- Quark Propagator:

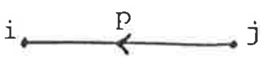

Figure A.2: Quark Propagator

$$S^{ij}_F(p) = \frac{\delta^{ij}}{(m - \not{p})} \qquad (A.9)$$

where, in this context, the indices (i, j) denote the components of the representation matrices of the gauge group $SU(N)$.

- Three Gluon Vertex:

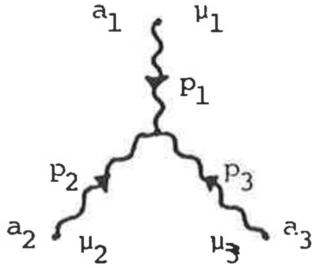

Figure A.3: Three Gluon Vertex

$$- igf^{a_1a_2a_3}\left[g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3}+g_{\mu_2\mu_3}(p_2-p_3)_{\mu_1}+g_{\mu_3\mu_1}(p_3-p_1)_{\mu_2}\right] . \quad (A.10)$$

- Four Gluon Vertex:

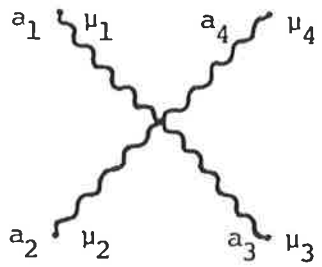

Figure A.4: Four Gluon Vertex

$$-g^2[(f^{a_1a_3a}f^{a_2a_4a}-f^{a_1a_4a}f^{a_3a_2a})g_{\mu_1\mu_2}g_{\mu_3\mu_4}+ \quad (A.11)$$
$$(f^{a_1a_2a}f^{a_3a_4a}-f^{a_1a_4a}f^{a_2a_3a})g_{\mu_1\mu_3}g_{\mu_2\mu_4}+(f^{a_1a_3a}f^{a_4a_2a}-f^{a_1a_2a}f^{a_3a_4a})g_{\mu_1\mu_4}g_{\mu_3\mu_2}]$$

- Fermion-Gluon Vertex:

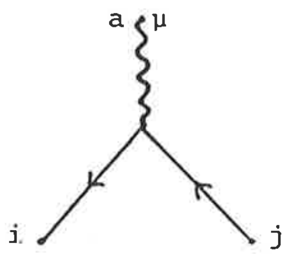

Figure A.5: Fermion-Gluon Vertex

$$g\gamma_\mu T^a_{ij} \quad (A.12)$$

- Gluon Loop:

$$\frac{1}{i}\int\frac{d^{2\omega}p}{(2\pi)^{2\omega}}\delta^{ab}g^{\mu\nu} \quad (A.13)$$

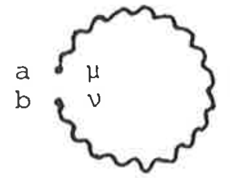

Figure A.6: Gluon Loop

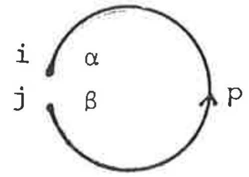

Figure A.7: Quark Loop

- Quark Loop:

$$-\frac{1}{i}\int\frac{d^{2\omega}p}{(2\pi)^{2\omega}}\delta^{ij}g^{\alpha\beta} \tag{A.14}$$

  where α and β are spinor indices.

- Symmetry Factors: The gauge boson loop and tadpole diagrams both have symmetry factors of $1/(2!) = 1/2$.

A.3 Feynman Rules for Static Wilson Loop Amplitudes

The above rules apply for diagrams within the loop amplitude. Extra rules are required in order to 'attach' the diagram to the rectangular loop.

- Gluon Attaches to Horizontal Rungs:

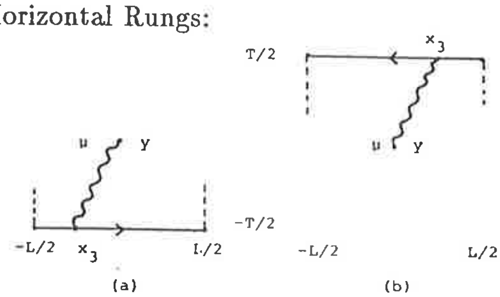

Figure A.8: Gluon Attached to Horizontal Rungs: (a) Lower, (b) Upper

$$(a)\quad \int_{-L/2}^{L/2} dx_3 D^{ab}_{3\mu}(-T/2, x_3; y_0, \mathbf{y}) \tag{A.15}$$

$$(b) \quad \int_{L/2}^{-L/2} dx_3 D^{ab}_{\mu 3}(y_0, \mathbf{y}; T/2, x_3) \tag{A.16}$$

- Gluon Attaches to Vertical Rungs:

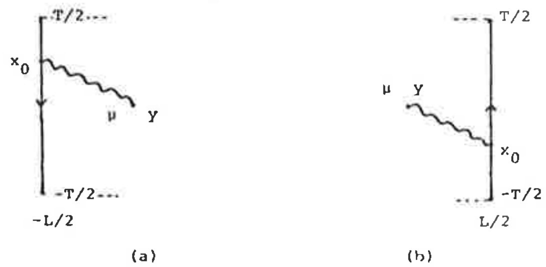

Figure A.9: Gluon Attached to Vertical Rungs: (a) Left, (b) Right

$$(a) \quad \int_{T/2}^{-T/2} dx_0 D^{ab}_{\mu 0}(y_0, \mathbf{y}; x_0, -L/2) \tag{A.17}$$

$$(b) \quad \int_{-T/2}^{T/2} dx_0 D^{ab}_{0\mu}(x_0, L/2; y_0, \mathbf{y}) \tag{A.18}$$

In the temporal gauge these last two rules are unnecessary as $D_{0\mu} = D_{\mu 0} = 0$. That is, such diagrams will vanish.

- Group Factors: As well as the above rules, a representation matrix T^a is present at each vertex between the gluon and the loop contour. These matrices must be multiplied in the order in which they attach to the loop in the direction of the oriented contour. The trace of the resulting matrix in the final amplitude must be taken. For order g^2 diagrams the trace generates an overall 'Abelian' factor:

$$Tr[T^a T^b]\delta^{ab} = C_2(R) \tag{A.19}$$

where the δ^{ab} comes from the single propagator in the diagram.

A useful rule for simplifying the sum of double propagator diagrams such as those in figure [A.10] may be derived from the properties of the gauge group algebra.

This diagram involves the expression

$$Tr[T^a T^b T^c T^d][\delta^{ab}\delta^{cd} I_1 + \delta^{ac}\delta^{bd} I_2 + \delta^{ad}\delta^{bc} I_3] \tag{A.20}$$

where I_2 is always the integral corresponding to the 'crossed-propagator' diagram. After some group algebra, the expression can be rewritten as

$$C_2(R)^2[I_1 + I_2 + I_3] - 1/2 C_2(R) C_2(G) I_2 \tag{A.21}$$

where the first term here is the pure Abelian contribution to second order, and the second is the strictly non-Abelian contribution. Thus the non-Abelian contribution arises purely in terms of the crossed diagram (2).

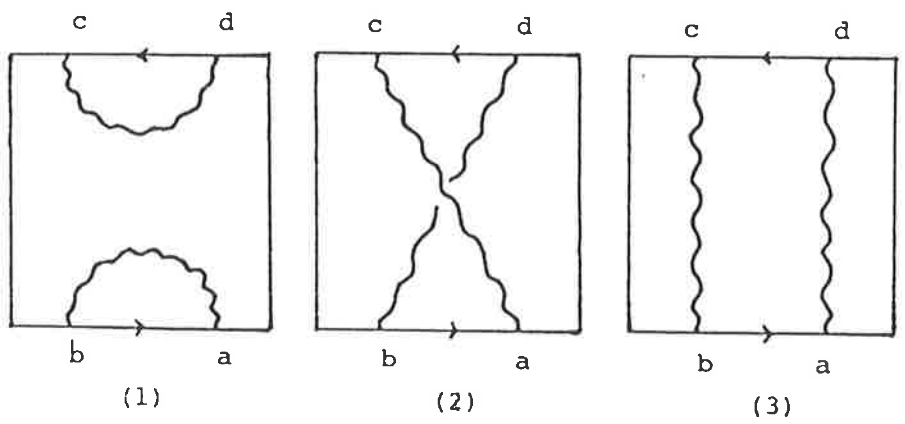

Figure A.10: Double Propagator Diagrams: (1) + (2) + (3).

Appendix B

Feynman Integrals

To simplify expressions the following notations have been introduced:

$$\bar{I} = \frac{i}{16\pi^2(2-\omega)} \tag{B.1}$$

$$[q.n]^2 = (q.n)^2 + \alpha^2(n^2)^2 \tag{B.2}$$

$$T_{[\mu_1,...,\mu_n]} = \textit{symmetrise } T \textit{ w.r.t. indices } \mu_1, ..., \mu_n \tag{B.3}$$

$$Dq = \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \tag{B.4}$$

B.1 Integrals with Linear Noncovariant Denominator.

The following integrals complement the results given by Konetschny [128] who deals with zero or one Lorentz indices - that is, zero or one momenta in the numerator of the integrand - and has the equivalent of the parameter $\alpha = 0$. Here I keep α nonzero and give results up to four momenta in the numerator.

B.1.1 Complete Integrals

I begin with integrals for which the complete results have been determined - divergent and UV convergent parts. The result for no Lorentz indices is given in chapter 6 in 6.13, and the notation developed there will be used here - with the additional simplification of writing $(p \cdot n - i\alpha)$ as $p \cdot n$ in the result. This is to avoid confusing α in the prescription for the $p \cdot n$ pole with the exponents α and β in the integrand of the expressions below.

- Two Lorentz Indices:

$$\int Dq \frac{q_\mu q_\nu}{(q^2 + 2p \cdot q - L + i\epsilon)^\alpha (q \cdot n + i\alpha)^\beta} = \sum_{j=1}^{4} T^{(j)}_{\mu\nu}(p) S^{(2)}_j(\alpha, \beta; \omega, p) \tag{B.5}$$

where the $T^{(j)}$ denote the following tensor structures:

$$T^{(1)}_{\mu\nu} = g_{\mu\nu} \qquad T^{(2)}_{\mu\nu} = p_\mu p_\nu \tag{B.6}$$
$$T^{(3)}_{\mu\nu} = p_{[\mu}n_{\nu]} \qquad T^{(4)}_{\mu\nu} = n_\mu n_\nu$$

and the functions S are given by:

$$\begin{aligned} S_1^{(2)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta+1}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-3/2}(n^2)^{\alpha-\omega-1}(2p\cdot n)^{-(2\alpha+\beta-2\omega-2)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega-1)\Gamma(2\alpha+\beta-2\omega-2)}{\Gamma(\alpha)\Gamma(2\alpha+2\beta-2\omega-2)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-1,\alpha+\frac{\beta}{2}-\omega-\frac{1}{2};\alpha+\beta-\omega-\frac{1}{2};1-\frac{p^2+L}{p_L^2}) \end{aligned} \tag{B.7}$$

$$\begin{aligned} S_2^{(2)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega}(n^2)^{\alpha-\omega}(2p\cdot n)^{-(2\alpha+\beta-2\omega)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega)}{\Gamma(\alpha)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega,\alpha+\frac{\beta}{2}-\omega+\frac{1}{2};\alpha+\beta-\omega+\frac{1}{2};1-\frac{p^2+L}{p_L^2}) \end{aligned} \tag{B.8}$$

$$\begin{aligned} S_3^{(2)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-1/2}(n^2)^{\alpha-\omega-1}(2p\cdot n)^{-(2\alpha+\beta-2\omega-1)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega-1)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-\frac{1}{2},\alpha+\frac{\beta}{2}-\omega;\alpha+\beta-\omega+\frac{1}{2};1-\frac{p^2+L}{p_L^2}) \end{aligned} \tag{B.9}$$

$$\begin{aligned} S_4^{(2)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-1}(n^2)^{\alpha-\omega-2}(2p\cdot n)^{-(2\alpha+\beta-2\omega-2)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega-2)\Gamma(\beta+2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-1,\alpha+\frac{\beta}{2}-\omega-\frac{1}{2};\alpha+\beta-\omega+\frac{1}{2};1-\frac{p^2+L}{p_L^2}) \end{aligned} \tag{B.10}$$

- Three Lorentz Indices:

$$\int Dq \frac{q_\mu q_\nu q_\lambda}{(q^2+2p\cdot q-L+i\epsilon)^\alpha(q\cdot n+i\alpha)^\beta} = \sum_{j=1}^{6} T^{(j)}_{\mu\nu\lambda}(p) S_j^{(3)}(\alpha,\beta;\omega,p) \tag{B.11}$$

where we now have the six tensor structures:

$$T^{(1)}_{\mu\nu\lambda} = g_{[\mu\nu}p_{\lambda]} \qquad T^{(2)}_{\mu\nu\lambda} = g_{[\mu\nu}n_{\lambda]} \tag{B.12}$$
$$T^{(3)}_{\mu\nu\lambda} = p_\mu p_\nu p_\lambda \qquad T^{(4)}_{\mu\nu} = p_{[\mu}p_\nu n_{\lambda]}$$
$$T^{(5)}_{\mu\nu\lambda} = p_{[\mu}n_\nu n_{\lambda]} \qquad T^{(6)}_{\mu\nu\lambda} = n_\mu n_\nu n_\lambda \,.$$

The six $S_j^{(3)}$ are:

$$\begin{aligned} S_1^{(3)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-3/2}(n^2)^{\alpha-\omega-1}(2p\cdot n)^{-(2\alpha+\beta-2\omega-2)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega-1)\Gamma(2\alpha+\beta-2\omega-2)}{\Gamma(\alpha)\Gamma(2\alpha+2\beta-2\omega-2)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-1, \alpha+\frac{\beta}{2}-\omega-\frac{1}{2}; \alpha+\beta-\omega-\frac{1}{2}; 1-\frac{p^2+L}{p_L^2}) \end{aligned} \quad (B.13)$$

$$\begin{aligned} S_2^{(3)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-2}(n^2)^{\alpha-\omega-2}(2p\cdot n)^{-(2\alpha+\beta-2\omega-3)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega-1)\Gamma(2\alpha+\beta-2\omega-3)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega-2)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-\frac{3}{2}, \alpha+\frac{\beta}{2}-\omega-1; \alpha+\beta-\omega-\frac{1}{2}; 1-\frac{p^2+L}{p_L^2}) \end{aligned} \quad (B.14)$$

$$\begin{aligned} S_3^{(3)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta+1}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega}(n^2)^{\alpha-\omega}(2p\cdot n)^{-(2\alpha+\beta-2\omega)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega)}{\Gamma(\alpha)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega, \alpha+\frac{\beta}{2}-\omega+\frac{1}{2}; \alpha+\beta-\omega+\frac{1}{2}; 1-\frac{p^2+L}{p_L^2}) \end{aligned} \quad (B.15)$$

$$\begin{aligned} S_4^{(3)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta+1}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-1/2}(n^2)^{\alpha-\omega-1}(2p\cdot n)^{-(2\alpha+\beta-2\omega-1)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega-1)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-\frac{1}{2}, \alpha+\frac{\beta}{2}-\omega; \alpha+\beta-\omega+\frac{1}{2}; 1-\frac{p^2+L}{p_L^2}) \end{aligned} \quad (B.16)$$

$$\begin{aligned} S_5^{(3)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta+1}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-1}(n^2)^{\alpha-\omega-2}(2p\cdot n)^{-(2\alpha+\beta-2\omega-2)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega-2)\Gamma(\beta+2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-1, \alpha+\frac{\beta}{2}-\omega-\frac{1}{2}; \alpha+\beta-\omega+\frac{1}{2}; 1-\frac{p^2+L}{p_L^2}) \end{aligned} \quad (B.17)$$

$$\begin{aligned} S_6^{(3)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta+1}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-3/2}(n^2)^{\alpha-\omega-3}(2p\cdot n)^{-(2\alpha+\beta-2\omega-3)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega-3)\Gamma(\beta+3)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-\frac{3}{2}, \alpha+\frac{\beta}{2}-\omega-1; \alpha+\beta-\omega+\frac{1}{2}; 1-\frac{p^2+L}{p_L^2}) . \end{aligned} \quad (B.18)$$

- Four Lorentz Indices:

$$\int Dq \frac{q_\mu q_\nu q_\lambda q_\rho}{(q^2+2p\cdot q-L+i\epsilon)^\alpha (q\cdot n+i\alpha)^\beta} = \sum_{j=1}^{9} T^{(j)}_{\mu\nu\lambda\rho}(p) S^{(4)}_j(\alpha,\beta;\omega,p)\,. \quad \text{(B.19)}$$

There are now nine possible tensor structures to deal with this, and nine corresponding coefficient functions $S^{(4)}_j$. This is the most complex integral necessary for the calculations in this work.

$$\begin{aligned} T^{(1)}_{\mu\nu\lambda\rho} &= g_{[\mu\nu}g_{\lambda\rho]} & T^{(2)}_{\mu\nu\lambda\rho} &= g_{[\mu\nu}p_\lambda p_{\rho]} \\ T^{(3)}_{\mu\nu\lambda\rho} &= g_{[\mu\nu}p_\lambda n_{\rho]} & T^{(4)}_{\mu\nu} &= g_{[\mu\nu}n_\lambda n_{\rho]} \\ T^{(5)}_{\mu\nu\lambda\rho} &= p_\mu p_\nu p_\lambda p_\rho & T^{(6)}_{\mu\nu\lambda\rho} &= p_{[\mu}p_\nu p_\lambda n_{\rho]} \\ T^{(7)}_{\mu\nu\lambda\rho} &= p_{[\mu}p_\nu n_\lambda n_{\rho]} & T^{(8)}_{\mu\nu\lambda\rho} &= p_{[\mu}n_\nu n_\lambda n_{\rho]} \\ T^{(9)}_{\mu\nu\lambda\rho} &= n_\mu n_\nu n_\lambda n_\rho \end{aligned} \quad \text{(B.20)}$$

$$\begin{aligned} S^{(4)}_1(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-3}(n^2)^{\alpha-\omega-2}(2p\cdot n)^{-(2\alpha+\beta-2\omega-4)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega-2)\Gamma(2\alpha+\beta-2\omega-4)}{\Gamma(\alpha)\Gamma(2\alpha+2\beta-2\omega-4)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-2, \alpha+\frac{\beta}{2}-\omega-\frac{3}{2}; \alpha+\beta-\omega-\frac{3}{2}; 1-\frac{p^2+L}{p_L^2}) \end{aligned} \quad \text{(B.21)}$$

$$\begin{aligned} S^{(4)}_2(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta+1}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-3/2}(n^2)^{\alpha-\omega-1}(2p\cdot n)^{-(2\alpha+\beta-2\omega-2)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega-1)\Gamma(2\alpha+\beta-2\omega-2)}{\Gamma(\alpha)\Gamma(2\alpha+2\beta-2\omega-2)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-1, \alpha+\frac{\beta}{2}-\omega-\frac{1}{2}; \alpha+\beta-\omega-\frac{1}{2}; 1-\frac{p^2+L}{p_L^2}) \end{aligned} \quad \text{(B.22)}$$

$$\begin{aligned} S^{(4)}_3(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta+1}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-2}(n^2)^{\alpha-\omega-2}(2p\cdot n)^{-(2\alpha+\beta-2\omega-3)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega-1)\Gamma(2\alpha+\beta-2\omega-3)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega-2)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-\frac{3}{2}, \alpha+\frac{\beta}{2}-\omega-1; \alpha+\beta-\omega-\frac{1}{2}; 1-\frac{p^2+L}{p_L^2}) \end{aligned} \quad \text{(B.23)}$$

$$\begin{aligned} S^{(4)}_4(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta+1}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-5/2}(n^2)^{\alpha-\omega-3}(2p\cdot n)^{-(2\alpha+\beta-2\omega-4)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega-1)\Gamma(2\alpha+\beta-2\omega-4)\Gamma(\beta+2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega-2)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-2, \alpha+\frac{\beta}{2}-\omega-\frac{3}{2}; \alpha+\beta-\omega-\frac{1}{2}; 1-\frac{p^2+L}{p_L^2}) \end{aligned} \quad \text{(B.24)}$$

$$\begin{aligned} S_5^{(4)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega}(n^2)^{\alpha-\omega}(2p\cdot n)^{-(2\alpha+\beta-2\omega)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega)}{\Gamma(\alpha)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega,\alpha+\frac{\beta}{2}-\omega+\frac{1}{2};\alpha+\beta-\omega+\frac{1}{2};1-\frac{p^2+L}{p_L^2}) \end{aligned} \tag{B.25}$$

$$\begin{aligned} S_6^{(4)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-1/2}(n^2)^{\alpha-\omega-1}(2p\cdot n)^{-(2\alpha+\beta-2\omega-1)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega-1)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-\frac{1}{2},\alpha+\frac{\beta}{2}-\omega;\alpha+\beta-\omega+\frac{1}{2};1-\frac{p^2+L}{p_L^2}) \end{aligned} \tag{B.26}$$

$$\begin{aligned} S_7^{(4)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-1}(n^2)^{\alpha-\omega-2}(2p\cdot n)^{-(2\alpha+\beta-2\omega-2)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega-2)\Gamma(\beta+2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-1,\alpha+\frac{\beta}{2}-\omega-\frac{1}{2};\alpha+\beta-\omega+\frac{1}{2};1-\frac{p^2+L}{p_L^2}) \end{aligned} \tag{B.27}$$

$$\begin{aligned} S_8^{(4)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-3/2}(n^2)^{\alpha-\omega-3}(2p\cdot n)^{-(2\alpha+\beta-2\omega-3)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega-3)\Gamma(\beta+3)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-\frac{3}{2},\alpha+\frac{\beta}{2}-\omega-1;\alpha+\beta-\omega+\frac{1}{2};1-\frac{p^2+L}{p_L^2}) \end{aligned} \tag{B.28}$$

$$\begin{aligned} S_9^{(4)}(\alpha,\beta;\omega,p) &= \frac{i(-1)^{\alpha+\beta}}{(4\pi)^\omega} 4^{\alpha+\beta-\omega-2}(n^2)^{\alpha-\omega-4}(2p\cdot n)^{-(2\alpha+\beta-2\omega-4)} \\ &\times \frac{\Gamma(\alpha+\beta-\omega)\Gamma(2\alpha+\beta-2\omega-4)\Gamma(\beta+4)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+2\beta-2\omega)} \\ &\times {}_2F_1(\alpha+\frac{\beta}{2}-\omega-2,\alpha+\frac{\beta}{2}-\omega-\frac{3}{2};\alpha+\beta-\omega+\frac{1}{2};1-\frac{p^2+L}{p_L^2}) \end{aligned} \tag{B.29}$$

B.1.2 UV Divergent Parts

I now give the divergent parts of the basic noncovariant integrals used in deriving results for the alpha prescription. The parameter α has been kept finite, so these complement the results given in the appendices of the review by Leibbrandt [90]. Pole or divergent parts of integrals are now assumed from herein.

- Massless Integrals:

$$\int Dq \frac{1}{q^2(q-p)^2(q\cdot n+i\alpha)} = 0 \tag{B.30}$$

$$\int Dq \frac{q_\mu}{q^2(q-p)^2(q\cdot n+i\alpha)} = \frac{n_\mu}{n^2}\bar{I} \tag{B.31}$$

$$\int Dq \frac{q_\mu q_\nu}{q^2(q-p)^2(q\cdot n+i\alpha)} = \frac{p\cdot n+2i\alpha}{2n^2}[g_{\mu\nu} + \frac{p_{[\mu}n_{\nu]}}{p\cdot n+i\alpha} - \frac{2}{n^2}n_\mu n_\nu]\bar{I} \tag{B.32}$$

$$\begin{aligned}\int Dq \frac{q_\mu q_\nu q_\lambda}{q^2(q-p)^2(q\cdot n+i\alpha)} &= [(\frac{p\cdot n}{3}+\frac{i\alpha}{2})\frac{p_{[\mu}g_{\nu\lambda]}}{n^2} \\ &- (\frac{(p\cdot n)^2}{3} + i\alpha p\cdot n - \alpha^2 + \frac{p^2n^2}{12})\frac{n_{[\mu}g_{\nu\lambda]}}{(n^2)^2} \\ &+ \frac{1}{3n^2}p_{[\mu}p_\nu n_{\lambda]} - (\frac{2p\cdot n}{3}+i\alpha)\frac{p_{[\mu}n_\nu n_{\lambda]}}{(n^2)^2} \\ &+ (\frac{4(p\cdot n)^2}{3} + 4i\alpha p\cdot n - 4\alpha^2 + \frac{p^2n^2}{6})\frac{n_\mu n_\nu n_\lambda}{(n^2)^3}]\bar{I}\end{aligned} \tag{B.33}$$

$$\begin{aligned}\int Dq \frac{q_\mu q_\nu q_\lambda q_\rho}{q^2(q-p)^2(q\cdot n+i\alpha)} &= [-\frac{1}{6}(\frac{(p\cdot n)^3}{2} + 2i\alpha(p\cdot n)^2 - 3\alpha^2 p\cdot n \\ &- 2i\alpha^3 + \frac{p^2n^2p\cdot n}{4} + \frac{i\alpha p^2n^2}{2})\frac{g_{[\mu\nu}g_{\lambda\rho]}}{(n^2)^4} \\ &+ (\frac{p\cdot n}{4}+\frac{i\alpha}{3})\frac{g_{[\mu\nu}p_\lambda p_{\rho]}}{n^2} \\ &- (\frac{(p\cdot n)^2}{4} + \frac{2i\alpha p\cdot n}{3} - \frac{\alpha^2}{2} + \frac{p^2n^2}{24})\frac{g_{[\mu\nu}p_\lambda n_{\rho]}}{(n^2)^2} \\ &+ \frac{1}{3}((p\cdot n)^3 + 4i\alpha(p\cdot n)^2 - 6\alpha^2 p\cdot n - 4i\alpha^3 \\ &+ \frac{p^2n^2p\cdot n}{4} + \frac{i\alpha p^2n^2}{2})\frac{g_{[\mu\nu}n_\lambda n_{\rho]}}{(n^2)^3} + \frac{1}{4}\frac{p_{[\mu}p_\nu p_\lambda n_{\rho]}}{n^2} \\ &- 2(\frac{p\cdot n}{4}+\frac{i\alpha}{3})\frac{p_{[\mu}p_\nu n_\lambda n_{\rho]}}{(n^2)^2} \\ &+ ((p\cdot n)^2 + \frac{8i\alpha p\cdot n}{3} - 2\alpha^2 + \frac{p^2n^2}{12})\frac{p_{[\mu}n_\nu n_\lambda n_{\rho]}}{(n^2)^3} \\ &- (2(p\cdot n)^3 + 8i\alpha(p\cdot n)^2 - 12\alpha^2 p\cdot n - 8i\alpha^3 \\ &+ \frac{p\cdot np^2n^2}{3} + \frac{2i\alpha p^2n^2}{3})\frac{n_\mu n_\nu n_\lambda n_\rho}{(n^2)^4}]\bar{I}\end{aligned} \tag{B.34}$$

$$\begin{aligned}\int Dq \frac{q_\mu q_\nu q_\lambda}{(q-p)^2(q\cdot n+i\alpha)} &= [-\frac{2}{3}(p\cdot n+i\alpha)^3\frac{g_{[\mu\nu}p_{\lambda]}}{(n^2)^2} \\ &+ \frac{2}{3}(p\cdot n+i\alpha)^4\frac{g_{[\mu\nu}n_{\lambda]}}{(n^2)^3} + 2(p\cdot n+i\alpha)\frac{p_\mu p_\nu p_\lambda}{n^2} \\ &- 2(p\cdot n+i\alpha)^2\frac{p_{[\mu}p_\nu n_{\lambda]}}{(n^2)^2} + \frac{8}{3}(p\cdot n+i\alpha)^3\frac{p_{[\mu}n_\nu n_{\lambda]}}{(n^2)^3} \\ &- 4(p\cdot n+i\alpha)^4\frac{n_\mu n_\nu n_\lambda}{(n^2)^4}]\bar{I}\end{aligned} \tag{B.35}$$

$$\int Dq \frac{q_\mu q_\nu q_\lambda q_\rho}{(q-p)^2(q\cdot n+i\alpha)} = \frac{2}{15}(p\cdot n+i\alpha)^5\frac{g_{[\mu\nu}g_{\lambda\rho]}}{(n^2)^3}$$
$$-\frac{2}{3}(p\cdot n+i\alpha)^3\frac{g_{[\mu\nu}p_\lambda p_{\rho]}}{(n^2)^2}+\frac{2}{3}(p\cdot n+i\alpha)^4\frac{g_{[\mu\nu}p_\lambda n_{\rho]}}{(n^2)^3}$$
$$-\frac{4}{5}(p\cdot n+i\alpha)^5\frac{g_{[\mu\nu}n_\lambda n_{\rho]}}{(n^2)^4}+2(p\cdot n+i\alpha)\frac{p_\mu p_\nu p_\lambda p_\rho}{n^2}$$
$$-2(p\cdot n+i\alpha)^2\frac{p_{[\mu}p_\nu p_\lambda n_{\rho]}}{(n^2)^2}+\frac{8}{3}(p\cdot n+i\alpha)^3\frac{p_{[\mu}p_\nu n_\lambda n_{\rho]}}{(n^2)^3}$$
$$-4(p\cdot n+i\alpha)^4\frac{p_{[\mu}n_\nu n_\lambda n_{\rho]}}{(n^2)^4}+\frac{32}{5}(p\cdot n+i\alpha)^5\frac{n_\mu n_\nu n_\lambda n_\rho}{(n^2)^5}]\bar{I} \quad \text{(B.36)}$$

- Massive Integrals:

$$\int Dq \frac{1}{q^2[(q-p)^2-m^2](q\cdot n+i\alpha)} = 0 \quad \text{(B.37)}$$

$$\int Dq \frac{q_\mu}{q^2[(q-p)^2-m^2](q\cdot n+i\alpha)} = Same\ as\ result\ for\ m=0. \quad \text{(B.38)}$$

$$\int Dq \frac{q_\mu q_\nu}{q^2[(q-p)^2-m^2](q\cdot n+i\alpha)} = Same\ as\ result\ for\ m=0. \quad \text{(B.39)}$$

$$\int Dq \frac{q_\mu q_\nu q_\lambda}{q^2[(q-p)^2-m^2](q\cdot n+i\alpha)} = [(\frac{p\cdot n}{3}+\frac{i\alpha}{2})\frac{g_{[\mu\nu}p_{\lambda]}}{n^2} \quad \text{(B.40)}$$
$$-(\frac{(p\cdot n)^2}{3}+i\alpha p\cdot n-\alpha^2+\frac{p^2n^2}{12}-\frac{m^2n^2}{4})\frac{g_{[\mu\nu}n_{\lambda]}}{(n^2)^2}$$
$$+\frac{1}{3}\frac{p_{[\mu}p_\nu n_{\lambda]}}{n^2}+(4(\frac{(p\cdot n)^2}{3}+i\alpha p\cdot n-\alpha^2)$$
$$+\frac{p^2n^2}{6}-\frac{m^2n^2}{2})\frac{n_\mu n_\nu n_\lambda}{(n^2)^3}]\bar{I}$$

The following integral is useful for the non-Abelian correction to the fermion-gluon vertex:

$$\int Dq \frac{q_\mu q_\nu q_\lambda q_\rho}{q^2[(q-p)^2-m^2][(q-k)^2-m^2](q\cdot n+i\alpha)} = \quad \text{(B.41)}$$
$$[\frac{1}{2}(\frac{p\cdot n}{3!}+\frac{k\cdot n}{3!}+\frac{i\alpha}{2})\frac{g_{[\mu\nu}g_{\lambda\rho]}}{n^2}+\frac{1}{3!}\frac{1}{2}(\frac{g_{[\mu\nu}p_\lambda n_{\rho]}}{n^2}+\frac{g_{[\mu\nu}k_\lambda n_{\rho]}}{n^2})$$
$$-(\frac{p\cdot n}{3!}+\frac{k\cdot n}{3!}+\frac{i\alpha}{2})\frac{g_{[\mu\nu}n_\lambda n_{\rho]}}{(n^2)^2}$$
$$-\frac{1}{3!}(\frac{p_{[\mu}n_\nu n_\lambda n_{\rho]}}{(n^2)^2}+\frac{k_{[\mu}n_\nu n_\lambda n_{\rho]}}{(n^2)^2})$$
$$+4(\frac{p\cdot n}{3!}+\frac{k\cdot n}{3!}+\frac{i\alpha}{2})\frac{n_\mu n_\nu n_\lambda n_\rho}{(n^2)^3}]\bar{I}$$

The results for integrals with the same denominator as in the integrand of this expression, but fewer momenta in the numerator vanish - consistent with power-counting. All of these results have more general applicability than the alpha prescription.

B.2 Alpha Prescription Integrals

B.2.1 Massless Integrals

$$\int Dq \frac{q_\mu q_\nu}{q^2(q-p)^2[q.n]^2} = -[\frac{g_{\mu\nu}}{n^2} - 2\frac{n_\mu n_\nu}{(n^2)^2}]\bar{I} \tag{B.42}$$

$$\int Dq \frac{q_\mu q_\nu q_\lambda}{q^2(q-p)^2[q.n]^2} = -[\frac{p_{[\mu}g_{\nu\lambda]}}{2n^2} - p.n\frac{n_{[\mu}g_{\nu\lambda]}}{(n^2)^2} - \frac{p_{[\mu}n_\nu n_{\lambda]}}{(n^2)^2} + 4p.n\frac{n_\mu n_\nu n_\lambda}{(n^2)^3}]\bar{I} \tag{B.43}$$

$$\begin{aligned}\int Dq \frac{q_\mu q_\nu q_\lambda q_\rho}{q^2(q-p)^2[q.n]^2} = [&\frac{1}{3}((p.n)^2 - \alpha^2(n^2)^2 + p^2n^2/4)\frac{g_{[\mu\nu}g_{\lambda\rho]}}{(n^2)^2} \\ &- \frac{g_{[\mu\nu}p_\lambda p_{\rho]}}{(n^2)} + \frac{2p.n}{3}\frac{g_{[\mu\nu}p_\lambda n_{\rho]}}{(n^2)^2} \\ &- \frac{2}{3}(2(p.n)^2 - 2\alpha^2(n^2)^2 + p^2n^2/4)\frac{g_{[\mu\nu}n_\lambda n_{\rho]}}{(n^2)^3} \\ &+ \frac{2}{3}\frac{p_{[\mu}p_\nu n_\lambda n_{\rho]}}{(n^2)^2} - \frac{8p.n}{3}\frac{p_{[\mu}n_\nu n_\lambda n_{\rho]}}{(n^2)^3} \\ &+ 4(2(p.n)^2 - 2\alpha^2(n^2)^2 + p^2n^2/6)\frac{n_\mu n_\nu n_\lambda n_\rho}{(n^2)^4}]\bar{I}\end{aligned} \tag{B.44}$$

$$\int Dq \frac{q_\mu q_\nu q_\lambda q_\rho}{q^2(q-p)^2[q.n]^2[(q-p).n]^2} = [\frac{1}{3}\frac{g_{[\mu\nu}g_{\lambda\rho]}}{(n^2)^2} - \frac{4}{3}\frac{g_{[\mu\nu}n_\lambda n_{\rho]}}{(n^2)^3} + 8\frac{n_\mu n_\nu n_\lambda n_\rho}{(n^2)^4}]\bar{I} \tag{B.45}$$

$$\int Dq \frac{q_\mu q_\nu q_\lambda}{(q-p)^2[q.n]^2[(q-p).n]^2} = [\frac{2}{3}\frac{p_{[\mu}g_{\nu\lambda]}}{(n^2)^2} - \frac{4p.n}{3}\frac{n_{[\mu}g_{\nu\lambda]}}{(n^2)^3} - \frac{8}{3}\frac{p_{[\mu}n_\nu n_{\lambda]}}{(n^2)^3} + 8p.n\frac{n_\mu n_\nu n_\lambda}{(n^2)^4}]\bar{I} \tag{B.46}$$

$$\begin{aligned}\int Dq \frac{q_\mu q_\nu q_\lambda q_\rho}{(q-p)^2[q.n]^2[(q-p).n]^2} = -[&\frac{2}{15}(3(p.n)^2 - 2\alpha^2(n^2)^2)\frac{g_{[\mu\nu}g_{\lambda\rho]}}{(n^2)^3} \\ &+ \frac{2}{3}\frac{g_{[\mu\nu}p_\lambda p_{\rho]}}{(n^2)^2} - \frac{4p.n}{3}\frac{g_{[\mu\nu}p_\lambda n_{\rho]}}{(n^2)^3} \\ &+ \frac{4}{5}(3(p.n)^2 - 2\alpha^2(n^2)^2)\frac{g_{[\mu\nu}n_\lambda n_{\rho]}}{(n^2)^4} \\ &- \frac{8}{3}\frac{p_{[\mu}p_\nu n_\lambda n_{\rho]}}{(n^2)^3} + 8p.n\frac{p_{[\mu}n_\nu n_\lambda n_{\rho]}}{(n^2)^4} \\ &- \frac{32}{5}(3(p.n)^2 - 2\alpha^2(n^2)^2)\frac{n_\mu n_\nu n_\lambda n_\rho}{(n^2)^5}]\bar{I}\end{aligned} \tag{B.47}$$

B.2.2 Massive Integrals

$$\int Dq \frac{q_\mu q_\nu}{q^2((q-p)^2 - m^2)[q.n]^2} = Same\ as\ Result\ for\ m = 0 \tag{B.48}$$

$$\int Dq \frac{q_\mu q_\nu q_\lambda}{q^2((q-p)^2 - m^2)[q.n]^2} = Same\ as\ Result\ for\ m = 0 \tag{B.49}$$

$$\int Dq \frac{q_\mu q_\nu q_\lambda q_\rho}{q^2((q-p)^2 - m^2)((q-k)^2 - m^2)[q.n]^2} = -[\frac{1}{4}\frac{g_{[\mu\nu}g_{\lambda\rho]}}{n^2} - \frac{1}{2}\frac{g_{[\mu\nu}n_\lambda n_{\rho]}}{(n^2)^2} + 2\frac{n_\mu n_\nu n_\lambda n_\rho}{(n^2)^3}]\bar{I}\,. \tag{B.50}$$

These integrals suffice to generate all other required divergent parts using the reducing methods discussed in the main body of this work. One observes that replacing the mass by zero introduces no IR divergences, thus divergent parts are the same as those for the corresponding massless integrals.

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ERRATA

Page 4, line 23: "gauranteed" should be replaced by "guaranteed".

Page 9, para. 3, line 3: "irrepairable" should be replaced by "irreparable".

Page 10, para. 3, line 5: insert [125] after "Przeszowski".

Page 10, para. 3, line 10: "safegaurding" should be replaced by "safeguarding".

Page 15, after equation (2.15): replace entire line with "where $\mathcal{L}$ is given by (2.4), J^{μ} is a c-number source for the gauge field, and χ and $\bar{\chi}$ are a-number sources for the fermion".

Page 16, last para., line 5: after "dimensional regularisation[102]." insert sentence "Of course then, in any regularisation scheme other than dimensional ghosts do not decouple for the inhomogeneous axial gauge.".

Page 17, para. 4, line 5: between "the BRST transformations are a" and "subset of the gauge transformations" insert "(global)".

Page 18, section 2.4, line 2: delete comma in "define, by".

Page 19, equations (2.28) and (2.29): insert brackets "(" and ")" around the operator sum $D_i^{ab}(x)\pi_i^b(x) + g\bar{\psi}\gamma_0 T^a\psi$.

Page 32, equation (3.3): "$x - y$" in denominator should be replaced by "$\mathbf{x} - \mathbf{y}$".

Page 34, equation (3.13): "1/2" should be replaced by "$\frac{1}{2}$".

Page 37, equation (3.31): "1/2" should be replaced by "$\frac{1}{2}$".

Page 44, line 6: "gaurantee" should be replaced by "guarantee".

Page 44, para. 5, line 2: "accomodate" should be replaced by "accommodate".

Page 45, equation (4.7): insert factor g^2 in second term.

Page 45, second last para., line 11: " 'prescript' " should be replaced by "prescribe". Next line: "unprescripted" should be replaced by "unprescribed".

Page 46, two lines after equation (4.10): "unprescripted" should be replaced by "unprescribed". Two lines after equation (4.11): "prescripted" should be replaced by "prescribed".

Page 46, section 4.2.2: "specfically" should be replaced by "specifically".

Page 46, section 4.2.2, line 4: "and nor" should be replaced by "and so".

Page 50, line 7: "invariance" should be replaced by "noninvariance".

Page 54, equation (4.36): all factors "1/2" should be replaced by "$\frac{1}{2}$".

Page 57, after equation (4.76): between "and has" and "only contributions" insert "(unsurprisingly)".

Page 62, after equation (5.3): replace line with "where the number of space-time dimensions is continued from four to 2ω. The general form (5.3) is consistent with both my result, to be discussed in the next chapter,".

Page 63, two lines before section 5.3: "occurence" should be replaced by "occurrence".

Page 66, line 3: delete the sentence "The problem with ... in the result.". (The sentence only confuses and the intended meaning contributes nothing extra).

Page 67, second line after equation (5.36): "assymmetry" should be replaced by "asymmetry".

Page 73, end of first para.: add "The specific problem the generalised LM prescription suffers of relevance here is that of nonlocal momentum dependence. Green's functions in the temporal gauge with LM prescription suffer from terms with external momentum dependence $1/(p \cdot n)$. Now in the light cone gauge these consistently occur with a factor n_μ so that the unamputated graphs vanish and the renormalisation is viable[49]. In the temporal gauge this does not occur and so no renormalisation program has been developed for this gauge. The question is whether with the alpha prescription matters are any better." .

Page 76, after equations (6.14), (6.16) and (6.17): "(6.13)" should be replaced by "(6.12)".
Page 85, para. 3, line 4: delete commas in "number of, well-defined,".
Page 87, last para., line 2: "suggets" should be replaced by "suggests".
Page 93, equation (A.21): "1/2" should be replaced by "$\frac{1}{2}$".
Page 100, line 2: "herein" should be replaced by "hereon".
Page 109, reference [96]: should be replaced by "A.C. Kalloniatis, R.J. Crewther, 'Propagators for the Fully-Fixed Temporal Gauge', Adelaide University preprint ADP-92-186/T116. Submitted to Nuclear Physics B.".